Precession of Electrostatic Gyroscopes
Due to Internal Fields

A Thesis

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Pin-chieh Wang

August, 1972

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ABSTRACT

The centrifugally induced electric field within a rotating metallic sphere is considered by utilizing the correspondence of gravitational acceleration to centrifugal acceleration. We consider a spherical rotor placed in an inhomogeneous electric field specially arranged to support the weight of the sphere. The torque on the rotor due to all electrical interactions is determined. From the magnitude of the resulting precession we may predict the value of the rotationally induced electric field. The calculated torque is $\tau = \frac{5 V m \omega a^4}{108 I e d}$, where V is the potential difference of the electrodes, m, ω, a, I are, respectively, the mass, angular velocity, radius and moment of inertia of the rotor, e is the electric charge and d is the distance between the electrodes and the rotor. For a spherical shell 2 inches in diameter and spinning at 200 cycles/sec, the internal field should produce a precession of $\sim 0.2\ \widehat{\text{min}}$/hour which is observable.

Contents

CHAPTER I

INTRODUCTION

In order to determine the sign of the mass of antimatter, the acceleration of electrons and positrons in the earth's gravitational field has been studied in a drift tube by F. C. Witteborn (1). This experiment had raised the question as to whether or not a metallic shield produces an electric field that affects the falling particles due to the influence of the earth's gravitation on the metal. In 1966, Schiff and Barnhill (2) published the results of a quantum mechanical calculation of the resulting field induced by gravity in a metallic object. Based on this theory, they concluded that the field should be proportional to the gradient of the ground state energy eigenvalue of the metallic system with respect to the position of a perturbing test charge located at that field point. The resulting field they obtained was a uniform field and the magnitude was found to be $\frac{mg}{e}$, directed so as to exert an upward force on an electron. Here m and e are the electron mass and charge respectively, and g is the acceleration of gravity. Thus a gravitationally induced electric field $E \sim 10^{-12}\ V/cm$ was predicted. They concluded their theory by stating that free electrons are not expected to fall under gravity, while positrons will with acceleration $2g$, if they are placed in the interior of a hollow conducting cylinder.

In 1967, Witteborn and Fairbank (3) published the results of their experiment which purported to measure the acceleration of free electrons in a hollow conducting cylinder. Their findings were in aggrement with the result of Schiff and Barnhill. During the same year, another theoretical treatment of the same problem was presented by Dessler and Michel (4). Their theory is based on a statistical model for electrons in the metal in which their long range electrostatic interaction with a differentially compressed lattice of positive ions is taken into account. The field predicted by Dessler and Michel was in the upward direction and was estimated to be of the order of magnitude $E \approx \frac{Mg}{e}$; where M is the atomic ion mass. For the case of copper, $E \sim 10^{-8}$ V/cm. The electric field obtained was about four orders of magnitude bigger than that of Schiff. In 1968, Herring (5) showed that Schiff's treatment of lattice compression was in error and the corrected result was found to be in agreement with Dessler's prediction. However, at the same time Peshkin (6) discussed the possibility that electrons on the surface could shield the lattice ions to such an extent that the compression effect would not contribute appreciably, a result in favor of Schiff.

In 1969, Beams (7) made use of a rapidly spinning rotor to obtain centrifugal acceleration which is much in excess of that due to gravity. He found that the resulting electric field seemed to be of the order of that expected from Dessler's prediction. Later in 1970, Schiff [8] published another paper related to the same object of this gravitation-induced electric field. In that paper he resolved the question of whether or not the surface electrons provide high degree of shielding of the ions that is needed if the ionic effect is to be suppressed. Finally, he agreed with Dessler's theory and disagreed with Peshkin as well as Herring's assumption (9).

It is the purpose of this paper to calculate some effects resulting from this induced field. In particular the precession rate of the rotor of a working electrostatic gyroscope is investigated. If one assumes a correspondence of centrifugal acceleration to gravitational acceleration, there will be a charge distribution on the rotor. By considering the interaction of these charges with the suspending electric field, the resulting torque can be calculated. The resulting precession rate is then determined. This result then determines the feasibility of an experiment to measure the induced electric field.

In the second chapter a simple model of a metal under the influence of a gravitational field is discussed. It is found that this model predicts results in agreement with Dessler. In the third chapter, the correspondence of the centrifugal acceleration to the gravitational acceleration is applied. The charge distributions of both solid and shell type spherical rotors are calculated. For the case of solid sphere, the charge density inside is found to be constant, and on the surface to be quadrupolar. For the shell rotor, the charge density is found to be constant for both interior and the inside surface of the rotor. The same quadrupolar distribution is found on the outside surface. Because of this charge distribution, when the sphere is rotating in an inhomogeneous electric field, there will exist a torque on the sphere. In the fourth chapter this torque is evaluated for a geometry appropriate for operating certain electrostatic gyroscopes. When a sphere is rotating in an inhomogeneous electric field there are two kinds of charge distributions. One is inertially induced, and the other is due to electrostatic induction by the electrodes. Actually only the interaction of the induced charge with the electric field generated by the rotational charge is necessary to be considered in calculating the torque as is shown in chapter four.

Since a uniform magnetic field can be used to align the spin axis of the rotor, the equation of motion of a sphere rotating in a uniform magnetic field will be derived in chapter five, and this equation solved for the resulting motion. In the final chapter, our conclusion are presented.

CHAPTER II

GRAVITATIONALLY INDUCED ELECTRIC FIELDS IN METALS

We now discuss a very simple model which indicates the origin and the magnitude of the gravitationally induced electric field in a metal.

Suppose we have a metal in equilibrium which we visualize as consisting of two parts, these being the electrons and the ionic lattice. Let the electrons (ions) have mass m (M), charge -e (e), and density n_e (n_i). If this system is placed in a gravitational field, from the macroscopic point of view equilibrium results from the equality of body forces and pressure gradient. For each component of our model we then have

$$\left.\begin{aligned} \nabla p_i &= n_i M \vec{g} + n_i e \vec{E} \\ \nabla p_e &= n_e m \vec{g} - n_e e \vec{E} \end{aligned}\right\} . \tag{2-1}$$

Since the gas pressure is proportional to the energy density, or $p = a\epsilon n$, we have $\nabla p = a\epsilon\nabla n$. Here ϵ represents the average energy of one particle and a is an appropriate constant, namely 2/3 for electrons and 1/3 for ions [10].

Substituting this relation into Eq.(2-1), the equilibrium equation for the electrons is

$$+\left(\frac{2}{3}\right)\epsilon_e \nabla n_e - n_e m g + n_e e E = 0 \tag{2-2}$$

and for the ions

$$+\left(\frac{1}{3}\right)\epsilon_i \nabla n_i - n_i M g - n_i e E = 0. \tag{2-3}$$

Writing $\vec{g}$ and $\vec{E}$ as gradients of gravitational potential V_g and electric potential V_E respectively, Eq.(2-2) can be written

$$+\left(\frac{2}{3}\right)\epsilon_e \nabla n_e + n_e m \nabla V_g - n_e e \nabla V_E = 0 \tag{2-4}$$

which is factored to give

$$\nabla\left(+\frac{2}{3}\epsilon_e \ln n_e + m_e V_g - e V_E\right) = 0. \tag{2-5}$$

We conclude that

$$+\frac{2}{3}\epsilon_e \ln n_e + m V_g - e V_E = const. \tag{2-6}$$

Suppose when both gravitational field and electric field are zero the electron density and the ion density are both n_o. That is to say when $V_g = V_E = 0$, then $n = n_o$.
Eq.(2-6) thus becomes

$$+\frac{2}{3}\epsilon_e \ln\frac{n_e}{n_o} + m V_g - e V_e = 0. \tag{2-7}$$

In general, we expect the charge small departures from the unperturbed density so that $n_e = n_o + \delta n_e$ with $\delta n_e \ll n_o$. By using this relation and approximation $\ln(1+\frac{\delta n_e}{n_o}) \approx \frac{\delta n_e}{n_o}$, Eq.(2-7) can be simplified to give

$$\left(\frac{2}{3}\right) \epsilon_e \frac{\delta n_e}{n_o} + m V_g - e V_e = 0$$

which rearrange to give

$$\delta n_e = -\frac{3 n_o}{2 \epsilon_e} (m V_g - e V_E).$$

For the ions we obtain by simular means

$$\delta n_i = -\frac{n_o}{2 \epsilon_i} (M V_g + e V_E)$$

Since the net charge in the metal is $e\,\delta(n_i - n_e)$ we obtain from Poisson's equation

$$\nabla^2 V_E = -4\pi e (\delta n_i - \delta n_e)$$

$$= -\frac{6\pi e n_o}{\epsilon_i \epsilon_e}\left[\epsilon_i (m V_g - e V_E) - 2 \epsilon_e (M V_g + e V_E)\right].$$

The electrons will screen all positive charge except near the surfaces and in such regions we have

$$\epsilon_i (m V_g - e V_E) - 2 \epsilon_e (M V_g + e V_E) = 0$$

$$V_E = -\frac{2 M \epsilon_e + m \epsilon_i}{e \epsilon_i + 2 e \epsilon_e} V_g$$

From this equation the self-consistent induced electric field is obtained. We find

$$\vec{E} = -\frac{2M\epsilon_e + m\epsilon_i}{e\epsilon_i + 2e\epsilon_e} \vec{g} = -\frac{2M + m\frac{\epsilon_i}{\epsilon_e}}{e\frac{\epsilon_i}{\epsilon_e} + 2e} \vec{g}$$

$$\vec{E} = \frac{-2M\epsilon_e + m\epsilon_i}{e\epsilon_i + 2e\epsilon_e}\,\vec{g} = \frac{-2M + m\frac{\epsilon_i}{\epsilon_e}}{e\frac{\epsilon_i}{\epsilon_e} + 2e}\,\vec{g}$$

For common materials $\frac{m}{M} \sim 10^{-4}$, $\frac{\epsilon_i}{\epsilon_e} \sim 10^{-2}$ so $\vec{E} \approx -\frac{M\vec{g}}{e}$.

From the above discussion, it is clear that the direction of the electric field is opposite to the gravitational field and the magnitude of E is proportional to the mass of the ion. This result agree with that of Dessler. In the case of a sphere, which forms the rotor of a gyroscope, the effective gravitational field is $\vec{g} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$, where $\vec{\omega}$ is the net angular velocity of the rotor and $\vec{r}$ is the position of a point relative to the center of the sphere. So, the induced electric field inside the rotor is $\vec{E} = \frac{M}{e}\vec{\omega} \times (\vec{\omega} \times \vec{r})$.

CHAPTER III

THE CHARGE DISTRIBUTION OF THE ROTOR

The results of chapter I tell us that inside a rotating metallic sphere the electric field is given by

$$\vec{E}_< = \frac{M}{e}\,\vec{a} = \frac{M}{e}\,\vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{3-1}$$

Here $\vec{\omega}$ is the angular velocity and $\vec{r}$ is the distance from the center of the sphere to the point being considered.

If coordinates are chosen to make the spin axis parallel to the z-direction, or $\vec{\omega} = \omega\vec{\epsilon}_z$, we have

$$\vec{E}_< = \frac{M}{e}\left[(\vec{\omega}\cdot\vec{r})\,\vec{\omega} - \omega^2\vec{r}\right]$$

$$= -\frac{E_0}{a}\left(x\,\vec{\epsilon}_x + y\,\vec{\epsilon}_y\right) \tag{3-2}$$

where $E_0 = \frac{aM\omega^2}{e}$ and a is the radius of the sphere. The inside electric potential is thus found to be

$$\phi_< = \phi_0 + \frac{E_0}{2a}(x^2 + y^2) = \phi_0 + \frac{E_0}{2a}\,r^2\sin^2\theta. \tag{3-3}$$

On the other hand, since the outside potential is a solution of Laplace's Equation, and after considering the continuity of potential at the surface, the required solution of Laplace Equation can be written as

$$\phi_> = \frac{A}{r^3}\,P_2(\cos\theta) = \frac{A}{r^3}(2 - 3\sin^2\theta). \tag{3-4}$$

Here $\phi_>$ is the potential function outside of the sphere, as distinguished from the inside solution $\phi_<$. Since the potential must be continuous at the boundary surface, we may evaluate all coefficients and obtain

$$\phi_0 = -\frac{1}{3} E_0 a \quad , \qquad A = -\frac{1}{6} E_0 a^4 .$$

We than have

$$\phi_> = -\frac{E_0 a^4}{6 r^3} \left(2 - 3 \sin^2\theta \right)$$

and

$$\phi_< = -\frac{1}{3} E_0 a + \frac{E_0}{2a} r^2 \sin^2\theta .$$

From these results, the charge densities on the surface and within the sphere can be derived:

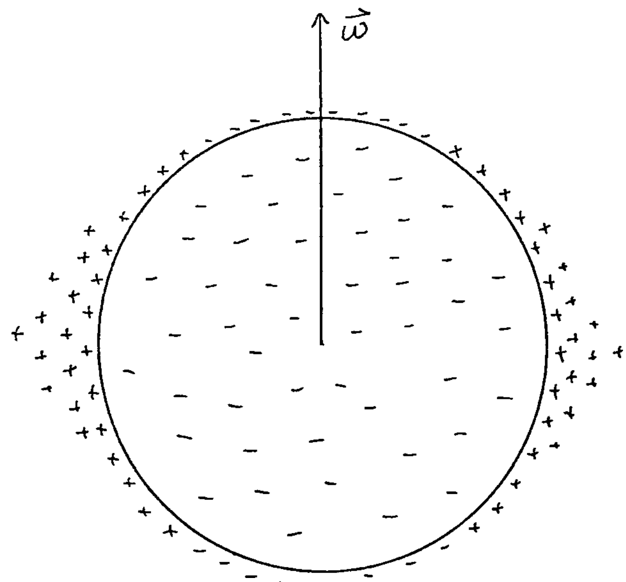
$$4\pi\sigma = \vec{n}\cdot\vec{\nabla}\left(\phi_< - \phi_>\right)\Big|_{r=a} = -E_0\left[1 - \frac{5}{2}\sin^2\theta\right] ,$$

$$\sigma = \frac{E_0}{4\pi}\left(\frac{5}{2}\sin^2\theta - 1\right). \tag{3-6}$$

The volume distribution inside the sphere is found to be

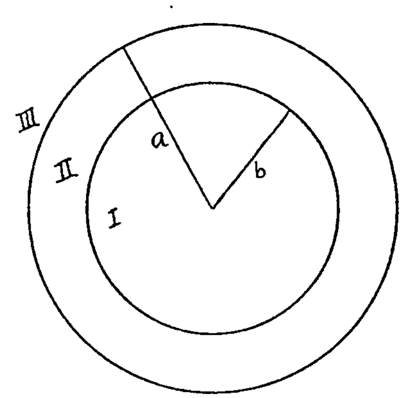
$$\rho_< = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} = -\frac{E_0}{2\pi a} . \tag{3-7}$$

From these results it is obvious that the surface charge distribution is quadrupolar while the inside volume distribution is constant, as indicated in the diagram.



In the previous paragraphs the charge distribution of a rotational solid sphere has been calculated. However in order to minimize the weight of the electrically suspended rotor, it is practical to design the rotor as a thin shell.

Consider now a metallic shell with inner radius b, and with outer radius a, as shown in the figure.



Let the shell rotate with an angular velocity $\vec{\omega}$ along the z-axis. Here, we have three potentials to be matched on the two boundary surfaces. We label these regions I, II, III.

In the II and III regions the potentials can be obtained from previous calculations:

$$\Phi_{II} = -\frac{E_0 a}{3} + \frac{E_0}{2a} r^2 \sin^2\theta \,, \tag{3-8}$$

$$\Phi_{III} = -\frac{E_0 a^4}{6 r^3} (2 - 3 \sin^2\theta). \tag{3-9}$$

Since there is no charge in region I, the potential Φ_I can be expanded in Legendre Polynomials. By comparing the order of $\sin\theta$ in Φ_{II} and by requiring the continuity of potential functions, Φ_I can be written

$$\Phi_I = \frac{E_0 (b^2 - a^2)}{3a} - \frac{E_0 r^2}{6a} (2 - 3 \sin^2\theta). \tag{3-10}$$

From the results for Φ_I and Φ_{II} we can calculate the charge density on the inner surface, namely

$$4\pi \rho_{in} = \vec{n} \cdot \nabla (\Phi_I - \Phi_{II}) \tag{3-11}$$

where $\vec{n}$ is the unit vector points radially outward. By substituting Eq.(3-8) and Eq.(3-10) into Eq.(3-11) we find

$$4\pi \rho_{in} = -\frac{E_0 b}{3a} (2 - 3 \sin^2\theta) - \frac{E_0 b}{a} \sin^2\theta$$

$$= -\frac{2E_0 b}{3a}$$

or

$$\rho_{in} = -\frac{E_0 b}{6\pi a}.$$

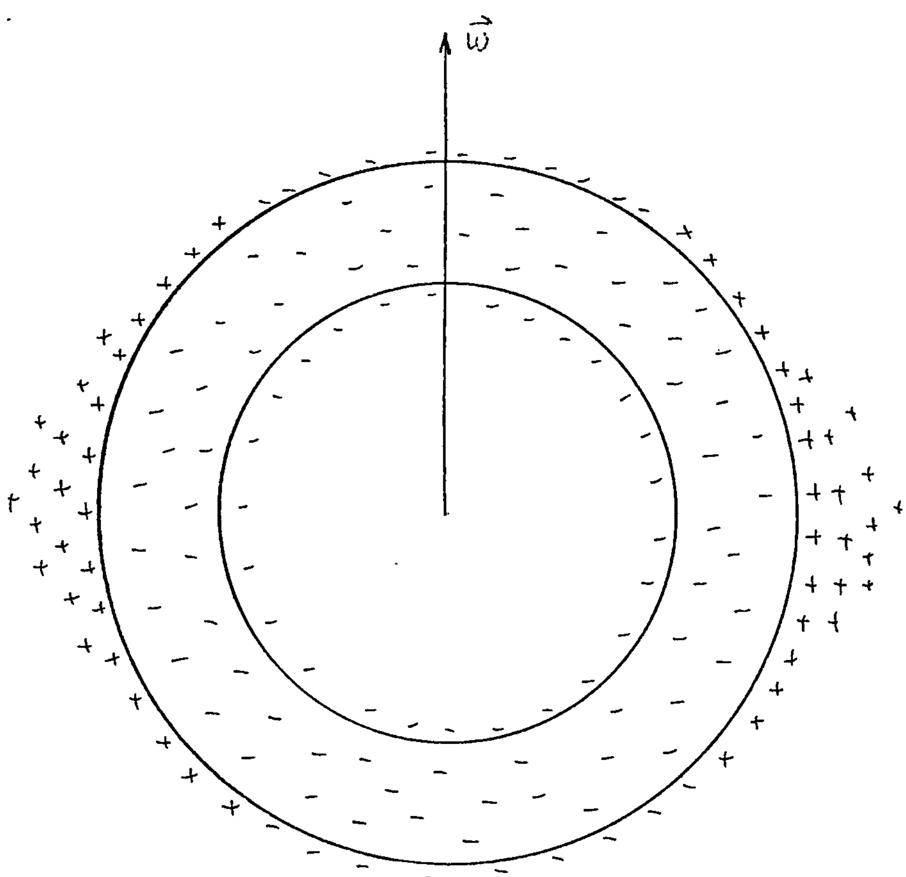
The outside charge density and volume charge density are again

$$\sigma_> = \frac{E_0}{4\pi} \left(\frac{5}{2} \sin^2\theta - 1 \right)$$

and

$$\rho = - \frac{E_0}{2 \pi b} \quad .$$

One can see that the charge on the external surface of a rotating shell is identical with that of the solid sphere and that the inner surface acquires a constant charge density, the total charge of which is exactly that of the portion of the solid sphere removed to construct the shell. The resulting charge distribution for a rotating shell is shown in the following figure.

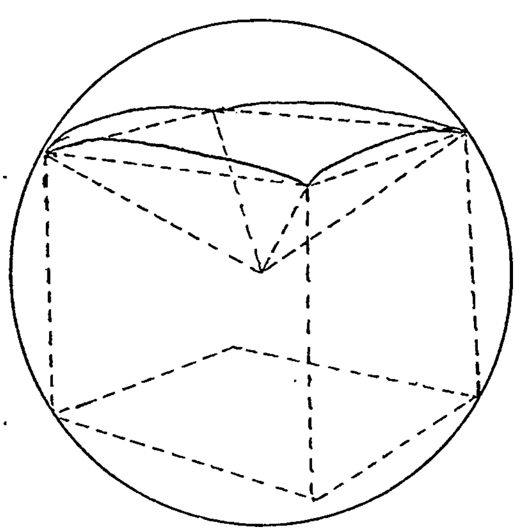


CHAPTER IV

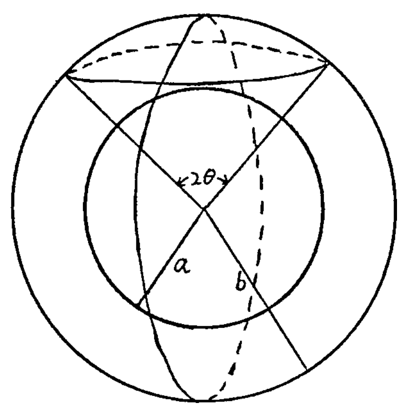
THE TORQUE ON A ROTATING SPHERE

In this chapter the torque on a rotating sphere in an external electric field is considered. We consider a field configulation appropriate for an operating gyroscope at the Jet Propulsion Laboratory for which there are six electrodes which together essentially form a complete sphere.

If a cube is circumscribed by a spherical shell as shown below, the top electrode is formed by that part of the shell which remains when cut by the four planes formed by the sphere and the top edges of the cube.



The resulting electric field can be obtained approximately by assuming that the electrode, being 1/6 the surface area of the sphere, is formed by cutting the shell with an extention angle 2θ from the center, as shown in the figure



The potential of the electrodes is adjusted such that the weight of the sphere can be balanced by the resulting electric stresses on the sphere. It is clear that only the top and the bottom electrodes need to be considered here.

For simplicity, the top electrode with constant potential V_0 is considered; assuming all others have zero potential. The potential of the region between the electrodes and the rotor, having no charge, is a solution of Laplace's Equation

$$\Phi = \sum_{\ell} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) . \tag{4-1}$$

If the rotor is zero potential, then $V=0$ when $r=a$ and

$$\Phi(a) = 0 = \sum_{\ell} \left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$\Phi(b,\theta) = V(\theta) = \sum_{\ell} \left(A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} \right) P_\ell(\cos\theta) \tag{4-2}$$

Comparing the coefficients of $P_\ell(\cos\theta)$, we have

$$A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = 0$$

$$A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} = g_\ell \equiv \frac{2\ell+1}{2} \int V(\theta) P_\ell(\cos\theta) \, d(\cos\theta) \tag{4-3}$$

Solving the above equations, we find

$$A_\ell = \frac{g_\ell}{b^\ell \left[1 - \left(\frac{a}{b}\right)^{2\ell+1} \right]}$$

$$B_\ell = - \frac{\left(\frac{a}{b}\right)^\ell a^{\ell+1}}{1 - \left(\frac{a}{b}\right)^{2\ell+1}} g_\ell$$

Substituting A_ℓ and B_ℓ into Eq.(4-1) we obtain

$$\phi = \sum_{\ell} \frac{g_\ell}{1 - \left(\frac{a}{b}\right)^{2\ell+1}} \left[\left(\frac{r}{b}\right)^\ell - \left(\frac{a}{b}\right)^\ell \left(\frac{a}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta). \tag{4-5}$$

From the potential function, we have the electric field at the surface of the sphere

$$E_n\Big|_{r=a} = -\frac{\partial \phi}{\partial r}\Big|_{r=a}$$

$$= -\frac{1}{a}\sum_{\ell} \frac{g_\ell\left(\frac{a}{b}\right)^{\ell}}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \; (2\ell+1)\; P_\ell(\cos\theta)$$

$$= 4\pi\sigma. \tag{4-6}$$

Thus the charge density induced on the surface is

$$\sigma = -\frac{1}{4\pi a}\sum_{\ell} \frac{g_\ell\left(\frac{a}{b}\right)^{\ell}}{1-\left(\frac{a}{b}\right)^{2\ell+1}} \; (2\ell+1)\; P_\ell(\cos\theta). \tag{4-7}$$

In addition to this charge there is also that resulting from rotation. The torque acting on the sphere is

$$\vec{\tau} = \int_s \sigma\, \vec{r} \times \vec{E}\; ds. \tag{4-8}$$

In which the integration is over the surface. Here the electric field $\vec{E}$ is the summation of the rotational field $\vec{E}_{rot}$ and the induced field $\vec{E}_{in}$. The surface charge density is the summation of the rotational charge and the induced charge. Thus Eq.(4-8) becomes

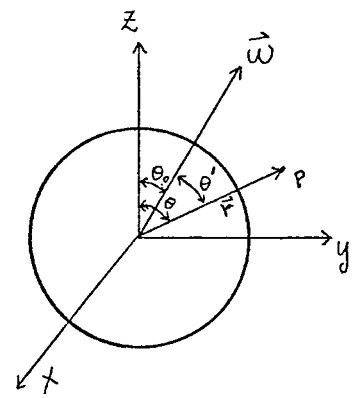
$$\vec{\tau} = \int_s \sigma\, \vec{r} \times \vec{E}\; ds = \int (\sigma_{rot} + \sigma_{in})\; \vec{r} \times (\vec{E}_{in} + \vec{E}_{rot})\; ds.$$

Since the induced electric field is radially directed at the surface, $\vec{r}\times\vec{E}_{in}$ is identically zero. Further, it is obvious that the $\vec{E}_{in}$ will not interact with σ_{in}, since a body can never make a torque on itself. The expression is thus simplified to

$$\vec{\tau} = \int \vec{r} \times (\sigma_{in}\; \vec{E}_{rot})\; ds.$$

From the results of the last chapter, the electric potential resulting from rotation is

$$\Phi = -\frac{1}{6} E_0 a \left(\frac{a}{r}\right)^3 (3\cos^2\theta' - 1) \quad ; \quad \theta' = \theta - \theta_0 .$$



Here θ_0 is the angle between the rotating axis and the z-axis and $\vec{\omega}$ is chosen to coincide with the z'-axis in the prime coordinate system.

Since $\cos\theta' = \frac{z'}{r}$, we have

$$\Phi_{rot} = -\frac{1}{6} E_0 a \left(\frac{a}{r}\right)^3 \left[3\left(\frac{z'}{r}\right)^2 - 1 \right] \tag{4-9}$$

But $z' = y\sin\theta_0 + z\cos\theta_0$ and

$$\left(\frac{z'}{r}\right)^2 = \left(\frac{y}{r}\right)^2 \sin\theta_0 + \left(\frac{z}{r}\right)^2 \cos^2\theta_0 + \frac{2yz}{r^2}\sin\theta_0\cos\theta_0$$

after substitution, Eq.(4-9) becomes

$$\Phi_{rot} = -\frac{1}{6} E_0 a \left(\frac{a}{r}\right)^3 \Big[3\sin^2\theta \sin^2\phi \sin^2\theta_0 + 3\cos^2\theta\cos^2\theta_0$$

$$+ 3\sin\theta\cos\theta\sin\phi\sin 2\theta_0 - 1 \Big] .$$

From this expression, the polar components of $\vec{E}_{rot}$ is found to be

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi_{rot}}{\partial\theta}$$

$$= E_0 \big[\sin\theta\cos\theta\sin^2\phi\sin^2\theta_0 - \sin\theta\cos\theta\cos^2\theta_0 + (\cos^2\theta - \sin^2\theta)\sin\phi\sin\theta_0\cos\theta_0 \tag{4-10}$$

while the azimuthal component is

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}$$

$$= E_0 [\sin\theta \sin\phi \cos\phi \sin^2\theta_0 + \cos\theta \cos\phi \sin\theta_0 \cos\theta_0]. \qquad (4\text{-}11)$$

Substituting E_θ, E_ϕ and Eq.(4-7) into Eq.(4-8) and transfering the reference system into cartesian coordinates by the relations

$$\vec{\epsilon}_r = \vec{\epsilon}_x \sin\theta \cos\phi + \vec{\epsilon}_y \sin\theta \sin\phi + \vec{\epsilon}_z \cos\theta$$

$$\vec{\epsilon}_\phi = -\vec{\epsilon}_x \sin\phi + \vec{\epsilon}_y \cos\phi$$

$$\vec{\epsilon}_\theta = \vec{\epsilon}_x \cos\theta \cos\phi + \vec{\epsilon}_y \cos\theta \sin\phi - \vec{\epsilon}_z \sin\theta$$

we obtain

$$\tau_x = -a \int \sigma(\theta) \, (E_\theta \sin\phi + E_\phi \cos\theta \cos\phi) \, ds \qquad (4\text{-}12)$$

$$= -a^3 E_0 \int \sigma(\theta) [\cos^2\theta - \sin^2\theta \sin^2\phi] \sin\theta_0 \cos\theta_0 \, d\Omega$$

$$= 2\pi a^3 E_0 \int d(\cos\theta) \, \sigma(\theta) \left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} \right) \sin\theta_0 \cos\theta_0$$

which may be written

$$\tau_x = -2\pi a^3 E_0 \int d(\cos\theta) \, P_2(\cos\theta) \, \sigma(\theta) \, \sin\theta_0 \cos\theta_0 . \qquad (4\text{-}13)$$

Now from Eq.(4-7), $\sigma(\theta)$ is expressed as a Legendre series of the form $\sigma_\theta = \sum_\ell \sigma_\ell P_\ell(\cos\theta)$

with
$$\sigma_\ell = -\frac{1}{4\pi a} \frac{g_\ell \left(\frac{a}{b}\right)^\ell}{1-\left(\frac{a}{b}\right)^{2\ell+1}} (2\ell+1). \tag{4-14}$$

Substituting σ into Eq.(4-13), it is seen that only $\ell=2$ gives a non-vanishing contribution, so that

$$\tau_x = -2\pi a^3 E_0 \sigma_2 \frac{2}{5} \sin\theta_0 \cos\theta_0$$

$$= -\frac{4\pi}{5} a^3 E_0 \sigma_2 \sin\theta_0 \cos\theta_0$$

$$= \frac{E_0 a^2}{2} \frac{g_2 \left(\frac{a}{b}\right)^2}{1-\left(\frac{a}{b}\right)^5} \sin 2\theta_0 .$$

But

$$g_2 = \frac{5}{2}\int V(\theta) P_2(\cos\theta)\, d(\cos\theta)$$

$$= \frac{5}{2} V_0 \int_{\theta=0}^{\theta=\alpha} \frac{1}{2}(3\cos^2\theta - 1)\, d(\cos\theta) \tag{4-15}$$

$$= -\frac{5}{4} V_0 \sin^2\alpha \cos\alpha .$$

It has been assumed that the area of one electrode is one sixth that of the sphere, hence

$$A_c = 2\pi a^2 \int_{\cos\alpha}^{1} d(\cos\alpha) = 2\pi a^2 (1-\cos\alpha) = \frac{4\pi a^2}{6} .$$

Thus
$$\cos\alpha = \frac{2}{3} \quad ; \quad \sin\alpha = \frac{\sqrt{5}}{3} .$$

Inserting these results into Eq.(4-14) and Eq.(4-15), we obtain

$$q_2 = -\frac{5}{4} V_0 \left(\frac{5}{9}\right)\left(\frac{2}{3}\right) = \frac{25}{54} V_0$$

$$\sigma_2 = -\frac{5}{4\pi a} \left(\frac{25}{54} V_0\right) \frac{\left(\frac{a}{b}\right)^2}{1-\left(\frac{a}{b}\right)^5}$$

and

$$\tau_x = \frac{25}{108} E_0 a^2 V_0 \frac{\left(\frac{a}{b}\right)^2}{1-\left(\frac{a}{b}\right)^5} \sin 2\theta_0 \quad . \tag{4-16}$$

If θ_0 is chosen to be 45°, we have for the maximum torque conditions

$$\sin 2\theta_0 = 1 .$$

We now write $b=a(1+\frac{d}{a})$ where d is the rotor-electrode separation satisfying $\frac{d}{a} \ll 1$.

We then have

$$1-\left(\frac{a}{b}\right)^5 = 1-\frac{1}{\left(1+\frac{d}{a}\right)^5} \approx 1-\left(1-\frac{5d}{a}\right) = \frac{5d}{a}$$

and

$$\frac{\left(\frac{a}{b}\right)^2}{1-\left(\frac{a}{b}\right)^5} \approx \frac{a}{5d}$$

Eq.(4-16) can now be simplified to

$$\tau_x = -\frac{5}{108} V_0 \frac{a^3 E_0}{d}$$

The precessional angular velocity then can be obtained from the dynamical law

$$\tau = I \Omega \omega$$

Where I is the moment of inertia of the shell, which is $\frac{2}{5} m \frac{r_0^5 - r_i^5}{r_0^3 - r_i^3}$; Ω is the precessional angular velocity.

From the above relations, it is clear that

$$\Omega = -\frac{5}{108} V_0 \frac{m \omega a^4}{I e d} .$$

Appropriate parameters for the JPL gyroscope are

$$m = 1.6 \times 10^{-23} \text{ gm} , \quad a = 2.54 \text{ cm} , \quad V = 108 \text{ Volts}$$

$$\omega = 200 \text{ rev/sec} , \quad a/d \sim 10^3 , \quad I = 10^3 \text{ c.g.s. Unit}$$

The magnitude of the expected precession is found to be

$$\Omega = 0.19 \ \widehat{\text{min}}/\text{hour} .$$

CHAPTER V

A ROTATING SPHERE IN A UNIFORM MAGNETIC FIELD

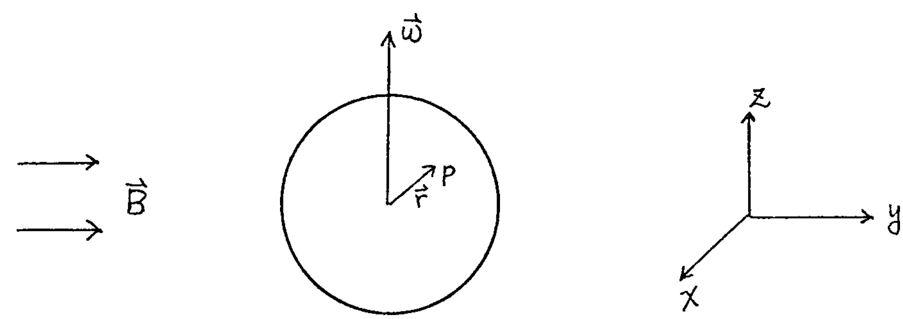
Due to the desirability in experiments to align the axis of the spinning rotor and to calibrate the magnitude of the polarized charges, a sphere rotating in a magnetic field is considered. These arise from two effects due to the magnetic field. First, the component of $\vec{B}$ parallel to the spin axis produces a charge distribution analagous to that produced by the inertial forces. It is, then, appropriate to use this effect to null out or enhance the charge as desired. Second, the component of $\vec{B}$ perpendicular to the spin axis produces eddy currents and hence torques which tend to damp the motion as well as change the spin orientation.

For example, if the field $\vec{B}$ is parallel to the spin axis, the electrons of the rotor respond to the field $\vec{E}_i$ to produce a charge distribution and hence an electrostatic field

$$\vec{E}_s = -\vec{E}_i = \frac{1}{c} \vec{B} \times (\vec{\omega} \times \vec{r})$$

Since by assumption $\vec{B} = \pm B \vec{\epsilon}_s$ we see by comparition with Eq.(3-1) that $\vec{B} = \frac{cM\vec{\omega}}{e}$ will produce identical effects to those arising from inertially produced electric fields.

For simplicity, we consider the case of the spin axis being perpendicular to the $\vec{B}$ field as shown in the figure.



When the sphere is rotating with an angular velocity $\vec{\omega}$ there are two kinds of electric fields inside the rotor. These are an induced electric field $\vec{E}_i$ and an electrostatic field $\vec{E}_s$. The induced electric field can be obtained from the Lorentz Transformation directly.

$$\vec{E}_i = \frac{1}{c}\vec{V}\times\vec{B} = \frac{1}{c}(\vec{\omega}\times\vec{r})\times\vec{B}$$

$$= \frac{1}{c}\left[(\vec{\omega}\cdot\vec{B})\vec{r} - (\vec{B}\cdot\vec{r})\vec{\omega}\right]$$

$$= -\frac{B\omega}{c}\, y\, \vec{\epsilon}_z$$

$$= -\frac{E_0 y}{a}\, \vec{\epsilon}_z \qquad (5\text{-}1)$$

where $E_0 = \frac{B\omega}{c} a$.

There is no current flow through the surface of the rotor and hence $\vec{J}\cdot\vec{n} = 0$ at the surface. Hence according to Ohm's Law, the net electric field at the surface of the rotor should lie in the tangential plane, or $(\vec{E}_i+\vec{E}_s)\cdot\vec{n}=0$. Here $\vec{n}$ is the unit vector radially outward from the surface. That is to say

$$\vec{E}_s\cdot\vec{n} = \frac{B\omega}{c}\, y\, \vec{\epsilon}_z\cdot\vec{n} = \frac{B\omega}{c}\, a \sin\theta \cos\theta \sin\phi. \tag{5-2}$$

When equilibrium is reached, the net charge inside is zero and the electrostatic potential inside is a solution of the Laplace Equation. Thus

$$V_s = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\varphi).$$

When $r=0$, V_s is finite, which implies that $B_{lm}=0$ and by comparing with Eq.(5-2) only the term $\ell=2$ is non-vanishing. Then

$$V_s = A r^2 \sin\theta \cos\theta \sin\phi = A y z$$

and (5-3)

$$E_s = -\nabla V_s = A\,(y\,\vec{\epsilon}_z + z\,\vec{\epsilon}_y)$$

By applying the condition that the normal component of $\vec{E}=\vec{E}_s+\vec{E}_i$ is zero on the surface, and assuming $A=E_0\beta$,

$$\vec{E} = \vec{E}_s + \vec{E}_i = E_0\left[\left(\beta-\frac{1}{a}\right) y\,\vec{\epsilon}_z + \beta z\,\vec{\epsilon}_y\right]$$

and

$$\vec{E}\cdot\vec{n}\Big|_{r=a} = E_0\left[\left(\beta-\frac{1}{a}\right) a \sin\theta \sin\phi \cos\theta + \beta a \sin\theta \cos\theta \sin\phi\right]=0$$

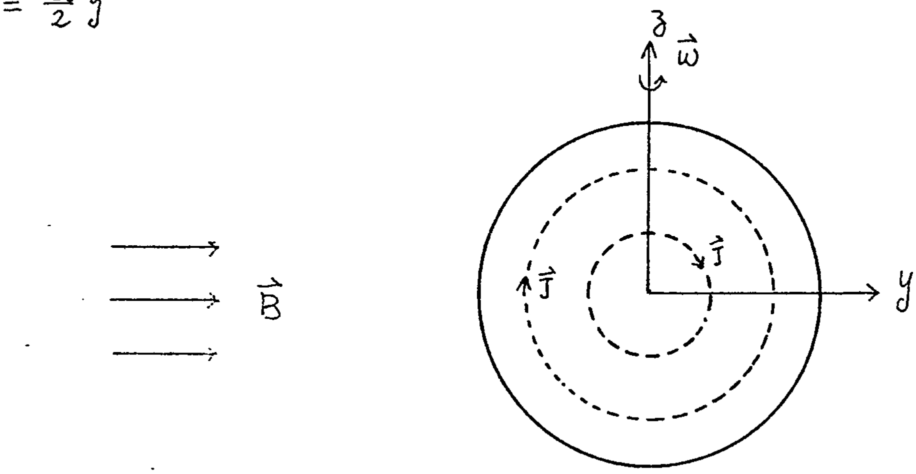
We then conclude that

$$\beta = \frac{1}{2a} \qquad \text{and} \qquad \vec{E}_s = \frac{E_0}{2a}\left[y\,\vec{\epsilon}_z + z\,\vec{\epsilon}_y\right] \tag{5-4}$$

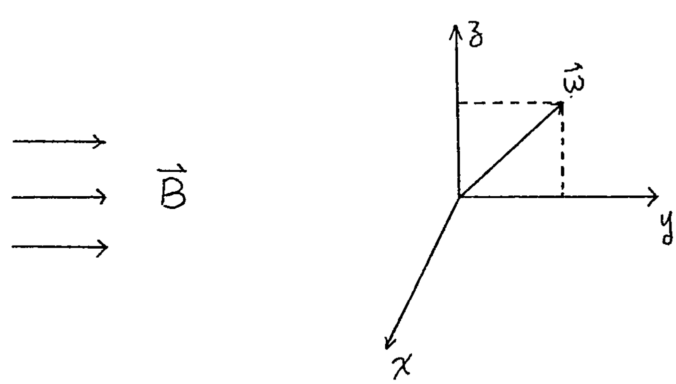
Thus, the total electric field is

$$\vec{E} = \frac{E_0}{2a}\left(z\,\vec{\epsilon}_y - y\,\vec{\epsilon}_z\right).$$

The current resulting from this field is, from Ohm's Law, $\vec{J} = \frac{J_0}{a}\left(z\,\vec{\epsilon}_y - y\,\vec{\epsilon}_z\right)$ where $J_0 = \frac{E_0}{2} g$



In general the magnetic field can be applied in an arbitrary direction. For this case we can generalize the above discussion simply by considering $\vec{B}$ along the z-direction and $\vec{\omega}$ in the y-z plane, such that $\vec{B}\times\vec{\omega}$ is still along the x-direction as shown in the figure.



In this case the current will be

$$\vec{J} = \frac{J_0}{a} \vec{r} \times (\hat{B} \times \hat{\omega}) = \frac{g}{2c} \vec{r} \times (\vec{B} \times \vec{\omega}) \tag{5-5}$$

where $\hat{B}$ and $\hat{\omega}$ are unit vectors of $\vec{B}$ and $\vec{\omega}$ respectively. The force element on the sphere can thus be calculated from the current density as

$$d\vec{F} = \frac{1}{c} \vec{J} \times \vec{B} \, d^3x = -\frac{J_0}{ac} (\vec{r} \cdot \vec{B})(\hat{\omega} \times \hat{B}) \, d^3x .$$

The net torque due to $\vec{B}$ is thus

$$\begin{aligned} \vec{\tau} &= -\frac{J_0}{ac} \int (\vec{r} \cdot \vec{B}) \vec{r} \times (\hat{\omega} \times \hat{B}) \, d^3x \\ &= -\frac{4\pi}{30} \frac{g a^5}{c^2} \vec{B} \times (\vec{\omega} \times \vec{B}) \\ &= -\frac{2\pi}{15} \frac{g a^5}{c^2} \left[(\vec{B} \cdot \vec{\omega}) \vec{B} - B^2 \vec{\omega} \right]. \end{aligned} \tag{5-6}$$

From this result the equation of motion of the rotor is

$$I \frac{d\vec{\omega}}{dt} = \frac{2\pi}{15} \frac{g a^5}{c^2} B^2 \omega \left[(\hat{B} \cdot \hat{\omega}) \hat{B} - \hat{\omega} \right] \tag{5-7}$$

which can be solved to satisfy the initial condition $\vec{\omega}(0) = \vec{\omega}_0$.

One finds

$$\omega(t) = \omega_0 \left[\hat{\omega}_0 e^{-\Omega t} + (\hat{\omega}_0 \cdot \hat{B})(1 - e^{-\Omega t}) \hat{B} \right] \tag{5-8}$$

where the damping constant Ω is given by

$$\Omega = \frac{2\pi}{15} \frac{g a^5}{I c^2} B^2 . \tag{5-9}$$

From Eq.(5-8) we can see that for sufficiently long time the angular velocity approches

$$\lim_{t\to\infty} \vec{\omega}(t) = \omega_0 (\hat{\omega}_0 \cdot \hat{B}) \hat{B}.$$

This means that the spin axis can be aligned along the external magnetic field after a characteristic time of

$$\frac{1}{\Omega} = \frac{15}{2\pi} \frac{I c^2}{g a^5 B^2},$$

CHAPTER VI

CONCLUSION

The calculations we have done above shows that the ideal case of a gyroscope rotating in a suspending electric field will be acted upon by a torque. The magnitude of the torque is proportional to the voltage of the electrodes and to the spinning velocity. Hence, the gravitationally induced precession should be easier to observe if we increase the spinning velocity as well as the voltage supply. In particular, for the JPL configulation, the precession will be increased by an factor 10 if the rotor spin could be increased from 200 cycles/sec. to 400 cycles/sec. and the field increased from 100 volts to 500 volts.

The results for the magnetic field effect shows that it is useful for calibration purposes particulary for a gyro system which is gymballed so that the spin axis remains fixed relative to an externally imposed magnetic field. Otherwise one has difficulties separating the effects of the gravitationally induced field and the magnetic field.

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