Rayleigh–Taylor instability for adiabatically stratified fluids

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A linear analysis of the effects of compressibility on the stability of two superposed isentropic fluids is presented. The results of the analysis, which differ from those available in the literature for other unperturbed stratifications, are illustrated with several numerical examples. It is found that, in the present conditions, compressibility has a stabilizing effect at small wavelengths and a destabilizing effect at long wavelengths. The magnitude of these effects is, however, small in most circumstances. A physical basis for the interpretation of the results is also described in qualitative terms. This discussion sheds some light on the nature of the differences between the present results and those relative to the case of isothermal stratification.

I. INTRODUCTION

In recent years a number of papers dealing with the effect of compressibility on the classical Rayleigh–Taylor instability have appeared in *The Physics of Fluids*.[1–4] In the study of this problem one has a great latitude in the selection of the unperturbed state, and it may be expected that different unperturbed states will have different stability characteristics. References 2 and 4 address the case in which the two fluids are both isothermally stratified. Here, we study the case of an adiabatic stratification, such as may be encountered in laser fusion, in which the density nonuniformity is produced by a compression that is too rapid to change the initial entropy distribution. Another example is that of liquids far away from the critical temperature, for which the pressure–density relationship is only weakly dependent upon the entropy.

As expected, our results are different from those of Refs. 2 and 4. The heuristic discussion of the effects of compressibility given in Sec. V enables us to clarify the origin of the differences and, more generally, to shed light on the physical effects at play in this problem.

The field equations are reduced to a confluent hypergeometric function from which the characteristic equation is obtained in closed form. The results are illustrated with a number of numerical examples, from which it appears that the overall effect of compressibility on the instability of adiabatically stratified fluids is small. Over a broad band of disturbance wavelengths, the compressible growth rates practically coincide with the incompressible ones. Only for very long wavelengths the differences can be as large as $10\% \sim 20\%$. However, the growth rate of the instability associated to these waves is extremely small compared with that of shorter waves. It may be concluded, therefore, that these long waves, where the differences between the compressible and incompressible dynamics is most marked, play at most a small role in the actual dynamics of the instability.

II. MATHEMATICAL FORMULATION

The unperturbed situation consists of two compressible, inviscid fluids at rest, separated by the horizontal plane $z = 0$. We refer to the upper and lower fluids by using the subscripts 1 and 2, respectively. In general, the fluids may be taken to satisfy an equation of state of the form $p = p(\rho,s)$, where p is the pressure, ρ is the density, and s is the entropy. In each fluid the unperturbed pressure distribution satisfies the equation

$$\nabla p = -\rho(p) g \mathbf{e}_3, \qquad (1)$$

where $\mathbf{e}_3$ is the unit vector in the vertical direction and g is the acceleration, or other constant force field, that acts in the direction $-\mathbf{e}_3$.

The equilibrium equation (1) and the equation of state are two relations between the three state variables p, ρ, and s, and therefore the unperturbed equilibrium condition is not unique. The case considered in Ref. 2 is that of two exponentially stratified fluids, which, for perfect gases, implies an isothermal configuration. Here we consider instead a situation in which the dependence of the equation of state upon s can be effectively ignored by considering fluids and transformations that are adequately described by an equation of state of the form

$$[(p+B)/(p_r+B)]^{1/\gamma} = \rho/\rho_r. \qquad (2)$$

Here ρ_r is the density at the reference pressure p_r and B and γ ($\geqslant 1$) are characteristic constants of the fluid. The use of constant values of B and γ throughout each fluid implies either an isentropic state or a negligible dependence of p on s, at least in the range of values relevant to the problem.

By taking $B = 0$ and γ equal to the ratio of the specific heats, Eq. (2) describes the pressure–density relation of a perfect gas. The choice $B = 3049.13$ bar, $\gamma = 7.15$ makes it applicable to water at normal temperatures up to hundreds of kilobar pressures. The limit $\gamma \to \infty$ reproduces the incompressible situation. The isothermal gas case is found for $\gamma = 1$, $B = 0$. Since (2) is assumed to hold both for the unperturbed and the perturbed states, in this case the evolution of the perturbation would also be isothermal. These results coincide with those of Ref. 2 for $\gamma = 1$.

If we choose as reference pressure the unperturbed pressure at the interface p_0, the solution of (1) is

$$\frac{p_i(z) + B_i}{p_0 + B_i} = \left(1 - \frac{\gamma_i - 1}{\gamma_i}\frac{\rho_i(0)}{p_0 + B_i} g z\right)^{\gamma_i/(\gamma_i - 1)}, \qquad (3)$$

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and the corresponding density distribution is

$$\frac{\rho_i(z)}{\rho_i(0)} = \left(1 - \frac{\gamma_i - 1}{\gamma_i}\frac{\rho_i(0)}{p_0 + B_i} gz\right)^{1/(\gamma_i - 1)}. \qquad (4)$$

Here and in the following the index i takes the values 1 and 2. For the particular case $\gamma_i = 1$, Eq. (3) reduces to

$$\frac{p_i(z) + B_i}{p_0 + B_i} = \exp\left(-\frac{\rho_i(0)}{p_0 + B_i} gz\right),$$

i.e., Rayleigh's "exponential-atmosphere."

It is clear from the preceding relations that the thickness h for the upper layer of fluid cannot exceed a critical value h_c, given by

$$h_c = [\gamma_1/(\gamma_1 - 1)][(p_0 + B_1)/g\rho_1(0)]. \qquad (5)$$

As γ_1 decreases from ∞ (incompressible fluid) to 1 (fluid with constant sound speed), the critical thickness h_c increases from $(p_0 + B_1)/[g\rho_1(0)]$ to ∞. Therefore, in general, the upper fluid occupies a finite layer, a complication that can be ignored in the incompressible case. In the following we shall consider both the cases of a rigid and of a free surface bounding the upper fluid at $z = h(\leqslant h_c)$.

We now introduce a small disturbance in the system and linearize around the unperturbed state. Denoting perturbed quantities by a prime, we can write the continuity equation as

$$\frac{\partial \rho_i'}{\partial t} + \nabla\cdot(\rho_i \mathbf{u}') = 0, \qquad (6)$$

the momentum equation as

$$\rho_i \frac{\partial \mathbf{u}_i'}{\partial t} = -\nabla p_i' - \rho_i' g \mathbf{e}_3, \qquad (7)$$

and the equation of state as

$$p_i' = c_i^2 \rho_i', \qquad (8)$$

where $c_i^2 = dp_i/d\rho_i$ is the square of the velocity of linear pressure waves in the ith fluid. From Eqs. (3) and (4) it follows that

$$c_i^2(z) = \gamma_i\{[p_i(z) + B_i]/\rho_i(z)\} = c_i^2(0) - (\gamma_i - 1)gz. \qquad (9)$$

Following Ref. 1, by use of the relation $\rho_i = \rho_i(p_i)$, it is possible to rewrite the continuity equation as an equation for p_i',

$$c_i^{-2}\frac{\partial^2 p_i'}{\partial t^2} - \nabla^2 p_i' - c_i^{-2} g \mathbf{e}_3\cdot\nabla p_i' + g^2 \rho_i \frac{d^2\rho_i}{dp_i^2} p_i' = 0. \qquad (10)$$

For fluids that satisfy Eq. (2), the last term, $g^2\rho_i(d^2\rho_i/dp_i^2)p_i'$ further simplifies to

$$g^2[(1-\gamma_i)/c_i^4]p_i'.$$

The kinematic boundary condition demands that at the interface the velocity fields satisfy

$$w_i'|_{z=0} = \frac{\partial \xi}{\partial t}, \qquad (11)$$

where $\xi(x,y,t)$ is the displacement of the interface from equilibrium. Furthermore, since we shall neglect surface tension for simplicity, the pressure must be continuous, so that

$$p_1'|_{z=0} + \left.\frac{dp_1}{dz}\right|_{z=0}\xi = p_2'|_{z=0} + \left.\frac{dp_2}{dz}\right|_{z=0}\xi. \qquad (12)$$

The disturbance must also vanish as z tends to $-\infty$. Finally, if the upper fluid is bounded by a solid wall, the impermeability condition applies there and

$$w_1'|_{z=h} = 0. \qquad (13)$$

In the other case we consider, that of a free surface at $z = h + \eta(x,y,t)$, we impose the kinematic condition

$$w_1'|_{z=h} = \frac{\partial \eta}{\partial t}, \qquad (14)$$

and the continuity of pressure

$$p_1'|_{z=h} + \left.\frac{dp_1}{dz}\right|_{z=h}\eta = 0. \qquad (15)$$

Since only barotropic fluids are of concern here, no loss of generality is incurred by assuming the perturbations to be irrotational, as in Ref. 5. However, since we are going to separate out the time coordinate, it is just as simple to deal with the perturbed pressure field directly, and this is the route that we shall follow.

III. DISPERSION RELATION

We seek solutions of Eq. (10) in the form

$$p_i'(\mathbf{x},t) = \hat{p}_i(z)\exp i(nt + k_x x + k_y y),$$

with similar expressions for ρ_i', $\mathbf{u}_i'$, ξ, and η. The condition of continuity of pressure at the interface reduces to

$$\hat{p}_1(0) - \rho_1(0)g\hat{\xi} = \hat{p}_2(0) - \rho_2(0)g\hat{\xi}, \qquad (16)$$

whereas from the continuity of the normal velocity at the interface it follows that

$$\hat{w}_1(0) = \hat{w}_2(0). \qquad (17)$$

The kinematic condition (11) allows us to replace the amplitude of the surface displacement $\hat{\xi}$ with $-i\hat{w}_i/n$ so that, dividing Eq. (16) by Eq. (17), we obtain

$$\rho_1(0)\left(\frac{\hat{p}_1(0)}{in\rho_1(0)\hat{w}_1(0)} + \frac{g}{n^2}\right) = \rho_2(0)\left(\frac{\hat{p}_2(0)}{in\rho_2(0)\hat{w}_2(0)} + \frac{g}{n^2}\right). \qquad (18)$$

If $in\rho_i(0)\hat{w}_i(0)$ is expressed in terms of the pressure amplitude by evaluating the vertical component of Eq. (7) at $z=0$,

$$in\rho_i(0)\hat{w}_i(0) = -\{\hat{p}_i'(0) + [g/c_i^2(0)]\hat{p}_i(0)\},$$

Eq. (18) finally becomes

$$\rho_1(0)\left(\frac{1}{l_1} - \frac{gk}{n^2}\right) = \rho_2(0)\left(\frac{1}{l_2} - \frac{gk}{n^2}\right), \qquad (19)$$

where we have introduced the modulus of the wavenumber, $k = |\mathbf{k}|$, and the two constants

$$l_i = k^{-1}[g/c_i^2(0) + \hat{p}_i'(0)/\hat{p}_i(0)]. \qquad (20)$$

This equation is the dispersion relation. For the purposes of comparing the compressible with the incompressible case, past authors[1] have recast this relation in the form

$$n^2 \doteq n_0^2\{[\rho_1(0)+\rho_2(0)]/[\rho_2(0)/l_2-\rho_1(0)/l_1]\}, \quad (21)$$

where n_0 is the incompressible Rayleigh–Taylor growth rate for two unbounded fluids given by[6]

$$n_0^2 = \{[\rho_2(0)-\rho_1(0)]/[\rho_2(0)+\rho_1(0)]\}gk.$$

Although, as will be argued in the next section, a different scaling seems to better capture the difference between the two cases, we keep the equation in this form for the time being.

A. Evaluation of the l's. To determine the two constants l_1 and l_2, we solve the continuity equation (10), which, upon introduction of the normal modes, yields

$$\frac{d^2\hat{p}_i}{dz^2}+\frac{g}{c_i^2}\frac{d\hat{p}_i}{dz}+\left(\frac{n^2}{c_i^2}-k^2+g^2\frac{\gamma_i-1}{c_i^4}\right)\hat{p}_i=0. \quad (22)$$

By making the change of variable

$$s_i(z) = -2k\{z-c_i^2(0)/[g(\gamma_i-1)]\}$$
$$=2kc_i^2(z)/g(\gamma_i-1), \quad (23)$$

and introducing the transformation

$$\hat{p}_i|_{z=z(s_i)} = e^{-(1/2)s_i}s_i^{1/(\gamma_i-1)}f_i(s_i), \quad (24)$$

Eq. (22) becomes the confluent hypergeometric equation[7]

$$s_i\frac{d^2f_i}{ds_i^2}+\left(\frac{1}{\gamma_i-1}-s_i\right)\frac{df_i}{ds_i}-\frac{1}{2(\gamma_i-1)}\left(1-\frac{n^2}{gk}\right)f_i=0. \quad (25)$$

With the definitions

$$a_i=(b_i/2)(1-n^2/gk), \quad b_i=1/(\gamma_i-1), \quad (26)$$

the general solution of (25) is

$$f_i(s_i)=C_i\Phi(a_i,b_i,s_i)+D_i\Psi(a_i,b_i,s_i), \quad (27)$$

where Φ and Ψ are the two confluent hypergeometric functions and C_i and D_i are integration constants. The change of variable (23) maps the intervals

$$-\infty<z\leqslant 0, \quad 0\leqslant z\leqslant h,$$

into the intervals

$$\infty>s_2(z)\geqslant 2kc_2^2(0)/g(\gamma_2-1),$$
$$2kh_c\geqslant s_1(z)\geqslant 2k(h_c-h),$$

respectively, where h_c is the critical thickness previously introduced. Since $\gamma_i>1$ and $h\leqslant h_c$, s_1 and s_2 are always nonnegative.

In terms of the functions f_i, the l's are

$$l_i=1-2\{f_i'[s_i(0)]/f_i[s_i(0)]\}, \quad (28)$$

where use has been made of the expression for $d\hat{p}_i/dz$ at $z=z(s_i)$, given in the Appendix.

To complete the derivation we impose the conditions at the boundaries. The details of the calculations can be found in the Appendix. When the thickness of the upper layer is strictly less than h_c, the factor l_1 is equal to

$$l_1=1-2\frac{\Phi'(a_1,b_1,s_1(0))+Q\Psi'(a_1,b_1,s_1(0))}{\Phi(a_1,b_1,s_1(0))+Q\Psi(a_1,b_1,s_1(0))}, \quad (29)$$

where Q is defined as

$$Q=-\frac{R\Phi(a_1,b_1,s_1(h))-2\Phi'(a_1,b_1,s_1,(h))}{R\Psi(a_1,b_1,s_1(h))-2\Psi'(a_1,b_1,s_1(h))}. \quad (30)$$

When a solid wall bounds the upper layer of fluid, $R=1$. In the case of a free surface, however, $R=1-n^2/gk$. In the particular case in which the thickness h equals h_c, Eq. (29) reduces to

$$l_1=1-2\frac{\Phi'(a_1,b_1,s_1(0))}{\Phi(a_1,b_1,s_1(0))}. \quad (31)$$

The other factor l_2 is always equal to

$$l_2=1-2\frac{\Psi'(a_2,b_2,s_2(0))}{\Psi(a_2,b_2,s_2(0))}. \quad (32)$$

From the expressions given it is clear that the dispersion relation (19) is an implicit equation for the growth rate because l_1 and l_2 depend on n^2 through the parameters a_1, a_2, and R. Therefore it is not obvious that the roots n^2 will be found to be real. We consider this point in the Appendix, where it is shown on the basis of the self-adjointness of Eq. (25) that one can expect real values of n^2. This conclusion is supported by the numerical evidence to be presented in the next section.

It is shown in the Appendix that the dispersion relation (19) reduces to the known forms in the case of incompressible fluids and of fluids with a constant sound speed. In particular, in the latter case, in which both γ_1 and $\gamma_2\to 1$, with $h=h_c\to\infty$, one finds from (31) and (32),

$$l_1=-\left(1-\frac{n^2}{k^2c_1^2(0)}+\frac{g^2}{4k^2c_1^4(0)}\right)^{1/2}+\frac{g}{2kc_1^2(0)}, \quad (33)$$

$$l_2=\left(1-\frac{n^2}{k^2c_2^2(0)}+\frac{g^2}{4k^2c_2^4(0)}\right)^{1/2}+\frac{g}{2kc_2^2(0)}. \quad (34)$$

IV. RESULTS

The dispersion relation (19) is very complex and it is therefore useful to present some numerical results. Their calculation is not entirely trivial and is briefly explained in the Appendix.

The present system depends upon eight different parameters, namely, p_0, $\rho_1(0)$, $\rho_2(0)$, γ_1, γ_2, B_1, B_2, and h, and an exhaustive numerical study would be quite complicated and, in the end, of limited use. It is more interesting to try to bring out the basic effects of compressibility, as compared with the incompressible situation. This will enable us to understand the fundamental physics of the process, which we describe in qualitative terms in Sec. V.

In his study of compressibility effects for fluids with a constant speed of sound, Baker[3] found that (a) the most pronounced effect of compressibility is on the long wavelengths; (b) the instability is enhanced when the lighter fluid is sufficiently more compressible than the heavier one; and (c) the effect is reduced by large density differences.

Our study essentially confirms the last two conclusions for a larger class of fluids and demonstrates the effects of the polytropic index γ. The first conclusion is also confirmed, but to a smaller extent and for different reasons than in Ref. 3, as will be seen later on. Thus, in spite of a questionable aspect of his work,[8,9] Baker's findings are substantially correct.

In order to reduce the number of parameters we shall limit ourselves to gases, which, being much more compressible than liquids, should exhibit a maximum of compressibility effects. In this case $B_i = 0$ and $1 \leqslant \gamma_i < \frac{5}{3}$. In the presentation of the results we shall use the unit of length

$$L = p_0 / g\rho_1(0), \tag{35}$$

which is the limit value of the critical thickness h_c in the incompressible limit $\gamma_1 \to \infty$. Some comments on the physical meaning of this length will be made in Sec. V. Use of L eliminates the explicit dependence of the results upon the value of the pressure at the interface and the individual values of the interface densities that in the following will only enter through the ratio:

$$\epsilon = \rho_2(0)/\rho_1(0), \tag{36}$$

in terms of which the Atwood number may be written

$$\mathrm{At} = (1-\epsilon)/(1+\epsilon).$$

Further, we have

$$a_i = \tfrac{1}{2} b_i (1 - n^2/gk)$$

and

$$s_1(0) = \frac{4\pi}{\lambda^*}\left(\frac{\gamma_1}{\gamma_1 - 1}\right), \quad s_2(0) = \epsilon^{-1}\frac{4\pi}{\lambda^*}\left(\frac{\gamma_2}{\gamma_2 - 1}\right),$$

$$s_1(h) = \frac{4\pi}{\lambda^*}\left(\frac{\gamma_1}{\gamma_1 - 1} - h^*\right),$$

where $\lambda = 2\pi/k$ is the wavelength of the surface disturbance and the asterisk is appended to lengths nondimensionalized with respect to L. In order to further reduce the parameter space we shall only consider the case in which the thickness h of the fluid layer equals the critical value h_c. With this specification the quantity l_1 does not depend on h anymore and the dispersion relation takes the form

$$n^2/gk = G(n^2/gk, \lambda^*; \epsilon, \gamma_1, \gamma_2). \tag{37}$$

In earlier studies the values of the growth rate squared n^2 for the compressible case were contrasted with the square of the growth rate — At gk, valid in the case of two unbounded incompressible fluids. This procedure is not very meaningful since, in certain ranges of the perturbation wavelength, the fact that one is dealing with a finite layer of fluid rather than an infinite one is much more significant than compressibility. Therefore we shall show graphs of the growth rate divided by the incompressible growth rate for the case in which the upper fluid has a thickness L. *A priori*, this still leaves the two possibilities of an upper fluid with a rigid upper boundary, for which the growth rate is

$$n_w^2 = \frac{\rho_2(0) - \rho_1(0)}{\rho_1(0)\coth(kL) + \rho_2(0)}\, gk, \tag{38}$$

or with a free boundary, for which the growth rate is[10]

$$n_f^2 = \frac{\rho_2(0) - \rho_1(0)}{\rho_1(0) + \rho_2(0)\coth(kL)}\, gk. \tag{39}$$

We show in the Appendix that, if the incompressible limit $\gamma_1 \to \infty$ is taken on the compressible result maintaining $h = h_c$, $n \to n_f$ rather than n_w. This happens because, for $h = h_c$, the pressure condition at the upper boundary is automatically satisfied while the normal velocity is different from zero. These are the correct boundary conditions for a free surface. However, in these particular conditions, the appropriate boundary conditions for a rigid surface are also satisfied because the density and, therefore, the normal mass flux, also vanish. In order to identify more easily the specific effects of compressibility, it is therefore appropriate to present our results in terms of n^2/n_f^2. For smaller values of h and a rigid upper boundary, (38) may be a more meaningful reference.

To investigate the dependence of n on γ_1 and γ_2 we consider the values 1, $\frac{7}{5}$ (diatomic gases), $\frac{5}{3}$ (monatomic gases), and ∞ (incompressible limit). Since, according to Baker,[3] compressibility effects are larger for ϵ close to unity, we take $\epsilon = 0.9$ first.

Figure 1 shows the square of the growth rate normalized by the incompressible value (39) as a function of the wavelength of the surface perturbation. This figure refers to the limit case in which one of the two fluids is incompressible. It is clear that the fully incompressible results, as given by n_f, are approached both at long and short wavelengths. Figure 2 is similar, but now both fluids are compressible with $\gamma_1 = \gamma_2$. The curve for $\gamma_1 = \gamma_2 = 1$ corresponds to the case analyzed by Baker.[3] Finally, in Fig. 3, we show the growth rate for different finite values of γ_1 and γ_2.

If the growth rate were scaled by the unbounded, incompressible value n_0^2, as done in the past,[2,3] all the previous figures would have shown the ratio of the compressible and incompressible results to tend to zero for large λ. It is obvious that this behavior, far from being an effect of compressibility, is only a consequence of the finiteness of the upper fluid layer. In this connection it is perhaps remarkable that n^2 is reasonably close to n_f^2, even when the upper layer has an infinite thickness ($\gamma_1 = 1$). Evidently the important effect is the mass of the column of fluid 1 over a unit area of the interface, which is finite and equal to $\rho_1(0)\,L$ for each of the distributions (4). This mass is, of course, infinite for the classical configuration corresponding to n_0^2.

As could be anticipated, for very short waves compressibility is insignificant. With growing wavelength, it is seen

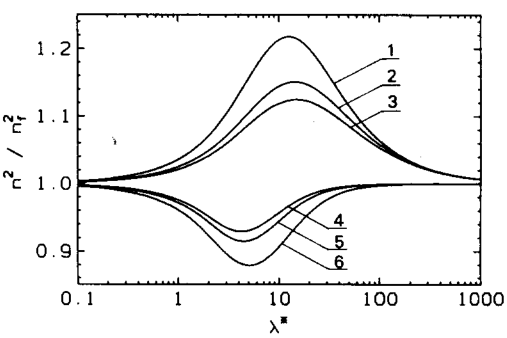

FIG. 1. Square of the growth rate normalized by the incompressible value (36) versus nondimensional disturbance wavelength λ^* for density ratio $\epsilon = 0.9$. The values of the adiabatic indices (γ_1, γ_2) for the curves 1–6 are, respectively, $(\infty, 1)$, $(\infty, \frac{7}{5})$, $(\infty, \frac{5}{3})$, $(\frac{5}{3}, \infty)$, $(\frac{7}{5}, \infty)$, $(1, \infty)$.

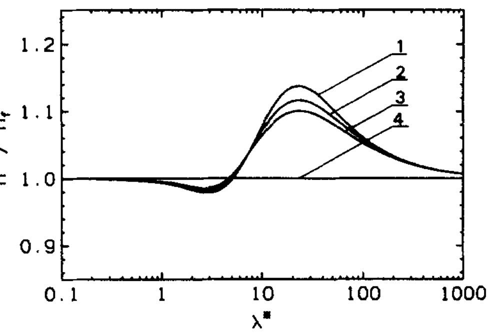

FIG. 2. Square of the growth rate normalized by the incompressible value (36) versus nondimensional disturbance wavelength λ^* for density ratio $\epsilon = 0.9$. The values of the adiabatic indices (γ_1, γ_2) for the curves 1–4 are, respectively, $(1,1)$ $(\frac{7}{5},\frac{7}{5})$, $(\frac{5}{3},\frac{5}{3})$, (∞,∞).

from Figs. 2 and 3 that compressibility has first a stabilizing effect and then a destabilizing one. This circumstance indicates that the result is determined by two opposing effects of compressibility, one stabilizing and one destabilizing. We shall return on this question in Sec. V. As the difference in the compressibilities of the two fluids becomes larger and larger, either the stabilizing or the destabilizing action tends to disappear. In particular, when the lower fluid is sufficiently more compressible than the upper one ($\gamma_2 < \gamma_1$) the growth rate curve shows only a maximum and the instability is enhanced at any wavelength. In the opposite case of lower fluid less compressible ($\gamma_1 < \gamma_2$) the value of the maximum becomes closer and closer to unity as γ_2 increases. Eventually, for $\gamma_2 \gg \gamma_1$, it disappears and compressibility exerts only a stabilizing action, as shown by curves 4–6 in Fig. 1. Finally, by comparing Figs. 1 and 2, it is seen that the effect on the growth rate is bigger when the two fluids have large differences in their compressibilities than when both are highly compressible.

Figures 4 and 5 are similar to Fig. 1 and Figs. 6 and 7 to Fig. 3, respectively. The density ratio for Figs. 4 and 6 is 0.1 and for Figs. 5 and 7 is 0.5. It is evident that the effect of compressibility decreases as the density ratio decreases. Furthermore, as ϵ approaches zero, the extremal values of the ratio are attained at longer and longer wavelengths while the deviations from 1 get smaller and smaller.

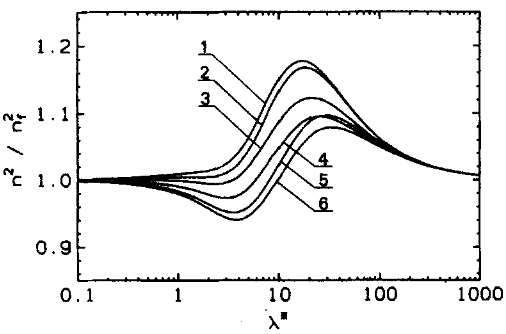

FIG. 3. Square of the growth rate normalized by the incompressible value (36) versus nondimensional disturbance wavelength λ^* for density ratio $\epsilon = 0.9$. The values of the adiabatic indices (γ_1,γ_2) for the curves 1–6 are, respectively, $(\frac{5}{3},1)$, $(\frac{7}{5},1)$, $(\frac{5}{3},\frac{7}{5})$, $(\frac{7}{5},\frac{5}{3})$, $(1,\frac{7}{5})$, $(1,\frac{5}{3})$.

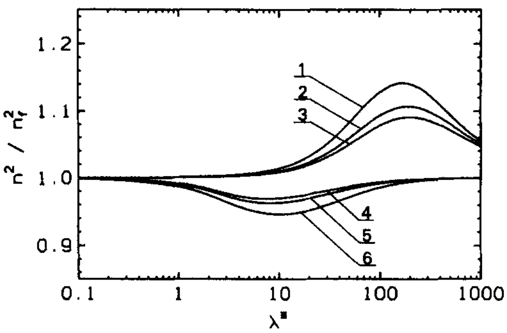

FIG. 4. Same as Fig. 1, but with density ratio $\epsilon = 0.1$.

V. PHYSICAL BASIS FOR THE EFFECT OF COMPRESSIBILITY

It is now possible to interpret qualitatively the results that have been discussed in Sec. IV. First of all, let us note that in the present problem in each fluid there are two length scales that are absent in the incompressible case with constant densities. The first one is a characteristic length resulting from the stratification and is given by

$$L_i = -\rho_i(0)\left(\frac{d\rho_i(0)}{dz}\right)^{-1} = -\rho_i(0)c_i^2(0)\left(\frac{dp_i(0)}{dz}\right)^{-1} = \frac{c_i^2(0)}{g},$$

where use has been made of Eq. (9) and the equilibrium condition (1). Up to factors of γ_1, L_1 is the length already introduced in Eq. (35), Sec. IV. A second length, characteristic of compressibility, is the distance Λ_i traveled by pressure perturbations during the characteristic time $|n|^{-1}$ and given by

$$\Lambda_i = c_i(0)/|n|.$$

If the wavelength of the surface disturbance is small compared with both of these lengths, the effects of compressibility will be negligible. In this case the estimate $n^2 \simeq -\mathrm{At}\, gk$ is applicable and we find

$$kL_i = kc_i^2/g, \quad k\Lambda_i = c_i(k/\mathrm{At}\, g)^{1/2}. \tag{40}$$

Since in practical applications At is not many orders of magnitude smaller than 1, of the two, the most stringent condition that ensures small compressibility effects is

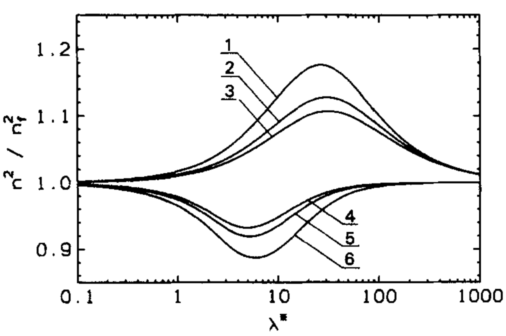

FIG. 5. Same as Fig. 1, but with density ratio $\epsilon = 0.5$.

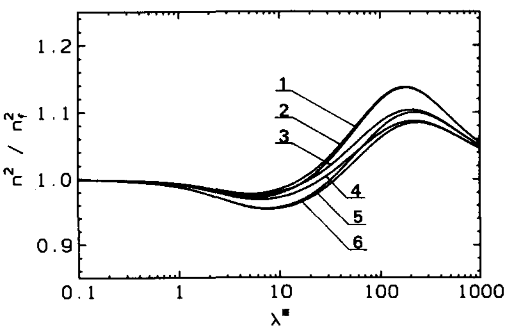

FIG. 6. Same as Fig. 3, but with density ratio $\epsilon = 0.1$.

$(kL_i)^{1/2} \gg 1$.

In order to understand the physical mechanisms through which compressibility affects the growth of disturbances, it is useful to note that the dispersion relation (19) is equivalent to the following equation of motion for the interface $z = \xi$:

$$[\rho_1(0)d_1 + \rho_2(0)d_2]\ddot{\xi} = [\rho_1(0) - \rho_2(0)]g\xi. \quad (41)$$

In the unbounded, incompressible case with the constant densities, $d_1 = d_2 = k^{-1}$ and the corresponding equation is

$$(1/k)[\rho_1(0) + \rho_2(0)]\ddot{\xi} = [\rho_1(0) - \rho_2(0)]g\xi. \quad (42)$$

A simple physical interpretation of this equation can be given in the following terms. Consider a wave of amplitude ξ. Per unit area, the fluid in its crest (which is lower fluid) has a weight of order $\rho_2(0)g\xi$ and is subject to a buoyancy force of the order of $\rho_1(0)g\xi$. The difference between these two terms accounts for the "force" acting on the wave, i.e., the right-hand side of this equation. One might say that this destabilizing action results from the *substitution* of a heavy fluid particle, of density ρ_1, with a lighter one of density ρ_2. As for the *effective inertia* of the wave (left-hand side of the equation) we note that the amplitude of the perturbation decreases exponentially at a rate k, so that only a depth of fluid of the order of k^{-1} is actually affected by the wave. The mass of this amount of fluid per unit area is therefore of the order of density $\times k^{-1}$ for the upper and lower fluids. This argument reduces Eq. (42) to a simple dynamical statement.

In the compressible case, the above substitution effect, responsible for the buoyancy force on the right-hand side of Eq. (41), remains the main destabilizing factor. Now, however, we must take into account a second contribution of buoyancy because of the fact that, throughout the fluids, particles with density $\rho + \rho'$ take the place of particles with density ρ. Three concurrent factors determine the density disturbance ρ'. Consider, for example, the fluid above and below a rising crest. This fluid comes from layers that are denser than the surroundings. Therefore, it brings a positive contribution to ρ', which is, at the first order,

$$-\frac{d\rho}{dz}\xi = -\left(\frac{d\rho}{dp}\right)_0 \frac{dp}{dz}\xi,$$

where the subscript 0 indicates that the derivative is taken in the unperturbed state. However, as the fluid moves through lighter layers, it expands because of the lower unperturbed pressure field. This second factor brings a negative contribution to ρ', which is

$$\left(\frac{d\rho}{dp}\right)_{\text{dynamic}} \frac{dp}{dz}\xi.$$

Now the derivative $d\rho/dp$ is taken following the particle motion and need not be the same as the one appearing in the previous equation because density and pressure in the base state do not need to satisfy the same equation of state satisfied by the dynamically evolving perturbation fields. For example, in the case of exponentially stratified fluids considered in Ref. 2, the net effect is

$$-\left(1 - \frac{1}{\gamma}\right)\frac{d\rho}{dz}\xi \gg 0,$$

and therefore it is always stabilizing. It evidently is maximum for $\gamma \to \infty$, i.e., the incompressible limit. This remark is at the basis of the result of Refs. 2 and 4 on the destabilizing effects of compressibility. In the case considered here, on the other hand, the equation of state (2) holds both for the unperturbed and the perturbed states, and the two contributions cancel each other. In this case the net density disturbance is due to a third contribution to ρ', which has the following origin. When a crest accelerates upward in an interval of time of order $|n|^{-1}$, it compresses the upper fluid above it up to a distance of order Λ_1. This leads to an increase in density, a restoring buoyancy force, and a retarding effect on the growth of the disturbance. On the other hand, an expansion wave travels downward in the lower fluid to a depth of order Λ_2 below the crest. The lower fluid thus expands, ρ' becomes negative, and the local buoyancy force pushes the fluid particles further up, enhancing the disturbance. From this argument it follows that, for what concerns this third effect, compressibility of the upper fluid tends to stabilize the system, while compressibility of the lower fluid plays a destabilizing role. This conclusion is clearly illustrated by Figs. 1, 4 and 5.

In view of the multiplicity of factors determining the density perturbation ρ', one may expect that, in general, at a given wavelength, the net effect on the stability character of the base state will depend on its detailed structure and on the dynamics of the disturbances.

Let us now put on a quantitative basis the previous argument for the particular base state of present concern. For the sake of simplicity, we consider fluids with a constant sound velocity. Equation (22) yields disturbances that decrease ex-

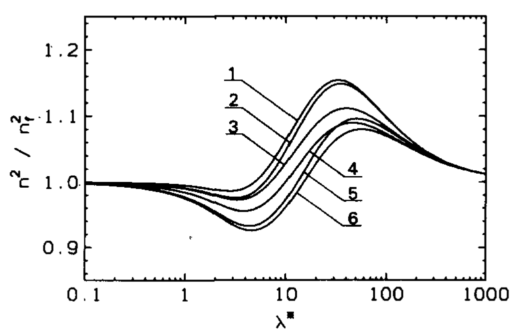

FIG. 7. Same as Fig. 3, but with density ratio $\epsilon = 0.5$.

ponentially with the distance from the interface at a rate D_i^{-1}. The distance D_i depends on the length scales of the problem as

$$D_1 = \left[\left(k^2 + \frac{1}{\Lambda_1^2} + \frac{1}{4L_1^2}\right)^{1/2} + \frac{1}{2L_1}\right]^{-1}, \quad (43)$$

$$D_2 = \left[\left(k^2 + \frac{1}{\Lambda_2^2} + \frac{1}{4L_2^2}\right)^{1/2} - \frac{1}{2L_2}\right]^{-1}. \quad (44)$$

Since we deal with the unstable configuration, from the vertical component of Eq. (7) evaluated at the interface,

$$\rho_i(0)\ddot{\xi} = -\left(\frac{\partial p_i'}{\partial z} + \frac{g}{c_i^2}p_i'\right)_{z=0}, \quad (45)$$

it follows that for a point on an upward accelerating crest $\ddot{\xi}$ is positive so that

$$\left(\frac{\partial p_1'}{\partial z} + \frac{g}{c_1^2}p_1'\right)_{z=0} = -\frac{1}{d_1}p_1'|_{z=0} < 0, \quad (46)$$

$$\left(\frac{\partial p_2'}{\partial z} + \frac{g}{c_2^2}p_2'\right)_{z=0} = \frac{1}{d_2}p_2'|_{z=0} < 0, \quad (47)$$

where d_1 and d_2 are defined by

$$d_1 = \frac{1}{D_1^{-1} - L_1^{-1}} = \left[\left(k^2 + \frac{1}{\Lambda_1^2} + \frac{1}{4L_1^2}\right)^{1/2} - \frac{1}{2L_1}\right]^{-1}, \quad (48)$$

$$d_2 = \frac{1}{D_2^{-1} + L_2^{-1}} = \left[\left(k^2 + \frac{1}{\Lambda_2^2} + \frac{1}{4L_2^2}\right)^{1/2} + \frac{1}{2L_2}\right]^{-1}. \quad (49)$$

It is apparent from Eqs. (46) and (47) that above a crest in the upper fluid the pressure and the density disturbances are positive while they are negative in the lower fluid, as expected. Since now the masses of fluid subject to an appreciable motion have depth D_i, we can write for the dynamical balance per unit horizontal area

$$[\rho_1(0)D_1 + \rho_2(0)D_2]\ddot{\xi} = [\rho_1(0) - \rho_2(0)]g\xi - \left(\frac{D_1}{L_1}p_1'|_{z=0} + \frac{D_2}{L_2}p_2'|_{z=0}\right). \quad (50)$$

If the values of the pressure disturbance on the two sides of the interface are expressed in terms of $\rho_i(0)$ and $\ddot{\xi}$ by means of Eqs. (45)–(49), the balance takes the form

$$[\rho_1(0)D_1 + \rho_2(0)D_2]\ddot{\xi} = [\rho_1(0) - \rho_2(0)]g\xi + [\rho_1(0)(D_1 - d_1) + \rho_2(0)(D_2 - d_2)]\ddot{\xi}, \quad (51)$$

which immediately yields Eq. (41).

When the disturbance wavelength is much smaller than the other two scales, by (40), d_1 and d_2 equal $k^{-1}(1 + 1/2kL_1)$ and $k^{-1}(1 - 1/2kL_2)$ up to terms of second order in $(kL_i)^{-1}$. For a given surface disturbance ξ, the force on the right-hand side of Eq. (41) is the same in the compressible and in the incompressible case. Therefore the ratio between the accelerations, which equals the ratio between the growth rates squared, is of the order

$$\frac{n^2}{n_0^2} \sim \frac{\rho_1(0) + \rho_2(0)}{\rho_1(0)(1 + 1/2kL_1) + \rho_2(0)(1 - 1/2kL_2)}. \quad (52)$$

Consider the configuration in which the upper fluid is compressible and the lower fluid is incompressible. Then $L_2 \to \infty$ and this relation may be rewritten as

$$n^2/n_0^2 \sim 1 - \rho_1(0)/2[\rho_1(0) + \rho_2(0)]kL_1 < 1, \quad (53)$$

which explicitly shows that compressibility of the upper fluid plays a stabilizing effect. As shown in Figs. 1, 4, and 5 the opposite configuration is more unstable than the incompressible case. Indeed, in this case Eq. (52) leads to

$$n^2/n_0^2 \sim 1 + \rho_2(0)/2[\rho_1(0) + \rho_2(0)]kL_2 > 1. \quad (54)$$

Since $L_2 = \epsilon^{-1}L_1$, the correction in the second case is ϵ^2 times smaller than the correction in the first case. This is why there is a net stabilizing effect at short wavelengths when both fluids are compressible, as shown by Fig. 2.

Let us now turn to the range of long disturbance wavelengths, $kL_i \ll 1$. In the compressible case the upper fluid has finite thickness equal to L_1. In writing the equation of motion for the interface the "inertia of the wave" is no longer $[\rho_1(0) + \rho_2(0)]k^{-1}\ddot{\xi}$, because the velocity profile depends on the type of upper boundary and does not decay exponentially. The easiest way to obtain an estimate of the "inertia" is to take the limit of the exact solution. Up to terms of the second order in kL_i, from Eq. (39) one finds

$$[\rho_1(0)d_1 + \rho_2(0)d_2]\ddot{\xi} = [\rho_2(0)/k^2L_1]\{1 + [\rho_1(0)/\rho_2(0)]kL_1\}\ddot{\xi}.$$

In the compressible case, we must first determine the behavior of Λ_i as k tends to zero. In order to obtain this information we substitute

$$L_i/k\,\mathrm{At}(n/n_0)^{-2},$$

for Λ_i^2 in d_i in the dispersion relation,

$$\frac{n^2}{n_0^2} = \frac{\rho_1(0) + \rho_2(0)}{k[\rho_1(0)d_1 + \rho_2(0)d_2]}. \quad (55)$$

If we assume that

$$n^2/n_0^2 \sim \alpha k^{\nu},$$

and we take the limit of both sides of Eq. (55), then the only consistent choice is found to be $\nu = 1$ and

$$\alpha = \{[\rho_1(0) + \rho_2(0)]/\rho_2(0)\}L_1.$$

Hence, for long disturbances, we obtain the estimate $\Lambda_i \sim \beta_i k^{-1}$, where

$$\beta_i = (\rho_2(0)/[\rho_1(0) + \rho_2(0)](L_i/L_1))^{1/2}. \quad (56)$$

The fact that $n^2/n_0^2 = o(1)$ for $k \to 0$ for other values of γ_i as well is confirmed by the numerical results shown in Figs. 1–7.

The "effective" depths of penetration of the disturbance, d_1 and d_2, therefore tend to

$$d_1 = [\rho_2(0)/\rho_1(0)k^2L_1][1 + O(k^2L_1^2)], \quad (57)$$

$$d_2 = L_2[1 + O(k^2L_2^2)], \quad (58)$$

since $(1 + \beta_1^2)/\beta_1^2 = \rho_1(0)/\rho_2(0)$. Hence the ratio of the interface accelerations in the compressible and incompressible cases approaches

$$\frac{n^2}{n_f^2} \sim \frac{1 + [\rho_1(0)/\rho_2(0)]kL_1}{1 + O(k^2L_1^2)} \sim 1 + \frac{\rho_1(0)}{\rho_2(0)} kL_1, \quad (59)$$

as kL_i tends to zero. At this order of approximation the above asymptotic expansion also holds when the upper fluid is incompressible and the lower one compressible. In the reverse configuration, however, the limit of d_1 is

$$d_1 = \frac{\rho_2(0)}{\rho_1(0)k^2L_1}\left(1 + \frac{\rho_1(0) - \rho_2(0)}{\rho_2(0)} kL_1 + O(k^2L_1^2)\right),$$

so that the ratio tends to

$$\frac{n^2}{n_f^2} \sim \frac{1 + [\rho_1(0)/\rho_2(0)]kL_1}{1 + [\rho_1(0)/\rho_2(0)]kL_1} \sim 1. \quad (60)$$

These limits show why in Figs. 1, 4, and 5 the lower curves approach 1 much faster than the others and why in Figs. 2, 3, 6, and 7 there is a net destabilizing effect at long wavelengths.

VI. SUMMARY AND CONCLUSIONS

The state of equilibrium of two superposed fluids in a constant force field is not uniquely specified. In the incompressible case an arbitrary density stratification may be assumed, while in the compressible case the distribution of any one of density, pressure, or one other thermodynamic variable can be prescribed. In both cases, different unperturbed states will exhibit different stability properties. The stability of an isothermal stratification has been considered in Refs. 2 and 4, where it was found that compressibility introduces a destabilizing effect. Here, we have studied the case of isentropic stratification. It is important to recognize that, in this case, the upper fluid must have a finite thickness (unless its speed of sound is constant). If the growth rate of the instability is compared with that for an incompressible upper fluid of infinite extent, many differences are found that are really dependent on the different thickness of the layers rather than on compressibility. When this factor is properly discounted, it is found that compressibility is stabilizing at short wavelengths and destabilizing at long wavelengths, although the effect is typically small (see Figs. 2, 3, 6, and 7). As is discussed at length in Sec. V, where a physical interpretation of our findings is given, this result is a consequence of the fact that compressibility of the upper fluid has a stabilizing effect, while compressibility of the lower fluid is destabilizing. The latter effect peaks around a wavelength that is about $\epsilon^{-1} = \rho_1(0)/\rho_2(0)$ times the wavelength at which the stabilizing effect attains its maximum. Upon decreasing the density ρ_2 of the lower fluid, therefore, compressibility effects are decreased and displaced toward longer wavelengths.

We have attempted to give in Sec. V a qualitative analysis of the factors affecting stability. There we have explicitly traced the origin of the differences between our results and those of Refs. 2 and 4 to the fact that a stabilizing mechanism is "short circuited" for the case of adiabatic stratification. That analysis may be useful in the interpretation of the stability of other unperturbed states as well.

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APPENDIX: DETAILS OF THE ANALYSIS

Some details of the analysis that were not included in the main body of the text, so as not to overly encumber the exposition, are given here for completeness.

1. Boundary conditions

The equation for the amplitude of the vertical velocity $\hat{w}_i(z)$ is easily derived from the z component of the momentum equation (7),

$$in\rho_i(z)\hat{w}_i(z) = \left(\frac{d\hat{p}_i}{dz}(z) + c_i^{-2}(z)\hat{p}_i(z)g\right). \quad (A1)$$

In terms of $s_i(z)$, the unperturbed density distribution is obtained by substituting z as given by (23) into Eq. (4),

$$\rho_i|_{z=z(s_i)} = \rho_i(0)\left(\frac{\gamma_i - 1}{2k}\frac{\rho_i(0)}{\gamma_i(p_0 + B_i)} g s_i\right)^{1/(\gamma_i - 1)}$$
$$= \rho_i(0)\left(\frac{s_i}{s_i(0)}\right)^{1/(\gamma_i - 1)}. \quad (A2)$$

The equation above, together with

$$\left.\frac{d\hat{p}_i}{dz}\right|_{z=z(s_i)} = 2ks_i^{1/(\gamma_i - 1)}e^{-(1/2)s_i} \times \left[\left(\frac{1}{2} - \frac{1}{s_i(\gamma_i - 1)}\right)f_i(s_i) - f_i'(s_i)\right], \quad (A3)$$

allows us to simplify (A1) to

$$\hat{w}_i|_{z=z(s_i)} = i\frac{k}{n\rho_i(0)}[s_i(0)]^{1/(\gamma_i - 1)}e^{-(1/2)s_i} \times [f_i(s_i) - 2f_i'(s_i)]. \quad (A4)$$

Consider the case in which the upper fluid is bounded by a solid plane at $z = h < h_c$. Note that $\hat{w}_1$ vanishes on this boundary; that is

$$f_1[s_1(h)] - 2f_1'[s_1(h)] = 0. \quad (A5)$$

Recalling the definition of f_1,

$$f_1(s_1) = C_1\Phi(a_1, b_1, s_1) + D_1\Psi(a_1, b_1, s_1), \quad (A6)$$

this condition implies

$$Q = \frac{D_1}{C_1} = -\frac{\Phi(a_1,b_1,s_1(h)) - 2\Phi'(a_1,b_1,s_1(h))}{\Psi(a_1,b_1,s_1(h)) - 2\Psi'(a_1,b_1,s_1(h))}. \quad (A7)$$

The quantity $l_1 = 1 - 2f_1'/f_1|_{s_1 = s_1(0)}$ is thus readily seen to be given by Eq. (29). When the upper fluid is bounded by a second free surface that oscillates about $z = h < h_c$, the amplitude of the oscillations is derived from Eq. (14) and is equal to $-i\hat{w}_1(h)/n$. Substitute this value into condition (15),

$$\hat{p}_1(h) + i p_1(h)[\hat{w}_1(h)/n]g = 0,$$

then express it in terms of $s_1(h)$ to obtain

$$(1 - n^2/gk)f_1[s_1(h)] - 2f_1'[s_1(h)] = 0. \quad (A8)$$

From the previous equation it follows that the ratio Q is now

$$Q = \frac{D_1}{C_1} = -\frac{(1 - n^2/gk)\Phi(a_1,b_1,s_1(h)) - 2\Phi'(a_1,b_1,s_1(h))}{(1 - n^2/gk)\Psi(a_1,b_1,s_1(h)) - 2\Psi'(a_1,b_1,s_1(h))}. \quad (A9)$$

Even though Q is different from the expression given in the previous case, l_1 has the same form as in Eq. (29).

The case in which the upper fluid has a thickness equal to the critical one requires a special treatment. The balance of mass at an impermeable wall is expressed by the equation

$$\rho \mathbf{u}\cdot\mathbf{n} = 0. \quad (A10)$$

This equation can be simplified into Eq. (13) if and only if the density does not vanish at any location on the wall. When $h = h_c$, ρ_1 does vanish at $z = h_c$ and therefore the condition that must be applied is not (13) but

$$\rho_1(h_c)\hat{w}_1(h_c) = 0. \quad (A11)$$

As z tends to h_c, s_1 tends to zero and

$$\Phi(a_1,b_1,s_1) \sim 1,$$

$$\Phi'(a_1,b_1,s_1) = (a_1/b_1)\Phi(a_1+1, b_1+1,s_1) \sim a_1/b_1,$$

$$\Psi(a_1,b_1,s_1) \sim q_1 + q_2 s_1^{1-b_1},$$

$$\Psi'(a_1,b_1,s_1) = -a_1\Psi(a_1+1,b_1+1,s_1) \sim a_1(q_3 + q_4 s_1^{-b_1}),$$

where

$$q_1 = \pi[\sin(\pi b_1)\Gamma(1+a_1-b_1)\Gamma(b_1)]^{-1},$$

$$q_2 = \pi[\sin(\pi b_1)\Gamma(a_1)\Gamma(2-b_1)]^{-1},$$

$$q_3 = \pi[\sin(\pi b_1)\Gamma(1+a_1-b_1)\Gamma(1+b_1)]^{-1},$$

$$q_4 = \pi[\sin(\pi b_1)\Gamma(1+a_1)\Gamma(1-b_1)]^{-1}.$$

From these asymptotic relations, it follows that

$$\rho_1\hat{w}_1 \sim i\frac{k}{n}\{O(1/s_1^{1/(\gamma_1-1)})C_1 - 2a_1q_4[1+O(s_1)]D_1\}, \quad (A12)$$

where we have substituted for b_1 its value $1/(\gamma_1 - 1)$. In order to satisfy condition (A11) we have to set $D_1 = 0$. Let us now turn to the other case in which the boundary at $h = h_c$ is a free surface. Continuity of pressure at the interface reduces to $\hat{p}_1(h_c) = 0$, which is automatically satisfied since

$$\hat{p}_1 \sim (C_1 + q_1 D_1)s_1^{1/(\gamma_1-1)} + q_2 D_1 s_1 \to 0,$$

as s_1 tends to zero for any $\gamma_1 > 1$. Nonetheless we must set $D_1 = 0$, otherwise the velocity $\hat{w}_1$ would become infinitely large, as follows from (A4),

$$\hat{w}_1 \sim i[k/n\rho_1(0)][s_1(0)]^{1/(\gamma_1-1)}\{O(1)C_1 - 2a_1q_4 s_1^{-1/(\gamma_1-1)}D_1[1+O(s_1)]\}.$$

In conclusion, in either case,

$$f_1(s_1) = C_1\Phi(a_1,b_1,s_1), \quad (A13)$$

so that Eq. (31) follows.

In the lower fluid the pressure disturbance must vanish as z tends to $-\infty$. In terms of the variable s_2, this is equivalent to requiring

$$\lim_{s_2\to\infty} e^{-(1/2)s_2} s_2^{b_2} f_2(s_2) = 0. \quad (A14)$$

For large s_2,

$$f_2(s_2) \sim C_2[\Gamma(b_2)/\Gamma(a_2)]e^{s_2}s_2^{a_2-b_2} + D_2 s_2^{-a_2},$$

therefore (A14) is satisfied only if $C_2 = 0$. From Eq. (28), Eq. (32) follows.

2. Reality of n^2

In terms of the variables s_i and f_i, introduced in Eqs. (23) and (24), the condition of continuity of the normal velocity at the interface is

$$\{[s_1(0)]^{1/(\gamma_1-1)}/\rho_1(0)\}e^{-(1/2)s_1(0)} \times\{f_1[s_1(0)] - 2f_1'[s_1(0)]\} = \{[s_2(0)]^{1/(\gamma_2-1)}/\rho_2(0)\}e^{-(1/2)s_2(0)} \times\{f_2[s_2(0)] - 2f_2'[s_2(0)]\}, \quad (A15)$$

whereas the condition of continuity of pressure at the interface becomes

$$[s_1(0)]^{1/(\gamma_1-1)}e^{-(1/2)s_1(0)}(f_1[s_1(0)] - (k/n^2)g\{f_1[s_1(0)] - 2f_1'[s_1(0)]\}) = [s_2(0)]^{1/(\gamma_2-1)}e^{-(1/2)s_2(0)}(f_2[s_2(0)] - (k/n^2)g\{f_2[s_2(0)] - 2f_2'[s_2(0)]\}). \quad (A16)$$

Let N be equal to n^2/gk and multiply Eq. (A16) by the complex conjugate of Eq. (A15). Since s_i and b_i are real, we obtain

$$[(s_1^{2b_1}/\rho_1)e^{-s_1}(N|f_1|^2 - 2Nf_1\bar{f}_1' + |f_1 - 2f_1'|^2)]_{s_1=s_1(0)} = [(s_2^{2b_2}/\rho_2)e^{-s_2}(N|f_2|^2 - 2Nf_2\bar{f}_2' + |f_2 - 2f_2'|^2)]_{s_2=s_2(0)}. \quad (A17)$$

Equation (25) can be written in the following form:

$$\frac{d}{ds_i}\left(s_i^{b_i}e^{-s_i}\frac{df_i}{ds_i}\right) = \frac{1-N}{2}b_i s_i^{b_i-1}e^{s_i}f_i, \quad (A18)$$

and this enables us to express the terms $Nf_i\bar{f}_i'|_{s_i=s_i(0)}$ in a different way by multiplying by $\bar{f}_i$ and integrating (A18) between $s_i(0)$ and $s_1(h)$, when $i=1$, or ∞, when $i=2$,

$$[s_i^{b_i}e^{-s_i}\bar{f}_i f_i']_{s_i=s_i(0)} = [s_i^{b_i}e^{-s_i}\bar{f}_i f_i']_{s_i=s_1(h)\text{ or }\infty} - J_i - [(1-N)/2]b_i I_i, \quad (A19)$$

where I_i and J_i denote the real integrals

$$I_1 = \int_{s_1(0)}^{s_1(h)} s_1^{b_1-1} e^{-s_1} |f_1|^2 \, ds_1,$$
$$I_2 = \int_{s_2(0)}^{\infty} s_2^{b_2-1} e^{-s_2} |f_2|^2 \, ds_2, \tag{A20}$$
$$J_1 = \int_{s_1(0)}^{s_1(h)} s_1^{b_1} e^{-s_1} |f_1'|^2 \, ds_1,$$
$$J_2 = \int_{s_2(0)}^{\infty} s_2^{b_2} e^{-s_2} |f_2'|^2 \, ds_2. \tag{A21}$$

When the upper fluid is bounded by a wall from (A5), it follows that

$$\bar{f}_1 f_1' = \tfrac{1}{2}|f_1|^2,$$

at $s_1 = s_1(h)$. On the other hand, when the upper boundary is a free surface, Eq. (A8) leads to

$$\bar{f}_1 f_1' = [(1-N)/2]|f_1|^2.$$

Hence, for $i = 1$, Eq. (A19) becomes

$$[2 s_1^{b_1} e^{-s_1} \bar{f}_1 f_1']_{s_1 = s_1(0)} = N(b_1 I_1 - d_1 K_1) - (2J_1 + b_1 I_1 - K_1), \tag{A22}$$

where $d_1 = 1$ for a free surface and $d_1 = 0$ for a wall, and K_1 is the real quantity

$$K_1 = [s_1^{b_1} e^{-s_1} |f_1|^2]_{s_1 = s_1(h)}.$$

When $i = 2$, the term $s_2^{b_2} e^{-s_2}|f_2|^2$ vanishes at $s_2 = \infty$ because of (A14), and (A19) reduces to

$$[2 s_2^{b_2} e^{-s_2} \bar{f}_2 f_2']_{s_2 = s_2(0)} = N b_2 I_2 - (2J_2 + b_2 I_2). \tag{A23}$$

By taking the complex conjugate of (A22) and (A23), substituting back into Eq. (A17), and then considering the imaginary part, we obtain

$$\mathscr{T}(N)[(s_1^{b_1}/\rho_1)(s_1^{b_1} e^{-s_1}|f_1|^2 + 2J_1 + b_1 I_1 - K_1)]_{s_1 = s_1(0)} = \mathscr{T}(N)[(s_2^{b_2}/\rho_2)(s_2^{b_2} e^{-s_2}|f_2|^2 + 2J_2 + b_2 I_2)]_{s_2 = s_2(0)}. \tag{A24}$$

The equation above can be simplified. Upon integration by parts,

$$b_1 I_1 + J_1 - K_1 = -[s_1^{b_1} e^{-s_1}|f_1|^2]_{s_1 = s_1(0)} + \int_{s_1(0)}^{s_1(h)} s_1^{b_1} e^{-s_1} |f_1 - f_1'|^2 \, ds_1$$
$$b_2 I_2 + J_2 = -[s_2^{b_2} e^{-s_2}|f_2|^2]_{s_2 = s_2(0)} + \int_{s_2(0)}^{\infty} s_2^{b_2} e^{-s_2} |f_2 - f_2'|^2 \, ds_2.$$

With these identities Eq. (A24) reduces to

$$\mathscr{T}(N)\left\{\frac{[s_1(0)]^{b_1}}{\rho_1(0)} \int_{s_1(h)}^{s_1(0)} s_1^{b_1} e^{-s_1}(|f_1 - f_1'|^2 + |f_1'|^2) ds_1 + \frac{[s_2(0)]^{b_2}}{\rho_2(0)} \int_{s_2(0)}^{\infty} s_2^{b_2} e^{-s_2}(|f_2 - f_2'|^2 + |f_2'|^2) ds_2\right\} = 0. \tag{A25}$$

Over the ranges of integration the integrands are positive or zero because neither s_1 nor s_2 are negative. Since $s_1(0) \geqslant s_1(h)$, the integrals are strictly positive, from which it follows that $\mathscr{T}(N) = 0$.

3. Limits of the dispersion relation

Consider the case in which the upper fluid, with a polytropic index γ_1, has thickness h equal to the critical thickness h_c. We now show that, in the limit $\gamma_i \to 1$, h_c tends to infinity and the expressions for l_1 and l_2, given by Eqs. (31) and (32), simplify to the expressions given in Eqs. (33) and (34) for the case of two semi-infinite fluids with constant speeds of sound.

As $\gamma_i \to 1$, the parameters and the argument of the confluent hypergeometric functions tend to infinity as

$$a_i = \frac{1}{2}\left(1 - \frac{n^2}{gk}\right)\nu_i, \quad b_i = \nu_i, \quad s_i(0) = \frac{2kc_i^2(0)}{g}\nu_i,$$

where $\nu_i = (\gamma_i - 1)^{-1}$. Therefore, as ν tends to infinity, we can apply the limit formulas of Ref. 14,

$$\Phi(a,b,s) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\left(\frac{2\pi}{u_-}\right)^{1/2} \times e^{-t_- s}(-t_-)^a(1+t_-)^{b-a}[1 + O(\nu^{-1})],$$
$$\Phi'(a,b,s) = -t_- \Phi(a,b,s),$$
$$\Psi(a,b,s) = [1/\Gamma(a)](2\pi/u_+)^{1/2} \times e^{-t_+ s} t_+^a (1+t_+)^{b-a}[1 + O(\nu^{-1})],$$
$$\Psi'(a,b,s) = -t_+ \Psi(a,b,s).$$

In the above equations we have set

$$u_\pm = (1 + t_\pm)(A + S t_\pm^2),$$

where

$$t_\pm = \frac{1}{2S}\{A + B - S \pm [(A + B - S)^2 + 4AS]^{1/2}\}, \tag{A26}$$

and A, B, and S are the asymptotic values of a/ν, $(b-a)/\nu$, and s/ν as $\nu \to \infty$, i.e.,

$$A_i = \frac{1}{2}\left(1 - \frac{n^2}{gk}\right), \quad B_i = \frac{1}{2}\left(1 + \frac{n^2}{gk}\right), \quad S_i = \frac{2kc_i^2(0)}{g}. \tag{A27}$$

From Eqs. (A26) and (A27), it follows that

$$\lim_{\gamma_1 \to 1} l_1 = 1 - 2(-t_-)_1 = -1\left(1 - \frac{n^2}{k^2 c_1^2(0)} + \frac{g^2}{4k^2 c_1^4(0)}\right)^{1/2} + \frac{g}{2kc_1^2(0)},$$
$$\lim_{\gamma_2 \to 1} l_2 = 1 - 2(-t_+)_2 = \left(1 - \frac{n^2}{k^2 c_2^2(0)} + \frac{g^2}{4k^2 c_2^4(0)}\right)^{1/2} + \frac{g}{2kc_2^2(0)},$$

which coincide with Eqs. (33) and (34).

The other interesting limit is the incompressible one. For the sake of simplicity we consider gases only, for which $B_i = 0$, and we take the thickness of the upper layer h to be less than L, defined in Eq. (35). If we take the limit $\gamma_i \to \infty$ and keep h constant, the parameter b_i tends to zero and

$$\lim_{\gamma_i \to \infty} \frac{a_i}{b_i} = \alpha_i, \quad \alpha_i = \frac{1}{2}\left(1 - \lim_{\gamma_1,\gamma_2 \to \infty} \frac{n^2}{gk}\right),$$

$$\lim_{\gamma_1 \to \infty} s_1(z) = 2k(L - z),$$

$$\lim_{\gamma_2 \to \infty} s_2(0) = 2k\epsilon^{-1}L.$$

When the parameters of the confluent hypergeometric functions tend to zero, but their ratio α remains finite, the following results apply[7]:

$$\Phi(\alpha b, b, s + o(1)) = 1 - \alpha + \alpha e^s + o(1),$$

$$\Phi'(\alpha b, b, s + o(1)) = \alpha e^s + o(1),$$

$$\Psi(\alpha b, b, s + o(1)) = 1 + o(1),$$

$$\Psi'(\alpha b, b, s + o(1)) = o(1).$$

By using the above equations we can derive the limit of Eq. (32),

$$\lim_{\gamma_2 \to \infty} l_2 = 1.$$

The quantity Q defined by Eq. (30) takes the limit value

$$\lim_{\gamma_1 \to \infty} Q = -1 + \alpha_1 + \alpha_1 e^{2k(L-h)}[(2-R)/R],$$

from which it follows that

$$\lim_{\gamma_1 \to \infty} l_1 = 1 - 2\{1 + e^{-2kh}[(2-R)/R]\}^{-1}. \quad \text{(A28)}$$

If a solid wall bounds the upper layer of fluid, $R = 1$ and Eq. (29) reduces to

$$\lim_{\gamma_1 \to \infty} l_1 = -\tanh(kh).$$

Upon substitution into the dispersion relation (19) we recover the incompressible result given by Eq. (38).

If the upper boundary is a free surface, $R = 1 - n^2/gk$ and Eq. (29) becomes

$$\lim_{\gamma_1 \to \infty} l_1 = \frac{-\sinh(kh) + (n^2/gk)\cosh(kh)}{\cosh(kh) - (n^2/gk)\sinh(kh)}. \quad \text{(A29)}$$

Substituting (A29) in (19) and dividing by R, we obtain the incompressible dispersion relation (39).

If, in taking the limit $\gamma_i \to \infty$, we let h be equal to h_c, then the thickness will not remain constant during the limit process, but will tend to L. Under these conditions, in calculating the limit of l_1, we must use Eq. (31) rather than Eq. (29), so that

$$\lim_{\gamma_1 \to \infty} l_1 = 1 - 2\frac{\alpha_1 e^{2kL}}{1 - \alpha_1 + \alpha_1 e^{2kL}} = \frac{-\sinh(kL) + (n^2/gk)\cosh(kL)}{\cosh(kL) - (n^2/gk)\sinh(kL)}. \quad \text{(A30)}$$

Also, in this case, l_2 tends to one as γ_2 tends to infinity. Hence the incompressible limit of the dispersion relation (19) is Eq. (39).

4. Numerical solution

The wavelength λ^* is increased in small steps. At each step a fixed-point iterative method is used to find the root of the dispersion relation (37), namely, the jth iterate is calculated as

$$(n^2/gk)_j = G[(n^2/gk)_{j-1}, \lambda^*; \epsilon, \gamma_1, \gamma_2]. \quad \text{(A31)}$$

As a starting guess, $(n^2/gk)_0$, we use the solution at the previous step. For small values of the argument s ($0 \leqslant s < 30$), the function Φ is evaluated using its power series

$$\Phi(a,b,s) = 1 + \sum_{j=1}^{\infty} \frac{(a)_j s^j}{(b)_j j!}, \quad \text{(A32)}$$

where $(a)_j = a(a+1)(a+2)\cdots(a+j-1)$. When $0 \leqslant s < 4$ the function Ψ is calculated from the definition

$$\Psi(a,b,s) = \frac{\pi}{\sin(\pi b)}\left(\frac{\Phi(a,b,s)}{\Gamma(1+a-b)\Gamma(b)} - s^{1-b}\frac{\Phi(1+a-b,2-b,s)}{\Gamma(a)\Gamma(2-b)}\right), \quad \text{(A33)}$$

valid for $b \neq 1,2,\ldots$, that is, for $\gamma \neq (m+1)/m$, $m = 1,2,\ldots$. At times, convergence in evaluating (A33) by means of (A32) is very slow. When this happens or when $4 \leqslant s < 30$, Ψ is computed using the integral representation

$$\Psi(a,b,s) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-st}t^{a-1}(1+t)^{b-a-1}\,dt, \quad a,b > 0. \quad \text{(A34)}$$

To perform the integral the 30-point Laguerre integration formula is used.

Because of the exponential behavior of Φ, difficulties arise in trying to evaluate the confluent hypergeometric functions for large values of the argument. In the particular case of thickness h equal to the critical value, the problem can be bypassed because what is needed are the logarithmic derivatives of the functions, i.e., Φ'/Φ and Ψ'/Ψ. If we introduce $g = f'/f$, where f is a solution of Kummer's equation

$$sf'' + (b-s)f' - af = 0, \quad \text{(A35)}$$

it can be shown that g satisfies the Riccati equation

$$g' + [(b-s)/s]\,g + g^2 = a/s. \quad \text{(A36)}$$

For large values of s the solutions of this equation can be approximated by a power series centered at infinity,

$$g(a,b,s) = \sum_{j=0}^{\infty} c_j s^{-j}, \quad \text{(A37)}$$

where

$$c_j = \frac{(b-j+1)c_{j-1} + \sum_{l=1}^{j-1} c_l c_{j-l}}{1 - 2c_0}, \quad j > 1, \quad \text{(A38)}$$

and

$$c_1 = (bc_0 - a)/(1 - 2c_0). \quad \text{(A39)}$$

When $c_0 = 1$, g is equal to Φ'/Φ, whereas it is equal to Ψ'/Ψ for $c_0 = 0$. We make use of the expansion (A37) when $s > 30$.

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[9]Reference 3 contains plots of the growth rates corresponding to different values of a parameter that is equivalent to the ratio between the square of the sound velocity in the lower and in the upper fluid. Since, however, B_1 and B_2 are taken to vanish and constant sound speeds are assumed, $\gamma_1 = \gamma_2 = 1$ and this ratio is equal to $c_2^2/c_1^2 = \rho_1(0)/\rho_2(0) = (1 + \mathrm{At})/(1 - \mathrm{At})$. Once the Atwood number At has been chosen, the ratio c_2^2/c_1^2 cannot be varied independently, as already pointed out in Ref. 8. Therefore most of Baker's results do not correspond to realizable cases. However, in each one of his families of curves, there is one for which the previous relation is satisfied. These correct curves are sufficient to derive the conclusions mentioned here.

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