THE ROLE OF MAGNITUDE PROCESSES AND WORKING MEMORY FOR LEARNING ALGEBRA

A Master's Thesis

Presented to

The Faculty of the Department

of Psychology

University of Houston

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

By

William Lacey

May 2017

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ABSTRACT

There has been much research into the predictors of early mathematics. In contrast, less information is available about how such predictors inform later skills such as algebra. Algebra is an important "gateway" to higher order mathematics, which is relevant given the increasing demand for workers in the STEM (Science, Technology, Engineering, and Math) fields. The present study investigates the role of both domain general and domain specific skills (including earlier developing math skills) for algebra. We focus on working memory, and magnitude processes (comparison and estimation), and contextualize their impact with fractions performance in 9th graders ($n = 90$). Fraction number line and fraction competency were found to predict end of year algebra performance as well as change across the 9th grade year in algebra performance. Working memory did not show a significant relationship to algebra performance. This study highlights the role that rational number skills play in the development and growth of later developing math skills.

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The Role of Magnitude Processes and Working Memory for Learning Algebra

Understanding mathematics is a subject of unique importance in society (Bouchey & Harter, 2005; Geary, 2013; Kilpatrick, Swafford, & Findell, 2001; Parsons & Bynner, 2008), and in recent years, literature on the predictors of mathematical ability has steadily expanded (Hansen et al., 2015; Tolarstudy, Lederberg & Fletcher, 2009). Evaluating these skills across development is critical because children who fall behind tend to stay behind (Geary, 2013), and because there is notable hierarchy of skill development in mathematics (Kilpatrick et al., 2001; Siegler & Lortie-Fogues, 2015). Among the earliest foundations of mathematical understanding is the informal (prior to schooling) development of numerosity, including magnitude skills (Dehaene, 2011; Gallistel & Gelman, 1992; Tosto et al., 2014) and the establishment of the mental number line for representing quantities (Moyer & Landauer, 1967; Restle, 1970). Through formal education, rudimentary algebraic knowledge continues with development of addition and subtraction operations and their relationships (Common Core Math Standards, 2015). Strong understanding of early formal mathematical skills (e.g. math facts) is integral to understand the next procedure and concept in the sequence (e.g., procedural computations, word problems; National Mathematics Advisory Panel, 2008).

There are by now numerous studies investigating the cognitive predictors of math for young students (Cirino, 2011; Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Geary, 2011; Hansen et al., 2015; Libertus, Feigenson, & Halberda, 2013). These studies document the role for both domain general skills, such as working memory (David, 2012; Swanson & Beebe-Frankenberger, 2004; Wilson & Swanson, 2001) and domain specific skills, such as magnitude understanding (Bonny & Lourenco, 2012; Schneider, Grabner, & Paetsch, 2009; Siegler & Booth, 2004; Siegler, Thompson, & Schneider, 2011; Toll & Van Luit, 2014).

Algebra is a critical step in mathematical education and development from grades 6 to 9 (Witzel, 2005), but empirical research in algebra lags behind that focused on the developing early whole number performance. Algebra is a skill of particular relevance as it pertains to STEM (science, technology, engineering and math) fields (Tyson, Lee, Borman, & Hanson, 2007) and continues to be more essential as STEM jobs become more prevalent (Asunda & Paul, 2015; NMAP, 2008). The transition to algebra from arithmetic is a particular area where children tend to struggle, specifically with the introduction of new concepts like the manipulation of symbols as variables (Kilpatrick et al., 2001; Lee & Wheeler, 1989; NMAP, 2008; Philipp & Schappelle, 1999; Stacey & MacGregor, 1989).

Algebra (and other cognitive and achievement domains) can be evaluated either procedurally or conceptually. Procedural understanding is defined as a sequence of actions performed, and conceptual knowledge as explicit or implicit understanding of the rules of a domain and their interrelations to other areas of a given domain that are able to be applied to other facets of that domain (Hiebert, 2013). At preschool ages, procedural and conceptual math understanding are difficult to dissociate (Sinclair & Sinclair, 2013) but as math develops and new concepts and procedures are introduced they become more separable (Hiebert, 2013). There is some argument that mathematics should be taught explicitly in a conceptual to procedural process (National Council of Teachers of Mathematics 2014), but without current consensus regarding such ordering effects. There is recent meta-analytic evidence that procedural skill and conceptual knowledge support one another and show bidirectional relationships with one another over time (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Schneider, & Star, 2015).

Recent studies have shown that working memory (Tolar et al., 2009), magnitude comparison (Geary et al., 2015), and magnitude estimation (Booth, Newton, & Twiss-Garrity, 2014) are all related to algebra. Fractions are particularly important for algebra (Brown & Quinn, 2007; DeWolf, Bassok & Holyoak, 2015; NMAP, 2008; Siegler et al., 2012; Wu, 2001), in part because they integrate a sense of magnitude comparison (Siegler et al., 2011) and magnitude estimation (Booth, Newton, & Twiss-Garrity, 2014; Gabriel, Szucs, & Content, 2013), as well as conceptual understanding of proportions (Kwon, Lawson, Chung, & Kim, 2000). Stronger fractions knowledge may also free working memory resources to focus on other components of algebra (e.g. new orders of operations; Jordan et al., 2013).

Recent evidence of the role of cognitive predictors and fractions in algebra is encouraging but the research base remains thin. Therefore, a goal of the present study is to fill the gap by positing specific roles for the relative contribution of domain specific skills of comparison and estimation in the context of working memory, as well as fractions competence. We do so in grade 9, when most students take Algebra I. We are particularly interested in the way these predictors work to aid in the learning of algebra, and therefore, we evaluate performance at the end of the year as well as evaluating change in understanding across the time that Algebra I is taken. Evaluating these factors, among others, in the context of one another can help consolidate current knowledge regarding the contributors to algebraic competency.

Working Memory

Among domain general predictors of math, working memory has received much of the focus in relevant research (LeFevre, DeStefano, Coleman, & Shanahan, 2005; Raghubar, Barnes, & Hecht, 2010). Working memory is a cognitive system involving the coordination of memory and attention with an outcome of complex cognition (Shipstead, Harrison, & Engle, 2015). The relationship between working memory and mathematics has been shown at many levels of early mathematics (Caviola, Mammarella, Cornoldi, & Lucangeli, 2012; Fuchs et al., 2014; Siegler et al., 2010; Swanson & Frankenberger, 2004), even when accounting for other cognitive measures like intelligence, age, short-term memory, reading, and processing speed (Berg, 2008; Bull & Johnston, 1997; Bull, Johnston, & Roy, 1999; Wilson & Swanson, 2001). The relationship between working memory and math was supported by a review by Friso-van den Bos, van der Ven, Kroesbergen, and van Luit (2013), who concluded that working memory is significantly related to arithmetic, word problems, and counting and concepts ability but not in algebra, which adds credence to the need to evaluate the role of working memory in algebra. Their analysis showed that visuospatial working memory skills, like the one used in the current study, significantly predicted all three-math outcomes (arithmetic, word problems, and counting and concepts) while being significantly more powerful predictor of arithmetic compared to word problems. Working memory also accounts for differences in mathematical skill between typically performing students versus children who struggle (Swanson & Sachse-Lee, 2001). Mathematical problem solving requires consciously maintaining and manipulating relevant information (Witzel, 2005), as well as for performing multistep computations (Swanson & Beebe-Frankenberger, 2004). In general, working memory resources are critical in processing newer information where there is a lack of familiarity, or in areas with which students struggle to learn; therefore, working memory can become a bottleneck for mathematical performance (Dehaene & Sigman, 2012). Algebra is particularly relevant in regard to the bottleneck because it is a new skill with which many students struggle. There is also growing empirical evidence that working memory is related specifically to algebra (Geary et al., 2015; Siegler et al., 2010; Tolar et al., 2009). While maintenance and manipulation of problem information for multistep problems occurs prior to algebra, the use of symbols and their use as variables with numbers are much more pronounced in algebra (Witzel, 2005). With the introduction of algebra specifically, the rapid increase in the use of symbols and variables and how systems of numbers are interrelated through functions and linear equations (Phillipp & Schappelle, 1999) places new demands on the working memory system.

In a randomized experimental study, Fuchs et al. (2014) found that fourth graders with weak working memory ability were still able to learn fractions better with conceptual instruction compared to fluency instruction in at risk learners. The conceptual instruction condition focused on shading and matching fraction regions while explaining their reasoning for doing so. The fluency instruction condition focused on answering problems on flash cards. Among those with weak working memory, students in the conceptual condition outperformed those in the fluency condition, implying a lesser role of working memory for conceptual learning. Friso-van den Bos, van der Ven, Kroesbergen, and van Luit (2013) found that working memory was more predictive of computation skills than word problem performance, which may also support the role of working memory in procedural over conceptual learning in mathematics, but it should be noted that there is not a one-to-one mapping of computations with procedural learning, and word problems with conceptual learning.

Magnitude and its Representation

Magnitude and its representation is a key domain specific skill related to mathematics (De Smedt, Noël, Gilmore, & Ansari, 2013; Ebersbach & Erz, 2014; Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008). Magnitude *comparison* and magnitude *estimation* are the primary ways that the representation and understanding of magnitude is assessed. Further, magnitude comparison can be segregated into symbolic and nonsymbolic tasks (Ebersbach & Erz, 2014; Skagerlund & Träff, 2016), and magnitude estimation can be observed with either whole or fraction number line tasks (DeWolf et al., 2015; Schneider et al., 2009; Siegler & Booth, 2004). The ability to more quickly and accurately identify the value of a number in *comparison* to others may assist in manipulating algebraic formulas and solving linear equations (Jang & Cho, 2015); more efficient processing may then allow more resources for the multi-step nature of algebraic processes. Firm magnitude representation and *estimation* skills may assist in gauging the appropriateness of an answer and so serve as a check for problem solving.

Feigenson, Dehaene, and Spelke (2004) describe two core number systems; the first core number system is that thee is a precise representation of small numerical values. The second is the approximate number system, which Feigenson et al. (2004) describe as a “noisy” approximate representation of numbers that allows for the discrimination between two quantities from infancy through adulthood in an imprecise fashion. It is the approximate system that is evaluated by most numeric comparison and estimation tasks. Early nonsymbolic skills differentiate between sounds and visual representations etc., which then expand to identify which Arabic numeral represents a larger quantity. It is thought that children intuitively map their representations onto a mental number line. The initial logarithmic number line has fixed variability but as children develop the number line develops to a more linear representation with scalar variability, the variability of estimations increases proportionally with mean estimations (Ebersbach & Erz, 2014). Another key component of the approximate number system is that it is ratio dependent, such that comparisons of larger ratios are more easily discriminable than are small ratios.

It has been suggested that humans have an innate mental number line representation (Ebersbach et al., 2008; Moyer & Landauer, 1967; Restle, 1970). One model of the number line representation states that it changes with development from a general number line (e.g. for time, space, and musical tonality) to more discrete representations (Case & Okamoto, 1996; Von Aster & Shalev, 2007). The Case and Okamoto (1996) model was further developed by Siegler and Booth (2004), who argued that a child's mental number line is, at least initially, logarithmic, meaning that larger numbers are represented as being closer together than smaller numbers. The number line then takes on a more linear representation as children develop and with experience (Friso-van den Bos, Kolkman, Kroesbergen, & Lesemean, 2014; Feigenson, Dehaene & Spelke, 2004; Mundy & Gilmore, 2009). Furthermore, a potential connection between magnitude comparison and magnitude estimation is that the mental number line linearity that develops is based around their the numbers children are familiar with at that point and their ability to use magnitude comparison skills (Dehaene, 2011; Ebersbach et al., 2008; Feigenson et al., 2004).

Comparison. As noted, comparison skills can be evaluated either nonsymbolically or symbolically. Nonsymbolic comparison tests the ability to discriminate between two magnitudes represented in non-numeric form; this skill is thought to develop informally and pre-verbally (Gallistel & Gelman, 1992). *Nonsymbolic* ability has been shown to be heritable (Tosto et al., 2014) and is even evident in animals (Dehaene, 2011); neuroimaging fMRI studies have shown bilateral activation in the intraparietal sulcus during tasks involving the nonsymbolic number system (De Smedt et al., 2013; Dehaene, Piazza, Pinel, & Cohen, 2003).

The role of nonsymbolic comparison in mathematics has been an issue of some debate to this point, with some studies finding a significant relationship between nonsymbolic and math (Bonny & Lourenco, 2013; De Smedt et al., 2013; Libertus et al., 2013; Matthews, Lewis, & Hubbard, 2015) and others finding no significant relationship (Lyons, Ansari, & Beilock, 2012; Holloway & Ansari, 2009; Sasanguie, Defever, Maertens, & Reynvoet, 2013). A recent meta-analysis by Chen and Li (2014) showed a moderate but significant relationship of $r = .20$ between nonsymbolic comparison and math skills. It also showed predictive value for future performance as well as retrospective performance. Another meta-analysis by Schneider et al (2016) showed a similar meta-correlation of .24, and a meta-correlation of .30 for symbolic comparison and mathematics. The role of nonsymbolic comparison has been shown at different ages but evidence appears stronger before the age of 6 (Fazio, Bailey, Thompson, & Siegler, 2014). Studies have also shown that older students perform better than younger students on nonsymbolic tasks (Halberda et al., 2012).

Symbolic comparison requires discriminating between two Arabic numerals, and therefore necessitates visual recognition of those numerals and their quantities; it is thought that this skill develops between the ages of 6 and 8 or with the onset of formal instruction utilizing Arabic numerals (Mundy & Gilmore, 2009). Accurate representation and identification of numerals is a precondition for written arithmetic and understanding their relationships. For example, simple addition utilizes symbolic comparison in that it requires an understanding that the sum must be larger than any of the addends. A recent review by De Smedt et al (2013) showed a clear relationship between symbolic comparison performance and mathematics. A recent study by Vanbinst, Ansari, Ghesquière, and De Smedt (2016) even posits that symbolic comparison is as important to arithmetic as phonological awareness is to reading.

An indirect role for the nonsymbolic number system has been shown for fractions performance (Cirino, Tolar, Fuchs, & Huston-Warren, 2016; Siegler, Thompson, & Schneider, 2011). The role of the nonsymbolic system in conceptual algebra is less clear as there is currently little research on the matter. Studies have shown an indirect relationship between nonsymbolic comparison and algebra through the symbolic number system (Geary, Hoard, Nugent, & Rouder, 2015; Matthews et al., 2015). The present study expects a similar pattern, with symbolic and nonsymbolic comparison both relating to algebra but with symbolic comparison having a much stronger relationship.

Geary et al. (2015) demonstrated a relationship between comparison skills and algebra in 9th graders. That study found more support for the role of nonsymbolic comparison for procedural algebra relative to conceptual algebra, but also that the nonsymbolic effects were not significant after accounting for symbolic comparison. The present study differs from Geary et al. (2015) in its diverse population (see Methods), its simultaneous consideration of nonverbal working memory, and its evaluation of growth. Overall, there is very little information on the role of symbolic comparison and algebra compared to other forms of mathematics, but symbolic comparison should relate strongly to procedural algebra given its demonstrated relevance to earlier developed computational skill (Vanbinst et al., 2016).

Estimation. Both whole number line estimation (Booth et al., 2014; Hansen et al., 2015, LeFevere, Greenham, & Waheed, 1993; Siegler & Booth, 2004) and fraction number line estimation (Booth & Newton, 2012; Siegler, Thompson, & Schneider, 2011) have been strongly implicated as a predictor of mathematic ability. DeWolf and Vosniadou (2013) provide evidence that whole number and fraction estimation are two separate abilities; for example, in number line tasks, participants use the fraction as its own representation, rather than discriminating only among the whole number portions. They evaluated these different representations by finding a significant role for fractions number line, decimal number line, and fractions performance separate form each other.

Recent studies have found that fraction number line and whole number line knowledge are both uniquely predictive of algebraic ability as well as improvement in algebraic ability both conceptually and procedurally (Booth & Newton, 2012; Booth et al., 2014). Whole number line tasks have been shown to relate to procedural mathematics knowledge (Fuchs et al., 2010). However, DeWolf et al. (2015) found a relationship between fraction number line and algebra but not with whole number line and algebra. A child's ability to recognize the value of a fraction in the context of whole numbers and missing variables is an important mechanistic feature in algebra (e.g. slope, distributive property; NMAP, 2008). Therefore, examination of both fraction and whole number line estimation abilities as it relates to algebra will add new knowledge in an understudied population.

Current Study

The goal of the current study is to evaluate the roles of magnitude processes and working memory and their relation to end of year algebra performance in the context of fractions skills. Filling the empirical gap around the role of magnitude processes and working memory can have both theoretical and practical implications. Multiple studies have evaluated algebra-like skills prior to formal instruction (Booth, Newton, & Twiss-Garrity, 2013; Geary et al., 2015), but fewer evaluate algebra content after its deliberate instruction (DeWolf et al., 2015). The current study will contribute to knowledge regarding the role of domain general skills such as working memory interact with the development of magnitude skills (Jordan et al., 2014). Practically, knowledge regarding the relative roles of working memory and magnitude process, and how these operate in the context of proximal mathematical skills (e.g., fractions) may inform earlier mathematics instruction and the scaffolding of the factors that allow a smoother transition at this critical time point.

Hypotheses

Hypothesis 1: Multiple Regression: Each Algebra outcome will be predicted by the above variables (fraction competency, fraction number line, whole number line, symbolic comparison, nonsymbolic comparison, and working memory) to different degrees.

a. We expect that the set of predictors together will more strongly predict procedural algebra over conceptual algebra performance at the end of the year due to the potential role of working memory (Fuchs et al., 2014) and symbolic comparison (Geary et al., 2015) in procedural relative to conceptual skills.

b. For conceptual algebra, we hypothesize that fraction competency will be the strongest significant predictor (Brown & Quinn, 2007; DeWolf et al., 2015), followed by fraction number line (Booth & Newton, 2012), whole number line (Cirino et al., 2016; Tolar, Cirino, & Fuchs, 2016), working memory (Lee, Ng, Ng, & Lim, 2004), and symbolic comparison (Cirino et al., 2016; Geary et al., 2015), though not nonsymbolic comparison due to its previously shown mediation by symbolic comparison (Geary et al., 2016; Matthews et al., 2016).

c. For procedural algebra, we hypothesize that fraction competency will be the strongest significant predictor (Brown & Quinn, 2007; DeWolf et al., 2015), followed by fraction number line (DeWolf et al., 2015), whole number line (Booth & Newton, 2012; Cirino et al., 2016; Tolar, Cirino, & Fuchs, 2016), symbolic comparison (Cirino et al., 2016; Geary et al., 2015), and then working memory (Siegler et al., 2010), but not nonsymbolic comparison due to its previously shown mediation by symbolic comparison (Geary et al., 2016; Matthews et al., 2016).

Hypothesis 2: Multivariate Regression: Predictors will vary in the extent to which they predict procedural vs. conceptual algebra.

a. Working memory is hypothesized to significantly predict both procedural and conceptual algebra (Geary et al., 2015; Siegler et al., 2010) at the end of the year, but we expect its impact to be significantly greater for procedural algebra relative to conceptual algebra based on patterns from the research on earlier math (Fuchs et al., 2014, Friso-van den Bos et al., 2013).

b. We also expect whole and fraction number line to predict both procedural and conceptual algebra (Booth & Newton, 2012; Booth et al., 2014), but we expect that its impact will be significantly greater for conceptual relative to procedural algebra. We also predict that the impact of fraction number line will be greater than whole number line for both procedural and conceptual algebra based on the significant role of fractions in algebra (Siegler et al., 2012).

c. We expect symbolic comparison (Geary et al., 2015) and fraction performance competency (Brown & Quinn, 2007; DeWolf et al., 2015) to significantly predict both procedural and conceptual algebra to an equivalent degree

Hypothesis 3: Predictors will predict growth across the year.

We expect that the pattern of prediction demonstrated for end of year algebra performance will be maintained, after consideration of beginning of the year algebra performance.

Method

Participants

Ninety-six ninth grade students from an urban district in a large southwestern city participated in the current study. Some of the participants and some of the measures have been reported on previously (Cirino et al., 2016; submitted), though the focus of those studies was entirely different (predicting 6th grade math performance; longitudinal follow up of students from grade 6 through grade 9). The current participants are the only ones who received the measures required to address the aims of this paper. Their average age was 15.03 (SD = 0.62). All students were instructed in English, and schools reported that 11% of subjects had limited proficiency in the English language. Fifty-four percent were male, and with regard to ethnicity, 67% were Hispanic/Latino, 26% Black, and 6% Caucasian. Most students were in regular classes, though 10% were receiving special education services. Eighty-two students were recruited in Fall of their 9th grade year, and there was an attempt to re-assess these students at the end of the year (76 students were evaluated at both time points). As recruitment was difficult, further attempts were made to increase sample size, resulting in 14 students who were assessed only in Spring. This resulted in the sample of 94 unique students seen at least one time. The 14 students, for whom data was available only in Spring, were compared on the above measures, to the 80 who received cognitive and number measures in Fall. T-tests compared the 80 students to the 14 students and there were no significant differences in performance except for symbolic comparison, $t(79) = 2.39$, $p = .005$.

Measures

Working Memory. Working memory was measured with Automated Symmetry Span (Conway, Kane, Bunting, Hambrick, Wilhelm, & Engle, 2005), which requires students to recall spatial locations while making judgments about symmetry. The students are instructed to quickly and accurately make a decision on symmetry, maintaining at last 85% accuracy. There are 12 trials, with three trials each at span lengths of 2, 3, 4, and 5; the order of each trial is randomly generated. Instead of the less reliable all-or-nothing scoring (Conway et al., 2005), the primary measure is a "partial credit-load" score that assigns points for each trial according to its length. The maximum score is 42. Cronbach's alpha was found to be 0.74 in the current study.

Magnitude Comparison. The nonsymbolic comparison task used was Panamath (Halberda, Ly, Wilmer, Naiman, & Germine, 2012; www.panamath.com). In the Panamath task students see dots on a computer screen, yellow on the left and blue on the right, for 600ms, followed by a backward mask, and then a fixation symbol. Students quickly decide which side has more dots and press the corresponding key on the right or left side of the keyboard. Dot size, area, and the ratio between the dot set-sizes systematically vary. Two hundred trials were administered with variations in dot size and area. The primary measure utilized is the W score, which reflects discriminability ratios (lower values represent finer discrimination). Previous studies have found split-test reliability of 0.73 in 11 to 17 year old children (Halberda et al., 2012).

Symbolic Comparison. The measure used was developed specifically for this project (e.g., Cirino et al., 2016). The Symbolic Comparison task has 91 trials in which participants see two single or double-digit Arabic numerals, and must decide which number is greater as quickly and accurately as possible. The ratio of the two numbers was designed to maximally overlap with those of the nonsymbolic presentation of dots from Panamath. Available measures include accuracy and response time, though accuracy is expected to be quite high, and individual differences are more likely for response time to correct responses, which was used here. Internal consistency for response time was $\alpha = 0.98$.

Magnitude Estimation. The current study uses a task developed by Siegler and Booth (2004), Whole Number Line Estimation. In the Whole Number Line Estimation task there is a line on the screen bound by 0 on one end and then 1,000 on the other. The number line is also displayed with an Arabic numeral above it. The score utilized was the mean deviation (in centimeters) from the numbers' actual line position (Siegler & Booth, 2004). Alpha was 0.91 in the current study.

For Fraction Number Line Estimation, the current study uses a modified version of the Bailey, Siegler, and Geary (2007) model. It is similar to the whole number version but is instead bounded by anchors of 0 to 2 and the stimulus is a fraction presented above the line. Again, participants place a mark where on the number line the fraction line should appear, and again, the primary measure the deviation (in cm) from the number's actual line position. Cronbach's alpha was 0.73 for the current study.

Mathematical Measures. The Brown and Quinn Fraction Competency Test (Brown & Quinn, 2007) is a 25-item measure of fractions to assess conceptual and procedural knowledge presented through word problems. The same items are used from the original but are ordered by difficulty from the original study. Elementary algebra students were 52% accurate on the Brown and Quinn Fraction Competency Test, and it correlated $r = .58$ with algebra course final score (Brown & Quinn, 2007). Cronbach's alpha in the current study was 0.70.

The Algebra Procedural Skill and Conceptual Knowledge Test was developed as part of the larger project (e.g., Tolar, Cirino, & Fuchs, under review) and includes 40 procedural and conceptual algebraic equations. For purposes of the present study, items were identified as focused on procedures, or concepts, or both, while noting that the distinction can be difficult to make (Rittle-Johnson, Schneider, & Star, 2015). Procedural items involve knowing the order of operations to complete a problem (e.g. solve for x: $xy + yz = 2y$). Conceptual items are about understanding the underlying concepts of a problem and how to use that information (e.g. "which of A through D CANNOT be true about the slope of a linear function"). The measure as a whole showed $\alpha = 0.84$, whereas a subset of 25 procedural items showed $\alpha = 0.88$, but the 10 conceptual items showed $\alpha = 0.62$. Due to the undesirable reliability of the conceptual items ($\alpha < 0.65$; DeVellis, 1991) the conceptual measure was deemed unfit for analysis. Half of the conceptual problems were in the last seven items of the test, which most children did not finish. The mean score on the conceptual test was 1.94, which leaves little room to evaluate differences to evaluate.

Procedures

Trained examiners assessed students, in two sessions at their schools (each ~1 hour). Group administration occurred where feasible or efficient (e.g., math tasks), and other measures were administered individually (e.g., those requiring a computer). The measures reported here are a subset of those given to students in the project as a whole. Examiners were trained and supervised by project investigators as well as master's level staff with 20 years of assessment experience in schools. Data administration, collection, and scoring were quality controlled both in the field, as well as off-site, before electronic data processing. Optical scanning and automatized quality checks flagged any further issues in coding, which was followed by additional manual data verification. Most students were initially assessed in Fall of their ninth grade year on number and cognitive measures, as well as an algebra pretest; in Spring of the same school year the algebra test was repeated. As noted above, some students ($n = 14$) were assessed only in Spring of their ninth grade year. The test order was quasi-randomized in order to balance any impacts that order effects may occur, while being efficient logistically in the school settings. The University Institutional Review Board reviewed all procedures, and school permission and both parental consent and student assent were obtained.

Analyses

Prior to the analyses that addressed hypotheses, we evaluated relevant variables for properties that impacted assumptions of the planned analyses, which included regression and change score analyses. Assumptions of regression analyses include: (a) normality; (b) linearity; (c) little or no multicollinearity; and (d) homoscedasticity (Osborne & Waters, 2002). Normality was addressed by examining distributional properties (e.g., skew, kurtosis, outliers); in addition to Q-Q plots (comparing actual to expected-normal distributions) and the Anderson Darling (Stephens, 1974) test for normality. Across all variables examined, Q-Q plots showed initial unusual distributions but when outliers at three standard deviations were removed the distributions were normal and all Anderson Darling tests were non-significant, $p > .05$. There were however a total of 7 outliers (three for Panamath, two for Symbolic Comparison, three for Whole Number Line Estimation, and one for the Algebra posttest, but none for the remaining five variables). Analyses were run with and without these seven individuals, and the pattern of predictive values were highly similar. Therefore, their data points were retained for the analyses presented below. Regression diagnostics were utilized to evaluate other analytic properties, including homoscedasticity (via the pattern of residuals of predicted values was consistent), undue influence (via leverage values), and multicollinearity (via variance inflation and tolerance statistics). These procedures did not yield further impediments to the assumptions of the regression analyses. Five data points were found to be outliers of the regression model but did not interrupt the assumptions of the regression analysis. When the model was run with and without these outliers there were no significant differences in the regression models. There was no further information on these subjects that would give credence to removing them from the results.

Other than regression, the other primary analyses involved latent change scores, evaluated in a structural framework (e.g., McArdle, 2001). The advantage of such a procedure is the creation of a latent variable representing error free growth in the variable(s) of interest, which can then be predicted. These models were evaluated in MPLUS 7.1 (Muthén & Muthén, 2007), utilizing maximum likelihood estimation. Typically, model fits are evaluated via multiple metrics, including comparative fit index (CFI; Bentler, 1990), root mean square error of approximation (RMSEA; Steiger & Lind, 1980), both of which are widely recommended for latent growth models (e.g., Curran, Obeidat, & Losardo, 2010; Preacher, Wichman, MacCallum, & Briggs, 2008). The results were bootstrapped on each of 1000 simulated data sets (Efron & Tibshirani, 1993). Bootstrapping has been shown to be more accurate than normal theory tests (MacKinnon, Lockwood & Williams, 2004). This is more accurate because it gives an estimate of the true score by running multiple samples. Bootstrapping provides confidence intervals, which are then examined for their overlap with 0 (if it does not, then the result is significant). Given the sample size and potential for over fitting models, we also evaluated change via repeated measures ANOVA, as well as ANCOVA. A repeated measures ANOVA is analogous to the prediction of change, but does so on manifest variables, whereas ANCOVA is most similar to the primary regression analyses, but one in which the pretest is used as a covariate as a more stringent test of the unique impact of the set of prediction variables in a regression-based analyses.

A number of covariates were considered for the above models, including age, sex, ethnicity, SES, proficiency in English, or placement in special education or gifted classes due to a previously shown relation to mathematical outcomes (Kao & Thompson, 2003; Ladson-billings & Madison, 1997; Lubienski, 2002; McGraw, Lubienski, & Strutchens, 2006; Tate, 1997). However, in the present study, none of these variables were significantly related to algebra outcomes (all $p > .05$). While there is precedent for the relation of such variables to similar outcomes, the present sample showed a restriction in range in most of these variables (e.g., all but one student qualified for free/reduced lunch).

Results

Preliminary Results

Descriptive statistics for the predictor and outcome variables appear in Table 1. Planned analyses included prediction of performance on procedural versus conceptual algebra items with multivariate regression. However, as noted above, conceptual items showed poor reliability. It is possible that the limited number of pure conceptual problems in combination with the sample size impacted results. Given these difficulties, it would be imprudent to proceed with our original analytic plan (comparing the predictors of conceptual versus procedural problem types). However, the procedural algebra measure did have strong reliability, as did the test as a whole (that included additional items that focused on concepts, or with mixed emphasis on concepts and procedures). Therefore, we evaluated two sets of analyses – one for the algebra measure as a whole, and then a second set for just the procedural items, with a focus on how the substantive results may differ. While less than ideal, it still allows for some evaluation of potential differences, or at least those specific to procedural algebraic skill.

- Insert Table 1 about here –

- Insert Table 2 about here -

Table 3 shows zero-order correlations among all outcome and predictor variables. End of year algebra significantly correlated ($p < 0.05$) with procedural algebra performance, fraction performance, working memory, whole number line, and fraction number line. End of year procedural algebra significantly correlated ($p < 0.05$) with fraction performance, working memory, whole number line, and fraction number line.

- Insert table 3 about here -

Multiple Regression Analyses

Multiple regression results with all cognitive predictors included appear below, for both the whole algebra test (see Table 4) as well as the procedural only items (see Table 5). Effect size is reported through eta-squared ($\hat{n}^2$). The overall model for whole algebra performance was significant, $F(6,64) = 11.28$, $p < .001$, $R^2 = 0.51$. There was a unique predictive effect for fractions performance, $t(1, 64) = 3.40$, $p = .001$, $\hat{n}^2 = 0.09$, indicating a positive relationship between students' scores on the fractions performance measure and the algebra test. There was a significant negative relationship of fraction number line performance with the algebra test, $t(1, 64) = -4.45$, $p < .001$, $\hat{n}^2 = 0.15$; the negative effect makes sense given that fraction number line performance was a deviation (error) score.

The model for procedural items was highly similar to that of the whole algebra test. The overall model was significant, $F(6, 71) = 7.09$, $p < .001$ and was uniquely predicted by fractions performance, $t(1, 64) = 2.84$, $p = .0003$, $\hat{n}^2 = 0.13$ and fraction number line performance, $t(1, 64) = -2.03$, $p = 0.046$, $\hat{n}^2 =0.04$. Inclusion of covariates did not change substantive interpretations for either the algebra test as a whole, or for the procedural items.

- Insert Table 4 about here -

- Insert Table 5 about here

Change Analyses

Latent change analysis results appear in Figure 1. The change score first regressed post-test on pre-test, and then created a latent variable with a single loading (on post-test), which in turn regressed the change onto pre-test. This results in a just identified model (with 0 degrees of freedom), and therefore no model fit. Subsequently, the change variables were additionally regressed on the predictor variables (fraction performance, working memory, whole-number and fraction estimation, symbolic and nonsymbolic comparison), which resulted in a poor model fit, X2(6) = 31.89, $p < .001$, RMSEA = .23, CFI = .459, TLI = .368, SRMR = .117. Among predictor variables, bootstrapped results (1000 iterations) showed effects for fraction performance (b = 0.58; 0.18 to 1.03), whole number line estimation (b = -0.95; -1.86 to -0.10), and fraction number line estimation (b = 1.24; -1.90 to -0.69). The source of the lack of model fit was that the model as specified above assumed that pre-test performance was unrelated to the other predictor variables. When these covariances were allowed, the result was a just identified model (0 degrees of freedom). Notably, however, results were similar to the previous model, with bootstrapped results identifying fraction performance (b = 0.58; 0.16 to 1.01), whole number line estimation (b = -0.95; -1.91 to -0.15), and fraction number line estimation (b = -1.19; -1.83 to -0.63) as significant predictors. Other parameters appear in Figure 1.

- Insert Figure 1 about here -

For clarity, the results of the latent change model were compared to more typical approaches to evaluating the impact of earlier knowledge on later performance (repeated measures ANOVA as well as ANCOVA). For repeated measures ANOVA, run on participants with complete data, there was a significant effect for time from pre-test to post test (Wilks' $\lambda = 0.78$, $F(1,68) = 18.77$, $p < .0001$). When additional predictor variables were added, time no longer showed significant change from pre to post test, Wilks' $\lambda = .99$, $F(1,77) <1$, with similar means at pre 9.73 (3.77) and post, 10.06 (5.63); it was however, the case that fraction number line was a unique predictor, $t(6, 68) = -2.51$, $p = .0145$. The ANCOVA on the procedural algebra items revealed a full model with $F(7,70) = 6.01$, $R^2 = 0.37$, $p < .001$. In this case, only fractions performance was a unique contributor, $t(1,70) = 3.55$, $p < .001$, $\hat{n}^2 = 0.11$.

- Insert Table 6 about here -

Discussion

The goal of the current study was to evaluate essential predictors of algebra learning and their relative roles when assessed in the context of one another. Key predictors included domain specific magnitude (estimation and comparison) and domain general cognitive factors (working memory), which were evaluated in the context of key proximal achievement measures (fractions). Measures involving rational numbers (fractions performance and fraction number line estimation) were clearly the strongest contributors to final algebra performance. A unique contribution of the present study is that these predictors were also clearly relevant for growth in skill over the course of Algebra I. These results advance our understanding of the skills needed to effect change in algebraic understanding and expand a relatively small body of work in this area.

Predictors Relating to Algebra

We originally hypothesized that fraction competency would most strongly predict algebra, followed by fraction number line, whole number line, working memory, and symbolic comparison. Consistent with these hypotheses, fraction competency and fraction number line were each significant predictors, though fraction number line was the stronger predictor. In contrast, whole number line, symbolic comparison, and working memory were not unique predictors, along with nonsymbolic comparison (though the impact of nonsymbolic comparison was not expected to be strong).

Fraction competency has been consistently shown to be one of the strongest predictors of algebra (NMAP, 2008) and lower level math skills are generally found to be the strongest predictors of other math skills due to their hierarchical nature (Kilpatrick et al., 2001). Fraction competency can be connected to algebra through basic understandings of algebra properties (e.g. the distributive property). The current study continues to add to this research base, showing the important role of rational numbers in early success of algebra performance separated from magnitude performance (DeWolf, Bassok, & Holyoak, 2015). We empirically supported the findings of Booth et al. (2014) that fraction number line performance was predictive of algebra performance but whole number line performance was not. The current study also adds to the theory that a key link between rational numbers and algebra is fraction magnitude. The connection between rational numbers and magnitude is evident from examination of algebra problems. Students learn that you can take things from one side of an equation to another through division, which creates a fraction. Magnitude skills are used to help make these decisions by identifying the relations one fraction may have to another.

However, our prediction regarding whole number line performance was not supported. Whole number line estimation has been shown to consistently relate to math performance (Hansen et al., 2015), but did not show a significant relationship with algebra performance at the end of the year in the current study. Booth, Newton, and Twiss-Garrity (2013) also found that fraction magnitude knowledge and not whole number knowledge predicted components of algebra performance. A potential reason for the relatively greater effect of the fraction vs. whole number line is that rational numbers are more complex than whole numbers. The specificity of the fraction number line task implies an increased knowledge in the relations between numbers. For algebra this higher order estimation skill appears to have greater import in this study supporting the previous work by Booth, Newton, and Twiss-Garrity (2013).

There is limited research into the role of symbolic comparison and algebra. What evidence exists does show a relationship (Geary et. al., 2015) although there is a clearer role in early mathematics. Schneider et al (2015) found that as children age and learn more complex math skills the role of symbolic comparison decreases. Due to the role of symbolic comparison in the development of the mental number line, magnitude estimation skills may take on a greater role as this development coincides with more complex math skills (e.g. algebra) and the role of symbolic comparison decreases.

Our prediction regarding working memory performance being significantly related to algebra performance was not supported. Our hypothesis was based on the growing literature of the role of working memory in algebra performance (Fuchs et al., 2014; Tolar et al., 2009). Studies have found that working memory performance is related to fraction competency (Cirino et al., 2016) but these effects were indirect. When working memory was evaluated in multiple regression with nonsymbolic comparison or symbolic comparison, working memory was a sifnificant predictor in those models. When fraction number line, whole number line, or fraction performance were added to the model then working memory was not significant.

Despite the fact that previous research shows that nonsymbolic comparison relates to math performance (De Smedt et al., 2013; Matthews et al., 2015) it is still an area of debate, as other research has found no significant relationship (Lyons et al., 2012; Holloway & Ansari, 2009). As it pertains to algebra, the very limited research on the relationship between nonsymbolic comparison and algebra has shown that it can be eliminated when accounting for symbolic comparison (Geary et. al., 2015). In addition, a recent meta-analysis demonstrates a decrease in the role of magnitude comparison skills as children age (Schneider et al., 2015).

There was significant evidence of change related to time from pre- to post-test algebra through repeated measures ANOVA. When cognitive predictors were added to the model there was no longer a significant effect of time, which may imply that these cognitive skills are what drive change. With all cognitive factors in the model the only significant predictor of change was the fractions number line task. Even when accounting for pre-test algebra in the model the only significant predictor was still the fraction number line performance. The role of fraction number line in the repeated measures ANOVA reinforces the critical role that fraction magnitude skills may play in algebra performance and learning. The results from the ANCOVA show that fraction performance and fraction number line significantly predict change. The role of fraction performance and fraction number line estimation found in the ANCOVA are consistent in with the multiple regression results from the end of the year algebra. These results show a significant relationship with fractions and fraction number line and a moderate, but not significant, relationship between whole number line and algebra performance at the end of the year. The ANCOVA, repeated measures ANOVA, and latent change results add support to findings that fraction magnitude skills are important for learning algebra (Booth, Newton, & Twiss-Garrity, 2013). Booth and Newton (2012) found that fraction magnitude skills relation with algebra involves more than just automaticity of algebra knowledge. A fraction, for example ¼, occupies a position on the number line, and other fractions (e.g., ¾) similarly have a position, and these quantities can be evaluated and compared, which may allow for increased understanding of an algebraic concept like slope. As a child becomes more familiar with these relationships, this will contribute to faster processing of this information.

For the algebra test, procedural and conceptual problems could not be reliably separated. Rittle-Johnson, Schneider, and Star (2015) were clear about the bidirectional relationship between procedural and conceptual knowledge. They called their relationships "interwoven" and "mutually reinforcing." These results are also supported by NMAP (2008). There still may be a need to identify whether procedural or conceptual instruction should come first in instruction but the results of this study imply their overlap is so great that they will continue to mutually reinforce each other. Developing effective interventions and instructions in both methods may be more effective for learning. This is specifically relevant considering that fraction magnitude estimation contains both procedural and conceptual components. Training skills that are important for both may allow for more efficient intervention or instruction for children as they prepare for algebra.

From the standpoint of an educator these results can help guide focus in the instruction of rational numbers and algebra (e.g., helping students to learn relationships to fractions through explicit instruction of fraction magnitude through the use of number lines). This will allow for increased procedural and conceptual knowledge of fractions but also potentially increase preparedness for algebra. Perhaps explicit teaching of such skills at earlier points might sharpen the representations of the mental number line, leading to a stronger preparation for algebra learning.

Limitations

The current study did have some limitations that may have impacted results. First, sample size was relatively small; however, many of the zero-order correlations in Table 3 are consistent with prior work (e.g., Tolar et al., (under review); Geary et al., 2015), and our models were not over-parameterized, and a number of related studies utilized similar sized samples (DeWolf et al., 2015; Booth & Newton, 2012). Thus, while it would be important to replicate these results in larger and/or different populations (particularly those with more diversity on the more or less uniform covariates examined here), they are more likely to complement than contradict these results. Second, not all students were assessed at both the beginning and end of year. However, as noted above, the differences in these groups was small, and the change analyses (which focused on students with both time points) found a similar set of unique predictors. Third, each predictive and outcome construct was defined with a single variable, and thus is subject to its particular psychometric characteristics. However, the properties of the utilized variables were good, and a reasonable pattern of relationships was obtained. Also, the measures chosen were consistent with the measures of prior studies (e.g., Geary et al., 2015; Hansen et al., 2015). Nonetheless, broader latent variable analyses may offer more precision for future studies. Likely most significant, the present study was not successful in differentiating conceptual from procedural algebra items. Being able to differentiate items along such lines would no doubt have led to potentially more impactful conclusions. However, as noted above, separating these skills is difficult (NMAP, 2008) and combined with their bidirectional relationships (Rittle-Johnson, Schneider, & Star), it is possible that similar results may have been obtained for both.

Conclusion

Despite the previously mentioned limitations, the current study found evidence of significant contributions of fractions performance and fraction number line to predicting end of year algebra performance. These results were robust in the context of other relevant predictors, as well as when beginning of the year algebraic knowledge was taken into account. The results emphasize the role that clear understanding of rational numbers plays in the development of mathematics, supporting much recent work, but also extending this impact specifically to algebraic learning across a school year. It is likely that instructional techniques and cognitive supports that encourage an understanding of rational numbers and how these are implemented in algebra, prior to or during Algebra I will be beneficial for enhancing students' understanding of algebra and beyond.

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Table 1

Participant Demographics

Variable	**Mean (SD) / Frequency**
Female	44.7%
Male	53.2%
Not Reported	2.1%
Age	15.03 (0.62)
Race	
White	5.32%
Black	24.5%
Hispanic	61.7%
Other	1.1%
Not Disclosed	7.4%
Special Education	
Yes	9.6
No	83.0
Not Disclosed	7.4%
Lunch Status	
Free Lunch	91.5%
Reduced Lunch	1.1%
Not Disclosed	7.4%

Note: SD= Standard Deviation

Table 2

Descriptive Statistics for All Variables

Variable	N	Mean	Std Dev	Kurtosis	Skewness
Academic Tests					
Algebra	86	14.63	6.38	0.75	0.69
Brown/Quinn Fractions	94	5.62	2.88	0.46	0.70
Cognitive Tests					
Symmetry Span	91	23.21	7.94	0.54	-0.40
Panamath	89	0.21	0.07	1.89	1.18
Symbolic	91	651.7	134.61	1.61	1.22
Comparison	90	1	1.19	13.43	3.30
Whole Number Line	80	1.90	2.17	-0.20	0.66
Fraction Number Line		3.82			

Note: N = number of participants; Std Dev = standard deviation

Table 3

Correlational Matrix for All Variables

	Alg	P Alg	Frac	SS	Pan	SC	WNL	**FNL**
Alg	1.00	-	-	-	-	-	-	-
P Alg	0.94**	1.00	-	-	-	-	-	-
Frac	0.50**	0.44**	1.00	-	-	-	-	-
SS	0.28*	0.31**	0.10	1.00	-	-	-	-
Pan	-0.10	-0.06	-0.01	-0.17	1.00	-	-	-
SC	-0.20	-0.21	0.13	-0.34**	0.21	1.00	-	-
W NL	-0.34**	-0.37**	-0.07	-0.29**	0.17	0.30**	1.00	-
F NL	-0.61**	-0.62**	-0.36**	-0.15	-0.03	0.35**	0.32**	1.00

Note: *=p<.05, **=p<.01; Alg = Algebra; P Alg = Procedural Algebra; SS = Symmetry Span (working memory); SC = Symbolic Comparison; WNL = Whole Number Line; FNL = Fraction Number Line.

Table 4
Multiple Regression Results for Working Memory, Magnitude Comparison, Magnitude Estimation, and Fractions for End of Year Algebra

	B	SE (B)	β	p	$\hat{n}^2$
Working Memory					
Symmetry Span	1.79	0.07	0.13	0.08	0.02
Magnitude Comparison					
Panamath	-1.19	9.91	-11.80	0.24	0.01
Symbolic Comparison	1.63	0.01	0.01	0.11	0.02
Magnitude Estimation					
Whole Number Line	-1.65	0.52	-0.86	0.10	0.02
Fraction Number Line	-4.45	0.32	-1.40	<0.0001	0.15
Academic					
Fractions	3.40	0.21	0.71	0.001	0.09

Note: B = unstandardized regression coefficient; β = standardized regression coefficient; $\hat{n}^2$ = eta-squared, a measure of effect size

Table 5
Multiple Regression Results for Working Memory, Magnitude Comparison, Magnitude Estimation, and Fractions for End of Year Procedural Algebra

	B	SE (B)	β	p	$\hat{n}^2$
Executive Function					
Symmetry Span	1.38	0.07	0.10	0.17	0.02
Magnitude Comparison					
Panamath	-1.12	9.22	-10.3	0.27	0.01
Symbolic Comparison	0.64	0.00	0.00	0.52	0.00
Magnitude Estimation					
Whole Number Line	-1.97	0.47	-0.92	0.052	0.03
Fraction Number Line	-2.03	0.29	-0.59	0.046	0.04
Academic					
Fractions	3.85	0.19	0.73	0.000	0.13

Note: B = unstandardized regression coefficient; β = standardized regression coefficient; $\hat{n}^2$ = eta-squared, a measure of effect size

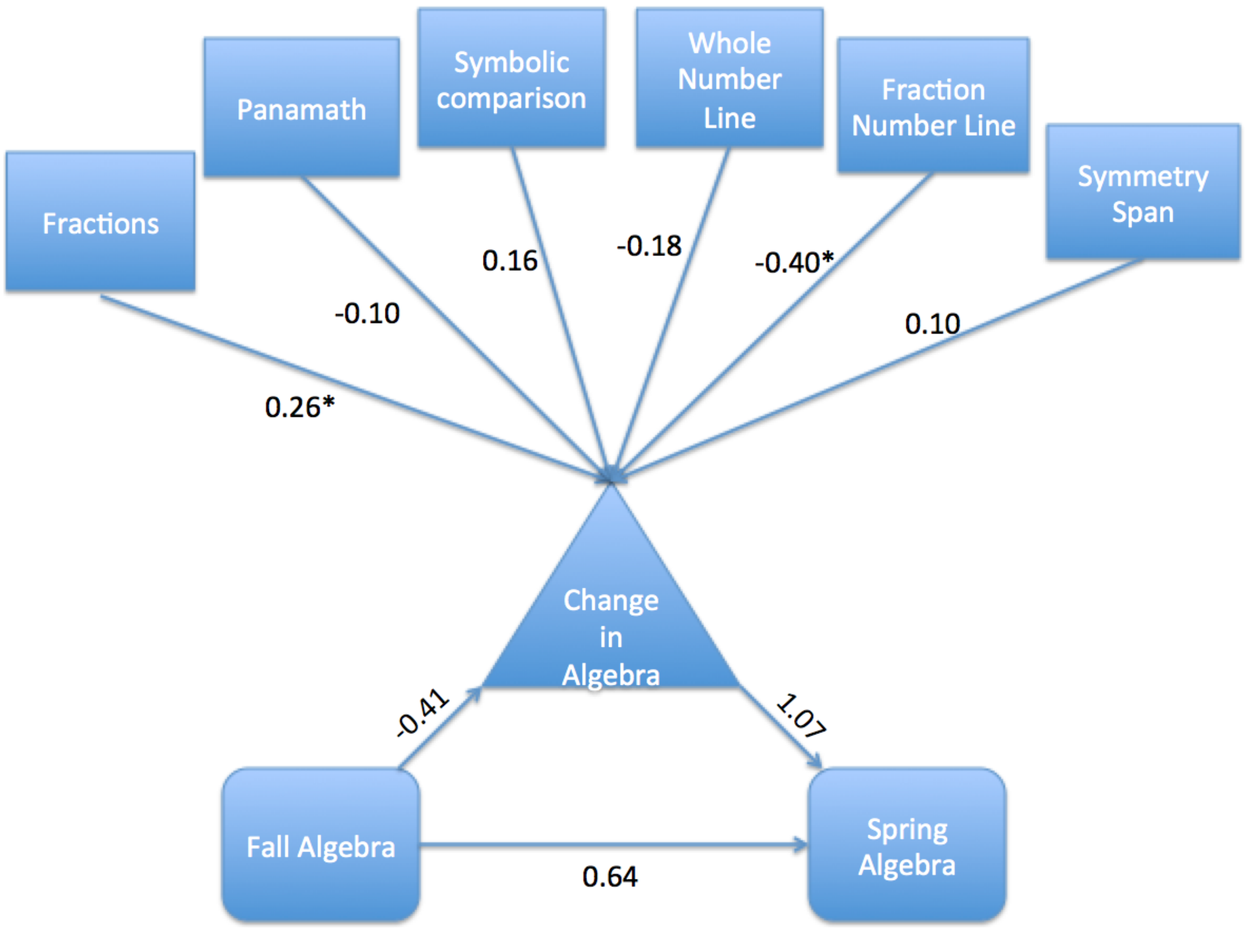

Figure 1. Standardized Estimates for end of year algebra performance model for latent change score. *Note* *= $p < 0.05$. The standardized estimates mean that for each standard deviation of change for fractions, the change in algebra will be 0.26 in the same direction.