SMALL SEMILATTICES AND COSTABILITY

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Yiu-Wa Lau

August, 1971

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ABSTRACT

This dissertation centers its attention firstly on topological semilattices with small semilattices and some equivalences of that property. This property is highly related to finite dimensional locally connected compact topological semilattices and the extension property with respect to finite subsemilattices.

A topological semigroup S is said to be costable with respect to a class of semigroups if for each semigroup T in the class and given a continuous homomorphism from T onto S, then cd T $\geq$ cd S where cd is the codimension function. The basic example of a compact semilattice which is costable with respect to compact semilattices is a 1-dimensional compact semilattice that fails to have small semilattices. With this basic example, we can construct higher dimensional costable semilattices.

Although costability stems from looking at semilattices, there is no reason why we cannot consider compact semigroups in general. Chapter 4 is intended to illustrate various classes of costable semigroups, and hopefully, these classes can characterize some of the commutative costable semigroups.

CONTENTS

Chapter 1

Introduction

We shall first establish some topological preliminaries. The topological spaces that will be used will almost always be compact Hausdorff spaces.

A continuum is a compact connected Hausdorff space, and a P-continuum is a locally connected continuum. A Peano continuum is a metric P-continuum.

Suppose $\{S_\sigma\}_{\sigma \varepsilon D}$ is a collection of spaces indexed by a directed set D, and for each $\sigma < \beta$ in D, there exists a continuous function $f^\beta_\sigma : S_\beta \to S_\sigma$. We define inverse limit of S_σ as

$$\text{Inv Lim } S_\sigma = \left\{ (X_\sigma)_{\sigma \varepsilon D} \middle| f^\beta_\sigma (X_\beta) = X_\sigma \right\}$$

We shall have occasions to use some cohomology [3], which will be the Alexander-Wallace-Spanier cohomology with some fixed nontrivial abelian group as coefficients.

A compact space X is said to have codimension $\leq$ n [3] if for each closed subset A, $i : A \to X$, $i(a) = a$, then $i^*: H^n(X) \to H^n(A)$ is onto. This is also equivalent to $H^{n+1}(X, A) = 0$ for each closed subset A of X. If $i:(A, B) \to (C, D)$ is an inclusion map and e is an element of $H^n(C, D)$, then we shall write $e|(A, B)$ for $i^*(e)$.

The codimension (cd) of X is equal to n if cd X $\leq$ n and cd X $\nleq$ n - 1. In the text, the dimension function will always be codimension, unless specified. A space X is <u>acyclic</u> if $H^n(X) = 0$ for each n. We shall also use the equivalence that given a compact Hausdorff space Y, then Y is 0-dimensional if and only if Y is totally disconnected (i.e., every component is a singleton set).

In Hurewicz and Wallman [5], inductive dimension (ind) is defined, and it is known that cd X $\leq$ ind X for every compact Hausdorff space X. Also if $f:X \to Y$ is a continuous function from a compact Hausdorff space X to another such space Y, then

$$\text{ind } X \leq \text{ind } Y + \text{ind } f \quad ,$$

where

$$\text{ind } f = \max \{ n \mid \text{ind } f^{-1}(p) = n, p \in Y \}$$

We shall establish similar results for codimension.

<u>Theorem</u> 1.1. Let $f:X \to Y$ be a continuous function from a compact Hausdorff space X to another such space Y. Then

$$\text{cd } X \leq \text{ind } Y + \text{cd } f.$$

<u>Proof</u> If ind Y = -1, then it is vacuously satisfied since Y is assumed to be nonempty. Let ind Y = n $\geq$ 0. Induct on n. Let cd f = m, and we shall show cd X $\leq$ n + m.

Suppose cd $X \nleq n + m$. Then there exists $e \neq 0$, $e \in H^{n+m+1}(X, A)$ for some closed subset A. By the Hausdorff Maximality Principle, we can find a closed subset F of X such that $e|_{(F,\ F\cap A)} \neq 0$ and if K is a proper closed subset of F, then $e|_{(K,\ K\cap A)} = 0$. Let $h = e|_{(F,\ F\cap A)}$.

We can see that f(F) is not a point; otherwise cd $f \neq m$. Since ind $Y = n$, then ind $f(F) \leq n$, and it follows from the definition that there exists P and Q, closed subsets of Y such that $f(F) = P\cup Q$ and $\text{ind}(P\cap Q) \leq n - 1$. Let $R = F\cap f^{-1}(P)$ and $S = F\cap f^{-1}(Q)$.

Note $S\cap R = F\cap f^{-1}(P\cap Q)$. Hence by induction $\text{cd}(S\cap R) \leq n + m - 1$.

Consider the Meyer-Vietoris sequence[15]

$$H^{n+m}(S\cap R,\ S\cap R\cap A) \to H^{n+m+1}(F,\ F\cap A)$$

$$\xrightarrow{J^*} H^{n+m+1}(S,\ S\cap A) \times H^{n+m+1}(R,\ R\cap A)$$

But $H^{n+m}(S\cap R,\ S\cap R\cap A) = 0$. Therefore J^* is one to one, and so $J^*(h) \neq (0, 0)$. But

$$J^*(h) = (h|_{(S,\ S\cap A)},\ h|_{(R,\ R\cap A)}) = (0, 0) \quad ,$$

since R and S are proper closed subsets in F. This is a contradiction. Hence cd $X \leq m + n$.

<u>Corollary</u> 1.2. If X is a compact Hausdorff space, then cd $X \leq$ ind X.

<u>Proof</u>. Take f to be the identity function on X.

Corollary 1.3. $cd(A \times B) \leq ind\ A + cd\ B$ where A, B are compact Hausdorff spaces.

Proof. Take the projection function $\pi(a, b) = a$ for each (a, b) in $A \times B$.

In fact, we can conclude a stronger result for lower dimensional spaces [3].

Theorem 1.4. Let cd X = n, ind Y = 1, where X, Y are compact Hausdorff spaces. Then $cd(X \times Y) = 1 + n$. Furthermore, if ind Y = 0 (i.e., Y is totally disconnected) then $cd(X \times Y) = n$.

The various notions of boundary points have been studied [4]. We call a point p in a space X a marginal point if for each open set U containing p, there exists an open set V such that $p \in V \subseteq U$ and

$$H^n(X, X\backslash V) = 0 \quad \text{for each } n.$$

Also, p is a peripheral point if for each open set U containing p, there exists an open set V such that $p \in V \subseteq U$ and if $i : (X, X\backslash U) \to (X, X\backslash V)$ is the inclusion map, then

$$i^* : H^n(X, X\backslash V) \to H^n(X, X\backslash U)$$

is a zero homomorphism for each n. A point is called an inner point if it is not peripheral.

These concepts have been studied by Lawson and Madison [12]. The following results are of great interest.

Theorem 1.5. Let cd X = n where X is a compact Hausdorff space. Then if K is a compact subset of X such that each point of K is peripheral in X, then cd K $\leq$ n - 1.

A partial order is a reflexive, transitive, and antisymmetric relation on X.

If $\leq$ is a partial order on X and A is a subset of X, then

$L(A) = \{x \in X \mid x \leq a \text{ for some } a \text{ in } A\}$,

$M(A) = \{x \in X \mid a \leq x \text{ for some } a \text{ in } A\}$,

$L(x) = L(\{x\})$ for any x in X.

and $M(x) = M(\{x\})$.

An arc chain is an arc in X such that for any two points a, b in the arc, either a $\leq$ b or b $\leq$ a.

Theorem 1.6. Let (X, $\leq$) be a compact partially ordered space where

(1) L(x) is acyclic for each x in X ,

(2) X has a least element .

Then a maximal element in the partial order is marginal in X.

We shall call a function f from a topological space to another space a light function if $f^{-1}(x)$ is totally disconnected for each x whenever $f^{-1}(x)$ is nonempty. Furthermore, a function f is called a monotone function if $f^{-1}(x)$ is connected for each x.

Let us insert another topological lemma which is useful for later purposes.

Lemma 1.7. If Y is a compact Hausdorff space, then Y is the continuous image of a totally disconnected compact Hausdorff space.

Proof. Since Y is a compact Hausdorff space, it can be embedded into a product of some family of copies of the interval [0, 1], $\mathbb{P}I_\sigma$. But for each σ, we can find C_σ, the canonical Cantor set, and a continuous function f from $\mathbb{P}C_\sigma$ onto $\mathbb{P}I_\sigma$. Hence, $f|f^{-1}(Y)$ is a continuous function from a totally disconnected compact Hausdorff space onto Y.

Let us now turn to some semigroup preliminaries. A topological semigroup S is a Hausdorff space endowed with an associative continuous operation $\cdot : S \times S \to S$. We shall write E(S) to be the collection of all elements of S with the property $x^2 = x$ (x is an idempotent). We shall denote by M(S) the minimal ideal of S. All other notions of compact semigroups can be found in [4]. In particular, we use the notions of Rees quotient, Green's relations, maximal subgroups, and compact simple semigroups.

A topological semilattice (TSL) is a topological semigroup with S = E(S) and xy = yx for all x, y in the semigroup (i.e., commutative). There is a partial order defined by $x \leq y$ if xy = x. When we write L(A), we mean this specific partial order.

There is another partial order which is of special interest. We define $x \Delta y$ if $x = y$ or there exists an arc chain from x to y where $x < y$. An element which is maximal in this order is called a semimaximal element.

It is easy to see that every maximal element ($\leq$) is a semimaximal element. Also, since the lower set (with respect to Δ) of a point is contractible (thus acyclic) and zero is arc-chainable to every point, then an application of theorem 1.6 yields that every semimaximal element is marginal.

A TSL S has small semilattices at a point p in S if for each open set U containing p, there exists a neighborhood N of p such that N is a subsemilattice contained in U. The property of having small semilattices (at each point) for compact TSL's is preserved under subsemilattices, homomorphic images, and topological products [10]. There are also some equivalences of having small semilattices which will be stated in chapter 2. Examples of compact TSL's having small semilattices are compact 0-dimensional TSL's, finite dimensional P-continuum TSL's [11], and compact TSL's with countable number of semimaximal points [10].

It is also important to note that if S is a compact TSL and L(x) has small semilattices at x, then S has small semilattices at x[10]. Hence every compact TSL always has small semilattices at the zero element.
It is not so surprising that it has small semilattices at zero since for each open set U containing zero, $L_0(U)$ is an open subsemilattice (in fact, an ideal).

In Lawson's dissertation [10], the following conditions are shown to be equivalent.

Theorem 1.7. Let S be a compact TSL. These are equivalent:

(a) For each x in S, M(x) is connected.

(b) If $y < z$, then there exists an arc chain from y to z.

(c) S is order dense.

It is tempting to ask the question: "Does every continuum TSL have small semilattices?" Lawson [7] has constructed a one dimensional compact connected TSL without small semilattices at its identity. In fact, the hyperspace of the closed ideals of that example is a lattice (thus locally connected) and it does not have small semilattices.

Let us look at some semigroups on the interval [0 ,1]. If 0 acts as a zero and 1 acts as an identity, then for following examples are "essentially" all the semigroups [4] that we could put on [0, 1], namely the usual real number multiplication (called the usual interval), xy = min of x and y (called the min interval), or the usual interval modulo the closed ideal $[0, \frac{1}{2}]$ (called a nil interval).

We say that a semigroup S acts on a space Z if there exists a continuous function $f: S \times Z \to Z$ where $f(st, z) = f(s, f(t,z))$ for all s, t in S and z in Z.

CHAPTER 2

Small Semilattices

In this chapter, we shall show some equivalences of the property of having small semilattices. It is known [10] that, for compact TSL's, having small semilattices is equivalent to the embeddability of the compact TSL into a product of min intervals iseomorphically, where an iseomorphism is an isomorphism which is also a homeomorphism. We shall denote an iseomorphism by $\cong$.

Let us define Hom (S, I) to be the collection of all continuous homomorphisms from the TSL S into the min interval I. We shall say that Hom (S, I) separates points if given x, y in S, $x \neq y$, there exists f in Hom (S, I) such that $f(x) \neq f(y)$. In fact, Hom (S, I) separating points is equivalent to having small semilattices in the case of compact TSL's.

It is known that a finite dimensional compact locally connected TSL has small semilattices, but is it true that in some way we could "make" TSL's having small semilattices from finite dimensional compact locally connected TSL's? The following theorem shows a positive answer to the question.

Theorem 2.1. Let S be a compact TSL. Then S has small semilattices if and only if $S \cong \text{Inv Lim } S_\sigma$ where S_σ is a finite dimensional locally connected compact TSL.

Proof. Suppose $S \cong \text{Inv Lim } S_\sigma$. Since S_σ is a finite dimensional locally connected compact TSL, S_σ has small semilattices. Since $\text{Inv Lim } S_\sigma$ is contained in S_σ, then it has small semilattices.

So let us assume S has small semilattices. Then $S \cong T$ for some compact TSL T contained in a product of min intervals. We can actually take

$$T \subseteq \prod_{\sigma \varepsilon A} I_\sigma$$

where A is Hom (S,I) and I_σ is the min interval.

Let C be the collection of all nonempty finite subsets of A. Then C can be made into a directed set by inclusion. We shall denote n(F) to be the number of elements in F for each F in C. If n = n(F), subdivide I_σ into $\frac{1}{2^n}$ length subintervals for each σ in F. We can form n-cells contained in $\prod_{\sigma \varepsilon F} I_\sigma$ by taking product of subintervals from each I_σ. Then for each F, keep the n-cells that meet with $\pi_F(T)$ and call the union of these cells K_F. Since K_F is a finite union of n-cells, it is locally connected and compact; it also turns out to be a TSL under the induced multiplication of the product of min intervals.

We can consider $\text{Inv Lim } K_F$ by taking the continuous homomorphism from K_G to K_F $(F \subseteq G)$ to be the projection function.

Let $h: T \to \prod_{F \varepsilon C} K_F$ be given by $h(t) = (\pi_F(t))_{F \varepsilon C}$.

Clearly h embeds T iseomorphically into $\text{Inv Lim } K_F$. The difficult part is to show $h(T) = \text{Inv Lim } K_F$.

Let (k_F) be an element of Inv Lim K_F. We shall show first that $k_F \varepsilon \pi_F(T)$ for each F.

Suppose there exists F such that k_F is not in $\pi_F(T)$. Then $d(k_F, \pi_F(T)) = c > o$, where d is the Euclidean metric on $\prod_{\sigma \varepsilon F} I_\sigma$.

Choose G in C such that $F \leq G$ and $n(G) = n$ and the diameter of each cell in $\prod_{\sigma \varepsilon G} I_\sigma$ is less than c.

Since k_G belongs to some n-cell meeting $\pi_G(T)$, then $d(k_G, \pi_G(t)) < c$ for some t in T.

$$d(k_F, \pi_F(t)) = \sqrt{\sum_{\sigma \varepsilon F} (k_\sigma - \pi_\sigma(t))^2}$$

$$\leq \sqrt{\sum_{\sigma \varepsilon G} (k_\sigma - \pi_\sigma(t))^2}$$

$$= d(k_G, \pi_G(t))$$

$$< c \quad .$$

But this is a contradiction to $d(k_F, \pi_F(T)) = c$.

Therefore we can assume $k_F \varepsilon \pi_F(T)$ for each F. Then $k_F = \pi_F(t_F)$ for some t_F in T.

Let t_F cluster to some t in T. Fix G in C. We shall show that $\pi_G(t) = k_G$.

Notice $\pi_G(t_F)$ clusters to $\pi_G(t)$ since π_G is a continuous function.

We shall prove that $\pi_G(t_F)$ converges to k_G. (In fact, $\pi_G(t_F)$ is residually equal to k_G).

Let U be an open set containing k_G. Consider $F \cup G$. For each H in C such that $F \cup G \subseteq H$,

$$\pi_H(t_H) = k_H \quad ;$$

$$\pi_G \, \pi_H(t_H) = \pi_G(k_H) \quad ;$$

$$\pi_G(t_H) = k_G \quad .$$

Thus $\pi_G(t_H)$ belongs to U.

Hence

$$k_G = \pi_G(t) \text{ for each } G.$$

$$h(t) = (k_F)_{F \varepsilon C} \quad .$$

Therefore, h is an iseomorphism from T onto Inv Lim S_σ. The theorem is proved.

The next step might be to try to characterize all finite dimensional P-continuum TSL's. It turns out that such a semilattice is not necessarily the inverse limit of compact TSL's with M(x) connected for each x because of the following theorem.

Theorem 2.2. If $S = \text{Inv Lim}\,(S_\sigma, f_\sigma^\beta)$ where for each x_σ in S_σ, $M(x_\sigma)$ is connected, then $M(x)$ is connected for each x in S.

Proof. Suppose $S = \text{Inv Lim}\, S_\sigma$ as in the hypothesis. Let x be in S. Then $x = (x_\sigma)$ where x_σ is in S_σ.

If $\sigma < \beta$, and y_β is in $M(x_\beta)$,

then

$$y_\beta\, x_\beta = x_\beta \quad ;$$

$$f_\sigma^\beta(y_\beta\, x_\beta) = f_\sigma^\beta\,(x_\beta) \quad ;$$

$$f_\sigma^\beta(y_\beta) \cdot x_\sigma = x_\sigma \quad .$$

hence

$$x_\sigma \leq f_\sigma^\beta(y_\beta) \quad .$$

Therefore $(M(x_\sigma), f_\sigma^\beta|_{M(x_\beta)})$ forms an inverse system.

Let $x \leq y$. Then y_σ belongs to $M(x_\sigma)$. Thus x, y belong to Inv Lim $M(x_\sigma)$ which is connected and contained in $M(x)$. Thus $M(x)$ is connected.

Another equivalence of having small semilattices is that Hom(S, I) separates points. But if we look at the latter property as an "extension property" with respect to some sort of finite sets to the whole space, we can also get another characterization of having small semilattices.

We shall say that S has the extension property with respect to finite subsemilattices if for each finite subsemilattice K and for each f in

Hom(K, I), there exists F in Hom(S, I) such that $F|_K = f$. This property is interesting in the sense that an algebraic homomorphism on K is extendable to a topological homomorphism on all of S.

Theorem 2.3. Let S be a compact TSL. Then S has small semilattices if and only if S has the extension property with respect to finite subsemilattices.

Proof. Suppose S has the extension property. Let $x \neq y$ be in S. We will show that Hom(S, I) separates x and y. We can assume that $x \nleq y$, i.e., $xy \neq x$. Then {xy, x, y} is a finite subsemilattice.

Let $f:\{xy, x, y\} \to I$ be given by

$$f(xy) = f(y) = 0 ;$$
$$f(x) = 1$$

It is easy to see that f is a homomorphism. Then we can extend f to F on S. Hence F separates x and y. But if Hom(S, I) separates points, then S has small semilattices.

Let us now assume that S has small semilattices. Let K be a finite subsemilattice of S and f be in Hom(K, I).

Induct on $|f(K)|$, the number of elements in f(K).

Suppose $|f(K)| = 2$. Then $f(K) = \{a, b\}$, where $a < b$. Let $A = L(f^{-1}(a))$ and $y = \inf f^{-1}(b)$. Since S has small semilattices, there exists a continuous homorphism F from S to [a, b] such that $F(A) = a$ and $F(y) = b$. Since $F(A) = a$ and $F(y) = b$, then $F|_K = f$.

We can assume $|f(K)| = m > 2$. Let $a = \inf f(K)$, $c = \max f(K)$, and $b = \max (f(K)\backslash c)$, and $a < b < c$. Note that $|f(f^{-1}[a, b])| < m$. Then there exists g from S to $[a, c]$ such that $g|_{f^{-1}[a, b)} = f$ and $g(f^{-1}(b)) = c$.

Also, $|f(f^{-1}[b, c])| < m$. Then we can find $h:S \to [b, c]$ such that $h|_{f^{-1}[b, c]} = f$. Now let $F(x) = g(x)\, h(x)$ for each x in S. It is easy to show $F|_K = f$.

It is natural to ask the question whether every compact TSL with small semilattices has the extension property with respect to closed subsemilattices. This turns out to be false in general. In fact, if that were true, then each arc chain (metric) in such a compact TSL is a retract of the TSL if we take the identity function on the chain and extend it. And there are folklore examples which do not have such a property.

But if a compact TSL does not have small semilattices at a point, then "how many" other points are there at which it does not have small semilattices? The following theorem and corollary answer that question.

Theorem 2.4. Suppose S is a compact TSL with identity and does not have small semilattices at 1. Then there exists an open set U containing 1 such that S does not have small semilattices at x for each x in U.

Proof. Let us assume the contrary. Then there exists a net $\{x_\sigma\}_{\sigma \in D}$ converging to 1 such that S has small semilattices at x_σ for each σ.

Consider $f:S \to \prod_{\sigma \in D} L(x_\sigma)$

by $f(x) = (xx_\sigma)_{\sigma \in D}$.

Then f is a continuous homomorohism. Also, $L(x_\sigma)$ has small semilattices at x_σ. Since S does not have small semilattices at 1, there exists V, an open set containing 1, such that any neighborhood of 1 contained in V is not a subsemilattice.

Since $S \backslash V$ is compact, $f(S \backslash V)$ is closed. We will show $(x_\sigma) \in f(S) \backslash f(S \backslash V)$.

Suppose $(x_\sigma) \in f(S \backslash V)$. Then there exists y in $S \backslash V$ such that $f(y) = (x_\sigma)$.

Hence $(yx_\sigma) = (x_\sigma)$;

$yx_\sigma = x_\sigma$ for each σ.

But x_σ converges to 1; y = 1. But 1 is in V, a contradiction.

Therefore, (x_σ) belongs to $f(S) \backslash f(S \backslash V)$ which is open in f(S). But f(S) has small semilattices at $(x\sigma)$. Choose W, a neighborhood of (x_σ) and a subsemilattice contained in $f(S) \backslash f(S \backslash V)$.

Then 1 belongs to $f^{-1}(W)$ which is clearly contained in V. We have a contradiction.

Corollary 2.5. Suppose S is a compact TSL which does not have small semilattices at a point. Then S does not have small semilattices at an infinite number of points.

Proof. Suppose S does not have small semilattices at x. Then L(x) does not have small semilattices at x, and x is the identity in L(x). By the previous theorem, there exists a set U open in L(x) and containing x such that L(x) does not have small semilattices at any point in U. But U is not a finite set; otherwise {x} would be open in L(x) and L(x) has small semilattices at x.

Since L(x) does not have small semilattices at each y in U, L(y) does not have small semilattices at each point y in U. Hence, S does not have small semilattices at each point of U, and U is an infinite set.

Because of the above theorem, we can get some nice examples. Suppose we have a compact TSL S which does not have small semilattices. We can cone S over the min interval to make S connected and still the cone does not have small semilattices. Thus, S can be assumed to a continuum.

Since S does not have small semilattices at some point x, we can get an example not having small semilattices at its identity by looking at L(x), which is a retract of S. We may assume S has an identity.

Now we can get an open set U containing 1 as in theorem 2.4. Then T = L(S\U) is a closed ideal in S.

Thus, S/T is a continuum TSL where Hom (S/T, I) does not separate x and 0 for each x $\neq$ 0. Also, S/T does not have small semilattices at any non-zero point.

To conclude this chapter, we shall state the following theorem which gives some equivalences of having small semilattices.

<u>Theorem</u> 2.6. Let S be a compact TSL. These are equivalent:

(a) S has small semilattices;

(b) Hom (S, I) separates points;

(c) $S \cong \text{Inv Lim } S_\sigma$ where S_σ is a finite dimensional compact locally connected TSL;

(d) S has the extension property with respect to finite subsemilattices.

CHAPTER 3

Costability Concerning TSL's

This chapter is about the concept of costability and some examples of costable semigroups. The most basic example is that of a 1-dimensional compact TSL which does not have small semilattices at some point. It has a special property with respect to dimension.

Theorem 3.1. If S is a compact one dimensional TSL which does not have small semilattices, then there is no 0-dimensional compact TSL T with a continuous homomorphism from T onto S.

Proof. Suppose there is $f:T \to S$. Since T is 0-dimensional, then T is the inverse limit of finite TSL's [14], thus having small semilattices. But S is a homomorphic image of T; S has small semilattices. But this is a contradiction.

From the above theorem, any compact TSL which is mapped onto S must be at least one dimensional, which leads to the definition of costability.

Suppose S is a compact semigroup. Then we say S is costable with respect to a class of semigroups if for each semigroup T in the class and a continuous homomorphism from T onto S, then $\text{cd } S \leq \text{cd } T$.

Therefore, another way to look at theorem 3.1 is to say that S is costable with respect to compact TSL's. In fact, S is costable with respect to compact semigroups. Before we do that, we shall prove a lemma.

Lemma 3.2. If S is a compact totally disconnected semigroup and S has an identity, then S has small semigroups at 1.

Proof. Since S has those properties, S is iseomorphic to an inverse limit of finite semigroups, each of which has an identity, i.e., $S \triangleq \text{Inv Lim } S_\sigma$ where S_σ is a finite semigroup. Let 1 belong to $U \cap \text{Inv Lim } S_\sigma$ where U is open in $\mathbb{P}S_\sigma$. Then we can find $1 \times \ldots \times 1 \times \mathbb{P}S_\sigma$ contained in U by the construction of the product topology. The $1 \times \ldots \times 1 \times \mathbb{P}S_\sigma \cap \text{Inv Lim } S_\sigma$ is an open semigroup in the inverse limit. Hence S has small semigroups at 1 .

Theorem 3.3. Suppose S is a one dimensional compact TSL. Then S is costable with respect to compact semigroups if and only if S does not have small semilattices.

Proof. Suppose S is costable with respect to compact semigroups. We want to show S does not have small semilattices. Suppose S has small semilattices at every point; then S is embeddable into some product of min intervals. Let $S \subseteq \mathbb{P}I_\sigma$ where I_σ is a min interval. Let C_σ be the canonical Cantor set with the min multiplication. Then there exists a continuous homomorphism from C_σ onto I_σ , thus inducing a continuous homomorphism f from $\mathbb{P}C_\sigma$ onto $\mathbb{P}I_\sigma$. But $f^{-1}(S)$ is also 0-dimensional and it goes onto S, which is a contradiction of S being costable. Hence, S does not have small semilattices.

Now let us suppose that S does not have small semilattices at some point x in S. Let f be a continuous homomorphism from a compact semigroup T onto S. Then $f^{-1}(x)$ is a subsemigroup of T, and we can find a minimal idempotent e in $f^{-1}(x)$ (in fact, any idempotent in the minimal ideal of $f^{-1}(x)$).

We can restrict our attention to eTe and L(x). We will assume $1 \varepsilon T$, $1 \varepsilon S$ and $f^{-1}(1) = H(1)$. Suppose T is 0-dimensional. Since S does not have small semilattices at 1, there exists an open set U containing 1 such that for each neighborhood V containing 1 and contained in U, V is not a subsemilattice.

Since $f^{-1}(U)$ is open and contains 1, we can find an open subsemigroup W such that $1 \varepsilon W \subseteq f^{-1}(U)$ by the previous lemma. Then $H(1) \cdot W$ is open. Notice $f(H(1) \cdot W) = f(H(1)) \cdot f(W) = f(W)$ which is a subsemigroup of S and $1 \varepsilon f(W) \subseteq U$.

Now all we have to show is that f(W) is a neighborhood of 1. But notice that $1 \varepsilon S \ f(T \ H(1) \cdot W) \subseteq f(W)$ and $f(T \backslash H(1) \cdot W)$ is a closed set. Then f(W) is a neighborhood of 1 contained in U and it is also a subsemigroup, thus a subsemilattice. This is a contradiction, giving us that S is costable with respect to compact semigroups.

<u>Theorem</u> 3.4. Let S be a compact 1-dimensional TSL. The following are equivalent statements:

(a) S does not have small semilattices;

(b) S is costable with respect to compact TSL's;

(c) S is costable with respect to compact semigroups.

Proof. We have shown that (a) and (c) are equivalent. It is obvious that (c) implies (b). In order to prove (b) implies (a), we can adopt the proof in the previous theorem as (c) implies (a).

So far we have only seen 1-dimensional costable TSL's. One would like to produce higher dimensional costable TSL's.

Theorem 3.5. Suppose S_i is a compact TSL costable with respect to compact TSL's, cd $S_i = 1$, cd $\prod_{i=1}^{n} S_i = n$. Then $\prod_{i=1}^{n} S_i$ is costable with respect to compact TSL's.

Proof. We would like to reduce S_i to have an identity by looking at x_i, at which S_i does not have small semilattices. Then $L(x_i)$ has codimension $\leq n$ because it is contained in S_i. Since $L(x_i)$ is not zero dimensional, we can find a nontrivial component in $L(x_i)$, thus a nontrivial arc chain in $L(x_i)$. So we have a product of n arc chains contained in $\prod_{i=1}^{n} L(x_i)$. Hence cd $\prod_{i=1}^{n} L(x_i) \geq n$. We have cd $\prod_{i=1}^{n} L(x_i) = n$. So let us assume each S_i has an identity.

Let $f: S \to \prod S_i$ be an onto continuous homomorphism. We can assume f to be light because we can monotone-light factor f [4], and a monotone homomorphism for compact TSL's does not raise codimension of the domain [1].

For each i, let

$$Q_i = f^{-1}(1 \times \cdots \times 1 \times S_i \times 1 \times \cdots \times 1)$$

and Q_i is at least one dimensional. Thus, we can find an arc chain A_i in Q_i. Let Z_i be the zero in A_i.

Consider m: $A_i \to S$ by the multiplication function. To prove cd $S \geq n$ [10], it is enough to show that $m^{-1}\, m(Z_1, \ldots, Z_n) = (Z_1, \ldots, Z_n)$.

Let b_i be in A_i such that $m(b_1, \ldots, b_n) = m(Z_1, \ldots, Z_n)$. Then $b_1 \ldots b_n = Z_1 \ldots Z_n$.

Let $1 \leq i \leq n$.

Then $\pi_i\, f(b_1 \ldots b_n) = \pi_i\, f(Z_1 \ldots Z_n)$, where π_i is the i^{th} projection function.

But $\pi_i\, f(b_1 \ldots b_n) = \pi_i\, f(b_i)$ since $\pi_i\, f(b_j) = 1$ if $j \neq i$. Similarly, $\pi_i\, f(Z_1 \ldots Z_n) = \pi_i\, f(Z_i)$.

Therefore, $\pi_i\, f(b_i) = \pi_i\, f(Z_i)$. $f(Z_i) = f(b_i)$ since $\pi_j\, f(Z_i) = 1 = \pi_j\, f(b_i)$ for $j \neq i$.

If $Z_i \neq b_i$, then the sub-arc B of A_i from Z_i to b_i is a nondegenerate continuum and $f(B) = f(Z_i) = f(b_i)$. But this contradicts the lightness of f. Hence, $Z_i = b_i$, which yields cd $S \geq n$. So S_i is costable with respect to compact TSL's.

There is another product theorem concerning compact costable TSL's.

Theorem 3.6. Suppose C is an n-dimensional compact TSL with identity and costable with respect to compact TSL's; S is a compact TSL of inductive dimension 1 and costable with respect to compact TSL's. The C x S is costable with respect to compact TSL's.

Proof. Since cd C = n, ind S = 1, we have cd(C x S) = 1 + n. Also using the same technique as before, we can assume S has an identity.

Let $f:T \to C \times S$ be a continuous onto homomorphism from T, a compact TSL.

Let $B = f^{-1}(C \times 1)$ and A be an arc chain in $f^{-1}(1 \times S)$. Let d be the identity in the arc chain A.

Let K = Bd. Then

$$\begin{aligned} f(K) &= f(B) \cdot f(d) \\ &= (C \times 1)\ (1, e) \text{ where } f(d) = (1, e), \\ &= C \times e\ . \end{aligned}$$

But C x e is iseomorphic to C. Thus, cd K $\geq$ n.

We can consider A as a semigroup acting on the space KA by way of multiplication. There is a natural quasi-order on KA by $x \leq y$ if $Ax \subseteq Ay$. Then it is easy to check that K is a set of maximal points in this quasi-order. But under this action, K will be a set of marginal points. Applying theorem 1.5, we can conclude that cd KA $\geq$ n. Hence, cd T $\geq$ n + 1.

Theorem 3.7. Let S be a compact TSL with identity, cd S = n. Then there exists a continuous homomorphism from C onto S, where C is a compact TSL with identity costable with respect to compact TSL's and cd C = n.

Proof. Let N be the collection of all non-negative integers j such that there exists $g:K \to S$ a continuous homomorphism from K a compact TSL of dimension j onto S. Let m be the minimal integer of N. We will show that this K is costable.

Let $f:T \to K$ be a continuous homomorphism from T a compact TSL onto K. Pick e in $f^{-1}(1)$. Then $g \circ f:eTe \to S$ is a continuous onto homomorphism. Since m is minimal with respect to this property, cd eTe $\geq$ n. Hence, K is costable with respect to compact TSL's.

$$\text{Let } C = K \times \underbrace{L \times \cdots \times L}_{n-m \text{ copies}} .$$

Let $\pi:C \to K$ be the projection function $\pi(x, \ell_1, \ldots, \ell_{n-m}) = x$, and let L be an inductive dimension one compact TSL with identity and costable with respect to compact TSL's.

By theorem 3.6, we know C is costable with respect to compact TSL's and $g \circ \pi:C \to S$ is a continuous onto homomorphism.

The previous theorem indicates that any compact TSL is the homomorphic image of some costable TSL of the same dimension.

CHAPTER 4

Costability in General

It is natural to ask if there are other kinds of costable semigroups. We shall establish some such examples. Even with these semigroups, we will not be able to characterize all costable compact semigroups.

First, we prove a lemma.

Lemma 4.1. Let S be an n-dimensional compact semigroup. Then S is costable with respect to compact semigroups if and only if S contains a closed n-dimensional subsemigroup T which is costable with respect to compact semigroups.

Proof. Suppose S has such a semigroup T. Let $f:K \to S$ be a continuous homomorphism from a compact semigroup K onto S. The $f^{-1}(T)$ is mapped onto T; cd $f^{-1}(T) \geq$ cd T = n. But $f^{-1}(T)$ is contained in K; hence, cd K $\geq$ n.

The converse is clear.

Lemma 4.2. Let S be an n-dimensional compact semigroup and $g:S \to T$ where T is an n-dimensional compact semigroup costable with respect to compact semigroups. Then S is costable with respect to compact semigroups.

Proof. It is straightforward.

<u>Theorem</u> 4.3. The nil thread is costable with respect to compact semigroups.

<u>Proof</u>. Let N be a nil thread, and $f:S \to N$ be a continuous homomorphism from a compact semigroup S onto N. Let e be a minimal idempotent in $f^{-1}(1)$ and reduce S to eSe. Thus, we can assume S has an identity and $H(1) = f^{-1}(1)$.

Clearly, H(1) is not open in S; otherwise $f(S\backslash H(1))$ would be closed and $\{1\} = N\backslash f(S\backslash H(1))$ would be open in N. Also, H(1) has an idempotent-free neighborhood (i.e., a neighborhood having 1 as its only idempotent) since N has one at 1. Thus, we can produce a non-trivial arc starting from 1 in S [4]. Hence, cd $S \geq 1$.

<u>Corollary</u> 4.4. The usual thread is costable with respect to compact semigroups.

<u>Proof</u>. This is an application of theorems 4.3, 4.2.

<u>Corollary</u> 4.5. Let S be a compact semigroup with identity and of dimension one such that H(1) is not open and has an idempotent-free neighborhood. Then S is costable with respect to compact semigroups.

<u>Proof</u>. Since S has those properties, we can produce an arc from 1. Let a be a non-identity point of the arc.

Then S/SaS contains a nil thread. By applying 4.3, 4.2, 4.1, we have that S is costable with respect to compact semigroups.

Theorem 4.6. Suppose N_i is a nil thread, $i = 1, 2, \ldots, n$. Then $\prod_{i=1}^{n} N_i$ is costable with respect to compact semigroups.

Proof. Let $f: S \to \prod_{i=1}^{n} N_i$ be a continuous homomorphism. Let

$$B = \left\{ (x_i) \in \prod_{i=1}^{n} N_i \mid (x_i) \text{ has at most one non-zero entry} \right\} .$$

Then B is an ideal and we can consider $f: S \to \prod_{i=1}^{n} N_i/B$. Also we can assume S has a zero by looking as $S/f^{-1}(B)$. For each i, we can find a nil (or usual) thread

$$U_i \subseteq f^{-1}(1 \times \ldots \times 1 \times N_i \times 1 \times \ldots \times 1) .$$

Let $A = U_2 U_3 \ldots U_n$;

$$Y = \bigcup_{j=2}^{n} U_1 U_2 \ldots \hat{U}_j \ldots U_n$$

where $\hat{U}_j$ means delete U_j from the product, and

$$Z = \bigcup_{j=2}^{n} U_2 U_3 \ldots \hat{U}_j \ldots U_n .$$

By finite induction, we can assume $H^{n-2}(Z) \neq 0$. By computation, we can show

$$A \cap Y = Z .$$

Consider the Meyer Vietoris sequence:

$$H^{n-2}(A) \times H^{n-2}(Y) \to H^{n-2}(Z) \to H^{n-1}(A \cup Y).$$

Since A is contractible (using U_2) and Y is contractible (using U_1), both A and Y are acyclic; thus, $H^{n-2}(A) = 0 = H^{n-2}(Y)$. Hence, $H^{n-2}(Z) \neq 0$ implies $H^{n-1}(A \cup Y) \neq 0$.

But $A \cup Y \subseteq U_1 U_2 \ldots U_n$ and the latter is acyclic. This gives us

$$cd(U_1 U_2 \ldots U_n) \nleq n - 1.$$

Hence $cd\ S \geq n$.

It is easy to see that a product of usual threads is costable with respect to compact semigroups. But it is interesting to note that a product of min intervals is not costable.

We shall now turn our attention to another area. It is known [18] that if we use inductive dimension for compact groups, then a homomorphism cannot raise its inductive dimension. We shall look at this from a codimension point of view.

Lemma 4.7. If G is an n-dimensional compact group, then G is costable with respect to compact groups.

Proof. Let $f: H \to G$ be a continuous homomorphism from a compact group H onto G. But G is iseomorphic to an inverse limit of compact Lie groups G_i where i belongs to some directed set [18]. But $cd\ G \leq \sup cd\ G_i$, by an application of the continuity axiom [15]. Hence we have $cd\ G \leq cd\ G_i$ for some i. Then we have an onto homomorphism $g: H \to G_i$ by composing f with the natural homomorphism of G onto G_i. Let $cd\ G_i = m$.

But G_i is a Lie group and contains an m-cell which is liftable to H [18]. Hence, cd H $\geq$ m; consequently, cd H $\geq$ n.

Theorem 4.8. An n-dimensional compact group G is costable with respect to compact semigroups.

Proof. Let $f: S \to G$ be a continuous homomorphism from a compact semigroup S onto G. Let e be an idempotent in M(S). Then H(e) = eSe [4].

But f(e) = 1 since f(e) is an idempotent. Since f(H(e)) = f(eSe) = f(e) f(S) f(e) = G, by the previous lemma, cd H(e) $\geq$ n. Hence, cd S $\geq$ n.

We shall see the importance of the costability of compact groups. It turns out that this determines the costability of completely simple semigroups.

Theorem 4.9. Let S be an n-dimensional compact semigroup with S = M(S). Then S is costable with respect to compact semigroups if and only if S contains an n-dimensional compact group.

Proof. Suppose S is costable with respect to compact semigroups. Let G be a maximal subgroup of S. Then we can assume S = L x G x R, in which L, R are compact Hausdorff spaces and the multiplication is defined by

$$(1, g, r)\ (x, h, y) = (1, g\sigma(r, x)\ h, y)$$

where σ is a continuous function from R x L to G.

Suppose cd G is less than n. By lemma 1.7, we can find Y and Z, both 0-dimensional compact Hausdorff spaces, and $f:Y \to L$, $g:Z \to R$ continuous onto functions.

Let T = Y x G x Z and define a Rees multiplication on T by virtue of the function

$$\sigma o(g \times f) : Z \times Y \to G.$$

It follows that f x 1 x g is a continuous homomorphism from Y x G x z onto L x G x R, where 1 is the identity function on G. But the codimension of Y x G x Z is that of the group G, which is less than n. This contradicts the costability of S. Hence S contains an n-dimensional group.

The converse is a straightforward application of theorem 4.8 and 4.1.

We now turn to another class of semigroups called the cylindrical semigroups [4] which include the usual threads and groups.

<u>Theorem</u> 4.10. If S is an n-dimensional cylindrical semigroup, then S is costable with respect to compact semigroups.

<u>Proof</u>. Let $f:\sum \times G \to S$ be a continuous homomorphism from the product of the universal solenoidal semigroup and a compact group onto the semigroup S.

Also, let $g:T \to S$ be a continuous homomorphism from a compact semigroup T onto S. We shall show cd T $\geq$ n.

Using the same technique as before, we can assume T has an identity and $g(H(1)) = H(1)$. We shall denote the minimal ideal of Σ by K.

Case 1. Suppose S has an n-dimensional compact group. Then it is clear that S is costable with respect to compact semigroups.

Case 2. Suppose S does not have an n-dimensional group. Then $cd(f(K \times G)) < n$ since $K \times G$ is a group.

Let $P = S/f(K \times G)$; note that P also has codimension equal to n. We have the following analytic diagram:

$$\begin{array}{ccc} U \times G & \xrightarrow{h} & P = S/f(K \times G) \\ \sigma \times 1 \uparrow & & \uparrow j \\ \Sigma \times G & \xrightarrow{f} & S \end{array}$$

where U is the usual interval;

σ is the natural map of Σ onto Σ/K;

1 is the identity map on G;

j is the natural map of S onto P;

and h is the induced map.

Now, let us look at P/H where H is the H-relation. Then P/H is an interval [4]. By theorem 1.1,

$$\begin{aligned} cd\ P &\leq ind\ P/H + \sup cd\ H(x) \\ &= 1 + \sup cd\ H(x) \\ &\leq 1 + cd\ H(1), \end{aligned}$$

since cd $H(x) \leq$ cd $H(1)$ for each x in P.

Hence $n - 1 \leq$ cd $H(1)$.

Consider $j_{\circ}g:T \rightarrow P$ and let G be the maximal group of the identity in T. Then cd $G \geq$ cd $H(1)$.

Hence cd $G \geq n - 1$.

But G is not open in T since H(1) is not open in P(P is connected), and G has an idempotent-free neighborhood in T. Hence we can get an arc leaving from 1 to some point a in $T\backslash G$ [4].

Then T/TaT is a compact connected semigroup containing an iseomorphic copy of G. By a theorem in Hofmann and Mostert [4], we have cd T/TaT > cd G.

Hence cd $T/TaT \geq n$.

But cd $T \geq$ cd $T/TaT \geq n$.

CHAPTER 5

Appendix

In this chapter, we shall raise some questions. The first one concerns the characterization of commutative costable semigroups. Are they "made" out of usual (nil) threads, compact groups and costable semilattices? Here, "made" means containing a costable compact semigroup of a homomorphic pre-image of a costable semigroup.

The next question is about separation. We know that if S is a compact TSL with small semilattices, we can separate points in S by Hom(S, I). One wonders if we can use Hom(S, L) to separate points in a continuum TSL where L is some suitable TSL like that of Lawson's example. Since L contains an arc chain, we know Hom(S, L) separates points of a compact TSL with small semilattices.

We have noticed that if S is a compact 1-dimensional costable TSL, we can reduce it to L(x) where x is the point at which S does not have small semilattices. It is not known whether any TSL costable with respect to some class of semigroups is reducible to a subsemilattice with an identity and of the same dimension.

In fact, it is not known if a compact TSL of codimension n has a point x in it so that cd L(x) = n. We can provide some partial answer to this question.

Lemma 5.1. Let S be an n-dimensional compact metric TSL such that for each inner point x, M(x) has non-empty interior. Then there exists x in S such that cd L(x) = n.

Proof. Since S is compact metric, we can find $\{x_i\}$ a countable dense subset of S. But cd S = n implies that there exists a closed subset A of S such that the natural homomorphism $H^{n-1}(S) \to H^{n-1}(A)$ is not onto.

Let K be the image of $H^{n-1}(S)$ and let R be a K-roof [12]. For each x in R\A, x is an inner point.

Since M(x) has interior, M(x) contains some x_i . Then x belongs to $L(x_i)$.

Hence $R \backslash A \subseteq \bigcup L(x_i)$.

Consider $H^{n-1}(R) \to H^{n-1}(A) \to H^n(R, A)$, where the first map is not onto.

Hence $H^n(R, A) \neq 0$; cd $R/A \not\leq n - 1$. Thus $cd(R \backslash A) \not\leq n - 1$ [3].

But R\A is contained in a countable union of $L(x_i)$'s. Therefore, cd $L(x_i) \not\leq n - 1$ for some i. Thus cd $L(x_i) = n$.

Lemma 5.2. Let S be an n-dimensional P-continuum TSL. Then there exists x in S such that cd L(x) = n.

Proof. Since S is n-dimensional and has small semilattices, we can find a continuous homomorphism from S onto I^n where I^n is a product of n copies of I [9].

Let x be a point of $f^{-1}(1)$. Then $f(L(x)) = I^n$.

But $L(x)$ is a P-continuum and f cannot raise its dimension.

Hence, cd $L(x) \geq n$. But $L(x) \subseteq S$ yields cd $L(x) = n$.

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