UNIVERSITY of HOUSTON
COLLEGE of NATURAL SCIENCES & MATHEMATICS

Green's function in Viscoelastic Media

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i. What are Viscoelastic Materials?

Materials

Elastic	Newtonian Viscous Fluid	Viscoelastic Material
No Dissipation; No Memory	Dissipation; No Memory	Dissipation + Memory

How to detect Viscoelastic media:

Green's function: G(x)

- Characterized by nature of their response to a suddenly applied force (uniform distribution of surface tractions)

Stress 2 (time: t +δt)

Stress 1(time: t)

Viscoelastic Material

Response at time : t +δt

Combined Effect Stress1 and Stress 2

On the basis of deformed states:

Isotropic solids → Only one deformed state for a particular stress

$\lim_{t\to\infty} G_1(t) \to$ *non−zero constant*

Viscoelastic Fluids → Infinite deformed states for a particular stress

$\lim_{t\to\infty} G_1(t) \to 0$

ii. What is Green's function?

Fig. 1.

Green function (tensor) is the response of medium on the force action

$\psi = G*f$

$\theta\psi(x) = f(x)$

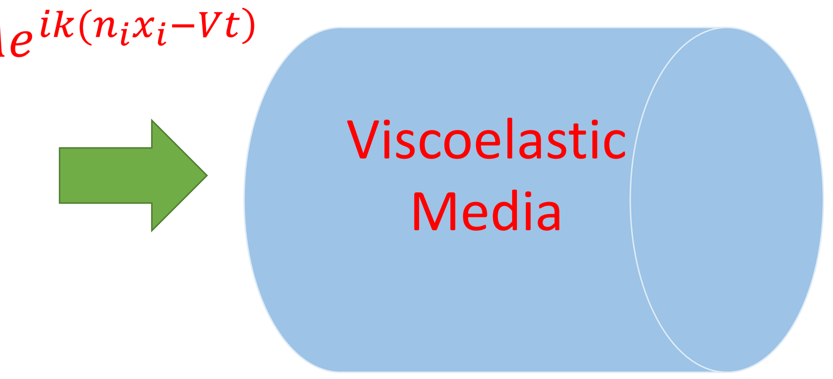
Then if, $\theta G(x) = \delta(x)$

Then $\psi(x) = \int f(x')G(x-x')d^3x'$

iii. The problem to be dealt with:

Plane Wave:

$u_i = Ae^{ik(n_i x_i - Vt)}$



iv. Procedure

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + \eta_{ijkl}\frac{\partial \varepsilon_{kl}}{\partial t}$$

Wave Equation: $\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$

Modified Wave Equation:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \eta_{ijkl}\frac{\partial^3 u_{kl}}{\partial x_j \partial x_l \partial t}$$

$$u_i = Ae^{ik(n_i x_i - Vt)}$$

Modified Green Christoffel Equation:

$$\left(\Gamma_{ik} - i\omega D_{ik} - \rho V^2 \delta_{ik}\right)u_k = 0$$

Where, $V = \frac{\omega}{k}$ V is the Phase Velocity and thus $\Rightarrow kV = \omega$

And $U_i = U_k\delta_{ik}$ (Internal Tensor Product)

And , $\Gamma_{ik} = C_{ijkl}n_j n_l,\ D_{ik} = \eta_{ijkl}n_j n_l$

Now Converting to Fourier Space:

$$\tilde{G}_{km}(t,x) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\tilde{G}_{km}(\omega,\boldsymbol{k})e^{-i\omega t + i\boldsymbol{k}x}$$

$$\delta(t)\delta^3(x) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}e^{-i\omega t + i\boldsymbol{k}x}$$

$$\left(\Gamma_{ik} - i\omega D_{ik} - \rho V^2\delta_{ik}\right)\tilde{G}_{km}(\omega,\boldsymbol{k}) = \delta_{im}$$

$$\Rightarrow \tilde{G}_{km}(\omega,\boldsymbol{k}) = \left(\Gamma_{ik} - i\omega D_{ik} - \rho V^2\delta_{ik}\right)^{-1}$$

$$\boxed{\tilde{G}_{km}(\omega,\boldsymbol{k}) = \left(\Pi_{ik} - \gamma.I\right)^{-1}}$$

$$\Pi_{ik} - \gamma.I = \begin{pmatrix}\Pi_{11}-\gamma & \Pi_{12} & \Pi_{13}\\ \Pi_{21} & \Pi_{22}-\gamma & \Pi_{23}\\ \Pi_{31} & \Pi_{32} & \Pi_{33}-\gamma\end{pmatrix}$$

Where: $\left(\Gamma_{ik} - i\omega D_{ik}\right) = \Pi_{ik}$ $\rho V^2 = \gamma$

$$(\Pi_{ik} - \gamma.I)^{-1} = \frac{1}{\Delta(\gamma)}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \frac{\tilde{A}(\gamma)}{\Delta(\gamma)}$$

Solving for the Green's function in K-space:

Case I: 3 different roots

$$\tilde{G}(\omega,\boldsymbol{k}) = \frac{\tilde{A}(\gamma_1)(\gamma_2-\gamma_3)}{Q(\gamma_1-\gamma)} + \frac{\tilde{A}(\gamma_2)(\gamma_3-\gamma_1)}{Q(\gamma_2-\gamma)} + \frac{\tilde{A}(\gamma_3)(\gamma_1-\gamma_2)}{Q(\gamma_3-\gamma)}$$

Case II: 2 common roots

$$\tilde{G}(\omega,\boldsymbol{k}) = \frac{\tilde{A}(\gamma_1)}{(\gamma_1-\gamma_2)^2}\left(\frac{1}{(\gamma_1-\gamma)}\right) + \left(1 - \frac{\tilde{A}(\gamma_1)}{(\gamma_1-\gamma_2)^2} + \frac{\tilde{A}(\gamma_2)}{(\gamma_1-\gamma_2)}\left(-\frac{\partial}{\partial\gamma_2}\right)\right)\left(\frac{1}{(\gamma_2-\gamma)}\right)$$

Case III: 3 common roots

$$\tilde{G}(\omega,\boldsymbol{k}) = \left(\Pi_{ik} - \gamma.I\right)^{-1} = \frac{\hat{I}}{\gamma_1-\gamma} - \frac{\tilde{A}'(\gamma_1)}{(\gamma_1-\gamma)^2} + \frac{\tilde{A}(\gamma_1)}{(\gamma_1-\gamma)^3}$$

v. Results:

$$\eta = \begin{bmatrix}\eta_1+2\eta_2 & \eta_1 & \eta_1 & 0 & 0 & 0\\ \eta_1 & \eta_1+2\eta_2 & \eta_1 & 0 & 0 & 0\\ \eta_1 & \eta_1 & \eta_1+2\eta_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \eta_2 & 0 & 0\\ 0 & 0 & 0 & 0 & \eta_2 & 0\\ 0 & 0 & 0 & 0 & 0 & \eta_2\end{bmatrix}$$

$$C = \begin{bmatrix}(\lambda+2\mu)(1-\Delta_N-i\Delta_N^I) & (1-\Delta_N-i\Delta_N^I)\lambda & (1-\Delta_N-i\Delta_N^I)\lambda & 0 & 0 & 0\\ (1-\Delta_N-i\Delta_N^I)\lambda & (\lambda+2\mu)\left\{1-\frac{(\Delta_N+i\Delta_N^I)\lambda^2}{(\lambda+2\mu)^2}\right\} & \lambda\left\{1-\frac{(\Delta_N+i\Delta_N^I)\lambda}{(\lambda+2\mu)}\right\} & 0 & 0 & 0\\ (1-\Delta_N-i\Delta_N^I)\lambda & \lambda\left\{1-\frac{(\Delta_N+i\Delta_N^I)\lambda}{(\lambda+2\mu)}\right\} & (\lambda+2\mu)\left\{1-\frac{(\Delta_N+i\Delta_N^I)\lambda^2}{(\lambda+2\mu)^2}\right\} & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & (1-\Delta_T-i\Delta_T^I)\mu & 0\\ 0 & 0 & 0 & 0 & 0 & (1-\Delta_T-i\Delta_T^I)\mu\end{bmatrix}$$

We get the Eigenvalues as: (2 common roots case)

$$\begin{bmatrix}(1-\Delta_T-i\Delta_T^I)\mu - i\eta_2\omega\\ (1-\Delta_T-i\Delta_T^I)\mu - i\eta_2\omega\\ (1-\Delta_N-i\Delta_N^I)(\lambda+2\mu) - i(\eta_1+2\eta_2)\omega\end{bmatrix}$$

The Green's function in Fourier space:

$$G(\omega,k) = \begin{bmatrix}-\frac{k^2}{\rho\omega^2+k^2\{(\lambda+2\mu)(1-\Delta_N-i\Delta_N^I)+2(1-\Delta_T-i\Delta_T^I)\mu+i(\eta_1+2\eta_2)\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}}\\ -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}}\\ -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}} & -\frac{k^2}{\rho\omega^2+k^2\{(-1+\Delta_T+i\Delta_T^I)\mu+i\eta_2\omega\}}\end{bmatrix}$$

vi. Conclusion:

Green's function for the viscoelastic media is found, which can useful to see how a wave is attenuated in an anisotropic viscoelastic media. For the future we will consider wave propagation in multi-layered anisotropic viscoelastic media.