Real Operator Spaces, Real Operator Algebras, and Real Jordan Operator Algebras

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Abstract

The theory of operator spaces has been intensively studied with spaces over the complex field. In this study, we would like to investigate corresponding theory on spaces over the real field which included real operator spaces, real operator algebras and real Jordan operator algebras.

Contents

Chapter 1

Introduction

The study of operator spaces has been focused on spaces over the complex field ($\mathbb{C}$). However, the theory of C^*-algebras over the real field ($\mathbb{R}$) has also been studied and can be found in few references such as [13], [18], [20], and [22]. In addition, real W^* algebras is also studied in [10] and [17]. The fundamental theory of Jordan algebras, JB-algebras and JBW algebras can be found in [14]. The basic theory of real operator spaces has been initiated in [23], [24], and [25].

We would like to continue to investigate further in this program to obtain analogous results for real operator spaces.

A difficulty of studying real spaces is that real spaces may lack of properties that complex spaces have. For example, for $T \in B(H)$ where H is a real Hilbert space, the condition $\langle T\xi, \xi \rangle \geq 0$ for every $\xi \in H$ is not enough to tell that $T \geq 0$. In addition, in complex C^*-algebras A, $a \in A$ is positive if and only if $s(a) \geq 0$ for all states s on A. But this is not true in real C^*-algebras. However, there are many properties that both real and complex spaces have. For example both real C^*-algebras and complex C^*-algebras have contractive approximation identities. Most of the time, we can obtain some results about real spaces from results from complex spaces.

The main technique that we use throughout this study is the technique of complexification. We investigate how real operator spaces and their complexification are related. Usually, properties of real operator spaces are closely related to corresponding properties of their complexification.

Therefore, many results about real operator spaces can be obtained from their complexification.

In Chapter 2, we further study real operator spaces. We first study properties of a real C^*-algebra. Many facts that are true in complex C^*-algebras are as well true in real C^*-algebras. For example, p is a projection in a real C^*-algebra if and only $\|p\| = 1$ and $p^2 = p$. We also investigate properties of positive functionals and real states on a real C^*-algebra. An analog of the well known fact in complex C^*-algebras which states that for a homomorphism between C^*-algebras, being contractive is equivalent to being a $*$-homomorphism has been proved. In addition, we investigate the operator space complexification of a real operator space. We show that all reasonable norms of a real Banach space come from real operator space structures that are given to a real Banach space. We provide a contractive linear map between real operator spaces whose complex extension is not contractive.

In Chapter 3, we investigate real operator algebras. We prove that a unitization of a real operator algebra is unique up to isometric homomorphism. In addition, we also study positivity and real positivity in a real operator algebras.

In Chapter 4, we investigate real Jordan operator algebras. We obtain analogous results to [7] for real Jordan operator algebras. For instance, the unitization of a real Jordan operator algebra is unique up to isometric Jordan homomorphism but not up to completely isometric Jordan homomorphism.

Chapter 2

Real Operator Spaces

2.1 Real Spaces

We review some background about real spaces (see e.g. [13, 16, 18, 19]). Let $(X, \|\cdot\|)$ be a real normed space and define $X_c = X \times X = \{(x_1, x_2) : x_1, x_2 \in X\}$. We denote (x_1, x_2) by $x_1 + ix_2$ and consider $X = \{x + i0 = (x, 0) : x \in X\}$ as a subset of X_c. If $A, B \subseteq X$, we denote $A + iB = \{a + ib : a \in A, b \in B\}$. Using this notation, we can write $X_c = X + iX$. Addition on X_c is defined in a natural way by

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) = (x_1 + x_2) + i(y_1 + y_2),$$

and complex scalar multiplication is defined on X_c by

$$(\alpha + i\beta) \cdot (x + iy) = (\alpha + i\beta) \cdot (x, y) = (\alpha x - \beta y, \beta x + \alpha y) = (\alpha x - \beta y) + i(\beta x + \alpha y)$$

for $(x_1, y_1), (x_2, y_2), (x, y) \in X_c$ and $\alpha, \beta \in \mathbb{R}$. Then X_c is a complex vector space. Observe that any $z \in X_c$ can be uniquely written as $x + iy$ for $x, y \in X$. A complex norm $\|\cdot\|_c$ which is defined on X_c is called a *reasonable norm* if

(i) $\|x\|_c = \|x\|$ for all $x \in X$, and

(ii) $\|x_1 + ix_2\| = \|x_1 - ix_2\|$ for all $x_1, x_2 \in X$.

We call X_c equipped with a reasonable norm, $\|\cdot\|_c$ a *reasonable complexification* of X. For a real Banach space X with a norm $\|\cdot\|$, there are many reasonable norms that can be defined on X_c. For example, for $p \in [1, \infty)$, define $|x + iy|_p = (\|x\|^p + \|y\|^p)^{1/p}$. Then

$$\|x + iy\|_p = \frac{\sup\left\{|e^{i\theta}(x+iy)|_p : \theta \in [0, 2\pi]\right\}}{\sup\left\{\Big(|\cos\theta|^p + |\sin\theta|^p\Big)^{1/p} : \theta \in [0, 2\pi]\right\}}$$

is a reasonable norm on X_c (see [18] or [23]).

Note that $(\mathbb{R}, |\cdot|)$ has a unique complexification which is $\mathbb{C}$. Therefore the complex norm $\|\cdot\|_p$ above coincides with the absolute value of complex numbers, i.e., for $\alpha, \beta \in \mathbb{R}$ and $p \in [1, \infty)$, $\|\alpha + i\beta\|_p = |\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}$. In addition, for a net $(\alpha_t + i\beta_t)$ in $\mathbb{C}$ where $\alpha_t, \beta_t, \alpha, \beta \in \mathbb{R}$, $\alpha_t + i\beta_t \to \alpha + i\beta$ if and only if $\alpha_t \to \alpha$ and $\beta_t \to \beta$.

2.1.1 Remark. Assume that $(X, \|\cdot\|)$ is a real normed space and $(X_c, \|\cdot\|_c)$ is its reasonable complexification. We list some important facts.

(i) $\|x\| = \|x\|_c \leq \|x + iy\|_c$ and $\|y\| = \|y\|_c \leq \|x + iy\|_c$ for all $x, y \in X$ (see Proposition 1 in [19]).

(ii) Let (x_t) and (y_t) be nets in X and $x + iy \in X_c$. By the inequalities in (i) and the triangle inequality, we have

$$\max\{\|x_t - x\|, \|y_t - y\|\} \leq \|(x_t + iy_t) - (x + iy)\|_c = \|(x_t - x) + (y_t - y)\|_c \leq \|x_t - x\| + \|y_t - y\|.$$

Therefore, $x_t + iy_t$ uniformly converges to $x + iy$ in X_c if and only if x_t and y_t uniformly converge to x and y in X respectively.

(iii) $B_X(r) = \{x \in X : \|x\| < r\} \subseteq B_{X_c}(r) = \{x + iy : \|x + iy\|_c < r\}$. In addition, $B_X(r) \subseteq B_{X_c}(r) \subseteq B_X(r) + iB_X(r)$. To see this, if $x \in B_X(r)$, then $\|x + i0\|_c = \|x\| \leq r$. In addition, if $x+iy \in B_{X_c}(r)$ i.e., $\|x+iy\| \leq r$, then from the fact (i) $\|x\| \leq r$ and $\|y\| \leq r$. Thus, $x+iy \in B_X(r) + iB_X(r)$.

(iv) If $(X, \|\cdot\|)$ is a real Banach space, the smallest reasonable norm that can be defined on X_c is the Taylor norm (see Proposition 3 in [19]) and can be described in few different ways as follows:

$$\|x+iy\|_T = \sup\{\|\alpha x + \beta y\| : \alpha^2 + \beta^2 = 1\} = \sup\{\sqrt{\phi(x)^2 + \phi(y)^2} : \phi \in X^*, \|\phi\| = 1\}.$$

Let X and Y be real Banach spaces and let $T : X \to Y$ be a real linear map. Define $T_c : X_c \to Y_c$ as

$$T_c(x+iy) = T(x) + iT(y).$$

Then T_c is a complex linear map and is called the *complex extension* of T. One interesting fact that has been proved in Proposition 4 in [19] is that T_c has the same norm as T if the Taylor norm is given to both X_c and Y_c. In [19], the authors define *a natural complexification procedure* to be a way of defining a reasonable norm on any X_c and Y_c so that the complex extension $T_c \in B(X_c, Y_c)$ has the same norm as T for every $T \in B(X,Y)$. Thus the procedure of the Taylor norm is a natural complexification procedure.

The following proposition tells us that a complex linear functional $\psi \in (X_c)^*$ is uniquely written as $\chi + i\rho$ where $\chi, \rho \in X^*$ and if $\phi \in X^*$, its complex extension ϕ_c has the same norm as the norm of ϕ. This lemma is crucial and will be used throughout this dissertation. It can be found in Proposition 1.4.1 in [18] or Proposition 7 in [19].

2.1.2 Proposition. *Let X be a real Banach space and X_c be its reasonable complexification.*

(i) Let $\phi \in X^*$. Define $\phi_c : X_c \to \mathbb{C}$ by $\phi_c(x+iy) = \phi(x) + i\phi(y)$ for $x+iy \in X_c$. Then $\phi_c \in (X_c)^*$ and $\|\phi_c\|_{X_c} = \|\phi\|_X$.

(ii) $(X_c)^*$ is a reasonable complexification of X^*.

We mentioned in the remark 2.1.1 (ii) that a net $x_t + iy_t$ uniformly converges to $x + iy$ if and only if x_t and y_t uniformly converge to x and y respectively. This fact holds for weak convergence and weak* convergence too.

2.1.3 Lemma. *Let $(X_c, \|\cdot\|_c)$ be a reasonable complexification of $(X, \|\cdot\|)$ and $(x_t + iy_t)$ be a net in X_c. Then $x_t + iy_t$ weakly converges to $x + iy$ in X_c if and only if x_t and y_t weakly converges to x and y in X respectively.*

Proof. First, assume that $x_t + iy_t$ weakly converges to $x + iy$. Let $\phi \in X^*$. Then $\phi(x_t) + i\phi(y_t) = \phi_c(x_t + x_t) \to \phi_c(x + iy) = \phi(x) + i\phi(y)$. Thus, $\phi(x_t) \to \phi(x)$ and $\phi(y_t) \to \phi(y)$. Conversely, assume that x_t and y_t weakly converge to x and y respectively. Let $\psi \in (X_c)^*$. We can write $\psi = \chi + i\rho$ where $\chi, \rho \in X^*$. By weak convergence of x_t and y_t in X,

$$\begin{aligned}\phi_c(x_t + iy_t) &= \Big(\chi(x_t) - \rho(y_t)\Big) + i(\chi(y_t) + \rho(x_t)) \\ &\to \Big(\chi(x) - \rho(y)\Big) + i(\chi(y) + \rho(x)) \\ &= \psi(x + iy).\end{aligned}$$

□

2.1.4 Lemma. *Let $(X_c, \|\cdot\|_c)$ be a reasonable complexification of $(X, \|\cdot\|)$ and $(\xi_t + i\eta_t)$ be a net in $(X_c)^*$ where $\xi_t, \eta_t \in A^*$. Then $\xi_t + i\eta_t$ weak* converges to $\xi + i\eta$ in $(X_c)^*$, where $\xi, \eta \in A^*$, if and only if ξ_t and η_t weak* converge to ξ and η in X^* respectively.*

Proof. Assume that $\xi_t + i\eta_t$ weak* converges to $\xi + i\eta$ in $(X_c)^*$ and $x \in X$. Then $\xi_t(x) + i\eta_t(x) = (\xi_t + i\eta_t)(x) \to (\xi + i\eta)(x) = \xi(x) + i\eta(x)$. Thus, $\xi_t(x) \to \xi(x)$ and $\eta_t(x) \to \eta(x)$. Conversely, assume that ξ_t and η_t weak* converge to ξ and η respectively in X^* and $x + iy \in X_c$. Then $(\xi_t + i\eta_t)(x + iy) = \xi_t(x) - \eta_t(y) + i(\xi_t(y) + \eta_t(x)) \to \xi(x) - \eta(y) + i(\xi(y) + \eta(x)) = (\xi + i\eta)(x + iy)$. □

2.1.5 Remark. As the previous lemmas, a net in a reasonable complexification of X converges in uniform topology, weak topology or weak* topology if and only the real part and complex part of the net converge in uniform topology, weak topology or weak* topology respectively. Thus the topology on a reasonable complexification in any of these is the product topology or Tychonoff topology on $X_c = X \times X$. Therefore, if $x_t + iy_t$ converges to $x + iy$ in any of these typologies, we conclude that x_t converges to x and y_t converges to y in that topology. We may use this fact in many places without referring to these two lemmas.

A *Banach algebra* is a Banach space $(A, \|\cdot\|)$ together with a product $\cdot : A \times A \to A$ which satisfies $\|a \cdot b\| \leq \|a\|\|b\|$ for all $a, b \in A$. If $(A, \|\cdot\|)$ is a real Banach algebra, define a product on A_c to be

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Assume that there is a reasonable norm $\|\cdot\|_c$ on A_c that makes A_c a complex Banach algebra with this product. If A has an identity e, then A_c has also the same identity e. Define the *spectrum* of $a \in A$ to be the spectrum of $a = a + i0$ in A_c, i.e., $\sigma_A(a) = \sigma_{A_c}(a)$. If an algebra A does not have an identity, define $A^1 = A \times \mathbb{R}$ with the product $(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta)$. By denoting (a, α) to be $a + \lambda$, we have $A^1 = \{a + \alpha : a \in A, \alpha \in \mathbb{R}\}$ and the product on A^1 is $(a + \alpha) \cdot (b + \beta) = (ab + \alpha b + \beta a) + \alpha\beta$ for $a + \alpha, b + \beta \in A^1$. Then 1 is the identity of A^1. Define the the spectrum of element a in a nonunital real Banach algebra A as the spectrum of $a + 0 \in A^1$, i.e., $\sigma_A(a) = \sigma_{A^1}(a)$. In either cases, the following fact holds.

2.1.6 Lemma. *Let A be a real Banach algebra and $a, b \in A$. Then*

$$\sigma_{A_c}(a - ib) = \{\bar{\lambda} : \lambda \in \sigma_{A_c}(a + ib)\}.$$

Proof. If $\lambda = \alpha + i\beta \notin \sigma_{A_c}(a + ib)$, $a + ib - \alpha - i\beta$ has an inverse say $(x + iy)$. Thus,

$$(a + ib - \alpha - i\beta)(x + iy) = (ax - by - \alpha x + \beta y) + i(ay + bx - \alpha y - \beta x) = 1.$$

So, $ax - by - \alpha x + \beta y = 1$ and $ay + bx - \alpha y - \beta x = 0$. Therefore,

$$(a - ib - \alpha + i\beta)(x - iy) = (ax - by - \alpha x + \beta y) - i(ay + bx - \alpha y - \beta x) = 1.$$

That is $\overline{\lambda} = \alpha - i\beta \notin \sigma_{A_c}(a - ib)$. □

A *Banach $*$-algebra* A is a Banach algebra with an *involution* $* : A \to A$ which is a conjugate linear map satisfying $(ab)^* = b^*a^*$ and $(a^*)^* = a$. Let A_c be a complexification of a real Banach $*$-algebra A. We can naturally define an involution on A_c as

$$(a + ib)^* = a^* - ib^*.$$

Assume that there is a reasonable norm on A_c that makes A_c a complex Banach algebra. Then A_c is a complex Banach $*$-algebra. A complex C^*-algebra is a complex $*$-algebra such that $\|aa^*\| = \|a\|^2$ for any element a. We call a real Banach $*$-algebra A a *real C^*-algebra* if there is a reasonable norm on A_c which makes A_c a complex C^*-algebras. Note that if A is a real C^*-algebra, there is a unique reasonable norm that makes A_c a complex C^*-algebra. There are equivalence conditions for being a real C^*-algebra (see Lemma 1.1 in [10] or Proposition 5.1.2 in [18]). The most important one is that A is a real C^*-algebra if and only if A can be isometrically $*$ isomorphic to a uniformly closed $*$-subalgebra of $B(H)$ where H is a real Hilbert space. In particular, one can define a real unital C^*-algebra to be a unital Banach $*$-algebra such that $\|aa^*\| = \|a\|^2$ and $1 + aa^*$ is invertible for every $a \in A$ (Chapter 8 in [13]). Note that $\mathbb{C}$ with the identity involution, i.e., $x^* = x$, is not a real C^*-algebra. There are analogous properties of real C^*-algebras as the complex C^*-algebras which can be found in [13] and [18], for example.

Let A be a real C^*-algebra and

$$M_n(A) = \{[a_{ij}]_{n\times n} : a_{ij} \in A, 1 \leq i, j \leq n\}$$

for $n \in \mathbb{N}$. Then there is a unique C^*-norm on $M_n(A)$ such that $M_n(A)$ is a real C^*-algebra (see

Proposition 5.1.10 in [18]). Let B be a real C^*-algebra and $T : A \to B$ be a linear map from A to B. The (n-th) amplification of T is the associated map $T_n : M_n(A) \to M_n(B)$ which

$$T([a_{ij}]_{n\times n}) = [T(a_{ij})]_{n\times n}.$$

We say that T is *completely bounded* if $\|T\|_n < \infty$ for all $n \in \mathbb{N}$ and define

$$\|T\|_{cb} = \sup\{\|T\|_n : n \in \mathbb{N}\}.$$

We say that T is *completely isometric* if T_n is isometric for all $n \in \mathbb{N}$. We say that T is *completely contractive* if $\|T\|_{cb} \leq 1$. And we say that T is *completely isometric* if T_n is isometric for all $n \in \mathbb{N}$. In particular, $M_n(B(H)) = B(H^n)$ for a real Hilbert space H. Since a real C^*-algebra A can be isometrically $*$ isomorphic to a real C^*-subalgebra of $B(H)$, $M_n(A) \subseteq M_n(B(H)) = B(H^n)$ (see section 4 in [24]). Note that $B(H)_c = B(H_c) \subseteq B(H^2)$ (see page 1051 in [23]) and

$$\|a + ib\|_{B(H)_c} = \left\| \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\|_{B(H^2)}.$$

Thus A_c can be realized as a subspace of $B(H_c)$ and

$$\|a + ib\|_{A_c} = \left\| \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\|_{M_2(A)}.$$

In addition, A^{**}, the bidual of A, is a real C^*-algebra with the Arens product, and A is a C^*-subalgebra of A^{**} (see Theorem 1.6 in [10] or Theorem 5.5.3 and Theorem 5.5.4 in [18]). Moreover, if (η_t) is a net in A^{**} converging weak* to η, we have $\eta_t\nu$ converges weak* to $\eta\nu$ and $\nu\eta_t$ converges weak* to $\nu\eta$ for all $\nu \in A^{**}$. That is the Arens product on A^{**} is separately weak* continuous.

Let A be a real C^*-algebra and $a \in A$. We call a a *positive element* of A if $\sigma_A(a) \subseteq [0, \infty)$. We write $a \geq 0$ if a is positive and denote A^+ to be the set of all positive elements in A. We say that

$p \in A$ is a *projection* if $p = p^* = p^2$. If in addition, $pap = pa = ap$ for all $a \in A$, we call p a *central projection* on A.

2.1.7 Lemma. *Let A be a real C^*-algebra and $a \in A$. Then $a \geq 0$ if and only if $a = xx^* + yy^*$ for some $x, y \in A$.*

Proof. If $a \geq 0$ in A, then $a \geq 0$ in A_c. By the corresponding fact in complex C^*-algebras, $a = (x + iy)(x + iy)^* = (x + iy)(x^* - iy^*) = xx^* + yy^* + i(xy^* - yx^*)$. Therefore, $a = xx^* + yy^*$. Conversely, $xx^* + yy^*$ is positive in A_c and thus positive in A. □

2.1.8 Remark. (i) For a unital real C^*-algebra A, $a \in A$ is positive if and only if $a = xx^*$. If A is a nonunital real C^*-algebra, its unitization A^1 is unique. Let $a \in A \subseteq A^1$ be positive. Then $a = (x + \lambda)^*(x + \lambda) = xx^* + \lambda x + \lambda x^* + \lambda^2$. This implies $\lambda^2 = 0$ and so $a = xx^*$. Therefore the above lemma can be restated that $a \geq 0$ if and only if $a = xx^*$ for some $x \in A$.

(ii) In the case of a complex C^*-algebra, a $*$-homomorphism maps positive elements to positive elements. This facts holds in a real C^*-algebra as well. Let $\pi : A \to B$ be a $*$-homomorphism between real C^*-algebras A and B. Then $\pi(a) \geq 0$ for all $a \geq 0$. To see this, we write $a = xx^*$. Then $\pi(a) = \pi(xx^*) = \pi(x)\pi(x)^* \geq 0$ in B.

2.1.9 Lemma. *Let A be a real C^*-algebra and $a, b \in A$. If $a + ib$ is positive in A_c, then a and $a - ib$ are positive in A.*

Proof. Since $a + ib$ is positive, we have $a + ib = (a + ib)^* = a^* - ib^*$. Thus, $a^* = a$ and $b^* = -b$. Hence, $a - ib$ is also selfadjoint. By Lemma 2.1.6, $\sigma_{A_c}(a - ib) = \{\overline{\lambda} : \lambda \in \sigma_{A_c}(a + ib)\}) \subseteq [0, \infty)$. Thus, both $a + ib$ and $a - ib$ are positive and so $2a = a + ib + a - ib$ is positive. Therefore, a and $a - ib$ are positive. □

A real C^*-subalgebra A of $B(H)$, where H is a real Hilbert space, is called real *W^*-algebra* if A is weak* closed in $B(H)$ (see also [10] and [17]). We obtain an equivalence of $x + iy$ being positive in A_c for a W^*-algebra A. Thank to Dr. Blecher who provides the proofs of the fact.

2.1.10 Lemma. *Let A be a real W^*-algebra. Assume that bounded nets (z_ε) weak* converges to z and (x_t) SOT converges to x. Then $(x_t z_\varepsilon x_t)$ weak* converges to xzx.*

Proof. We may assume that A is a real W^*-subalgebra of $B(H)$ where H is a real Hilbert space. Let $\xi, \eta \in H$. Then

$$\begin{aligned}|\langle x_t z_\varepsilon x_t \xi, \eta\rangle - \langle xzx\xi, \eta\rangle| &\leq |\langle x_t z_\varepsilon x_t \xi, \eta\rangle - \langle x z_\varepsilon x_t \xi, \eta\rangle| + |\langle x z_\varepsilon x_t \xi, \eta\rangle - \langle xzx_t\xi, \eta\rangle| + |\langle xzx_t\xi, \eta\rangle - \langle xzx\xi, \eta\rangle| \\ &\leq \|z_\varepsilon x_t \xi\| \|x_t \eta - x\eta\| + \|x_t\xi\| \|z_\varepsilon x\eta - zx\eta\| + \|x_t \xi\| \|x_t zx\eta - xzx\eta\| \\ &\to 0.\end{aligned}$$

□

2.1.11 Proposition. *Let A be a real W^*-algebra and A_c be its complexification. Let $a, b \in A$. Then $a + ib$ is positive in A_c if and only if x is positive and $b = rzr$ where r is the root of a and z is anti symmetric contraction in A.*

Proof. Assume that x is invertible. Since $a + ib \geq 0$, then $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is positive. Hence

$$-\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \leq \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \leq \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

By multiplying $\begin{bmatrix} a^{-1/2} & 0 \\ 0 & a^{-1/2} \end{bmatrix}$ to both left and right of the inequality, we have

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leq \begin{bmatrix} 0 & -a^{-1/2}ba^{-1/2} \\ a^{-1/2}ba^{-1/2} & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies $\|a^{-1/2}ba^{-1/2}\| \leq 1$ and $b = a^{1/2}zb^{1/2}$ where $z = a^{-1/2}ba^{-1/2}$ and $\|z\| \leq 1$. Since $b^* = -b$, we have $z^* = -z$. Thus z is contractive antisymetric. Now, assume that a is not invertible. Let $a_{1/n} = a + \frac{1}{n}$ for $n \in \mathbb{N}$. Then $a_{1/n}$ is invertible and $a_{1/n} \to a$. Hence $a_{1/n}^{1/2} \to a^{1/2}$.

By the above, $b_{1/n} = a_{1/n}^{1/2} z_n a_{1/n}^{1/2}$ where z_n is antisymetric. By taking a weak* limit of a subnet of (z_n) and using the above lemma, we obtain that $b = a^{1/2} z a^{1/2}$ and z is antisymmetric. □

of a

We state a fact about projections in a real C^*-algebra. We see that a projection p is positive since $p = p^2 = pp^*$ by Lemma 2.1.7. In addition, we also have the following fact which is an analogous fact of the fact in complex C^*-algebras.

2.1.12 Lemma. *Let A be a real C^*-algebra and A_c be its C^*-algebra complexification. If $p \in A$ is a projection in A then p is a projection in A_c. Also, p is a projection in A if and only if $p^2 = p$ and $\|p\| \leq 1$.*

Proof. If p is a projection in A, then $(p + i0)^* = p + i0 = (p + i0)^2$ is a projection in A_c. By considering p as a projection in A_c, and using a fact in complex C^*-algebras, p is a projection if and only if $p^2 = p$ and $\|p\| \leq 1$. □

2.1.13 Remark. If $p + iq$ is a projection in A_c, we easily conclude from previous facts that $p - iq$ is a projection. In addition, we have that $p^* = p, q^* = -q$, $\|p\| \leq 1, \|q\| \leq 1$, and p is positive. But $Re(p + iq) = p$ might not be a projection, as in the following example. Let $p = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$ be elements in $B(\mathbb{R}^2)$. Then $p + iq = \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix}$ is a projection in $B(\mathbb{C}^2) = B(\mathbb{R}^2)_c$. We can see that p is not a projection of $B(\mathbb{R}^2)$.

2.1.14 Lemma. *If p is a central projection in a unital real C^*-algebra A, then $1 - p$ is a central projection and*

$$\|a\| = \max\{\|pa\|, \|a - pa\|\}.$$

Proof. Assume that $p \in B(H)$ is a central projection and thus $I_H - p \in B(H)$ is a central projection. Let $x \in B(H)$. Then $x = px + (I_H - p)x$. Since $H = pH \oplus (I - p)H$, we have $\|x\| = \max\{\|pa\|, \|a - pa\|\}$. □

A *contractive approximation identity* or *cai* is a contractive net (e_t) in a real or complex C^* algebra A such that $e_t a$ and ae_t converge to a for all $a \in A$. Now, let A be a real C^*-algebra. If (e_t) is a cai in A, then $e_t(a+ib) = e_t a + ie_t b \to a+ib$. Thus, any cai in A is a cai in A_c. Moreover, if $(E_t) = \big(Re(E_t) + iIm(E_t)\big)$ is a cai in A_c, we have that $E_t a = Re(E_t)a + iIm(E_t)a \to a$. Thus, $Re(E_t)a \to a$. Hence $\big(Re(E_t)\big)$ is a cai in A. In addition, if A has a cai (e_t), (e_t^*) is also a cai and $\big((e_t + e_t^*)/2\big)$ is a selfadjoint cai. Any real or complex C^*-algebra A always has a cai (e_t) (see Proposition 5.2.4 in [18]). Moreover, a cai in A converges weak* to an identity $e \in A^{**}$. We state this as the following lemma.

2.1.15 Lemma. *Let (e_t) be a cai in a real C^*-algebra A and e be its weak* limit in A^{**}. Then e is the identity of A^{**}. Therefore, any cai in a real C^*-algebra converges weak* to the identity of A^{**} and for all $\phi \in A^*$, $\phi(e_t) \to \phi(e)$.*

A real state on a unital real C^*-algebra is a linear functional $\phi \in A^*$ such that $\phi(a) = \phi(a^*)$ and $\phi(e) = \|\phi\| = 1$ where $a \in A$ and e is the identity of A (see [13]). In general, we can define a real state on a real C^*-algebra (whether unital or nonunital) using a cai (follow from the definition of states in a complex approximate operator algebra in [5]). Let A be a real C^*-algebra and $\phi \in A^*$. We say ϕ is *positive* if $\phi(a^*) = \phi(a)$ for all $a \in A$ and $\phi(b) \geq 0$ for all $b \geq 0$. We say that ϕ is a *real state* on A if ϕ is positive and $\phi(e_t) \to 1$ for a cai (e_t) in A (for all cai (e_t) in A, by the previous lemma). This definition will be equivalent to the definition of real state as defined in Definition 5.2.5 in [18].

2.1.16 Lemma. *Let A be a real C^*-algebra and $\phi \in A^*$. If $\phi(e_t) \to \|\phi\|$ for all cai (e_t) in A, then $\phi \geq 0$.*

Proof. We consider ϕ as a linear functional on A^{**}. Since A^{**} is a real von Neumann algebra containing the identity e and $\phi(e) = \lim_t \phi(e_t) = \|\phi\|$, by Lemma 4.5.5 in [18], ϕ is positive in A^{**}. Since A is a real C^*-subalgebra of A^{**}, ϕ is positive in A. $\square$

The following lemma gives equivalent conditions for $\phi \in A^*$ to be a real state. Note that the condition (iv) in the following lemma coincides with the definition of real state in Definition 5.2.5

in [18]. Some of the facts in the next three lemmas can be found in Proposition 5.2.6 [18]. We provide an additional fact and proof here.

2.1.17 Lemma. *Let A be a real C^*-algebra, $\phi \in A^*$ and (e_t) be a cai in A. The following are equivalent.*

(i) ϕ is a real state on A.

(ii) ϕ is a real state on A^{**}.

(iii) ϕ_c is a state on A_c.

(iv) $\phi(e_t) \to \|\phi\| = 1$.

Moreover, if A is unital with the identity 1 and ϕ is a a positive linear functional on A, we have that ϕ is a real state on A if and only if $\phi(1) = 1$.

Proof. $(i) \Rightarrow (ii)$ We consider ϕ as a real functional on A^{**}. Then $\phi(e_t) \to \phi(e) = 1$ where e is the identity of A^{**}. Thus, by Lemma 4.5.5 in [18], ϕ is positive on A^{**}. Since (e_t) is also a cai in A^{**}, ϕ is a real state on A^{**}.

$(ii) \Rightarrow (i)$ Since A is a real C^*-subalgebra of A^{**}, ϕ is a real state on A.

$(i) \Rightarrow (iii)$ Since (e_t) is a cai for A_c, ϕ_c satisfies the conditions of being a state of a complex C^*-algebra.

$(iii) \Rightarrow (iv)$ Since ϕ_c is a state on a complex C^*-algebra A_c, $\|\phi_c\| = 1$. By Proposition 2.1.2, $\|\phi\| = \|\phi_c\| = 1$. Also, $\phi(e_t) = \phi_c(e_t) \to 1$. Thus, ϕ is a real state on A.

$(iv) \Rightarrow (i)$ Since $\lim_t \phi(e_t) = \|\phi\|$, $\phi \geq 0$ by Lemma 2.1.16. Thus, ϕ is a real state.

The last statement follows since 1 is also a cai in a unital real C^*-algebra A. Thus for a positive functional ϕ on A, ϕ is a state if and only if $\phi(1) = 1$. □

2.1.18 Lemma. *If ψ is a state of A_c, then $Re(\psi)$ is a real state of A.*

Proof. By Proposition 2.1.2, we can write $\psi = \chi + i\rho$ where $\chi, \rho \in A^*$. If (e_t) is a real cai in A, then (e_t) is a cai in A_c and thus $\psi(e_t) = \chi(e_t) + i\rho(e_t) \to 1$. Thus, $\chi(e_t) \to 1$. Also, if $x \in A \subseteq A_c$, then

$\chi(x) + i\rho(x) = \psi(x) = \psi(x^*) = \chi(x^*) + i\rho(x^*)$. Thus, we have $\chi(x) = \chi(x^*)$. Therefore, $Re(\psi) = \chi$ is a real state. ☐

The following lemma is an exercise in [13]. We provide a proof of this fact as follows.

2.1.19 Lemma. *In a real C^*-algebra A, $x^* = -x$ if and only if $s(x) = 0$ for every real states s on A.*

Proof. Assume that $x^* = -x$ and s is a real state on A. By definition of a real state and linearity, $s(x) = s(x^*) = s(-x) = -s(x)$. Thus, $s(x) = 0$.

Conversely, assume that $s(x) = 0$ for all real states s and $x^* \neq x$. Note that $x + x^*$ is selfadjoint. We consider $x + x^*$ in the complexification A_c. There exists a state ψ on A_c such that $\psi(x + x^*) = \|x + x^*\|$ by a fact for complex C^*-algebras. By the above lemma, $Re(\psi)$ is a real state on A and so $Re(\psi(x + x^*)) = \|x + x^*\|$. Since $x^* \neq -x$, $x^* + x \neq 0$ and $\|x + x^*\| \geq 0$. Hence, there is a real state $\phi = Re(\psi)$ such that $\|x + x^*\| = \phi(x + x^*) = \phi(x) + \phi(x^*) = 2\phi(x)$. This contradicts the assumption. ☐

2.1.20 Lemma. *Let A be a real C^*-algebra, s be a real state on $B(H)$ where H is a real Hilbert space, and $\theta : A \to B(H)$ be a contractive homomorphism. Then $s \circ \theta$ is a real state on A.*

Proof. Without loss of generality, we assume that $\overline{(\theta(A)(H))} = H$. If (e_t) is a cai of A, $\theta(e_t) \to I_H$ weak* and thus $s \circ \theta(e_t) = s(\theta(e_t)) \to 1$. Also, $\|s \circ \theta\| \leq \|s\|\|\theta\| \leq 1$. Since $s \circ \theta(e_t) \to 1$, $\|s \circ \theta\| = 1$. Thus, $s \circ \theta$ is a real state on A. ☐

The following theorem is the analog of a well known fact in complex C^*-algebras.

2.1.21 Theorem. *Let A be a real C^*-algebra and let $\theta : A \to B(H)$ be a contractive homomorphism. Then*

(i) $\theta(h)$ is selfadjoint if h is selfadjoint, and

(ii) $\theta(k)$ is skew symmetric if k is skew symmetric ($k^* = -k$).

Indeed, $\theta(x^*) = \theta(x)^*$ for all $x \in A$.

Proof. (i) Let $h \in A$ be selfadjoint. Then $\theta^{**} : A^{**} \to B(H)^{**}$ is a contractive homomorphism between real W^*-algebras. Then by 4.3.4 in [18], h is approximable by real linear combination of projections $p \in A^{**}$. Thus $\theta(h) = \theta^{**}(h)$ is approximated by a linear combination of $\theta^{**}(p)$. But $\theta^{**}(p)^2 = \theta^{**}(p), \|\theta^{**}(p)\| \leq 1$. Hence $\theta(p)$ is selfadjoint by Lemma 2.1.12. So $\theta^{**}(h)$ is selfadjoint in $B(H)^{**}$. Since $\theta^{**}(h) = \theta^{**}|_A(h) = \theta(h)$, $\theta(h)$ is selfadjoint in A.

(ii) Let $k \in A$ be skew symmetric, i.e., $k^* = -k$ and s be a real state on $B(H)$. Then $s \circ \theta$ is a real state on A by Lemma 2.1.20. Since $k^* = -k$ and by Lemma 2.1.19, $s(\theta(k)) = s \circ \theta(k) = 0$ for all real state s on $B(H)$. By the converse direction of Lemma 2.1.19, $\theta(k)^* = -\theta(k)$.

Now, let $x \in A$. We can write $x = h + k$ where $h = (x + x^*)/2$ and $k = (x - x^*)/2$. Then $\theta(h)$ is selfadjoint and $\theta(k^*) = -\theta(k)$. So, $\theta(x^*) = \theta(h^* + k^*) = \theta(h - k) = \theta(h) - \theta(k) = \theta(h)^* + \theta(k)^* = \theta(h + k)^* = \theta(x)^*$. □

It is well know that a homomorphism between complex C^*-algebra is contractive if and only if $*$-homomorphic. This fact holds in real C^*-algebras.

2.1.22 Corollary. *If $\pi : A \to B$ be a homomorphism between real C^*-algebras, then π is contractive if and only if π is $*$-homomorphism.*

Proof. Assume that π is contractive. Since B is a real C^*-algebra, there is an isometric $*$-homomorphism $\theta : A \to B(H)$ for some real Hilbert space H. Then $\theta \circ \pi : A \to B(H)$ is a contractive homomorphism. By the previous theorem, $\theta \circ \pi(x^*) = \theta \circ \pi(x)^* = \theta(\pi(x))^* = \theta(\pi(x)^*)$. By injectivity of θ, $\pi(x^*) = \pi(x)^*$. Conversely, if π is a $*$-homomorphism, $\pi_c : A_c \to B_c$ is a $*$-homomorphism. By the corresponding fact for complex C^*-algebras, π_c is contractive. Thus, $\pi|_A = \pi$ is contractive. □

2.1.23 Remark. Let A and B be real C^*-algebras. If $\pi : A \to B$ is a contractive homomorphism, by the above corollary π is $*$-homomorphism. Then $\pi_c : A_c \to B_c$ is a $*$-homomorphism. Thus, π_c is completely contractive by a fact in complex C^*-algebras. Since $\pi_c|_A = \pi$, π is completely contractive. Therefore, we have that a contractive homomorphism between real C^*-algebras is completely contractive.

Next, we consider the unitization of a nonunital real C^*-algebra.

2.1.24 Corollary. *Let A be a nonunital real C^*-algebra. Define a product on $A^1 = A \oplus \mathbb{R}$ to be*

$$(a + \alpha)(b + \beta) = (ab + \beta a + \alpha b) + \alpha\beta, \text{ and}$$

$$(a + \alpha)^* = a^* + \alpha$$

where $a, b \in A$ and $\alpha, \beta \in \mathbb{R}$. Then there is a unique real norm on A^1 which makes A^1 a unital real C^*-algebra.

Proof. Let $\pi : A \to B(H)$ be an isometric $*$-homomorphism, where H is a real Hilbert space. Then we can identify A^1 as $\{\pi(a) + \alpha I_H : a \in A, \alpha \in \mathbb{R}\}$, which is a unital real C^*-subalgebra of $B(H)$. Thus there is a norm on A^1 such that A^1 is a real unital C^*-algebra. Suppose $\|\cdot\|$ is a norm on A^1 that makes A^1 a real unital C^*-algebra. Let $\rho : A^1 \to \pi(A^1) \subseteq B(H)$ be defined as $\rho(a + \lambda) = \pi(a) + \lambda I_H$. Then ρ is a $*$-homomorphism. Thus, it is contractive by Corollary 2.1.22. For the same reason, ρ^{-1} is contractive. Thus, ρ is isometric. Therefore $\|\cdot\|$ is equal to the norm inherited from $B(H)$. $\square$

The previous fact is also mentioned in the remark after Proposition 5.2.4 in [18]. In fact, if (e_t) is a cai of a nonunital C^*-algebra A, the C^*-norm of $a + \lambda \in A^1$ is as follows:

$$\|a + \lambda\| = \lim_t \|e_t a + \lambda e_t\| = \lim_t \|a e_t + \lambda e_t\| = \sup\{\|ax + \lambda x\| : x \in A, \|x\| \leq 1\}.$$

Moreover, a real state on A can be extended to a real state on A^1 in a natural way. The proof of the following fact uses the Cauchy-Schwarz inequality. To obtain the Cauchy-Schwarz inequality for a real C^*-algebra, we can consider A as a subset of its complexification A_c. For a real state on A, ϕ_c is a state on A_c by Lemma 2.1.17. Thus, for $a, b \in A \subseteq A_c$,

$$\phi(a^*b)^2 = \phi_c^2(b^*a) \leq \phi_c(a^*a)\phi_c(b^*b) = \phi(a^*a)\phi(b^*b).$$

2.1.25 Proposition. *Let A be a nonunital real C^*-algebra and $\phi \in A^*$ be a real state. Then $\phi^\circ(a+\lambda) = \phi(a) + \lambda$ is a real state on A^1.*

Proof. First, we show that ϕ° is a positive linear functional on A^1. Let $a \in A$ and $\lambda \in \mathbb{R}$. We claim that $\phi(a^*a) \geq \phi(a)^2$. To see this, let (e_t) be a cai of A. Then $\phi(ae_t) \to \phi(a)$ and by the Cauchy–Schwarz inequality,

$$|\phi(ae_t)|^2 \leq |\phi(aa^*)\phi(e_t^*e_t)| \leq |\phi(a^*a)|\|\phi\|\|e_t^*\|\|e_t\| \leq |\phi(a^*a)|.$$

Therefore, $\phi(a)^2 \leq \phi(a^*a)$. As a consequence,

$$\phi^\circ((a+\lambda)^*(a+\lambda)) = \phi(a^*a) + 2\lambda\phi(a) + \lambda^2 \geq \phi(a)^2 + 2\lambda\phi(a) + \lambda^2 = (\phi(a)+\lambda)^2 \geq 0.$$

This shows that ϕ° is positive. Moreover, it is trivial that $\phi^\circ(1) = 1$. Therefore, ϕ° is a real state. $\square$

2.1.26 Remark. In a complex unital C^*-algebra, a positive element in A can be determined by states, i.e., $a \geq 0$ in A if and only if $s(a) \geq 0$ for all $s \in S(A)$. However, this is not true in a real C^*-algebra. For example, $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \in B(\mathbb{R}^2)$. Let $s \in S(B(\mathbb{R}^2))$. Then $s\left(\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 2 + \phi\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$. Since $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and s is selfadjoint, $s\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 0$. Hence, $s\left(\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = 2 \geq 0$ for all states $s \in B(\mathbb{R}^2)$. However, $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is not positive. Therefore, the space of states on a real C^*-algebra does not suffice to determine positive elements. However for a selfadjoint $a \in A$, a is positive if and only if $s(a) \geq 0$ for all $s \in S(A)$. The direct implication is obvious. To prove the converse, we consider $a \in A_c$. Since a is selfadjoint, $\sigma_{A_c}(a) \subseteq \mathbb{R}$. Assume that there is a negative real number $\lambda \in \sigma_{A_c}(a)$.

Then there is a state $\psi \in S(A_c)$ such that $\psi(a) = \lambda < 0$. Since $Re(\psi)$ is a real state on A, we have $Re(\psi)(a) = \lambda < 0$. Therefore, if $a = a^*$ and $s(a) \geq 0$ for all $s \in S(A)$, $a \geq 0$.

If B is a complex C^*-algebra, a *real form* of B is a real C^*-algebra such that $B = A_c$. In [11], the author provides a counter example of a complex C^*-algebra which does not admit a real form. In [22], the author investigates real forms of a complex C^*-algebra. Another question we can ask is that can a real C^*-algebra be a complex C^*-algebra? That is there a complex scalar multiplication extending a real scalar multiplication which makes it a complex C^*-algebra. Obviously, $\mathbb{R}$ has no complex scalar multiplication thus can not be a complex C^*-algebra. Therefore, not all real C^*-algebra can be given a complex multiplication so that it turns to be a complex C^*-algebra.

A *complex structure* on a real vector space X is a linear map $J : X \to X$ satisfying $J^2(x) = -x$. Note that if J is a complex structure on X, $-J$ is also a complex structure on X. If a real vector space X admits a complex structure J, then one can define a complex scalar multiplication on X to be

$$(\alpha + i\beta)x = \alpha x + \beta J(x).$$

Then X with this complex scar is a complex vector space. The following proposition gives conditions when a real C^*-algebra can be given a complex scalar so that it is a complex C^*-algebra.

2.1.27 Proposition. *Let A be a real C^* algebra. Then A has a complex structure which makes it a complex C^* algebra if and only if there is an $\mathbb{R}$-linear map $J : A \to A$ such that*

(i) $J^2(x) = -x$

(ii) $J(x^*) = -J(x)^*$

(iii) $\|\alpha(x) + \beta J(x)\| = (\sqrt{\alpha^2 + \beta^2})\|x\|$

Proof. If A is a complex C^* algebra, by defining $J(x) = ix$, J satisfies all the above properties. Now, assume that A is a real C^* algebra and admits a linear map J satisfying the above properties. Then, by defining $(\alpha + i\beta)a = \alpha a + \beta J(a)$, A is a complex vector space and the original norm

$\|\cdot\|$ on A is a complex norm with respect to this complex structure and makes A_c a complex C^*-algebra. $\square$

2.1.28 Remark. (i) Define $J : B(\mathbb{R}^2) \to B(\mathbb{R}^2)$ to be

$$J(A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A.$$

Then, we have that $J^2(A) = -A$ but $J(A^*) \neq -J(A)^*$. This complex structure does not makes $B(\mathbb{R}^2)$ which is a real C^*-algebra to be a complex C^*-algebras. Also, one can show that there is no complex structure on $B(\mathbb{R}^2)$ which makes $B(\mathbb{R}^2)$ a complex C^*-algebra.

(ii) If A is a complex C^* -algebra and $\pi : A \to B$ be an $\mathbb{R}$-linear isometric homomorphism into a real C^*-algebras B. Then, $\pi(A) \subseteq B$ admits a complex structure which makes it a complex C^*-agebra and $\pi : A \to \pi(A)$ is a $\mathbb{C}$-linear isometric isomorphism. This is easily seen by defining a complex structure $J : \pi(A) \to \pi(A)$ to be $J(\pi(a)) = \pi(ia)$. That is a real C^*-algebra might contain a subset which is a complex C^*-algebra. We provide an example. Let H be a real Hilbert space and $B(H)_c$ be its complexification of $B(H)$. The map $\pi : B(H)_c \to B(H^2)$ is defined to be

$$\pi(x + iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Then J is an $\mathbb{R}$-linear isometric homomorphism. Therefore $B(H^2)$, which is a real C^*-algebra, has a subset which is a complex C^*-algebra.

A complex C^*-algebra A can always be considered as a real C^*-algebra denoted by $A_{\mathbb{R}}$ which has a complex structure $J(x) = ix$. Obviously, $A_{sa} = (A_{\mathbb{R}})_{sa}$. We will use the following lemma (Proposition 11.22 in [9]) to obtain relationship between A and $A_{\mathbb{R}}$.

2.1.29 Lemma. *Let X be a complex Banach space and $X_{\mathbb{R}} = X$ as a real Banach space. Then for a complex bounded linear functional $\psi \in X^*$, $Re(\psi) = (\psi + \overline{\psi})/2 \in X^*_{\mathbb{R}}$ with $\|\psi\| = \|Re(\psi)\|$. In* *addition, for a real bounded linear functional* $\phi \in X_{\mathbb{R}}^*$, $\psi(x) = \phi(x) - i\phi(ix)$ *is a bounded complex linear functional which* $\|\psi\| = \|\phi\|$.

Define

$$\mathfrak{r}_A = \{x \in A : x + x^* \geq 0\}.$$

An element in $\mathfrak{r}_A$ is called real positive. The following states that positivity and real positivity on A and $A_{\mathbb{R}}$ are closely related.

2.1.30 Proposition. *Let* A *be a complex* C^**-algebra and* $A_{\mathbb{R}} = A$ *be considered as a real* C^**-algebra. The following hold.*

(i) $x \geq 0$ *in* A *if and only if* $x \geq 0$ *in* $A_{\mathbb{R}}$.

(ii) If $\psi \in S(A)$ *then* $\psi(x) = \phi(x) - i\phi(ix)$ *for some* $\phi \in S(A_{\mathbb{R}})$.

(iii) If $\phi \in S(A_{\mathbb{R}})$ *then* $\phi = Re(\psi)$ *for some* $\psi \in S(A)$.

(iv) $\mathfrak{r}_A = \mathfrak{r}_{A_{\mathbb{R}}}$.

Proof. (i) This follows from the fact that positive elements are of the form xx^* for some $x \in A$.

(ii) Let $\phi(x) = Re(\psi)$. Then ϕ is a real linear functional on $A_{\mathbb{R}}$. In addition, $\phi(xx^*) = Re(\psi(xx^*)) = \psi(xx^*) \geq 0$ for all $x \in A$. Thus $Re(\psi)$ is positive. By the lemma above, $\|\psi\| = \|Re(\psi)\| = 1$. Thus $Re(\psi) \in S(A_{\mathbb{R}})$.

(iii) Define $\psi(x) = \phi(x) - i\phi(ix)$. By the above lemma, $\|\psi\| = \|\phi\|$. Moreover, $\psi(xx^*) = \phi(xx^*) - i\phi(i(xx^*))$. Observe that $(ixx^*)^* = -ixx^*$. Since ϕ is positive, $\phi(ixx^*) = \phi((ixx^*)^*) = -\phi(ixx^*)$. Thus, $\phi(ixx^*) = 0$. Hence $\psi(xx^*) = \phi(xx^*) \geq 0$. Thus $\psi \in S(A)$.

(iv) By (i), $x + x^* \geq 0$ in A if and only if $x + x^* \geq 0$ in $A_{\mathbb{R}}$. Therefore, $\mathfrak{r}_A = \mathfrak{r}_{A_{\mathbb{R}}}$. □

2.1.31 Remark. Let A be a unital complex C^*-algebra. One can define

$$\mathfrak{r}_A = \{x \in A : Re(\psi) \geq 0, \psi \in S(A)\}.$$

To see this let $x \in A$ be such that $x + x^* \geq 0$, and $\psi \in S(A)$. Then $\psi(x + x^*) \geq 0$. Since $\psi(x^*) = \overline{\psi(x)}$, $\psi(x + x^*) = 2Re(\psi(x)) \geq 0$. Conversely, let $Re(\psi(x)) \geq 0$ for all $\psi \in S(A)$. Since $\psi(x^*) = \overline{\psi(x)}$, $\psi(x + x^*) = 2Re(\psi(x)) \geq 0$ for all $\psi \in S(A)$. This implies $x + x^* \geq 0$. In the case when we consider A as a real C^*-algebra, $\mathfrak{r}_{A_{\mathbb{R}}} = \{x \in A_{\mathbb{R}} : \phi(x) \geq 0, \phi \in S(A_{\mathbb{R}})\}$ is equal to $\mathfrak{r}_A$.

Denote $B(H)_{sa}$ to be the space of selfadjoint operators on a real or complex Hilbert space H. A JC-algebra is a uniformly closed subspace of $B(H)_{sa}$ which is closed under the Jordan product $a \circ b = (ab + ba)/2$. A weakly closed JC-algebra is called *JW-algebra*. These two algebras have been studied for example in [12] and [26]. A real *JC^*-algebra* A is a closed selfadjoint subspace ($a^* \in A$ if $a \in A$) of a real C^*-algebra B such that $ab + ba \in A$ for all $a, b \in A$ (equivalently, $a^2 \in A$ for all $a \in A$). A JC-algebra is an example of real JC^*-algebras. If A is a real JC^*-algebra, then A_c will be a complex JC^*-subalgebra of B_c.

In general, a real Banach space A with an involution $* : A \to A$ and a commutative bilinear map $\circ : A \times A \to A$ is a JC^*-algebra if there exists an isometric $*$-Jordan homomorphism $\pi : A \to B$ ($\pi(a^2) = \pi(a)^2$) where B is a real C^*-algebra. We call such a real C^*-algebra B a *C^*-algebra container* of A. Since the complex JC^*-algebra A_c has a cai $(E_t) = (e_t + ie'_t)$, (e_t) will be a cai in A. Thus, a real JC^*-algebra has a cai. Also, A^{**} is a real JC^*-algebra which A is a JC^*-subalgebra of A^{**}. We define positivity in a JC^*-algebra by declaring an element $a \in A$ is positive if a is positive in B. A functional $\phi \in A^*$ is positive if $\phi(a^*) = \phi(a)$ and $\phi(b) \geq 0$ for all positive element b. A positive functional ϕ is a real state on A if $\phi(e_t) \to 1$. We have an analog of Lemma 2.1.17 for real JC^*-algebras.

2.1.32 Lemma. *Let $\pi : A \to B$ be a $*$-Jordan homomorphism between real JC^*-algebra A and a real C^*-algebra B. If $a \geq 0$ in A, $\pi(a) \geq 0$ in B.*

Proof. Let $x \geq 0$. Then $x = x^*$ and $C^*(x) = JC^*(x)$. We have that $\pi|_{C^*(x)}$ is a $*$-homomorphism between real C^*-algebras. Thus, $\pi|_{C^*(x)}$ maps a positive element in $C^*(x)$ to a positive element in B. Therefore, $\pi|_{C^*(x)}(x) = \pi(x) \geq 0$. □

The previous lemma implies that the set of positive elements in a JC^*-algebra A is independent of the choice of real C^*-algebra container.

2.1.33 Lemma. *Let A be a real JC^*-algebra and $\phi \in A^*$. The following are equivalent.*

1. *ϕ is a real state on A.*

2. *ϕ is a real state on A^{**}.*

3. *ϕ_c is a state on A_c.*

4. *$\|\phi\| = 1$ and $\phi(e_t) \to 1$.*

Moreover, if A is unital JC^*-algebra with the identity 1_A and ϕ is a a positive linear functional, then ϕ is a real state if and only if $\phi(1_A) = 1$.

Proof. We assume that A is a JC^*-subalgebra of a real C^*-algebra B and A generates B. If (e_t) is a cai of A, we have (e_t) is a cai of B. By the Hahn-Banach Theorem, ϕ extends to a functional $\tilde{\phi}$ on B which preserves the norm. Thus, $\tilde{\phi}$ is a real state on B. Apply Lemma 2.1.17 with $\tilde{\phi}$ and obtain the result by restricting $\tilde{\phi}|_A = \phi$. □

2.1.34 Corollary. *Let A be a real JC^*-algebra and $\theta : A \to B(H)$ be a Jordan homomorphism. Then θ is contractive if and only if θ is selfadjoint.*

Proof. Since $x = h + k$ where $h = (x + x^*)/2$ and $k = (x - x^*)/2$. Then h and k are normal. Therefore, $C^*(h) = JC^*(h)$ and $C^*(k) = JC^*(k)$ are real C^* algebras. By consider restrictions of θ on $C^*(h)$ and $C^*(k)$, both are contractive homomorphism between real C^*-algebras. By Corollary 2.1.22, $\theta(x^*) = \theta(h^* + k^*) = \theta(h)^* + \theta(k)^* = \theta(h + k)^* = \theta(x)^*$. Conversely, if θ is selfadjoint i.e., θ is a real $*$-Jordan homomorphism, then $\theta_c : A_c \to B(H)_c$ is a complex $*$ Jordan homorphism. By the corresponding fact for complex JC^*-algebras, θ_c is contractive and thus $(\theta_c)|_A = \theta$ is contractive. □

2.2 Real Operator Spaces and their Complexifications

A *concrete operator space* is a (real or complex) Banach space X together with a (real or complex) linear isometric embedding $\pi : X \to B$ where B is a (real or complex) C^*-algebra. We call such a C^*-algebra, a *C^*-algebra container* of A. Thus, we may consider a concrete operator space X as a closed subspace $\pi(X)$ of a C^*-algebra B. An operator space comes naturally with the matrix norm $\|\cdot\|_n$ on $M_n(X)$ which is obtained from $M_n(\pi(X))$ as a closed subspace of the C^*-algebra $M_n(B)$. The collection of norms $\{\|\cdot\| : n \in \mathbb{N}\}$ satisfies the following two conditions.

(R1) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$ for all $n \in \mathbb{N}, x \in M_n(X)$ and $\alpha, \beta \in M_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

(R2) $x \in M_n(X)$ and $\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$ for all $m, n \in \mathbb{N}$, $x \in M_m(X)$ and $y \in M_n(Y)$.

These two conditions are referred as *Ruan's axioms*. Conversely, if X is a (real or complex) vector space with a collection of norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ which are defined on $M_n(X)$ for each $n \in \mathbb{N}$, Ruan's Theorem states that X can be embedded completely isometrically into a (real or complex) C^*-algebra if and only if the Ruan's axioms hold. Thus an *abstract operator space* can be defined as a vector space X that is given a collection of norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ which satisfies Ruan's axioms. Such a collection of norms is called a (real) *operator space structure of X*.

Let X and Y be operator spaces and $T : X \to Y$ be a linear map. For $n \in \mathbb{N}$, define $T_n : M_n(X) \to M_n(Y)$ to be $T_n([x_{ij}]) = [T(x_{ij})]$. We call T_n the (nth) *amplification of T*. We say T is a *completely bounded map* if $T_n : M_n(X) \to M_n(Y)$ is bounded for all $n \in \mathbb{N}$ and denote $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T_n\|$. A map T is *completely isometric* if T_n is isometric for all $n \in \mathbb{N}$. If T is a completely isometric bijection, we say X and Y are completely isometrically isomorphic and we consider X and Y as the same operator space.

Without loss of generality, we may replace B by $B(H)$ where H is a (real or complex) Hilbert space. This follows from the fact that any isometric $*$-homomorphism between C^*-algebras is completely contractive and a C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $B(H)$ for a real Hilbert space H. Therefore, the operator spaces determined by either B or $B(H)$ are completely isometrically isomorphic.

There are many operator space structures we can define on a given Banach space X. One can identify X as a subspace of $C(Ball(X^*))$ by $\pi(x)(\phi) = \phi(x)$ for $x \in X$ and $\phi \in Ball(X^*)$. This is called a *minimal operator space structure* of X and we denote this operator space as $Min(X)$. The norm of $[x_{ij}]_{n\times n} \in M_n(X)$ with this operator space structure is given by

$$\|[x_{ij}]\|_n = \sup_{\phi \in Ball(X^*)} \left\|[\phi(x_{ij})]\right\|_{M_n(\mathbb{R})}.$$

Another operator space structure that can be defined on X is given by the following norm:

$$\|[x_{ij}]\|_n = \sup\{\|[u(x_{ij})]\| : u \in Ball(B(X,Y)) \text{ for all operator spaces } Y\}.$$

This operator space is denoted by $Max(X)$.

Now, we consider a complexification of a real operator space. Assume that X is a real closed subspace of a real C^*-algebra B. The complexification X_c of X is obtained the norm from B_c or $M_2(B)$. That is

$$\|x+iy\|_{X_c} = \|x+iy\|_{B_c} = \left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\| M_2(B).$$

The norm of X_c that is obtained this way is is reasonable. In particular, if X with the operator space structure given by norms $\|\cdot\|_n$ on $M_n(X)$, we identify

$$x+iy = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \in M_2(X)$$

where the complex scalar multiplication is defined to be

$$(\alpha+i\beta)(x+iy) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} \alpha x - \beta y & -\beta x - \alpha y \\ \beta x + \alpha y & \alpha x - \beta y \end{bmatrix} \in M_2(X)$$

And simillary, $[x_{jk} + iy_{jk}]_{n\times n} \in M_n(X_c) \subseteq M_{2n}(X)$. Then $M_n(X_c)$ which is obtained the norm from $M_{2n}(X)$ by this identification, is a complex Banach space. We denote the complex norm on $M_n(X_c)$ by $(\|\cdot\|_c)_n = \|\cdot\|_{2n}|_{M_n(X_c)}$ and $(\|\cdot\|_c)_1 = \|\cdot\|_c$. Ruan proved that X_c with the collection of norms $\{(\|\cdot\|_c)_n : n \in \mathbb{N}\}$ is a complex operator space (see [23]). Ruan also showed that there is a unique complexification of a real operator space up to completely isometrically isomorphism, and this can be obtained by the above procedure. The operator space complex norm on X_c is referred as the *cannonical reasonable complex extension* of the matrix norm on X. We call a complex operator space X_c that is obtained from the complexification of a real operator space X, the *operator space complexification* of X.

A reasonable complexification of a real Banach space X can be obtained from a real operator space structure on X. Different operator space structures on X may define the same reasonable norm on X_c. For example, real row operator space and real column operator space on $B(\mathbb{R}^n)$ are different, but their operator space complexifications provide the same reasonable complex norm on X_c (but not on $M_n(X_c)$ for $n \geq 2$). Also, different operator space structures on a real Banach space may define different reasonable complexifications.

2.2.1 Example. Let $X = \mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ with the real norm $\|(x_1, x_2)\| = \sqrt{|x_1|^2 + |x_2|^2}$. One can identify X as a subspace of $B(\mathbb{R}^2)$ by the isometric map

$$(x_1, x_2) \mapsto \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix} \in B(\mathbb{R}^2).$$

The complexification norm of X_c obtaining from this identification ($B(\mathbb{R}^2)_c = B(\mathbb{C}^2)$) is

$$\left\| \begin{bmatrix} x_1 + iy_1 & x_2 + iy_2 \\ 0 & 0 \end{bmatrix} \right\| = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}.$$

Now, consider a complexification of X obtaining from $Min(X)$ i.e., X is embedded into $C(Ball(X^*))$ where $\Omega = Ball(X^*)$ equipped with the weak*-topology. Note that $C_{\mathbb{C}}(\Omega)$ is the unique complex C^*-algebra which is a complexification of $C_{\mathbb{R}}(\Omega)$. Also, a real functional on X is of the form

$\varphi_{s,t}(x_1, x_2) = sx_1 + tx_2$ where $s, t \in \mathbb{R}$ and

$$\|\varphi_{s,t}\| = \sqrt{s^2 + t^2}.$$

Thus, $Ball(X^*) = \{\varphi_{s,t} : s^2 + t^2 \leq 1\}$. We denote the norm obtained from $C(Ball(X^*))$ as $\|\cdot\|_{Min}$. Thus,

$$|((1,0) + i(0,1))(\varphi_{s,t})| = |\varphi_{s,t}(1,0) + i\varphi_{s,t}(0,1)| = |s + it| = \sqrt{s^2 + t^2}.$$

This shows that $\|(1,0) + i(0,1)\|_{Min} = 1$. However, $\|(1,0) + i(0,1)\|_{B(\mathbb{C}^2)} = \left\| \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \right\|_{B(\mathbb{C}^2)} = \sqrt{2}$.

2.2.2 Remark. The Taylor norm that we mentioned in 2.1.1 (iv) is the smallest reasonable complex norm that can be given to a real Banach space X. This norm coincides with the reasonable norm coming from the operator space $Min(X)$.

For a real Banach space X, X_c can be obtained a reasonable norm from a real operator space structure that is given to X. The converse of this is also true. That is any reasonable norm on X_c is coming from a real operator space structure on X.

2.2.3 Lemma. *Let $a, b, c, d \in \mathbb{R}$. Then*

$$\left\| \begin{bmatrix} a + ib & -c - id \\ c + id & a + ib \end{bmatrix} \right\|_{M_2(C)} = \max\{\sqrt{(a-d)^2 + (c+d)^2}, \sqrt{(a+d)^2 + (c-b)^2}\}.$$

Proof. Let $A = \begin{bmatrix} a + ib & -c - id \\ c + id & a + ib \end{bmatrix}$. Then

$$AA^* = \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & -2i(ad - bc) \\ 2i(ad - bc) & a^2 + b^2 + c^2 + d^2 \end{bmatrix}.$$

Then eigenvalues of AA^* are $a^2+b^2+c^2+d^2+2ad-2bc = (a+d)^2+(b-c)^2$ and $a^2+b^2+c^2+d^2-2ad+2bc = (a-d)^2+(b+c)^2$. Therefore, the norm of A is the bigger value of $\sqrt{(a-d)^2 + (c+d)^2}$

and $\sqrt{(a+d)^2+(c-b)^2}$. □

2.2.4 Proposition. *Let $(X, \|\cdot\|)$ be a real Banach space and $(X_c, \|\cdot\|_c)$ be a reasonable complexification of X. There is a real isometric embedding $\pi : X \to B$ where B is a real C^*-algebra such that $\pi_c : X_c \to B_c$ is isometric.*

Proof. Let $i : X_c \to C(Ball(X_c^*))$ be a canonical embedding of X_c into the space of continuous functions on $Ball(X_c^*))$. Let $\pi = i|_X$ and $B = C(Ball(X_c^*)$ where we consider $C(Ball(X_c^*))$ as a real C^*-algebra. We need to show that $\|x+iy\|_c = \left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\|_{M_2(B)}$. We know that

$$\|x+iy\|_c = \sup\{|\psi(x+iy)| : \psi \in Ball((X_c)^*)\}$$

and

$$\left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\|_{M_2(B)} = \sup\{\left\| \begin{bmatrix} \psi(x) & -\psi(y) \\ \psi(y) & \psi(x) \end{bmatrix} \right\|_{M_2(\mathbb{C})} : \psi \in Ball((X_c)^*)\}.$$

If $\psi \in (X_c)^*$, then $\psi = \chi + i\rho$ where $\chi, \rho \in X^*$. Thus, we can write

$$\begin{aligned} \|x+iy\|_c &= \sup\{|(\chi+i\rho)(x+iy)| : \chi+i\rho \in Ball(X_c^*)\} \\ &= \sup\{\sqrt{(\chi(x)-\rho(y))^2+(\rho(x)+\chi(y))^2} \qquad (2.1) \\ &\quad : \chi+i\rho \in Ball(X_c^*)\}. \end{aligned}$$

On the other hand,

$$\left\| \begin{bmatrix} \psi(x) & -\psi(y) \\ \psi(y) & \psi(x) \end{bmatrix} \right\|_{M_2(\mathbb{C})} = \left\| \begin{bmatrix} \chi(x)+i\rho(x) & -\chi(y)-i\rho(y) \\ \chi(y)+i\rho(y) & \chi(x)+i\rho(x) \end{bmatrix} \right\|_{M_2(\mathbb{C})}.$$

By the previous lemma ($a = \chi(x), b = \rho(x), c = \chi(y), d = \rho(y)$),

$$\left\| \begin{bmatrix} \psi(x) & -\psi(y) \\ \psi(y) & \psi(x) \end{bmatrix} \right\|_{M_2(\mathbb{C})} = \max\Big\{ \sqrt{(\chi(x) - \rho(y))^2 + (\rho(x) + \chi(y))^2}, \\ \sqrt{(\chi(x) + \rho(y))^2 + (\rho(x) - \chi(y))^2} \Big\}.$$

Therefore,

$$\left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\|_{M_2(B)} = \sup \Bigg\{ \max\Big\{ \sqrt{(\chi(x) - \rho(y))^2 + (\rho(x) + \chi(y))^2}, \\ \sqrt{(\chi(x) + \rho(y))^2 + (\rho(x) - \chi(y))^2} \Big\} \\ : \chi + i\rho \in Ball(X_c^*) \Bigg\}. \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain that $\|x+iy\|_c \leq \left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\|_{M_2(B)}$. Since $\|\chi+i\rho\| = \|\chi-i\rho\|$, we also have the converse inequality. Therefore,

$$\|x + iy\|_c = \left\| \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right\|_{M_2(B)}.$$

□

Consequently, we obtain the following corollary.

2.2.5 Corollary. *Let X be a real Banach space. Let $(X_c, \|\cdot\|_{max})$ be a reasonable complexification of X which is obtained from $Max(X)$. Then*

$$\|x + iy\|_T \leq \|x + iy\|_c \leq \|x + iy\|_{max}$$

for any reasonable norm $\|\cdot\|_c$ on X_c.

Proof. The first inequality is proved in [19]. Since $\|\cdot\|_{max}$ is the biggest norm among reasonable complexifications from all real operator spaces given to X, and by the previous proposition, it is also the biggest reasonable complex norm that can be defined on X_c as a complex Banach space. $\square$

2.2.6 Remark. Due to the intensive study of complex operator spaces, there are many known operator space structures which can be defined on a complex Banach space. Let X be a real Banach space and X_c be a reasonable complexification of X. Assume that there is an isometric embedding $\pi : X_c \to B$ where B is a complex C^*-algebra. Then $\pi|_X : X \to B$ is a real isometric embedding where we consider $B = B_{\mathbb{R}}$ as a real C^*-algebra. Then we have the operator space complexification of X which is obtained from this embedding. If the norm on X_c and the norm that is obtained by the complex operator space X_c are identical, we call such a complex operator space a *natural complex operator space* of X_c. From the proposition, we see that $Min(X_c)$ is a natural complex operator space. Assume X_c is a subspace of a complex C^*-algebra B and there is a real C^*-algebra $Re(B)$ such that $B = Re(B)_c$. If $X \subseteq Re(B)$, then the operator space X_c is a natural complex operator space. This can be a topic that we can investigate later in the future.

2.3 Real Unital Operator Spaces

An operator space A with an isometry $\pi : A \to B(H)$ is called a *unital operator space* if there is an element $e \in A$ such that $\pi(e) = I_H$. Concretely, a unital operator space is a closed subspace of $B(H)$ which contains the identity $e = I_H$ of $B(H)$. An element $a \in A$ is *positive* if $\pi(a)$ is positive in $B(H)$. A linear functional $\phi : A \to \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) on a unital operator space A is called a *(real) state* if $\|\phi\| = \phi(e) = 1$. Denote the set of all real states of A as $S(A)$. If A is a complex operator space, define

$$\mathfrak{r}_A = \{x \in A : Re(\phi(x)) \geq 0 \text{ for all } \phi \in S(A)\}.$$

In the case when A is a real unital operator space,

$$\mathfrak{r}_A = \{x \in A : \phi(x) \geq 0 \text{ for all } \phi \in S(A)\}.$$

An element in $\mathfrak{r}_A$ is called *real positive*. We investigate the relation of states and $\mathfrak{r}_A$ of a unital real operator space and $\mathfrak{r}_{A_c}$ of its operator space complexification.

2.3.1 Remark. (i) Let A be a unital subspace of $B(H)$ and $\pi : A \to B(K)$ be isometric linear map such that $\pi(I_H) = I_K$. If $a \geq 0$ in A. By the equivalent definition of positivity in $B(H)_c$, there is $t \geq \|a\|$ such that $\|tI_H - a\| \leq t$. Since π is isometric, we have $t \geq \|\pi(a)\|$ and

$$\|tI_K - \pi(a)\| = \|\pi(tI_H - (a))\| = \|tI_H - a\| \leq t.$$

Thus $\pi(a)$ is positive. This shows that positivity in a unital real unital operator space is independent of a choice of representation.

(ii) Let $\phi \in A^*$ be a real state on a unital real subspace of a real C^*-algebra $B(H)$ where H is a real Hilbert space. By the Hahn-Banach extension theorem, there is an extension $\tilde{\phi} \in B^**$ of ϕ such that $\|\phi\| = \|\tilde{\phi}\| = 1$. Since $\tilde{\phi}(I_H) = \phi(I_H) = 1$, $\tilde{\phi}$ is a real state on $B(H)$. Therefore, a real state on A is a restriction of a state on $B(H)$.

2.3.2 Lemma. *Let A be a real unital operator space and A_c be the operator space complexification of A. If $\phi \in S(A)$ then $\phi_c \in S(A_c)$, and if $\psi \in S(A_c)$ then $Re(\psi) \in S(A)$. In addition, $S(A) \subseteq S(A_c)$ and $\{Re(\psi) : \psi \in S(A_c)\} = S(A)$.*

Proof. It is obvious that $\phi_c(e) = 1$. By Proposition 2.1.2, $\|\phi_c\| = \|\phi\| = 1$. Thus, $\phi_c \in S(A_c)$. For $\psi \in S(A_c)$, we can write $\psi = Re(\psi) + iIm(\psi)$ where $Re(\psi), Im(\psi) \in A^*$. We have that $1 = Re(\psi)(e) + iIm(\psi)(e)$. This implies $Re(\psi)(e) = 1$. Thus, $1 \leq \|Re(\psi)\| \leq \|\psi\| = 1$ and so $\|Re(\psi)\| = Re(\psi)(e) = 1$. Therefore $Re(\psi) \in S(A)$. □

2.3.3 Proposition. *Let A_c be the operator space complexification of a real unital operator space A. Then $\mathfrak{r}_A = A \cap \mathfrak{r}_{A_c}$.*

Proof. Let $x \in \mathfrak{r}_A$ and $\psi \in S(A_c)$. Thus, $Re(\psi) \in S(A)$ by the previous lemma. Then $Re(\psi(x)) \geq 0$ by the definition of $\mathfrak{r}_A$. On the other hand, let $x \in A \cap \mathfrak{r}_{A_c}$ and $\phi \in S(A)$. Then $\phi_c \in S(A_c)$ by the previous lemma and thus $\phi(x) = \phi_c(x) = Re(\phi_c(x)) \geq 0$. □

2.3.4 Lemma. *Le A be a real unital operator space and $\pi : A \to B(K)$ be an isometric linear map which $\pi(e) = I_K$. Then $\phi \in S(\pi(A))$ if and only if there is $\phi \circ \pi \in S(A)$.*

Proof. Let $\phi \in S(\pi(A))$. Then $\phi \circ \pi \in A^*$. We also have $\phi(\pi(e)) = \phi(I_K) = 1$ and $\|\phi\| \leq \|\phi \circ \pi\| \leq \|\phi\| \cdot \|\pi\| \leq 1$. Thus, $\phi \circ \pi \in S(A)$. Conversely, $\phi(\pi(e)) = \phi(I_K) = 1$. Since π is isometric,

$$\|\phi\|_{\pi(A)^*} = \sup_{\|\pi(x)\|=1} |\phi(\pi(x))| = \sup_{\|x\|=1} |(\phi \circ \pi)(x)| = 1.$$

□

2.3.5 Proposition. *Let A be a real unital subspace of $B(H)$ where H is a real Hilbert space. Let $\pi : A \to B(K)$ be an isometric linear map which K is a real Hilbert space and $\pi(e) = I_K$. Then $x \in \mathfrak{r}_A$ if and only if $\pi(x) \in \mathfrak{r}_{\pi(A)}$.*

Proof. Let $x \in \mathfrak{r}_A$ and $\phi \in S(\pi(A))$. Then $\phi(\pi(x)) = \phi \circ \pi(x) \geq 0$ since $\phi \circ \pi \in S(A)$ by the above lemma. Thus, $\pi(x) \in \mathfrak{r}_{\pi(A)}$. The converse implication is obtained by the same proof applying to $\pi^{-1} : \pi(A) \to A \subseteq B(H)$. □

Therefore, we have proved that $\mathfrak{r}_A$ is independent of a choice of real Hilbert space.

2.3.6 Proposition. *Let A be a real unital subspace of $B(H)$ where H is a real Hilbert space. Then $x \in \mathfrak{r}_A$ if and only if $x + x^* \geq 0$.*

Proof. Let $x \in \mathfrak{r}_A$. To show $x + x^* \geq 0$, we may show that $x + x^* \geq 0$ in $B(H_c)$. Since $x \in \mathfrak{r}_A \subseteq \mathfrak{r}_{A_c} \subseteq B(H_c)$, by an equivalence of $x \in \mathfrak{r}_{A_c}$ in complex case, $x + x^* \geq 0$. So, $x + x^* \geq 0$ in $B(H)$ as well. Conversely, let $x + x^* \geq 0$ and $\phi \in S(A)$. Then ϕ extends to a real state $\tilde{\phi}$ on $B(H)$ and so $\tilde{\phi}(x + x^*) \geq 0$. Since $\tilde{\phi}(x) = \tilde{\phi}(x^*)$,

$$2\phi(x) = 2\tilde{\phi}(x) = \tilde{\phi}(x) + \tilde{\phi}(x^*) = \tilde{\phi}(x + x^*) \geq 0.$$

□

Therefore, under a unital linear isometry, $\mathfrak{r}_A$ is independent of a choice of representation and can be alternatively defined as $\mathfrak{r}_A = \{x : x + x^* \geq 0\}$.

Next, we consider Arveson's Theorem (Proposition 1.2.8 in [1]) for the real case.

2.3.7 Proposition. *Let A be a unital real subspace of a real C^*-algebra B. Let $\pi : A \to B(H)$ be completely contractive and $\pi(e) = I_H$. Then π has a bounded selfadjoint linear extension $\tilde{\pi} : A + A^* \to B(H)$ which is completely positive. Indeed $\tilde{\pi}$ is completely contraction.*

Proof. Since π is completely contractive, $\pi_c : A_c \to B(H_c)$ is completely contractive. Apply Proposition 1.2.8 in [1], there is a unique extension $\tilde{\pi_c} : A_c + A_c^* \to B(H_c)$ which is selfadjoint and completely positive. Let $a, b \in A$. Then $\tilde{\pi_c}(a + b^*) = \pi_c(a) + \pi_c(b)^* = \pi(a) + \pi(b)^* \in B(H)$. Thus, $\tilde{\pi} = \tilde{\pi_c}|_{A+A^*}$ is the desired map. Since $A + A^*$ is an real operator system, Proposition 4.1 in [23], $\tilde{\pi}$ is also completely contractive. □

2.3.8 Remark. (i) Let A be a unital operator space and $\pi : A \to B(H)$ be a unital isometry. By the previous proposition, $\pi|_{\Delta(A)}$, where $\Delta(A) = \{a \in A : a^* \in A\}$, is selfadjoint and isometric. If $\tilde{\pi}(a+b^*) = \tilde{\pi}(c+d^*)$, $\pi(a-c) = \pi((b-d)^*)$. Since π is isometric, $a-c = (b-d)^*$ and so $a + b^* = c + d^*$. Thus π is completely isometric. Hence $A + A^*$ is independent of a choice of a representation.

(ii) The operator space $(A + A^*) \subseteq B(H)$ has the operator space complexification $(A + A^*)_c \subseteq B(H_c)$. In addition, $A_c + A_c^*$ is also a reasonable operator space complexification of $A + A^*$. By the uniqueness of the operator space complexification, $A_c + A_c^* = (A + A^*)_c$.

(iii) One can also study *approximately real unital operator spaces A*. This is defined to be an operator subspace A of a real C^*-algebra B where A contain a net (e_t) such that $e_t a$ and $a e_t$ converge weak* to a for all $a \in A$.

2.4 Complexification of Real Contractive Maps

Let X and Y be operator spaces. If $T : X \to Y$ is a completely contractive map, then $T_c : X_c \to Y_c$ (which is defined as $T_c(x+iy) = T(x) + iT(y)$) is completely contractive. However, if $T : X \to Y$ is only contractive, then T_c may not be contractive. This depends on the operator space structures that are given to X and Y.

2.4.1 Proposition. *Let X and Y be real Banach spaces and $T : X \to Y$ be a bounded linear map. We have the following.*

(i) If X is given the maximal operator space structure, i.e., $X = Max(X)$, then $\|T_c\| = \|T\|$.

(ii) If Y is given the minimal operator space structure i.e. $Y = Min(Y)$, then $\|T_c\| = \|T\|$.

Proof. (i) Without loss of generality, we assume that $\|T\| = 1$. Let $X = Max(X)$ and Y is an operator space. Then $\|T_c(x+iy)\| = \left\| \begin{bmatrix} T(x) & -T(y) \\ T(y) & T(x) \end{bmatrix} \right\|_{M_2(Y)}$. Since $T \in Ball(B(X,Y))$,

$$\left\| \begin{bmatrix} T(x) & -T(y) \\ T(y) & T(x) \end{bmatrix} \right\|_{M_2(Y)} \leq \sup \left\{ \left\| \begin{bmatrix} u(x) & -u(y) \\ u(y) & u(x) \end{bmatrix} \right\|_{M_2(Y')} : u \in Ball(B(X,Y')) \text{ for an operator space } Y' \right\}.$$

Therefore, $\|T_c(x+iy)\| \leq \|x+iy\|_{max}$.

(ii) Now, let $Y = Min(Y)$ and X be a real operator space. Assume that $\|T\| = 1$. We find the norm of $T_c(x+iy) = T(x)+iT(y)$. Since the norm of $Min(Y)_c$ is obtained from $C(Ball(Y^*))_c$, we identify $T(x) + iT(y) \in C(Ball(Y^*))_c$. For $\phi \in Ball(Y^*)$,

$$|T_c(x+iy)(\varphi)| = |(T(x)+iT(y))(\phi)| = |(\phi \circ T)_c(x+iy)| \leq \|(\phi \circ T)_c\| \|x+iy\|.$$

Since $\phi \circ T$ is a contractive linear real functional, $(\phi \circ T)_c$ is still contractive. So, $|T_c(x+iy)(\phi)| \leq \|x+iy\|$ which proves $\|T_c\| = \|T\|$. □

Observe that $\mathbb{R}$ has only one complexification, and thus $Max(\mathbb{R})_c = Min(\mathbb{R})_c = \mathbb{C}$. If $\phi : X \to \mathbb{R}$ or $\phi : \mathbb{R} \to X$ is a contractive linear map, then ϕ_c is contractive.

Let X and Y be operator spaces. We say that (X, Y) is *contractive preservable* if for any contractive map $T : X \to Y$, its complex extension $T_c : X_c \to Y_c$ is contractive. We list some obvious examples here.

2.4.2 Examples. (i) By the previous proposition, $(Z, Min(Y))$ and $(Max(X), Z)$ are contractive preservable for any real operator space Z.

(ii) As we mentioned above $(X, \mathbb{R})$ is contractive preservable. This coincides with Proposition 2.1.2. Also, $(\mathbb{R}, X)$ is contractive preservable.

(iii) Let Ω be a locally compact Hausdorff space and $X = C_0(\Omega, \mathbb{R})$. Then $C_0(\Omega, \mathbb{C})$ is a complexification of X. Then $(X, \mathbb{R})$ is contractive preservable. This is a special case of the above fact. We provide an alternative proof of this by using a fact in measure theory. Note that $\|f + ig\| = \sup_{x \in X} \left\| \begin{bmatrix} f(x) & -g(x) \\ g(x) & f(x) \end{bmatrix} \right\| = \sup_{x \in X} \sqrt{f(x)^2 + g(x)^2}$.

Proof. Let $T : C_0(\Omega) \to \mathbb{R}$ be contractive. There is a Radon measure μ such that $|\mu|(X) = \|T\|$ and $T(f) = \int f d\mu$. Then

$$\begin{aligned} |T(f+ig)| = |\int f + ig d\mu| \leq \int |f + ig| d|\mu| \leq \int \sqrt{f^2 + g^2} d|\mu| \\ \leq \sup_{x \in X} \{\sqrt{f(x)^2 + g(x)^2}\} \cdot |\mu|(X) = \|f + ig\| \cdot \|T\|. \end{aligned}$$

□

In general, a pair of operator spaces X and Y might not be contractive preservable. We provide an example. Let $T : Min(l_2^1) \to Max(l_2^1)$ be the identity map. Obviously, T and T^{-1} are isometric. We claim that $T_c : Min(l_2^1)_c \to Max(l_2^1)_c$ is not isometric. Denote $Min = Min(l_2^1)$ and $Max = Max(l_2^1)$. Thus, we need to show that there is some $(a, b) + i(c, d) \in Min(l_2^1)_c$ such that $\|(a, b) + i(b, c)\|_{Min_c} \neq \|(a, b) + i(b, c)\|_{Max_c}$. We choose $(a, b) = (1, 1)$ and $(c, d) = (-1, 1)$. Then

$$\|(a,b)+i(b,c)\|_{Min_c} = \left\| \begin{bmatrix} (1,1) & (1,-1) \\ (-1,1) & (1,1) \end{bmatrix} \right\|_{M_2(Min)}$$

and

$$\|(a,b)+i(b,c)\|_{Max_c} = \left\| \begin{bmatrix} (1,1) & (1,-1) \\ (-1,1) & (1,1) \end{bmatrix} \right\|_{M_2(Max)}.$$

To find $\left\| \begin{bmatrix} (1,1) & (1,-1) \\ (-1,1) & (1,1) \end{bmatrix} \right\|_{M_2(Min)}$, let $(\alpha,\beta) \in (l_2^1)^* = l_2^\infty$, such that $\|(\alpha,\beta)\| = \max\{|\alpha|,|\beta|\} = 1$. Then

$$\left\| \begin{bmatrix} (\alpha,\beta)(1,1) & (\alpha,\beta)(1,-1) \\ (\alpha,\beta)(-1,1) & (\alpha,\beta)(1,1) \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha+\beta & \alpha-\beta \\ -\alpha+\beta & \alpha+\beta \end{bmatrix} \right\|.$$

The norm of $\begin{bmatrix} \alpha+\beta & \alpha-\beta \\ -\alpha+\beta & \alpha+\beta \end{bmatrix}$ is the square root of the biggest absolute eigenvalues of

$$\begin{bmatrix} \alpha+\beta & \alpha-\beta \\ -\alpha+\beta & \alpha+\beta \end{bmatrix} \begin{bmatrix} \alpha+\beta & -\alpha+\beta \\ \alpha-\beta & \alpha+\beta \end{bmatrix} = \begin{bmatrix} 2(\alpha^2+\beta^2) & 0 \\ 0 & 2(\alpha^2+\beta^2) \end{bmatrix},$$

which is $\sqrt{2(\alpha^2+\beta^2)}$. Thus, $\|(a,b)+i(c,d)\|_{Min} = 2$.

To find $\left\| \begin{bmatrix} (1,1) & (1,-1) \\ (-1,1) & (1,1) \end{bmatrix} \right\|_{M_2(Max)}$, we need the following facts. Define a complexification norm for $(l_2^1)_c$ to be

$$\begin{aligned} \|(a,b)+i(c,d)\| &= \sup\{\|\alpha(a,b)+\beta(c,d)\| : \alpha^2+\beta^2 \leq 1\} \\ &= \sup\max\{|\alpha a+\beta c|, |\alpha d+\beta d|\} \\ &= \max\{a^2+c^2, b^2+d^2\} \\ &= \|(a+ic, b+id)\|_{l^\infty(\mathbb{C})}. \end{aligned}$$

With this complexification, we obtain that

$$l_2^\infty(\mathbb{R})_c = l_2^\infty(\mathbb{C}).$$

We obtain the following lemma.

2.4.3 Lemma. $Max(l_2^1(\mathbb{R}))_c = Max(l_2^1(\mathbb{C}))$.

Proof.

$$\begin{aligned} Max(l_2^1(\mathbb{R}))_c &= (Min(l_2^\infty(\mathbb{R}))^*)_c \quad (\text{ by Proposition 2.6 in [25]}) \\ &= (Min(l_2^\infty(\mathbb{R}))_c)^* \quad (\text{ by Proposition 2.3 in [25]}) \\ &= Min(l_2^\infty(\mathbb{C})) \quad (\text{ by the fact mentioned above}) \\ &= Max(l_2^\infty(\mathbb{C})^*) \quad (\text{by 1.4.12 in [3] }) \\ &= Max(l_2^1(\mathbb{C})) \quad (\text{ by the fact mentioned above}). \end{aligned}$$

□

By the previous lemma, we obtain

$$\left\| \begin{bmatrix} (1,1) & (1,-1) \\ (-1,1) & (1,1) \end{bmatrix} \right\|_{M_2(Max)} = \|(1,1) + i(1,-1)\|_{Max(l_2^1(\mathbb{C}))} = \|(1+i, 1-i)\|_{Max(l_2^1(\mathbb{C}))} = 2\sqrt{2}.$$

Now, we will construct real operator algebras X and Y which are not algebra contractive preservable. Let X be a real operator space. Thus, we can regard X as a closed subspace of $B(H)$, for some real Hilbert space H. Define $U(X) \subseteq B(H^2)$ to be the space of elements of the form

$$A = \begin{bmatrix} \alpha & x \\ 0 & \beta \end{bmatrix}$$

where $\alpha = \alpha I_H, \beta = \beta I_H \in B(H)$ and $x \in X \subseteq B(H)$. Note that

$$\|A\| = \sup\{\|A(\zeta, \eta)\| : \|\zeta\|^2 + \|\eta\|^2 = 1, \zeta, \eta \in H\}.$$

Following the proof of Proposition 2.2.11 in [3], we obtain that

$$\|A\|^2 = \sup\{(|\alpha|\sqrt{1-t^2} + \|x\|t)^2 + |\beta t|^2 : t \in [0,1]\}. \tag{2.3}$$

From this equation, we can easily see that

$$\left\| \begin{bmatrix} \alpha & x \\ 0 & \beta \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|x\| \\ 0 & \beta \end{bmatrix} \right\|$$

If $A = \begin{bmatrix} \alpha_1 & x \\ 0 & \beta_1 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_2 & y \\ 0 & \beta_2 \end{bmatrix}$ are elements in $U(X)$, define

$$AB = \begin{bmatrix} \alpha_1 & x \\ 0 & \beta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 & y \\ 0 & \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 & \alpha_1 y + \beta_2 y \\ 0 & \beta_1\beta_2 \end{bmatrix}.$$

This makes $U(X)$ a real operator algebra.

2.4.4 Lemma. *Let $T : X \to Y$ be a linear contraction (resp. isometry) from operator spaces X to Y. Then $\theta_T : U(X) \to U(Y)$ defined by*

$$\theta_T\left(\begin{bmatrix} \alpha & x \\ 0 & \beta \end{bmatrix} \right) = \begin{bmatrix} \alpha & T(x) \\ 0 & \beta \end{bmatrix}$$

is a contractive (resp. isometry) homomorphism.

Proof. It is routine to show that θ_T is a homomorphism. If T is a contraction, then $\|Tx\| \leq \|x\|$.

Thus, $(|\alpha|\sqrt{1-t^2}+\|Tx\|t)^2+|\beta t|^2 \le (|\alpha|\sqrt{1-t^2}+\|x\|t)^2+|\beta t|^2$. By (2.3),

$$\left\|\begin{bmatrix}\alpha & T(x)\\ 0 & \beta\end{bmatrix}\right\| \le \left\|\begin{bmatrix}\alpha & x\\ 0 & \beta\end{bmatrix}\right\|.$$

□

Let $X = Min(l_2^1)$ and $Y = Max(l_2^1)$ and $T : X \to Y$ be the identity map. By the previous lemma, $\theta_T : U(X) \to U(Y)$ and $\theta_T^{-1} : U(Y) \to U(X)$ are contractive. But $T_c^{-1} : U(Y)_c \to U(X)_c$ is not contractive. Since $X = Min(l_2^1)$ is a subspace of $U(X)$ and $Y = Min(l_2^2)$ is a subspace of $U(Y)$ and we showed that $\|(1,1)+i(-1,1)\|_{Max} = 2\sqrt{2}$ and $\|(1,1)+i(-1,1)\|_{Min} = \sqrt{2}$. Thus,

$$\left\|\begin{bmatrix}0 & (1,1) & 0 & (-1,1)\\ 0 & 0 & 0 & 0\\ 0 & (1,-1) & 0 & (1,1)\\ 0 & 0 & 0 & 0\end{bmatrix}\right\|_{(M_2(U(X))} < \left\|\begin{bmatrix}0 & (1,1) & 0 & (-1,1)\\ 0 & 0 & 0 & 0\\ 0 & (1,-1) & 0 & (1,1)\\ 0 & 0 & 0 & 0\end{bmatrix}\right\|_{(M_2(U(Y))},$$

where $\|\cdot\|_{M_2(U(X))}$ and $\|\cdot\|_{M_2(U(Y)}$ are the norms on $M_2(U(X)$ and $M_2(U(Y))$ respectively. This implies T_c^{-1} is not contractive. We conclude as the following corollary.

2.4.5 Corollary. *There is an a real operator algebra X and Y and a contractive (resp. isometry) homomorphism $T : X \to Y$ whose canonical complexification map $T_c : X_c \to Y_c$ is not contractive.*

2.5 The Universal Real C^*-Algebras of Real Operator Spaces

The universal complex C^*-algebra of a complex operator space is introduced in Theorem 8.14 in [21]. Following the proof of Theorem 8.1.4 in [21], we also have such a universal real C^*-algebra for a real operator space.

2.5.1 Theorem. *Let X be a real operator space. There is a unique real C^*-algebra, $C^*(X)$, and a completely isometric embedding $i : X \to C^*(X)$ with the following universal properties:*

(i) For any real C^*-algebra B and any completely contractive map $j : X \to B$, there is a $*$-homomorphism $\pi : C^*(X) \to B$ such that $\pi \circ i = j$.

(ii) $i(X)$ generates $C^*(X)$.

The unitization of $C^*(X)$ is denoted by $C^*_u(X)$. Also, $C^*_u(X)$ has a universal property that for any unital C^*-algebra B and a completely contractive map $j : X \to B$, there is a unital $*$-homomorphism $\pi : C^*_u(X) \to B$ such that $\pi \circ j = i$. The unitization of a real operator space X is defined to be $X^1 = \{i(x) + \lambda u : x \in X, \lambda \in \mathbb{R}\}$ as a subspace of $C^*_u(X)$ where u is the identity of $C^*_u(X)$. We investigate the complexification of $C^*(X)$ and $C^*_u(X)$ and see that the complexification of $C^*(X)$ and $C^*(X)$ are $C^*(X_c)$ and $C^*_u(X_c)$ respectively.

2.5.2 Proposition. *Let X be a real operator space and X_c be the operator space complexification of X. Then, $C^*(X)_c = C^*(X_c)$ and $C^*_u(X)_c = C^*_u(X_c)$.*

Proof. We show that $(C^*(X)_c, i_c)$ has the universal property as in Theorem 8.14 in [21]. Let B be a complex C^*-algebra and $J : X_c \to B$ be a completely contractive map. By considering B as a real C^*-algebra, $j = J|_X : X \to B$ is a real completely contractive map. By the universal property of $C^*(X)$, there is a $*$-homomorphism $\pi : C^*(X) \to B$. Define $\pi_c : C^*(X)_c \to B$ by $\pi_c(x+iy) = \pi(x)+i\pi(y)$ for $x, y \in C^*(X)$. Then, π_c is a complex $*$-homomorphism and $\pi_c \circ J = i_c$. By the uniqueness of a unitization of a C^*-algebra, we also have $C^*_u(X)_c = C^*_u(X_c)$. □

2.5.3 Remark. For a real unital operator space X, a C^*-extension of X is defined in Definition 4.11 in [25]. A *C^*-extension* of a unital real operator space X is defined to be a pair (B, j) consisting of a unital real C^*algebra B and a unital complete isometry $j : X \to B$ such that $j(X)$ generates B as a real C^*-algebra. There is a unique C^*-extension of X, $(C^*_e(X), i)$, with the following universal property: If (A, j) is a C^*-extension of X, there exists a $*$-homomorphism $\pi : A \to C^*_e(X)$ (unique and surjective) such that $\pi \circ j = i$. The C^*-algebra $C^*_e(X)$ is called the *C^*-envelope* of X. Let (B, j) be a complex C^*-extension of X_c. If we could show that B is a complexification of a real C^*-algebra and $j(X) \subseteq A$, then we could prove that $C^*_e(X)_c$ is the complex C^*-envelope of X_c.

However, the statement might not be true and $C_e^*(X)_c$ might not be the complex C^*-envelope of X_c.

Chapter 3

Real Operator Algebras

3.1 Definitions and Fundamental Facts of Real Operator Algebras

A *concrete real operator algebra* is a real Banach algebra A together with a real isometric homomorphism $\pi : A \to B$ where B is a real C^*-algebra. In other words, a concrete real operator algebra is a closed subalgebra of a real C^*-algebra. Obviously, a concrete real operator algebra is a concrete real operator space. Therefore, it comes naturally with the matrix norm $\|\cdot\|_n$ on $M_n(A)$ which satisfies Ruan's axioms. A real operator space which is also a real Banach algebra is called an *abstract real operator algebra* if there exists a completely isometric homomorphism $\pi : A \to B$ where B is a real C^*-algebra. Since the definition of a concrete real operator algebra and the definition of an abstract real operator algebra are equivalent, when we say that A is a real operator algebra, we already assume that A comes with norm $\|\cdot\|_n$ on $M_n(A)$ and a completely isometric homomorphism from A into a real C^*-algebra B or $B(H)$ where H is a real Hilbert space. A characterization of operator algebras (which is called the BRS theorem) can be found in [3] for the complex case and in [25] for the real case.

Let A be a real operator algebra. The operator space complexification A_c of A which is described in Chapter 2.2 is a complex operator space. The product on A_c is defined in a natural way as

$$(x + iy)(u + iv) = xu - yv + i(xv + yu),$$

for $x, y, u, v \in A$. The complex operator space norm on A_c, together with this product, is a complex operator algebra (see Section 3 in [25]). We state this fact as the following lemma.

3.1.1 Lemma. *Let A be a real operator algebra. Then the operator space complexification A_c of A is a complex operator algebra.*

Assume that A is a real operator subalgebra of $B(H)$ where H is a real Hilbert space. A *contractive approximation identity* or a *cai* is a contractive net (e_t) in A such that $e_t a \to a$ and $ae_t \to a$ for all $a \in A$. If A admits a cai, we call A an *approximately real unital operator algebra.* Denote $A^1 = span\{A, I_H\}$ when $I_H \notin A$, and let $C^*_{B(H)}(A)$ be the real C^*-algebra generated by A inside $B(H)$. It is trivial that if A is an (approximately) unital real operator algebra, A_c is also (approximately) unital complex operator algebra. The following lemma shows that a cai of a real operator algebra A is a cai of the C^*-algebra generated by A.

3.1.2 Lemma. *If A is a real operator subalgebra of a real C^*-algebra B and $C^*_B(A)$ is the C^*-subalgebra of B generated by A, then a cai of A is a cai of $C^*_B(A)$.*

Proof. Let (e_t) be a cai of A. Then (e_t) is a cai of A_c as a subalgebra of B_c. By the fact from the complex case, (e_t) is a cai of $C^*_{B_c}(A_c)$ where $C^*_{B_c}(A_c)$ is the complex C^*-subalgebra of B_c generated by A_c (see equation (1.1) in [6]). Since $C^*_B(A)$ is a real C^*-subalgebra of $C^*_{B_c}(A_c)$, (e_t) is a cai of $C^*_B(A)$ as well. $\square$

3.1.3 Remark. The C^*-algebra generated by an algebra A may depend on the C^*-algebra container of A. However, the previous lemma shows that a cai in A will be a cai of the C^*-algebra generated by an algebra A for any C^*-algebra container.

In Chapter 2, we defined a real state for a real unital operator space. Now, we will define a real state for an approximately unital operator algebra. Let A be an approximately unital real operator subalgebra of a real C^*-algebra B containing a cai (e_t). We call a linear functional $\phi : A \to \mathbb{R}$ a *real state* on A if $1 = \|\phi\| = \lim_t \phi(e_t)$.

3.1.4 Lemma. *Let ϕ be a real state on an approximately unital real operator algebra A. Then $\lim_t \phi(e_t) = 1$ for every cai $e_t \in A$.*

Proof. This follows from the fact that all cai converge weak* to $e \in A^{**}$ and thus $\lim_t \phi(e_t) = \phi(e) = 1$. □

3.1.5 Remark. Let A be a real operator subalgebra of a real C^*-algebra B. Let $\pi : A \to B(K)$ be an isometric homomorphism where K is a real Hilbert space. Denote

$$\Delta(A) = \{a : a \in A \text{ and } a^* \in A\}.$$

Thus A^* is a real C^*-algebra and $\pi|_{\Delta(A)} : \Delta(A) \to B(K)$ is isometric homomorphism. By Corollary 2.1.22, $\pi|_{\Delta(A)}$ is selfadjoint. Therefore, $\pi|_{\Delta(A)}(a^*) = \pi|_{\Delta(A)}(a)^* \in A$. Therefore the range of $\pi|_{\Delta(A)}$ is $\Delta(\pi(A))$. This shows that $\Delta(A)$ is well defined and independent of a choice of representation. As a consequence, $\tilde{\pi} : A + A^* \to B(H)$ where $\tilde{\pi}(a + b^*) = \pi(a) + \pi(b)^*$ is well defined. To see this, let $a+b^* = c+d^*$. Thus $a-c = (b-d)^*$. Hence $a+c \in \Delta(A)$ and $\pi(a-c) = \pi((b-d)^*) = \pi(b)^* - \pi(d)^*$. Thus, $\pi(a+b^*) = \pi(c+d^*)$. In addition $\tilde{\pi}$ is injective. To see this, assume that $\tilde{\pi}(a+b^*) = \tilde{\pi}(c+d^*)$. So $\pi(a - c) = \tilde{\pi}(a - c) = \pi(b^* - d^*) = \pi(b^* - d^*)$. Since $a - c, b^* - d^* \in A$ and π is isometric, $a - c = d^* - b^*$, i.e., $a + b^* = c + d^*$. Therefore, $A + A^*$ is independent of a choice of representation.

3.1.6 Proposition. *If ϕ is a real state on a real approximation unital subalgebra A of a real C^*-algebra B, then ϕ extends uniquely to a state on $A + A^*$.*

Proof. By the Hahn-Banach Theorem, ϕ extends to a linear functional $\tilde{\phi}$ on $C^*_B(A)$ with $\|\tilde{\phi}\| = \|\phi\| = 1$. Since a cai of A is a cai of $C^*_B(A)$ by Lemma 3.1.2, $\tilde{\phi}(e_t) = \phi(e_t) \to 1$ and so $\tilde{\phi}$ is a real state on $C^*_B(A)$. Therefore, $\tilde{\phi}|_{A+A^*}$ is a real state extension of ψ to $A+A^*$. If ψ is a real state extension of ϕ to $A+A^*$, this can be extended to a real state $\tilde{\psi}$ on $C^*_B(A)$. Thus $\tilde{\phi}$ and $\tilde{\psi}$ agree on A. Let $x \in A$. Since a real state on a real C^*-algebra is selfadjoint, $\tilde{\phi}(x^*) = \tilde{\phi}(x) = \phi(x) = \psi(x) = \tilde{\psi}(x) = \tilde{\phi}(x^*)$. Thus, any real state extensions of ϕ agree on $A + A^*$. Therefore, it is unique. □

3.1.7 Lemma. *If $\psi : A_c \to \mathbb{C}$ is a state on the complexification of A, then $Re(\psi)$ is a real state on A.*

Proof. We have $Re(\psi(e_t)) \to 1$. Also, $\|Re(\psi)\| \leq \|\psi\| = 1$. Since $Re(\psi(e_t)) \to 1$, $\|Re(\psi)\| = 1$. □

3.1.8 Lemma. *ϕ is a real state on A if and only if ϕ_c is a state on A_c.*

Proof. We have that $\|\phi_c\| = \|\phi\|$ by Proposition 2.1.2. If ϕ is a state on A, then $\phi_c(e_t) = \varphi_c(e_t) \to 1$ for a cai $e_t \in A$. Conversely, if ϕ_c is a state on A_c and E_t is a cai in A_c. Then $Re(E_t)$ is a cai in A and $\phi(Re(E_t)) \to 1$. □

The following is a real analogous fact of Proposition 2.1.18 in [3].

3.1.9 Proposition. *Let ϕ be a linear functional on an approximately unital operator algebra $A \subseteq B(H)$. The following are equivalent:*

(i) ϕ is a real state on A.

(ii) ϕ^1 defined by $\phi^1(a + \lambda 1) = \phi(a) + \lambda$ is a state on A^1.

(iii) $\phi(e) = 1$ where e is the identity of A^{**} where here we consider $\phi \in A^{***}$.

Proof. This follows by Lemma 3.1.7 and Lemma 3.1.8 above and analogous facts in the complex case (see Proposition 2.1.18 [3]). □

3.1.10 Remark. If A is a nonunital approximately real operator algebra and ϕ is a real state on A, there is a unique real state on A^1 which extends ϕ, namely $\phi^1(a + \lambda) = \phi(a) + \lambda$.

3.2 Unitization of Real Operator Algebras

The unitization of a complex operator algebra is unique up to isometric (respective completely isometric) homomorphism due to the Meyer's theorem. In addition, the unitization of a real operator algebra is unique up to completely isometric homomorphism (see Theorem 3.5 in [25]). Now, we will show that a unitization of real operator algebra is unique up to isometric homomorphism.

If A is a real operator subalgebra of $B(H)$ where H is a real Hilbert space and A_c be its operator space complexification. Thus A_c can be considered as an operator subalgebra of $B(H_c)$. An element in $B(H_c)$ is of the form $a + ib$ where $a, b \in B(H)$. Thus, we can regard an element of the form $a + i0 \in B(H_c)$ where $a \in A$ as an element of $B(H)$. In the other word, $B(H) = B(H) + i\{0\} \subseteq B(H_c)$.

The following lemma is a specific case for $n = 1$ of Lemma 2.1.12 from [3].

3.2.1 Lemma. *Let $A \subseteq B(H)$ be a real operator algebra and $A_c \subseteq B(H_c)$ be its complexification where H is a real Hilbert space. Assume that $I_H \notin A$. Then for $a, b \in A$ and $\lambda \in \mathbb{C}$, we have*

$$|\lambda| \leq \|(a + ib) + \lambda I_H\|_c.$$

Proof. Since $I_H \notin A$, then $I_H = I_{H_c} \notin A_c$. Then A_c satisfies the conditions in Lemma 2.1.12 from [3]. By this lemma,

$$|\lambda| \leq \|a + ib + \lambda I_{H_c}\|_c = \|a + ib + \lambda I_H\|_c$$

for $a + ib \in A_c$ and $\lambda \in \mathbb{R}$. □

The following facts are mentioned in 2.1.14 in [3] in the complex case.

3.2.2 Lemma. *Let K be a complex Hilbert space.*

(1) $S \in B(K)$ is strictly accretive ($S + S^*$ is positive and invertible) if and only if $-1 \notin \sigma(S)$ and the Cayley transform $c(S) = (S - I)(S + I)^{-1}$ is a strict contraction ($\|T\| < 1$).

(2) $T \in B(K)$ is a strict contraction if and only if $1 \notin \sigma(T)$ and the inverse Cayley transform $d(T) = (I + T)(I - T)^{-1}$ is strictly accretive.

If $T \in B(H)$, we can regard $T = T_c \in B(H_c)$. Thus, the previous lemma applies to $T = T_c \in B(H_c)$ (by considering $K = H_c$).

3.2.3 Theorem (Meyer)**.** *Let A be a real operator subalgebra of $B(H)$ where H is a real Hilbert space, and assume that $I_H \notin A$. Let $\pi : A \to B(K)$ be a contractive homomorphism for a real Hilbert* *space K. Let $A^1 = span_{\mathbb{R}}\{A, I_H\} \subseteq B(H)$ and define $\pi^o : A^1 \to B(K)$ by $\pi^o(a+\lambda I_H) = \pi(a)+\lambda I_K$. Then π^0 is a contractive homomorphism.*

Proof. We follow the proof of the Meyer's theorem for a complex operator algebra (see Theorem 2.1.13 in [3]) using the fact that A has a complexification which is a complex operator algebra.

Let $T = a + \lambda I_H \in A^1$ for some $a \in A$ and $\lambda \in \mathbb{R}$ and $\|T\| < 1$. We claim that $\|\pi^\circ(T)\| < 1$.

Step 1: Showing that $|\lambda| < 1$. Consider

$$T_c = T + i0 = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} a+\lambda I_H & 0 \\ 0 & a+\lambda I_H \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \lambda \begin{bmatrix} I_H & 0 \\ 0 & I_H \end{bmatrix} = a + \lambda I_{H_c}.$$

By the property of complexification, $\|T_c\|_{B(H_c)} = \|T\|_{B(H)} < 1$. By applying Lemma 3.2.1, $|\lambda| < 1$.

Step 2: We regard $T_c = T$ and $I = I_H = I_{H_c}$. Here, we consider everything sits in $B(H_c)$. Since T is strictly contractive, by the second lemma, $(I+T)(I-T)^{-1}$ is strictly accretive. Set $\alpha = (1+\lambda)/(1-\lambda)$. Then $\alpha > 0$ and

$$\theta = \frac{1}{\alpha}(I+T)(I-T)^{-1} = I + \frac{1}{\alpha}\Big((I+T)(I-T)^{-1} - (I+\lambda)(I-\lambda)^{-1}\Big)$$

is also strictly accretive. Note that $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$. Since A^1 is norm closed, $(I-T)^{-1} \in A_c^1$. In addition, $(I+T)(I-T)^{-1} = (I-T)^{-1}(I+T)$, and we can write

$$\begin{aligned}(I+T)(I-T)^{-1} - (I+\lambda)(I-\lambda)^{-1} &= (I-T)^{-1}\Big((I+T)(I-\lambda) - (I-T)(I+\lambda)\Big)(I-\lambda)^{-1} \\ &= 2(I-T)^{-1}a(I-\lambda)^{-1} \\ &= \frac{2}{1-\lambda}(I-T)^{-1}a.\end{aligned}$$

Since A_c is an ideal of A_c^1, $\theta - I = \alpha^{-1}(I+T)(I-T)^{-1} - (I+\lambda)(I-\lambda)^{-1} \in A_c = A + iA$. Since $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k \in A + i\{0\}$, we have $\theta - I \in A + i\{0\}$. Also $(\theta+I)^{-1} \in A_c^1$. We may assume that $(\theta+I)^{-1} = S_1 + iS_2$. Then we must have that $S_2 = 0$. Therefore, $(\theta+I)^{-1} \in A^1 + i\{0\}$. Since A_c is a ideal of A_c^1, $(\theta - I)(\theta+I)^{-1} \in A_c$. Also, $(\theta-I)(\theta+I)^{-1} \in A + i\{0\}$.

Step 3: Since π_c^o is a unital homomorphism and $\theta + I$ is invertible, $\pi_c^o(\theta + I) = \pi_c^o(\theta) + I$ is invertible and $\pi_c((\theta + I)^{-1}) = (\pi_c^o(\theta) + I))^{-1}$. Thus,

$$\pi_c^o((\theta - I)(\theta + I)^{-1}) = (\pi_c^0(\theta) - I)(\pi_c^o(\theta) + I)^{-1}.$$

We know that θ is strictly accretive, thus $(\theta - I)(\theta + I)^{-1}$ is contractive and is an element in $A + i\{0\} \subseteq A_c$. Since $\pi_c^o|_A = \pi$, $\pi_c^\circ((\theta - I)(\theta + I)^{-1}) \in B(K) + i\{0\}$. By the complexification property,

$$\|(\pi_c^0(\theta) - I)(\pi_c^o(\theta) + I)^{-1}\|_{B(K_c)} = \|\pi_c^\circ((\theta - I)(\theta + I)^{-1})\|_{B(K_c)} = \|\pi^\circ((\theta - I)(\theta + I)^{-1})\|_{B(K)} < 1.$$

Thus, $\pi_c^o(\theta)$ is strictly accretive in $B(K_c)$. Thus,

$$\alpha\pi_c^0(\theta) = \pi_c^o((I + T)(I - T)^{-1}) = (I + \pi_c^o(T))(I - \pi_c^o(T))^{-1}$$

is also strictly accretive. Therefore $\pi_c^o(T)$ is strictly contractive, i.e., $\|\pi_c^\circ(T)\|_{B(K_c)} = \|\pi^\circ(T)\|_{B(K)} < 1$. □

By the above real version of Meyer's Theorem, we can show that a unitization of a real operator algebra is unique.

3.2.4 Theorem (Meyer Theorem for Real Operator Algebra)**.** *Let $A \subseteq B(H)$ be a nonunital real operator algebra where H be a real Hilbert space. Let $\pi : A \to B(K)$ be an isometric homomorphism for a real Hilbert space K. Then the unital homomorphism $\pi^\circ : A^1 \to B(K)$ where $\pi^\circ(a + \lambda I_H) = \pi(a) + \lambda I_K$ is an isometry.*

Proof. Since A is nonunital, $I_H \notin A$. Suppose that there is $a \in A$ such that $\pi(a) = I_K$. However a is not an identity. Thus, there is $b \in A$ such that $ab \neq b$. Thus, $\pi(ab) = \pi(a)\pi(b) = \pi(b)$. This contradicts the fact that π is an isometry. Therefore, $I_K \notin \pi(A)$. Also, π° is injective. We can regard $\pi : A \to \pi(A)$. Then both π and π^{-1} are isometric. Thus, π° and $(\pi^\circ)^{-1}$ are contractive by

Meyer's theorem. Thus, π° is an isometry. $\square$

3.3 Real Positivity on Real Operator Algebras

In [2], the authors study real positivity on complex operator algebras. We examine the case of real positivity on real operator algebras.

Let A be a real operator subalgebras of a real C^*-algebras B. We say $a \in A$ is positive if a is positive in B. We say that a is real positive if a is real positive in B. If A is unital, by the fact of unital operator spaces, positivity and real positivity of real operator algebra are independent of a choice of a representation. If A is nonunital, by the uniqueness of the unitization of A, positivity and real positivity of real operator algebra are independent of a choice of a representation. Therefore, positivity and real positivity of real operator algebra are independent of a choice of representation. Therefore, we may define

$$\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}.$$

Let $T : A \to B$ be a completely bounded map. Then T is called positive if $T(a) \geq 0$ for all $a \geq 0$ and T is call real positive if $T(a) \in \mathfrak{r}_B$ for all $a \in \mathfrak{r}_A$. We call T completely positive if T_n is positive for every $n \in \mathbb{N}$ and we call T completely real positive if T_n is real positive for every $n \in \mathbb{N}$.

It might not be true that $T_c : A_c \to B_c$, the complex extension of a positive or real positive map $T : A \to B$, is positive or real positive respectively. However if T is completely positive or completely real positive, T_c is completely positive and completely real positive respectively.

3.3.1 Lemma. *Let A be a real unital C^*-algebra and A_c be its C^*-algebra complexification. Let $a, b \in A$. Then following hold.*

(i) $a + ib$ is invertible in A_c if and only if $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible in $M_2(A)$.

(ii) Assume that $a+ib$ is selfadjoint. Then $\lambda \in \sigma_{A_c}(a+ib)$ if and only if $\lambda \in \sigma_{M_2(A)}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$.

(iii) $a+ib$ *is positive in* A_c *if and only if* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *is positive in* $M_2(A)$.

(iv) $a+ib$ *is real positive in* A_c *if and only if* $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *is real positive in* $M_2(A)$.

Proof. (i) Let $x+iy$ be the inverse of $a+ib$. Then it is easy to see that $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Conversely, assume that $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

From the above equation, we have $ay - bw = 0 = -xb + ya$ and so $yay - ybw = 0 = -xby + yay$. Thus $ybw = xby$. In addition, we obtain from above equation that $by + aw = 1$ and $xa + yb = 1$. Hence $xby + xaw = x$ and $xaw + ybw = w$. Since $xby = ybw$, $x = ybw + xaw = w$. Replacing $w = x$, we have

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x & y \\ z & x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & x \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

From the above equation, we have $ay-bx = 0 = -xb+ya$. Then $xay-xbx = 0 = -xbx+yax$ and so $xay = yax$. We also have $ax - bz = 1 = -zb + xa$. So $yax - ybz = y = -zby + xay$. Combining these equations, we obtain $ybz = zby$. Therefore $y = -ybz + xay$. Again from the above equation, $by + ax = 1$ and so $zby + zax = z$. Moreover $za + xb = 0$ and so $zax + xbx = 0$. Thus $zax = -xbx$. As a consequence, $z = zby + zax = zby - xbx$. In addition, $ay - bx = 0$ and so $ay = bx$. Therefore, $z = zby - xay$. Since $ybz = zby$, $z = ybz - xay = -(-ybz + xay) = -y$. Thus, we have that $x = w$ and $y = -z$. Therefore, $x - iy$ is the inverse of $a + ib$.

(ii) It is easy to see that $a + ib$ is selfadjoint if and only if $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is selfadjoint. Thus,

$\sigma_{A_c}(a+ib) \subseteq \mathbb{R}$ and $\sigma_{M_2(A)}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) \subseteq \mathbb{R}$. Assume that $\lambda \in \mathbb{R} \setminus \sigma_{A_c}(a+ib)$. Then $(a-\lambda)+ib$ has an inverse $x+iy$. Then the inverse of $\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}$ is $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. Conversely if $\lambda \in \mathbb{R} \setminus \sigma_{M_2(A)}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$. Then $\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}$ has an inverse. By (i), its inverse is of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. Thus $x+iy$ is the inverse of $(a-\lambda)+ib$.

(iii) Since $a+ib$ is positive, it is selfadjoint. By (ii), $\sigma_{A_c}(a+ib) = \sigma_{M_2(A)}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Therefore, we have that $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is positive in $M_2(A)$. The converse is similar.

(iv) From (iii), we have that $(a+ib)+(a+ib)^* \geq 0$ if and only if $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^*$ is positive. $\square$

The statements (iii) and (iv) in the previous lemma also hold when A is a real unital operator space or a real operator algebra.

3.3.2 Lemma. *Let A be a real unital operator space or real operator algebra and A_c be its complexification. Let $a, b \in A$. Then following hold.*

(i) $a+ib$ is positive in A_c if and only if $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is positive in $M_2(A)$.

(ii) $a+ib$ is real positive in A_c if and only if $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is real positive in $M_2(A)$.

Proof. Since $x+iy$ is positive (resp. $x+iy$ is real positive) in a A_c if $x+iy$ is positive (resp. $x+iy$ is real positive) in the complexification of a real C^*-algebra containing A. Then apply the above lemma. $\square$

3.3.3 Proposition. *Let A and B be real operator algebras and $T : A \to B$ be completely positive (completely real positive). Then $T_c : A_c \to B_c$ is completely positive (completely real positive).*

Proof. Let $[a_{jk} + ib_{jk}]$ be positive in $M_n(A_c)$. By the previous lemma, $\begin{bmatrix} [a_{jk}] & -[b_{jk}] \\ [b_{jk}] & [a_{jk}] \end{bmatrix}$ is positive in $M_{2n}(A)$. Since T is completely positive (completely real positive), $T_{2n}\left(\begin{bmatrix} [a_{jk}] & -[b_{jk}] \\ [b_{jk}] & [a_{jk}] \end{bmatrix}\right) = \begin{bmatrix} T([a_{jk}]) & -T([b_{jk}]) \\ T([b_{jk}]) & T([a_{jk}]) \end{bmatrix}$ is positive (real positive). That is $T([a_{jk}]) + iT([b_{jk}])$ is positive (real positive) in $M_n(B_c)$. □

We may obtain analogous results of from [2] in the real case.

3.3.4 Proposition (Real case of Theorem 2.4 in [2])**.** *If $T : A \to B$ is a linear map between real C^*-algebras, then T is completely positive if and only if T is completely real positive.*

Proof. This follows imedialy from Theorem 2.4 in [2] and the above proposition. □

3.3.5 Proposition (Real case of Theorem 2.5 in [2])**.** *If $T : A \to B(H)$ is linear completely real positive on a unital real operator space A, then the canonical extension $\tilde{T} : A \to A^* \to B(H) : x + y^* \mapsto T(x) + T(y)^*$ is well defined and completely positive.*

Proof. This follows immediately from Theorem 2.5 in [2] and Proposition 3.3.3. □

3.4 Universal Algebras of Operator Algebras

Let A be a real operator algebra. A *C^*-cover* of A is a pair (B, j) where B is a C^*-algebra and $j : A \to B$ is a (completely) isometric homomorphism such that $j(A)$ generates B as a C^*-algebra. That is $C^*_B(j(A)) = B$.

We obtain a universal C^*-algebra, $C^*_{max}(A)$, of a real operator algebra A by the same method as for a complex operator algebra (see Proposition 2.4.2 in [3]), which has the universal property as follows.

3.4.1 Theorem. *Let A be a real operator algebra. Then there exists a C^*-cover $(C^*_{max}(A), j)$ of A with the universal property: if $\pi : A \to D$ is any contractive homomorphism into a real C^*-algebra D, then there exists a $*$-homomorphism $\tilde{\pi} : C^*_{max}(A) \to D$ such that $\tilde{\pi} \circ j = \pi$.*

The following fact shows that the complex norm of A_c obtained from $(C^*_{max}(A), j)$ is the biggest norm among all complex norms of A_c obtained from real operator algebras containing A.

3.4.2 Proposition. *Let A be a real Banach algebra. Suppose that $\pi : A \to B(H)$ is isometric homomorphism from A to $B(H)$ where H is a real Hilbert space. We have*

$$\|\pi(a) + i\pi(b)\|_{B(H_c)} \leq \|j(a) + i(j(b))\|_{C^*max(A)},$$

where $\|\cdot\|_{B(H_c)}$ and $\|\cdot\|_{C^*_{max(A)}}$ are the reasonable norms of A_c that are obtained by the complexification of $B(H)$ and $C^*_{max}(A)$ respectively.

Proof. By a universal property of $C^*_{max}(A)$, there is a $*$-homomorphism $\tilde{\pi} : C^*_{max}(A) \to B(H)$ such that $\tilde{\pi} \circ j = \pi$. Then $\tilde{\pi}_c : C^*_{max}(A)_c \to B(H_c)$ is also $*$-homomorphism thus contractive. Hence,

$$\|\pi(a) + i\pi(b)\|_{B(H_c)} = \|\tilde{\pi}_c(j(a) + ij(b))\|_{B(H_c)} \leq \|j(a) + ij(b)\|_{C^*_{max}(A)}.$$

$\square$

3.4.3 Proposition. *Let $(C^*_{max}(A), j)$ be the universal real C^*-algebra of a real operator algebra A. Let A_c be obtained the norm from $(C^*_{max}(A))_c$. Then $(C^*_{max}(A)_c, j_c)$ is the universal complex C^*-algebra of A_c.*

Proof. Let $\rho : A_c \to D$ be a contractive homomorphism from A_c to a complex C^*-algebra D. By considering D as a real C^*-algebra, $\pi = \rho|_A$ is a contractive homomorphism from A to D. By the universal property, there exists $\tilde{\pi} : C^*_{max}(A) \to D$ such that $\tilde{\pi} \circ j = \pi$. Then define $\tilde{\pi}_c : C^*_{max}(A)_c \to D$ by $\tilde{\pi}_c(a + ib) = \tilde{\pi}(a) + i\tilde{\pi}(b)$. It is simple to check that $\tilde{\pi}_c$ is a *-homomorphism and $\tilde{\pi}_c \circ j_c = \rho$. $\square$

Chapter 4

Real Jordan Operator Algebras

4.1 Definitions

Let X be an algebra over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). The *Jordan product* on X is defined as

$$a \circ b = \frac{1}{2}(ab + ba)$$

for $a, b \in X$. A subspace A of X which is closed under Jordan product, i.e., $a \circ b \in A$ for all $a, b \in A$ (or equivalently $a^2 \in A$ for all $a \in A$), is called a *Jordan subalgebra* of X. If A and B are Jordan subalgebras (of algebras X and Y), a linear map $\pi : A \to B$ which satisfies $\pi(a \circ b) = \pi(a) \circ \pi(b)$ is called a *Jordan homomorphism.* It is simple fact that a linear map between Jordan subalgebras is a Jordan homomorphism if and only if $\pi(a^2) = \pi(a)^2$ for all $a \in A$. Obviously, a homomorphism is a Jordan homomorphism. But a Jordan homomorphism might not be a homomorphism for example the transpose map on $B(\mathbb{R}^n)$ or $B(\mathbb{C}^n)$.

Complex Jordan operator algebras have been studied in [7]. In this chapter, we investigate Jordan operator algebras over the real field, or real Jordan operator algebras.

A concrete real Jordan operator algebra is a real Banach space A with a bilinear map $\circ : A \times A \to A$ which is commutative, i.e., $a \circ b = b \circ a$ (does not have to be associative), together with an isometric Jordan homomorphism $\pi : A \to B$ (i.e., $\pi(a \circ b) = \pi(a) \circ \pi(b) = (\pi(a)\pi(b) + \pi(b)\pi(a))/2$), where B

is a real C^*-algebra or $B = B(H)$ for a real Hilbert space H. The bilinear map $\circ$ is called a *Jordan product*. A concrete real Jordan operator algebra can be realized as a real Jordan subalgebra of a real C^*-algebra. An *abstract real Jordan operator algebra* is a real operator space A with a bilinear map $\circ : A \times A \to A$ (does not have to be associative) on A and there exists a completely isometric Jordan homomorphism $\pi : A \to B(H)$ (i.e., $\pi(a \circ b) = \pi(a) \circ \pi(b) = (\pi(a)\pi(b) + \pi(b)\pi(a))/2$) where H is a real Hilbert space.

4.1.1 Example. Consider $\mathbb{R}^2$ with the Euclidean norm. Define $(a, b) \circ (c, d) = (ac, (ad + bc)/2)$ on $\mathbb{R}^2$ and $\pi : \mathbb{R}^2 \to B(\mathbb{R}^2)$ by $\pi(a, b) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$. We can check that A with the map π is a concrete real Jordan operator algebra.

If A is a real Jordan operator algebra, define the Jordan product on A_c in a natural way as

$$(a + ib) \circ (c + id) = (a \circ c - b \circ d) + i(a \circ d + b \circ d).$$

Then the operator space complexification A_c of A with this product is a complex Jordan operator algebra.

4.1.2 Proposition. *Let A be a real Jordan operator algebra. Then the operator space complexification A_c of A is a complex Jordan operator algebra.*

Proof. By the definition, there is a completely isometric Jordan homomomorphism $\pi : A \to B(H)$. Then $\pi_c : A_c \to B(H_c)$ is completely isometric. Consider

$$\begin{aligned}
\pi_c\big((a + ib) \circ (c + id)\big) &= \pi_c\big((a \circ c - b \circ d) + i(a \circ d + b \circ d)\big) \\
&= \pi(a \circ c) - \pi(b \circ d) + i\pi(a \circ d + i\pi(b \circ d) \\
&= \pi(a) \circ \pi(c) - \pi(b) \circ \pi(d) + i(\pi(a) \circ \pi(d) + \pi(b) \circ \pi(d) \\
&= \pi(a + ib) \circ \pi(c + id).
\end{aligned}$$

Thus π_c is a Jordan homomorphism. □

Note that the complexification of a real Jordan operator algebra is unique up to complete isometry since it is a real operator space and the operator space complexification is unique up to complete isometry.

Generally, Jordan operator algebras are different from operator algebras, but the Jordan operator algebra generated by a singleton is a commutative operator algebra.

4.1.3 Proposition. *Let A be a Jordan operator subalgebra of a C^*-algebra B and $a \in A$. Then the Jordan operator subalgebra generated by a, denoted by $joa_B(a)$, is the operator subalgebra generated by a, denoted by $oa_B(a)$. In addition, if $\pi : A \to B(H)$ is a Jordan homomorphism. Then the restriction $\pi : oa_B(x) \to B(H)$ is a homomorphism.*

Proof. We can see that a^n is in both $joa_B(a)$ and $oa_B(a)$ for all $n \in \mathbb{N}$. And both $joa_B(a)$ and $oa(a)$ are the closure of the span of a^n. Thus, $joa_B(a) = oa_B(a)$. Since $\pi : A \to B(H)$ is a Jordan homomorphism, $\pi(a^n) = \pi(a)^n$. Therefore, the restriction of π to $oa_B(a)$ is a homomorphism. □

For a real Jordan operator subalgebra A of a real C^*-algebra B, define

$$\Delta(A) = \{a \in A : a^* \in A\}.$$

Then $\Delta(A)$ is a real JC^*-subalgebra of B. As the complex case, $\Delta(A)$ is independent of a real C^*-algebra container.

4.1.4 Proposition. *Let A be a real Jordan operator subalgebra of a real C^*-algebra B. If $\pi : A \to B'$ is a contractive Jordan homomorphism from A to a real C^*-algebra B', $\pi|_{\Delta(A)} : \Delta(A) \to \Delta(\pi(A))$ is a Jordan $*$-homomorphism. In addition, if π is isometric, $\pi|_{\Delta(A)}$ is an isometric Jordan $*$-homomorphism between $\Delta(A)$ and $\Delta(\pi(A))$.*

Proof. Obviously $\pi|_{\Delta(A)} : \Delta(A) \to B'$ is a Jordan homomorphism (we consider a real C^*-algebra B' as a JC^*-algebra). We know that a contractive Jordan homomorphism between JC^*-algebras is a Jordan $*$-homomorphism (Lemma 2.1.32). Thus, $\pi|_{\Delta(A)}$ is a Jordan $*$-homomorphism. Therefore if $a \in \Delta(A)$, $\pi(a^*) = \pi(a)^* \in \pi(A)$. Thus, $\pi(\Delta(A)) \subseteq \Delta(\pi(A))$. Hence, $\pi|_{\Delta(A)} : \Delta(A) \to \Delta(\pi(A))$

is a Jordan $*$- homomorphism. Therefore, if π is isometric, then $\pi|_{\Delta(A)}$ is an isometric Jordan $*$-homomomorphism between $\Delta(A)$ and $\Delta(\pi(A))$. $\square$

Let A be a real Jordan operator algebra and $p \in A$. If $p \circ p = p$ and $\|p\| = 1$, we call p a *projection* in A. We call a projection p a *central projection* in A (with respect to B) if $pxp = p \circ x$ for all $x \in A$. We will see from the following lemma that if A is a Jordan subalgebra of a real C^*-algebra B, then p is also a usual projection in a real C^*-algebra B. Also, the central projection is independent of the choice of a real C^*-algebra container.

4.1.5 Proposition. *If A is a real Jordan subalgebra of a real C^*-algebra B and p is a projection in A, then p is a projection in B ($p^2 = p = p^*$) and $p \in \Delta(A)$. Also, p is a central projection of A if and only if $p \circ x = pxp = px = xp = pxp$. In addition, the definition of central projection is independent of a choice of real C^*-algebra container.*

Proof. Since $p \circ p = (pp + pp)/2 = p^2$ in B and $\|p\|_B = \|p\| = 1$, by Lemma 2.1.12, p is a projection in B. Thus, $p = p^* = p^2$. Hence, $p \in \Delta(A)$. The fact that p is a central projection if and only if $p \circ p = pxp = px = xp$ is simple. Next, we show that the definition of central projection is independent of the choice of C^*-algebra container of A. Let $\pi : A \to B'$ be an isometric Jordan homomorphism between A and a real C*-algebra B', and p be a central projection on A with respect to B. It is obvious that $\pi(p)$ is a projection in B'. Let $x \in \pi(A)$. We need to show that $\pi(p)$ is a central projection of $\pi(A)$, i.e., $\pi(p) \circ \pi(x) = \pi(p)\pi(x)\pi(p)$. Since $p \circ x = pxp = px = xp$, we have that $\pi(p \circ x) = \pi(pxp) = \pi(px) = \pi(xp)$. By the property of a Jordan homomorphism, $(\pi(p)\pi(x) + \pi(x)\pi(p))/2 = \pi(p) \circ \pi(x) = \pi(p \circ x) = \pi(px)$. Also, $\pi(p) \circ \pi(px) = \pi(p \circ px) = \pi(px)$. Therefore, $\pi(p) \circ (\pi(p)\pi(x) + \pi(x)\pi(p))/2 = \pi(p) \circ (px)$. Thus, $(\pi(p)\pi(x)\pi(p) + \pi(p) \circ \pi(x))/2 = \pi(px) = \pi(p) \circ \pi(x)$. This implies $\pi(p)\pi(x)\pi(p) = \pi(p) \circ \pi(x)$. $\square$

By the above proposition, a projection in a Jordan operator algebra is a projection in its C^*-algebra container. Moreover, a central projection in a Jordan operator algebra is a central projection in its C^*-algebra container. The proof of the following lemma using this fact and will be identical with its corresponding fact in the case when A is a real C^*-algebra in Lemma 2.1.14 (and also the corresponding complex case).

4.1.6 Lemma. *Let A be a unital real Jordan operator algebra and p be a central projection of A. Then $1-p$ is a central projection and for $a \in A$,*

$$\|a\| = \max\{\|pa\|, \|a - pa\|\}.$$

Proof. Since a central projection p in a real Jordan operator algebra is a central projection in its C^*-algebra container, this will follow from Lemma 2.1.14. □

4.2 Bidual of Jordan Operator Algebras

Let A be a real Jordan operator subalgebra of a real C^*-algebra B. We know that B^{**} is a real C^*-algebra with the Arens product which extends the original product of B (see Theorem 5.5.3 in [18]). The Jordan Arens product is described in the complex case in [7] toward the end of Chapter 1. Now, we describe the Jordan Arens product on A^{**} in the real case.

An element $\eta \in A^{**}$ is a weak* limit of a net (a_s) in A by Goldstine's Theorem, and for any $\varphi \in A^*$,

$$\eta(\varphi) = \lim_s \varphi(a_s).$$

Therefore, for $\eta, \nu \in A^{**}$, there exist nets (a_s) and (b_t) such that a_s weak* converges to η and b_t weak* converges to ν. Then the Jordan Arens product on A^{**} can be defined as

$$\eta \circ \nu(\varphi) = \lim_s \lim_t \varphi(a_s \circ b_t) = \Big(\lim_s \lim_t \varphi(a_s b_t + b_t a_s)\Big)/2.$$

for a functional $\varphi \in A^{**}$. In addition, when we consider η, ν as elements of B^{**}, their product on B^{**} is described as

$$\eta\nu(\psi) = \lim_s \lim_t \psi(a_s b_t)$$

where $\psi \in B^{**}$. We know that B is Arens regular (Theorem 5.5.4 in [18]). Therefore, by the weak* continuity of addition and the property of Arens regularity of B (Theorem 2.1 in [15]), the Jordan product on B^{**} can be described as

$$\eta \circ \nu(\psi) = \big(\lim_s \lim_t \psi(a_s b_t) + \lim_s \lim_t \psi(b_t a_s)\big)/2 = \lim_s \lim_t \psi(a_s \circ b_t).$$

4.2.1 Lemma. *Let A be a real Jordan operator algebra and A^{**} be the bidual of A. For $\nu \in A^{**}$, the map $\eta \mapsto \eta \circ \nu$ is weak* continuous, i.e., if (η_t) is a net in A^{**} weak* converges to η, then $\eta_t \circ \nu$ weak* converges to $\eta \circ \nu$.*

Proof. We assume that A is a real Jordan subalgebra of a real C^*-algebra B and thus A^{**} is a real Jordan subalgebra of B^{**}. Then $\eta_t \circ \nu = (\eta_t \nu + \nu \eta_t)/2$. Since addition is weak* continuous and the Arens product on B^{**} is separately weak* continuous, we have $\eta_t \circ \nu = (\eta_t \nu + \nu \eta_t)/2$ weak* converges to $(\eta\nu + \nu\eta)/2 = \eta \circ \nu$. □

4.2.2 Proposition. *Let $\pi : A \to M$ be a contractive Jordan homomorphism between a real Jordan operator algebra A and a weak* closed real Jordan operator algebra M. Then π extends uniquely to a weak* continuous contractive Jordan homomorphism $\tilde{\pi} : A^{**} \to M$.*

Proof. We have that $\pi_c : A_c \to M_c$ is a completely contractive complex Jordan homomorphism between complex A_c and M_c. Thus π_c is contractive by the fact in the complex case. Therefore, π_c extends uniquely to a weak* continuous contractive Jordan homomorphism $\tilde{\pi}_c : A_c^{**} \to M_c$. We claim that $\tilde{\pi}_c|(A^{**})$ is a subset of M. Let (x_t) be a net in A which weak* converges to $\eta \in A^{**}$. Then $\tilde{\pi}_c(x_t) = \pi_c(x_t) = \pi(x_t) \in A$ weak* converges to $\pi(\eta)$. Since $(\pi(x_t))$ is a net in M, its weak* limit must be in M. Therefore, $\tilde{\pi}_c|_{A^{**}} : A^{**} \to M$. This completes the proof. □

4.3 A Characterization of Unital Jordan Operator Algebras

Let X, Y and Z be operator spaces. A bilinear map $T : X \times Y \to Z$ is bounded if $\|T(x,y)\| \leq C\|x\|\|y\|$ for all $(x,y) \in X \times Y$. Define

$$\|T\| = \inf\{C : \|T(x,y)\| \leq C\|x\|\|y\| \text{ for all } (x,y) \in X \times Y\}.$$

Then $T_n : M_n(X) \times M_n(Y) \to M_n(Z)$ is also a bilinear map. Define

$$\|T\|_{cb} = \sup\{\|T_n\| : n \in \mathbb{N}\}.$$

If $\|T\|_{cb} \leq 1$, then T is called completely contractive in the sense of Christensen and Sinclair. If $m : A \times A \to B$ be a bilinear map. Then

$$\begin{aligned} m_c(a+ib, c+id) &= m(a,c) - m(b,d) + i(m(a,d) + m(b,c)) \\ &= \begin{bmatrix} m(a,c) - m(b,d) & m(a,d) + m(b,c) \\ -(m(a,d) + m(b,c)) & m(a,c) - m(b,d) \end{bmatrix} \\ &= m\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \right) \\ &= m_2(a+ib, c+id). \end{aligned}$$

Thus, we can see that $m_2|_{A_c} = m_c$. We obtain the following lemma.

4.3.1 Lemma. *Let A be a real unital operator space with a bilinear map $m : A \times A \to B$ where B is a unital operator space containing A as a unital-subspace completely isometrically ($1_A = 1_B \in A$). If m is completely contractive in the sense of Christensen and Sinclair, then $m_c : A_c \times A_c \to B_c$ is completely contractive in the sense of Christensen and Sinclair.*

Proof. Since $A_c = \{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in M_2(A)\} \subseteq M_2(A)$ and $B_c = \{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in M_2(B)\} \subseteq M_2(B)$. Then

$m_c = m_2|_{A_c \times A_c}$ where $m_2 : M_2(A) \times M_2(A) \to M_2(B)$ is the amplification of m. Since m_2 is also real completely contractive in the sense of Christensen and Sinclair, $\|m_c\|_{cb} \leq 1$. □

4.3.2 Theorem (Real case of Theorem 2.1 in [7])**.** *Let A be a unital operator space (resp. operator system) with a bilinear map $m : A \times A \to B$ which is completely contractive in the sense of Christensen and Sinclair. Here B is a unital operator space containing A as a unital subspace completely isometrically. Define $a \circ b = \frac{1}{2}(m(a,b) + m(b,a))$ and suppose that A is closed under this operation. Assume also that $m(1,a) = m(a,1) = a$ for all $a \in A$. Then A is a unital Jordan operator algebra (resp. JC^*-algebra) with Jordan product $a \circ b$.*

Proof. By the above lemma, $m_c : A_c \times A_c \to B_c$ is complex completely contractive in the sense of Christensen and Sinclair. Define

$$(a + ib) \circ (c + id) = \frac{1}{2}(m(a + ib, c + id) + m(c + id, a + ib)).$$

We can easily see that this extends $a \circ b$ on A. By Theorem 2.1 in [7], A_c is a unital Jordan operator algebra with Jordan product $(a+ib) \circ (c+id)$. Since A is closed under Jordan product $\circ$ and $1 \in A$, A is a real unital Jordan operator algebra. □

A real or complex *approximately unital operator space* is a real or complex operator subspace A of a real or complex C^*-algebra B which contains a net (e_t) in A with the property that $e_t a \to a$ for all $a \in A$. Such a net is called a *contractive approximate identity* or a *cai* of A with respect to B.

4.3.3 Lemma. *Let A be a real approximately unital operator space. Let $m : A \times A \to B$ be a bilinear map and (e_t) be a cai in B such that $m(e_t, a) \to a$ (resp. $m(a, e_t) \to a$) for all $a \in A$. Then $m_c(e_t, a + ib)$ (resp. $m_c(a + ib, e_t)$) $\to a + ib$ for all $a + ib \in A_c$.*

Proof. We have $m(e_t, a + ib) = m(e_t, a) + im(e_t, b) \to a + ib$. □

4.3.4 Theorem (Real case of Theorem 2.3 in [7])**.** *Let A be a real approximately unital operator space (resp. operator system) containing a cai (e_t) for an operator algebra B. Let $m : A \times A \to B$*

be a completely contractive bilinear map in the sense of Christensen and Sinclair. Define $a \circ b = \frac{1}{2}(m(a,b) + m(b,a))$ *and suppose that* A *is closed under this operation. Assume also that* $m(e_t, a) \to a$ *and* $m(a, e_t) \to a$ *for all* $a \in A$*. Then* A *is a real Jordan operator algebra (resp real* JC^**-algebra) with Jordan product* $a \circ b$ *and* $e_t \circ a \to a$ *for all* $a \in A$*.*

Proof. By complexification, this follows from the lemma above and Theorem 2.3 in [7] in the complex case. □

4.4 Unitization of Real Jordan Operator Algebras

Recall the fact that a unitization of complex operator algebra is unique up to (completely) isometric homomorphism (Theorem 2.1.13 in [3]). For a real operator algebra, this corresponding fact is proved in [25] in the case of completely isometric homomorphisms. We prove the case of isometric homomorphism in 3.2.3. We now investigate this fact for a real Jordan operator algebra. Let A be a real Jordan operator subalgebra of $B(H)$ where H is a real Hilbert space and $I_H \notin A$. Define $A^1 = \{a + \lambda I_H : a \in A, \lambda \in \mathbb{R}\}$. Then

$$(a + \alpha I_H) \circ (b + \beta I_H) = \Big((a + \alpha I_H)(b + \beta I_H) + (b + \beta I_H)(a + \alpha I_H)\Big)/2$$
$$= \Big(ab + ba + 2\alpha b + 2\beta a + 2\alpha\beta I_H)\Big)/2 = a \circ b + \alpha b + \beta a + \alpha\beta I_H \in A^1.$$

Thus, A^1 is closed under Jordan product. We investigate if A^1 is unique up to isometric, that is the real Jordan operator algebra version of Proposition 2.4 in [7].

4.4.1 Proposition. *Let* $A \subseteq B(H)$ *be a real Jordan operator algebra and suppose that* $I_H \notin A$*. Let* $\pi : A \to B(K)$ *be a contractive (resp. isometric) Jordan homomorphism, where* K *is a real Hilbert space. We extend* π *to* $\pi^\circ : A^1 \to B(K)$ *by* $\pi^\circ(a + \lambda I_H) = \pi(a) + \lambda I_K$*. Then* π° *is a contractive (resp. isometric) Jordan homomorphism.*

Proof. First, we show that π° is a Jordan homomorphism. Let $a, b \in A$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
\pi^\circ((a+\alpha I_H)\circ(b+\beta I_H)) &= \pi(a\circ b+\alpha b+\beta a+\alpha\beta I_H) \\
&= \pi(a\circ b)+\alpha\pi(b)+\beta\pi(a)+\alpha\beta I_K \\
&= \pi(a)\circ\pi(b)+\alpha\pi(b)+\beta\pi(a)+\alpha\beta I_K \\
&= \big(\pi(a)\pi(b)+\pi(b)\pi(a)+2\alpha\pi(b)+2\beta\pi(a)+2\alpha\beta I_K\big)/2 \\
&= \big(\pi(a)+\alpha I_K)(\pi(b)+\beta I_K)+\pi(b)+\beta I_K)(\pi(a)+\alpha I_K)\big)/2 \\
&= \pi^\circ(a+\lambda I_H)\circ\pi^\circ(b+\beta I_H).
\end{aligned}$$

To see that π° is contractive, we follow the proof of Proposition 2.4 in [7]. Let $a \in A$. Then the restriction of π on $oa(a)$ is an algebra homomorphism into $oa(\pi(a))$. By applying Meyer's theorem for real operator algebra (see Theorem 3.2.4), we obtain that $\|\pi(a)+\lambda I_K\| \leq \|a+\lambda I_H\|$. □

By the above proposition, we obtain that a unitization of A is unique up to isometric Jordan homomorphism. We state the fact as the following corollary and this is a real Jordan operator space version of Corollary 2.5 in [7].

4.4.2 Corollary. *The unitization A^1 of a Jordan operator algebra is unique up to isometric Jordan isomorphism. In addition, $(A^1)_c = (A_c)^1$.*

Proof. We follow the proof of Corollary 2.5 in [7]. If A is nonunital, applying the above proposition. If A is unital and e is the identity of A, e is a central projection of A^1. By Lemma 4.1.6,

$$\|a+\lambda 1\| = \max\{\|e(a+\lambda 1\|, \|(1-e)(a+\lambda 1)\|\} = \max\{\|a+\lambda e\|, |\lambda|\}.$$

Since a unitization of complex Jordan operator algebra is also unique up to isometric Jordan homomorphism (see Corollary 2.5 [7]), $(A^1)_c = (A_c)^1$. □

4.4.3 Remark. A unitization of complex Jordan operator algebra is not unique up to completely isometric Jordan homomorphism as in Proposition 2.1 in [8]. The authors define

$$M_2 = \left\{ \begin{bmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subseteq B(\mathbb{C}^4),$$

and

$$F_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subseteq B(\mathbb{C}^6)$$

which are completely isometric. They show that M_2^2 and F_2^1 are not completely isometric. We adapt this example to obtain an example for a real operator space.

Let

$$M_2^{\mathbb{R}} = \left\{ \begin{bmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\} \subseteq B(\mathbb{R}^4),$$

and

$$F_2^{\mathbb{R}} = \left\{ \begin{bmatrix} 0 & 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\} \subseteq B(\mathbb{R}^6).$$

Then $M_2^{\mathbb{R}}$ and $F_2^{\mathbb{R}}$ are completely isometric and their operator space complexifications are M_2 and F_2 respectively. Let $(M_2^{\mathbb{R}})^1$ and $(F_2^{\mathbb{R}})^1$ be the unitizations of $M_2^{\mathbb{R}}$ and $F_2^{\mathbb{R}}$ respectively. Then $(M_2^{\mathbb{R}})_c^1$ is the unitization of M_2 and $F_2^{\mathbb{R}}$ is the unitization of F_2. Thus their operator space complexification of their unitizations are not completely isometric. If $(M_2^{\mathbb{R}})^1$ and $(F_2^{\mathbb{R}})^1$ are completely isometric, by Theorem 4.2 in [23], $(M_2^{\mathbb{R}})^1$ and $(F_2^{\mathbb{R}})^1$ are completely isometric. This contradicts the previous statement. Therefore, $(M_2^{\mathbb{R}})^1$ and $(F_2^{\mathbb{R}})^1$ are not completely isometric.

4.5 Universal Algebras of Real Jordan Operator Algebras

Let A be a real or complex Jordan subalgebra of a real C^*-algebra B or $B(H)$ where H is a real Hilbert space. A *C^*-cover* of A is a pair (B, j) where B is a real or complex C^*-algebra and $j : A \to B$ is a (completely) isometric Jordan homomorphism such that $j(A)$ generates B as a C^*-algebra. For a real Jordan operator algebra A, as in the complex case, there exists $(C^*_{max}(A), j)$, the maximal real C^* cover of A, with the universal property that for every (completely) contractive Jordan homomorphism $\pi : A \to B$ where B is a real C^*-algebra, there exists a unique $*$-homomorphism $\theta : C^*_{max}(A) \to B$ such that θ is a $*$-homomorphism and $\theta \circ j = \pi$. Note that we consider two types of C^*-cover. One when $j : A \to B$ is a completely isometric Jordan homomorphism and we simply denote this C^*-cover of A as $C^*_{max}(A)$. The other requires $j : A \to B$ to be an isometric Jordan homomorphism and we denote the maximal C^*-cover of A as $\tilde{C}^*_{max}(A)$ in this case.

In either case, we define $oa_{max}(A)$ to be the operator algebra generated by $j(A)$ in $C^*_{max}(A)$. Therefore, we have that $j(A) \subseteq oa_{max}(A) \subseteq C^*_{max}(A)$. The following facts are connections between $C^*_{max}(A)$ and $C^*_{max}(A_c)$.

4.5.1 Proposition. *Let A be a Jordan operator algebra. Then $(C^*_{max}(A)_c, j_c)$ is the maximal complex C^*-cover of A_c.*

Proof. By Theorem 3.1 in [23], the inherited complexification of A is unique and thus we can consider A_c with the norm from $C^*_{max}(A)_c$. Thus, $(C^*_{max}(A)_c, j_c)$ is a C^*-cover of A_c. Let $\pi : A_c \to B$ be a completely contractive Jordan homomorphism from A to a complex C^*-algebra

B. The restriction of $\pi|_A$ is a contractive real Jordan homomorphism from A to B where we consider B as a real C^*-algebra. By the universal property of $C^*_{max}(A)$, there is a $*$-homomorphism $\tilde{\pi}|_A : C^*_{max}(A) \to B$ such that $\tilde{\pi}|_A \circ j = \pi|_A$. Denote $\theta = \tilde{\pi}|_A$. Then $\theta_c : (C^*_{max}(A))_c \to B$ is a $*$-homomorphism and $\theta_c \circ j_c = \pi$. □

Note that if we consider this at the Banach space level, we must require A_c to obtain the complexification norm from $\tilde{C}^*_{max}(A)$ to ensure that j_c is an isometry. The following shows that this complexification norm of A_c is the maximal norm among all possible norms that makes A_c a complex Jordan operator algebra.

4.5.2 Proposition. *Let $\pi : A \to B$ be an isometric Jordan homomorphism where B is a real C^*-algebra. Let $\|\cdot\|_{B_c}$ and $\|\cdot\|_{max}$ be complexification norms of A_c obtained from B_c and $C^*_{max}(A)$ respectively. Then $\|a+ib\|_{B_c} \leq \|a+ib\|_{max}$.*

Proof. By the universal property of $C^*_{max}(A)$, there is a $*$-homomorphism $\tilde{\pi} : C^*_{max}(A) \to B$ which $\tilde{\pi} \circ j = \pi$. Then $\tilde{\pi}_c$ is also a $*$-homomorphism, thus contractive. Therefore, $\|\pi(a+ib)\|_{B_c} = \|\tilde{\pi}_c(j_c(a+ib))\|_{B_c} \leq \|j_c(a+ib)\|_{max}$. □

4.5.3 Proposition. *Let A_c obtain the complexification norm from $C^*_{max}(A)_c$. If $\pi : A \to B(H)$ is a contractive Jordan representation, then $\pi_c : A_c \to B(H_c)$ is a contractive Jordan representation.*

Proof. By the universal property of $C^*_{max}(A)$, there is a $*$-homomorphism $\tilde{\pi} : C^*_{max}(A) \to B(H)$ and $\tilde{\pi} \circ j = \pi$. Then $\tilde{\pi}_c : C^*_{max}(A)_c \to B(H_c)$ is a $*$-homomorphism and thus contractive. Also, $\pi_c = (\tilde{\pi} \circ j)_c = \tilde{\pi}_c \circ j_c$ which is a contractive Jordan homomorphism. □

4.5.4 Remark. Let X be a real unital operator space. A real C^*-extension of X and the real C^*-envelope of X are mentioned in Remark 2.5.3. Now, let A be a real unital Jordan operator algebra A with a completely isometric Jordan homomorphism $j : A \to B$ where B is a real C^*-algebra and let $(C^*(A), i)$ be the real C^*-envelope of A. We may assume that $j(X)$ generates B. Thus, (B, j) is a C^*-extension of A. By the universal property of C^*-envelope of A, there exists a $*$-homomorphism $\pi : B \to C^*_e(X)$ such that $\pi \circ j = i$. Since π is a $*$-homomorphism and j is a unital real Jordan homomorphism, we can see that $i = \pi \circ j$ is a unital Jordan homomorphism. And if A is a real unital JC^* operator algebra and j is a real unital Jordan $*$-homomorphism, i is also a unital real Jordan $*$-homomorphism. Therefore, we can consider X as a unital real Jordan operator subalgebra of $C_e^*(X)$.

4.6 Contractive Approximate Identities of Real Jordan Operator Algebras

Let A be a (real or complex) Jordan operator subalgebra of a (real or complex) C^*-algebra B and (e_t) be a net in $Ball(A)$. We define different types of contractive approximation identity for a real Jordan operator algebra.

(i) (e_t) is called a B-relative partial cai if $e_t a \to a$ and $a e_t \to a$ for all $a \in A$.

(ii) (e_t) is called a B-relative J-cai if $e_t \circ a \to a$ for all $a \in A$.

(iii) (e_t) is called a partial cai if $\pi(e_t)\pi(a) \to \pi(a)$ for every isometric Jordan homomorphism $\pi : A \to B'$ where B' is a (real or complex) C^*-algebra.

(iv) (e_t) is called a J-cai if $\pi(e_t) \circ \pi(a) \to \pi(a)$ for every isometric Jordan homomorphism $\pi : A \to B'$ where B' is a (real or complex) C^*-algebra.

Note that since π is an isometric Jordan homomorphism, $\pi(e_t) \circ \pi(a) - \pi(a) = \pi(e_t \circ a - a) \to \pi(0) = 0$ if (e_t) is a J-cai of A. Thus, (e_t) being a B-relative J-cai is equivalent to (e_t) being a J-cai. In the case of complex Jordan operator algebras, all of these are equivalent (see Theorem 2.6 in [7]). We will show, by using complexification of Jordan operator algebras and the fact from the complex case, that in the case of real Jordan operator algebras, all of these are equivalent as well.

4.6.1 Lemma. *Let A be a real Jordan subalgebra of a real C^*-algebra B. If (e_t) is a B-relative partial cai (respectively J-cai) of A, (e_t) is a B_c-relative partial cai (respectively J-cai) of A_c. In addition, if $(E_t) = (e_t + ie'_t)$ is a B_c-relative partial cai (respectively J-cai) of A_c, then (e_t) is a B-relative partial cai (respectively J-cai) of A.*

Proof. The first statement is obvious. Now, let $a \in A$ and $(E_t) = (e_t + ie'_t)$ is a B_c-relative partial cai (respectively J-cai). Then $E_t a = e_t a + ie'_t a \to a$ (respectively, $E_t \circ a = e_t \circ a + ie'_t \circ a \to a$). This implies that $e_t a \to a$ (respectively, $e_t \circ a \to a$). □

4.6.2 Lemma. *Let A be a real Jordan operator subalgebra of a real C^*-algebra B and A_c is its complexification which will be a complex Jordan operator subalgebra of a complex C^*-algebra B_c. The following facts hold.*

(i) A has a B-relative partial cai if and only if A_c has a B_c-relative partial cai.

(ii) A has a J-cai if and only if A_c has a J-cai.

(iii) A_c has a partial cai if and only if A has a partial cai.

Proof. (i) If (e_t) is a $B-$relative partial cai of A, then $E_t = e_t + i0$ is a B_c-relative partial cai of A_c. Conversely, if $(E_t) = (e_t + ie'_t)$ is a B_c-partial cai of A_c, then by the previous lemma (e_t) is a B-partial cai of A.

(ii) If A has a J-cai (e_t), then $(E_t) = (e_t + i0)$ is a J-cai for A_c. Conversely, by the previous lemma if A_c has a J-cai $(E_t) = (e_t + ie'_t)$, then (e_t) is a J-cai for A.

(iii) Let A_c have a partial cai and A be a Jordan subalgebra of a real C^*-algebra B. Then A_c is a Jordan subalgebra of a complex C^*-algebra B_c. Therefore, A_c has a B_c-relative partial cai $(E_t) = (e_t + ie'_t)$. By the previous lemma, (e_t) is a B-relative partial cai of A. Conversely, assume that A has a partial cai. Let $\pi : A_c \to B'$ be a complex Jordan isometric homomorphism from A_c to a complex C^*-algebra B'. Consider A as a Jordan subalgebra of $C^*_{max}(A)$. Then A has a $C^*_{max}(A)$-relative partial cai (e_t). Since $C^*_{max}(A)_c$ is the maximal C^* cover of A_c, by the universal property of $C^*_{max}(A)_c$, there is a $*$-homomorphism $\theta : C^*_{max}(A)_c \to B$ such that $\pi(a+ib) = \theta(a+ib)$ for $a + ib \in A_c$. Thus, $\pi(e_t a) = \theta(e_t a) \to \theta(a) = \pi(a)$, i.e., A_c has a partial cai. □

We obtain the real version of Lemma 2.6 in [7] by applying the previous lemma and Lemma 2.6.

4.6.3 Lemma (Real version of Lemma 2.6 in [7]). *If A is a Jordan operator subalgebra of a C^*-algebra B, then the following are equivalent.*

(i) A has a partial cai.

(ii) A has a $B-$relative partial cai.

(iii) A has a $J-$cai.

(iv) A^{**} has an identity p of norm 1 with respect to the Jordan Arens product on A^{**} which coincides on A^{**} with the restriction of the usual product in B^{**}. Indeed p is the identity of the real von Neumann subalgebra of B^{**} generated by A.

Proof. By the previous lemma, A has a partial cai if and only if A_c has a partial cai. By Lemma 2.6 in [7], this is equivalent to A_c having a B_c-relative partial cai and A_c having a J-cai. By the previous lemma again, A has a B-relative partial cai and a J-cai. Thus we have proved (i), (ii) and (iii) are equivalent. The proof of $(iii) \to (iv)$ is identical to complex case. Now, assume (iv). Then A_c has an identity p. By Lemma 2.6 (iv) in [7], A_c has a partial cai $(E_t) = (e_t + ie'_t)$. Therefore, (e_t) will be a partial cai of A. $\square$

Let A be a real Jordan operator algebra. Define

$$\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$$

and

$$\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}.$$

4.6.4 Theorem (Real case of Theorem 2.8 of [7]). *If A is an approximately unital real Jordan operator algebra then A is an M-ideal in A^1. Also, $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$ and $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. Finally, A has a partial cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. We follow the proof that A is an M-ideal as the complex case in Theorem 2.8 [7]. Since A is approximately unital, a cai (e_t) weak* converges to the identity e of A^{**}. Now e is a central projection in $(A^1)^{**}$, and the multiplication by e is an M-projection from $(A^1)^{**}$ to A^{**}. This shows A is an M-ideal of A^1. Now, we will use a complexification to obtain the rest. Let $(A_c, \|\cdot\|_c)$ be the operator space complexification of A. Then $\mathfrak{F}_{A_c}$ is weak* dense in $\mathfrak{F}_{A_c^{**}}$. Let $x \in \mathfrak{F}_{A^{**}}$. Then $\|1-x\| = \|1-x\|_c \leq 1$ and thus must be in $\mathfrak{F}_{A^{**}}$. By the density in the complex case (see Theorem 2.8 in [7]), there is a net $(a_t + ib_t)$ weak* converges to x which implies a_t weak* converges to x. Since $\|a_t - 1\| \leq \|a_t + ib_t - 1\|_c \leq 1$, $a_t \in \mathfrak{F}_A$. This shows that $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$.

Similarly, if $x \in \mathfrak{r}_{A^{**}}$, then $x \in \mathfrak{r}_{A_c^{**}}$ and there is a net $(a_t + ib_t)$ in $\mathfrak{r}_{A_c}$ weak* converging to x. Since $(a_t + ib_t) + (a_t + ib_t)^* \geq 0$, then $a_t + a_t^* \geq 0$ (see Lemma 2.1.9) i.e., $a_t \in \mathfrak{r}_A$. Moreover, a_t weak* converges to x.

Finally, by the corresponding fact in the complex case, A_c has a partial cai $\{E_t = e_t + ie'_t\}$ in $\frac{1}{2}\mathfrak{r}_{A_c}$. Thus, (e_t) is a partial cai in A. Since $\|1 - \frac{1}{2}e_t\| \leq \|1 - \frac{1}{2}(e_t + ie'_t)\| \leq 1$, $e_t \in \frac{1}{2}\mathfrak{r}_A$. □

4.6.5 Proposition. *Let A be an approximately unital real Jordan operator algebra. Then the set of contractions in $\mathfrak{r}_A$ is weak* dense in the set of contractions in $\mathfrak{r}_{A^{**}}$.*

Proof. Let $x \in \mathfrak{r}_{A^{**}}$ and $\|x\| \leq 1$. By the corresponding fact in the complex case (see Proposition 2.10 in[7]), there is a net $(X_t) = (a_t + ib_t)$ in $\mathfrak{r}_{A_c} \cap Ball(A_c)$ which weak* converges to x. Then (a_t) weak* converges to x. In addition, $\|a_t\| \leq 1$ and $a_t + a_t^* \geq 0$. Therefore, we have proved the statement. □

4.6.6 Proposition. *If A is a Jordan operator algebra with a countable Jordan cai, then A has a countable partial cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. If A has a countable Jordan cai, then it is a countable Jordan cai of A_c. Therefore, by the complex case (see Corollary 2.11 in [7]), there is a countable partial cai $(E_n) = (e_n + if_n)$ in $\frac{1}{2}\mathfrak{F}_{A_c}$. Since $e_n \in \frac{1}{2}\mathfrak{F}_A$ and is as well a partial cai, there is a countable partial cai in $\mathfrak{F}_A$. □

A unitization of a real or complex Jordan operator algebra may not be unique up to completely isometry Jordan isomorphism (see Proposition 2.1 in [8]). However, the unitization of an approximately Jordan algebra is unique up to completely isometric isomorphism.

4.6.7 Proposition (Real case of Proposition 2.12 in [7]). *If A is a nonunital approximately unital real Jordan operator algebra then the unitization A^1 is well defined up to completely isometric Jordan isomorphism and the matrix norms are*

$$\|[a_{ij}+\lambda_{ij}1]\| = \sup\left\{\left\| \begin{bmatrix} a_{ij}\circ x+\lambda x & -(a_{ij}\circ y+\lambda y) \\ -(a_{ij}\circ y+\lambda y) & a_{ij}\circ x+\lambda x \end{bmatrix} \right\|_{M_{2n}(A)} : \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \in Ball(M_2(A))\right\}.$$

Proof. Let A be a nonunital approximately unital real Jordan operator algebra. Then A_c be a nonunital approximately unital complex Jordan operator algebra. By Proposition 2.12 in [7], $(A_c)^1$ is unique up to completely isometric Jordan isomorphism. If A^1 is a unitization of A, then $(A^1)_c$ is a unitization of A_c. By the uniqueness, $(A^1)_c = (A_c)^1$. Therefore, the norm on A^1 can be given by the norm of $(A_c)^1$. Since the norm of $M_n(A_c)$ is obtained from $M_{2n}(A)$, by applying the formula in Proposition 2.12 in [7] and identifying a_{ij} as $\begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix} \in M_2(A)$, we obtain the formula above. □

Let A be a nonunital approximately unital real Jordan operator algebra. By the previous proposition, the unitization of a nonunital approximately unital real Jordan operator algebra is unique up to completely isometric Jordan homomorphism. Thus, we define the real C^*-envelope of A to be the real C^*-algebra generated by $j(A)$ where $(C^*_e(A^1), j)$ is the C^*-envelope of A^1. We denote the C^*-envelope of A as $C^*_e(A)$.

We also obtain the analogous of Proposition 2.14 in [7] for a nonunital approximately unital real Jordan operator algebra.

4.6.8 Proposition (Real case of Proposition 2.14 in [7]). *Let A be an approximately unital Jordan operator algebra. Let $(C^*_e(A), j)$ be the $C^* -$ envelope of A. Then $j|_A$ is a Jordan homomorphism onto a Jordan subalgebra of $C^*_e(A)$, and $C^*_e(A)$ has the following universal property: Given any C^*-cover (B, i) of A, there exists a (necessarily unique and surjective) $*$-homomorphism $\theta : B \to C^*_e(A)$ such that $\theta \circ i = j$.*

Proof. The proof is the same as the complex case. □

We investigate the weak operator topology and strong operator topology of $B(H)$ and $B(H_c)$.

4.6.9 Lemma. *Let H be a real Hilbert space and H_c be its complexification. Then*

(i) $T_t + iS_t \in B(H_c)$ WOT converges to $T + iS$ in $B(H_c)$ if and only if T_t and S_t WOT converge to S and T in $B(H)$ respectively, and

(ii) $T_t + iS_t \in B(H_c)$ SOT converges to $T + iS$ if and only if T_t and S_t SOT converge to S and T respectively.

Proof. (i) $T_t + iS_t$ WOT converge to $T + iS$ in $B(H_c)$ respectively. Let $x \in H$. Then $\langle (T_t + iS_t)x, y\rangle = \langle T_t x, y\rangle + i\langle S_t x, y\rangle \to \langle Tx, y\rangle + i\langle Sx, y\rangle$. This implies that $\langle T_t x, y\rangle \to \langle Tx, y\rangle$ and $\langle S_t x, y\rangle \to \langle Sx, y\rangle$. Conversely, assume that T_t and S_t WOT converge to S and T in $B(H)$ respectively. Let $x + iy, u + iv \in H_c$. Then $\langle (T_t + iS_t)(x + iy), u + iv\rangle = \langle T_t x, u\rangle - \langle S_t y, u\rangle - \langle T_t y, v\rangle - \langle S_t x, v\rangle + i(\langle T_t x, v\rangle + -\langle S_t y, v\rangle\langle T_t y, u\rangle\langle S_t x, u\rangle) \to 0$.

(ii) Let $T_t + iS_t \in B(H_c)$ WOT converges to $T + iS$ and $x \in H$. Then $\|(T_t + iS_t)x - (T + iS)x\| = \|T_t x - Tx + i(S_t x - Sx)\| \to 0$. Since $\|T_t x - Tx\| \leq \|T_t x - Tx + i(S_t x - Sx)\|$ and $\|T_t x - Tx\| \leq \|T_t x - Tx + i(S_t x - Sx)\|$, T_t and S_t SOT converge to S and T respectively. Conversely, let T_t and S_t SOT converge to S and T in $B(H)$ respectively. Then $\|(T_t + iS_t)(x + iy) - (T + iS)(x + iy)\| \leq \|T_t x - Tx\| + \|T_t y - Ty\| + \|S_t x - Sx\| + \|S_t y - Sy\| \to 0$. Therefore, $T_t + iS_t$ SOT converges to $T + iS$. $\square$

Now, by using the two lemmas above, we obtain analogous fact of Lemma 2.19 in [7].

4.6.10 Proposition (Real case of Proposition 2.19 in [7])**.** *Let A be an approximately unital real Jordan operator algebra and let $\pi : A \to B(H)$ be a contractive Hilbert space Jordan representation. We let P be the projection onto $K = [\pi(A)H]$. Then $\pi(e_t) \to P$ in the weak* (and WOT) topology of $B(H)$ for any J-cai (e_t) for A. Moreover, for $a \in A$, we have $\pi(a) = P\pi(a)P$, and the compression of π to K is a contractive Hilbert space Jordan representation. Also, if (e_t) is a partial cai for A, then $\pi(e_t)\pi(a) \to \pi(a)$ and $\pi(a)\pi(e_t) \to \pi(a)$. In particular, $\pi(e_t)|_K \to I_K$ SOT in B(K).*

Proof. Apply the lemma above with the complexification of A. $\square$

4.6.11 Lemma (Real case of Lemma 2.20 in [7]). *Let A be a real approximately unital Jordan operator algebra with a partial cai (e_t). Denote the identity of A^1 by 1. The following facts hold.*

(i) If $\psi : A^1 \to \mathbb{R}$ is a functional on A^1, then $\lim_t \psi(e_t) = \psi(1)$ if and only if $\|\psi\| = \|\psi|_A\|$.

(ii) Let $\varphi : A \to \mathbb{R}$ be any functional on A. Then φ uniquely extends to a functional on A^1 of the same norm.

Proof. (i) If $\psi : A^1 \to \mathbb{R}$ is a functional on A^1, then $\psi_c(a + ib + \alpha + i\beta) = \psi(a + \alpha) + i\psi(b + \beta)$ is a functional on $(A^1)_c = (A_c)^1$ (see Corollary 4.4.2) and $\|\psi\| = \|\psi_c\|$ by Proposition 2.1.2. By Lemma 2.20 in [7], $\lim_t \psi(e_t) = \lim_t \psi_c(e_t) = \psi_c(1) = \psi(1)$ if and only if $\|\psi_c\| = \|\psi\| = \|\psi|_A\| = \|(\psi|_A)_c\|$.

(ii) Assume that φ extends to φ^1 in A^1. Then φ_c extends uniquely to φ_c^1 on A_c^1 with the same norm by Lemma 2.20 [7]. If (e_t) is a cai in A then (e_t) is a cai in A_c and thus by (1), $\varphi_c(1) = \lim_t \varphi_c(e_t) \in \mathbb{R}$. Therefore, φ can extend to a functional φ^1 on A^1 with the same norm and $\varphi^1 = \varphi_c|_{A^1}$. For the uniqueness, if ψ is a functional on A^1 that extends φ of the same norm. Then ψ_c is a functional that extends φ_c on A^1 of the same norm. By the uniqueness in the complex case, $(\varphi^1)_c = \psi_c$ and thus $\varphi^1 = (\psi_c)_{A^1}$. $\square$

We also obtain the real case of Lemma 2.21 in [7] and its proof is identical to the complex case.

4.6.12 Lemma (Real case of Lemma 2.21 in [7]). *For a norm 1 functional φ on an approximately unital Jordan operator algebra A, the following are equivalent:*

1. *φ extends to a state on A^1.*
2. *$\varphi(e_t) \to 1$ for every partial cai $e_t \in A$.*
3. *$\varphi(e_t) \to 1$ for some partial cai for A.*
4. *$\varphi(e) = 1$ where e is the identity of A^{**}.*
5. *$\varphi(e_t) \to 1$ for every Jordan cai for A.*
6. *$\varphi(e_t) \to 1$ for some Jordan cai for A.*

4.7 Multiplier Algebras

In [7], the authors define the multiplier algebra of an approximately unital complex Jordan operator algebra. This procedure can be done with an approximately unital real Jordan operator algebra. Let A be an approximately unital real Jordan operator algebra and $(C_e^*(A), j)$ be its C^*-envelope. Using the product from $C_e^*(A)$, we define

$$LM(A) = \{\eta \in A^{**} : \eta A \subseteq A\},$$

$$RM(A) = \{\eta \in A^{**} : A\eta \subseteq A\}, \text{ and}$$

$$M(A) = LM(A) \cap RM(A).$$

Note that the above definitions are independent of the choice of C^*-algebra container. To see this let A be a real Jordan subalgebra of B where B is a real C^*-algebra. We may assume that A generates B. By the universal property of $C_e^*(A)$, there is a $*$-homomorphism $\theta : B \to C_e^*(A)$ such that $\theta = j$ on A. Let $a, b \in A$. If $ab \in A$, $j(a)j(b) = \theta(a)\theta(b) = \theta(ab) = j(ab)$. Therefore, the product in B matches the product in $C_e^*(A)$ for $a, b \in A$ and $ab \in A$.

First, we prove the real case of Lemma 2.1.6 in [3].

4.7.1 Lemma. *Suppose that a is an element of $B(H)$ where H is a real Hilbert space and that (e_t) is a net of contractions in $B(H)$ such that $ae_t \to a$. Then $ae_te_t^* \to a$, $ae_t^*e_t \to a$ and $ae_t^* \to a$.*

Proof. If $ae_t \to a$ in $B(H)$, then $ae_t \to a$ in $B(H)_c = B(H_c)$. By Lemma 2.1.6 in [3], we have $ae_te_t^* \to a$, $ae_t^*e_t \to a$ and $ae_t^* \to a$ in $B(H_c)$. Since $a, e_t \in B(H)$, $ae_te_t^* \to a$, $ae_t^*e_t \to a$ and $ae_t^* \to a$ in $B(H)$. □

Then we can apply the above lemma and follow the proof of Lemma 2.23 in [7] to obtain the real case version of this lemma.

4.7.2 Lemma (Real case of Lemma 2.23 in [7])**.** *Let A be an approximately unital real Jordan operator algebra. If p is a projection in $LM(A)$ then $p \in M(A)$. Moreover $\Delta(LM(A)) \subseteq M(A)$.*

If A is an approximately unital real Jordan operator algebra, the *Jordan multiplier algebra* of A is defined to be

$$JM(A) = \{\eta \in A^{**} : \eta a + a\eta \in A, \forall a \in A\}.$$

We follow the proof of Lemma 1.2 in [4] to obtain that $JM(A)$ is a real Jordan operator algebra. Since A is approximately unital $JM(A)$ contains the identity of A^{**}. Thus, $JM(A)$ is a unital real Jordan operator algebra and A is an approximately unital Jordan ideal of $JM(A)$.

4.8 Hereditary Subalgebras and Open Projections

Let A be a real Jordan operator algebra. We know that A^{**} is as well a real Jordan operator algebra. Denote $X^{\perp} = \{\varphi \in A^* : \varphi(x) = 0, \forall x \in X\}$ for $X \subseteq A$. It is known that $X^{\perp}$ is a weak* closed subspace of A^*. Thus, $X^{\perp\perp} = (X^{\perp})^{\perp}$ is a weak* closed subspace of A^{**} and $X^{\perp\perp} = \overline{span(X)}^{w^*}$. If X is a subspace of A, an element of $\eta \in X^{\perp\perp}$ is a weak* limit of a net $a_t \in X$. If in addition, X is a real Jordan subalgebra, $X^{\perp\perp}$ is a real Jordan subalgebra of A^{**}. This follows by the formula

$$\eta \circ \nu(\varphi) = \lim_s \lim_t \varphi(a_s \circ b_t) = \Big(\lim_s \lim_t \varphi(a_s b_t + b_t a_s)\Big)/2$$

for $\varphi \in A^*$. If $\eta \in X^{\perp\perp}$, there is a net $\{x_t\}$ in X such that $x_t \to \eta$. Then

$$\eta^2(\varphi) = \lim_t \lim_{t'} \varphi(x_t \circ x'_t) = \Big(\lim_t \lim_{t'} \varphi(x_t x_{t'} + x_{t'} x_t)\Big)/2 = 0$$

if $\varphi \in X^{\perp}$.

Let A be a real Jordan operator algebra. A projection p in A^{**} is called an *A-open projection* if $p \in (pA^{**}p \cap A)^{\perp\perp}$. Let p be A-open and set $D = pA^{**}p \cap A = \{a \in A^{**} : a = pap\}$. Then D is a closed real Jordan subalgebra of A and thus $D^{\perp\perp}$ is a Jordan subalgebra of A^{**}. In addition, p is an identity of $D^{\perp\perp}$ (by continuity of $x \mapsto a \circ x$ for fixed $a \in A$ in Lemma 4.2.1). Thus, D is an approximately unital real Jordan operator algebra which contains a net (e_t) weak* converging to p. Such a Jordan operator subalgebra D is called a *hereditary subalgebra (HSA)*. The A-open projection p in A^{**} such that $D = pA^{**}p \cap A$ is called the *support projection* of D. We immediately obtain the following.

4.8.1 Lemma. *For any real Jordan operator algebra A, a projection $p \in A^{**}$ is A-open if and only if p is a support projection of a HSA in A.*

If A is an approximately unital real Jordan operator algebra, there is an identity $e \in A^{**}$. We call a projection q *A-closed* if $q^\perp = e - q$ is A-open.

4.8.2 Proposition. *Let A be a real Jordan operator algebra. A projection in A^{**} is open if and only if it is open in $(A^1)^{**}$.*

Proof. If p is A-open, there is a net (x_t) in $A \subseteq A^1$ such that $px_tp \to p$ weak*. Conversely, if p is A^1-open, there is a net $(x_t + \lambda_t)$ in A^1 where $x_t \in A$ and $\lambda_t \in \mathbb{R}$ such that $x_t + \lambda_t = px_tp + p\lambda_tp \to p$ weak* in A^1. This forces $\lambda_t \to 0$ and $px_tp \to p$ weak*. $\square$

4.8.3 Proposition. *If D is a HSA in A, then D_c is a HSA in A_c. In addition, a projection $p \in A^{**}$ is A-open if and only if p is A_c-open. Also, a projection $p \in A^{**}$ is A-closed if and only if p is A_c-closed.*

Proof. If p is the support projection of D, then $D = pA^{**}p \cap A$. Then $D_c = pA_c^{**}p \cap A_c$. Thus, D_c is a HSA. Obviously p is A-open, so p is A_c-open. Conversely, if p is A_c-open, $p \in (p(A^{**})_c p \cap A_c)^{\perp\perp}$. Since $p \in A^{**}$, we have $p \in (p(A^{**})p \cap A)^{\perp\perp}$. $\square$

4.8.4 Proposition. *For any approximately unital real Jordan operator algebra A, every projection in $JM(A)$ is A-open and A-closed.*

Proof. We can follow the proof of complex case. However, we provide a proof by using operator space complexification. Let $p \in JM(A)$ be a projection. Then $pa + ap = a$ for all $a \in A$. This implies that $p(a + ib) + (a + ib)p = a + ib$ for all $a, b \in A$, i.e., $p \in JM(A_c)$. By Lemma 3.2 in [7], p is A_c-open and A_c-closed. Since $p \in A^{**}$, by Lemma 4.8.1, p is A-open and A-closed. $\square$

4.8.5 Lemma. *Let A be an approximately unital real Jordan operator algebra and $D \subseteq A$ be a hereditary subalgebra of A. Then D_c is a hereditary subalgebra of A_c*

Proof. Let $p \in A^{**}$ be a hereditary subalgebra of A and $x + iy \in D_c$. Then $p(x + iy)p = x + iy$. Therefore, $D_c = p(A_c)^{**}p \cap A_c$. Hence D_c is an hereditary subalgebra of A_c with the support projection p. □

4.8.6 Remark. The converse of the previous lemma may not be true. If we know that $D \subseteq A$ and D_c is a hereditary subalgebra of A_c with a support projection $p \in A_c^{**}$, we still do not know that $p \in A^{**}$. Therefore we may not conclude that D is a hereditary subalgebra of A.

4.8.7 Theorem (Real Case of Theorem 3.5 in [7])**.** *Suppose that D is a hereditary subalgebra of an approximately unital real Jordan operator algebra A. Then every $f \in D$ has a unique Hahn-Banach extension to a functional in A^* of the same norm.*

Proof. By the lemma above, D_c is a hereditary subalgebra of A_c. Therefore, $f_c \in (D_c)^*$ has a unique Banach extension, namely $\hat{f}_c$ by Theorem 3.5 in [7]. Note that $\|f\| = \|f_c\| = \|\hat{f}_c\|$ by Proposition 2.1.2. Let $\tilde{f} \in A^*$ be a real Hanh-Banach extension of f Then $\tilde{f}_c$ is a functional on $(A_c)^*$ and $\|\tilde{f}_c\| = \|\tilde{f}\| = \|f\|$ by Proposition 2.1.2. Thus, $\tilde{f}_c$ is a Hahn-Banach extension of f_c. By the uniqueness $\hat{f}_c = \tilde{f}_c$. Thus, $\tilde{f} = \hat{f}|_A$. □

4.9 Real Positivity in Real Jordan Operator Algebras

Let A be a real Jordan operator subalgebra of a real C^*-algebra B. Recall that $\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}$. Define $\mathfrak{r}_A$-ordering on A to be the order $\preceq$ induced by $\mathfrak{r}_A$, i.e., $b \preceq a$ if and only if $a - b \in \mathfrak{r}_A$.

4.9.1 Lemma. *Let A be a real Jordan operator algebra which generates a real C^*-algebra B and A_c be the operator space complexification of A. Let $\mathcal{U}_A = \{a \in A : \|a\| < 1\}$ and $\mathcal{U}_{A_c} = \{a + ib \in A_c : \|a + ib\| < 1\}$. Then A is approximately unital if and only if for any positive $b \in \mathcal{U}_B$ there exists $a \in \mathfrak{r}_A$ with $b \preceq a$.*

Proof. Let A be approximately unital and b be positive in $\mathcal{U}_B$. Then A_c is approximately unital. Since $\|b\|_c = \|b\|$ and a positive element in A is defined to be positive in A_c, $b \in \mathcal{U}_{B_c}$ and is positive. By Theorem 4.1 in [7], there exists $x + iy \in \mathfrak{r}_{A_c}$ such that $b \preceq x + iy$. Therefore, $(x - b) + iy \in \mathfrak{r}_{A_c}$, i.e., $(x - b) + (x - b)^* + i(y - y^*) \geq 0$. By Lemma 2.1.9, $(x - b) + (x - b)^* \geq 0$. Thus, $x - b \in \mathfrak{r}_A$ and thus $b \preceq x$.

Conversely, suppose that for any positive $b' \in \mathcal{U}_B$ there exists $x' \in \mathfrak{r}_A$ with $b \preceq x$. Let $a + ib$ be positive in $\mathcal{U}_{B_c}$. Then a is positive in $\mathcal{U}_B$. By the assumption, there exists $x \in \mathfrak{r}_A$ such that $a \preceq x$. Since bb^* is positive in B, by Proposition 5.2.2 (2) in [18], there exists a unique $c \in B^+$ such that $c^2 = bb^*$. Thus $\|c\| = \|b\| \leq \|a + ib\| < 1$. Thus c is positive in $\mathcal{U}_B$. By the assumption there exists $y \in \mathfrak{r}_A$ such that $c \preceq y$. Since (ib) is self adjoint, there exists $h_+, h_- \in (B_c)^+$ such that $ib = h_+ - h_-$ and $h_+h_- = 0$. Then $h_+ + h_-$ is the positive root of $(ib)(ib)^* = bb^*$. By the uniqueness of positive root, $c = h_+ + h_-$. Therefore, $ib \leq c$. Since both $x, y \in \mathfrak{r}_A$, $x + y \in \mathfrak{r}_A \subseteq \mathfrak{r}_{A_c}$. Now, we have $a \preceq x$, $ib \leq c$ and $c \preceq y$. Since $ib \leq c$ implies $ib \preceq c$, we have $ib \preceq y$. Therefore, $a + ib \preceq x + y$. By Theorem 4.1 in [7], A_c is approximately unital and so A is approximately unital by Lemma 4.6.2. □

From the proof above, for a positive element $a + ib$ in B_c, we can even pick $x \in \mathfrak{r}_A$ such that $a + ib \preceq x$. Therefore, if a complex Jordan operator subalgebra A' of a complex C^*-algebra B is a complexification of a real Jordan operator algebra A, A' is approximately unital if and only if for every positive element $a + ib$ in B', there is $x \in \mathfrak{r}_A$ such that $a + ib \preceq x$.

4.9.2 Proposition. *Let A be an approximately unital real Jordan operator algebra. Then $A = \mathfrak{r}_A - \mathfrak{r}_A$ and $A = \mathfrak{c}_A - \mathfrak{c}_A$.*

Proof. If A is approximately unital, then A_c is approximately unital. By Theorem 4.1 in [7], $A_c = \mathfrak{r}_{A_c} - \mathfrak{r}_{A_c}$. Let $a \in A \subseteq A_c$. Then $a = (x + iy) - (z + iw)$ where $x + iy, z + iw \in \mathfrak{r}_{A_c}$. Thus $a = x + z$. Since $Re(x + iy) = x$ and $Re(z + iw) \in \mathfrak{r}_A$, $A = \mathfrak{r}_A - \mathfrak{r}_A$. Similarly, $A_c = \mathfrak{c}_{A_c} - \mathfrak{c}_{A_c}$ by Theorem 4.1 in [7]. So $a = \alpha(x + iy) - \beta(z + iw) = (\alpha x - \beta z) + i(\alpha y - \beta w)$ where $\alpha, \beta \in [0, \infty)$. If $x + iy \in \mathfrak{F}_{A_c}$, then $Re(x + iy) = x \in \mathfrak{F}_A$. Thus, $x, z \in \mathfrak{F}_A$ and so $a = \alpha x - \beta z \in \mathfrak{c}_{A_c} - \mathfrak{c}_{A_c}$. □

4.9.3 Remark. In the Remark 3.1.5, we see that, for a real operator algebra A, $\Delta(A) = \{a \in A : a^* \in A\}$ and $A + A^*$ are well defined and independent of a choice of representation. This holds if A is a real Jordan operator algebra. Let A be a real Jordan subalgebra of $B(H)$ and $\Delta(A) = \{a \in A : a^* \in A\}$. This is a real JC^*-subalgebra of $B(H)$. Let $\pi : A \to B(K)$ be an isometric Jordan homomorphism. Then $\pi|_{\Delta(A)} : \Delta(A) \to B(K)$ is an isometric Jordan homomorphism between real JC^*-algebras. Therefore, $\pi|_{\Delta(A)}$ is selfadjoint by Corollary 2.1.34. Therefore $\Delta(A)$ is well defined and independent of a choice of representation. By following the same proof as in the Remark 3.1.5, we also obtain that $A + A^*$ is independent of a choice of representation.

4.9.4 Proposition. *Let A be a real operator system and $T : A \to B(H)$. Then T is real positive (resp. completely real positive) if and only if T is completely positive (resp. completely positive).*

Proof. If T is real positive and $a \geq 0$, $T(a + a^*) = T(2a) = 2T(a) \geq 0$. Conversely, if T is positive and $a + a^* \geq 0$. Then $T(a + a^*) \geq 0$, i.e., T is real positive. □

4.9.5 Remark. A positive map $\phi : A \to B(H)$ where A is an operator system does not imply ϕ is selfadjoint (see [24]). From Proposition 4.1 in [24] and the above proposition, if $T : A \to A$ is selfadjoint, we have that T is completely real positive if and only if T is completely contractive.

The following is an analog of Stinespring Dilation for completely positive map in real unital Jordan operator algebra.

4.9.6 Proposition. *Let A be a real unital Jordan operator subalgebra of a real unital C^*-algebra B and $T : A \to B(H)$ is unital completely positive and $T(a^*) = T(a)^*$ for every $a \in \Delta(A)$. Then T has a selfadjoint unital completely positive extension $\tilde{T} : B \to B(H)$. In addition there is a $*$-representation $\pi : B \to B(K)$ for a real Hilbert space K and a contraction $s \in B(H, K)$ such that*

$$\tilde{T}(a) = s^* \pi(a) s$$

for all $a \in B$.

Proof. The map $\pi|_{\Delta(A)} : \Delta(A) \to B(H)$ is selfadjoint and completely positive on $\Delta(A)$ which is a real operator system. By Proposition 4.2 in [23], there is a selfadjoint unital completely positive extension $\tilde{\pi} : B \to B(H)$. We claim that $\tilde{\pi}(a) = \pi(a)$ for $a \in A$. Since $A \subseteq A + A^* \subseteq B$, $\tilde{\pi}|_{A+A^*}$ is selfadjoint extension of π which is unique. Therefore, $\tilde{\pi}(a) = \tilde{\pi}|_{A+A^*}(a) = \pi(a)$ for every $a \in A$. Therefore, $\tilde{\pi}$ is an extension of π. By Theorem 4.3 in [24], there is a $*$-representation $\pi : B \to B(K)$ where K is a real Hilbert space and bounded operator $s \in B(H, K)$ such that $\|\tilde{T}\|_{cb} = \|s\|^2$ and

$$T(a) = s^* \pi(a) s$$

for all $a \in B$. $\square$

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