AN INVESTIGATION OF THE STABILITY OF A LINEAR VISCOELASTIC BAR

A Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

College of Engineering

University of Houston

In Partial Fulfillment

of the Requirement for the Degree

Master of Science

---

by

Carl Dennis Faust

August 1970

ACKNOWLEDGEMENTS

The author is indebted to those who were instrumental in the preparation of this thesis, particularly to Dr. Ray Nachlinger for the patience and advice which he gave throughout this study. Sincerest appreciation is extended to Mrs. Jan Foley for the typing of the manuscript. Also I wish to extend by deepest appreciation to my wife, Mary Anne. Without her complete understanding and encouragement, this thesis would never have been completed.

AN INVESTIGATION OF THE STABILITY
OF A LINEAR VISCOELASTIC BAR

An Abstract of a Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

College of Engineering

University of Houston

In Partial Fulfillment

of the Requirement for the Degree

Master of Science

---

by

Carl Dennis Faust

August 1970

ABSTRACT

The intent of this study is to investigate the stability of an initially straight, simply supported, linear viscoelastic bar subjected to a harmonically varying axial load. Infinitesimal viscoelasticity is assumed through the study.

The study is based upon the concepts of continuum mechanics developed by W. Noll. A Volterra integral of the second kind is derived and solved for the deflection of the bar as a function of time. The solution of the equation is based on the assumption that the stress relaxation function can be approximated as a series of monotonically decreasing exponentials.

The study proceeds by defining the concept of the systems stability. Based on the general solution, two lemmas and a theorem are proved with respect to stability. The theorem proves that two conditions are necessary and sufficient for stability.

A numerical example is presented for which four cases are investigated. The stress relaxation function, $c(\tau)$, is assumed to be a single exponential. The lemmas and theorem are applied to determine the system's stability.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Symbol:

$A(t)$	= time dependent portion of deflection
a	= coefficient of $\phi(\tau)$, Equation 5.1
a_j	= $a_j'b$
a_j'	= coefficient of Equation 3.4
$B(t)$	= loading function, Equations 2.8 and 2.9
b	= coefficient of $B(t)$, Equation 2.9
$c(\tau)$	= modified relaxation function, Equation 2.8
d	= coefficient of $B(t)$, Equation 2.9
$E(t)$	= strain tensor
$E(t-\tau)$	= strain histories
$F(s)$	= $1/(s^2 + c(s) + d)$
$F(t)$	= inverse LaPlace transform of $F(s)$
g	= static portion of harmonic load
h	= coefficient of harmonic load
I	= column moment of inertia
i	= $\sqrt{-1}$
$K^{(n)}(t,\tau)$	= iterative kernel, Equation 3.3
l	= column length
M	= bending monent
n	= counter 1, 2, ...
$P(t)$	= axial load

Symbol:

$S(t,\tau)$	= solvent kernel, Equation 3.2
$T(t)$	= stress tensor
t	= time
$v(x, t)$	= deflection
x	= spatial coordinate
α	= damping parameter
β	= roots to Equation 2.11
ε	= initial condition on $\dot{A}(o)$
ξ	= dummy integration variable
Π	= PI, 3.14159
ρ	= material density
τ	= time for histories
$\phi(\tau)$	= relaxation modulus
$\phi(o)$	= material instantaneous modulus
$\phi(\infty)$	= material equilibrium modulus
$\dot{\phi}(\tau)$	= relaxation function
ω	= forcing frequency

Chapter 1

INTRODUCTION

The subject of linear viscoelasticity appears to have been first investigated in 1874 by Boltzmann (1). He originally studied the three-dimensional case, but restricted it to isotropic materials. Since the original work was done, many people have investigated the various aspects of the linear theory of viscoelasticity. Many of the studies formulated the governing equations based on models constructed from springs and dashpots; the Kelvin model, the Maxwell model, and combinations of each. In 1958, Noll (2) published his paper on the mathematical theory of mechanical behavior of continuous media. In 1961, Coleman and Noll (3) presented the general constitutive equation for a linear, viscoelastic material, assuming infinitesimal viscoelasticity. They re-examined the basic hypotheses of the classical linear theory based on continuum mechanics. In this way, the general constitutive equation could be derived without the aid of spring and dashpot models. The investigation presented herein is based on the results of their paper and on the work of Coleman and Misel (4).

The problem presented in this paper is the investigation of the stability of a linear viscoelastic bar, assuming infinitesimal viscoelasticity, subjected to a harmonically varying axial load. The bar is considered initially straight and simply supported at each end. The general governing equation is derived and a solution presented. Then, stability is defined and several lemmas and a theorem are proved.

Lastly, a numerical problem is solved and the derived lemmas and theorem are applied to determine the stability of the problem.

Chapter 2

DERIVATION OF GOVERNING EQUATION

The constitutive equation for a simple material with fading memory was derived in Reference 3. By taking the reference configuration as the natural state, we obtain the equation used in Reference 4. This equation (Eq. 2.1) relates the stress tensor, T(t), to the strain tensor, E(t) and its histories, E(t-τ).

$$T(t) = \phi(o)\ E(t) + \int_o^\infty \dot{\phi}(\tau)\ E(t-\tau)\ d\tau \qquad 2.1$$

From beam theory and using the Bernoulli-Euler hypothesis, the strain is related to the deflection as shown below

$$E(t) = -y \frac{\partial^2 v(x,t)}{\partial x^2}$$

Substituting this relationship into the constitutive equation and integrating over the area, we obtain equation 2.2

$$\phi(o) \frac{\partial^2 v(x,t)}{\partial x^2} + \int_o^\infty \dot{\phi}(\tau) \frac{\partial^2 v(x,t-\tau)}{\partial x^2} d\tau = - \frac{1}{I} M - \frac{P(t)}{I} v(x,t) \qquad 2.2$$

Taking the second partial derivative of equation 2.2 with respect to x and knowing that

$$\frac{\partial^2 M}{\partial x^2} = \rho \frac{\partial^2 v(x,t)}{\partial t^2}$$

The governing equation, Equation 2.3, is obtained.

$$- \frac{\rho}{I} \frac{\partial^2 v(x,t)}{\partial t^2} = \phi(o) \frac{\partial^4 v(x,t)}{\partial x^4} + \int_o^t \dot{\phi}(\tau) \frac{\partial^4 v(x,t-\tau)}{\partial x^4} d\tau + \frac{P(t)}{I} \frac{\partial^2 v(x,t)}{\partial x^2} \qquad 2.3$$

The physical model, a simply supported bar, suggests that the form of the deflection curve is

$$v(x,t) = A(t) \sin \frac{n\Pi}{l} x \qquad 2.4$$

The boundary and initial conditions shown in Equation 2.5 will be used through this study.

$$\begin{array}{lll} v(0,t) = 0 & v(1,t) = 0 & M(0,t) = 0 \\ v(x,0) = 0 & \dot{v}(x,0) = \varepsilon(x) & M(1,t) = 0 \end{array} \qquad 2.5$$

Substitution of the deflection curve, Equation 2.4, into Equation 2.3 yields the following equation:

$$- \frac{\rho}{I} \ddot{A}(t) \sin \frac{n\Pi}{l} x = \phi(o) \frac{n^4\Pi^4}{l^4} A(t) \sin \frac{n\Pi}{l} x + \int_o^t \dot{\phi}(\tau) \frac{n^4\Pi^4}{l^4} A(t-\tau) \sin\frac{n\Pi}{l} x \, d\tau - \frac{P(t)}{I} \frac{n^2\Pi^2}{l^2} A(t) \sin\frac{n\Pi}{l} x \qquad 2.6$$

But, the integral is not spatially dependent; therefore, the sine term can be factored out of the equation. Doing this we obtain

$$\ddot{A}(t) + B(t)A(t) + \int_o^t c(\tau)A(t-\tau)d\tau = 0 \qquad 2.7$$

where

$$B(t) = \frac{I}{\rho}\frac{n^2\Pi^2}{1^2}\left[\frac{n^2\Pi^2}{1^2}\phi(o) - \frac{P(t)}{I}\right]$$

$$c(\tau) = \frac{In^4\Pi^4}{\rho 1^4}\dot{\phi}(\tau) \qquad 2.8$$

Now we must solve Equation 2.7 for A(t). The axial load imposed on the bar is a harmonically varying load and can be represented as

$$P(t) = g + he^{i\omega t}.$$

Substituting P(t) into Equation 2.8 the following form of B(t) is obtained.

$$B(t) = \frac{In^2\Pi^2}{\rho 1^2}\left\{\frac{n^2\Pi^2}{1^2}\phi(o) - \frac{g}{I}\right\} - \frac{n^2\Pi^2}{\rho 1^2}\ he^{i\omega t}$$

or

$$B(t) = d + be^{i\omega t} \qquad 2.9$$

Substituting for B(t) and taking the LaPlace transform of Equation 2.7 we obtain, using the boundary and initial conditions of Equation 2.5

$$A(s) + bF(s)A(s-i\omega) = \varepsilon F(s) \qquad 2.10$$

with

$$F(s) = \frac{1}{s^2 + c(s) + d} \qquad 2.11$$

On taking the inverse transformation of Equation 2.10, we obtain

$$A(t) = \varepsilon F(t) - b\int_0^t F(t-\tau)e^{i\omega\tau}A(\tau)d\tau \qquad 2.12$$

This equation may be rearranged into a more useful form.

$$\varepsilon F(t) = A(t) + \int_o^t F(t-\tau) b e^{i\omega\tau} A(\tau) d\tau \qquad 2.13$$

We have thus reduced the initial problem, a fourth order integro-differential equacion with variable coefficients, to the solution of Equation 2.13, which is a Volterra integral equation of the second kind.

Chapter 3

SOLUTION OF GOVERNING EQUATION

Since Equation 2.13 is a Volterra integral of the second kind, by Reference 5, its solution has the form

$$A(t) = \varepsilon F(t) + \varepsilon \int_0^t S(t,\tau)F(\tau)d\tau \qquad 3.1$$

Where $S(t,\tau)$ is called the solvent kernel. The solvent kernel is related to the iterative kernels, $K^{(n)}(t,\tau)$ by:

$$S(t,\tau) = \sum_{n=1}^{\infty} K^{(n)}(t, \tau) \qquad 3.2$$

The iterative kernels are evaluated by the following relationships.

$$\begin{aligned} K^{(1)}(t,\tau) &= -F(t-\tau)be^{i\omega\tau} \\ K^{(2)}(t,\tau) &= \int_\tau^t K^{(1)}(t,\xi)K^{(1)}(\xi,\tau)d\xi \qquad 3.3 \\ K^{(3)}(t,\tau) &= \int_\tau^t K^{(1)}(t,\xi)K^{(2)}(\xi,\tau)d\xi \\ &\vdots \\ K^{(n)}(t,t) &= \int_\tau^t K^{(1)}(t,\xi)K^{(n-1)}(\xi,\tau)d\xi \end{aligned}$$

Before we can proceed with the solution of Equation 2.13 it is necessary to determine the general form of $F(t)$, $F(\tau)$, and $F(t-\tau)$.

From Equation 2.11, we know

$$F(s) = \frac{1}{s^2 + c(s) + d}$$

The term c(s) in the un-transformed form is called the stress relaxation function. Physically, $c(\tau)$ represents the material's ability to remember its near past better than its distant past. The name "Fading Memory" has been given to this material phenomenon. If, as is usually done, we assume that the stress relaxation function can be approximated by a series of decaying exponentials, we can write

$$c(\tau) = \sum_{j=1}^{n} c_j \, e^{-\alpha_j \tau} \qquad 3.4$$

Thus, an approximation to the inverse of F(s) is of the form

$$F(t-\tau) = \sum_{j=1}^{n} a_j' \, e^{-\beta_j (t-\tau)} \qquad 3.5$$

Where the β_j's are the roots to the denominator of Equation 2.11. By examining Equations 2.11 and 3.4, we see that the roots are dependent on d, the static portion of B(t), c_j, and α_j, a damping type material property.

Now, substituting Equation 3.5 into Equation 3.3, with $a_j = b\, a_j'$, we obtain the iterative kernels.

$$K^{(1)}(t,\tau) = -\sum_{j=1}^{n} a_j \, e^{-\beta_j t + (\beta_j + i\omega)\tau} \qquad 3.6$$

$$K^{(2)}(t,\tau) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j\, a_k}{(\beta_j-\beta_k+i\omega)} \left\{ e^{(-\beta_k+i\omega)t+(\beta_k+i\omega)\tau} \right. \tag{3.7}$$

$$\left. - e^{-\beta_j t+(\beta_j+2i\omega)\tau} \right\}$$

$$K^{(3)}(t,\tau) = -\sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{a_j a_l a_m}{(\beta_l-\beta_m+i\omega)} \left\{ \frac{1}{(\beta_j-\beta_m+2i\omega)} e^{(-\beta_m+2i\omega)t+(\beta_m+i\omega)\tau} \right.$$

$$- \frac{1}{(\beta_j-\beta_l+i\omega)} e^{(-\beta_l+i\omega)t+(\beta_l+2i\omega)\tau} \tag{3.8}$$

$$\left. + \frac{\beta_l-\beta_m+i\omega}{(\beta_j-\beta_l+i\omega)(\beta_j-\beta_m+2i\omega)} e^{-\beta_j t+(\beta_j+3i\omega)\tau} \right\}$$

.

.

.

$$K^{(n)}(t,\tau) = (-1)^n \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \cdots \sum_{n=1}^{n} \frac{a_a a_b a_c \cdots a_n}{(\beta_{n-1}-\beta_n+i\omega)}$$

$$\left\{ \frac{e^{(-\beta_n+[n-1]i\omega)t+(\beta_n+i\omega)\tau}}{(\beta_a-\beta_n+[n-1]i\omega)(\beta_b-\beta_n+[n-2]i\omega)\cdots(\beta_{n-2}-\beta_n+2i\omega)} \right. \tag{3.9}$$

$$- \frac{e^{(-\beta_{n-1}+[n-2]i\omega)t+(\beta_{n-1}+2i\omega)\tau}}{(\beta_a-\beta_{n-1}+[n-2]i\omega)(\beta_b-\beta_{n-1}+[n-3]i\omega)\cdots(\beta_{n-3}-\beta_{n-1}+i\omega)} + \cdots$$

$$+ \left[\frac{-1}{(\beta_a-\beta_n+[n-1]i\omega)(\beta_b-\beta_n+[n-2]i\omega)\cdots} \right.$$

$$\left. \left. + \frac{1}{(\beta_a-\beta_{n-1}+[n-2]i\omega)(\beta_b-\beta_{n-1}+[n-3]i\omega)\cdots} - \cdots \right] e^{-\beta_a t+(\beta_a+ni\omega)\tau} \right\}$$

The solution for A(t) is now obtained by substituting into Equation 3.1. This yields,

$$A(t) = \varepsilon \sum_{j=1}^{n} a_j' e^{-\beta_j t} + \varepsilon \int_0^t \Bigg\{ \Bigg(- \sum_{j=1}^{n} a_j \; e^{[-\beta_j t + (\beta_j + i\omega)\tau]}$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j a_k}{(\beta_j - \beta_k + i\omega)} \left[e^{[(-\beta_k + i\omega)t + (\beta_k + i\omega)\tau]} - e^{[-\beta_j t + (\beta_j + 2i\omega)\tau]} \right]$$

$$- \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{a_j a_l a_m}{(\beta_l - \beta_m + i\omega)} \Bigg[\frac{1}{(\beta_j - \beta_m + 2i\omega)} \; e^{[(-\beta_m + 2i\omega)t + (\beta_m + i\omega)\tau]} \qquad 3.10$$

$$- \frac{1}{(\beta_j - \beta_l + i\omega)} \; e^{[(-\beta_l + i\omega)t + (\beta_l + 2i\omega)\tau]}$$

$$+ \frac{(\beta_l + \beta_m + i\omega)}{(\beta_j - \beta_l + i\omega)(\beta_j - \beta_m + 2i\omega)} e^{[-\beta_j t + (\beta_j + 3i\omega)\tau]} \Bigg]$$

$$+ \cdots + K^{(n)}(t,\tau) \Bigg) \Bigg(\sum_{j=1}^{n} a_j' e^{-\beta_j \tau} \Bigg) \Bigg\} \; d\tau$$

If we assume that F(t) is a bounded function of time, we can write Equation 3.1 in the form of the inequality shown in Equation 3.11. This form of the equation shows that a stable solution can be obtained if $S(t,\tau)$ is also a bounded function.

$$A(t) \leq \varepsilon \left\{ 1 + \int_0^t \; |S(t,\tau)| \; d\tau \right\} \text{MAX} \; |F(t)| \qquad 3.11$$

Substituting the solvent kernel into Equation 3.11 and performing the integration yields,

$$A(t) \le \varepsilon\left[1 + \left|-\sum_{j=1}^{n} \frac{a_j}{(\beta_j+i\omega)}\left\{e^{i\omega t} - e^{(-\beta_j+i\omega)t}\right\}\right.\right.$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j a_k}{(\beta_j-\beta_k+i\omega)(\beta_k+i\omega)(\beta_j+2i\omega)}\left\{(\beta_j-\beta_k+i\omega)e^{2i\omega t}\right.$$

$$\left. - (\beta_j+2i\omega)e^{(-\beta_k+i\omega)t} + (\beta_k+i\omega)e^{-\beta_j t}\right\}$$

$$+ \cdots + (-1)^n \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\cdots\sum_{n=1}^{n} \frac{a_a a_b a_c \cdots a_n}{(\beta_{n-1}-\beta_n+i\omega)}$$

$$\left\{\left[\frac{1}{(\beta_a-\beta_n+[n-1]i\omega)(\beta_b-\beta_n+[n-2]i\omega)\cdots(\beta_{n-2}-\beta_n+2i\omega)(\beta_n+i\omega)} + \cdots\right]e^{ni\omega t}\right.$$

$$- \frac{1}{(\beta_a-\beta_n+[n-1]i\omega)(\beta_b-\beta_n+[n-2]i\omega)\cdots(\beta_{n-2}-\beta_n+2i\omega)(\beta_n+i\omega)}\, e^{(-\beta_n+[n-1]i\omega)t}$$

$$+ \cdots + \left[\frac{1}{(\beta_a-\beta_n+[n-1]i\omega)(\beta_b-\beta_n+[n-2]i\omega)\cdots(\beta_{n-2}-\beta_n+2i\omega)(\beta_a+ni\omega)}\right.$$

$$\left.\left.\left.\left. + \cdots\right] e^{-\beta_a t}\right\}\right|\right] \mathrm{MAX}\left|\sum_{j=1}^{n} a_j' \, e^{-\beta_j t}\right| \qquad 3.12$$

Substitution of Equation 3.12 into the assumed deflection curve, Equation 2.4, yields the general solution of the deflection, $v(x,t)$, when the function $F(t)$ is bounded.

CHAPTER 4

STABILITY CRITERIA

The concept of stability used in this study is defined as follows:

Definition

An initially straight, simply supported viscoelastic bar subjected to a harmonically varying end load, is said to be stable, if and only if, the deflection of the bar is bounded as time increases without bound.

Using this stability definition, we can state and prove several theorems pertaining to the stability of the system under investigation. The stability of the system is taken with respect to time, thus, we will use, for simplicity, only the time dependent term of the deflection. This is permissible, since, the spatial term of v(x,t) is sine dependent and the sine is a bounded function.

Lemma 1

Let the governing stability equation be Equation 2.12

$$A(t) = \varepsilon F(t) + \varepsilon \int_0^t S(t,\tau)F(\tau)d\tau$$

If F(t) is an unbounded function of time, then the system is unstable.

Proof

Assume A(t) is bounded and F(t) is of the form indicated by Equation 3.4.

$$F(t) = \sum_{j=1}^{n} a_j' \, e^{-\beta_j t}$$

Now, if we have the negative of β_j, F(t) is an unbounded function of time. Thus,

$$A(t) = \varepsilon \sum_{j=1}^{n} a_j' e^{\beta_j t} + \varepsilon \int_o^t S(t,\tau) \sum_{j=1}^{n} a_j' e^{\beta_j \tau} d\tau$$

Now, A(t) is an unbounded function and by the definition the system is unstable.

Q.E.D.

Assume now, that F(t) is a bounded function of time.

Lemma 2

If F(t) is a bounded function, let the governing stability equation be Equation 3.12 and the roots to Equation 2.11 have at least one set of complex conjugate roots of the form

$$\beta_b = -b_j + \frac{1}{2} r i\omega$$

$$\beta_n = -b_j - \frac{1}{2} r i\omega$$

If r is a real positive integer, then at least one coefficient of the solvent kernel, $S(t,\tau)$, will be unbounded and the system will be unstable.

Proof

Assume A(t) is bounded and Equation 3.12 governs stability. Now, examine one coefficient, denoted as X below, in the n^{th} term of Equation 3.12.

$$X = \frac{1}{(\beta_a - \beta_n + [n-1]i\omega)(\beta_b - \beta_n + [n-2]i\omega)\ldots(\beta_{n-2} - \beta_n + 2i\omega)(\beta_n + i\omega)}$$

Substituting the two roots

$$X = \frac{1}{(\beta_a - \beta_n + [n-1]i\omega)\ldots(\beta_n + i\omega)(n-1-r)i\omega} .$$

Since, r is a real positive integer, let

$$r = n - 1$$

Thus, the coefficient, X, of $S(t,\tau)$ becomes unbounded. Therefore, the system is unstable.

Q.E.D.

Theorem

The system is unstable, if and only if, F(t) is an unbounded function of time and/or the roots to Equation 2.11 are such that

$$\beta_b = -b_j + \frac{1}{2} ri\omega$$

and

$$\beta_n = -b_j - \frac{1}{2} ri\omega$$

where r is a real positive integer.

Proof

a) Assume the governing stability equation for the system is Equation 3.4 and F(t) is an unbounded function. Then, by Lemma 1 the system is unstable.

b) Assume the governing stability equation for the system is Equation 3.12 and the roots to Equation 2.11 are such that

$$\beta_b = -b_j + \frac{1}{2} ri\omega$$

and

$$\beta_b = -b_j - \frac{1}{2} ri\omega$$

where r is a real positive integer. Then, by Lemma 2 the system is unstable.

c) Assume the system is unstable and Equation 3.4 governs the stability of the system. Thus

$$A(t) = \varepsilon F(t) + \int_0^t S(t,\tau)F(\tau)d\tau \ .$$

This equation can be written as an inequality of the form

$$A(t) \leq \varepsilon\{ 1 + \int_0^t |S(t,\tau)|d\tau\} \text{MAX}|F(t)|$$

From the definition of stability, the system is unstable, if and only if, A(t) is unbounded. A(t) is unbounded if F(t) and/or S(t,τ) is is unbounded. Examination of the form of F(t), shows that it is unbounded with time, only if the real portion of the roots to Equation 2.11 are positive. S(t,τ), by inspection, is unbounded only if the real portion of the roots are positive or there exists one set of complex conjugate roots of the form

$$\beta_b = -b_j + \frac{1}{2}ri\omega$$

and

$$\beta_b = -b_j - \frac{1}{2}ri\omega$$

where r is a real positive integer.

Q.E.D.

CHAPTER 5

NUMERICAL EXAMPLE

To illustrate the lemmas and theorem derived in this study, the following problem is posed.

Given the polyethylene bar in Figure 1, loaded by a harmonic load P(t), investigate its stability for the cases shown in Table 1. Also, assume that the stress relaxation function is obtained from the relaxation modulus in Equation 5.1.

$$\phi(\tau) = \phi(\infty)(1 + ae^{-\alpha\tau}) \tag{5.1}$$

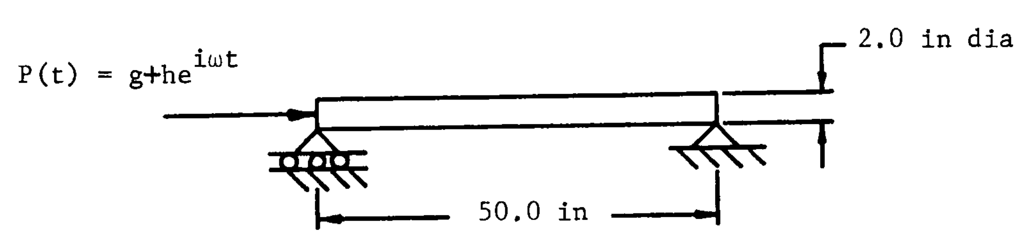

FIGURE 1

SAMPLE PROBLEM GEOMETRY

TABLE 1

SAMPLE PROBLEM CASES

Case No.	α	d	Q	Forcing Frequency ω
1	0.1	10.	5.	2.000000
2	0.1	10.	8.	3.162009
3	100.0	10.	5.	2.000000
4	100.0	20.	5.	1.118244

With Equation 5.1, Equation 2.11 becomes:

$$F(s) = \frac{s + \alpha}{s^3 + \alpha s^2 + ds + Q\alpha} \qquad 5.2$$

where

$$d = \frac{In^2\pi^2}{\rho l^2}\left\{\frac{n^2\pi^2}{l^2}\,\phi(0) - \frac{g}{I}\right\} \qquad 5.3$$

$$Q = \frac{In^2\pi^2}{\rho l^2}\left\{\frac{n^2\pi^2}{l^2}\,\phi(\infty) - \frac{g}{I}\right\} \qquad 5.4$$

The parameters $\phi(0)$ and $\phi(\infty)$ are called the instantaneous and equilibrium moduli of the material respectively. Figure 2 presents these parameters graphically. The modulus decreases exponentially to $\phi(\infty)$ with increasing time.

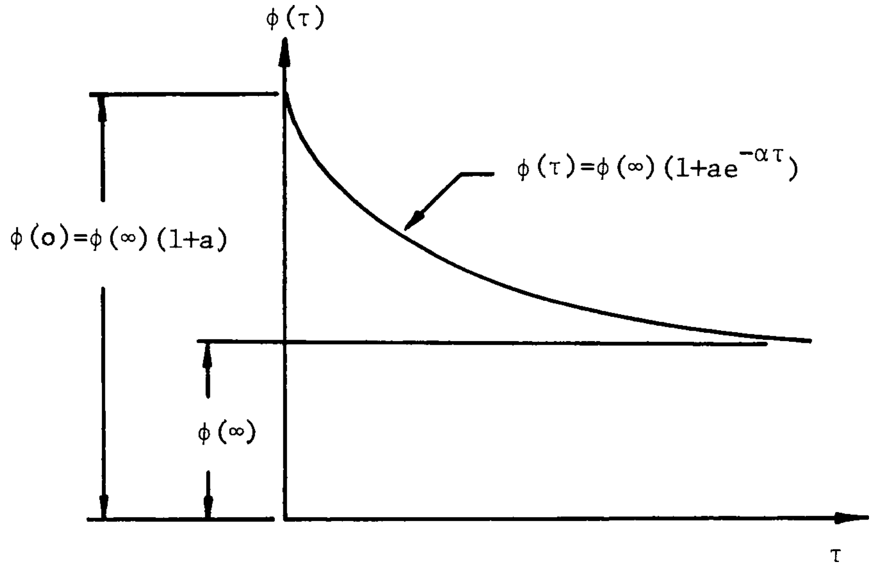

FIGURE 2

RELAXATION MODULUS

Examination of Equations 5.1 and 5.2 yields the following observations:

a. $\alpha \not\leq 0$; if $\alpha \leq 0$, the material would not exhibit viscoelastic material properties. $\alpha = 0$ would be elastic and $\alpha < 0$ would be contrary to test data.

b. $d > 0$; an analogy can be made between $\frac{In^2\pi^2}{1^2}\phi(0)$ and Euler's critical load, P_{cr}. Since, g is the static portion of the harmonic load, letting $d \leq 0$ would cause an instability.

c. $Q > 0$; the same analogy can be made here as for d.

This time consider $\phi(\infty)$ as Young's Modulus.

d. $d > Q$; $\phi(0) > \phi(\infty)$, thus d must be greater than Q.

Expressing Equation 5.2 in terms of the roots β_j, we obtain the following expression.

$$F(s) = \frac{s + \alpha}{(s + \beta_1)(s + \beta_2)(s + \beta_3)} \qquad 5.5$$

By varying the parameters α, d, and Q, and invoking the derived lemmas, the curve in Figure 3 can be obtained. The curve can be thought of as representing a stability boundary. The area above the boundary represents a region of general instability. Below the boundary, the stability is not known, except for the case where the forcing frequency is zero. Figure 3, thus, represents a necessary condition for stability but not a sufficient condition. Sufficiency is obtained from the thoerem.

Using the values of α, d, and Q in Table 1, the roots, β_j, can be calculated. These roots are shown in Table 2.

TABLE 2

ROOTS TO SAMPLE PROBLEM

Case	β_1	β_2	β_3
1	-.050012	-.024994 - 3.161784i	-.024994 + 3.161784i
2	-.080013	-.009994 - 3.162009i	-.009994 + 3.162009i
3	-99.94999	-.002499 - 2.236487i	-.002499 + 2.236487i
4	-99.84985	-.007508 - 2.236489i	-.007508 + 2.236489i

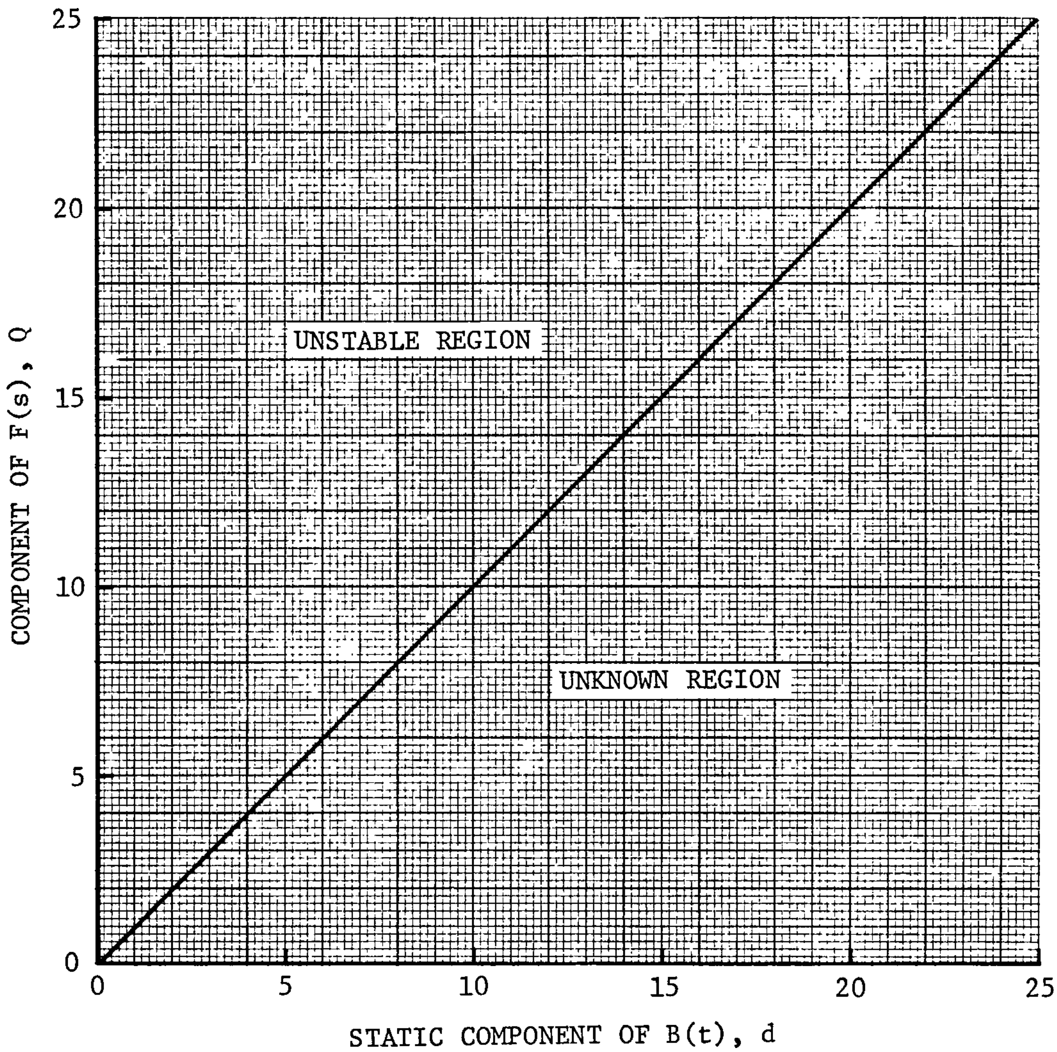

FIGURE 3. STABILITY BOUNDARY FOR A ONE TERM STRESS RELAXATION FUNCTION

It is seen that all real components of the roots are negative. Since, we take the inverse LaPlace transform of F(s), it is required that we have the following form (consider β_j as the magnitude only).

$$s = -\beta_j \longrightarrow s + \beta_j = 0$$

Taking the inverse transform of $\frac{1}{s + \beta_j}$ yields $e^{-\beta_j t}$. This is the desired form to obtain bounded functions.

From the data in Table 2 several trends can be seen. If α is increased with d and Q constant, the real root increases and the imaginary component decreases. This implies that if the forcing frequency was larger than the imaginary component, by increasing the damping parameter, α, a more stable condition would exist. If α and Q or α and d are held constant, the effect on the imaginary component is almost negligible.

Now investigate the stability of each case. Each case lies in the unknown region of Figure 3.

Case 1

Applying the theorem shows that both F(t) and S(t,τ) are bounded functions of time. Thus, the system is stable.

Case 2

The theorem is violated for this case. The imaginary component of the roots are integer multiples of the forcing frequency. Thus, by the theorem, the system is unstable.

Case 3

Case 3, like Case 1, is stable when the theorem is applied.

Case 4

Case 4 is unstable, since, the imaginary component of the roots are integer multiples of the forcing frequency.

From these four cases, it has been shown that a necessary condition for stability is that the combination of d and Q must lie in the unknown region of Figure 3. The sufficient condition for stability was then shown to be the theorem. Stated in terms of the roots, the following conditions must be met:

1. The real portions of the roots must be negative when written in the form

$$s = \beta_j = -x_j + iy_j$$

this yields F(t) as a bounded function of time.

2. The imaginary portion of the roots must not be an integral multiple of the forcing frequency.

REFERENCES

1. Boltzmann, S., Sitzber. Akad. Wiss. Wien, Math. Naturw. K L. 70, 275 (1874); Pogg. Ann. Phy. 7, 624 (1876).

2. Noll, W., "A Mathematical Theory of the Mechanical Behavior of Continuous Media," Archive of Rational Mechanics and Analysis, 2, 197 (1958).

3. Coleman, B. and W. Noll, "Foundations of Linear Viscoelasticity," Reviews of Modern Physics, 33, 239 (1961).

4. Coleman, B. and V. Mizel, "On the Stability of Solutions of Functional-Differential Equations," Archive of Rational Mechanics and Analysis, 30, 173 (1968).

5. Volterra, V., Theory of Functionals and of Integral and Integro-Differential Equations, Dover Publications, N.Y., 1959.