THEORY OF JORDAN OPERATOR ALGEBRAS AND OPERATOR $*$-ALGEBRAS

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Doctor of Philosophy

By

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THEORY OF JORDAN OPERATOR ALGEBRAS AND OPERATOR $*$-ALGEBRAS

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Abstract

An *operator algebra* is a closed subalgebra of $B(H)$, for a complex Hilbert space H. By a *Jordan operator algebra*, we mean a norm-closed *Jordan subalgebra* of $B(H)$, namely a norm-closed subspace closed under *Jordan product* $a \circ b = \frac{1}{2}(ab + ba)$. By an *operator $*$-algebra* we mean an operator algebra with an involution $\dagger$ making it a $*$-algebra with $\|[a_{ji}^{\dagger}]\| = \|[a_{ji}]\|$ for $[a_{ij}] \in M_n(A)$ and $n \in \mathbb{N}$. In this dissertation, we investigate the general theory of Jordan operator algebras and operator $*$-algebras.

In Chapter 3, we present Jordan variants of 'classical' facts from the theory of operator algebras. For example we begin Chapter 3 with general facts about Jordan operator algebras. We then give an abstract characterizations of Jordan operator algebras. We also discuss unitization and real positivity in Jordan operator algebras.

In Chapter 4, we study the hereditary subalgebras, open projections, ideals and M-ideals. We then develop the theory of real positive elements and real positive maps in the setting of Jordan operator algebras.

Chapter 5 is largely concerned with general theory of operator $*$-spaces.

In Chapter 6, we give several general results about operator $*$-algebras. For example we prove some facts about involutions on nonselfadjoint operator algebras and their relationship to the C^*-algebra they generate. We also discuss contractive approximate identities, Cohen factorizations for operator $*$-algebras etc.

In Chapter 7, we investigate hereditary subalgebras and ideals, noncommutative topology and peak projections in an involutive setting.

Contents

CHAPTER 1

Introduction

An (associative) *operator algebra* is a closed associative subalgebra of $B(H)$, for a complex Hilbert space H. By a *Jordan operator algebra* we mean a norm-closed *Jordan subalgebra* A of $B(H)$, namely a norm-closed subspace closed under the 'Jordan product' $a \circ b = \frac{1}{2}(ab + ba)$. Or equivalently, with $a^2 \in A$ for all $a \in A$ (that is, A is closed under squares; the equivalence uses the identity $a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2)$).

Selfadjoint Jordan operator algebras arose in the work of Jordan, von Neumann, and Wigner on the axiomatic foundations of quantum mechanics. One expects the 'observables' in a quantum system to constitute a (real) Jordan algebra, and if one also wants a good functional calculus and spectral theory one is led to such selfadjoint

Jordan algebras (known as JC*-algebras). Nowadays however the interest in Jordan algebras and related objects is almost exclusively from pure mathematicians (see e.g. [44]). Despite this interest, there seems to be only one paper in the literature that discusses Jordan operator algebras in our sense above, namely the excellent work of Arazy and Solel [2].

The first part of this thesis is a step in the direction of extending the selfadjoint Jordan theory to nonselfadjoint Jordan operator algebras. Our main discovery is that the theory of Jordan operator algebras is astonishingly similar to associative operator algebras. We were able to generalize many results, relevant to associative operator algebras, to the Jordan case with very minor exceptions. Since much of this parallels the huge existing theory of associative operator algebras there is quite a lot to do, and we map out here some foundational and main parts of this endeavor. Several of the more interesting questions and challenging aspects of the theory remain to be explored.

By an *operator $*$-algebra* we mean an operator algebra with an involution † making it a $*$-algebra with $\|[a_{ji}^{\dagger}]\| = \|[a_{ij}]\|$ for $[a_{ij}] \in M_n(A)$ and $n \in \mathbb{N}$. Here we are using the *matrix norms* of operator space theory (see e.g. [55]). This notion was first introduced by Mesland in the setting of noncommutative differential geometry [45], who was soon joined by Kaad and Lesch [42]. In several recent papers by these authors and coauthors they exploit operator $*$-algebras and involutive modules in geometric situations. Subsequently we noticed very many other examples of operator $*$-algebras, and other involutive operator algebras, occurring naturally in general operator algebra theory which seem to have not been studied hitherto. It is thus natural to investigate the general theory of involutive operator algebras, and this is the focus of the second part of this thesis. In a joint work with D. P. Blecher [25], we were able to include a rather large number of results since many proofs are similar to their operator algebra counterparts in the literature (see e.g. [14]). On the other hand, some of the main theorems about operator algebras do not have operator $*$-algebra variants, so some work is needed to disentangle the items that do work. Many of the results are focused around 'real positivity' in the sense of several recent papers of David. P. Blecher and his collaborators referenced in our bibliography.

My advisor gave me the projects of initiating and building the theories of Jordan operator algebras (extending the selfadjoint Jordan theory to not necessarily selfadjoint Jordan operator algebras) and involutive operator algebras. There are several papers (see e.g. [25, 26, 27]), which arose from these projects. In some of these papers the author did the ground work for the theory, building up a database of results.

CHAPTER 2

Overview and preliminaries

In this chapter, we fix our notation and provide some basic results about operator spaces and operator algebras that will be constantly used in this paper. We suggest the reader refer to [14, 30, 51, 55] for more details.

2.1 Operator spaces

Definition 2.1. A *concrete operator space* is a closed linear subspace X of $B(H, K)$, for Hilbert spaces H, K. An *abstract operator space* is a pair $(X, \{\|\cdot\|_n\}_{n\geq 1})$ consisting of a vector space X, and a norm on $M_n(X)$ for all $n \in \mathbb{N}$ such that there exists a linear complete isometry $u : X \to B(K, H)$. We call the sequence $\{\|\cdot\|_n\}_n$ an *operator space structure* on the vector space X.

Clearly subspaces of operator spaces are again operator spaces. We often identify two operator spaces X and Y if they are completely isometrically isomorphic. In this case we often write '$X \cong Y$ completely isometrically' or say '$X \cong Y$ as operator spaces.'

Proposition 2.2. *Let X be an operator space. Then*

(R1) $\|\alpha x \beta\| \leq \|\alpha\| \|x\|_n \|\beta\|$, *for all $n \in \mathbb{N}$ and $\alpha, \beta \in M_n$, and $x \in M_n(X)$.*

(R2) *For all $x \in M_m(X)$ and $y \in M_n(X)$, we have*

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max\{\|x\|_m, \|y\|_n\}.$$

Condition (R1) and (R2) above often called *Ruan's axioms.* The following result asserts that (R1) and (R2) characterize operator space structures on a vector space.

Theorem 2.3 (Ruan)**.** *Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\| \cdot \|_n$ on $M_n(X)$. Then X is linearly completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H, if and only if conditions (R1) and (R2) hold.*

Definition 2.4. An operator space Z is said to be *injective* if for any completely bounded linear map $u : X \to Z$ for any operator space Y containing X as a closed subspace, there exists a completely bounded extension $\hat{u} : Y \to Z$ such that $\hat{u}_{|X} = u$ and $\|\hat{u}\|_{cb} = \|u\|_{cb}$.

Definition 2.5. If X is an operator space in $B(K, H)$, then we define the *adjoint operator space* to be the space $X^\star = \{x^* : x \in X\} \subseteq B(H, K)$. As an abstract operator space $X^\star$ is independent of the particular representation of X on H and K.

Definition 2.6. Let X be an operator space. Then we define the *opposite operator space* X° to be the Banach space X with the 'transposed matrix norms' $|||[x_{ij}]||| = \|[x_{ji}]\|$.

Note that if A is a C*-algebra, then these matrix norms on A° coincide with the canonical matrix norms on the C*-algebra with its reverse multiplication. If X is a subspace of a C*-algebra A, then X° may be identified with completely isometrically with the associated subspace of the C*-algebra A°.

Definition 2.7. An *extension* of an operator space X is an operator space Y, together with a linear completely isometric map $i : X \to Y$. Y is called a *rigid* extension of X if I_Y is the only linear completely contractive map $Y \to Y$ which restricts to the identify map on $i(X)$. We say Y is an *essential* extension if whenever $u : Y \to Z$ is a completely contractive map into another operator space Z such that $u \circ i$ is a completely isometry, then u is a complete isometry. We say that (Y, i) is an *injective envelope* of X if Y is injective, and if there is no injective subspace of Y containing $i(X)$.

Lemma 2.8. *Let (Y, i) be an extension of an operator space X such that Y is injective. The following are equivalent:*

(1) *Y is an injective envelope of X,*

(2) *Y is a rigid extension of X,*

(3) *Y is an essential extension of X.*

Proof. See [14, Lemma 4.2.4]. □

Theorem 2.9. (*a*) *If X is a unital operator space (resp. unital operator algebra, approximately operator algebra), then there is an injective envelope $(I(X), j)$ for X such that $I(X)$ is a unital C^*-algebra and j is a unital map (resp. j is a unital homomorphism, j is a homomorphism).*

(*b*) *If A is an approximately unital operator algebra, and if (Y, j) is an injective envelope for A^1, then $(Y, j_{|_A})$ is an injective envelope for A.*

(*c*) *If A is an approximately unital operator algebra which is injective, then A is a unital C^*-algebra.*

Proof. See [14, Corollary 4.2.8] . □

2.2 Operator algebras

Definition 2.10. A *concrete operator algebra* is a closed subalgebra of $B(H)$, for some Hilbert space H. If A is an operator space and a Banach algebra, then we call A an *abstract operator algebra* if there exist a Hilbert space H and a completely isometric homomorphism $\pi : A \to B(H)$.

We often identify two operator algebras A and B which are *completely isometrically isomorphic*, that is, there eixsts a completely isometrically algebraic homomorphism from A onto B. In this case write '$A \cong B$ completely isometrically isomorphically' or '$A \cong B$ as operator algebras'.

Definition 2.11. If A is an operator algebra, then we say that a net (e_t) in $\mathrm{Ball}(A)$ is a contractive approximate identity (cai for short) for A if $e_t a \to a$ and $a e_t \to a$ for all $a \in A$.

Definition 2.12. An operator algebra A is *unital* if it has an identity of norm 1. We call A *approximately unital* if A possess a cai.

Every C^*-algebra is an approximately unital operator algebra.

Definition 2.13. If S is a subset of a C^*-algebra B, then $C^*_B(S)$ denotes the smallest C^*-subalgebra of B containing S. A *C^*-cover* of an operator algebra A is a pair (B, j) consisting of a C^*-algebra B, and a completely isometric homomorphism $j : A \to B$, such that $j(A)$ generates B as a C^*-algebra i.e. $C^*_B(j(A)) = B$.

Lemma 2.14. *Let A be an operator algebra and suppose that B is a C^*-cover of A. If A is approximately unital, then every cai for A is a cai for B. If A is unital, then 1_A serves as an identity for B.*

Proof. See [14, Lemma 2.1.7]. □

Remark 2.15. A C^*-cover (B, j) of an approximately unital operator algebra A is a unital C^*-algebra if and only if A is unital. Indeed, if A is unital with unit 1_A, then $j(1_A)$ is a unit for B by Lemma 2.14. Conversely, if B is unital with unit 1_B, then by Lemma 2.14 we see that any cai (e_t) for A satisfies $j(e_t) = j(e_t)1_B \to 1_B$. Since $j(A)$ is closed, we have $1_B \in j(A)$.

Often problems concerning an operator algebra A are solved by first tackling the case where A is unital; and then in general case considering the *unitization* A^1 of A. If A is nonunital operator algebra, then a unitization of A is also an operator algebra. The following result shows that up to completely isometric isomorphism, this unitization does not depend on the embedding $A \subset B(H)$.

Theorem 2.16. [14, Theorem 2.1.13]. *Let $A \subset B(H)$ be a nonunital operator algebra, and let $\pi : A \to B(K)$ be an isometric (resp. completely isometric) homomorphism. Then the unital homomorphism from* Span$\{A, I_H\}$ *into $B(K)$ extending π is an isometry (resp. complete isometry).*

Although the unitization A^1 is now defined unambiguously, it is in general very difficult to describe its norm explicitly. However if A is an approximately unital operator algebra, and if $A \subset B(H)$ nondegenerately, then for any integer $n \geq 1$ and for any matrices $[a_{ij}] \in M_n(A)$ and $[\lambda_{ij}] \in M_n$, we have

$$\|[a_{ij} + \lambda_{ij} I_H]\| = \sup\{\|[a_{ij}c + \lambda_{ij}c]\| : c \in A, \|c\| \leq 1\}.$$

The following result usually referred to the *BRS theorem* gives a criterion for a unital (or more generally an approximately unital) Banach algebra with an operator space structure to be an operator algebra.

Theorem 2.17 (BRS Theorem)**.** *Let A be an operator space which is also an approximately unital Banach algebra. Let $m : A \times A \to A$ denote the multiplication on*

A. The following are equivalent:

(i) *The mapping* $m : A \otimes_h A \to A$ *is completely contractive.*

(ii) *For any* $n \geq 1$, $M_n(A)$ *is a Banach algebra. That is,*

$$\left\| \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right] \right\|_{M_n(A)} \leq \|[a_{ij}]\|_{M_n(A)} \|[b_{ij}]\|_{M_n(A)}$$

for all $[a_{ij}]$ *and* $[b_{ij}]$ *in* $M_n(A)$.

(iii) *A is an operator algebra, that is, there exist a Hilbert space* H *and a completely isometric homomorphism* $\pi : A \to B(H)$.

Proof. See [14, Theorem 2.3.2]. □

Definition 2.18. Let A be an operator algebra. We say J is an *ideal* of A if it is a closed two-sided ideal of A.

Corollary 2.19. [14, Theorem 2.3.4] *Let* J *be an ideal in an operator algebra* A. *Then* A/J *is an operator algebra.*

2.3 Universal algebras of an operator algebra

There are minimal and maximal C^*-algebras generated by an operator algebra. Let us first look at C^*-envelope or minimal C^*-cover. References in this section may be found in [14].

Definition 2.20. Let X be a unital operator space. We define a C^**-envelope* of X to be any C^*-extension (B, i) with the universal property of the next theorem.

Theorem 2.21 (Arverson-Hamana)**.** *If X is a unital operator space, then there exists a C^*-extension (B, i) of X with the following universal property: Given any C^*-extension (A, j) of X, there exists a (necessarily unique and surjective) $*$-homomorphism $\pi : A \to B$, such that $\pi \circ j = i$.*

Remark 2.22. The C^*-envelope of X may be taken to be any C^*-extension (A, j) of X for which there exists no nontrivial closed two-sided ideal I of A such that $q \circ j$ is completely isometric on X, where q is the quotient map from A to A/I.

Definition 2.23. Let A be a nonunital operator algebra. Then C^**-envelope* of A is a pair (B, i), where B is the C^*-subalgebra generated by the copy $i(A)$ of A inside a C^*-envelope $(C_e^*(A^1), i)$ of the unitization A^1 of A.

Proposition 2.24. *Let A be an operator algebra, and let $(C_e^*(A), i)$ be a C^*-envelope of A. Then $C^*(A)$ has the following universal property: Given any C^*-cover (B, j) of A, there exists a necessary unique and surjective $*$-homomorphism $\pi : B \to C_e^*(A)$ such that $\pi \circ j = i$.*

Now we turn to the maximal C^*-algebra case.

Definition 2.25. Let A be an operator algebra. We define the *maximal* or *universal* C^*-algebra to be the C^*-cover $(C^*_{\max}(A), i)$ with the universal property of the next proposition.

Proposition 2.26. [14, Proposition 2.4.2] *Let A be an operator algebra. Then there exists a C^*-cover $(C^*_{\max}(A), j)$ of A with the following universal property: if $\pi : A \to$*

D is any completely contractive homomorphism into a C^*-algebra D, then there exists a (necessarily unique) $*$-homomorphism $\tilde{\pi} : C^*_{\max}(A) \to D$ such that $\tilde{\pi} \circ j = i$.

CHAPTER 3

General theory of Jordan operator algebras

3.1 JC*-algebras

The selfadjoint variant of a Jordan operator algebra (that is, a closed selfadjoint subspace of a C^*-algebra which is closed under squares) is exactly what is known in the literature as a JC*-algebra. We describe some basic background and results about JC*-algebras (see also e.g. the texts [44, 31, 56, 61, 62, 57] for more details and background to some of the objects mentioned here in passing).

Definition 3.1 (Kaplansky). Let A be a complex Banach space and a complex

Jordan algebra equipped with an involution $*$. Then A is a *Jordan C^*-algebra* if the following four conditions are satisfied.

(i) $\|x \circ y\| \leq \|x\|\|y\|$ for all x and y in A.

(ii) $\|x\| = \|x^*\|$ for all x in A.

(iii) $\|\{xx^*x\}\| = \|x\|^3$ for all x in A, where $\{xx^*x\}$ is the Jordan triple product defined by $\{abc\} = (a \circ b) \circ c + a \circ (b \circ c) - c \circ (a \circ b)$.

A Jordan C^*-algebra is said to be a *JC*-algebra* if it is isometrically $*$-isomorphic to a norm-closed, Jordan $*$-subalgebra of $B(H)$ for some Hilbert space H. A JC*-algebra is a complexifications of its selfadjoint part, which is a JC-algebra in the sense of Topping [60]. Conversely, Wright has shown (see [43]) that the complexification of any JB-algebra is a JB*-algebra, in the sense that there is a JB*-algebra norm on the complexification. JC*-algebras are very close to C^*-algebras, for example they have a positive and increasing (in the usual senses) Jordan contractive approximate identity; see [31, Proposition 3.5.23].

Any selfadjoint element x in a JC*-algebra A generates a C^*-algebra and hence is a difference of two positive elements. Thus $A = \mathrm{Span}(A_+)$ in this case. J*-algebras containing the identity operator are just the unital JC*-algebras, as may be seen using the fact that J*-algebras are closed under the operation xy^*x (see [37]). There is a similar statement for J*-algebras containing a unitary in the sense of [38] (see e.g. [38, Proposition 7.1]).

A contractive Jordan morphism T from a JC*-algebra A into $B(H)$ preserves the adjoint $*$. Indeed for any selfadjoint element x in a JC*-algebra, $T_{|C^*(x)}$ is a $*$-homomorphism so $T(x)$ is selfadjoint. Thus T is easily seen to be a J*-homomorphism on A in the sense of [38] (i.e. $T(aa^*a) = T(a)T(a)^*T(a)$, for any $a \in A$). A Jordan $*$-homomorphism T between JC^*-algebras is contractive, and if it is one-to-one then it is isometric. A linear surjection between JC^*-algebras is an isometry if and only if it is preserves the triple product xy^*x, and if and only if it is preserves 'cubes' xx^*x. These results are due to Harris [37, 38] in the more general case of J^*-algebras. Such a surjection is a Jordan $*$-isomorphism if and only if it is approximately unital, and if and only if it is positive.

A Jordan ideal of a Jordan algebra A is a subspace E with $\eta \circ \xi \in E$ for $\eta \in E, \xi \in A$. The Jordan ideals of a JC*-algebra A coincide with the JB*-ideals of A when A is regarded as a JC*-triple [38]. That is, if J is a closed subspace of a JC*-algebra A such that $a \circ J \subset J$, then $ab^*c + cb^*a \in J$ whenever $a, b, c \in A$ and at least one of these are in J. This implies that $J = J^*$ (take $a = c$ in a Jordan approximate identity for A, recalling that every JC*-algebra has a positive Jordan approximate identity [31, Proposition 3.5.23]).

An interesting class of JC*-algebras are the Cartan factors of type IV (spin factors, see e.g. [37]): these are selfadjoint operator spaces A in $B(H)$ such that $x^2 \in \mathbb{C}I_H$ for all $x \in A$. They are isomorphic to a Hilbert space, and contain no projections except I. They may be constructed by finding a set of selfadjoint unitaries $\{u_i : i \in S\}$ with $u_i \circ u_j = 0$ if $i \neq j$ and setting $A = \mathrm{Span}\{I, u_i : i \in S\}$.

3.2 General facts about Jordan operator algebras

Jordan subalgebras of commutative (associative) operator algebras are ordinary (commutative associative) operator algebras on a Hilbert space, and the Jordan product is the ordinary product. In particular if a is an element in a Jordan operator algebra A inside a C^*-algebra B, then the closed Jordan algebra generated by a in A equals the closed operator algebra generated by a in B. We write this as $\mathrm{oa}(a)$.

Associative operator algebras and JC*-algebras are of course Jordan operator algebras. So is $\{a \in A : a = a^T\}$, for any subalgebra A of M_n. More generally, given a homomorphism π and an antihomomorphism θ on an associative operator algebra A, $\{a \in A : \pi(a) = \theta(a)\}$ is a Jordan operator algebra. As another example we mention the Jordan subalgebra $\{(x, q(x)) : x \in B(H)\}$ of $B(H) \oplus^\infty Q(H)^{\mathrm{op}}$. Here $q : B(H) \to Q(H)$ is the canonical quotient map onto the Calkin algebra. This space is clearly closed under squares. This example has appeared in operator space theory, for example it is an operator space with a predual but no operator space predual, and hence is not representable completely isometrically and weak* homeomorphically as a weak* closed space of Hilbert space operators.

If A is a Jordan operator subalgebra of $B(H)$, then the *diagonal* $\Delta(A) = A \cap A^*$ is a JC*-algebra. If A is unital then as a a JC*-algebra $\Delta(A)$ is independent of the Hilbert space H. That is, if $T : A \to B(K)$ is an isometric Jordan homomorphism for a Hilbert space K, then T restricts to an isometric Jordan $*$-homomorphism from $A \cap A^*$ onto $T(A) \cap T(A)^*$. This follows from a fact in the last section about contractive Jordan morphisms between JC*-algebras preserving the adjoint. An element q in a Jordan operator algebra A is called a *projection* if $q^2 = q$ and $\|q\| = 1$ (so these are just the orthogonal projections on the Hilbert space A acts on, which are in A). Clearly $q \in \Delta(A)$. A projection q in a Jordan operator algebra A will be called *central* if $qxq = q \circ x$ for all $x \in A$. For $x \in A$, using the 2×2 'matrix picture' of x with respect to q one sees that

$$qxq = q \circ x \qquad \text{if and only if} \qquad qx = xq = qxq \tag{3.2.1}$$

with the products qx and xq taken in any C^*-algebra containing A as a Jordan subalgebra). Note that this implies and is equivalent to that q is central in any generated (associative) operator algebra, or in a generated C^*-algebra. This notion is independent of the particular generated (associative) operator algebra since it is captured by the intrinsic formula $qxq = q \circ x$ for $x \in A$.

In a Jordan operator algebra we have the Jordan identity

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x).$$

For a, b, c in a Jordan operator algebra we have

$$abc + cba = 2[(a \circ b) \circ c + a \circ (b \circ c)] - 2(a \circ c) \circ b. \tag{3.2.2}$$

Hence if we define a Jordan ideal to be a subspace J of a Jordan algebra A such that $A \circ J \subset J$, then $abc + cba \in J$ whenever $a, b, c \in A$ and at least one of these are in J. Thus A/J is a Jordan algebra, but we do not believe it is in general a Jordan operator algebra without extra conditions on J. Putting $a = c$ in the identity (3.2.2) above gives $2aba = (ab + ba)a + a(ab + ba) - [a^2b + ba^2] \in A$, or

$$aba = 2(a \circ b) \circ a - a^2 \circ b. \tag{3.2.3}$$

By a C^*-*cover* of a Jordan operator algebra we mean a pair (B, j) consisting of a C^*-algebra B generated by $j(A)$, for a completely isometric Jordan homomorphism $j : A \to B$.

Let A be a Jordan subalgebra of a C^*-algebra B. Then we may equip the second dual A^{**} with a Jordan Arens product as follows. Consider $a \in A, \varphi \in A^*$ and $\eta, \nu \in A^{**}$. Let $a \circ \varphi$ $(= \varphi \circ a)$ be the element of A^* defined by

$$\langle a \circ \varphi, b\rangle = \langle \varphi, \frac{ab + ba}{2}\rangle$$

for any $b \in A$. Then let $\eta \circ \varphi (= \varphi \circ \eta)$ be the element of A^* defined by

$$\langle \eta \circ \varphi, a\rangle = \langle \eta, a \circ \varphi\rangle.$$

By definition, the Arens Jordan product on A^{**} is given by

$$\langle \eta \circ \nu, \varphi\rangle = \langle \eta, \nu \circ \varphi\rangle.$$

This is equal to the Jordan product in A^{**} coming from the associative Arens product in B^{**}, and we have $\eta \circ \nu = \nu \circ \eta$. Indeed suppose that $a_s \in A$, $b_t \in A$ such that $a_s \to \eta$

and $b_t \to \nu$ in the weak* topology. For any $\varphi \in A^*$ let $\hat{\varphi} \in B^*$ be a Hahn-Banach extension. Note that

$$\begin{aligned} 2\langle \eta \circ \nu, \varphi \rangle = 2\langle \eta, \nu \circ \varphi \rangle &= \lim_s \langle \nu \circ \varphi, a_s \rangle \\ &= 2 \lim_s \lim_t \langle b_t \circ \varphi, a_s \rangle = \lim_s \lim_t \langle \varphi, a_s b_t + b_t a_s \rangle. \end{aligned}$$

Note that

$$\begin{aligned} 2\langle \eta \circ_B \nu, \hat{\varphi} \rangle &= \langle (\eta\nu + \nu\eta), \hat{\varphi} \rangle \\ &= \langle \eta\nu, \hat{\varphi} \rangle + \langle \nu\eta, \hat{\varphi} \rangle \\ &= \langle \eta, \nu\hat{\varphi} \rangle + \langle \nu, \eta\hat{\varphi} \rangle \\ &= \lim_s \langle \nu\hat{\varphi}, a_s \rangle + \lim_s \langle \eta\hat{\varphi}, b_t \rangle \\ &= \lim_s \lim_t \langle \hat{\varphi} a_s, b_t \rangle + \lim_s \lim_t \langle \hat{\varphi} b_t, a_s \rangle \\ &= \lim_s \lim_t \langle \hat{\varphi}, a_s b_t + b_t a_s \rangle \\ &= \lim_s \lim_t \langle \varphi, a_s b_t + b_t a_s \rangle. \end{aligned}$$

Thus the Jordan product in A^{**} agrees on A^{**} with the Jordan product coming from the associative Arens product in B^{**}. We also see from this that $\eta \circ \nu = \nu \circ \eta$. In any case, the bidual of a Jordan operator algebra is a Jordan operator algebra, and may be viewed as a Jordan subalgebra of the von Neumann algebra B^{**}.

JW*-algebras (that is, weak* closed JC*-algebras) are closed under meets and joins of projections (see [60, Theorem 6.4] or [36, Lemma 4.2.8]; one may also see this since meets and joins may be defined in terms of limits formed from

$$q_1 \cdots q_{n-1} q_n q_{n-1} \cdots q_1 \text{ and } (q_1 + \cdots q_n)^{\frac{1}{n}},$$

both of which make sense in any Jordan Banach algebra). Since for any Jordan operator algebra A we have that A^{**} is a Jordan operator algebra with diagonal $\Delta(A^{**})$ a JW*-algebra, it follows that A^{**} is also closed under meets and joins of projections. In particular $p \vee q$ is the weak* limit of the sequence $(p+q)^{\frac{1}{n}}$, for projections $p, q \in A^{**}$.

By the analogous proof for the operator algebra case (see 2.5.5 in [14]), any contractive (resp. completely contractive) Jordan homomorphism from a Jordan operator algebra A into a weak* closed Jordan operator algebra M extends uniquely to a weak* continuous contractive (resp. completely contractive) Jordan homomorphism $\tilde{\pi} : A^{**} \to M$.

3.3 A characterization of unital Jordan operator algebras

The following is an operator space characterization of unital (or approximately unital) Jordan operator algebras (resp. JC*-algebras). It references however a containing operator space B, which may be taken to be a C^*-algebra if one wishes.

Theorem 3.2. *Let A be a unital operator space (resp. operator system) with a bilinear map $m : A \times A \to B$ which is completely contractive in the sense of Christensen and Sinclair (see e.g. the first paragraph of* 1.5.4 *in [14]). Here B is a unital operator space containing A as a unital-subspace (so $1_B \in A$) completely isometrically. Define $a \circ b = \frac{1}{2}(m(a,b) + m(b,a))$, and suppose that A is closed under this operation. Assume also that $m(1,a) = m(a,1) = a$ for $a \in A$. Then A is a unital Jordan operator algebra (resp. JC*-algebra) with Jordan product $a \circ b$.*

Proof. We will use the injective envelope $I(A)$ and its properties (see e.g. Lemma 2.8 and Theorem 2.9). By injectivity, the canonical morphism $A \to I(A)$ extends to a unital completely contractive $u : B \to I(A)$. By injectivity again, i.e. the well known extension theorem for completely contractive bilinear maps/the injectivity of the Haagerup tensor product, and the universal property of that tensor product, we can use $u \circ m$ to induce a linear complete contraction $\tilde{m} : I(A) \otimes_h I(A) \to I(A)$. It is known that $I(A)$ is a unital C^*-algebra by Theorem 2.9. By rigidity of the injective envelope, $\tilde{m}(1,x) = x = \tilde{m}(x,1)$ for all $x \in I(A)$. By the nonassociative case of the BRS theorem (see e.g. 4.6.3 in [14]), together with the Banach-Stone theorem for operator algebras (see e.g. 8.3.13 in [14]), $\tilde{m}$ must be the canonical product map. Hence for $a, b \in A$, $u(m(a,b))$ is the product taken in $I(A)$. Since

$$u(m(a,b)) + u(m(b,a)) = u(m(a,b) + m(b,a)) = m(a,b) + m(b,a) = 2a \circ b \in A,$$

A is a unital Jordan subalgebra of $I(A)$. If in addition A is an operator system then the embedding of A in $I(A)$ is a complete order embedding by e.g. 1.3.3 in [14], so that A is a JC*-subalgebra of $I(A)$. □

Remark 3.3. The unwary reader might have expected a characterization in terms of a bilinear map $m : A \times A \to A$ on a unital operator space A such that m is completely contractive in the sense of Christensen and Sinclair, and makes A a Jordan algebra in the algebraic sense. However it is easy to prove that under those hypotheses A is completely isometrically isomorphic to a commutative operator algebra. To see this notice that the nonassociative case of the BRS theorem mentioned in the proof shows that A is an associative operator algebra. Since $m(a, b) = m(b, a)$ for $a, b \in A$ we see that A is commutative.

There is an 'approximately unital' analogue of Theorem 3.2.

Definition 3.4. An *approximately unital* operator space A is a subspace of an approximately unital operator algebra B, such that A contains a cai (e_t) for B. The hypothesis that $m(e_t, a) \to a$ and $m(a, e_t) \to a$ for $a \in A$ is shown in the later result Lemma 3.13 to be a reasonable one: in a Jordan operator algebra with a cai satisfying $e_t \circ a \to a$, one can find another cai satisfying $e_t a \to a$ and $a e_t \to a$ with products here in any C^*-algebra or approximately unital operator algebra containing A as a closed Jordan subalgebra.

Theorem 3.5. *Let A be an approximately unital operator space (resp. operator system) containing a cai (e_t) for an operator algebra B as above. Let $m : A \times A \to B$ be a completely contractive bilinear map in the sense of Christensen and Sinclair. Define $a \circ b = \frac{1}{2}(m(a, b) + m(b, a))$, and suppose that A is closed under this operation. Assume also that $m(e_t, a) \to a$ and $m(a, e_t) \to a$ for $a \in A$. Then A is a Jordan* *operator algebra (resp. JC*-algebra) with Jordan product $a \circ b$, and $e_t \circ a \to a$ for $a \in A$.*

Proof. Let $e = 1_{B^{**}} \in A^{**}$. We consider the canonical weak* continuous extension $\tilde{m} : A^{**} \times A^{**} \to B^{**}$. By standard approximation arguments $\tilde{m}(e, a) = \tilde{m}(a, e) = a$ for all $a \in A^{**}$, and $\tilde{m}(a, b) + \tilde{m}(b, a) \in A^{**}$ for all $a, b \in A^{**}$. By Theorem 3.2 we have that A^{**} is a unital Jordan operator algebra (resp. JC*-algebra) with Jordan product $\frac{1}{2}(\tilde{m}(a, b) + \tilde{m}(b, a))$. Hence A is a Jordan operator algebra with Jordan product $a \circ b$. Clearly $e_t \circ a \to a$. $\square$

3.4 Meyer's theorem, unitization and real positive elements

The following follows from Meyer's theorem on the unitization of operator algebras (see e.g. 2.1.13 and 2.1.15 in [14]).

Proposition 3.6. *If A and B are Jordan subalgebras of $B(H)$ and $B(K)$ respectively, with $I_H \notin A$, and if $T : A \to B$ is a contractive (resp. isometric) Jordan homomorphism, then there is a unital contractive (resp. isometric) Jordan homomorphism extending T from $A + \mathbb{C}\, I_H$ to $B + \mathbb{C}\, I_K$ (for the isometric case we also need $I_K \notin B$).*

Proof. It is only necessary to show that if $a \in A$ is fixed, then Claim: $\|T(a) + \lambda 1_K\| \leq \|a + \lambda 1_H\|$ for $\lambda \in \mathbb{C}$. However the restriction of T to $\mathrm{oa}(a)$ is an algebra homomorphism into $\mathrm{oa}(T(a))$, and so the Claim follows from Meyers result. $\square$

Corollary 3.7 (Uniqueness of unitization for Jordan operator algebras)**.** *The unitization A^1 of a Jordan operator algebra is unique up to isometric Jordan isomorphism.*

Proof. If A is nonunital then this follows from Proposition 3.6. If A is unital, and A^1 is a unitization on which A has codimension 1, then since the identity e of A is easily seen to be a central projection in A^1 we have $\|a + \lambda 1\| = \max\{\|a + \lambda e\|, |\lambda|\}$. $\square$

Because of Corollary 3.7, for a Jordan operator algebra A we can define unambiguously $\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$. The diagonal $\Delta(A) = A \cap A^* = \Delta(A^1) \cap A$ is a JC^*-algebra, as is easily seen, and now it is clear that as a JC^*-algebra $\Delta(A)$ is independent of the particular Hilbert space A is represented on (since this is true for $\Delta(A^1)$ as we said in Subsection 3.2). That is, if $T : A \to B(K)$ is an isometric Jordan homomorphism for a Hilbert space K, then T restricts to an isometric Jordan $*$-homomorphism from $A \cap A^*$ onto $T(A) \cap T(A)^*$. Every JC*-algebra is approximately unital (see [31, Proposition 3.5.23]).

If A is a unital Jordan operator subalgebra of $B(H)$, with $I_H \in A$ then $A_{\rm sa}$ makes sense, and is independent of H, these are the hermitian elements in A (that is, $\|\exp(ith)\| = 1$ for real t; or equivalently $\varphi(h) \in \mathbb{R}$ for all states φ of A, where by 'state' we mean a unital contractive functional). Similarly, $\mathfrak{r}_A$, the real positive or accretive elements in A, may be defined as the set of $h \in A$ with Re $\varphi(h) \geq 0$ for all states φ of A. This is equivalent to all the other usual conditions characterizing accretive elements (see e.g. [10, Lemma 2.4 and Proposition 6.6]; some of these use the fact that the Jordan algebra generated by a single element and 1 is an algebra).

If A is a possibly nonunital Jordan operator algebra we define $\mathfrak{r}_A$ to be the elements with positive real part–we call these the *real positive* or *accretive* elements of A. Since the unitization is well defined by Proposition 3.6, so is $\mathfrak{r}_A$. Alternatively, note that $A^1 + (A^1)^*$, and hence $A + A^*$, is well defined as a unital selfadjoint subspace independently (up to unital (positive) isometry) of the particular Hilbert space that A^1 is represented isometrically and nondegenerately, by [4, Proposition 1.2.8]. That is, a unital Jordan isometry $T : A^1 \to B(K)$ extends uniquely to a unital positive Jordan isometry $A^1 + (A^1)^* \to T(A^1) + T(A^1)^*$. Thus a statement such as $a + b^* \geq 0$ makes sense whenever $a, b \in A$, and is independent of the particular H on which A is represented as above. This gives another way of seeing that the set $\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}$ is independent of the particular Jordan representation of A too.

We have $x \in \mathfrak{c}_A = \mathbb{R}_+ \mathfrak{F}_A$ if and only if there is a positive constant C with $x^*x \leq C(x+x^*)$ (to see this note that $\|1-tx\|^2 \leq 1$ if and only if $(1-tx)^*(1-tx) \leq 1$). Also, $\mathfrak{r}_A$ is a closed cone in A, hence is Archimidean (that is, x and $x + ny \in \mathfrak{r}_A$ for all $n \in \mathbb{N}$ implies that $y \in \mathfrak{r}_A$). On the other hand $\mathfrak{c}_A = \mathbb{R}_+ \mathfrak{F}_A$ is not closed in general, but it is a proper cone (that is, $\mathfrak{c}_A \cap (-\mathfrak{c}_A) = (0)$). This follows from the proof of the analogous operator algebra result in the introduction to [22], since 1 is an extreme point of the ball of any unital Jordan algebra A since e.g. A is a unital subalgebra of a unital C^*-algebra B and 1 is extreme in $\mathrm{Ball}(B)$.

If A is a nonunital Jordan subalgebra of a unital C^*-algebra B then we can identify A^1 with $A + \mathbb{C}\, 1_B$, and it follows that $\mathfrak{F}_A = \mathfrak{F}_B \cap A$ and $\mathfrak{r}_A = \mathfrak{r}_B \cap A$. Hence if A is a Jordan subalgebra of a Jordan operator algebra B then $\mathfrak{r}_A = \mathfrak{r}_B \cap A$ and

$\mathfrak{r}_A = \mathfrak{r}_B \cap A$.

3.5 Universal algebras of a Jordan operator algebra

There are maximal and minimal associative algebras generated by a Jordan operator algebra A. Let us first look at maximal C^*-algebra case.

Consider the direct sum ρ of 'all' contractive (resp. completely contractive) Jordan representations $\pi : A \to B(H_\pi)$. There are standard ways to avoid the set theoretic issues with the 'all' here–see e.g. the proof of [14, Proposition 2.4.2]. Let $C^*_{\max}(A)$ be the C^*-subalgebra of $B(\oplus_\pi H_\pi)$ generated by $\rho(A)$. For simplicity we describe the 'contractive' case, that is the Banach space rather than operator space variant of $C^*_{\max}(A)$. The compression map $B(\oplus_\pi H_\pi) \to B(H_\pi)$ is a $*$-homomorphism when restricted to $C^*_{\max}(A)$. It follows that $C^*_{\max}(A)$ is the 'biggest' C^*-cover of A.

Theorem 3.8. *Let A be a Jordan operator algebra. Then $C^*_{\max}(A)$ has the universal property that for every contractive Jordan representation $\pi : A \to B(H_\pi)$, there exists a unique $*$-homomorphism $\theta : C^*_{\max}(A) \to B(H_\pi)$ with $\theta \circ \rho = \pi$.*

We define $\mathrm{oa}_{\max}(A)$ to be the operator algebra generated by $\rho(A)$ inside $C^*_{\max}(A)$. Again we focus on the Banach space rather than operator space variant, if one needs the operator space version, simply replace word 'contractive' by 'completely contractive' below. This has the universal property.

Proposition 3.9. *For every contractive Jordan representation $\pi : A \to B(H_\pi)$, there exists a unique contractive homomorphism $\theta : \mathrm{oa}_{\max}(A) \to B(H_\pi)$ with $\theta \circ \rho = \pi$.*

It follows that if A is a Jordan subalgebra of an approximately unital operator algebra C (resp. of a C^*-algebra B), such that A generates C as an operator algebra (resp. B as a C^*-algebra), then there exists a unique contractive homomorphism θ from $\mathrm{oa}_{\max}(A)$ (resp. $C^*_{\max}(A)$) into C (resp. onto B) with $\theta(\rho(a)) = a$ for all $a \in A$. Similarly, if $j : A \to C$ is a contractive Jordan homomorphism and C is a closed Jordan subalgebra of a C^*-algebra B, then there exists a unique contractive homomorphism θ from $\mathrm{oa}_{\max}(A)$ (resp. $C^*_{\max}(A)$) into C (resp. into B) with $\theta(\rho(a)) = j(a)$ for all $a \in A$.

We now turn to the C^*-envelope, or 'minimal' C^*-cover of A.

Proposition 3.10. *If A be a unital Jordan operator algebra, then the $(C^*_e(A), i)$ has the following universal property: Let $j : A \to B$ be a completely isometric Jordan homomorphism into a C^*-algebra B generated by $j(A)$. Then there exists a necessarily unique and surjective $*$-homomorphism $\pi : B \to C^*_e(A)$ such that $\pi \circ j = i$.*

Proof. This follows directly from Proposition 2.21. □

If A is an approximately unital Jordan operator algebra we define $C^*_e(A)$ to be the C^*-algebra D generated by $j(A)$ inside $(C^*_e(A^1), j)$, where A^1 is the unitization. We will discuss this further after we have studied the unitization in the approximately unital case in the next section. Define $\mathrm{oa}_e(A)$ to be the operator algebra generated by $j(A)$ in $C^*_e(A)$.

Finally, there are universal JC*-algebra envelopes of a Jordan operator algebra A. Namely, consider the JC*-subalgebra of $C^*_{\max}(A)$ generated by A. This clearly has the universal property that for every contractive (again there is a completely contractive version that is almost identical) Jordan representation $\pi : A \to B(H_\pi)$, there exists a unique contractive Jordan $*$-homomorphism $\theta : C^*_{\max}(A) \to B(H_\pi)$ with $\theta \circ \rho = \pi$. If A is also approximately unital we may also consider the JC*-subalgebra of $C^*_e(A)$ generated by A. This will have a universal property similar to that a few paragraphs up, or in Proposition 3.22 below, but addressing JC*-algebras B generated by a completely isometric Jordan homomorphic copy of A.

3.6 Contractive approximate identities and consequences

Definition 3.11. If A is a Jordan operator subalgebra of a C^*-algebra B then we say that a net (e_t) in $\text{Ball}(A)$ is a *B-relative partial cai* for A if $e_t a \to a$ and $ae_t \to a$ for all $a \in A$. Here we are using the usual product on B, which may not give an element in A, and may depend on B. We say that a net (e_t) in $\text{Ball}(A)$ is a *partial cai* for A if for every C^*-algebra B containing A as a Jordan subalgebra, $e_t a \to a$ and $ae_t \to a$ for all $a \in A$, using the product on B. We say that A is *approximately unital* if it has a partial cai.

Nonetheless the existence of such a cai is independent of B, as we shall see.

Definition 3.12. If A is an operator algebra or Jordan operator algebra then we say that a net (e_t) in $\mathrm{Ball}(A)$ is a *Jordan cai* or *J-cai* for A if $e_t a + a e_t \to 2a$ for all $a \in A$.

Lemma 3.13. *If A is a Jordan operator subalgebra of a C^*-algebra B, then the following are equivalent:*

(i) *A has a partial cai.*

(ii) *A has a B-relative partial cai.*

(iii) *A has a J-cai.*

(iv) *A^{**} has an identity p of norm 1 with respect to the Jordan Arens product on A^{**}, which coincides on A^{**} with the restriction of the usual product in B^{**}. Indeed p is the identity of the von Neumann algebra $C^*_B(A)^{**}$.*

If these hold then p is an open projection in B^{**} in the sense of Akemann [1], *and any partial cai (e_t) for A is a cai for $C^*_B(A)$ (and for $\mathrm{oa}_B(A)$), and every J-cai for A converges weak* to p.*

Proof. That (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv) If p is a weak* limit point of (e_t), then in the weak* limit we have $pa + ap = 2a$ for all $a \in A$, hence for all $a \in A^{**}$. Thus p is an identity for the Jordan product of A^{**}. In particular, $p^2 = p$, and so p is an orthogonal projection in B^{**}. It then follows from (3.2.1) that $\eta p = p\eta = \eta$ for all $\eta \in A^{**}$ in the B^{**} product. So p is an identity in the B^{**} product on A^{**} (which may not map into A^{**} if A is not an operator algebra). By topology it now follows that every J-cai for A converges weak* to p.

(iv) $\Rightarrow$ (i) Suppose that p is an orthogonal projection in A^{**}. Then by (3.2.1), $\eta p = p\eta = \eta$ for all $\eta \in A^{**}$ iff p is an identity in the Jordan product on A^{**}. We may replace B with $D = C^*_{\max}(A)$, letting ρ be the usual inclusion of A in D. We may then follow a standard route to obtain a cai, see e.g. the last part of the proof of [14, Proposition 2.5.8]. That is we begin by choosing a net (x_t) in Ball(A) with $e_t \to p$ weak*. In D we have $ae_t \to a$ and $e_t a \to a$ weakly. Thus for any finite set $F = \{a_1, \cdots, a_n\} \subset A$ the zero vector is in the weak and norm closure in $D^{(2n)}$ of

$$\{(a_1 u - a_1, \cdots, a_n u - a_n, ua_1 - a_1, \cdots, ua_n - a_n) : u \in \Lambda\}.$$

From this one produces, by the standard method in e.g. the last part of the proof of [14, Proposition 2.5.8], a D-relative partial cai (e_t) for A formed from convex combinations. Suppose that B is any C^*-algebra containing A as a Jordan subalgebra via a completely isometric inclusion $i : A \to B$, such that $B = C^*_B(i(A))$. Then the existence of a canonical $*$-homomorphism $\theta : C^*_{\max}(A) \to B$ with $\theta \circ \rho = i$, gives $i(e_t)i(a) = \theta(\rho(e_t)\rho(a)) \to \theta(\rho(a)) = i(a)$ for $a \in A$, and similarly $i(a)i(e_t) \to i(a)$. So (e_t) is a partial cai for A.

If these hold, and if A is a Jordan subalgebra of a C^*-algebra B, then since $e_t = e_t p \to p$ weak*, we have as in the operator algebra case that p is open in B^{**}. Note that $C^*_B(A)$ is a C^*-algebra with cai (e_t) by [14, Lemma 2.1.6], and so $p = 1_{C^*_B(A)^{**}}$. $\square$

Remark 3.14. *It follows from the last result that any C^*-cover (B, j) of an approximately unital Jordan operator algebra A is a unital C^*-algebra if and only if A is unital. Indeed if B is unital then $1_B = \lim_t e_t \in A$. Similarly $oa_B(A)$ is unital if and only if A is unital.*

If A is a Jordan operator algebra in a C^*-algebra (resp. operator algebra) B and (e_t) is a partial cai for A, then it follows from the above that $\{T \in B : Te_t \to T, e_tT \to T\}$ is a C^*-algebra with cai (e_t) containing A.

Theorem 3.15. *If A is an approximately unital Jordan operator algebra then A is an M-ideal in A^1. Also $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$ and $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. Finally, A has a partial cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. As in the operator algebra case, if e is the identity of A^{**} viewed as a (central) projection in $(A^1)^{**}$, then multiplication by e is an M-projection from $(A^1)^{**}$ onto A^{**}. This gives the first assertion and the assertions about weak* density are identical to the proof in [19, Theorem 5.2]. One does need to know that for $x \in \mathfrak{r}_A$ we have $x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$, but this is easy as we can work in the operator algebra $\mathrm{oa}(x)$.

Note that the identity of A^{**} is in $\frac{1}{2}\mathfrak{F}_{A^{**}}$. Hence by the just established weak* density, it may be approximated by a net in $\frac{1}{2}\mathfrak{F}_A$. From this as in Lemma 3.13 (but taking $\Lambda = \frac{1}{2}\mathfrak{F}_A$) one may construct a cai in $\frac{1}{2}\mathfrak{F}_A$ in a standard way from convex combinations. Alternatively, one may copy the proof of Read's theorem in [9] to get this. $\square$

Remark 3.16. Indeed as in [20, Theorem 2.4] by taking nth roots one may find in any approximately unital Jordan operator algebra, a partial cai which is *nearly* *positive* in the sense described in the introduction of [22].

The following is a 'Kaplansky density' result for $\mathfrak{r}_{A^{**}}$:

Proposition 3.17. *Let A be an approximately unital Jordan operator algebra. Then the set of contractions in $\mathfrak{r}_A$ is weak* dense in the set of contractions in $\mathfrak{r}_{A^{**}}$.*

Proof. We showed in Theorem 3.15 that $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. The bipolar argument in [19, Proposition 6.4] (but replacing appeals to results in that paper by appeals to the matching results in the present paper) shows that $\mathrm{Ball}(A) \cap \mathfrak{r}_A$ is weak* dense in $\mathrm{Ball}(A^{**}) \cap \mathfrak{r}_{A^{**}}$. $\square$

Corollary 3.18. *If A is a Jordan operator algebra with a countable Jordan cai (f_n), then A has a countable partial cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. By Theorem 3.15, A has a partial cai (e_t) in $\frac{1}{2}\mathfrak{F}_A$. Choose t_n with $\|f_n e_{t_n} - f_n\| \vee \|e_{t_n} f_n - f_n\| < 2^{-n}$. It is easy to see that (e_{t_n}) is a countable partial cai in $\frac{1}{2}\mathfrak{F}_A$. $\square$

Proposition 3.19. *If A is a nonunital approximately unital Jordan operator algebra then the unitization A^1 is well defined up to completely isometric Jordan isomorphism; the matrix norms are*

$$\|[a_{ij} + \lambda_{ij} 1]\| = \sup\{\|[a_{ij} \circ c + \lambda_{ij} c\|_{M_n(A)} : c \in \mathrm{Ball}(A)\}, \qquad a_{ij} \in A, \lambda_{ij} \in \mathbb{C}\,.$$

Proof. Suppose that $\pi : A \to B(H)$ is a completely isometric Jordan homomorphism. Let B be the C^*-subalgebra of $B(H)$ generated by $\pi(A)$, let $K = \overline{BH}$, and let

$e = P_K$ be the projection onto K. It will be clear to C^*-algebraists, and follows easily from e.g. [14, Lemma 2.1.9], that $\|\pi(b) + \lambda I_H\| = \max\{|\lambda|, \|\pi(b)_{|K} + \lambda I_K\|\}$ for $b \in B, \lambda \in \mathbb{C}$, and similarly at the matrix level. Thus we may suppose that $H = K$. Note that $I_K \notin B$, since if it were then the identity e of $\pi(A)^{**}$ is the identity of B^{**} by Lemma 3.13, which is $I_K \in B$, so that $e \in B \cap \pi(A)^{**} = \pi(A)$, contradicting that A is nonunital. By Lemma 3.13, for any partial cai (e_t) for A, $(\pi(e_t))$ is a partial cai for $\pi(A)$ and hence it (and also $(\pi(e_t)^*)$) is a cai for B by Lemma 3.13. Since B acts nondegenerately on H it is clear that $\pi(e_t) \to I_H$ and $\pi(e_t)^* \to I_H$ strongly on H. It is easy to see that $\|[\pi(a_{ij}) + \lambda_{ij} I_H]\|$ equals

$$\sup\{|\sum_{i,j} \langle (\pi(a_{ij}) + \lambda_{ij} I_H)\zeta_j, \eta_i \rangle| = \sup\{\lim_t \langle (\pi(a_{ij}) \circ \pi(e_t) + \lambda_{ij}\pi(e_t))\zeta_j, \eta_i \rangle |\},$$

supremum over $\zeta, \eta \in \mathrm{Ball}(H^{(n)})$. This is dominated by

$$\sup_t \{\|[\pi(a_{ij} \circ e_t + \lambda_{ij} e_t)]\| \leq \sup \|[a_{ij} \circ c + \lambda_{ij} c]\| : c \in \mathrm{Ball}(A)\}.$$

In turn the latter equals

$$\sup \|[\pi(a_{ij}) \circ \pi(c) + \lambda_{ij}\pi(c)]\| : c \in \mathrm{Ball}(A)\} \leq \|[\pi(a_{ij}) + \lambda_{ij} I_H]\|.$$

This proves the assertion. □

Remark 3.20. (1) By the proof one may replace $\circ c$ in the last statement with $\circ e_t$, and take the supremum over all t.

(2) A similar argument, but replacing a term above by $\sup\{\lim_t \langle (\pi(a_{ij})\pi(e_t) +$

$\lambda_{ij}\pi(e_t))\zeta_j, \eta_i\rangle|$, shows that

$$\|[a_{ij} + \lambda_{ij}1]\| = \sup\{\|[a_{ij}c + \lambda_{ij}c]\|_{M_n(A)} : c \in \mathrm{Ball}(A)\}$$

where the product $a_{ij}c$ is with respect to (any fixed) containing C^*-algebra. One also has similar formulae with ac replaced by ca.

One question raised here is that is the unitization of a general Jordan operator algebra unique up to completely isometric isomorphism? The answer is in the negative, as one sees in the following result.

Consider the set E_2 of 4×4 matrices

$$\begin{bmatrix} 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \alpha, \beta \in \mathbb{C}\,.$$

This is not an associative algebra but is a Jordan operator algebra with zero Jordan product. That is, $xy+yx = 0$ for any two such matrices, which is an anticommutation relation. Let G be the space of the same matrices but with first column and last row removed, so that $G \subset M_3$. Let F_2 be the set of matrices in $M_2(G) \subset M_6$ which are zero in all of the first three columns and all of the last three rows. This is an operator algebra with zero product, which is linearly completely isometric to G. So $E_2 \cong F_2$ completely isometrically isomorphically as Jordan operator algebras, but not of course as associative operator algebras.

Proposition 3.21. *The two unitizations $\mathbb{C}\, I_4 + E_2$ and $\mathbb{C}\, I_6 + F_2$ of the Jordan operator algebra E_2 above, are not completely isometrically isomorphic as Jordan operator algebras, nor even as unital operator spaces.*

Proof. The C^*-envelope of $\mathbb{C}\, I_4 + E_2$ is at most 16 dimensional since $\mathbb{C}\, I_4 + E_2 \subset M_4$ (we shall not need this but it is easy to see that $C^*_e(\mathbb{C}\, I_4 + E_2)$ is M_4). We shall show that the dimension of the C^*-envelope of $\mathbb{C}\, I_6 + F_2$ is 18. Indeed $\mathbb{C}\, I_6 + F_2$ is (via a switch of columns and rows) completely isometrically isomorphic to the unital subalgebra A of $M_3 \oplus M_3$ consisting of matrices

$$\begin{bmatrix} \lambda & \alpha & \beta \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \lambda & 0 & -\beta \\ 0 & \lambda & \alpha \\ 0 & 0 & \lambda \end{bmatrix} \in M_3 \oplus M_3, \qquad \lambda, \alpha, \beta \in \mathbb{C}\,.$$

We claim that $C^*_e(A) = M_3 \oplus M_3$. Indeed the $*$-algebra generated by A contains $E_{12} + E_{56}, E_{13} - E_{46}, (E_{12} + E_{56})(E_{31} - E_{64}) = -E_{54}$, hence also $E_{45}, E_{45}(E_{12} + E_{56}) = E_{46}$, and thus E_{56}. Indeed it contains E_{ij} for $4 \leq i, j \leq 6$. Hence also $E_{12}, E_{1,3}$, indeed E_{ij} for $1 \leq i, j \leq 3$. So A generates $M_3 \oplus M_3$. Let J be a nontrivial ideal in $M_3 \oplus M_3$, such that the canonical map $A \to (M_3 \oplus M_3)/J$ is a complete isometry. We may assume $J = 0 \oplus M_3$, the contrary case being similar. That is the canonical compression of A to its first 3×3 block (that is, multiplication on A by $I_3 \oplus 0$) is completely isometric. Note that this implies that $R_2 \cap C_2$ is completely isometric to R_2, which is known to be false (e.g. [55, Theorem 10.5]). Thus $J = (0)$, so that by e.g. the last lines of 4.3.2 in [14], $C^*_e(A) = M_3 \oplus M_3$ as desired. $\square$

If A is an approximately unital Jordan operator algebra we define the C^*-*envelope* $C_e^*(A)$ to be the C^*-algebra D generated by $j(A)$ inside $(C_e^*(A^1), j)$, where A^1 is the unitization.

Proposition 3.22. *Let A be an approximately unital Jordan operator algebra, and let $C_e^*(A)$ and j be as defined above. Then $j_{|A}$ is a Jordan homomorphism onto a Jordan subalgebra of $C_e^*(A)$, and $C_e^*(A)$ has the following universal property: Given any C^*-cover (B, i) of A, there exists a (necessarily unique and surjective) $*$-homomorphism $\theta\colon B \to C_e^*(A)$ such that $\theta \circ i = j_{|A}$.*

Proof. Any completely isometric Jordan homomorphism $i : A \to B$ into a C^*-algebra B generated by $i(A)$, extends by the uniqueness of the unitization (see Proposition 3.19) to a unital completely isometric Jordan homomorphism $i^1 : A^1 \to B^1$. If $\theta : B^1 \to C_e^*(A^1)$ is the $*$-homomorphism coming from the universal property of $C_e^*(A^1)$ (see e.g. [14, Theorem 4.3.1]), then $\theta_{|B} : B \to D$ is a surjective $*$-homomorphism with $j_{|A} = \theta \circ i$. It also follows that $j(A)$ is a Jordan subalgebra of $C_e^*(A^1)$, and $j_{|A}$ is an approximately unital Jordan isomorphism onto $j(A)$. $\square$

3.7 Cohen factorization for Jordan modules

The Cohen factorization theorem is a crucial tool for Banach and operator algebras, and their modules. In this section we prove a variant that works for Jordan operator algebras and their 'modules'.

Definition 3.23. Let A be a Jordan operator algebra. A Banach space (resp. operator space) X together with a contractive (resp. completely contractive) bilinear map $A \times X \to X$ is called a left *Jordan Banach (resp. operator) A-premodule.* If A is approximately unital then we say that X is nondegenerate if $e_t x \to x$ for $x \in X$, where (e_t) is a cai for A (in this section when we say 'cai' we mean 'partial cai'. It will follow from the next theorem that if one cai for A works here then so will any other cai). Similar definitions hold in the 'right premodule' case, and a *Jordan Banach (resp. operator) A-prebimodule* is both a left and a right Jordan Banach (resp. operator) A-premodule such that $a(xb) = (ax)b$ for all $a, b \in A, x \in X$.

We remark that this definition is not related to the classical notion of a Jordan module due to Eilenberg (cf. [40, p. 512]) . A good example to bear in mind is the case where $X = C^*_e(A)$ or $X = C^*_{\max}(A)$.

If X is a nondegenerate Jordan Banach A-premodule (resp. A-prebimodule) then X is a Jordan Banach A^1-premodule (resp. A^1-prebimodule) for the natural unital 'action'. For example, if $b = a + \lambda 1 \in A^1$, and $x \in X$, then $\|(e_t \circ a)x - ax\| \leq \|x\|\|e_t \circ a - a\| \to 0$, and so

$$\|(e_t \circ b)x - bx\| \leq \|(e_t \circ a)x - ax\| + \|\lambda(e_t x - x)\| \to 0.$$

Hence $\|bx\| = \lim_t \|(e_t \circ b)x\| \leq \|x\|\|e_t \circ b\| \leq \|x\|\|b\|$. Similarly $\|xb\| \leq \|x\|\|b\|$ in the prebimodule case.

We say that such X is a (left) *Jordan Banach A-module* if

$$(a_1 a_2 \cdots a_m)x = a_1(a_2(\cdots(a_m x))\cdots)$$

for all $m \in \mathbb{N}$ and $a_1, \cdots, a_m \in A^1$ such that $a_1 a_2 \cdots a_m \in A^1$. The latter product is the one on some fixed containing C^*-algebra, for example $C^*_{\max}(A^1)$. In fact we shall only need the cases $m = 2, 3$ in the results below. Similar notation holds in the bimodule case, or for Jordan operator A-modules and bimodules.

The condition $a(xb) = (ax)b$ often holds automatically:

Proposition 3.24. *Suppose that X is an operator space, A is an approximately unital Jordan operator algebra, and that there are completely contractive bilinear maps $A \times X \to X$ and $X \times A \to X$ which are nondegenerate in the sense that $e_t x \to x$ and $x e_t \to x$ for $x \in X$, where (e_t) is a cai for A. Then $a(xb) = (ax)b$ for all $a, b \in A, x \in X$.*

Proof. The 'actions' are oplications in the sense of [14, Theorem 4.6.2], and by that theorem there are linear complete contractions $\theta : A \to \mathcal{M}_l(X)$ and $\pi : A \to \mathcal{M}_r(X)$ such that $ax = \theta(a)(x)$ and $xb = \pi(b)x$ for all $a, b \in A, x \in X$. Since left and right multipliers commute (see 4.5.6 in [14]), $(ax)b = a(xb)$ for such a, b, x. $\square$

The following is a Jordan algebra version of the Cohen factorization theorem:

Theorem 3.25. *If A is an approximately unital Jordan operator algebra, and if X is a nondegenerate Jordan Banach A-module (resp. A-bimodule), and if $b \in X$ then there exists an element $b_0 \in X$ and an element $a \in \mathfrak{F}_A$ with $b = ab_0$ (resp. $b = ab_0a$).*

Moreover if $\|b\| < 1$ then b_0 and a may be chosen of norm < 1. Also, b_0 may be chosen to be in the closure of $\{ab : a \in A\}$ (resp. $\{aba : a \in A\}$).

Proof. We follow the usual Cohen method as in the proof of e.g. 4.4 and 4.8 of [19]. Suppose that $b \in X$ with $\|b\| < 1$. Given any $\varepsilon > 0$, let $a_0 = 1$. Choose $f_1 \in \frac{1}{2}\mathfrak{F}_A$ from the cai such that

$$\|(ba_0^{-1})(1-f_1)\| + \|(1-f_1)(a_0^{-1}b)\| < 2^{-2}\varepsilon.$$

Let $a_1 = 2^{-1}f_1 + 2^{-1}$, then $a_1 \in \mathfrak{F}_{A^1}$. By the Neumann lemma a_1 is invertible in $\mathrm{oa}(1, a_1)$, and has inverse in A^1 with $\|a_1^{-1}\| \leq 2$. Similarly, choose $f_2 \in \frac{1}{2}\mathfrak{F}_A$ such that

$$\|(ba_1^{-1})(1-f_2)\| + \|(1-f_2)(a_1^{-1}b)\| < 2^{-4}\varepsilon.$$

By induction, let $a_n = \sum_{k=1}^{n} 2^{-k}f_k + 2^{-n}$. We have

$$\|1 - a_n\| = \|\sum_{k=1}^{n} 2^{-k}(1-f_k)\| \leq \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}.$$

By the Neumann lemma a_n is invertible in $\mathrm{oa}(1, a_n)$, and has inverse in A^1 with $\|a_n^{-1}\| \leq 2^n$. Choose $f_{n+1} \in \frac{1}{2}\mathfrak{F}_A$ such that

$$\|(ba_n^{-1})(1-f_{n+1})\| + \|(1-f_{n+1})(a_n^{-1}b)\| < 2^{-2(n+1)}\varepsilon.$$

Note that $a_{n+1}^{-1} - a_n^{-1} = a_n^{-1}(a_n - a_{n+1})a_{n+1}^{-1} = 2^{-n-1}a_n^{-1}(1-f_{n+1})a_{n+1}^{-1}$, and similarly $a_{n+1}^{-1} - a_n^{-1} = 2^{-n-1}a_{n+1}^{-1}(1-f_{n+1})a_n^{-1}$. Set $x_n = a_n^{-1}b$ (resp. $x_n = a_n^{-1}ba_n^{-1}$). We continue in the bimodule case, the left module case is similar but easier. We have

$$\begin{aligned} x_{n+1} - x_n = a_{n+1}^{-1} b a_{n+1}^{-1} - a_n^{-1} b a_n^{-1} &= a_{n+1}^{-1} b (a_{n+1}^{-1} - a_n^{-1}) + (a_{n+1}^{-1} - a_n^{-1}) b a_n^{-1} \\ &= 2^{-n-1} (a_{n+1}^{-1} b (a_n^{-1} (1 - f_{n+1}) a_{n+1}^{-1}) + (a_{n+1}^{-1} (1 - f_{n+1}) a_n^{-1}) b a_n^{-1}). \end{aligned}$$

Because of the relations $x(abc) = ((xa)b)c$ and $(abc)x = a(b(cx))$, $\|x_{n+1} - x_n\|$ is dominated by

$$\begin{aligned} &2^{-n-1} (\|a_{n+1}^{-1}\| \|(b a_n^{-1})(1 - f_{n+1})\| \|a_{n+1}^{-1}\| + \|a_{n+1}^{-1}\| \|(1 - f_{n+1})(a_n^{-1} b)\| \|a_n^{-1}\|) \\ &\leq 2^{-n-1} (\|a_{n+1}^{-1}\|^2 + \|a_{n+1}^{-1}\| \|a_n^{-1}\|) 2^{-2(n+1)} \varepsilon \\ &\leq 2^{-3(n+1)} (2^{2(n+1)} + 2^{2n+1}) \varepsilon < 2^{-n} \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Let $b_0 = \lim_n x_n$ and $a = \sum_{k=1}^{+\infty} 2^{-k} f_k$, then $a \in \frac{1}{2} \mathfrak{F}_A$. Hence, $b = a b_0 a$ since $b = a_n x_n a_n$ and $a_n \to a$ and $x_n \to b_0$. Also,

$$\|x_n - b\| \leq \sum_{k=1}^{n} \|x_k - x_{k-1}\| \leq 2\varepsilon,$$

so that $\|b - b_0\| \leq 2\varepsilon$. Thus $\|b_0\| \leq \|b\| + 2\varepsilon$, and this is < 1 if $2\varepsilon < 1 - \|b\|$. Choose some $t > 1$ such that $\|tb\| < 1$. By the argument above, there exists $a \in \frac{1}{2} \mathfrak{F}_A$ and $b_0 \in B$ of norm < 1 such that $tb = a b_0 a$. Let $a' = \frac{a}{\sqrt{t}}$, then $b = a' b_0 a'$. Then $\|a'\| < 1$ and $\|b_0\| < 1$. □

Corollary 3.26. *If A is an approximately unital Jordan operator subalgebra of a C^*-algebra B, and if B is generated as a C^*-algebra by A, then if $b \in B$ there exists an element $b_0 \in B$ and an element $a \in \mathfrak{F}_A$ with $b = a b_0 a$. Moreover if $\|b\| < 1$ then*

b_0 and a may be chosen of norm < 1. Also, b_0 may be chosen to be in the closure of $\{aba : a \in A\}$.

Proof. This follows immediately from the Cohen type Theorem 3.25 above. $\square$

There is a similar one-sided result using the one-sided version of our Cohen factorization result.

If A is a Jordan operator subalgebra of a C^*-algebra B then we say that a net (e_t) in $\mathrm{Ball}(A)$ is a *left B-partial cai* for A if $e_t a \to a$ for all $a \in A$. Here we are using the usual product on B, which may not give an element in A, and may depend on B. We then can factor any $b \in C^*_B(A)$ as ab_0 for a, b_0 as above, using the one-sided Cohen factorization result above. We remark that by a modification of the proof of Lemma 3.13 and Theorem 3.15 one can show that the following are equivalent:

(i) A has a left B-partial cai.

(ii) A^{**} has a left identity p of norm 1 with respect to the usual product in B^{**}.

(iii) A has a left B-partial cai in $\frac{1}{2}\mathfrak{F}_A$.

If these hold then p is an open projection in B^{**} in the sense of Akemann [1] (that is, is the weak* limit of an increasing net in B).

3.8 Jordan representations

Following the (associative) operator algebra case we have:

Lemma 3.27. *Let A be an approximately unital Jordan operator algebra and let $\pi : A \to B(H)$ be a contractive Hilbert space Jordan representation. We let P be the projection onto $K = [\pi(A)H]$. Then $\pi(e_t) \to P$ in the weak* (and WOT) topology of $B(H)$ for any J-cai (e_t) for A. Moreover, for $a \in A$ we have $\pi(a) = P\pi(a)P$, and the compression of π to K is a contractive Hilbert space Jordan representation. Also, if (e_t) is a partial cai for A, then $\pi(e_t)\pi(a) \to \pi(a)$ and $\pi(a)\pi(e_t) \to \pi(a)$. In particular, $\pi(e_t)_{|K} \to I_K$ SOT in $B(K)$.*

Proof. Let A be an approximately unital Jordan operator algebra and let $\pi : A \to B(H)$ be a contractive Hilbert space Jordan representation. If (e_t) is a J-cai for A then by the proof of Lemma 3.13 we have that $e_t \to p$ weak*, where p is an identity for A^{**}. The canonical weak* continuous extension $\tilde{\pi} : A^{**} \to B(H)$ takes p to a projection P on H, and $\pi(e_t) \to P$ WOT. Note that

$$\tilde{\pi}(p)\pi(a) + \pi(a)\tilde{\pi}(p) = P\pi(a) + \pi(a)P = 2\pi(a), \qquad a \in A,$$

so that as in the proof of Lemma 3.13 we have $P\pi(a) = \pi(a)P = \pi(a)$ for $a \in A$. We have $\pi(e_t)\pi(a) \to \tilde{\pi}(p)\pi(a) = \pi(a)$ WOT. If Q is the projection onto $[\pi(A)H]$ it follows that $PQ = Q$, so $Q \leq P$. If $\eta \perp \pi(A)H$ then $0 = \langle \pi(e_t)\zeta, \eta\rangle \to \langle P\zeta, \eta\rangle$, so that $\eta \perp P(H)$. Hence $P(H) \subset [\pi(A)H]$ and so $P \leq Q$ and $P = Q$. It is now evident that the compression of π to $[\pi(A)H]$ is a contractive Hilbert space Jordan representation.

Suppose that $\rho : A \to C^*_{\max}(A)$ is the canonical map. If (e_t) is a partial cai for A, then $\rho(e_t)\rho(a) \to \rho(a)$ and $\rho(a)\rho(e_t) \to \rho(a)$. If $\theta : D = C^*_{\max}(A) \to B(H_\pi)$ is the

$*$-homomorphism with $\theta \circ \rho = \pi$ then

$$\pi(e_t)\pi(a) = \theta(\rho(e_t)\rho(a)) \to \theta(\rho(a)) = \pi(a).$$

Similarly $\pi(a)\pi(e_t) \to \pi(a)$. $\square$

Definition 3.28. A *nondegenerate Jordan representation* of an approximately unital Jordan operator algebra on H is a contractive Hilbert space Jordan representation $\pi : A \to B(H)$ such that $\pi(A)H$ is dense in H.

Proposition 3.29. *Let $\pi : A \to B(H)$ be a Jordan representation. Then the canonical weak* continuous extension $\tilde{\pi} : A^{**} \to B(H)$ is unital iff π is nondegenerate.*

Proof. By the last result, we know that $\tilde{\pi}(1) = I_H$ iff $\pi(e_t) \to I_H$ weak* in $B(H)$ for any partial cai (e_t) of A, that is iff $\pi(e_t) \to I_H$ WOT. $\square$

Let H, K, π be as in Lemma 3.27. If we regard $B(K)$ as a subalgebra of $B(H)$ in the natural way (by identifying any T in $B(K)$ with the map $T \oplus 0$ in $B(K \oplus K^{\perp}) = B(H)$), then the Jordan homomorphism π is valued in $B(K)$. Note that π is nondegenerate when regarded as valued in $B(K)$, since $\pi(e_t)\pi(a) \to \pi(a)$ WOT. As in the (associative) operator algebra case [14], this yields a principle whereby to reduce a possibly degenerate Jordan homomorphism to a nondegenerate one.

Corollary 3.30. *For any approximately unital Jordan operator algebra A, there exist a Hilbert space H and a nondegenerate completely isometric Jordan homomorphism $\pi : A \to B(H)$.*

Corollary 3.31. *Let B be a C^*-cover of an approximately unital Jordan operator algebra A. If $\pi \colon B \to B(H)$ is a $*$-representation, then π is nondegenerate if and only if its restriction $\pi_{|A}$ is nondegenerate.*

Proof. Indeed, π is nondegenerate iff $\pi(e_t) \to I_H$ WOT where (e_t) is a partial cai for A, since then (e_t) is a cai for B. $\square$

3.9 Approximate identities and functionals

Following [14, Proposition 2.1.18] we have:

Lemma 3.32. *Let A be an approximately unital Jordan operator algebra with a partial cai (e_t). Denote the identity of A^1 by 1*

(1) *If $\psi : A^1 \to \mathbb{C}$ is a functional on A^1, then $\lim_t \psi(e_t) = \psi(1)$ if and only if $\|\psi\| = \|\psi_{|A}\|$.*

(2) *Let $\varphi : A \to \mathbb{C}$ be any functional on A. Then φ uniquely extends to a functional on A^1 of the same norm.*

Proof. (1) Suppose that $\psi : A^1 \to \mathbb{C}$ satisfies $\lim_t \psi(e_t) = \psi(1)$. For any $a \in A$ and $\lambda \in \mathbb{C}$, we have $\lim_t \psi(a \circ e_t + \lambda e_t) = \psi(a + \lambda 1)$, and so

$$|\psi(a+\lambda 1)| \leq \|\psi_{|A}\| \lim_t \|a \circ e_t + \lambda e_t\| \leq \|\psi_{|A}\| \sup_t \|a \circ e_t + \lambda e_t\| = \|\psi_{|A}\| \|a + \lambda 1\|.$$

(We have used Proposition 3.19.) Hence $\|\psi\| = \|\psi_{|A}\|$.

Conversely, suppose that $\|\psi\| = \|\psi_{|A}\|$, which we may assume to be 1. We may extend ψ to $C^*(A^1)$, and then there exists a unital $*$-representation $\pi : C^*(A^1) \to B(H)$ and vectors $\xi, \eta \in \mathrm{Ball}(H)$ with $\psi(x) = \langle \pi(x)\xi, \eta\rangle$ for any $x \in A^1$. Let $K = [\pi(A)\xi]$, and let p be the projection onto K. For any $a \in A$, we have $\langle \pi(a)\xi, \eta\rangle = \langle p\pi(a)\xi, \eta\rangle$, and so

$$|\langle \pi(a)\xi, \eta\rangle| = |\langle \pi(a)\xi, p\eta\rangle| \le \|\pi(a)\xi\|\|p\eta\| \le \|a\|\|p\eta\|.$$

This implies that $1 = \|\psi_{|A}\| \le \|p\eta\|$, so that $\eta \in K$. By Lemma 3.27 we have that $(\pi(e_t))$ converges WOT to the projection onto $[\pi(A)H]$, and so

$$\psi(e_t) = \langle \pi(e_t)\xi, \eta\rangle \to \langle \xi, p\eta\rangle = \langle \xi, \eta\rangle = \psi(1).$$

(2) If $\varphi \in A^*$ then similarly to the above there exists a nondegenerate $*$-representation $\pi : C^*(A) \to B(H)$ and vectors $\xi, \eta \in \mathrm{Ball}(H)$ with $\psi(x) = \langle \pi(x)\xi, \eta\rangle$ for any $x \in A$. We have $\psi(e_t) = \langle \pi(e_t)\xi, \eta\rangle \to \langle \xi, \eta\rangle$. We may now finish as in the proof of [14, Proposition 2.1.18 (2)]. $\square$

Define a *state* on an approximately unital Jordan operator algebra to be a functional satisfying the conditions in the next result.

Lemma 3.33. *For a norm* 1 *functional* φ *on an approximately unital Jordan operator algebra* A*, the following are equivalent:*

(1) φ *extends to a state on* A^1.

(2) $\varphi(e_t) \to 1$ *for every partial cai for* A.

(3) $\varphi(e_t) \to 1$ *for some partial cai for* A.

(4) $\varphi(e) = 1$ *where* e *is the identity of* A^{**}.

(5) $\varphi(e_t) \to 1$ *for every Jordan cai for* A.

(6) $\varphi(e_t) \to 1$ *for some Jordan cai for* A.

Proof. That (1) $\Rightarrow$ (2) follows from Lemma 3.32. That (3) implies (4), (6) $\Rightarrow$ (1), and (4) $\Leftrightarrow$ (5) follows from the last assertion of Lemma 3.13, that any Jordan cai for A converges to $1_{A^{**}}$. Clearly, (2) implies (3), and (5) implies (6). $\square$

Corollary 3.34. *Let* A *be an approximately unital Jordan operator algebra. Then any injective envelope of* A^1 *is an injective envelope of* A. *Moreover this may be taken to be a unital* C^**-algebra* $I(A)$ *containing* $C^*_e(A)$ *as a* C^**-subalgebra, and hence containing* A *as a Jordan subalgebra.*

Proof. This follows just as in [14, Corollary 4.2.8 (1) and (2)], but appealing to Lemma 3.32 above in place of the reference to 2.1.18 there. Note that $I(A^1)$ may be taken to be a unital C^*-algebra containing $C^*_e(A^1)$ as a unital C^*-subalgebra. The proof of Proposition 3.22 shows that the inclusion $A^1 \to C^*_e(A^1)$ is a Jordan morphism, so $I(A^1)$ contains $C^*_e(A)$ as a C^*-subalgebra, and A as a Jordan subalgebra. $\square$

It follows as in the introduction to [22] that states on A are also the norm 1 functionals that extend to a state on any containing C^*-algebra generated by A.

It follows from facts in Section 3.4 that for any Jordan operator algebra A, $x \in \mathfrak{r}_A$ iff $\mathrm{Re}(\varphi(x)) \geq 0$ for all states φ of A^1. Indeed, such φ extend to states on $C^*(A^1)$.

3.10 Multiplier algebras

Let A be an approximately unital Jordan operator algebra and let $(C_e^*(A), j)$ be its C^*-envelope. Let $i : A \to B$ be a completely isometric Jordan morphism into a C^*-algebra. Suppose that $a, b \in A$ and that $i(a)i(b) \in i(A)$. If $\theta : C_B^*(i(A)) \to C_e^*(A)$ is the $*$-homomorphism coming from the universal property then $j(a)j(b) = \theta(i(a)i(b)) \in \theta(i(A)) = j(A)$, and

$$j^{-1}(j(a)j(b)) = j^{-1}(\theta(i(a)i(b))) = i^{-1}(i(a)i(b)).$$

This shows that the 'product' in B of elements in A, if it falls in A, matches the product in $C_e^*(A)$. With this in mind we define the left multiplier algebra $LM(A)$ to be the set $\{\eta \in A^{**} : \eta A \subset A\}$, where the product here is the one in $C_e^*(A)^{**}$. We will soon see that this is in fact an (associative) algebra. We define the right multiplier algebra $RM(A)$ and multiplier algebra $M(A)$ analogously. If A is unital then these algebras are contained in A.

Lemma 3.35. *Let A be an approximately unital Jordan operator algebra. If p is a projection in $LM(A)$ then $p \in M(A)$. More generally, the diagonal $\Delta(LM(A)) \subset M(A)$.*

Proof. See [11, Lemma 5.1] for the operator algebra case. Let (e_t) be a partial cai for A. In $C_e^*(A)^{**}$ we have by [14, Lemma 2.1.6] that

$$ap = \lim_t ae_t^* p = \lim_t a(p^* e_t)^* \in C_e^*(A), \qquad a \in A.$$

Also $a \circ p \in A^{\perp\perp}$ so $ap = 2a \circ p - pa \in A^{\perp\perp}$. So $ap \in A^{\perp\perp} \cap C_e^*(A) = A$, and hence $p \in M(A)$. The same proof works if $p \in \Delta(LM(A))$. □

Theorem 3.36. *Let A be an approximately unital Jordan operator algebra and let $B = C_e^*(A)$, with A considered as a Jordan subalgebra. Then $LM(A) = \{\eta \in B^{**} : \eta A \subset A\}$. This is completely isometrically isomorphic to the (associative) operator algebra $\mathcal{M}_\ell(A)$ of operator space left multipliers of A in the sense of e.g.* [14, Section 4.5]*, and is completely isometrically isomorphic to a unital subalgebra of $CB(A)$. Also, $\|T\|_{\rm cb} = \|T\|$ for $T \in LM(A)$ thought of as an operator on A. Finally, for any nondegenerate completely isometric Jordan representation π of A on a Hilbert space H, the algebra $\{T \in B(H) : T\pi(A) \subset \pi(A)\}$ is completely isometrically isomorphic to a unital subalgebra of $LM(A)$, and this isomorphism maps onto $LM(A)$ if π is a faithful nondegenerate $*$-representation of B (or a nondegenerate completely isometric representation of $\mathrm{oa}_e(A)$).*

Proof. Obviously $LM(A) \subset \{\eta \in B^{**} : \eta A \subset A\}$. Conversely, if η is in the latter set then $\eta e_t \in A$, where (e_t) is partial cai for A. Hence $\eta \in A^{**}$, since by Lemma 3.13 (e_t) is a cai for B. So $LM(A) = \{\eta \in B^{**} : \eta A \subset A\}$.

Recall from Corollary 3.34 that $I(A)$ is a unital C^*-algebra. It follows from 4.4.13 and the proof of Theorem 4.5.5 in [14] that $\mathcal{M}_\ell(A)$ is completely isometrically isomorphic to $\{T \in I(A) : Tj(A) \subset j(A)\}$. Note that

$$Tj(a)^* = \lim_t Tj(e_t)j(a)^* \in j(A)j(A)^* \subset C_e^*(A).$$

Hence $T \in LM(C_e^*(A))$, and we may view $\mathcal{M}_\ell(A)$ as $\{T \in LM(C_e^*(A)) : Tj(A) \subset j(A)\}$. If $\eta \in C_e^*(A)^{**}$ and $\eta j(A) \subset j(A)$ then as in the last centered formula and the line after it, we have $\eta \in LM(C_e^*(A))$. So $\mathcal{M}_\ell(A) \cong \{\eta \in C_e^*(A)^{**} : Tj(A) \subset j(A)\}$. Thus from the last paragraph $LM(A) \cong \mathcal{M}_\ell(A)$. We remark that this may also be deduced from e.g. 8.4.1 in [14]. It also follows that for any $u \in \mathcal{M}_\ell(A)$, $\text{w}^*\lim_t u(e_t)$ exists in A^{**}, and equals $\sigma(u)$ where $\sigma : \mathcal{M}_\ell(A) \to LM(A)$ is the isomorphism above.

The canonical map $L : LM(A) \to CB(A)$ is a completely contractive homomorphism. On the other hand for $[\eta_{ij}] \in M_n(LM(A))$ we have

$$\|L(\eta_{ij})]\|_{M_n(CB(A))} \geq \|[\eta_{ij}e_t]\|.$$

It follows by Alaoglu's theorem that in the weak* limit with t, $\|[\eta_{ij}]\| \leq \|L(\eta_{ij})]\|_{M_n(CB(A))}$. Thus $LM(A)$ is completely isometrically isomorphic to a unital subalgebra of $CB(A)$. Note that

$$\|[\eta_{ij}a_{kl}]\| = \lim_t \|[\eta_{ij}e_t a_{kl}]\| \leq \sup_t \|[\eta_{ij}e_t]\|, \qquad [a_{kl}] \in \text{Ball}(M_m(A)),$$

so the last supremum equals the cb norm of $[\eta_{ij}]$ thought of as an element of $M_n(CB(A))$.

Let $LM(\pi) = \{T \in B(H) : T\pi(A) \subset \pi(A)\}$. There is a canonical complete contraction $LM(\pi) \to \mathcal{M}_\ell(A)$. Composing this with the map $\sigma : \mathcal{M}_\ell(A) \to LM(A)$ above gives a homomorphism $\nu : LM(\pi) \to LM(A)$. The canonical weak* continuous extension $\tilde{\pi} : A^{**} \to B(H)$ is a completely contractive Jordan homomorphism, and

$$\tilde{\pi}(\nu(T)) = \mathrm{w}^*\mathrm{lim}_t \pi(\pi^{-1}(T\pi(e_t))) = T, \qquad T \in LM(\pi),$$

by the nondegeneracy of π. It follows that ν is completely isometric.

If π is a faithful nondegenerate $*$-representation of B or a nondegenerate completely isometric representation of $\mathrm{oa}_e(A)$, and $T \in B(H)$ with $T\pi(A) \subset \pi(A)$ then as in the second paragraph of the proof above we have $T\pi(B) \subset \pi(B)$ or $T\pi(\mathrm{oa}_e(A)) \subset \pi(\mathrm{oa}_e(A))$. Thus in the first case we may identify $LM(\pi)$ with $\{\eta \in B^{**} : \eta A \subset A\}$, which we saw above was $LM(A)$. A similar argument works in the second case. $\square$

Definition 3.37. If A is a Jordan operator algebra, the *Jordan multiplier algebra* of A is

$$JM(A) = \{\eta \in A^{**} : \eta a + a\eta \in A, \forall a \in A\}.$$

This is a unital Jordan operator algebra in which A is an approximately unital Jordan ideal, follows by using the identity (3.2.2) in the obvious computation).

Remark 3.38. Presumably there is also a variant of this definition in terms of operators in $B(H)$, if $A \subset B(H)$ nondegenerately.

Note that $A = JM(A)$ if A is unital. If a projection $p \in A^{**}$ is in $JM(A)$ then $pAp \subset A$. This follows from the identity (3.2.3). Of course $M(A) \subset JM(A)$.

CHAPTER 4

Hereditary subalgebras, ideals, and open projections

4.1 Hereditary subalgebras and open projections

Through this section A is a Jordan operator algebra (possibly not approximately unital). Then A^{**} is a Jordan operator algebra.

Definition 4.1. A projection in A^{**} is *open* in A^{**}, or A-*open* for short, if $p \in (pA^{**}p \cap A)^{\perp\perp}$. That is, if and only if there is a net (x_t) in A with

$$x_t = px_tp \to p \ \text{ weak}^*.$$

This is a derivative of Akemann's notion of open projections for C^*-algebras, a key part of his powerful variant of noncommutative topology (see e.g. [1, 52]). If p is open in A^{**} then clearly

$$D = pA^{**}p \cap A = \{a \in A : a = pap\}$$

is a closed Jordan subalgebra of A, and the Jordan subalgebra $D^{\perp\perp}$ of A^{**} has identity p (note $x_t \in D$). By Lemma 3.13 D has a partial cai (even one in $\frac{1}{2}\mathfrak{F}_A$ by Theorem 3.15). If A is also approximately unital then a projection p in A^{**} is A-closed if $p^\perp$ is A-open.

We call such a Jordan subalgebra D a *hereditary subalgebra* (or HSA for short) of A, and we say that p is the *support projection* of D. It follows from the above that the support projection of a HSA is the weak* limit of any partial cai from the HSA. One consequence of this is that a projection in A^{**} is open in A^{**} if and only if it is open in $(A^1)^{**}$.

We remark that if A is a JC*-algebra then the net (x_t) above may be taken to be positive in the definition of hereditary subalgebra, or of open projections, and in many of the results below one may provide 'positivity proofs' as opposed to working with real positive elements.

Corollary 4.2. *For any Jordan operator algebra A, a projection $p \in A^{**}$ is A-open if and only if p is the support projection of a HSA in A.*

Proposition 4.3. *For any approximately unital Jordan operator algebra A, every projection p in the Jordan multiplier algebra $JM(A)$ (see* 3.37*) is A-open and also*

A-closed.

Proof. Indeed, if A is approximately unital and (e_t) is a partial cai of A, then $pA^{**}p \cap A = pAp$ by Remark 3.38, and $(pe_tp) \subset A$ has weak* limit p. So p is A-open. Similarly, $p^\perp$ is A-open since $p^\perp \in JM(A)$. $\square$

If B is a C^*-algebra containing a Jordan operator algebra A and p in A^{**} is A-open then p is open as a projection in B^{**} (since it is the weak* limit of a net (x_t) with $x_t = px_tp$, see [11]).

Definition 4.4. A Jordan subalgebra J of a Jordan operator algebra A is an *inner ideal* in the Jordan sense if for any $b, c \in J$ and $a \in A$, then $bac + cab \in J$ (or equivalently, $bAb \subset J$ for all $b \in J$).

Proposition 4.5. *A subspace D of a Jordan operator algebra A is a HSA if and only if it is an approximately unital inner ideal in the Jordan sense. In this case $D^{\perp\perp} = pA^{**}p$, where p is the support projection of the HSA. Conversely if p is a projection in A^{**} and E is a subspace of A^{**} such that $E^{\perp\perp} = pA^{**}p$, then E is a HSA and p is its A-open support projection.*

Proof. If D is a HSA, with $D = \{b \in A : pbp = b\}$, for some A-open projection $p \in A^{**}$, then for any $b, c \in D$ and $a \in A$, we have

$$bac + cab = pbacp + pcabp = p(bac + cab)p.$$

Hence $bac + cab \in D$. Thus D is an approximately unital inner ideal.

Conversely, if J is an approximately unital inner ideal, then $J^{\perp\perp}$ is a Jordan operator algebra with identity p say which is a weak* limit of a net in J. Clearly $J^{\perp\perp} \subseteq pA^{**}p$. By routine weak* density arguments $J^{\perp\perp}$ is an inner ideal, and so $J^{\perp\perp} = pA^{**}p$, and $J = pA^{**}p \cap A$. Hence p is open and J is an HSA. The last statement is obvious since by functional analysis $E = pA^{**}p \cap A$. □

Proposition 4.6. *If D_1, D_2 are HSA's in a Jordan operator algebra A, and if p_1, p_2 are their support projections, then $D_1 \subset D_2$ if and only if $p_1 \leq p_2$.*

Proof. ($\Rightarrow$) Note that $p_1 \in (D_1)^{\perp\perp} \subset (D_2)^{\perp\perp} = p_2A^{**}p_2$ and p_2 is an identity for D_2, so that $p_1 \leq p_2$.

($\Leftarrow$) If $p_1 \leq p_2$, then $D_1 = p_1A^{**}p_1 \cap A \subset D_2 = p_2A^{**}p_2 \cap A$. □

Proposition 4.7. *If J is an approximately unital closed Jordan ideal in a Jordan operator algebra A, then J is a HSA.*

Proof. Since J satisfies $axb + bxa \in J$ for $a, b \in J, x \in A$, then J is an inner ideal by Proposition 4.5. So J is a HSA. □

Corollary 4.8. *Let A be a Jordan operator algebra and (e_t) is a net in* $\mathrm{Ball}(A)$ *such that $e_te_s \to e_t$ and $e_se_t \to e_t$ with t (product in some C^*-algebra containing A). Then*

$$\{x \in A : xe_t \to x, e_tx \to x\}$$

is a HSA of A. Conversely, every HSA of A arises in this way.

Proof. Denote $J = \{x \in A : xe_t \to x, e_tx \to x\}$, product in some C^*-algebra D

containing A. It is easy to see that J is a Jordan subalgebra of A and (e_t) is a D-relative partial cai of J. So J is approximately unital. For any $x, y \in J$ and $a \in A$, then

$$\|(xay + yax)e_t - (xay + yax)\| = \|xa(ye_t - y) + ya(xe_t - x)\| \to 0.$$

Similarly, $e_t(xay + yax) \to (xay + yax)$. Hence, $xay + yax \in J$. By Proposition 4.5, J is a HSA.

Conversely, suppose that $D = pA^{**}p \cap A$, where p is an A-open projection. There exists a partial cai (e_t) of D with weak* limit p. Denote

$$J = \{x \in A : xe_t \to x, e_t x \to x\},$$

with product in some C^*-algebra containing A. Then $J \subset D = \{x \in A : pxp = x\}$. However clearly $D \subset J$. $\square$

As in [11, Theorem 2.10] we have:

Theorem 4.9. *Suppose that D is a hereditary subalgebra of an approximately unital Jordan operator algebra A. Then every $f \in D^*$ has a unique Hahn-Banach extension to a functional in A^* (of the same norm).*

Proof. Follow the proof of [11, Theorem 2.10], viewing A as a Jordan subalgebra of a C^*-algebra B, and working in B^{**}. If p is the support projection of D then since A^{**} is a unital Jordan algebra we have $p\eta(1-p) + (1-p)\eta p \in A^{**}$ for $\eta \in A^{**}$. The argument for [11, Theorem 2.10] then shows that $g(p\eta(1-p)+(1-p)\eta p)=0$ for any Hahn-Banach extension g of f. The rest of the proof is identical. $\square$

The analogue of [11, Proposition 2.11] holds too. For example, we have:

Corollary 4.10. *Let D be a HSA in an approximately unital Jordan operator algebra A. Then any completely contractive map T from D into a unital weak* closed Jordan operator algebra N such that $T(e_t) \to 1_N$ weak* for some partial cai (e_t) for D, has a unique completely contractive extension $\tilde{T} : A \to N$ with $\tilde{T}(f_s) \to 1_N$ weak* for some (or all) partial cai (f_s) for A.*

Proof. The canonical weak* continuous extension $\hat{T} : D^{**} \to N$ is unital and completely contractive, and can be extended to a weak* continuous unital complete contraction $\Phi(\eta) = \hat{T}(p\eta p)$ on A^{**}, where p is the support projection of D. This in turn restricts to a completely contractive $\tilde{T} : A \to N$ with $\tilde{T}(f_s) \to 1_N$ weak* for all partial cai (f_s) for A. For uniqueness, any other such extension $T' : A \to N$ extends to a weak* continuous unital complete contraction $\Psi : A^{**} \to N$, and $\Psi(p) = \lim_t \Phi(e_t) = 1_N$. Then Ψ extends further to a unital completely positive $\hat{\Psi} : R \to B(H)$ where R is a C^*-algebra containing A^{**} as a unital subspace, and where $N \subset B(H)$ unitally. Then for $\eta \in A^{**}$ we have, using Choi's multiplicative domain trick, that

$$\Psi(\eta) = \hat{\Psi}(p)\hat{\Psi}(\eta)\hat{\Psi}(p) = \hat{\Psi}(p\eta p) = \hat{T}(p\eta p).$$

Thus $T'(a) = \Phi(a) = \tilde{T}(a)$ for $a \in A$. $\square$

4.2 Support projections and HSA's

If p is a Hilbert space projection on a Hilbert space H, and x is any operator on H with $px + xp = 2x$, then $px = xp = x$ by (3.2.1). It follows that the 'Jordan support' (the smallest projection with $px + xp = 2x$) of a real positive operators x on H is the usual support projection of x in $B(H)$ if that exists (which means that the right and left support projections in $B(H)$ agree). This support projection does exist for real positive operators x, as is shown in [5, Section 3]. Indeed for a Jordan operator algebra A on H, if $x \in \mathfrak{r}_A$ and $px = x$ or $x = px$ for a projection p on H then it is easy to see that $pxp = x$ $(= px = xp)$. Thus the left and right support projections on H agree, and this will also be the smallest projection with $pxp = x$.

If x is an element of a Jordan subalgebra A we may also consider the *Jordan support projection* in A^{**}, if it exists, namely the smallest projection $p \in A^{**}$ such that $px + xp = 2x$. Recall that if the left and right support projections of x in A^{**} (that is the smallest projection in A^{**} such that $px = x$ or $xp = x$ respectively) coincide, then we call this the *support projection* of x, and write it as $s(x)$. If this holds, then $s(x)$ clearly also equals the Jordan support projection in A^{**}.

The following result is a Jordan operator algebra version of results in [20, Section 2].

Lemma 4.11. *For any Jordan operator algebra A, if $x \in \mathfrak{r}_A$, with $x \neq 0$, then the left support projection of x in A^{**} equals the right support projection equals the Jordan support projection, and also equals $s(\mathfrak{F}(x))$ where $\mathfrak{F}(x) = x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$. This also is the weak* limit of the net $(x^{\frac{1}{n}})$, and is an A-open projection in A^{**}, and is* *open in B^{**} in the sense of Akemann* [1] *if A is a Jordan subalgebra of a C^*-algebra B. If A is a Jordan subalgebra of $B(H)$ then the left and right support projection of x in H are also equal, and equal the Jordan support projection there.*

Proof. Viewing $\mathrm{oa}(x) \subset A$, as in the operator algebra case the identity e of $\mathrm{oa}(x)^{**}$ is a projection, and $e = \mathrm{w}^*\lim x^{\frac{1}{n}} \in \overline{xAx}^{w^*} \subset A^{**}$ with $ex = xe = x$. Also, any projection in B^{**} with $px = x$ or $xp = x$ satisfies $pe = e$. So e is the support projection $s(x)$ in A^{**} or in B^{**}, and by the discussion above the lemma also equals the Jordan support projection. It is A-open and open in the sense of Akemann since $x^{\frac{1}{n}} = ex^{\frac{1}{n}}e \to e$ weak*. To see the support projection equals $s(x(1+x)^{-1})$, simply note that $px = x$ if and only if $px(1+x)^{-1} = x(1+x)^{-1}$. That $\mathfrak{F}(x) = x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$ is as in the argument above [22, Lemma 2.5].

Suppose that $\pi : A^{**} \to B(H)$ is the natural weak*-continuous Jordan homomorphism extending the inclusion map on A. Then $\pi(e)$ is an orthogonal projection in $B(H)$ with

$$\pi(e)x + x\pi(e) = \pi(ex + xe) = \pi(2x) = 2x.$$

Then $\pi(e)x = x\pi(e) = x$ by (3.2.1), so that $P \leq \pi(e)$ where P is the Jordan support projection of x in $B(H)$. If $x_t \to p$ weak* with $x_t \in xAx$, then

$$P\pi(e) = \lim_t Px_t = \lim_t x_t = \pi(\lim_t x_t) = \pi(e),$$

so $\pi(e) \leq P$. Hence $\pi(e) = P$. That the left and right support projection of x in H are also equal to P for real positive x is discussed above the lemma. $\square$

Corollary 4.12. *If A is a closed Jordan subalgebra of a C^*-algebra B, and $x \in \mathfrak{r}_A$, then the support projection of x computed in A^{**} is the same, via the canonical embedding $A^{**} \cong A^{\perp\perp} \subset B^{**}$, as the support projection of x computed in B^{**}.*

If $x \in \mathfrak{F}_A$ for any Jordan operator algebra A then $x \in \mathrm{oa}(x)$, the closed (associative) algebra generated by x in A, and $x = \lim_n x^{\frac{1}{n}} x$, so that $\overline{xAx} = \overline{xA^1x}$.

Lemma 4.13. *For any Jordan operator algebra A, if $x \in \mathfrak{r}_A$, with $x \neq 0$, then $\overline{xAx}$ is a HSA, $\overline{xAx} = s(x)A^{**}s(x) \cap A$ and $s(x)$ is the support projection of $\overline{xAx}$. If $a = \mathfrak{F}(x) = x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$ then $\overline{xAx} = \overline{aAa}$. This HSA has $(x^{\frac{1}{n}})$ as a partial cai, and this cai is in $\mathfrak{r}_A$ (resp. in $\mathfrak{F}_A$, in $\frac{1}{2}\mathfrak{F}_A$) if x is real positive (resp. in $\mathfrak{F}_A$, in $\frac{1}{2}\mathfrak{F}_A$). If also $y \in \mathfrak{r}_A$ then $\overline{xAx} \subset \overline{yAy}$ if and only if $s(x) \leq s(y)$.*

Proof. Any cai (e_t) for $\mathrm{oa}(x)$ serves as a cai for the closure of xAx and the weak* limit of this cai is $s(x)$, then clearly $\overline{xAx} \subset s(x)A^{**}s(x) \cap A$. Since (e_t) converges weak* to $s(x)$, if $b \in s(x)A^{**}s(x) \cap A$ we have $e_t b e_t \to b$ weakly. By Mazur's theorem, a convex combination converges in norm, so $b \in \overline{xAx}$. As in operator algebra case, we know that $s(x) = s(a)$, which means that $\overline{xAx} = \overline{aAa}$.

The last assertion follows from the above and the remark after Proposition 4.5.

$\square$

Lemma 4.14. *Let A be an approximately unital Jordan operator algebra. If $x \in \mathfrak{F}_A$, then for any state φ of A, $\varphi(x) = 0$ if and only if $\varphi(s(x)) = 0$.*

Proof. Let $B = C^*(A)$, then states φ on A are precisely the restrictions of states on B. Continuing to write φ for canonical extension to A^{**}, if $\varphi(s(x)) = 0$, then by

Cauchy-Schwarz,

$$|\varphi(x)| = |\varphi(s(x)x)| \leq \varphi(s(x))^{\frac{1}{2}}\varphi(x^*x)^{\frac{1}{2}} = 0.$$

Conversely, if $\varphi(x) = 0$ then $\varphi(x^*x) \leq \varphi(x + x^*) = 0$, since $x \in \mathfrak{F}_A$. By Cauchy-Schwarz, $\varphi(ax) = 0$ for all $a \in A$. Since any cai for $\mathrm{oa}(x)$ converges to $s(x)$ weak* we have $\varphi(s(x)) = 0$. □

Lemma 4.15. *Let A be an approximately unital Jordan operator algebra. For $x \in \mathfrak{r}_A$, consider the conditions*

(i) $\overline{xAx} = A$.

(ii) $s(x) = 1_{A^{**}}$.

(iii) $\varphi(x) \neq 0$ *for every state of* A.

(iv) $\varphi(\mathrm{Re}(x)) > 0$ *for every state* φ *of* $C^*(A)$.

Then (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i). *If $x \in \mathfrak{F}_A$ all these conditions are equivalent.*

Proof. This as in [20, Lemma 2.10] and the discussion of the $\mathfrak{r}_A$ variant of that result above [22, Theorem 3.2]; part of it following from Lemma 4.14. □

An element in $\mathfrak{r}_A$ with $\mathrm{Re}(x)$ strictly positive in the C^*-algebraic sense as in (iv) of the previous result will be called *strictly real positive*. Many of the results on strictly real positive elements from [20, 22] will be true in the Jordan case, with the same proof. For example we will have the Jordan operator algebra version of [22, Corollary 3.5] that if x is strictly real positive then so is $x^{\frac{1}{k}}$ for $k \in \mathbb{N}$.

Lemma 4.16. *Let A be a Jordan operator algebra, a subalgebra of a C^*-algebra B.*

(1) *The support projection of a HSA D in A equals $\vee_{a \in \mathfrak{F}_D} s(a)$ (which equals $\vee_{a \in \mathfrak{r}_D} s(a)$).*

(2) *The supremum in B^{**} (or equivalently, in the diagonal $\Delta(A^{**})$) of any collection $\{p_i\}$ of A-open projections is A-open, and is the support projection of the smallest HSA containing all the HSA's corresponding to the p_i.*

Proof. Let $\{D_i : i \in I\}$ be a collection of HSA's in a Jordan operator algebra A. Let C be the convex hull of $\cup_{i \in I} \mathfrak{F}_{D_i}$, which is a subset of $\mathfrak{F}_A$. Let D be the closure of $\{aAa : a \in C\}$. Since any $a \in \mathfrak{F}_{D_i}$ has a cube root in $\mathfrak{F}_{D_i}$, $\mathfrak{F}_{D_i}$ and C are subsets of $\{aAa : a \in C\} \subset D$. We show that D is a subspace. If $a_1, a_2 \in C$ then $a = \frac{1}{2}(a_1 + a_2) \in C$. We have $s(a_1) \vee s(a_2) = s(a)$, since this is true with respect to $\mathrm{oa}(A)$ (this follows for example from [20, Proposition 2.14] and Corollary 2.6), and A^{**} is closed under meets and joins. Hence $a_1Aa_1 + a_2Aa_2 \subseteq s(a)A^{**}s(a) \cap A = \overline{aAa}$. So D is a subspace. Moreover D is an inner ideal since $abaAaba \subset aAa$ for $a \in C, b \in A$.

For any finite set $F = \{a_1, \cdots, a_n\} \subset \{aAa : a \in C\}$, a similar argument shows that there exists $a \in C$ with $F \subset aAa$. Hence $a^{\frac{1}{n}} a_k \to a_k$ and $a_k a^{\frac{1}{n}} \to a_k$ for all k. It follows that D is approximately unital, and is a HSA.

Clearly D is the smallest HSA containing all the D_i, since any HSA containing all the D_i would contain C and $\{aAa : a \in C\}$. If p is the support projection of

D then $f = \vee_{a \in C} s(a) \leq p$. Conversely, if $a \in C$ then $aAa \subset fA^{**}f$. Hence D and $D^{\perp\perp} = pA^{**}p$ are contained in $fA^{**}f$, so that $p \leq f$ and $p = f$. Of course $D = pA^{**}p \cap A$.

In particular, when I is singleton we see that the support projection of a HSA D equals $\vee_{a \in \mathfrak{F}_D} s(a)$. This proves (1) (using also the fact from Lemma 4.11 that $s(a) = s(\mathfrak{F}(a))$ for $a \in \mathfrak{r}_D$).

For (2), if p_i is the support projection of D_i above then $r = \vee_{i \in I} p_i \leq p$ clearly. On the other hand, if $a \in C$ is a convex combination of elements of $\mathfrak{F}_{D_{i_j}}$ for $j = 1, \cdots, m$, then $rar = a$, so that $s(a) \leq r$. This implies by the above that $p \leq r$ and $p = r$. So suprema in B^{**} of collections of A-open projections are A-open. The last assertion is clear from the above. $\square$

Remark 4.17. *The intersection of two Jordan inner ideals in a Jordan operator algebra A is a Jordan inner ideal, and is the largest Jordan inner ideal contained in the two (this is not true with 'inner ideals' replaced by HSA's, not even in the associative operator algebra case where this would correspond to a false statement about right ideals with left approximate identities–see [28, Section 5.4]).*

As in the operator algebra case, we may use $\mathfrak{r}_A$ and $\mathfrak{F}_A$ somewhat interchangably in most of the next several results. This is because of facts like: if $a \in \mathfrak{r}_A$ then a Jordan subalgebra of A contains a if and only if it contains $x = a(1+a)^{-1} \in \frac{1}{2}\mathfrak{F}_A$. Indeed $x \in \mathrm{oa}(a)$ and since $x + xa = a$ we have $a = x(1-x)^{-1} \in \mathrm{oa}(x)$ as in the proof of [22, Lemma 2.5] (the power series for $(1-x)^{-1}$ converges by the Neumann lemma since $\|x\| < 1$, as follows from [22, Lemma 2.5] with A replaced by $\mathrm{oa}(a)$).

Also, $\overline{xAx} = \overline{aAa}$ by Lemma 4.13.

Lemma 4.18. *For any Jordan operator algebra A, if $E \subset \mathfrak{r}_A$ then the smallest hereditary subalgebra of A containing E is $pA^{**}p \cap A$ where $p = \vee_{x \in E} s(x)$.*

Proof. By Lemma 4.16, $pA^{**}p \cap A$ is a hereditary subalgebra of A, and it contains (a_i). Conversely if D is a hereditary subalgebra of A containing (a_i) then $D^{\perp\perp}$ contains p by the usual argument, so $pA^{**}p \subset D^{\perp\perp}$ and $pA^{**}p \cap A \subset D^{\perp\perp} \cap A = D$. □

As in [20] (above Proposition 2.14 there), the correspondence between a HSA D and its support projection is a bijective order embedding from the lattice of HSA's of a Jordan operator algebra A and the lattice of A-open projections in A^{**} (see e.g. [47] for a JB-algebra variant of this). Write $Q(A)$ for the quasistate space of A, that is the set of states multiplied by numbers in $[0, 1]$. In the next several results we will be using facts from Section 3.9, namely that states on an approximately unital Jordan subalgebra A are restrictions of states on a containing C^*-algebra B.

Theorem 4.19. *Suppose that A is an approximately unital Jordan subalgebra of a C^*-algebra B. If p is a nontrivial projection in $A^{\perp\perp} \cong A^{**}$, then the following are equivalent:*

(i) *p is open in B^{**}.*

(ii) *The set $F_p = \{\varphi \in Q(A) : \varphi(p) = 0\}$ is a weak* closed face in $Q(A)$ containing 0.*

(iii) *p is lower semicontinous on $Q(A)$.*

These all hold for A-open projections in A^{**}, and for such projections $F_p = Q(A) \cap D^{\perp}$ where D is the HSA in A supported by p.

Proof. The first assertions are just as in [11, Theorem 4.1] (and the remark above that result), using the remark above the present theorem and fact that A-open projections are open with respect to a containing C^*-algebra. For the last assertion, if p is A-open then $Q(A) \cap D^{\perp} \subset F_p$ since a cai for D converges weak* to p. Conversely if $\varphi \in F_p$ then by the fact above the theorem φ extends to a positive functional on a C^*-cover of A. One may assume that this is a state, and use the Cauchy-Schwarz inequality to see that $|\varphi(x)| = |\varphi(px)| \leq \varphi(p)^{\frac{1}{2}} \varphi(x^*x)^{\frac{1}{2}} = 0$ for $x \in D$. □

If A is unital then there is a similar result and proof using the state space $S(A)$ in place of $Q(A)$.

Proposition 4.20. *Let A be an approximately unital Jordan operator algebra The correspondence $p \mapsto F_p$ is a one-to-one order reversing embedding from the A-open projections into the lattice of weak* closed faces of $Q(A)$ containing 0, thus $p_1 \leq p_2$ if and only if $F_{p_2} \subset F_{p_1}$ for A-open projections p_1, p_2 in A^{**}. Similarly there is a one-to-one order reversing embedding $D \mapsto F_p$ from the HSA's in A into the lattice of faces of $Q(A)$ above, where p is the support projection of the HSA D.*

Proof. Indeed an argument similar to the last part of the last proof shows that $F_{p_1} \subset F_{p_2}$ if $p_2 \leq p_1$. Conversely, if $F_{p_1} \subset F_{p_2}$, then by the argument above [20, Proposition 2.14] this implies a similar inclusion but in $Q(B)$, which by the C^*-theory gives $p_2 \leq p_1$. The last assertion follows from the first and the bijection between HSA's and their support projections. □

Corollary 4.21. *Let A be any Jordan operator algebra (not necessarily with an identity or approximate identity.) Suppose that (x_k) is a sequence in $\mathfrak{F}_A$, and that $\alpha_k \in (0,1]$ and $\sum_{k=1}^{\infty} \alpha_k = 1$. Then the HSA generated by all the $\overline{x_k A x_k}$ equals $\overline{zAz}$ where $z = \sum_{k=1}^{\infty} \alpha_k x_k \in \mathfrak{F}_A$. Equivalently (by Lemma 4.16), $\vee_k\, s(x_k) = s(z)$.*

Proof. This follows similarly to the operator algebra case in [20, Proposition 2,14]. If $x \in \mathfrak{F}_A$ then $\overline{xAx} = \overline{xA^1x}$ as we said above Lemma 4.13. So we may assume that A is unital. As in the operator algebra case $F_{s(z)} = \cap_{k=1}^{\infty} F_{s(x_k)}$, which implies by the lattice isomorphisms above the corollary that $\vee_k s(x_k) \leq s(z)$, and that the smallest HSA D containing all the $\overline{x_k A x_k}$ contains $\overline{zAz}$. Conversely, $z \in \sum_k \overline{x_k A x_k} \subset D$, so that $s(z) \leq \vee_k s(x_k)$, and so we have equality. $\square$

Theorem 4.22. (1) *If A is an associative operator algebra then the HSA's (resp. right ideals with left contractive approximate identities) in A are precisely the sets of form $\overline{EAE}$ (resp. $\overline{EA}$) for some $E \subset \mathfrak{r}_A$. The latter set is the smallest HSA (resp. right ideal with left approximate identity) of A containing E.*

(2) *If A is a Jordan operator algebra then the HSA's in A are precisely the sets of form $\overline{\{xAx : x \in \mathrm{conv}(E)\}}$ for some $E \subset \mathfrak{r}_A$. The latter set equals*

$$\overline{\{xay + yax : x, y \in \mathrm{conv}(E), a \in A\}},$$

and is the smallest HSA of A containing E (c.f. Lemma 4.18).

Proof. (1) One direction is obvious by taking E to be a real positive cai for the HSA or right ideal. For the other direction, we may assume that $E \subset \frac{1}{2}\mathfrak{F}_A$ by the argument in the first lines of the just mentioned proof. Note that $D = \overline{EAE}$ (resp. $\overline{EA}$) satisfies $DAD \subset D$ (resp. is a right ideal). For any finite subset $F \subset E$ if a_F is the average of the elements in F then $F \subset \overline{a_F A a_F}$ (resp. $F \subset \overline{a_F A}$) since $s(a_F) = \vee_{x \in E}\, s(x)$ by the operator algebra variant of Corollary 4.21. By a standard argument (e.g. seen in Lemma 4.16), it is easy to see that $(a_F^{\frac{1}{n}})$ will serve as the approximate identity we seek. Or we can find the latter by the method in the next paragraph. The last assertion is fairly obvious.

(2) If $x, y \in \mathrm{conv}(E)$ and $a, b \in A$ then $xax + yay \in \overline{zAz}$ where $z = \frac{1}{2}(x+y)$ by Corollary 4.21. So $D = \overline{\{xAx : x \in \mathrm{conv}(E)\}}$ is a closed inner ideal of A. It follows by the remark before Proposition 4.5 that if $x, y \in \mathrm{conv}(E)$ then $xaxbycy + ycybxax \in D$. Since $x \in \overline{xAx}$ and $y \in \overline{yAy}$ we have $D = \overline{\{xay + yax : x, y \in \mathrm{conv}(E), a \in A\}}$. If F, a_F are as in the proof of (1) then $a_F \in \mathrm{conv}(E)$, and $F \subset \overline{a_F A a_F} \subset D$. The HSA $\overline{a_F A a_F}$ has a J-cai in D, so that there exists $d_{\epsilon,F} \in \mathrm{Ball}(D)$ such that $\|d_{\epsilon,F} x + x d_{\epsilon,F} - 2x\| < \epsilon$ for all $x \in F$. Hence D has $(d_{\epsilon,F})$ as a J-cai. Again the final assertion is obvious. $\square$

Theorem 4.23. *Let A be a Jordan operator algebra (not necessarily with an identity or approximate identity.) The HSA's in A are precisely the closures of unions of an increasing net of HSA's of the form $\overline{xAx}$ for $x \in \mathfrak{r}_A$ (or equivalently, by an assertion in Lemma 4.13 for $x \in \mathfrak{F}_A$).*

Proof. Suppose that D is a HSA. The set of HSA's $\overline{a_F A a_F}$ as in the last proof, indexed by the finite subsets F of $\mathfrak{F}_D$, is an increasing net. Lemma 4.16 shows that the closure of the union of these HSA's is D. $\square$

A HSA D is called $\mathfrak{F}$-*principal* if $D = \overline{xAx}$ for some $x \in \mathfrak{F}_A$. By an assertion in Lemma 4.13 we can also allow $x \in \mathfrak{r}_A$ here. Corollary 4.21 says that the HSA generated by a countable number of $\mathfrak{F}$-principal HSA's is $\mathfrak{F}$-principal.

Theorem 4.24. *Let A be any Jordan operator algebra (not necessarily with an identity or approximate identity.) Every separable HSA or HSA with a countable cai is $\mathfrak{F}$-principal.*

Proof. If D is a HSA with a countable cai, then D has a countable partial cai $(e_n) \subset \frac{1}{2}\mathfrak{F}_D$. Also D is generated by the HSA's $\overline{e_nAe_n}$, and so D is $\mathfrak{F}$-principal by the last result. For the separable case, note that any separable approximately unital Jordan operator algebra has a countable cai. $\square$

Corollary 4.25. *If A is a separable Jordan operator algebra, then the A-open projections in A^{**} are precisely the $s(x)$ for $x \in \mathfrak{r}_A$.*

Proof. If A is separable then so is any HSA. So the result follows from Theorem 4.24, Lemma 4.13, and Corollary 4.12. $\square$

Theorem 4.26. *Let A be any approximately unital Jordan operator algebra. The following are equivalent*

(i) *A has a countable Jordan cai.*

(ii) *There exists $x \in \mathfrak{r}_A$ such that $A = \overline{xAx}$.*

(iii) *There is an element x in $\mathfrak{r}_A$ with $s(x) = 1_{A^{**}}$.*

(iv) *A has a strictly real positive element in $\mathfrak{r}_A$.*

If A is separable then these all hold.

Proof. The equivalence of (ii), (iii), and (iv) comes from Lemma 4.15 and the reasoning for [22, Theorem 3.2]. These imply (i) since (a scaling of) $(x^{\frac{1}{k}})$ is a countable partial cai. The rest follows from Theorem 4.24 applied to $A = D$. $\square$

Remark 4.27. We remark again that one may replace $\mathfrak{r}_A$ by $\mathfrak{F}_A$ in the last several results.

Theorem 4.28. *An approximately unital Jordan operator algebra with no countable Jordan cai, has nontrivial HSA's.*

Proof. If A has no countable cai then by Theorem 4.26 for any nonzero $x \in \mathfrak{F}_A$, we have $A \neq \overline{xAx}$. The latter is a nontrivial HSA in A. $\square$

We recall that an element x in a unital Jordan algebra A is invertible if there exists $y \in A$ with $x \circ y = 1$ and $x^2 \circ y = x$. If A is an associative algebra and $x \circ y = \frac{1}{2}(xy + yx)$ then it is known that this coincides with the usual definition. For a Jordan operator algebra the spectrum $\mathrm{Sp}_A(x)$ is defined to be the scalars λ such that $\lambda 1 - x$ is invertible in A^1, and as in the Banach algebra case can be shown to be a compact nonempty set, on whose complement $(\lambda 1 - x)^{-1}$ is analytic as usual, and the spectral radius is the usual $\lim_n \|x^n\|^{\frac{1}{n}}$. These facts are all well known in the theory of Jordan Banach algebras. For a general (possibly nonunital) Jordan operator algebra we say that x is *quasi-invertible* if $1 - x$ is invertible in A^1.

Theorem 4.29. *For a Jordan operator algebra A, if $x \in \mathfrak{r}_A$, the following are equivalent:*

(i) $x\,\mathrm{joa}(x)\,x$ *is closed.*

(ii) $\mathrm{joa}(x)$ *is unital.*

(iii) *There exists* $y \in \mathrm{joa}(x)$ *with* $xyx = x$.

(iv) xAx *is closed.*

(v) *There exists* $y \in A$ *such that* $xyx = x$. *Also, the latter conditions imply*

(vi) 0 *is isolated in or absent from* $\mathrm{Sp}_A(x)$.

Finally, if further $\mathrm{joa}(x)$ *is semisimple, then condition* (i)-(vi) *are all equivalent.*

Proof. If A is unital, $x \in \mathfrak{r}_A$, and x is invertible in A, then $s(x) = 1 \in \mathrm{joa}(x)$. Thus (ii) is true, and in this case (i)-(vi) are all obvious. So we may assume that x is not invertible in A. The equivalence of (i)–(iii) is the same as the operator algebra case (since $\mathrm{joa}(x) = \mathrm{oa}(x)$); and these imply (v).

(iv) $\Rightarrow$ (v) Suppose that xAx is closed. Now $x = (x^{\frac{1}{3}})^3$, and since $x = \lim_n x^{\frac{1}{n}}\, x x^{\frac{1}{n}}$ and $x^{\frac{1}{n}} \in \mathrm{oa}(x)$ we see that

$$x^{\frac{1}{3}} \in \mathrm{oa}(x) = \overline{x\mathrm{oa}(x)x} \subset \overline{xAx} = xAx,$$

and so $xyx = x$ for some $y \in A$.

(v) $\Rightarrow$ (iv) Similarly to the original proof of [20, Theorem 3.2],

$$xAx = (xyx)A(xyx) \subseteq xyAyx \subseteq xAx,$$

so that $xAx = xyAyx$. Since xy is an idempotent (in any containg C^*-algebra), xAx is closed.

Clearly (v) implies the same assertion but with A replaced by $C^*(A)$. This, by the theorem we are generalizing, implies (i)–(iii).

That (v) implies (vi) is as in [20, Theorem 3.2]: If 0 is not isolated in $\mathrm{Sp}_A(x)$, then there is a sequence of boundary points in $\mathrm{Sp}_A(x)$ converging to 0. Since $\partial \mathrm{Sp}_A(x) \subset \mathrm{Sp}_{B(H)}(x)$ as in the Banach algebra case, we obtain a contradiction. Similarly (vi) implies (ii) if $\mathrm{oa}(x)$ is semisimple: if the latter is not unital, then 0 is an isolated point in $\mathrm{Sp}_A(x)$ if and only if 0 is isolated in $\mathrm{Sp}_{\mathrm{oa}(x)}(x)$, and so we can use the original proof. $\square$

Theorem 4.30. *For a unital Jordan operator algebra A, the following are equivalent:*

(i) *A has no nontrivial HSA's (equivalently, A^{**} has no nontrivial open projections).*

(ii) $a^n \to 0$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\,1$.

(iii) *The spectral radius* $r(a) < \|a\|$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\,1$.

(iv) *The numerical radius* $v(a) < \|a\|$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\,1$.

(v) $\|1 + a\| < 2$ *for all* $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\,1$.

(vi) $\mathrm{Ball}(A) \setminus \mathbb{C}\,1$ *consists entirely of elements x which are quasi-invertible in A.*

If A has a partial cai then the following are equivalent:

(a) *A has no nontrivial HSA's.*

(b) A^1 *has one nontrivial HSA.*

(c) $\mathrm{Re}(x)$ *is strictly positive for every* $x \in \mathfrak{F}_A \setminus \{0\}$.

Proof. Clearly (ii)–(v) are equivalent by the theorem we are generalizing applied to $\mathrm{oa}(a)$.

(i) $\Rightarrow$ (vi) The HSA's are in bijective correspondence with the open projections in A^{**}, and these are the same thing as suprema of support projections of elements in $\mathfrak{F}_A$ (or equivalently of real positive elements in A by Lemma 4.16 (1). Thus, A has no nontrivial HSA's if and only if $s(x)$ is an identity for A^{**} for all $x \in \mathfrak{F}_A \setminus \{0\}$. Let $B = \mathrm{oa}(1, a)$ for $a \in \mathrm{Ball}(A) \setminus \{1\}$. Then $B \subset \overline{(1-a)A(1-a)} = A$, so that $(1-a)^{\frac{1}{n}}$ is an approximate identity for B, which must therefore converge to 1. It follows that $B = \mathrm{oa}(1-a)$, and so by the Neumann lemma (approximating 1) we have $1-a$ is invertible in B, hence in A, so that (vi) holds.

(vi) $\Rightarrow$ (i) Conversely $a \in \mathrm{Ball}(A) \setminus \{1\}$ quasi-invertible, with quasi-inverse a', implies that

$$(1-a)(1-a')A(1-a')(1-a) = A \subset (1-a)A(1-a) \subset A,$$

so that $s(1-a) = 1$. Hence (vi) implies that every nonzero element in $\mathfrak{F}_A$ has support projection 1.

(iii) $\Rightarrow$ (vi) If $r(a) < \|a\|$ for all $a \in \mathrm{Ball}(A) \setminus \mathbb{C}\,1$, then a is quasi-invertible, and its quasi-inverse is in $\mathrm{oa}(1, a)$.

(i) $\Rightarrow$ (v) If there exists $a \in \text{Ball}(A) \setminus \mathbb{C}\,1$ such that $\|1 + a\| = 2$, then $\|a\| = 1$. By the Hahn-Banach Theorem, $\exists\, \varphi \in \text{Ball}(A^*)$ such that $\varphi(1+a) = 2$, which means that $\varphi(1) = \varphi(a) = 1$. Hence φ is a state and $\varphi(1-a) = 0$. This implies $s(1-a) \neq 1$ by Lemma 4.14, which contradicts (i) by the relationship between HSA's and $s(x)$ mentioned at the start of the proof.

That (b) implies (a) is obvious. For the converse note that if $e = 1_{A^{**}}$ is the (central) support projection of A then for an open projection p in $(A^1)^{**}$, ep is open by [18, Proposition 6.4]. (We may write the 'non-Jordan expression' ep since e is central; thus if one likes such expessions below may be evaluated in any containing generated C^*-algebra.) Note that $1 - e$ is a minimal projection in $(A^1)^{**}$ since $(1-e)(a + \lambda 1) = \lambda(1-e)$ for all $a \in A, \lambda \in \mathbb{C}$. So if (a) holds then $ep = e$ or $ep = 0$, whence $p = e$ or $p = 1$, or $p = 0$ or $p = 1 - e$. The last of these is impossible, since if $x_t = (1-e)x_t \to 1 - e$ with $x_t \in A^1$, then $ex_t = 0$, which implies that x_t is in the kernel of the character on A^1 that annihilates A. So $x_t \in \mathbb{C}\,1$, giving the contradiction $1 - e \in \mathbb{C}\,1$. That (a) is equivalent to (c) directly follows from [26, Lemma 3.11] (which gives $\text{Re}(x)$ is strictly positive if and only if $s(x) = 1$) and the relationship between HSA's and $s(x)$ mentioned at the start of the proof. This equivalence holds for unital A too. $\square$

Remark 4.31. The last result has similar corollaries as in the associative algebra case. For example one may deduce that an approximately unital Jordan operator algebra with no countable Jordan cai, has nontrivial HSA's. To see this note that if A has no countable partial cai then by Theorem 4.15 there is no element $x \in \mathfrak{r}_A$ with $s(x) = 1_{A^{**}}$. Thus by the previous proof there are nontrivial HSA's in A.

The following are Jordan variants of Theorem 7.1 and Corollary 7.2 in [20].

Theorem 4.32. *If A is an approximately unital Jordan operator algebra which is a closed two-sided Jordan ideal in an operator algebra B, then $\overline{xAx}$ is a HSA in A for all $x \in \mathfrak{r}_B$.*

Proof. For any $x \in \mathfrak{r}_B$, then by the operator algebra case we know that $\mathrm{oa}(x)$ has a countable cai (e_n) say. Let $J = \overline{xAx}$, then $yAy \subset J$ for all $y \in J$. Suppose that (f_t) is a cai for A, then $(e_n f_t e_n) \in J$ is a Jordan cai for J by routine techniques. So $\overline{xAx}$ is a HSA in A by [26, Proposition 3.3]. $\square$

Corollary 4.33. *If A is an approximately unital Jordan operator algebra, and if η is a real positive element in the Jordan multiplier algebra of A, then $\overline{\eta A \eta}$ is a HSA in A.*

The following is the Jordan version of [20, Corollary 2.25]:

Corollary 4.34. *Let A be a unital Jordan subalgebra of C^*-algebra B and let $q \in A^{**}$ be a closed projection associated with an HSA D in A (that is, $D = \{a \in A : q^\perp a q^\perp = a\}$). Then an explicit Jordan cai for D is given by $x_{(u,\epsilon)} = 1 - a$, where a is an element which satisfies the conclusions of the noncommutative Urysohn theorem* 9.5 *in* [18], *for an open projection $u \geq q$, and a scalar $\epsilon > 0$. This Jordan cai is indexed by such pairs (u, ϵ), that is, by the product of the directed set of open projections $u \geq q$, and the set of $\epsilon > 0$. This Jordan cai is also in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. Certainly $x_{(u,\epsilon)} q = (1-a)q = q - q = 0$, and similarly $q x_{(u,\epsilon)} = 0$, so that $x_{(u,\epsilon)} \in D$. Also, $\|x_{(u,\epsilon)}\| \leq 1$, indeed $\|1 - 2x_{(u,\epsilon)}\| \leq 1$. The proof in [20, Corollary 2.25] shows that $x_{(u,\epsilon)}b \to b$ in B for $b \in \mathrm{Ball}(D)$. Similarly $bx_{(u,\epsilon)} \to b$. So $x_{(u,\epsilon)}b+bx_{(u,\epsilon)} \to 2b$ in A. Therefore, $(x_{(u,\epsilon)})$ is a Jordan cai for D. $\square$

4.3 M-ideals

Theorem 4.35. *Let A be an approximately unital Jordan operator algebra.*

(1) *The M-ideals in A are the complete M-ideals. These are exactly the closed Jordan ideals in A which are approximately unital.*

(2) *The M-summands in A are the complete M-summands. These are exactly the sets Ae for a projection e in $JM(A)$ (or equivalently in $M(A)$) such that e commutes with all elements in A). If A is unital then these are the closed Jordan ideals in A which possess a Jordan identity of norm 1.*

(3) *The right M-ideals in A are of the form $J = pA^{**} \cap A$, where p is a projection in $M(A^{**})$ with $J^{\perp\perp} = pA^{**}$. Each right M-ideal in A is a Jordan subalgebra with a left $C^*_e(A)$-partial cai.*

(4) *The right M-summands in A are exactly the sets pA for an idempotent contraction $p \in M(A)$.*

Proof. (4) By 4.5.15 in [14], the left M-projections are the projections in the left multiplier algebra $\mathcal{M}_\ell(A)$ of [14]. Hence, the right M-summands in A are exactly the sets pA for an idempotent contraction $p \in M_l(A) = LM(A)$. So p may be regarded as a projection in $LM(A)$. However by Lemma 3.35 any projection in $LM(A)$ is in $M(A)$.

(3) If J is a right M-ideal then $J^{**} = J^{\perp\perp} = \overline{J}^{w^*}$ is a right M-summand. Hence by (4), $J^{\perp\perp} = pA^{**}$, where p is a projection in $M(A^{**})$. Thus $J = J^{\perp\perp} \cap A = pA^{**} \cap A$. It follows that J is a Jordan subalgebra. Note that if $(e_t) \subset J$ with $e_t \to p$ weak*, then $e_t x \to px = x$ (products in $C_e^*(A)^{**}$) for all $x \in J$. Thus as in Lemma 3.13 a convex combination of the e_t are a left partial cai for J.

(2) If e is a projection in $M(A)$ commuting with A, then since $ea = eae \in A$ we see that left multiplication by e is in the algebra $M_l(A)$ mentioned above, and eA is a right M-summand by (4). Similarly $eA = Ae$ is a left M-summand by the left-handed version of (4). So eA is a complete M-summand by [14, Proposition 4.8.4 (2)].

Conversely, suppose that P is an M-projection on A. First suppose that A is unital. Set $z = P(1)$ and follow the proof of [14, Theorem 4.8.5 (2)], to see that z is Hermitian in A and $z^2 = z$, so that z is a projection in A. That argument goes on to show that if φ is any state with $P^*(\varphi) \neq 0$, and if $\psi = \frac{P^*(\varphi)}{\|P^*(\varphi)\|}$, then ψ is a state on A. As we said earlier, we can extend ψ to a state $\tilde{\psi}$ on some C^*-algebra generated by A. As in the argument we are following we obtain, for any $a \in A$, that $|\tilde{\psi}(a(1-z))|^2 \leq \tilde{\psi}(aa^*)\psi(1-z) = 0$, so that $\tilde{\psi}(a(1-z)) = 0$. Similarly, $\tilde{\psi}((1-z)a) = 0$. Hence, $\varphi(P(a(1-z) + (1-z)a)) = 0$. Since this holds for any state, we have $P(a(1-z) + (1-z)a) = 0$. Therefore, $(1-z) \circ A \subset (I-P)(A)$. By symmetry we have $z \circ A \subset P(A)$. If $a \in A$, then

$$a = \frac{az + za}{2} + \frac{a(1-z) + (1-z)a}{2},$$

so that $P(a) = \frac{az+za}{2} = z \circ a$. That P is idempotent yields the formula $zaz = z \circ a$. So z is central in A, and $az = za = zaz$, and $P(A) = zA$.

Next, if A is not unital consider the M-projection P^{**} on A^{**}. By the unital case $P^{**}(\eta) = z\eta = \eta z = z\eta z$ for all $\eta \in A^{**}$, for a central projection $z \in A^{**}$. We have $za + az = 2P(a) \in A$ for $a \in A$, so that $z \in M(A)$, and $P(A) = zA$.

Finally suppose that J is a closed Jordan ideal in A which possesses a Jordan identity e of norm 1. Then $ex = x = xe$ for all $x \in J$, as in the proof of Lemma 3.13. Also $eAe \subset J = eJe \subset eAe$. So $J = eAe$. Also $J = e \circ A$, so $ea + ae = eae + ae$, and so $ea = eae$. Similarly $ae = eae = ea$, so e is central in A. The rest is clear.

(1) If J is an approximately unital closed Jordan ideal in A, then $J^{\perp\perp}$ is by the usual approximation argument a unital weak* closed Jordan ideal in A^{**}. So by (2) we have $J^{\perp\perp}$ is the M-summand pA^{**} for a central projection $p \in A^{**}$. So J is an M-ideal. Conversely, if J is an M-ideal, then $J^{\perp\perp}$ is an M-summand in A^{**}. By (2), there exist a central projection $e \in A^{**}$ such that $J^{\perp\perp} = eA^{**}$ and $e \in M(A^{**})$. Note that $e \in J^{\perp\perp}$. By a routine argument similar to the associative case, J is a Jordan ideal with partial cai. $\square$

Corollary 4.36. *A subspace D in a Jordan operator algebra A is an approximately unital closed Jordan ideal in A if and only if there exists some open central projection p in A^{**}, such that $D = pA^{**}p \cap A$.*

Proof. If D is an approximately unital closed Jordan ideal in A then it is an approximately unital closed Jordan ideal in A^1. The proof of Theorem 4.35 (1) shows that $D^{\perp\perp} = pA^{**}p$, for a projection p in $D^{\perp\perp} \subset A^{**}$ (the weak* limit of a cai for D). Also p is central in $(A^1)^{**}$, hence in A^{**}. Clearly $D = pA^{**}p \cap A$, so p is open, and D is a HSA.

Conversely, if p is an open central projection in A^{**}, then p is an open central projection in $(A^1)^{**}$. Since $p \circ \eta = p\eta p \in A^{**}$ for $\eta \in A^{**}$, we have $D = pA^{**}p \cap A$ is a HSA and in particular is approximately unital. It is easy to see that D is a closed Jordan ideal since p is central. □

Proposition 4.37. *If J is an approximately unital closed two-sided Jordan ideal in a Jordan operator algebra A, then A/J is (completely isometrically isomorphic to) a Jordan operator algebra.*

Proof. Since $A/J \subset A^1/J$ we may assume that A is unital. By graduate functional analysis

$$A/J \subset A^{**}/J^{\perp\perp} = A^{**}/eA^{**} = e^{\perp}A^{**},$$

where e is the central support projection of J. The ensuing embedding $A/J \subset e^{\perp}A^{**}$ is the map $\theta(a + J) = e^{\perp}a$. Note that $\frac{1}{2}(ab + ba) + A$ maps to $\frac{1}{2}e^{\perp}(ab + ba) = \theta(a) \circ \theta(b)$. So θ is a completely isometric Jordan homomorphism into the Jordan operator algebra A^{**}, so A/J is completely isometrically isomorphic to a Jordan operator algebra. □

Clearly any approximately unital Jordan operator algebra A is an M-ideal in its unitization, or in $JM(A)$. As in [20, Proposition 6.1] we have:

Proposition 4.38. *If J is a closed Jordan ideal in a Jordan operator algebra A, and if J is approximately unital, then $q(\mathfrak{F}_A) = \mathfrak{F}_{A/J}$, where $q : A \to A/J$ is the quotient map.*

Proof. By Propositions 4.37 and 3.6 we can extend q to a contractive unital Jordan homomorphism from A^1 to a unitization of A/J, and then it is easy to see that $q(\mathfrak{F}_A) \subset \mathfrak{F}_{A/J}$.

For the reverse inclusion note that J is an M-ideal in A^1 by Theorem 4.35 (1). We may then proceed as in the proof of [20, Proposition 6.1]. Indeed suppose that $x \in A/J$ with $\|1 - x\| \leq 1$ in A^1/J. Since J is an M-ideal in A^1, it proximinal. Hence, there is an element $z = \lambda 1 + a$ in Ball $(A)^1$, with $\lambda \in \mathbb{C}$, $a \in A$ such that $\lambda 1 + a + J = 1 - x$. It is easy to see that $\lambda = 1$, and $a + J = -x$. Then $\|1 - (-a)\| = \|z\| \leq 1$, so $y \in \mathfrak{F}_A$, and $q(-a) = x$. $\square$

The following is the approximately unital Jordan version of [19, Corollary 8.9]. Below $W_A(x) = \{\varphi(x) : \varphi \in S(A)\}$ is the numerical range of x in A.

Proposition 4.39. *Suppose that J is an approximately unital Jordan ideal in a unital Jordan operator algebra A. Let $x \in A/J$ with $K = W_{A/J}(x)$. Then*

(i) *If K is not a nontrivial line segment in the plane, then there exists $a \in A$ with $a + J = x$, $\|a\| = \|x\|$, and $W_A(a) = W_{A/J}(x)$.*

(ii) *If $K = W_{A/J}(x)$ is a nontrivial line segment, let $\hat{K}$ be any thin triangle with*

K as one of the sides. Then there exists $a \in A$ with $a + J = x$, $\|a\| = \|x\|$, and $K \subset W_A(a) \subset \hat{K}$.

Proof. For (i) note that if the numerical range of x is not a singleton nor a line segment then this follows from [19, Theorem 8.8] since J is an M-ideal. If the numerical range of an element w in a unital Jordan algebra is a singleton then the same is true with respect to the generated C^*-algebra, so that $w \in \mathbb{C}\,1$.

If $W_{A/J}(x)$ is a nontrivial line segment K, this works just as in the proof of [19, Corollary 8.9]. Replace A by the unital Jordan algebra $B = A \oplus^\infty \mathbb{C}$, replace J by the approximately unital Jordan ideal $I = J \oplus (0)$. Then I is an M-ideal in B by Theorem 4.35 (1). For a scalar λ chosen as in the proof of [19, Corollary 8.9], $W((x, \lambda))$ is the convex hull of K and λ, hence has nonempty interior. Since I is an M-ideal in B, we may appeal to [19, Theorem 8.8] in the same way as in the proof to obtain our result. □

Corollary 4.40. *If J is an approximately unital Jordan ideal in any (not necessarily approximately unital) Jordan operator algebra A, then $q(\mathfrak{r}_A) = \mathfrak{r}_{A/J}$ where $q : A \to A/J$ is the quotient map.*

Proof. From Proposition 4.38, we know that $q(\mathfrak{F}_A) = \mathfrak{F}_{A/J}$, and $\mathfrak{r}_A = \overline{\mathbb{R}_+\,\mathfrak{F}_A}$ by the proof of [21, Theorem 3.3]. So $q(\mathbb{R}_+\,\mathfrak{F}_A) = \mathbb{R}_+\,\mathfrak{F}_{A/J}$. Taking closures we have $q(\mathfrak{r}_A) \subset \mathfrak{r}_{A/J}$.

The other direction uses Proposition 4.39. If A is unital the result immediately follows from Proposition 4.39 as in [19, Corollary 8.10]. If A is nonunital and A/J

is nonunital, then by Proposition 3.6 and Proposition 4.37 , we can extend q to a contractive unital Jordan homomorphism from A^1 to a unitization of A/J. Then J is still an M-ideal in A^1 by Theorem 4.35 (1). Therefore, again the result follows as in [19, Corollary 8.10] by applying the unital case to the canonical map from A^1 onto $(A/J)^1 = A^1/J$. (The latter formula following from Proposition 3.6.) If A/J is unital, then one may reduce to the previous case where it is not unital, by the trick for that in the proof of [19, Corollary 8.10]. □

Theorem 4.41. *Suppose that A is a Jordan operator algebra, $b \in A$, and p is an open projection in A^{**} commuting with b (see the introduction for the definition of this), such that $\|p^\perp b p^\perp\| \leq 1$ (here $p^\perp = 1-p$ where 1 is the identity of the unitization of A if A is nonunital). Suppose also that $\|p^\perp(1-2b)p^\perp\| \leq 1$. Then there exists $g \in \frac{1}{2}\mathfrak{F}_A \subset \mathrm{Ball}(A)$ commuting with p such that $p^\perp g p^\perp = p^\perp b p^\perp$. Indeed such g may be chosen 'nearly positive' in the sense of the introduction to* [22].

Proof. In the present setting the algebra D in the proof of [22, Theorem 4.10] is written as $\{x \in A \cap C : q \circ x = 0\}$ where $q = 1 - p$. So D equals, by facts mentioned in [18, Theorem 8.7],

$$\{x \in A \cap C : x = pxp\} = A \cap C \cap \tilde{D} = A \cap \tilde{D} = \{x \in A : x = pxp\},$$

the HSA in A with support p. So D is approximately unital. Now it is easy to chech the Jordan variant of the last few lines of the proof of [22, Theorem 4.10], using Proposition 4.37 in place of the analogous result used there. □

Lemma 4.42. *Suppose that A and B are closed Jordan subalgebras of unital Jordan* *operator algebras C and D respectively, with $1_C \notin A$ and $1_D \notin B$, and $q : A \to B$ is a 1-quotient map (that is, induces an isometry $A/\ker(q) \to B$) and Jordan homomorphism such that $\ker(q)$ is approximately unital. Then the unique unital extension of q to a unital map from $A + \mathbb{C}\,1_C$ to $B + \mathbb{C}\,1_D$ is a 1-quotient map.*

Proof. Let $J = \ker(q)$, let $\tilde{q} : A/J \to B$ be the induced isometry, and let $\theta : A + \mathbb{C}\,1_C \to B + \mathbb{C}\,1_D$ be the unique unital extension of q. This gives a one-to-one Jordan homomorphism $\tilde{\theta} : (A + \mathbb{C}\,1_C)/J \to B + \mathbb{C}\,1_D$ which equals $\tilde{q}$ on A/J. If B, and hence A/J, is not unital, then $\tilde{\theta}$ is an isometric Jordan isomorphism. Similarly, if B is unital, then $\tilde{\theta}$ is an isometric Jordan isomorphism by the uniqueness of the unitization of an already unital Jordan operator algebra. So in either case we may deduce that $\tilde{\theta}$ is an isometric Jordan isomorphism and θ is a 1-quotient map. $\square$

The following generalizes part of Corollary 4.40.

Theorem 4.43. (A noncommutative Tietze theorem) *Suppose that A is a Jordan operator algebra (not necessarily approximately unital), and that p is an open projection in A^{**}. Set $q = 1 - p \in (A^1)^{**}$. Suppose that $b \in A$ commutes with p, and $\|qbq\| \leq 1$, and that the numerical range of qbq (in $q(A^1)^{**}q$ or $(A^1)^{**}$) is contained in a compact convex set E in the plane. We also suppose, by fattening it slightly if necessary, that E is not a line segment. Then there exists $g \in \mathrm{Ball}(A)$ commuting with p, with $qgq = qbq$, such that the numerical range of g with respect to A^1 is contained in E.*

Proof. Similar remarks as in the proof of Theorem 4.41 apply here; except in addition one must use Proposition 4.39 and Lemma 4.42 in place of the analogous result used in the proof of [22, Theorem 4.12]. □

4.4 More on real positivity in Jordan operator algebras

The $\mathfrak{r}$-*ordering* is simply the order $\preccurlyeq$ induced by the above closed cone; that is $b \preccurlyeq a$ if and only if $a - b \in \mathfrak{r}_A$. If A is a Jordan subalgebra of a Jordan operator algebra B, we mentioned earlier that $\mathfrak{r}_A \subset \mathfrak{r}_B$. If A, B are approximately unital Jordan subalgebras of $B(H)$ then it follows from the fact that $A = \mathfrak{r}_A - \mathfrak{r}_A$ (see Theorem 4.44) and similarly for B that $A \subset B$ if and only if $\mathfrak{r}_A \subset \mathfrak{r}_B$. As in [20, Section 8], $\mathfrak{r}_A$ contains no idempotents which are not orthogonal projections, and no nonunitary isometries. In [21] it is shown that $\overline{\mathfrak{c}_A} = \mathfrak{r}_A$. Also $\mathfrak{r}_A$ contains no nonzero elements with square zero. Indeed if $(a + ib)^2 = a^2 - b^2 + i(ab + ba) = 0$ with $a \geq 0$ and $b = b^*$ then $a^2 = b^2$ so that a and b commute. Hence $ab = 0$ and $a^4 = a^2 b^2 = 0$. So $a = b = 0$.

Theorem 4.44. *Let A be a Jordan operator algebra which generates a C^*-algebra B, and let $\mathcal{U}_A$ denote the open unit ball $\{a \in A : \|a\| < 1\}$. The following are equivalent:*

(1) *A is approximately unital.*

(2) *For any positive $b \in \mathcal{U}_B$ there exists $a \in \mathfrak{r}_A$ with $b \preccurlyeq a$.*

(2') *Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$ and nearly positive.*

(3) *For any pair* $x, y \in \mathcal{U}_A$ *there exist nearly positive* $a \in \frac{1}{2}\mathfrak{F}_A$ *with* $x \preccurlyeq a$ *and* $y \preccurlyeq a$.

(4) *For any* $b \in \mathcal{U}_A$ *there exist nearly positive* $a \in \frac{1}{2}\mathfrak{F}_A$ *with* $-a \preccurlyeq b \preccurlyeq a$.

(5) *For any* $b \in \mathcal{U}_A$ *there exist* $x, y \in \frac{1}{2}\mathfrak{F}_A$ *with* $b = x - y$.

(6) $\mathfrak{r}_A$ *is a generating cone (that is,* $A = \mathfrak{r}_A - \mathfrak{r}_A$*).*

(7) $A = \mathfrak{c}_A - \mathfrak{c}_A$.

Proof. This is very similar to the proof of [22, Theorem 2.1] .

(1)$\Rightarrow$ (2') Let (e_t) be a partial cai for A in $\frac{1}{2}\mathfrak{F}_A$. By [14, Theorem 2.1.6], (e_t) is a cai for B, and hence so is (e_t^*), and $f_t = \mathrm{Re}(e_t)$. By Corollary 3.25, theorem, we may write $b^2 = zwz$, where $0 \leq w \leq 1$ and

$$z = \sum_{k=1}^{\infty} 2^{-k} f_{t_k} = \mathrm{Re}(\sum_{k=1}^{\infty} 2^{-k} e_{t_k}),$$

where f_{t_k} are some of the f_t. If $a = \sum_{k=1}^{\infty} 2^{-k} e_{t_k} \in \frac{1}{2}\mathfrak{F}_A$, then $z = \mathrm{Re}(a)$. Then $b^2 \leq z^2$, so that $b \leq z$ and $b \preceq a$. We also have $b \preceq a^{1/n}$ for each $n \in \mathbb{N}$ by[5, Proposition 4.7], which gives the nearly positive assertion.

(2')$\Rightarrow$ (3). By C^*-algebra theory there exists a positive $b \in \mathcal{U}_B$ with x and y dominated by b. Then apply (2').

(3)$\Rightarrow$ (4). Apply (3) to b and $-b$.

(4)$\Rightarrow$ (6). $b = \frac{a+b}{2} - \frac{a-b}{2} \in \mathfrak{r}_A - \mathfrak{r}_A$.

(6)$\Rightarrow$ (1). Suppose that A is a Jordan subalgebra of $B(H)$. Each $x \in \mathfrak{r}_A$ has a support projection $p_x \in B(H)$, which is just the weak* limit of $(x^{1/n})$, and hence is in A. Then $p = \vee_{x \in \mathfrak{r}_A} p_x$ is in A^{**} since A^{**} is closed under meets and joins of projections. For any $x \in \mathfrak{r}_A$ we have

$$pxp = xp = px = ps(x)x = s(x)x = x.$$

Since $\mathfrak{r}_A$ is generating, we have $px = x$ for all $x \in A$. It implies that A is unital, and hence A is approximately unital by Lemma 3.13.

(1) $\Rightarrow$(5). Assume that $\|x\| = 1$. Since $\mathfrak{F}_{A^{**}} = 1_{A^{**}} + \mathrm{Ball}(A^{**})$, $x = \eta - \xi$ for $\eta, \xi \in \frac{1}{2}\mathfrak{F}_{A^{**}}$. We may assume that A is nonunital. By Theorem 3.15, we deduce that x is the weak closure of the convex set $\frac{1}{2}\mathfrak{F}_A - \frac{1}{2}\mathfrak{F}_A$. Therefore it is in the norm closure, given $\varepsilon > 0$, there exists $a_0, b_0 \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0)\| < \frac{\varepsilon}{2}$. Similarly, there exists $a_1, b_1 \in \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0) - \frac{\varepsilon}{2}(a_1 - b_1)\| < \frac{\varepsilon}{2^2}$. Continuing in this manner, one produce sequences $(a_k), (b_k)$ in $\frac{1}{2}\mathfrak{F}_A$. Setting $a' = \sum_{k=1}^{\infty}(1/2^k)a_k$ and $b' = \sum_{k=1}^{\infty}(1/2^k)b_k$, which are in $\frac{1}{2}\mathfrak{F}_A$. We have $x = (a_0 - b_0) + \varepsilon(a' - b')$. Let $a = a_0 + \varepsilon a'$ and $b = b_0 + \varepsilon b'$. By convexity, we have $x = (a_0 - b_0) + \varepsilon(a' - b')$. Let $a = a_0 + \varepsilon a'$ and $b = b_0 + \varepsilon b'$. By convexity, $1/(1+\varepsilon)a \in \frac{1}{2}\mathfrak{F}_A$ and $1/(1+\varepsilon)b \in \frac{1}{2}\mathfrak{F}_A$.

If $\|x\| < 1$, choose $\varepsilon > 0$, with $\|x\|(1+\varepsilon) < 1$. Then $x/\|x\| = a - b$ as above, so that $x = \|x\|a - \|x\|b$. We have

$$\|x\|a = (\|x\|(1+\varepsilon)) \cdot (\frac{1}{1+\varepsilon}a) \in [0,1)\frac{1}{2}\mathfrak{F}_A \subset \frac{1}{2}\mathfrak{F}_A.$$

and similarly, $\|x\|b \in \frac{1}{2}\mathfrak{F}_A$.

It is obvious that (2') implies (2), and that (5) implies (7), which implies (6).

(2)⇒(6). If $a \in A$ then by C^*-algebra theory and (2) there exists $b \in B_+$ and $x \in \mathfrak{r}_A$ with $-x \preceq -b \preceq a \preceq b \preceq x$. Thus $a = \frac{a+x}{2} - \frac{x-a}{2} \in \mathfrak{r}_A - \mathfrak{r}_A$. $\square$

For the next results let A_H be the closure of the set $\{aAa : a \in \mathfrak{F}_A\}$. We shall show that A_H is the largest approximately unital Jordan subalgebra of A.

Corollary 4.45. *For any Jordan operator algebra A, the largest approximately unital Jordan subalgebra of A is*

$$\mathfrak{r}_A - \mathfrak{r}_A = \mathfrak{c}_A - \mathfrak{c}_A.$$

In particular these spaces are closed and form a HSA of A.

If A is a weak closed Jordan operator algebra then this largest approximately unital Jordan subalgebra is qAq where q is the largest projection in A. This is weak* closed.*

Proof. The proof of Lemma 4.16 (see also Theorem 4.22 (2) with $E = \mathfrak{F}_A$) yields A_H is the HSA $pA^{**}p \cap A$ where $p = \vee_{a \in \mathfrak{F}_A} s(a)$ is A-open. Similarly, A_H is the closure of the set $\{aAa : a \in \mathfrak{r}_A\}$. As in the proof of [21, Theorem 4.2 and Corollary 4.3] we have that A_H is the largest approximately unital Jordan subalgebra of A and $\mathfrak{F}_A = \mathfrak{F}_{A_H}$ and $\mathfrak{r}_A = \mathfrak{r}_{A_H}$. By Theorem 4.44 we have $A_H = \mathfrak{r}_{A_H} - \mathfrak{r}_{A_H} = \mathfrak{r}_A - \mathfrak{r}_A$, and similarly $A_H = \mathfrak{c}_{A_H} - \mathfrak{c}_{A_H} = \mathfrak{c}_A - \mathfrak{c}_A$.

The final assertion follows just as in [22, Corollary 2.2]. $\square$

As in [22, Lemma 2.3], and with the same proof we have:

Lemma 4.46. *Let A be any Jordan operator algebra. Then for every $n \in \mathbb{N}$,*

$$M_n(A_H) = M_n(A)_H \ , \quad \mathfrak{r}_{M_n(A)} = \mathfrak{r}_{M_n(A_H)} \ , \quad \mathfrak{F}_{M_n(A)} = \mathfrak{F}_{M_n(A_H)}.$$

If $S \subset \mathfrak{r}_A$, for a Jordan operator algebra A, write joa(S) for the smallest closed Jordan subalgebra of A containing S.

Proposition 4.47. *If S is any subset of $\mathfrak{r}_A$ for a Jordan operator algebra A, then* joa(S) *is approximately unital.*

Proof. Let $C =$ joa(S). Then $\mathfrak{r}_C = C \cap \mathfrak{r}_A$. If $x \in S$ then $x \in \mathfrak{r}_C = \mathfrak{r}_{C_H} \subset C_H$. So $C \subset C_H \subset C$, since C_H is a Jordan operator algebra containing S. Hence $C = C_H$, which is approximately unital. $\square$

Lemma 4.48. *For any Jordan operator algebra A, the $\mathfrak{F}$-transform $\mathfrak{F}(x) = 1 - (x + 1)^{-1} = x(x+1)^{-1}$ maps $\mathfrak{r}_A$ bijectively onto the set of elements of $\frac{1}{2}\mathfrak{F}_A$ of norm < 1. Thus $\mathfrak{F}(\mathfrak{r}_A) = \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$.*

Proof. This follows from part of the discussion above Lemma 4.18. $\square$

We recall that the positive part of the open unit ball of a C^*-algebra is a directed set, and indeed is a net which is a positive cai for B (see e.g. [52]). As in [22, Proposition 2.6 and Corollary 2.7], we have:

Proposition 4.49. *If A is an approximately unital Jordan operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set in the $\preccurlyeq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is an increasing partial cai for A.*

Corollary 4.50. *Let A be an approximately unital Jordan operator algebra, and B a C^*-algebra generated by A. If $b \in B_+$ with $\|b\| < 1$ then there is an increasing partial cai for A in $\frac{1}{2}\mathfrak{F}_A$, every term of which dominates b (where 'increasing' and 'dominates' are in the $\preccurlyeq$ ordering).*

Remark 4.51. *Any Jordan operator algebra A with a countable cai, and in particular any separable approximately unital Jordan operator algebra A, has a commuting partial cai which is increasing (for the $\preccurlyeq$ ordering), and also in $\frac{1}{2}\mathfrak{F}_A$ and nearly positive. Namely, by Theorem 4.26 we have $A = \overline{xAx}$ for some $x \in \frac{1}{2}\mathfrak{F}_A$, and $(x^{\frac{1}{n}})$ is a commuting partial cai which is increasing by [5, Proposition 4.7].*

A ($\mathbb{C}$-)linear map $T : A \to B$ between Jordan operator algebras is *real positive* if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$. We say that T is *real completely positive* or RCP if the nth matrix amplifications T_n are each real positive. It is clear from properties of $\mathfrak{r}_A$ mentioned earlier, that restrictions of real positive (resp. RCP) maps to Jordan subalgebras (or to unital operator subspaces) are again real positive (resp. RCP).

Corollary 4.52. *Let $T : A \to B$ be a linear map between approximately unital Jordan operator algebras, and suppose that T is real positive (resp. RCP). Then T is bounded (resp. completely bounded). Moreover T extends to a well defined positive map $\tilde{T} : A + A^* \to B + B^* : a + b^* \mapsto T(a) + T(b)^*$.*

Proof. This is as in [22, Corollary 2.9]. Note that T^{**} is real positive (using Theorem

3.15), and hence by the proof of [5, Theorem 2.5] it extends to a positive map on an operator system. Indeed it is completely positive, hence completely bounded, in the matrix normed case. Then restrict to $A + A^*$. $\square$

Remark 4.53. Similar results hold on unital operator spaces. With the same proof idea any real positive linear map on a unital operator space A extends to a well defined positive map on $A + A^*$. It is easy to see that a unital contractive linear map on a unital operator space is real positive (this follows e.g. from the fact that $\mathfrak{r}_A = \overline{\mathbb{R}_+(1 + \mathrm{Ball}(A))}$ in this case).

Theorem 4.54 (Extension and Stinespring Dilation for RCP Maps)**.** *If $T : A \to B(H)$ is a linear map on an approximately unital Jordan operator algebra, and if B is a C^*-algebra containing A, then T is RCP if and only if T has a completely positive extension $\tilde{T} : B \to B(H)$. This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : B \to B(K)$, and an operator $V : H \to K$. Moreover, this can be done with $\|T\| = \|T\|_{cb} = \|V\|^2$, and this equals $\|T(1)\|$ if A is unital.*

Proof. The structure of this proof follows the analogous results in [5, 20]. Indeed, in the proof of Corollary 4.52, T is completely bounded and T^{**} extends to a completely positive map $A^{**} + (A^{**})^* \to B(H)$. By Arveson's extension theorem [3], we may extend further to a completely positive map $\tilde{T} : B^{**} \to B(H)$ and $\tilde{T} = V^*\pi(\cdot)V$ for a $*$-representation $\pi : B^{**} \to B(K)$. Restricting $\tilde{T}$ and π to be we obtain the desired extension $\tilde{T} = V^*\pi_B(\cdot)V$.

The last assertion, about the norm follows immediately in the unital case, since it is well known for completely positive maps on C^*-algebras, and all of our extensions preserve norms. If A is an Jordan operator algebra with partial cai (e_t), and $B = C^*(A)$, then $T(e_t) \to \tilde{T}(1)$ weak*. Thus, $\|\tilde{T}(1)\| \leq \sup_t \|T(e_t)\|$ by Alaoglu's theorem. Consequently, by the unital space, $\|T\|_{cb} \leq \|\tilde{T}\|_{cb} = \|\tilde{T}(1)\| = \|V\|^2 \leq \|T\|$, and so $\|T\| = \|T\|_{cb} = \sup_t \|T(e_t)\|$. $\square$

An $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ is said to be *real positive* if $\varphi(\mathfrak{r}_A) \subset [0, \infty)$. By the usual trick, for any $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$, there is a unique $\mathbb{C}$-linear $\tilde{\varphi} : A \to \mathbb{C}$ with Re $\tilde{\varphi} = \varphi$, and clearly φ is real positive (resp. bounded) if and only if $\tilde{\varphi}$ is real positive (resp. bounded).

As in [22, Corollary 2.8], and with the same proof we have:

Corollary 4.55. *Let A be an approximately unital Jordan operator algebra, and B a C^*-algebra generated by A. Then every real positive $\varphi : A \to \mathbb{R}$ extends to a real positive real functional on B. Also, φ is bounded.*

We will write $\mathfrak{c}_{A^*}^{\mathbb{R}}$ for the real dual cone of $\mathfrak{r}_A$, the set of continuous $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ such that $\varphi(\mathfrak{r}_A) \subset [0, \infty)$. Since $\overline{\mathfrak{c}_A} = \mathfrak{r}_A$, we see that $\mathfrak{c}_{A^*}^{\mathbb{R}}$ is also the real dual cone of $\mathfrak{c}_A$. It is a proper cone; for if $\rho, -\rho \in \mathfrak{c}_{A^*}^{\mathbb{R}}$ then $\rho(a) = 0$ for all $a \in \mathfrak{r}_A$. Hence $\rho = 0$ by the fact above that the norm closure of $\mathfrak{r}_A - \mathfrak{r}_A$ is A.

Lemma 4.56. *Suppose that A is an approximately unital Jordan operator algebra. The real dual cone $\mathfrak{c}_{A^*}^{\mathbb{R}}$ equals $\{t \operatorname{Re}(\psi) : \psi \in S(A),\, t \in [0, \infty)\}$. It also equals the set of restrictions to A of the real parts of positive functionals on any C^*-algebra containing (a copy of) A as a closed Jordan subalgebra. The prepolar of $\mathfrak{c}_{A^*}^{\mathbb{R}}$, which equals its real predual cone, is $\mathfrak{r}_A$; and the polar of $\mathfrak{c}_{A^*}^{\mathbb{R}}$, which equals its real dual* *cone, is $\mathfrak{r}_{A^{**}}$. Thus the second dual cone of $\mathfrak{r}_A$ is $\mathfrak{r}_{A^{**}}$, and hence $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$.*

The following results are the approximately unital Jordan versions of most of 2-6–2.8 in [19].

Proposition 4.57. *Let A be an approximately unital nonunital Jordan operator algebra. Then $Q(A)$ is the weak* closure of $S(A)$. Also, a functional $f \in \mathfrak{c}_{A^*}$ if and only if f is a nonnegative multiple of a state. That is, an approximately unital nonunital Jordan operator algebra is scaled.*

Proof. That $Q(A)$ is the weak* closure of $S(A)$ will follow because this is true for C^*-algebras and because of the fact that states of A are precisely the restrictions to A of states on $C^*(A)$. The last assertion follows from the fact above that $Q(A)$ is weak* closed and the argument for [19, Lemma 2.7 (1)]. Alternatively, if f is real positive, then $\mathrm{Re}\, f$ is real positive (i.e. $\mathrm{Re}\, f(\mathfrak{r}_A) \subset [0, \infty)$), which implies by [26, Lemma 4.13] that $\mathrm{Re}\, f = \lambda\, \mathrm{Re}\, \varphi$ for some $\lambda \geq 0$ and $\varphi \in S(A)$. Therefore, $f = \lambda\varphi$ by the uniqueness of the extension of $\mathrm{Re}\, f$. $\square$

Corollary 4.58. *If A is an approximately unital nonunital Jordan operator algebra, then:*

(i) *$S(A^1)$ is the convex hull of trivial character χ_0 on A^1 (which annihilates A) and the set of states on A^1 extending states of A.*

(ii) $Q(A) = \{\varphi_{|A} : \varphi \in S(A^1)\}$.

Proof. These follow easily from the fact that they are true for C^*-algebras and states of A are precisely the restrictions to A of states on $C^*(A)$. Or one may deduce them e.g. from the fact that A is scaled as in the proof of [19, Lemma 2.7]. $\square$

Lemma 4.59. (Cf. [19, Lemma 6.6].) *Suppose A is an approximately unital Jordan operator algebra.*

(1) *The cones $\mathfrak{c}_{A^*}$ and $\mathfrak{c}_{A^*}^{\mathbb{R}}$ are additive (that is, the norm on A^* is additive on these cones).*

(2) *If (φ_t) is an increasing net in $\mathfrak{c}_{A^*}^{\mathbb{R}}$ which is bounded in norm, then the set converges in norm, and its limits is the least upper bound of the net.*

Proof. This is as in [19, Lemma 6.6], however one needs to appeal to [26, Lemma 4.13] in place of the matching result used there. $\square$

Corollary 4.60. *Let A be an approximately unital Jordan operator algebra. If $f \leq g \leq h$ in $B(A, \mathbb{R})$ in the natural 'dual ordering' induced by $\preceq$, then $\|g\| \leq \|f\| + \|h\|$.*

Proof. For any $x \in A$ with $\|x\| < 1$, then by Theorem 4.33 (5) there exists $a, b \in \frac{1}{2}\mathfrak{F}_A \subset \mathrm{Ball}(A)$ such that $x = a - b$. If $g(x) \geq 0$, then

$$g(x) = g(a) - g(b) \leq h(a) - f(b) \leq \|h\| + \|f\|.$$

If $g(x) \leq 0$, then

$$|g(x)| = g(b) - g(a) \leq h(b) - f(a) \leq \|h\| + \|f\|.$$

Therefore, $\|g\| \leq \|f\| + \|h\|$. $\square$

Following on Kadison's Jordan algebraic Banach–Stone theorem for C^*-algebras [41], many authors have proved variants for objects of 'Jordan type'. The following variant on the main result of Arazy and Solel [2] is a 'Banach–Stone type theorem for Jordan operator algebras'.

Proposition 4.61. *Suppose that $T : A \to B$ is an isometric surjection between approximately unital Jordan operator algebras. Then T is real positive if and only if T is a Jordan algebra homomorphism. If these hold and in addition T is completely isometric then T is real completely positive.*

Proof. If $T : A \to B$ is an isometric surjective Jordan algebra homomorphism between unital Jordan operator algebras, then T is unital hence real positive by a fact in Remark 4.53. If A and B are nonunital (possibly non-approximately unital) then T extends to a unital isometric surjective Jordan algebra homomorphism between the unitizations, hence is real positive again by the unital case.

Conversely, suppose that T is real positive. Again we may assume that A and B are unital, since by the usual arguments T^{**} is a real positive isometric surjection between unital Jordan operator algebras. Then the result follows from [16, Proposition 6.6]. If T is a completely isometric surjective Jordan algebra homomorphism then by Proposition 3.19, T extends to a unital completely isometric surjection between the unitizations, which then extends by Arveson's lemma e.g. [14, Lemma 1.3.6] to a unital completely contractive, hence completely positive, map on $A + A^*$. So T is real completely positive. $\square$

We close with a final Banach–Stone type theorem.

Proposition 4.62. *Suppose that $T : A \to B$ is a completely isometric surjection between approximately unital operator algebras. Then there exists a completely isometric surjective homomorphism $\pi : A \to B$, and a unitary u with $u, u^* \in M(B)$ with $T = u\pi(\cdot)$.*

Proof. See [14, Theorem 4.5.13]. □

Remark 4.63. The Jordan version of Banach-Stone type theorem in [26] was stated incorrectly and it will be fixed in [17].

CHAPTER 5

Operator $*$-spaces

5.1 Characterization of operator $*$-spaces

Definition 5.1. A Banach space X is a *Banach $*$-space* when it comes equipped with a conjugate linear involution $\dagger : X \to X$ such that

(a) $(x^\dagger)^\dagger = x, \quad \forall x \in X,$

(b) $\|x^\dagger\| = \|x\|, \quad \forall x \in X.$

Example 5.2. Let H^2 be the Hardy space $H^2(\mathbb{D})$, where $\mathbb{D}$ is the unit disc. The involution on H^2 is defined by

$$f^\dagger(z) = \overline{f(\overline{z})}, \text{ for any } f \in H^2.$$

Note that $(f^\dagger)^\dagger = f$ and $\|f^\dagger\| = \lim_{r\nearrow 1}\|f_r^\dagger\|_2 = \lim_{r\nearrow 1}\|f_r\|_2 = \|f\|$, so that H^2 becomes a Banach $*$-space.

Definition 5.3. An operator space X is an *operator $*$-space* when it comes equipped with an involution (which is conjugate linear) $\dagger : X \to X$ such that

(i) $(x^\dagger)^\dagger = x$;

(ii) The involution is completely isometric, i.e.,

$$\|[x_{ij}]^\dagger\| = \|[x_{ij}]\| \quad \text{for all } n \in \mathbb{N} \text{ and } [x_{ij}] \in M_n(X),$$

where $[x_{ij}]^\dagger = [x_{ji}^\dagger]$ for all $i, j \in \{1, \cdots, n\}$.

Example 5.4. Suppose X is a $*$-selfadjoint subspace of $B(H)$, where H is a Hilbert space. Denote

$$\mathcal{U}(X) = \left\{ \begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix} : x \in X, \lambda, \mu \in \mathbb{C} \right\},$$

then $\mathcal{U}(X)$ is a subspace of $B(H^{(2)})$. The involution on $\mathcal{U}(X)$ is defined by

$$\begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix}^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{\lambda} & x^* \\ 0 & \bar{\mu} \end{bmatrix} \in \mathcal{U}(X).$$

It is easy to see that $\mathcal{U}(X)$ with the involution defined above is an operator $*$-space. Moreover, variants such as $\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in X\right\}$ and $\left\{\begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix} : x \in X, \lambda \in \mathbb{C}\right\}$ are also operator $*$-spaces.

Definition 5.5. If X is an operator $*$-space, then we say that X is an *injective operator $*$-space* if X is injective in the sense of Definition 2.4.

Proposition 5.6. *Suppose that X, Y and Z are operator $*$-spaces and $X \subset Y$ is a $\dagger$-selfadjoint subspace. If Z is injective, then for any completely bounded $\dagger$-linear map $u : X \to Z$, there exists a completely bounded $\dagger$-linear extension $\hat{u} : Y \to Z$ such that $\|u\|_{cb} = \|\hat{u}\|_{cb}$.*

Proof. Suppose $u : X \to Z$ is a completely bounded $\dagger$-linear map. By injectivity of Z, there exists a completely bounded extension $v : Y \to Z$ such that $v_{|X} = u$ and $\|v\|_{cb} = \|u\|_{cb}$. Let $v^\dagger : Y \to Z$ be the map defined by $v^\dagger(y) = v(y^\dagger)^\dagger$. Then $v^\dagger_{|X} = u$ and $\|v^\dagger\|_{cb} = \|u\|_{cb}$. Let $\hat{u}$ denote be the map $\frac{v+v^\dagger}{2}$. Thus $\hat{u}$ is $\dagger$-linear, $\hat{u}_{|X} = u$ and $\|\hat{u}\|_{cb} = \|u\|_{cb}$. $\square$

Proposition 5.7. *Suppose that X is a vector space with an involution on X such that $(x^\dagger)^\dagger = x$. Suppose also that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$ satisfying the following conditions:*

(1) $\|\alpha[x_{ij}]^\dagger\beta\|_n \leq \|\alpha\|\|[x_{ij}]\|_n\|\beta\|$, *where* $[x_{ij}]^\dagger = [x_{ji}^\dagger]$ *for all* $n \in \mathbb{N}$ *and* $\alpha, \beta \in M_n$;

(2) *For all* $[x_{ij}] \in M_m(X)$ *and* $[y_{kl}] \in M_n(X)$, *we have*

$$\left\| \begin{bmatrix} [x_{ij}] & 0 \\ 0 & [y_{kl}] \end{bmatrix} \right\| = \max\{\|[x_{ij}]\|_m, \|[y_{kl}]\|_n\};$$

then X *is an operator* $*$*-space.*

Proof. Let α and β be I_n, then $\|[x_{ij}]^\dagger\|_n \leq \|[x_{ij}]\|_n$. By (i) of Definition 5.3, we get equality here. We also have $\|\alpha[x_{ij}]\beta\|_n \leq \|\alpha\|\|[x_{ij}]\|_n\|\beta\|$. By applying Ruan's Theorem (see e.g. theorem 2.3), we know that X is an operator space. Thus, X is an operator $*$-space. $\square$

Theorem 5.8. *Suppose that* X *is a vector space with an involution on* X *satisfying that* $(x^\dagger)^\dagger = x$ *and that for each* $n \in \mathbb{N}$ *we are given a norm* $\|\cdot\|_n$ *on* $M_n(X)$*. Then* X *is linearly completely isometrically* $*$*-isomorphic to a* $*$*-selfadjoint linear subspace of* $B(H)$ *for some Hilbert space* H*, if and only if* X *is an operator* $*$*-space.*

Proof. ($\Leftarrow$) Let X be an operator $*$-space. By Ruan's theorem, there exists a complete isometry $\pi : X \to B(K)$ such that X is completely isometrically isomorphic to $\pi(X) \subseteq B(K)$. Let $\rho : X \to B(K \oplus K)$ be the map defined by

$$\rho(x) = \begin{bmatrix} 0 & \pi(x) \\ \pi(x^\dagger)^* & 0 \end{bmatrix},$$

then $\rho(x^\dagger) = \rho(x)^*$, for all $x \in X$. Then the result follows immediately from the fact that ρ is completely isometric.

($\Rightarrow$) Suppose that $\rho : X \to B(H)$ is a completely isometric and $*$-linear map. Then X is an operator space and $\|[x_{ij}]^\dagger\| = \|[x_{ij}]\|$, for any $[x_{ij}] \in M_n(X)$. $\square$

Corollary 5.9. *If $Y \subset X$ is a closed linear subspace of an operator $*$-space X such that $Y^\dagger \subseteq Y$, then X/Y is an operator $*$-space with the matrix norm $M_n(X/Y)$ given by the formula*

$$\|[x_{ij}\dot{+}Y]\|_n = \inf\{\|[x_{ij} + y_{ij}]\|_n : y_{ij} \in Y\}.$$

Corollary 5.10. *Suppose that X is a vector space with an involution on X such that $(x^\dagger)^\dagger = x$. Furthermore, there exists a sequence $\rho = \{\rho_n\}_{n=1}^{\infty}$, where ρ_n is a seminorm on $M_n(X)$, satisfying* (1) *and* (2) *in Proposition 5.7. In this case, and if N is defined to be $\{x \in X : \rho_1(x) = 0\}$, then X/N is an operator $*$-space.*

Proof. By (1), we have that $x \in N$ if and only if $x^\dagger \in N$. Thus, the kernel of ρ_n is $M_n(N)$. By Proposition 5.7, X/N is an operator $*$-space. $\square$

Example 5.11 (Interpolation of operator $*$-spaces)**.** Suppose that (X_0, X_1) is a compatible couple of *two operator $*$-spaces.* Just like in the general operator space case let $\mathcal{S}$ be the strip of all complex numbers z with $0 \leq \mathrm{re}(z) \leq 1$ and let $\mathcal{F} = \mathcal{F}(X_0, X_1)$ be the space of all bounded and continuous functions $f : \mathcal{S} \to X_0 + X_1$ such that the restriction of f to the interior of $\mathcal{S}$ is analytic, and such that the maps $t \to f(it)$ and $t \to f(1 + it)$ belong to $C_0(\mathbb{R}; X_0)$ and $C_0(\mathbb{R}, X_1)$ respectively. For any $f \in \mathcal{F}$, the function $f^\dagger$ is defined by $f^\dagger(z) = \overline{f(\bar{z})} \in \mathcal{F}$. Then $\mathcal{F}(X_0, X_1)$ is an operator $*$-space with the involution $\dagger$.

For any $0 \leq \theta \leq 1$, let $\mathcal{F}_\theta(X_0, X_1)$ be the subspace of all $f \in \mathcal{F}$ for which $f(\theta) = 0$, which is $\dagger$-selfadjoint. And, the interpolation space $X_\theta = [X_0, X_1]_\theta$ is the subspace of $X_0 + X_1$ formed by all $x = f(\theta)$ for some $f \in \mathcal{F}$. As operator spaces, the interpolation space $X_\theta \cong \mathcal{F}(X_0, X_1)/\mathcal{F}_\theta(X_0, X_1)$ through the map $\pi : f \mapsto f(\theta)$. It is easy to see that π is $\dagger$-linear. Hence, X_θ is an operator $*$-space.

5.2 Some common operator $*$-space structures

5.2.1 Mapping spaces and dual

Let X, Y be operator $*$-spaces. The space $CB(X, Y)$ of completely bounded linear maps from X to Y is also an operator $*$-space. Indeed, if $u \in CB(X, Y)$, let $u^\dagger : X \to Y$ be the map defined by $u^\dagger(x) = u(x^\dagger)^\dagger$, for any $x \in X$. Thus, it is easy to see that $u^\dagger$ is also a completely bounded map with $\|u^\dagger\|_{cb} = \|u\|_{cb}$. Similarly, if $[u_{ij}] \in M_n(CB(X, Y))$, then $\|[u_{ij}]\|_{cb} = \|[u_{ij}]^\dagger\|_{cb}$. Thus, $CB(X, Y)$ is an operator $*$-space.

If $Y = \mathbb{C}$, then $X^* = CB(X, \mathbb{C})$ for any operator $*$-space X. If $\varphi \in X^*$ and the map $\varphi^\dagger : X \to \mathbb{C}$ is defined by $\varphi^\dagger(x) = \overline{\varphi(x^\dagger)}$, then X^* is an operator $*$-space (see more details later).

5.2.2 Minimal operator $*$-space

Let E be a Banach $*$-space and consider the canonical isometric inclusion of E in the commutative C^*-algebra $C(\mathrm{Ball}(E^*))$. Here E^* is equipped with the w^*-topology.

Recall that the matrix norms on Min(E) are given by

$$\|[x_{ij}]\|_n = \sup\{\|[\varphi(x_{ij})]\| : \varphi \in \mathrm{Ball}(E^*)\|, \text{ for } [x_{ij}] \in M_n(E)\}. \tag{5.2.1}$$

From (5.2.1), we claim that

$$\|[x_{ij}]\|_n = \|[x_{ij}]^\dagger\|_n, \ \forall\ [x_{ij}] \in M_n(E),\ n \in \mathbb{N}\,.$$

Indeed, for any $\varepsilon > 0$ there exists $\psi \in \mathrm{Ball}(E^*)$ such that $\|[x_{ij}]\|_n - \varepsilon < \|[\psi(x_{ij})]_n\|$. Moreover,

$$\|[x_{ij}]^\dagger\|_n = \|[x_{ji}^\dagger]\|_n \geq \|[\psi^\dagger(x_{ji}^\dagger)]\|_n = \|[\overline{\psi(x_{ji})}]\|_n = \|[\psi(x_{ij})]\|_n > \|[x_{ij}]\|_n - \varepsilon,$$

where $\psi^\dagger$ is defined similarly as above. This implies that $\|[x_{ij}]^\dagger\|_n \geq \|[x_{ij}]\|_n$. By symmetry, we know that $\|[x_{ij}]^\dagger\|_n = \|[x_{ij}]\|_n$. Thus every Banach ∗-space may be canonically considered to be an operator ∗-space. Also, for any bounded †-linear map u from an operator ∗-space Y into E, we have

$$\|u : Y \to \mathrm{Min}(E)\|_{cb} = \|u : Y \to E\|.$$

Suppose Ω is any compact space and τ is an order 2-homeomorphism on Ω. An involution on $C(\Omega)$ could be defined by

$$f^\dagger = \bar{f} \circ \tau,\ \forall f \in C(\Omega).$$

Obviously, $f^\dagger \in C(\Omega)$ and

$$\|f^\dagger\| = \sup\{|\bar{f}(\tau(\omega))| : \omega \in \Omega\} = \sup\{|\bar{f}(\omega)| : \omega \in \Omega\} = \|f\|.$$

By a routine argument, we know that $C(\Omega)$ is also an operator $*$-space with the involution defined above. Moreover, for any involution on $C(\Omega)$, then there exists τ is an order 2-homeomorphism on Ω such that $f^\dagger = \bar{f} \circ \tau,\ \forall f \in C(\Omega)$.

Furthermore, if $i : E \to C(\Omega)$ is a $\dagger$-isometry, then the matrix norms inherited by E from the operator $*$-structure of $C(\Omega)$ coincide with those in (5.2.1). Since $\mathrm{Ball}(E^*)$ has natural involution $\tau(\psi) = \psi^\dagger$, this means that the 'minimal operator $*$-spaces' are exactly the operator $*$-spaces completely isometrically $\dagger$-isomorphic to a $\dagger$-selfadjoint subspace of a $C(K)$-space where K is a compact space.

5.2.3 Maximal operator $*$-space

Let E be a Banach $*$-space. Then $\mathrm{Max}(E)$ is the largest operator space structure equipped on E. Recall that the matrix norms on $\mathrm{Max}(E)$ are given by the following formula:

$$\|[x_{ij}]\|_n = \sup\{\|[u(x_{ij})]\| : u \in \mathrm{Ball}(B(E,Y)),\ \text{all operator spaces } Y\}. \tag{5.2.2}$$

For any $\epsilon > 0$, there exists some operator space Y and $u \in \mathrm{Ball}(B(E,Y))$ such that

$$\|[x_{ij}]\|_n < \|u(x_{ij})\| + \epsilon.$$

Since Y is an operator space, then the adjoint $Y^\star$ is also an operator space. Let $u^\dagger : E \to Y^\star$ be the map defined by $u^\dagger(x) = u(x^\dagger)^* \in Y^\star$, for any $x \in E$. Note that

$$\|[x_{ij}]^\dagger\|_n = \|[x_{ji}^\dagger]\|_n > \|[u^\dagger(x_{ji}^\dagger)]\|_n = \|[u(x_{ji})^*]\|_n = \|[u(x_{ij})]^*\| > \|[x_{ij}]\|_n - \epsilon.$$

Thus, $\|[x_{ij}]^\dagger\|_n \geq \|[x_{ij}]\|_n$, and by symmetry we know that $\|[x_{ij}]^\dagger\|_n = \|[x_{ij}]\|_n$. Therefore, $\mathrm{Max}(E)$ is an operator $*$-space which is the largest operator $*$-space structure on E.

Proposition 5.12. *For any Banach $*$-space E,*

$$\mathrm{Min}(E)^* \cong \mathrm{Max}(E^*) \text{ and } \mathrm{Max}(E)^* \cong \mathrm{Min}(E^*),$$

completely $\dagger$-isometrically.

Proof. As operator spaces, $\mathrm{Min}(E)^* \cong \mathrm{Max}(E^*)$ and $\mathrm{Max}(E)^* \cong \mathrm{Min}(E^*)$. And, it is easy to see that the corresponding maps are $\dagger$-preserving. $\square$

5.2.4 The adjoint of an operator $*$-space

Let X be an operator $*$-space. The adjoint operator space $X^\star$ (see e.g. [14, 1.2.25]) can be equipped with involution $\dagger$. For any x, $(x^*)^\dagger$ is defined by $(x^\dagger)^*$, where the last $\dagger$ is the involution on X and the last $*$ is the one from the adjoint operator space. Note that

$$\|(x^*)^\dagger\| = \|(x^\dagger)^*\| = \|x^\dagger\| = \|x\| = \|x^*\|.$$

For the matrix norms,

$$\begin{aligned}\|[x_{ij}^*]^\dagger\| &= \|[(x_{ji}^*)^\dagger]\| = \|[(x_{ji}^\dagger)^*]\| = \|[x_{ij}^\dagger]^*]\| \\ &= \|[x_{ij}^\dagger]\| = \|[x_{ji}]\| = \|[x_{ij}^*]\|.\end{aligned}$$

Thus, $X^\star$ becomes an operator $*$-space.

5.3 Duality of operator $*$-spaces

Lemma 5.13. *Suppose that X is an operator $*$-space, then X^* is also an operator $*$-space.*

Proof. Suppose that $\varphi \in X^*$ and $\varphi^\dagger$ is defined earlier, then

$$(\varphi^\dagger)^\dagger(x) = \overline{\varphi^\dagger(x^\dagger)} = \varphi(x), \quad (\lambda\varphi)^\dagger(x) = \overline{\lambda\varphi(x^\dagger)} = \overline{\lambda}\varphi^\dagger(x).$$

Besides, for any $\varepsilon > 0$ there exists $x \in \mathrm{Ball}(X)$ such that $\|\varphi\| < |\varphi(x)| + \varepsilon$. Moreover, $\|\varphi^\dagger\| \geq |\varphi^\dagger(x^\dagger)| = |\overline{\varphi(x)}| > \|\varphi\| - \varepsilon$, which implies $\|\varphi^\dagger\| \geq \|\varphi\|$. Analogously, $\|\varphi^\dagger\| \leq \|\varphi\|$.

For the matrix norms, suppose $[\varphi_{ij}] \in M_n(X^*)$, then we identify $[\varphi_{ij}]^\dagger$ as $[\varphi_{ji}^\dagger] \in M_n(CB(X,\mathbb{C})) \cong CB(X, M_n(\mathbb{C}))$. By direct calculation, we get that

$$([\varphi_{ij}]^\dagger)^\dagger = [\varphi_{ji}^\dagger]^\dagger = [\varphi_{ij}],\ \|[\varphi_{ij}]^\dagger\| = \|[\varphi_{ij}]\|, \tag{5.3.3}$$

$$[\varphi_{ij}]_n^\dagger([x_{kl}]_m^\dagger) = ([\varphi_{ij}]_n([x_{kl}]_m))^*, \tag{5.3.4}$$

from this, it is fairly easy that X^* is an operator $*$-space. $\square$

Corollary 5.14. *Let X be an operator $*$-space, then X^{**} is also an operator $*$-space.*

Proposition 5.15. *Let X be an operator $*$-space. Then $\widehat{(x^\dagger)} = (\hat{x})^\dagger$, for any $x \in X$.*

Proof. For any $\varphi \in X^*$, then $\widehat{(x^\dagger)}(\varphi) = \varphi(x^\dagger)$. Besides,

$$(\hat{x})^\dagger(\varphi) = \overline{\hat{x}(\varphi^\dagger)} = \overline{\varphi^\dagger(x)} = \varphi(x^\dagger), \quad x \in X.$$

Thus, $\widehat{(x^\dagger)} = (\hat{x})^\dagger$. $\square$

Corollary 5.16. *Let X be an operator $*$-space. Then $X \subset X^{**}$ completely $\dagger$-isometrically via the canonical map i_X.*

Lemma 5.17. *Suppose that X is an operator $*$-space and $\eta_t \to \eta$ in the weak*-topology where $\eta_t, \eta \in X^{**}$. Then $\eta_t^\dagger \to \eta^\dagger$ in weak*-topology.*

Proof. Suppose that $\eta_t \to \eta$ in the weak*-topology. Then for any $\varphi \in X^*$, $\eta_t(\varphi) \to \eta(\varphi)$ which implies that $\eta_t^\dagger(\varphi) = \overline{\eta_t(\varphi^\dagger)} \to \overline{\eta(\varphi^\dagger)} = \eta^\dagger(\varphi)$. Thus, $\eta_t^\dagger \to \eta^\dagger$ in the weak*-topology. $\square$

Definition 5.18. An operator $*$-space Y is said to be a *dual operator $*$-space* if Y is a dual operator space such that the involution on Y is weak* continuous.

Lemma 5.19. *Any dual operator $*$-space is completely isometrically $*$-isomorphic via a homeomorphism for the w^*-topologies, to a w^*-closed selfadjoint subspace of $B(H)$, for some Hilbert space H.*

Proof. Since X is a dual operator space then there exists a Hilbert space K and a completely isometric map $\rho : X \to B(K)$ such that X is completely isometrically isomorphic to $\rho(X)$, which is w^*-closed. Let $H = K^{(2)}$ and

$$\theta(x) = \begin{bmatrix} 0 & \rho(x) \\ \rho(x^\dagger)^* & 0 \end{bmatrix} \in B(H).$$

Then θ is completely isometric $\dagger$-$*$-linear. Moreover, $\theta(X)$ is a w^*-closed self-adjoint subspace of $B(H)$. $\square$

If X and Y are two operator $*$-spaces and if $u : X \to Y^*$ is completely bounded, then its (unique) w^*-continuous linear extension $\tilde{u} = i_Y^* \circ u^{**} : X^{**} \to Y^*$ is completely bounded, with $\|\tilde{u}\|_{cb} = \|u\|_{cb}$. Note that $\tilde{u}$ is $\dagger$-linear if u is $\dagger$-linear. Moreover, as operator spaces, $CB(X, Y^*) \cong w^*CB(X^{**}, Y^*)$ completely isometrically via the mapping $u \mapsto \tilde{u}$.

Proposition 5.20. *Suppose X, Y are two operator $*$-spaces. Then the map θ : $CB(X, Y^*) \to CB(X^{**}, Y^*)$ defined by $\theta(u) = \tilde{u}$ is $\dagger$-linear. Moreover, $CB(X, Y^*) \cong w^*CB(X^{**}, Y^*)$ completely $\dagger$-isometrically.*

Proof. We only need to show the map θ is $\dagger$-preserving. Indeed, for any $u \in CB(X, Y^*)$, $\widetilde{u^\dagger} = i_Y^* \circ (u^\dagger)^{**}$. For any $\eta \in X^{**}$,

$$i_{Y^*} \circ (u^\dagger)^{**}(\eta) = i_Y^* \circ (u^{**})^\dagger(\eta) = i_Y^*(u^{**}(\eta^\dagger)^\dagger).$$

Moreover,

$$\tilde{u}^\dagger(\eta) = (i_Y^* \circ u^{**})^\dagger(\eta) = (i_Y^* \circ u^{**}(\eta^\dagger))^\dagger = i_Y^*(u^{**}(\eta^\dagger))^\dagger = i_Y^*(u^{**}(\eta^\dagger)^\dagger).$$

Hence, θ is $\dagger$-linear. $\square$

Suppose that X is an operator space and $\nu \in \mathbb{M}_n(X)$. Recall that $\|\cdot\|_1 : \mathbb{M}_n(X) \to [0, \infty)$ is given by

$$\|v\|_1 = \inf\{\|\alpha\|_2 \|\tilde{\nu}\| \|\beta\|_2 : \nu = \alpha\tilde{\nu}\beta\},$$

where $\alpha \in M_{n,r}$, $\beta \in M_{r,n}$ and $\tilde{\nu} \in M_r(X)$ with r arbitrary and $\|\cdot\|_2$ is the Hilbert-Schmidt norm. From [30, Lemma 4.1.1], we know that $\|\cdot\|_1$ is a norm on $\mathbb{M}_n(X)$. Let $T_n(X)$ denote the corresponding normed space.

Lemma 5.21. *Suppose that X is an operator $*$-space and $n \in \mathbb{N}$. Then*

$$T_n(X)^* \cong M_n(X^*), \ \dagger\text{-isometrically},$$
$$M_n(X)^* \cong T_n(X^*), \ \dagger\text{-isometrically},$$

through the scalar pairing in [30].

Proof. Suppose that $\varphi \in \mathrm{Ball}(M_n(X))^*$, then by [30, Lemma 2.3.3] there exist a mapping $\psi \in M_n(X^*)$ with $\|\psi\|_{cb} < 1$ and vectors in $\xi, \eta \in \mathbb{C}^{n^2}$ such that

$$\varphi([x_{ij}]) = \langle \psi_n([x_{ij}])\eta, \xi \rangle$$

$$
\begin{aligned}
&= \sum \psi_{kl}(x_{ij})\eta_{jl}\bar{\xi}_{ik} \\
&= \sum_{i,j} [\sum_{k,l} \bar{\xi}_{ik}\psi_{kl}\eta_{jl}](x_{ij}) \\
&= \langle \alpha\psi\beta, [x_{ij}]\rangle,
\end{aligned}
$$

where $\alpha_{ik} = \bar{\xi}_{ik}$ and $\beta_{lj} = \eta_{jl}$.

Moreover,

$$\varphi^{\dagger}([x_{ij}]) = \overline{\varphi([x_{ji}^{\dagger}])} = \overline{\langle \psi_n([x_{ji}^{\dagger}])\eta, \xi\rangle} = \langle \psi_n([x_{ji}^{\dagger}])\xi, \eta\rangle = \langle \beta^*\psi^{\dagger}\alpha^*, [x_{ij}]\rangle.$$

Thus, the map is $\dagger$-linear. Others follow from Lemma 4.1.1 in [30]. $\square$

Corollary 5.22. *If X is an operator $*$-space, then $M_n(X)^{**} \cong M_n(X^{**})$ completely $\dagger$-isometrically for all $n \in \mathbb{N}$.*

Proof. By Lemma 5.21, we know that for any $n \in \mathbb{N}$, $M_n(X^{**}) \cong T_n(X^*)^* \cong (M_n(X)^*)^*$, $\dagger$-isometrically. Similarly, we have that

$$M_m(M_n(X^{**})) \cong M_{mn}(X^{**}) \cong (T_{nm}(X^*))^* \cong (M_{mn}(X)^*)^* \cong (M_m(M_n(X))^{**}),$$

$\dagger$-isometrically. $\square$

5.4 Operator $*$-space tensor products

Let X and Y be operator $*$-spaces, and let $X \otimes Y$ denote their algebraic tensor product. Recall that any $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ can be associated with a map $\tilde{u} : Y^* \to X$ defined by $\tilde{u}(\psi) = \sum_{k=1}^n x_k \psi(y_k)$, for any $\psi \in Y^*$. If $u = \sum_{k=1}^n x_k \otimes \psi_k \in X \otimes Y^*$ then u can be associated with a map $\hat{u} : Y \to X$ defined by $\hat{u}(y) = \sum_k \psi(y) x_k$, for $y \in Y$. Both $\tilde{u}$ and $\hat{u}$ are completely bounded. Moreover, the map $u \to \tilde{u}$ is $\dagger$-linear. Indeed, if $u = \sum_{k=1}^n x_k \otimes y_k$, then $u^\dagger = \sum_{k=1}^n x_k^\dagger \otimes y_k^\dagger$. For any $\psi \in Y^*$, we have

$$(\tilde{u})^\dagger(\psi) = (\tilde{u}(\psi^\dagger))^\dagger = (\sum_k \psi^\dagger(y_k) x_k)^\dagger = \sum_k \psi(y_k^\dagger) x_k^\dagger = \widetilde{u^\dagger}(\psi).$$

Similarly, we know that the map $u \to \hat{u}$ is $\dagger$-linear. The *minimal tensor product* $X \otimes_{\min} Y$ may be defined to be (the completion of) $X \otimes Y$ in the matrix norms inherited from the operator $*$-space structure on $CB(Y^*, X)$. That is,

$$X \otimes_{\min} Y \hookrightarrow CB(Y^*, X) \quad \text{completely } \dagger\text{-isometrically.}$$

Explicitly, if $[w_{rs}] \in M_n(X \otimes_{\min} Y)$, then norm of $[w_{rs}]$ equals

$$\sup \left\{ \|[(\varphi_{kl} \otimes \psi_{ij})(w_{rs})]\| : [\varphi_{kl}] \in \mathrm{Ball}(M_m(X^*)), [\psi_{ij}] \in \mathrm{Ball}(M_s(Y^*)), m, s \in \mathbb{N} \right\}.$$

For any $\epsilon > 0$, there exists some $m, s \in \mathbb{N}$ such that

$$\|[w_{rs}]\|_{M_n(X \otimes_{\min} Y)} < \|[(\varphi_{kl} \otimes \psi_{ij})(w_{rs})]\| + \epsilon = \|[(\varphi_{lk}^\dagger \otimes \psi_{ji}^\dagger)(w_{sr}^\dagger)]\| + \epsilon.$$

This implies $\|[w_{rs}]\|_{M_n(X\otimes_{\min}Y)} \leq \|[w_{rs}]^\dagger\|_{M_n(X\otimes_{\min}Y)}$. Analogously, we know that $\|[w_{rs}]\|_{M_n(X\otimes_{\min}Y)} \geq \|[w_{rs}]^\dagger\|_{M_n(X\otimes_{\min}Y)}$. Therefore, $X\otimes_{\min}Y$ is an operator $*$-space.

Proposition 5.23. *For any operator $*$-spaces X, Y, we have*

$$X \otimes_{\min} Y^* \hookrightarrow CB(Y, X) \quad \textit{completely } \dagger\textit{-isometrically},$$

via the map $\wedge : u \to \hat{u}$.

Proof. Note that the map $\wedge : u \to \hat{u}$ is $\dagger$-linear. Then the result follows from the fact that as operator spaces, $X \otimes_{\min} Y^* \hookrightarrow CB(Y, X)$ completely isometrically. $\square$

Proposition 5.24. *For any operator $*$-space X,*

$$M_n \otimes_{\min} X \cong M_n(X) \quad \textit{completely } \dagger\textit{-isometrically.}$$

Proof. This is apparent from the following:

$$M_n \otimes_{\min} X \hookrightarrow w^*CB(X^*, M_n) \cong M_n(X),$$

where $w^*CB(X^*, M_n) \cong M_n(X)$ means that $w^*CB(X^*, M_n)$ is completely $\dagger$-isometrically isomorphic to $M_n(X)$. $\square$

Corollary 5.25. *Let X be an operator $*$-space. For any set I*

$$\mathbb{K}_I \otimes_{\min} X \cong \mathbb{K}_I(X) \quad \textit{completely } \dagger\textit{-isometrically.}$$

Proof. As operator spaces,

$$\mathbb{K}_I \otimes_{\min} X = \overline{\mathbb{M}_I^{\text{fin}} \otimes^{\min} X} \cong \overline{\mathbb{M}_I^{\text{fin}}(X)} = \mathbb{K}_I(X),$$

via $\dagger$-preserving maps. $\square$

Proposition 5.26. *Let E and F be Banach $*$-spaces, and let X be an operator $*$-space.*

(1) $\mathrm{Min}(E) \otimes_{\min} X = E \check{\otimes} X$ *as Banach $*$-spaces.*

(2) $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F) = \mathrm{Min}(E \check{\otimes} F)$ *as operator $*$-spaces.*

(3) *For any compact space Ω we have*

$$C(\Omega) \otimes_{\min} X \cong C(\Omega, X) \quad \textit{completely} \dagger\textit{-isometrically.}$$

Proof. Note that the map $u : \mathrm{Min}(E) \otimes_{\min} X \to E \check{\otimes} X$ is $\dagger$-linear. Also, as Banach spaces, $\mathrm{Min}(E) \otimes_{\min} X = E \check{\otimes} X$.

As operator spaces, $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F) \cong \mathrm{Min}(E \check{\otimes} F)$ completely isometrically. And, both $\mathrm{Min}(E) \otimes_{\min} \mathrm{Min}(F)$ and $\mathrm{Min}(E \check{\otimes} F)$ are operator $*$-spaces, then (2) follows from (1).

As operator spaces, $C(\Omega) \otimes_{\min} X \cong C(\Omega, X)$ completely isometrically. Since the canonical map preserves the involution, then (3) follows directly. $\square$

CHAPTER 6

Involutive operator algebras

6.1 Involutions on operator algerbras

By an *involution* we mean at least a bijection $\tau : A \to A$ which is of period 2: $\tau^2(a) = a$ for $a \in A$. A C^*-algebra B may have two kinds of extra involution: a period 2 conjugate linear $*$-antiautomorphism or a period 2 linear $*$-antiautomorphism. The former is just the usual involution $*$ composed with a period 2 $*$-automorphism of B. The latter is essentially the same as a 'real structure', that is if θ is the antiautomorphism then B is just the complexification of a real C^*-algebra $D = \{x \in B : x = \bar{x}\}$, where $\bar{x} = \theta(x)^*$. We may characterize $x \mapsto \bar{x}$ on B very simply as the map $a + ib \mapsto a - ib$ for $a, b \in D$.

By way of contrast, there are four distinct natural kinds of 'completely isometric involution' on a general operator algebra A. Namely, period 2 bijections which are

(1) conjugate linear antiautomorphisms $\dagger : A \to A$ satisfying $\|[a_{ji}^{\dagger}]\| = \|[a_{ij}]\|$,

(2) linear antiautomorphisms $\theta : A \to A$ satisfying $\|[a_{ji}^{\theta}]\| = \|[a_{ij}]\|$,

(3) conjugate linear automorphisms $- : A \to A$ satisfying $\|[\overline{a_{ij}}]\| = \|[a_{ij}]\|$,

(4) linear automorphisms $\pi : A \to A$ satisfying $\|[a_{ij}^{\pi}]\| = \|[a_{ij}]\|$.

Here $[a_{ij}]$ is a generic element in $M_n(A)$, the $n \times n$ matrices with entries in A, for all $n \in \mathbb{N}$. Class (1) is just the operator $*$-algebras mentioned earlier. Here we will call the algebras in class (2) *operator algebras with linear involution* θ, and write $\theta(a)$ as a^{θ}. We will not discuss (4) here, these are well studied and are only mentioned here because most of the results apply to all four classes. We will just say that this class is in bijective correspondence with the unital *completely symmetric projections* on A in the sense of [16], this correspondence is essentially Corollary 4.2 there. Similarly, for the same reasons we will not discuss class (3) here. By [58, Theorem 3.3], class (3) is essentially the same as 'real operator algebra structure', that is A is just the complexification of a real operator algebra $D = \{x \in B : x = \bar{x}\}$, and we may rewrite $\bar{x} = a - ib$ if $x = a + ib$ for $a, b \in D$. We also remark that if A is unital or approximately unital then one can easily show using the Banach-Stone theorem for operator algebras (see e.g. [14, Theorem 4.5.13]) that the matrix norm equality in (3) and (4) (resp. (1) and (2)) force the 'involution' to be multiplicative (resp. anti-multiplicative).

If A is a C^*-algebra then classes (1) and (4) are essentially the same after applying the C^*-algebra involution $*$. (Note that in this case the matrix norm equality in (1) or (4) follows from the same equality for 1×1 matrices, that is that the involution is isometric. Indeed it is well known that $*$-isomorphisms of C^*-algebras are completely isometric.) Similarly classes (2) and (3) essentially coincide if A is a C^*-algebra.

We will mostly focus on class (1) for specificity. In fact most of the results in the following apply to all four classes, however it would be too tedious to state several cases of each result. For example to get from case (1) to case (2) of results below one replaces $a^\dagger$ by a^θ, and †-*selfadjoint elements*, that is elements satisfying $a^\dagger = a$, by elements with $a^\theta = a$. We remark that if A is an operator algebra with linear involution θ, then $\{a \in A : a = a^\theta\}$ is a Jordan operator algebra in the sense of [26]. (We remark that these 'θ-*selfadjoint* elements' need not generate A, unlike for involutions of type (1).) Most of our discussion of class (2) involves finding interesting examples of such involutions. Indeed although classes (1)–(4) have similar theory, the *examples* of algebras in these classes are quite different in general.

Because of the ubiquity of the asterisk symbol in our area of study, we usually write the involution on an operator $*$-algebra as †, and refer to, for example, †-selfadjoint elements or subalgebras, and †-homomorphisms (the natural morphisms for $*$-algebras). By a *symmetry* we mean either a selfadjoint unitary operator, or a period 2 $*$-automorphism of a C^*-algebra, depending on the context.

6.2 Meyer's theorem and unitization

Definition 6.1. An operator algebra A is an *operator $*$-algebra* when it comes equipped with a conjugate linear involution $\dagger : A \to A$ such that

(i) $(a^\dagger)^\dagger = a$;

(ii) $(ab)^\dagger = b^\dagger a^\dagger$;

(iii) The involution is completely isometric, thus

$$\|[a_{ij}]^\dagger\| = \|[a_{ij}]\| \quad \text{for all } n \in \mathbb{N} \text{ and } [a_{ij}] \in M_n(A),$$

where $[a_{ij}]^\dagger = [a_{ji}^\dagger]$ for all $i, j \in \{1, \cdots, n\}$.

Example 6.2. $\mathcal{U}(X)$ defined in Example 5.4 is an operator $*$-algebra.

Example 6.3. Let $A(\mathbb{D})$ be the risk algebra on the unit disk. The involution on the disk algebra $A(\mathbb{D})$ is defined by $f^\dagger(z) = \overline{f(\overline{z})}$, for any $f \in A(\mathbb{D})$. Then $A(\mathbb{D})$ becomes an operator $*$-algebra.

Example 6.4 (Reduced C*-algebras[12, Examples 1.9])**.** Let $d \in \mathbb{N}$ and $G = \mathbb{Z}^d$. Let the operator algebra $C_r^{hol}(\mathbb{Z}^d)$ be the smallest closed subalgebra of the reduced group C^*-algebra, $C_r^*(\mathbb{Z}^d)$ such that

$$\lambda_g \in C_r^{hol}(\mathbb{Z}^d), \text{for all}\, g \in (\mathbb{N} \cup \{0\})^d.$$

Define the selfadjoint unitary operator

$$W : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d), \quad W(\delta_{(n_1,\cdots,n_d)}) = \delta_{(-n_1,\cdots,-n_d)}.$$

Then it is easy to see that

$$W\lambda_g^* W = \lambda_g, \text{for all } g \in (\mathbb{N} \cup \{0\})^d,$$

where the $*$-refer to the involution in $C_r^*(\mathbb{Z}^d)$.

We thus have a well-defined completely isometric involution

$$\dagger : C_r^{hol}(\mathbb{Z}^d) \to C_r^{hol}(\mathbb{Z}^d), \quad x^\dagger = Wx^*W,$$

and it follows that $C_r^{hol}(\mathbb{Z}^d)$ is an operator $*$-algebra.

Since $C_r^*(\mathbb{Z}^d)$ is isomorphic as a C^*-algebra to the continuous functions on the d-torus, via the isomorphism $\lambda(n_1, \cdots, n_d) \to z_1^{n_1} \cdot \cdots \cdot z_n^{n_d}$. Under this isomorphism, $C_r^{hol}(\mathbb{Z}^d)$ corresponds to the continuous functions $f : \mathbb{D}^d \to \mathbb{C}$ on the closed poly-disc that are holomorphic on the open poly-disk $(\mathbb{D}^o)^d$. Just as mentioned above, the involution is given by $f^\dagger(z) = \overline{f(\bar{z})}$.

The following result, the involutive variant of Meyer's theorem [14, Corollary 2.1.15], is useful in treating involutions on operator algebras with no identity or approximate identity.

Lemma 6.5. *Let $A \subset B(H)$ be a nonunital operator algebra. Suppose that there is* *an involution* $\dagger$ *on* A *making* A *an operator* $*$*-algebra. Then* $A+\mathbb{C}\, I_H$ *is an operator* $*$*-algebra with involution defined by* $(a+\lambda I_H)^\dagger = a^\dagger + \overline{\lambda} I_H$.

Proof. Define $\pi : A \to A^* \subset B(H)$ by $\pi(a) = (a^\dagger)^*$ where A^* is the adjoint of A, an operator algebra. It's easy to see that π is a completely isometric isomorphism. By Meyer's Theorem, there is a unital completely isometric isomorphism π^0 extending π from $A+\mathbb{C}\, I_H$ to $A^*+\mathbb{C}\, I_H$ by letting $\pi^0(a+\lambda I_H) = \pi(a) + \lambda I_H$. Notice that

$$\pi^0([a_{ij}+\lambda_{ij} I_H]) = \pi([a_{ij}]) + [\lambda_{ij} I_H] = ([a_{ij}]^\dagger)^* + [\lambda_{ij}] I_H = ([a_{ij}+\lambda_{ij} I_H]^\dagger)^*$$

Since π^0 is completely isometric, then

$$\|[a_{ij}+\lambda_{ij} I_H]\| = \|([a_{ij}+\lambda_{ij} I_H]^\dagger)^*\| = \|[a_{ij}+\lambda_{ij} I_H]^\dagger\|.$$

So that $A+\mathbb{C}\, I_H$ is an operator $*$-algebra. □

Proposition 6.6. *Let* A *and* B *be operator subalgebras of* $B(H)$ *and* $B(K)$ *respectively, with* $I_H \notin A$. *Also suppose that there exists involutions on* A *and* B *making them operator* $*$*-algebras correspondingly. Let* $\pi : A \to B$ *be a completely contractive (resp. completely isometric)* $\dagger$*-homomorphism, then there is a unital completely contractive (resp. completely isometric)* $\dagger$*-homomorphism extending* π *: from* $A+\mathbb{C}\, I_H$ *to* $B+\mathbb{C}\, I_K$ *(for the completely isometric case we also need* $I_k \notin B$*).*

Proof. By Lemma 6.5, we know that both $A+\mathbb{C}\, I_H$ and $B+\mathbb{C}\, I_K$ are operator $*$-algebras. Let $\pi^0(a+\lambda I_H) = \pi(a) + \lambda I_k$, $a \in A, \lambda \in \mathbb{C}$. We only need to show that π^0 is $\dagger$-preserving. Indeed if a and λ are fixed, then

$$\pi^0((a+\lambda I_H)^\dagger) = \pi^0(a^\dagger + \overline{\lambda} I_H) = \pi(a^\dagger) + \bar{\lambda} I_K = (\pi(a) + \lambda I_K)^\dagger = \pi^0(a+\lambda I_H)^\dagger. \quad \square$$

Remark 6.7. One may replace completely contractive (resp. completely isometric) by contractive (resp. isometric), then similar results could be obtained.

Corollary 6.8. *The unitization A^1 of an operator $*$-algebra is unique up to completely isometric $\dagger$-isomorphism.*

6.3 Universal algebras of an operator $*$-algebra

There are minimal and maximal C^*-algebras generated by an operator $*$-algebra. Let us first look at C^*-envelope or minimal C^*-cover.

Proposition 6.9. [12, Proposition 1.16] *Suppose that A is an operator $*$-algebra with completely isometric involution $\dagger : A \to A$. Then there exists an order two automorphism $\sigma : C_e^*(A) \to C_e^*(A)$ such that $\sigma(i(a^\dagger)) = i(a)^*$.*

Proof. Define the completely isometric algebra homomorphism $j : A \to C_e^*(A)$ by $j(a) = i(a^\dagger)^*$. Since $j(A) \subseteq C_e^*(A)$ generates $C_e^*(A)$ as a C^*-algebra, there exists a $*$-homomorphism $\sigma : C_e^*(A) \to C_e^*(A)$ such that $(\sigma \circ j)(a) = i(a)$. But then we have that

$$\sigma(i(a^\dagger)) = \sigma(j(a))^* = i(a)^* = i(a^*),$$

proving $\sigma(i(a^\dagger)) = i(a)^*$. Moreover, σ has order 2 since

$$\sigma^2(j(a)) = (\sigma \circ i)(a) = i(a^\dagger)^* = j(a),$$

and since $j(A) \subseteq C_e^*(A)$ generates $C_e^*(A)$ as a C^*-algebra. $\square$

Now we turn to the maximal C^*-algebra case.

Consider ρ, the direct sum of 'all' completely contractive representations $\pi : A \to B(H_\pi)$. There are standard ways to avoid the set theoretic issues with 'all' here-see [14, Proposition 2.4.2]. Let $C^*_{\max}(A)$ be the C^*-subalgebra of $B(\oplus_\pi H_\pi)$ generated by $\rho(A)$, then we have the following theorem:

Theorem 6.10. *Let A be an operator $*$-algebra. Then there exists an order two $*$-automorphism $\sigma : C^*_{\max}(A) \to C^*_{\max}(A)$ such that*

$$\sigma(\rho(A)) = \rho(A)^* \quad \textit{and} \quad \rho(a)^* = \sigma(\rho(a^\dagger)).$$

Proof. Let π be the completely isometrically isomorphism $\pi : A \to C^*_{\max}(A)$ defined by $\pi(a) = \rho(a^\dagger)^*$. By the universal property of $C^*_{\max}(A)$, there exists a unique $*$-homomorphism $\sigma : C^*_{\max}(A) \to C^*_{\max}(A)$ such that $\sigma(\rho(a)) = \pi(a) = \rho(a^\dagger)^*$ for any $a \in A$. Hence, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[d, "\rho"] \arrow[dr, "\pi"] & \\ C^*_{\max}(A) \arrow[r, "\sigma"] & C^*_{\max}(A) \end{tikzcd}$$

Moreover, σ has order 2 since

$$\sigma^2(\rho(A)) = \sigma(\rho(a^\dagger)^*) = \sigma(\rho(a^\dagger))^* = \rho(a),$$

and since $\rho(A) \subseteq C^*_{\max}(A)$ generates $C^*_{\max}(A)$ as a C^*-algebra. $\square$

Theorem 6.11. *If A is an operator $*$-algebra, then A^{**} is also an operator $*$-algebra. Indeed, if $(\xi_s) \in A^{**}$ converges to $\xi \in A^{**}$ in weak*-topology, then $\xi_s^\dagger \xrightarrow{w^*} \xi^\dagger$.*

Proof. There are two proofs given here.

(1) Suppose σ is the unique $*$-automorphism with order 2 in Theorem 6.10. Let $\theta : A \to A^*$ be the canonical map with $\theta(a) = (a^\dagger)^*$. Hence, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\rho"'] & A^* \arrow[d, "\rho^*"] \\ C^*_{\max}(A) \arrow[r, "\sigma"] & C^*_{\max}(A) \end{tikzcd}$$

here ρ is as above. Denote $C = C^*_{\max}(A)$ and let $\tilde{\sigma}$ be the w^*-continuous extension of σ on C^{**}, then $\tilde{\sigma}$ is a completely isometric isomorphism. Let $\eta^\dagger = \tilde{\sigma}(\eta)^*$ for any $\eta \in A^{**}$, then $\|\eta^\dagger\| = \|\eta\|$. Suppose that $(x_t) \in A$ and $x_t \to \eta$ in w^*-topology, then $\eta^\dagger = w^* - \lim \tilde{\sigma}(x_t)^* \in \overline{A}^{w^*} \cong A^{**}$. Therefore, A^{**} is an operator $*$-algebra.

If $(\xi_s) \in A^{**}$ converges to $\xi \in A^{**}$ in weak*-topology, then $\tilde{\sigma}(\xi_s)^* \xrightarrow{w^*} \tilde{\sigma}(\xi)^*$ which means that $a_s^\dagger \to \xi^\dagger$ in weak*-topology.

(2) From [14, Corollary 2.5.6], we know that A^{**} is an operator algebra. Moreover, A^{**} is an operator $*$-space from Corollary 5.14. We only need to show that A^{**} is a $*$-algebra. For any $\nu, \eta \in A^{**}$ and $\varphi \in A^*$, we let (a_α) and (b_β) be two nets in A converges to η and ν in w^*-topology of A^{**}. Then

$$\langle (\eta\nu)^\dagger, \varphi \rangle = \overline{\langle \eta\nu, \varphi^\dagger \rangle} = \lim_\alpha \lim_\beta \overline{\langle \varphi^\dagger, a_\alpha b_\beta \rangle}$$

$$
\begin{aligned}
&= \lim_\alpha \lim_\beta \langle \varphi, (a_\alpha b_\beta)^\dagger \rangle = \lim_\alpha \lim_\beta \langle \varphi, b_\beta^\dagger a_\alpha^\dagger \rangle \\
&= \langle \nu^\dagger \eta^\dagger, \varphi \rangle.
\end{aligned}
$$

Thus, $(\eta\nu)^\dagger = \nu^\dagger \eta^\dagger$. □

Corollary 6.12. *For any operator $*$-algebra A, then for any $n \geq 1$*

$$M_n(A)^{**} \cong M_n(A^{**}),$$

as operator $*$-algebras.

Definition 6.13. If the C^*-cover (B, j) generated by an operator $*$-algebra A is involutive and $j(a)^\dagger = j(a^\dagger)$, for any $a \in A$, we say that the involution on B is *compatible with A*.

Remark 6.14. The unwary reader might have expected every C^*-cover $(C^*(A), j)$ generated by an operator $*$-algebra A has an involution compatible with the one on A. However, that is false.

Example 6.15. The Toeplitz C^*-algebra is a well known C^*-cover of the disk algebra $A(\mathbb{D})$. We show that it is not compatible with the involution $\overline{f(\bar{z})}$ on $A(\mathbb{D})$. Let S be the unilateral shift on $\ell^2(\mathbb{N}_0)$ and $\mathrm{oa}(S)$ be the operator algebra generated by S. Then $\mathrm{oa}(S)$ is an operator $*$-algebra with trivial involution induced by $S^\dagger = S$. Suppose that the Toeplitz C^*-algebra $C^*(S)$ has an involution compatible with $\mathrm{oa}(S)$. Then there exists an order-2 $*$-isomorphism $C^*(S)$ such that $\sigma(S^\dagger) = S^*$. Moreover, we have

$$I = \sigma(I) = \sigma(S^*S) = \sigma(S)^*\sigma(S) = SS^* \neq I,$$

which is a contradiction.

Theorem 6.16. *Suppose that A is an operator $*$-algebra (possibly not approximately unital) and B is a C^*-cover generated by (A, j) where j is a completely isometric homomorphism. Then B has an involution compatible with A if and only if there exists an order 2 $*$-automorphism $\sigma : B \to B$ such that for any $a \in A$, $\sigma(j(a^\dagger)) = j(a)^*$.*

Proof. ($\Rightarrow$) If B has an involution compatible with A, then $j(a)^\dagger = j(a^\dagger)$, for all $a \in A$. Define $\sigma : B \to B$ by $\sigma(b) = (b^*)^\dagger$ for any $b \in B$. Then it is easy to see that σ is an order 2 $*$-automorphism.

($\Leftarrow$) The involution on B is defined by $b^\dagger = \sigma(b)^*$ for any $b \in B$. Then B is a C^*-algebra with involution which is compatible with A. $\square$

Example 6.17. Let A be a uniform algebra, then $A \subset C_e^*(A) = C(\partial A)$, where ∂A is Shilov boundary. Moreover, if A has involution, then there exists order-2 homeomorphism $\sigma : \partial A \to \partial A$ such that $f^\dagger(\omega) = \overline{f(\tau(\omega))}$ and $\overline{f \circ \tau} \in A$ for any $f = a \in A$.

6.4 Real positivity and the $\mathfrak{F}$-transform

Because of the uniqueness of unitization, for an operator algebra A we can define unambiguously $\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$. Then $\frac{1}{2}\mathfrak{F}_A = \{a \in A : \|1 - 2a\| \leq 1\} \subset \mathrm{Ball}(A)$. Similarly, $\mathfrak{r}_A$, the *real positive* or *accretive* elements in A, is $\{a \in A : a + a^* \geq 0\}$, where the adjoint a^* is taken in any C^*-cover of A. We write $\mathrm{oa}(x)$ for the operator algebra generated by an operator x. We write $x \preccurlyeq y$ if $y - x \in \mathfrak{r}_A$. The ensuing 'order theory' in the involutive case is largely similar to the operator algebra case.

Theorem 6.18. *Let A be an operator $*$-algebra which generates a C^*-algebra B with compatible involution $\dagger$, and let $\mathcal{U}_A = \{a \in A : \|a\| < 1\}$. The following are equivalent:*

(1) *A is approximately unital.*

(2) *For any $\dagger$-selfadjoint positive $b \in \mathcal{U}_B$ there exists $\dagger$-selfadjoint $a \in \mathfrak{c}_A$ with $b \preccurlyeq a$.*

(2') *Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$ and 'nearly positive' in the sense of the introduction to [22]: we can make it as close in norm as we like to an actual positive element.*

(3) *For any pair of $\dagger$-selfadjoint elements $x, y \in \mathcal{U}_A$ there exist nearly positive $\dagger$-selfadjoint $a \in \frac{1}{2}\mathfrak{F}_A$ with $x \preccurlyeq a$ and $y \preccurlyeq a$.*

(4) *For any $\dagger$-selfadjoint $b \in \mathcal{U}_A$ there exist nearly positive $\dagger$-selfadjoint $a \in \frac{1}{2}\mathfrak{F}_A$ with $-a \preccurlyeq b \preccurlyeq a$.*

(5) *For any $\dagger$-selfadjoint $b \in \mathcal{U}_A$ there exist $\dagger$-selfadjoint $x, y \in \frac{1}{2}\mathfrak{F}_A$ with $b = x - y$.*

(6) *$\mathfrak{r}_A$ is a generating cone, indeed any $\dagger$-selfadjoint element in A is a difference of two $\dagger$-selfadjoint elements in $\mathfrak{r}_A$.*

(7) *Same as (6) but with $\mathfrak{r}_A$ replaced by $\mathfrak{F}_A$.*

Proof. (1) $\Rightarrow$ (2') By the proof in [22, Theorem 2.1] for any $\dagger$-selfadjoint positive $b \in \mathcal{U}_B$ there exists $c \in \frac{1}{2}\mathfrak{F}_A$ and nearly positive with $b \leq \operatorname{Re} c$. Hence it is easy to see that $b \leq \operatorname{Re}(c^\dagger)$ and $b \leq \operatorname{Re} a$ where $a = (c + c^\dagger)/2$.

(2') $\Rightarrow$ (3) By C^*-algebra theory there exists positive $b \in \mathcal{U}_B$ with $\operatorname{Re} x$ and $\operatorname{Re} y$ both $\leq b$. It is easy to see that $b^\dagger = \sigma(b) \geq 0$. Then $\operatorname{Re} x \leq b^\dagger$, so that $\operatorname{Re} x \leq (b + b^\dagger)/2$. Similarly for y. Then apply (2') to obtain a from $(b + b^\dagger)/2$.

The remaining implications follow the proof in [22, Theorem 2.1] or Theorem 4.44 but using tricks similar to the ones we have used so far in this proof. $\square$

Definition 6.19. If $T \in B(H)$ for some Hilbert space H and $-1 \notin \operatorname{Sp}(T)$, then the *Cayley transform* is defined by $\kappa(T) = (T - I)(T + I)^{-1}$. The $\mathfrak{F}$-transform is $\mathfrak{F}(T) = \frac{1}{2}(1 + \kappa(T)) = T(1 + T)^{-1}$.

Proposition 6.20. *Let A be an operator $*$-algebra and σ be the associated $*$-automorphism on a (compatible) C^*-cover. If $x \in \mathfrak{r}_A$, then $x^\dagger$ is also real positive and $\kappa(x^\dagger) = \kappa(x)^*$.*

Proof. If x is real positive, then $x + x^* \geq 0$ which is equivalent to say that $\sigma(x) + \sigma(x)^* \geq 0$. This implies that $x^\dagger + (x^\dagger)^* = \sigma(x)^* + \sigma(x) \geq 0$.

Also, $\sigma(\kappa(x^\dagger)) = \sigma((x^\dagger - 1)(x^\dagger + 1)^{-1}) = (x^* - 1)(x^* + 1)^{-1} = \kappa(x)^*$. So $\kappa(x^\dagger) = \kappa(x)^\dagger$. $\square$

Lemma 6.21. [22, Lemma 2.5] *For any operator algebra A, the $\mathfrak{F}$-transform maps $\mathfrak{r}_A$ bijectively onto the set of elements of $\frac{1}{2}\mathfrak{F}_A$ of norm < 1. Thus $\mathfrak{F}(\mathfrak{r}_A) = \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$.*

Proof. First assume that A is unital. By Definition 6.19, $\mathfrak{F}(\mathfrak{r}_A)$ is contained in the set of elements of $\frac{1}{2}\mathfrak{F}_A$ whose spectrum does not contain 1. The inverse of the $\mathfrak{F}$-transform on this domain is $T(I-T)^{-1}$. To see for example that $T(I-T)^{-1} \in \mathfrak{r}_A$ if $T \in \frac{1}{2}\mathfrak{F}_A$ note that 2 $\mathrm{Re}(T(I-T)^{-1})$ equals

$$(I-T^*)^{-1}(T^*(I-T)+(I-T^*)T)(I-T)^{-1} = (I-T^*)(T+T^*-2T^*T)(I-T)^{-1},$$

which is positive since T^*T dominated by $\mathrm{Re}(T)$ if $T \in \frac{1}{2}\mathfrak{F}_A$. Hence for any (possible nonunital) operator algebra A the $\mathfrak{F}$-transform maps $\mathfrak{r}_A^1$ bijective onto the set of $\frac{1}{2}\mathfrak{F}_{A^1}$ whose spectrum does not contain 1. However, this equals the set of elements of $\frac{1}{2}\mathfrak{F}_{A^1}$ of norm < 1. Indeed, if $\|\mathfrak{F}(x)\| = 1$ then $\|\frac{1}{2}(1+\kappa(x))\| = 1$, and so $1-\kappa(x)$ is not invertible. Hence $1 \in \mathrm{Sp}_{A^1}(\kappa(x))$ and $1 \in \mathrm{Sp}_{A^1}(\mathfrak{F}(x))$. Since $\mathfrak{F}(x) \in A$ iff $x \in A$, we are done. □

Let A be an operator $*$-algebra and S a subset of A, we denote $H(S) = \{s \in S : s = s^\dagger\}$.

Corollary 6.22. *For any operator algebra A, the $\mathfrak{F}$-transform maps $H(\mathfrak{r}_A)$ bijectively onto the set of elements of $\frac{1}{2}H(\mathfrak{F}_A)$ of norm < 1.*

Thus in some sense we can identify $\mathfrak{r}_A$ with the strict contraction in $\frac{1}{2}\mathfrak{F}_A$, this for example induces an order on this set of strict contractions.

We recall that the positive part of the open unit ball of a C^*-algebra is a directed set, and indeed is a net which is a positive approximate identity for C^*-algebra. The following generalized this to operator $*$-algebras.

Proposition 6.23. *If A is an approximately unital operator $*$-algebra, then the* *seldadjoint part of* $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ *is a directed set in the* $\preceq$ *ordering and with this ordering the seldadjoint part of* $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ *is an increasing cai for* A.

Proof. By Corollary 6.22, we know $\mathfrak{F}(H(\mathfrak{r}_A)) = H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A)$. . By Theorem 6.18, $H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A)$ is directed by $\preceq$. So we may view $H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A)$ as a net (e_t). Given $x \in \frac{1}{2}H(\mathfrak{F}_A)$, choose n such that $\|\mathrm{Re}(x^{1/n})x - x\| < \varepsilon$ (note that $\mathrm{Re}(x^{1/n})$ is a cai for $C^*(\mathrm{oa}(x))$). If $z \in H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A)$ with $x^{1/n} \preceq z$ then

$$x^*|1 - z|^2 x \le x^*(1 - \mathrm{Re}(z))x \le x^*(1 - \mathrm{Re}(x^{1/n}))x \le \varepsilon.$$

Thus $e_t x \to x$ for all $x \in H(\frac{1}{2}\mathfrak{F}_A)$. Also, since $H(\frac{1}{2}\mathfrak{F}_A)$ generates the open unit ball, then we are done. $\square$

Corollary 6.24. *Let* A *be an approximately unital operator* $*$*-algebra, and* B *a compatible* C^**-algebra generated by* A*. If* $b \in B_+$ *with* $\|b\| < 1$ *then there is an increasing cai for* A *in* $H(\frac{1}{2}\mathfrak{F}_A)$*, every term of which dominates* b *(where 'incrasing' and 'dominates' are in the* $\preceq$ *ordering).*

Proof. Since $H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A)$ is a directed set, $\{a \in H(\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A) : b \preceq a\}$ is a subnet of this increasing cai in the last result. $\square$

6.5 Involutive ideals

An *involutive ideal* or †*-ideal* in an operator algebra with involution † is an ideal J with $J^\dagger \subset J$.

Proposition 6.25. *Let A be an operator $*$-algebra. Suppose J is a closed $\dagger$-ideal, then J and A/J are operator $*$-algebras.*

Proof. This follows from the matching fact for operator algebras [14, Proposition 2.3.4], and the computation

$$\|[a_{ji}^{\dagger} + J]\| \leq \|[a_{ji}^{\dagger} + x_{ji}^{\dagger}]\| \leq \|[a_{ij} + x_{ij}]\|, \qquad x_{ij} \in J,$$

so that $\|[a_{ji}^{\dagger} + J]\| \leq \|[a_{ij} + J]\|$ for $a_{ij} \in A$. Similarly, we have $\|[a_{ij} + J]\| \leq \|[a_{ji}^{\dagger} + J]\|$. □

Corollary 6.26 (Interpolation between operator $*$-algebras)**.** *Let (A_0, A_1) be a compatible couple of operator $*$-algebra, then $A_\theta = [A_0, A_1]$ is an operator $*$-algebra for any $\theta \in [0, 1]$.*

Proof. By Example 5.11, we know that $\mathcal{F}(A_0, A_1)$ is an operator $*$-space. Indeed, it is an operator $*$-algebra with the involution $\dagger$.

For any $0 \leq \theta \leq 1$, let $\mathcal{F}_\theta(A_0, A_1)$ be the two-sided closed ideal of all $f \in \mathcal{F}$ for which $f(\theta) = 0$. This is $\dagger$-selfadjoint. The interpolation space $A_\theta = [A_0, A_1]_\theta$ is the subspace of $A_0 + A_1$ formed by all $x = f(\theta)$ for some $f \in \mathcal{F}$. As operator spaces, the interpolation space $A_\theta \cong \mathcal{F}(A_0, A_1)/\mathcal{F}_\theta(A_0, A_1)$ through the map $\pi : f \mapsto f(\theta)$. It is easy to see that π is $\dagger$-linear. By Proposition 6.25, the quotient $A_\theta \cong \mathcal{F}(A_0, A_1)/\mathcal{F}_\theta(A_0, A_1)$ is an operator $*$-algebra. □

6.6 Contractive approximate identities

Lemma 6.27. *Let A be an operator $*$-algebra. Then the following are equivalent:*

(i) *A has a cai.*

(ii) *A has a $\dagger$-selfadjoint cai.*

(iii) *A has a left cai.*

(iv) *A has a right cai.*

(v) *A^{**} has an identity of norm* 1.

Proof. (i) $\Rightarrow$ (ii) If (e_t) is a cai for A, then $(e_t^\dagger)$ is also a cai for A. Let $f_t = (e_t+e_t^\dagger)/2$, then (f_t) is a $\dagger$-selfadjoint cai for A.

(iii) $\Rightarrow$ (iv) If (e_t) is a left cai for A, then $(e_t^\dagger)$ is a right cai. Analogously, it is easy to see that (iv) $\Rightarrow$ (iii.)

(iv) $\Rightarrow$ (v) Suppose that e is the limit of left cai (resp. right cai), then e (resp. right identity) of A^{**}. So $e = ef = f$. which implies that e is an identity for A with norm 1.

That (ii) $\Rightarrow$ (i), and (i) $\Rightarrow$ (iii), are obvious. That (v) $\Rightarrow$ (i) follows from Proposition 2.5.8 in [14]. □

Corollary 6.28. *If A is an operator $*$-algebra with a countable cai (f_n), then A has a countable $\dagger$-selfadjoint cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. By [20, Theorem 1.1], A has a cai (e_t) in $\frac{1}{2}\mathfrak{F}_A$. Denote $e'_t = \frac{e_t + e_t^\dagger}{2}$, then (e'_t) is also a cai in $\frac{1}{2}\mathfrak{F}_A$. Choosing t_n with $\|f_n e'_{t_n} - f_n\| + \|e'_{t_n} f_n - f_n\| < 2^{-n}$, it is easy to see that (e'_{t_n}) is a countable $\dagger$-selfadjoint cai in $\frac{1}{2}\mathfrak{F}_A$. $\square$

Corollary 6.29. *If J is a closed two-sided $\dagger$-ideal in an operator $*$-algebra A and if J has a cai, then J has a $\dagger$-selfadjoint cai (e_t) with $\|1 - 2e_t\| \leq 1$ for all t, which is also quasicentral in A.*

Proof. By the proof of Corollary 6.28, we know that J has a $\dagger$-selfadjoint cai, denoted (e_t), in $\frac{1}{2}\mathfrak{F}_A$. The weak* limit q of (e_t) is a central projection in A^{**}, and so $e_t a - a e_t \to 0$ weakly for all $a \in A$. A routine argument using Mazur's theorem shows that convex combination of the e_t comprise the desired cai, and they will still have the property of being $\dagger$-selfadjoint and in the convex set $\frac{1}{2}\mathfrak{F}_A$. $\square$

6.7 Cohen factorization for operator $*$-algebras

The Cohen factorization theorem is a crucial tool for Banach algebras, operator algebras and their modules. In this section we will give a variant that works for operator $*$-algebras and their modules.

Recall that if X is a Banach space and A is a Banach algebra then X is called a *Banach A-module* if there is a module action $A \times X \to X$ which is a contractive linear map. If A has a bounded approximate identity (e_t) then we say that X is *nondegenerate* if $e_t x \to x$ for $x \in X$. A *Banach A-bimodule* is both a left and a right Banach A-module such that $a(xb) = (ax)b$.

The following is an operator $*$-algebra version of the Cohen factorization theorem:

Theorem 6.30. *If A is an approximately unital operator $*$-algebra, and if X is a nondegenerate Banach A-module(resp. A-bimodule), if $b \in X$ then there exists an element $b_0 \in X$ and a $\dagger$-selfadjoint $a \in \mathfrak{F}_A$ with $b = ab_0$ (resp. $b = ab_0a$). Moreover if $\|b\| < 1$ then b_0 and a may be chosen of norm < 1.*

Proof. We adopt the idea in the proof of Cohen factorization Theorem(see, e.g.[53, Theorem 4.1]). Suppose that $b \in X$ with $\|b\| < 1$. Given any $\varepsilon > 0$, let $a_0 = 1$. Choose $\dagger$-selfadjoint $f_1 \in \frac{1}{2}\mathfrak{F}_A$ from the cai such that

$$\|ba_0^{-1}(1 - f_1)\| + \|(1 - f_1)a_0^{-1}b\| < 2^{-2}\varepsilon.$$

Let $a_1 = 2^{-1}f_1 + 2^{-1}$, then $a_1 \in \mathfrak{F}_{A^1}$. By Neumann lemma, a_1 is invertible in $\mathrm{oa}^*(1, a_1)$, and has inverse in A^1 with $\|a_1^{-1}\| \leq 2$. Similarly, choose $\dagger$-selfadjoint $f_2 \in \frac{1}{2}\mathfrak{F}_A$ such that $\|ba_1^{-1}(1 - f_2)\| + \|(1 - f_2)a_1^{-1}b\| < 2^{-4}\varepsilon$. By induction, let $a_n = \sum_{k=1}^{n} 2^{-k} f_k + 2^{-n}$, then We have

$$\|1 - a_n\| = \|\sum_{k=1}^{n} 2^{-k}(1 - f_k)\| \leq \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}.$$

By the Neumann lemma a_n is invertible in $\mathrm{oa}^*(1, a_n)$, and has inverse in A^1 with $\|a_n^{-1}\| \leq 2^n$. Choose $\dagger$-selfadjoint $f_{n+1} \in \frac{1}{2}\mathfrak{F}_A$ such that

$$\|ba_n^{-1}(1 - f_{n+1})\| + \|(1 - f_{n+1})a_n^{-1}b\| < 2^{-2(n+1)}\varepsilon.$$

Note that $a_{n+1}^{-1} - a_n^{-1} = a_n^{-1}(a_n - a_{n+1})a_{n+1}^{-1} = 2^{-n-1}a_n^{-1}(1 - f_{n+1})a_{n+1}^{-1}$ whereas

$2^{-n-1}a_{n+1}^{-1}(1-f_{n+1})a_n^{-1} = a_{n+1}^{-1}(a_n - a_{n+1})a_n^{-1} = a_{n+1}^{-1} - a_n^{-1}$. Set $x_n = a_n^{-1}b$ (resp. $x_n = a_n^{-1}ba_n^{-1}$). Now we just focus on the bimodule case, the left module case is similar but easier.

$$\begin{aligned}x_{n+1} - x_n &= a_{n+1}^{-1}ba_{n+1}^{-1} - a_n^{-1}ba_n^{-1} = a_{n+1}^{-1}b(a_{n+1}^{-1} - a_n^{-1}) + (a_{n+1}^{-1} - a_n^{-1})ba_n^{-1} \\ &= 2^{-n-1}a_{n+1}^{-1}ba_n^{-1}(1-f_{n+1})a_{n+1}^{-1} + 2^{-n-1}a_{n+1}^{-1}(1-f_{n+1})a_n^{-1}ba_n^{-1}.\end{aligned}$$

$\|x_{n+1} - x_n\|$ is dominated by

$$\begin{aligned}&2^{-n-1}(\|a_{n+1}^{-1}\|^2\|ba_n^{-1}(1-f_{n+1})a_{n+1}^{-1}]\| + \|a_{n+1}^{-1}\|\|a_n^{-1}\|\|(1-f_{n+1})a_n^{-1}b\|\| \\ &\leq 2^{-n-1}(\|a_{n+1}^{-1}\|^2 + \|a_{n+1}^{-1}\|\|a_n^{-1}\|)2^{-2(n+1)}\varepsilon \\ &\leq 2^{-3(n+1)}(2^{2(n+1)} + 2^{2n+1})\varepsilon < 2^{-n}\varepsilon.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Let $b_0 = \lim_n x_n$ and $a = \sum_{k=1}^{+\infty} 2^{-k}f_k$, which is †-selfadjoint. Then $a \in \frac{1}{2}\mathfrak{F}_A$. Hence, $b = ab_0a$ since $b = a_nx_na_n$ and $a_n \to a$ and $x_n \to b_0$. Also,

$$\|x_n - b\| \leq \sum_{k=1}^{n} \|x_k - x_{k-1}\| \leq 2\varepsilon,$$

so that $\|b - b_0\| \leq 2\varepsilon$. Thus $\|b_0\| \leq \|b\| + 2\varepsilon$, and this is < 1 if $2\varepsilon < 1 - \|b\|$. Choose some $t > 1$ such that $\|tb\| < 1$. By the argument above, there exists $a \in \frac{1}{2}\mathfrak{F}_A$ and $b_0 \in B$ of norm < 1 such that $tb = ab_0a$. Let $a' = \frac{a}{\sqrt{t}}$, then $b = a'b_0a'$. Then $\|a'\| < 1$ and $\|b_0\| < 1$. $\square$

6.8 Multiplier algebras

Theorem 6.31. *Let A be an approximately unital operator $*$-algebra. Then the following algebras are completely isometrically isomorphic:*

(i) $LM(A) = \{\eta \in A^{**} : \eta A \subset A\}$,

(ii) $LM(\pi) = \{T \in B(H) : T\pi(A) \subset \pi(A)\}$. *where π is a nondegenerate completely isometric representation of A on a Hilbert space H such that there exists an order 2 $*$-automorphism $\sigma : B(H) \to B(H)$ satisfying $\sigma(\pi(a))^* = \pi(a^\dagger)$ for any $a \in A$*

(iii) *the set of completely bounded right A-module maps $CB_A(A)$.*

Proof. See [14, Theorem 2.6.3]. □

Definition 6.32. Let A be an approximately unital operator $*$-algebra. Then we define

(i) $RM(A) = \{\xi \in A^{**} : A\xi \subset A\}$;

(ii) $RM(\pi) = \{S \in B(H) : \pi(A)S \subset \pi(A)\}$, for any nondegenerate completely isometric representation π of A on a Hilbert space H and there exists order-2 $*$-automorphism $\sigma : B(H) \to B(H)$ satisfies $\sigma(\pi(a))^* = \pi(a^\dagger)$ for any $a \in A$;

(iii) the set of completely bounded left A-module maps, which we denote as ${}_ACB(A)$.

Corollary 6.33. *Let A be an approximately unital operator $*$-algebra. Then*

(a) *$\eta \in LM(A)$ if and only if $\eta^\dagger \in RM(A)$, where $\eta, \eta^\dagger \in A^{**}$ and $\dagger$ is the involution in A^{**};*

(b) *$T \in LM(\pi)$ if and only if $T^\dagger \in RM(\pi)$, where $T^\dagger = \sigma(T)^*$;*

(c) *$L \in CB_A(A)$ if and only if $L^\dagger \in {}_ACB(A)$, where the map $L^\dagger$ is defined by $L^\dagger(a) = L(a^\dagger)^\dagger$.*

Proof. We just give the proof of (b). Suppose that $T \in LM(\pi)$, then

$$\pi(a)T^\dagger = \sigma(\pi(a^\dagger))^*\sigma(T)^* = (\sigma(T\pi(a^\dagger)))^* \in \sigma(\pi(A))^* \subset \pi(A).$$

Thus, $T^\dagger \in RM(\pi)$. Similarly, if $T^\dagger \in RM(\pi)$ then $T \in LM(\pi)$. $\square$

We consider pairs (D, μ) consisting of a unital operator $*$-algebra D and a completely isometric $\dagger$-homomorphism $\mu : A \to D$, such that $D\mu(A) \subset \mu(A)$, $\mu(A)D \subset \mu(A)$. We use the phrase *multiplier operator $*$-algebra* of A, and write $M(A)$, for any pair (D, μ) which is completely $\dagger$-isometrically A-isomorphic to $\mathcal{M}(A) = \{x \in A^{**} : xA \subset A \text{ and } Ax \subset A\}$. Note that by Corollary 5.16, the inclusion of A in A^{**} is a $\dagger$-homomorphism, hence the canonical map $i : A \to \mathcal{M}(A)$, is a $\dagger$-homomorphism. From this it follows that there is a unique involution on $M(A)$ for which i is a $\dagger$-homomorphism.

Proposition 6.34. *Suppose that A is an approximately unital operator $*$-algebra. If (D, μ) is a left multiplier operator algebra of A, then the closed subalgebra*

$$\{d \in D : \mu(A)d \subset \mu(A)\}$$

of D, together with the map μ, is a multiplier operator $*$-algebra of A.

Proof. Let E denote the set $\{d \in D : \mu(A)d \subset \mu(A)\}$. By [14, Proposition 2.6.8], we know that E is a multiplier operator algebra of A. Thus, there exists a completely isometric surjective homomorphism $\theta : \mathcal{M}(A) \to E$ such that $\theta \circ i_A = \mu$. Now we may define an involution on E by $d^\dagger = \theta(\eta^\dagger)$ if $d = \theta(\eta)$. Then it is easy to check that E is an operator $*$-algebra which is completely $\dagger$-isometrically A-isomorphic to $\mathcal{M}(A)$. $\square$

Example 6.35. Let $A = A_1(\mathbb{D})$, the functions in the disk algebra vanishing at 1, which is the norm closure of $(z-1)A(\mathbb{D})$, and let $B = \{f \in C(\mathbb{T}) : f(1) = 0\}$. By the nonunital variant of the Stone-Weierstrass theorem, B is generated as a C^*-algebra by A. Indeed $B = C^*_e(A)$, since any closed ideal of B is the set of functions that vanish on a closed set in the circle containing 1. Also for any $z_0 \in \mathbb{T}$, $z_0 \neq 1$, there is a function in A that peaks at z_0, if necessary by the noncommutative Urysohn lemma for approximately unital operator algebras [15]. So the involution on A descends from the natural involution on B. It is easy to see, for example by examining the bidual of B^{**} and noticing that A and B have a common cai, that $M(A) = \{T \in M(B) : TA \subset A\} = \{g \in C_b(\mathbb{T} \setminus \{1\}) : g(z-1) \in A(\mathbb{D})\}$. For such g, since the negative Fourier coefficients of $k = g(1-z)$ are zero, the negative Fourier coefficients of g are constant, hence zero by the Riemann-Lebesgue lemma. Thus g is in H^∞, and has an analytic extension to the open disk. Viewing g as a function h on $\bar{\mathbb{D}} \setminus \{1\}$ we have $h = k/(z-1)$ for some $k \in A(\mathbb{D})$. So $M(A)$ consists of the bounded continuous functions on $\bar{\mathbb{D}} \setminus \{1\}$ that are analytic in the open disk, with involution $\overline{f(\bar{z})}$.

Let A, B be approximately unital operator $*$-algebras. A completely contractive $\dagger$-homomorphism $\pi : A \to M(B)$ will be called a *multiplier-nondegenerate $\dagger$-morphism*, if B is a nondegenerate bimodule with respect to the natural module action of A on B via π. This is equivalent to saying that for any cai (e_t) of A, we have $\pi(e_t)b \to b$ and $b\pi(e_t) \to b$ for $b \in B$.

Proposition 6.36. *If A, B are approximately operator $*$-algebras, and if $\pi : A \to M(B)$ is a multiplier-nondegenerate $\dagger$-morphism then π extends uniquely to a unital completely contractive $\dagger$-homomorphism $\hat{\pi} : M(A) \to M(B)$. Moreover $\hat{\pi}$ is completely isometric if and only if π is completely isometric.*

Proof. Regard $M(A)$ and $M(B)$ as $\dagger$-subalgebras of A^{**} and B^{**} respectively. Let $\tilde{\pi} : A^{**} \to B^{**}$ be the unique w^*-continuous $\dagger$-homomorphism extending π. From [14, Proposition 2.6.12], we know that $\hat{\pi} = \tilde{\pi}(\cdot)_{|_{M(A)}}$ is the unique bounded homomorphism on $M(A)$ extending π, and $\hat{\pi}(M(A)) \subset M(B)$.

Let (e_t) be a $\dagger$-selfadjoint cai for A. Then for any $\eta \in M(A)$, $\eta e_t \in A$ and $\eta e_t \xrightarrow{w^*} \eta$. Hence

$$\hat{\pi}(\eta) = w^* - \lim_t \pi(\eta e_t).$$

On the other hand, $(\eta e_t)^\dagger \to \eta^\dagger$, which implies

$$\hat{\pi}(\eta^\dagger) = w^* - \lim_t \pi((\eta e_t)^\dagger) = w^* - \lim_t \pi((\eta e_t))^\dagger.$$

Since the involution on A^{**} is w^*-continuous, we get that

$$\hat{\pi}(\eta^\dagger) = \hat{\pi}(\eta)^\dagger.$$

The rest follows from [14, Proposition 2.6.12]. □

6.9 Dual operator $*$-algebras

Definition 6.37. Let M be a dual operator algebra and operator $*$-algebra such that the involution on M is weak* continuous. Then M is called a *dual operator $*$-algebra.*

We will identify any two dual operator $*$-algebras M and N which are w^*-homeomorphically and completely $\dagger$-isometrically isometric.

Proposition 6.38. *Let M be a dual (possibly nonunital) operator $*$-algebra.*

(1) *The w^*-closure of a $*$-subalgebra of M is a dual operator $*$-algebra.*

(2) *The unitization of M is also a dual operator $*$-algebra.*

Proof. For (1), the weak* -closure of $*$-subalgebra of M is a dual operator algebra by [14, Proposition 2.7.4 (4)].

For (2), suppose that M is a nonunital operator $*$-algebra and write I for the identity in M^1. Suppose that $(x_t)_t$ and $(\lambda_t)_t$ are nets in M and $\mathbb{C}$ respectively, with $(x_t + \lambda_t I)$ converging in w^*-topology. By Hahn-Banach theorem, it is easy to see that $(\lambda_t)_t$ converges in $\mathbb{C}$. It follows that $(x_t)_t$ converges in M in the w^*-topology. Thus, $(x_t + \lambda_t I)^\dagger$ converges in M^1, in the w^*-topology. The rest follows immediately from [14, Proposition 2.7.4 (5)]. □

Proposition 6.39. *Let A be an operator $*$-algebra, and I any cardinal. Then $\mathbb{K}_I(A)^{**} \cong \mathbb{M}_I(A^{**})$ as dual operator $*$-algebras.*

Proof. The canonical embedding $A \subset A^{**}$ induces a completely isometric †-homomorphism $\theta : \mathbb{K}_I(A) \to \mathbb{K}_I(A^{**}) \subset \mathbb{M}_I(A^{**})$. Notice that the involutions on $\mathbb{K}_I(A)^{**}$, $\mathbb{M}_I(A^{**})$ are w^*-continuous and $\mathbb{K}_I(A)^{**} \cong \mathbb{M}_I(A^{**})$ as operator algebras. Thus, $\mathbb{K}_I(A)^{**} \cong \mathbb{M}_I(A^{**})$ as dual operator $*$-algebras. $\square$

Lemma 6.40. *If X is a weak* closed selfadjoint subspace of $B(H)$ for a Hilbert space H, then $\mathcal{U}(X)$ as defined as in Example 6.2 is a dual operator $*$-algebra.*

Proof. By [14, Lemma 2.7.7(2)], we know that $\mathcal{U}(X)$ is a dual operator algebra. And, it is easy to see that the involution defined in Example 6.2 is weak*-continuous. So $\mathcal{U}(X)$ is an operator $*$-algebra. $\square$

The last result can be used to produce counterexamples concerning dual operator $*$-algebras, such as algebras with two distinct preduals, etc. Similarly one may use the $\mathcal{U}(X)$ construction to easily obtain an example of a dual operator algebra which is an operator $*$-algebra, but the involution is not weak*-continuous.

Recall that in [24] the *maximal W^*-algebra* $W^*_{\max}(M)$ was defined for unital dual operator algebras M. If M is a dual operator algebra but is not unital we define $W^*_{\max}(M)$ to be the von Neumann subalgebra of $W^*_{\max}(M^1)$ generated by the copy of M. Note that it has the desired universal property: if $\pi : M \to N$ is a weak* continuous completely contractive homomorphism into a von Neumann algebra N, then by the normal version of Meyer's theorem we may extend to a weak* continuous completely contractive unital homomorphism $\pi^1 : M^1 \to N$. Hence by the universal property of $W^*_{\max}(M^1)$, we may extend further to a normal unital $*$-homomorphism from $W^*_{\max}(M^1)$ into N. Restricting to $W^*_{\max}(M)$ we have shown that there exists a normal $*$-homomorphism $\tilde{\pi} : W^*_{\max}(M) \to N$ extending π.

Proposition 6.41. *Let $B = W^*_{\max}(M)$. Then M is a dual operator $*$-algebra if and only if there exists an order two $*$-automorphism $\sigma : B \to B$ such that $\sigma(M) = M^*$. In this case the involution on M is $a^\dagger = \sigma(a)^*$.*

Proof. This follows from a simple variant of the part of the proof of Theorem 6.10 that we did prove above, where one ensures that all maps there are weak* continuous. $\square$

Proposition 6.42. *For any dual operator $*$-algebra M, there is a Hilbert space H (which may be taken to be $K \oplus K$ if $M \subset B(K)$ as a dual operator algebra completely isometrically), and a symmetry (that is, a selfadjoint unitary) u on H, and a weak* continuous completely isometric homomorphism $\pi : M \to B(H)$ such that $\pi(a)^* = u\pi(a^\dagger)u$ for $a \in M$.*

Proof. This is a tiny modification of the proof in [12, Proposition 1.12], beginning with a weak* continuous completely isometric homomorphism $\rho : M \to B(K)$. Define $\pi : M \to B(K \oplus K)$ by $\pi(a) = \begin{pmatrix} \rho(a) & 0 \\ 0 & \rho(a^\dagger)^* \end{pmatrix}$ and let $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is easy to check that the $\pi : M \to B(H)$ is weak* continuous. $\square$

Proposition 6.43. *Let M be a dual operator $*$-algebra, and let I be a w^*-closed $\dagger$-ideal. Then M/I is a dual operator $*$-algebra.*

Proof. From [14, Proposition 2.7.11], we know that M/I is a dual operator algebra. As dual operator spaces, $M/I \cong (I_\perp)^*$, from which it is easy to see that the involution on M/I is w^*-continuous. $\square$

Lemma 6.44. *If A is an operator $*$-algebra then $\Delta(A) = A \cap A^*$ (adjoint taken in any containing C^*-algebra; see 2.1.2 in [14]), is a C^*-algebra and $\Delta(A)^\dagger = \Delta(A)$.*

Proof. That $\Delta(A)$ does not depend on the particular containing C^*-algebra may be found in e.g. 2.1.2 in [14]. as is the fact that it is spanned by its selfadjoint (with respect to the usual involution) elements. If A is also an operator $*$-algebra then $\Delta(A)$ is invariant under $\dagger$. Indeed suppose that B is a C^*-cover of A with compatible involution coming from a $*$-automorphism σ as usual. If $x = x^* \in \Delta(A)$ then $\sigma(x)$ is also selfadjoint, so is in $\Delta(A)$. This holds by linearity for any $x \in \Delta(A)$. So $\Delta(A)^\dagger = \Delta(A)$. $\square$

If M is a dual operator algebra then $\Delta(M) = M \cap M^*$, is a W^*-algebra (see e.g. 2.1.2 in [14]). If M is a dual operator $*$-algebra then $\Delta(M)$ is a dual operator $*$-algebra, indeed it is a W^*-algebra with an extra involution $\dagger$ inherited from M.

Proposition 6.45. *Suppose that M is a dual operator $*$-algebra. Suppose that $(p_i)_{i\in I}$ is a collection of projections in M. Then $(\wedge_{i\in I}\, p_i)^\dagger = \wedge_{i\in I}\, p_i^\dagger$ and $(\vee_{i\in I}\, p_i)^\dagger = \vee_{i\in I}\, p_i^\dagger$.*

Proof. By the analysis above the proposition we may assume that M is a W^*-algebra with an extra involution $\dagger$, which is weak* continuous and is of the form $x^\dagger = \sigma(x)^*$ for a weak* continuous period 2 $*$-automorphism σ of M. If p_i and p_j are two projections in M, then $p_i \wedge p_j = \lim_n (p_i p_j)^n = \lim_n (p_j p_i)^n$. By the weak*-continuity of involution on M we have

$$(p_i \wedge p_j)^\dagger = \lim_n [(p_j p_i)^n]^\dagger = \lim_n (p_i^\dagger p_j^\dagger)^n = p_i^\dagger \wedge p_j^\dagger.$$

Thus for any finite subset F of I, we have $(\wedge_{i\in F} p_i)^\dagger = \wedge_{i\in F} p_i^\dagger$. Note that the net $(\wedge_{i\in F} p_i)_F$ indexed by the directed set of finite subsets F of I, is a decreasing net with limit $\wedge_{i\in I} p_i$. We have

$$(\wedge_{i\in I} p_i)^\dagger = \lim_F (\wedge_{i\in F} p_i)^\dagger = \lim_F (\wedge_{i\in F} p_i^\dagger) = \wedge_{i\in I} p_i^\dagger.$$

by weak* continuity of involution. The statement about suprema of projections follows by taking orthocomplements. □

6.10 Involutive M-ideals

Definition 6.46. Let X be a Banach space.

(a) A linear projection P is called an *M-projection* if

$$\|x\| = \max\{\|Px\|, \|x - px\|\} \quad \text{for all } x \in X.$$

(b) A closed subspace $J \subset X$ is called an *M-summand* if it is the range of an M-prorjection.

Definition 6.47. If X is an operator space, then a linear idempotent $P : X \to X$

is said to be a *left M-projection* if the map

$$\sigma_P(x) = \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$$

is a complete isometry from $X \to C_2(X)$. A *right M-summand* of X is the range of a left M-projection on X. We say that Y is a *right M-ideal* if $Y^{\perp\perp}$ is a right M-summand in X^{**}. Similarly, a linear idempotent $Q : X \to X$ is said to be *right M-projection* if the map

$$\sigma_Q(x) = \begin{bmatrix} P(x) & x - P(x) \end{bmatrix}$$

is a complete isometry from $X \to R_2(X)$. Similar definitions pertain to *left M-ideals* and so on.

An M-projection P on a Banach $*$-space is called a *$\dagger$-M-projection* if P is $\dagger$-preserving. A subspace Y of a Banach $*$-space is called a *$\dagger$-M-summand* if Y is the range of a $\dagger$-M-projection. Such range is $\dagger$-closed. Indeed, if $y \in Y$, then $y = P(x)$ for some $x \in X$. Thus, $y^\dagger = P(x)^\dagger = P(x^\dagger) \in Y$. A subspace Y of E is called an *involutive M-ideal* or a *$\dagger$-M-ideal* in E if $Y^{\perp\perp}$ is a $\dagger$-M-summand in E^{**}. If X is an operator $*$-space, then an M-projection is called a *complete $\dagger$-M-projection* if the amplification P_n is a $\dagger$-M-projection on $M_n(X)$ for every $n \in \mathbb{N}$. Similarly, we could define *complete $\dagger$-M-summand*, *complete $\dagger$-M-ideal*, *left $\dagger$-M-projection*, *right $\dagger$-M-summand* and *right $\dagger$-M-ideal*.

Proposition 6.48. *Let X be an operator $*$-space.*

(1) *A linear idempotent $\dagger$-linear map $P : X \to X$. P is a left $\dagger$-M-projection if and only if it is a right $\dagger$-M-projection, and these imply P is a complete $\dagger$-M-projection.*

(2) *A subspace Y of X is a complete $\dagger$-M-summand if and only if it is a left $\dagger$-M-summand if and only if it is a right $\dagger$-M-summand.*

(3) *A subspace Y of X is a complete $\dagger$-M-ideal if and only if it is a left M-$\dagger$-ideal if and only if it is a right $\dagger$-M-ideal.*

Proof. (1) If P is a left $\dagger$-M-projection, then the map

$$\sigma_p(x) = \begin{pmatrix} P(x) \\ x - P(x) \end{pmatrix}$$

is a completely isometry from X to $C_2(X)$. Also,

$$\begin{aligned} \|x^\dagger\| = \|\sigma_P(x^\dagger)\| &= \left\| \begin{pmatrix} P(x^\dagger) \\ x^\dagger - P(x^\dagger) \end{pmatrix} \right\| = \left\| \begin{pmatrix} P(x^\dagger) & 0 \\ x^\dagger - P(x^\dagger) & 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} P(x)^\dagger) & 0 \\ x^\dagger - P(x)^\dagger & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} P(x) & x - P(x) \\ 0 & 0 \end{pmatrix}^\dagger \right\| \\ &= \left\| \begin{pmatrix} P(x) & x - P(x) \\ 0 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} P(x), x - P(x) \end{pmatrix} \right\| = \|x\|. \end{aligned}$$

One can easily generalize this to matrices, so that P is a right $\dagger$-M-projection. Similarly, if P is is a right $\dagger$-M-projection then P is a left $\dagger$-M-projection. By

Proposition 4.8.4 (1) in [14], we know that P is a complete †-M-projection.

(2) It follows from (1) and [14, Proposition 4.8.4 (2)]. Now (3) is also clear. □

Theorem 6.49. *Let A be an approximately unital operator $*$-algebra.*

(i) *The right †-M-ideals are the †-M-ideals in A, which are also the complete †-M-ideals. These are exactly the approximately unital †-ideals in A.*

(ii) *The right †-M-summands are the †-M-summands in A, which are also the complete †-M-summands. These are exactly the principal ideals Ae for a †-selfadjoint central projection $e \in M(A)$.*

Proof. (ii) By Proposition 6.48 (2), the right †-M-summands are exactly the complete †-M-summands. Moreover, by [14, Theorem 4.8.5 (3)], the M-summands in A are exactly the complete M-summands. If D is a †-M-summand, then D is a complete †-M-summand and there exists a central projection $e \in M(A)$ such that $D = eA$. Then $D^{\perp\perp} = eA^{**}$ and e is an identity for $D^{\perp\perp}$. Also, $e^{\dagger}$ serves as an identity in $D^{\perp\perp}$, so that $e = e^{\dagger}$.

(i) By a routine argument, the results follow as in [14, Theorem 4.8.5 (1)] and Proposition 6.48 (3). □

CHAPTER 7

Involutive hereditary subalgebras, ideals, and †-projections

7.1 Involutive hereditary subalgebras

Throughout this section A is an operator $*$-algebra (possibly not approximately unital). Then A^{**} is an operator $*$-algebra.

Definition 7.1. A projection in A^{**} is *open* in A^{**}, or A-open for short, if $p \in (pA^{**}p \cap A)^{\perp\perp}$. That is, if and only if there is a net (x_t) in A with

$$x_t = px_t = x_tp = px_tp \to p, \text{ weak}^*.$$

This is a derivative of Akemann's notion of open projections for C^*-algebras. If p is open in A^{**} then clearly

$$D = pA^{**}p \cap A = \{a \in A : a = ap = pa = pap\}$$

is a closed subalgebra of A, and the subalgebra $D^{\perp\perp}$ of A^{**} has identity p. By [14, Proposition 2.5.8] D has a cai. If A is also approximately unital then a projection p in A^{**} is closed if $p^\perp$ is open.

We call such a subalgebra D is a *hereditary subalgebra* of A (or HSA) and we say that p is the *support projection* of the HSA $pA^{**}p \cap A$. It follows from the above that the support projection of a HSA is the weak* limit of any cai from the HSA. If p is A-open, then $p^\dagger$ is also A-open. Indeed, if $x_t = px_t = x_tp = px_tp \to p$ weak*, then $x_t^\dagger = p^\dagger x_t^\dagger = x_t^\dagger p^\dagger = p^\dagger x_t^\dagger p^\dagger \to p^\dagger$ weak*, which means $p^\dagger$ is also open.

If p is †-selfadjoint and open, then we say p is †-*open* in A^{**}. That is, if and only if there exists a †-selfadjoint net (x_t) in A with

$$x_t = px_t = x_tp = px_tp \to p \text{ weak}^*.$$

If also A is approximately unital then we say that $p^\perp = 1 - p$ is †-closed. If p is †-open in A^{**} then clearly

$$D = pA^{**}p \cap A = \{a \in A : a = ap = pa = pap\}$$

is a closed †-subalgebra of A. We call such a †-subalgebra D is an *involutive hereditary* *subalgebra* or a †-*hereditary subalgebra* of A (or, †-HSA).

Definition 7.2. If J is a subspace of A, then we say J is an *inner ideal* in A if $acb \in J$ for any $a, b \in J$ and $c \in A$.

Proposition 7.3. [11, Proposition 2.2] *A subspace of an operator algebra is a HSA if and only if it is an approximately unital inner ideal.*

In the following statements, we often omit the proof details where are similar to the usual operator algebras (see e.g. [14], [20], [21]).

Proposition 7.4. *A subalgebra D of an operator $*$-algebra A is a HSA and $D^\dagger \subset D$ if and only if D is a †-HSA.*

Proof. One direction is trivial.

Conversely, if D is a HSA, $D = pA^{**}p \cap A$, for some open projection $p \in A^{**}$. Here, $p \in D^{\perp\perp}$ and p is an identity for $D^{\perp\perp}$. If also, D is †-selfadjoint, then $p^\dagger \in D^{\perp\perp}$ also serves as identity. By uniqueness of identity for $D^{\perp\perp}$, then $p = p^\dagger$. $\square$

Proposition 7.5. *A subspace of an operator $*$-algebra A is a †-HSA if and only if it is an approximately unital †-selfadjoint inner ideal.*

Proof. If J is a †-HSA, then J is an approximately unital †-selfadjoint inner ideal.

If J is an approximately unital †-selfadjoint inner ideal, then by Proposition 7.4 J is a HSA and †-selfadjoint which means that J is a †-HSA. $\square$

Remark. If J is an approximately unital ideal or inner ideal of operator $*$-algebra, we cannot necessarily expect J to be †-selfadjoint. For example, let $A(\mathbb{D})$

be the disk algebra and

$$A_i(\mathbb{D}) = \{f : f \in A(\mathbb{D}), f(i) = 0\}.$$

Then $A_i(\mathbb{D})$ is an approximately unital ideal but obviously it is not †-selfadjoint.

The following is another characterization of †-HSA's.

Corollary 7.6. *Let A be an operator $*$-algebra and suppose that (e_t) is a †-selfadjoint net in* $\mathrm{Ball}(A)$ *such that $e_t e_s \to e_s$ with t. Then*

$$\{x \in A : xe_t \to x, e_t x \to x\}$$

is a †-HSA of A. Conversely, every †-HSA of A arises in this way.

Proof. Let $J = \{x \in A : xe_t \to x, e_t x \to x\}$. Then it is easy to see that J is an inner ideal and $J^\dagger \subset J$. By Proposition 7.5, J is a †-HSA. Conversely, if D is a †-HSA and (e_t) is a †-selfadjoint cai for D, then

$$D = pA^{**}p \cap A = \{x \in A : xe_t \to x, e_t x \to x\},$$

where p is the weak* limit of (e_t). □

Definition 7.7. Closed right ideals J of an operator $*$-algebra A possessing a †-selfadjoint left cai will be called *r-†-ideals*. Similarly, closed left ideals J of an operator $*$-algebra A possessing a †-selfadjoint right cai will be called *l-†-ideals*.

Note that there is a bijective correspondence between r-†-ideals and l-†-ideals, namely $J \to J^\dagger$. For C^*-algebras r-ideals are precisely the closed right ideals, and there is an obvious bijective correspondence between r-ideals and l-ideals, namely $J \to J^*$.

Theorem 7.8. *Suppose that A is an operator $*$-algebra (possibly not approximately unital), and p is a $\dagger$-projection in A^{**}. Then the following are equivalent:*

(i) *p is $\dagger$-open in A^{**}.*

(ii) *p is the left support projection of an r-$\dagger$-ideal of A.*

(iii) *p is the right support projection of an l-$\dagger$-ideal of A.*

(iv) *p is the support projection of a $\dagger$-hereditary algebra of A.*

Proof. The equivalence of (i) and (iv) is just the definition of being $\dagger$-open in A^{**}.

Suppose (i), if p is $\dagger$-open then p is the support projection for some $\dagger$-HSA D. Let (e_t) be a $\dagger$-selfadjoint cai for D, then $p = w^*\text{-}\lim_t e_t$. Let

$$J = \{x \in A : e_t x \to x\},$$

then J is a right ideal of A with $\dagger$-selfadjoint left cai (e_t) and p is the left support projection of J.

Suppose (ii), if p is the left support projection of an r-$\dagger$-ideal J of A with $\dagger$-selfadjoint left cai (e_t), then $J = pA^{**} \cap A$. Therefore $J^\dagger = A^{**}p \cap A$, which is an l-$\dagger$-ideal and p is the right support projection of $J^\dagger$.

Suppose (iii), if p is the right support projection of an l-†-ideal of A with †-selfadjoint right cai (e_t), then $p = \text{weak*-}\lim_t e_t = p^\dagger$, which means that p is †-open.

Similarly we can get the equivalence between (i) and (iii). $\square$

If J is an operator $*$-algebra with an †-selfadjoint left cai (e_t), then we set

$$\mathcal{L}(J) = \{a \in J : ae_t \to a\}.$$

Corollary 7.9. *A subalgebra of an operator $*$-algebra A is †-hereditary if and only if it equals $\mathcal{L}(J)$ for an r-†-ideal J. Moreover the correspondence $J \mapsto \mathcal{L}(J)$ is a bijection from the set of r-†-ideals of A onto the set of †-HSA's of A. The inverse of this bijection is the map $D \to DA$. Similar results hold for the l-†-ideals of A.*

Proof. If D is a †-HSA, then by Corollary 7.6, we have

$$D = \{x \in A : xe_t \to x, e_t x \to x\},$$

where (e_t) is a †-selfadjoint cai for D. Set $J = \{x \in A : e_t x \to x\}$, then J is an r-†-ideal with $D = \mathcal{L}(J)$.

Conversely, if J is an r-†-ideal and (e_t) is a †-selfadjoint left cai for J, then

$$D = \{x \in A : xe_t \to x, e_t x \to x\}$$

is a †-HSA by Corollary 7.6, and $D = \mathcal{L}(J)$. The remainder is as in [11, Corollary 2.7]. $\square$

As in the operator algebra case [11, Corollary 2.8], if D is a $\dagger$-hereditary subalgebra of an operator $*$-algebra A, and if $J = DA$ and $K = AD$, then $JK = J \cap K = D$. Also as in the operator algebra case [11, Theorem 2.10], any $\dagger$-linear functional on a HSA D of an approximately unital operator $*$-algebra A has a unique $\dagger$-linear Hahn-Banach extension to A. This is because if φ is any Hahn-Banach extension to A, then $\overline{\varphi(x^\dagger)}$ is another, so these must be equal by [11, Theorem 2.10].

Proposition 7.10. *Let D be an approximately unital $\dagger$-subalgebra of an approximately unital operator $*$-algebra A. The following are equivalent:*

(i) *D is a $\dagger$-hereditary subalgebra of A.*

(ii) *Every completely contractive unital $\dagger$-linear map from D^{**} into a unital operator $*$-algebra B, has a unique completely contractive unital $\dagger$-extension from A^{**} into B.*

(iii) *Every completely contractive $\dagger$-linear map T from D into a unital weak* closed operator $*$-algebra B such that $T(e_t) \to 1_B$ weak* for some cai (e_t) for D has a unique completely contractive weakly $\dagger$-extension $\tilde{T}$ from A into B with $\tilde{T}(f_s) \to 1_B$ weak* for some(or all) cai (f_s) for A.*

Proof. We are identifying D^{**} with $D^{\perp\perp} \subset A^{**}$. Let e be the identity of D^{**}. Obviously, e is $\dagger$-selfadjoint.

(i)$\Rightarrow$(iii) The canonical weak* continuous extension $\hat{T} : D^{**} \to B$ is unital $\dagger$-preserving and completely contractive, and can be extended to a weak* continuous unital completely contractive $\dagger$-map $\Phi(\eta) = \hat{T}(e\eta e)$ on A^{**}. This in turn restricts to a completely contractive $\tilde{T} : A \to B$ with $\tilde{T}(f_s) \to 1_B$ weak* for all cai (f_s) for A. For the uniqueness, suppose that Ψ is such a †-extension of T, and $\tilde{\Psi}$ is the unique weak*-continuous extension of Ψ from A^{**} to B. It follows from the last remark in 2.6.16 of [14] that

$$\Phi(a) = \Phi(eae) = \hat{T}(eae) = \tilde{\Psi}(eae) = \tilde{\Psi}(e)\Psi(a)\tilde{\Psi}(e) = \Psi(a), a \in A.$$

(iii)⇒(i) If (iii) holds, then the inclusion from D to $D^{\perp\perp}$ extends to a unital complete †-contraction $T : A \to D^{**} \subset eA^{**}e$. The map $x \to exe$ on A^{**} is also a completely contractive unital †-extension of the inclusion map $D^{**} \to eD^{**}e$. It follows from the hypothesis that these maps coincide, and so $eA^{**}e = D^{**}$, which implies that D is a †-HSA.

The equivalence of (i) and (ii) could be obtained similarly. □

7.2 Support projections and †-HSA's

Definition 7.11. Let A be an operator algebra (possibly not unital). Then the *left (resp. right) support projection* of an element x in A is the smallest projection $p \in A^{**}$ such that $px = x$ (resp. $xp = x$), if such a projection exists (it always exists if A has a cai, see e.g. [20]). If the left and right support projection exist, and are equal, then we call it the *support projection*, written $s(x)$.

Definition 7.12. A closed right ideal J of an operator algebra A possessing a left cai is called *r-ideal*. Similarly, a closed left ideal J of an operator algebra A possessing a right cai will be called a *l-ideal*.

Lemma 7.13. [20, Lemma 2.5] *For any operator algebra A, if $x \in \mathfrak{F}_A$, with $x \neq 0$, then the left support projection of x equals the right support projection. If $A \subset B(H)$ via a representation π, for a Hilbert space H, such that the unique weak* continuous extension $\tilde{\pi} : A^{**} \to B(H)$ is (completely) isometric, then $s(x)$ also may be identified with the smallest projection p on H such that $px = x$ (and $xp = x$). That is, $s(x)H = \overline{\mathrm{Ran}(x)} = \ker(x)^{\perp}$. Also, $s(x)$ is an open projection in A^{**}. If A is a subalgebra of C^*-algebra B then $s(x)$ is open in B^{**} in the sense of Akemann [4].*

Corollary 7.14. [20, Corollary 2.6] *For any operator algebra A, if $x \in \mathfrak{F}_A$ with $x \neq 0$, then the closure of xA is an r-ideal in A and $s(x)$ is the support projection of this r-ideal. We have $\overline{xA} = s(x)A^{**} \cap A$. Also, $\overline{xAx}$ is the HSA matching $\overline{xA}$, and $x \in \overline{xAx}$.*

Theorem 7.15. [21, Corollary 3.4] *For any operator algebra A, if $x \in \mathfrak{r}_A$ and $x \neq 0$, then the left support projection of x equals the right support projection, and equals the weak* limit of $(a^{1/n})$. It also equals $s(y)$, where $y = x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$. Also, $s(x)$ is open in A^{**}.*

Corollary 7.16. *For any operator $*$-algebra A, if $x \in \mathfrak{r}_A$ is †-selfadjoint, then $a = \mathfrak{F}(x) = x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$ is †-selfadjoint, and $\overline{xA} = \overline{aA} = s(x)A^{**} \cap A$ is an r-†-ideal in A. Also, $\overline{xAx} = \overline{aAa}$ is the †-HSA matching $\overline{xA}$.*

Proof. It is easy to see that $a = x(1+x)^{-1}$ is †-selfadjoint, and is in $\frac{1}{2}\mathfrak{F}_A$ by Lemma 6.21. Since $(a^{1/n})$ is †-selfadjoint by a fact in the last proof, $(a^{1/n})$ serves as a †-selfadjoint left cai for $\overline{aA}$, which is a right ideal. Besides, $\overline{aAa}$ is †-selfadjoint and the weak* limit of $(a^{1/n})$ is $s(a)$. The rest follows from [21, Corollary 3.5]. Clearly $\overline{xA} \subset s(x)A^{**} \cap A$. On the other hand if $a \in s(x)A^{**} \cap A$ we have $x^{1/n}a \to a$ weakly. By Mazur's Theorem, a convex combination converges in norm, so $a \in \overline{xA}$. Moreover, $\overline{xAx}$ is a †-HSA matching $\overline{xA}$ by correspondence. $\square$

Proposition 7.17. *In an operator $*$-algebra A, $\mathfrak{F}(A)$ and $\mathfrak{r}_A$ are †-closed, and if $x \in \mathfrak{r}(A)$ we have $s(x)^\dagger = s(x^\dagger)$ and if $x \in \mathfrak{F}(A)$ then $s(x) \vee s(x^\dagger) = s(x + x^\dagger)$. In particular if x is †-selfadjoint then so is $s(x)$.*

Proof. Indeed applying † we see that $\|1 - x\| \leq 1$ implies $\|1 - x^\dagger\| \leq 1$. For the †-invariance of $\mathfrak{r}_A$ note that this is easy to see for a C^*-cover B with compatible involution (Definition 6.13), and then one may use the fact that $\mathfrak{r}_A = A \cap \mathfrak{r}_B$. Since $x^{1/n}$ may be written as a power series in $1 - x$ with real coefficients, it follows that $(x^\dagger)^{1/n} = (x^{1/n})^\dagger$. Then $s(x)^\dagger = (w^* \lim_n x^{1/n})^\dagger = s(x^\dagger)$. Let $x_1 = x$, $x_2 = x^\dagger$, (α_k) the sequence of positive scalars such that the sum of α_k is 1 and $\alpha_1 = \alpha_2$. By the proof of [20, Proposition 2.14], we know that $s(x + x^\dagger) = s(x) \vee s(x^\dagger)$. $\square$

Lemma 7.18. *If $x \in \mathfrak{F}_A$, with $x \neq 0$, then the operator $*$-algebra generated by x, denoted* $\mathrm{oa}^*(x)$*, has a cai. Indeed, the operator $*$-algebra* $\mathrm{oa}^*(x)$ *has a †-selfadjoint sequential cai belonging to $\frac{1}{2}\mathfrak{F}_A$.*

Proof. If $x \in \mathfrak{F}_A$, then $x^\dagger \in \mathfrak{F}_A$ as we proved above. Denote $B = C^*_e(A)$, then $p = s(x) \vee s(x^\dagger) = s(x + x^\dagger)$ in B^{**} is in $\mathrm{oa}^*(x)^{**}$. Clearly $px = xp = x$ and $px^\dagger = x^\dagger p = x^\dagger$. Therefore, p is an identity in $\mathrm{oa}^*(x)^{**}$. By [14, Theorem 2.5.8], $\mathrm{oa}^*(x)$ has a cai.

Moreover, since $\mathrm{oa}^*(x)$ is separable, by [20, Corollary 2.17], there exists $a \in \mathfrak{F}_A$ such that $s(a) = 1_{\mathrm{oa}^*(x)^{**}}$. Therefore $\mathrm{oa}^*(x)$ has a countable †-selfadjoint cai by applying to [20, Theorem 2.19] and Corollary 6.28. □

The following is a version of the Aarnes-Kadison Theorem for operator $*$-algebras.

Theorem 7.19 (Aarnes-Kadison type Theorem)**.** *If A is an operator $*$-algebra then the following are equivalent:*

(i) *There exists a †-selfadjoint $x \in \mathfrak{r}_A$ with $A = \overline{xAx}$.*

(ii) *There exists a †-selfadjoint $x \in \mathfrak{r}_A$ with $A = \overline{xA} = \overline{Ax}$.*

(iii) *There exists a †-selfadjoint $x \in \mathfrak{r}_A$ with $s(x) = 1_{A^{**}}$.*

(iv) *A has a countable †-selfadjoint cai.*

(v) *A has a †-selfadjoint and strictly real positive element.*

Indeed these are all equivalent to the same conditions with '†-selfadjoint' removed.

Proof. In (i)–(iii) we can assume that $x \in \mathfrak{F}_A$ by replacing it with the †-selfadjoint element $x(1+x)^{-1} \in \frac{1}{2}\mathfrak{F}_A$ (see [22, Section 2.2]). Then the equivalence of (i)–(iv) follows as in [20, Lemma 2.10 and Theorem 2.19], for (iv) using that $x^{\frac{1}{n}}$ is †-selfadjoint as we said in the proof of Corollary 7.16. Similarly (v) follows from these by [20, Lemma 2.10], and the converse follows since strictly real positive elements have support projection 1 (see [22, Section 3]). The final assertion follows since if A has a countable cai, then A has a †-selfadjoint countable cai (Lemma 6.27). □

Lemma 7.20. [20, Lemma 2.13] *If (J_i) is a family of r-ideals in an operator algebra A, with matching family of HSA's (D_i), and if $J = \overline{\sum_i J_i}$ then the HSA matching J is the HSA D generated by the (D_i) (that is, the smallest HSA in A containing all the D_i). Here 'matching' means the respect to the correspondence between r-ideals and HSA's.*

Proof. Let D' be the HSA generated by the (D_i) . Since $J_i \subset J$ we have $D_i \subset D$, and so $D' \subset D$. Conversely, since $D_i \subset D'$ we have $J_i \subset D'A$ so that $J \subset D'A$. Hence, $D \subset D'$. □

Lemma 7.21. *If (J_i) is a family of r-†-ideals in an operator $*$-algebra A, with matching family of †-HSA's (D_i), and if $J = \overline{\sum_i J_i}$ then the †-HSA matching J is the †-HSA D generated by the (D_i).*

Proof. This follows directly from Lemma 7.20, since every r-†-ideal is an r-ideal and any †-HSA is a HSA. □

Proposition 7.22. *Let A be an operator $*$-algebra (not necessarily with an identity or approximate identity). Suppose that (x_k) is a sequence of †-selfadjoint elements in $\mathfrak{F}_A$, and $\alpha_k \in (0,1]$ add to 1. Then the closure of the sum of the r-†-ideals $\overline{x_k A}$, is the r-†-ideal $\overline{zA}$, where $z = \sum_{k=1}^{\infty} \alpha_k x_k \in \mathfrak{F}_A$. Similarly, the †-HSA generated by all the $\overline{x_k A x_k}$ equals $\overline{zAz}$.*

Proof. As an r-ideal, $\overline{zA}$ is the closure of the sum of the r-ideals $\overline{x_k A}$. If $z \in \mathfrak{F}_A$ is †-selfadjoint then $\overline{zA}$ is an r-†-ideal. □

Lemma 7.23. *Let A be an operator $*$-algebra, a subalgebra of a C^*-algebra B.*

(i) *The support projection of a †-HSA D in A equals $\vee_{a\in H(\mathfrak{F}_D)} s(a)$ (which equals $\vee_{a\in H(\mathfrak{r}_D)} s(a)$).*

(ii) *The support projection of an r-†-ideal J in A equals $\vee_{a\in H(\mathfrak{F}_J)} s(a)$ (which equals $\vee_{a\in H(\mathfrak{r}_J)} s(a)$).*

Proof. (i) Suppose p is the support projection of D, then $p = \vee_{b\in\mathfrak{F}_D} s(b) = \vee_{b\in\mathfrak{r}_D} s(b)$ by the operator algebra variant of [26, Lemma 3.12]. Thus,

$$p \geq \vee_{a\in H(\mathfrak{r}_D)} s(a) \geq \vee_{a\in H(\mathfrak{F}_D)} s(a).$$

For any $b \in \mathfrak{F}_D$, we have $b^\dagger \in \mathfrak{F}_D$ and notice that $s(b) \vee s(b^\dagger) = s(\frac{b+b^\dagger}{2})$ (see e.g. Proposition 2.14 in [20]) and $(b+b^\dagger)/2 \in (\mathfrak{F}_D)_\dagger$. Hence,

$$p = \vee_{b\in\mathfrak{F}_D} s(b) \leq \vee_{a\in H(\mathfrak{F}_D)} s(a).$$

Therefore, $p \leq \vee_{a\in H(\mathfrak{F}_D)} s(a) \leq \vee_{a\in H(\mathfrak{r}_D)} s(a)$.

(ii) This is similar. $\square$

Lemma 7.24. *For any operator $*$-algebra A, if $E \subset (\mathfrak{r}_A)_\dagger$, then the smallest †-hereditary subalgebra of A containing E is $pA^{**}p \cap A$, where $p = \vee_{x\in E} s(x)$.*

Proof. By Lemma 7.23, $pA^{**}p \cap A$ is a †-hereditary subalgebra of A, and it contains E. Conversely, if D is a †-HSA of A containing E then $D^{\perp\perp}$ contains p by a routine argument, so $pA^{\perp\perp}p \subset D^{\perp\perp}$ and $pA^{\perp\perp}p \cap A \subset D^{\perp\perp} \cap A = D$. $\square$

Corollary 7.25. *For any operator $*$-algebra A, suppose that a convex set $E \subset \mathfrak{r}_A$ and $E^\dagger \subset E$. Then the smallest hereditary subalgebra of A containing E is $pA^{**}p \cap A$, where $p = \vee_{x \in H(E)} s(x)$. Indeed, this is the smallest †-HSA of A containing E.*

Proof. The smallest HSA containing E is $pA^{**}p \cap A$, where $p = \vee_{a \in E} s(a)$. For any $a \in E$, $\frac{a+a^\dagger}{2} \in E$ by convexity of E. Notice that $s(\frac{a+a^\dagger}{2}) \le p$ and $s(\frac{a+a^\dagger}{2}) \ge s(a)$, then $p = \vee_{x \in H(E)} s(x)$ and $pA^{**}p \cap A$ is a †-HSA. □

Theorem 7.26. *If A is an operator $*$-algebra then †-HSA's (resp. r-†-ideals) in A are precisely the sets of form $\overline{EAE}$ (resp. $\overline{EA}$) for some $E \subset (\mathfrak{r}_A)_\dagger$. The latter set is the smallest †-HSA (resp. r-†-ideal) of A containing E.*

Proof. If D is a †-HSA (resp. r-†-ideal) and taking E to be a †-selfadjoint cai for the †-HSA D (resp. a †-selfadjoint left cai for the r-†-ideal), then the results follows immediately.

Conversely for any $x \in (\mathfrak{r}_A)_\dagger$, we have $x(1+x)^{-1} \in (\frac{1}{2}\mathfrak{F}_A)_\dagger$ as we said in Corollary 7.16. Then as in [26, Theorem 3.18] we may assume that $E \subset (\frac{1}{2}\mathfrak{F}_A)_\dagger$. Note that $D = \overline{EAE}$ is the smallest HSA containing E by [26, Theorem 3.18] and D is †-selfadjoint, so that D is the smallest †-HSA containing E. Similarly, $\overline{EA}$ is the smallest right ideal with a †-selfadjoint left contractive identity of A containing E. Moreover, for any finite subset $F \subset E$ if a_F is the average of the elements in F, then $(a_F^{1/n})$ will serve as a †-selfadjoint left cai for $\overline{EA}$. □

In particular, the largest †-HSA in an operator $*$-algebra A is the largest HSA in A, and the largest approximately unital subalgebra in A (see [21, Section 4]), namely

$A_H = \overline{\mathfrak{r}_A A \mathfrak{r}_A} = \overline{H(\mathfrak{r}_A) A H(\mathfrak{r}_A)}$. The latter equality follows because A_H has a cai in $\mathfrak{r}_A$, hence has a cai in $H(\mathfrak{r}_A)$.

Theorem 7.27. *Let A be an operator $*$-algebra (not necessarily with an identity or approximate identity.) The †-HSA's (resp. r-†-ideals) in A are precisely the closures of unions of an increasing net of †-HSA's (resp. r-†-ideals) of the form $\overline{xAx}$ (resp. $\overline{xA}$) for $x \in (\mathfrak{r}_A)_\dagger$.*

Proof. Suppose that D is a †-HSA (resp. an r-†-ideal). The set of †-HSA's (resp. r-†-ideals) $\overline{a_F A a_F}$ (resp. $\overline{a_F A}$) as in the last proof, indexed by finite subsets F of $(\mathfrak{F}_D)_\dagger$, is an increasing net. Lemma 7.23 can be used to show, as in [26], that the closure of the union of these †-HSA's (resp. r-†-ideals) is D. □

As in the theory we are following, it follows that †-open projections are just the sup's of a collection (an increasing net if desired) of †-selfadjoint support projections $s(x)$ for †-selfadjoint $x \in \mathfrak{r}_A$.

Theorem 7.28. *Let A be any operator $*$-algebra (not necessarily with an identity or approximate identity). Every separable †-HSA or †-HSA with a countable cai (resp. separable r-†-ideal or r-†-ideal with a countable cai) is equal to $\overline{xAx}$ (resp. $\overline{xA}$) for some $x \in (\mathfrak{F}_A)_\dagger$.*

Proof. If D is a †-HSA with a countable cai, then D has a countable †-selfadjoint cai (e_n) in $\frac{1}{2}\mathfrak{F}_D$. Also, D is generated by the †-HSA's $\overline{e_n A e_n}$ so $D = \overline{xAx}$, where $x = \sum_{n=1}^{\infty} \frac{e_n}{2^n}$. For the separable case, note that any separable approximately unital operator $*$-algebra has a countable cai. For r-$\dagger$-ideals, the result follows from the same argument. $\square$

Corollary 7.29. *If A is a separable operator $*$-algebra, then the $\dagger$-open projections in A^{**} are precisely the $s(x)$ for $x \in (\mathfrak{r}_A)_\dagger$.*

Proof. If A is separable, then so is any $\dagger$-HSA. So the result follows from Theorem 7.28. $\square$

Corollary 7.30. *If A is a separable operator $*$-algebra with cai, then there exists an $x \in H(\mathfrak{F}_A)$ with $A = \overline{xA} = \overline{Ax} = \overline{xAx}$.*

7.3 Involutive compact projections

Throughout this section, A is an operator $*$-algebra.

Definition 7.31. A projection $q \in A^{**}$ is *compact relative to A* if it is closed and $q = qx$ for some $x \in \mathrm{Ball}(A)$. Furthermore, if q is $\dagger$-selfadjoint, we say that such q is an *involutive compact projection*, or is $\dagger$-*compact* in A^{**}.

Proposition 7.32. *A $\dagger$-projection q is compact if only if there exists a $\dagger$-selfadjoint element $a \in \mathrm{Ball}(A)$ such that $q = qa$.*

Proof. One direction is trivial. Conversely if q is compact, then there exists $a \in \mathrm{Ball}(A)$ such that $q = qa$. It is easy to argue from elementary operator theory that we have $aq = q$. Thus, $q = q(\frac{a+a^\dagger}{2})$. $\square$

Theorem 7.33. *Let A be an approximately unital operator $*$-algebra. If q is a $\dagger$-projection in A^{**} then the following are equivalent:*

(i) *q is a $\dagger$-closed projection in $(A^1)^{**}$.*

(ii) *q is $\dagger$-compact in A^{**}.*

(iii) *q is $\dagger$-closed such that there exists a $\dagger$-selfadjoint element $x \in \frac{1}{2}\mathfrak{F}_A$ such that $q = qx$.*

Proof. We may assume that A is nonunital

(iii)$\Rightarrow$ (ii). Obvious.

(ii)$\Rightarrow$ (i). If q is $\dagger$-compact projection in A^{**}, then there exists $y_t \in \mathrm{Ball}(A)$ with $y_t \to q$, and $y_t q = q$. Then $1 - y_t \to 1 - q$, and $(1 - y_t)(1 - q)$, and $y_t q = q$. Then $1 - y_t \to 1 - q$, and $(1 - y_t)(1 - q) = 1 - y_t$, so $1 - q$ is open in $(A^1)^{**}$. Hence, q is closed in $(A^1)^{**}$.

(i)$\Rightarrow$ (iii). Consider a projection $q \in A^{**}$ such that q is $\dagger$-closed in $(A^1)^{**}$. Then $q^{\perp}$ is $\dagger$-open in $(A^1)^{**}$; let $C = q^{\perp}(A^1)^{**}q^{\perp} \cap A^1$. Then $\dagger$-HSA in A^1 with support projection $q^{\perp}$. Note that since $eq = q$ we have $(1 - e)q^{\perp} = 1 - e$, and $f = 1 - e$ is a central minimal projection in $C^{**} = q^{\perp}(A^1)^{**}q^{\perp}$. Let D be the $\dagger$-HSA in A^1 with support projection $e - q = e(1 - q)$. This is an approximately unital ideal in C, indeed $D^{\perp\perp} = eC^{**}$. Note that

$$C^{**}/D^{\perp\perp} \cong C^{**}(q^{\perp} - q^{\perp}e) = C^{**}f = \mathbb{C}\, f \cong C.$$

The map implementing this isomorphism $C^{**}/D^{\perp\perp} \cong \mathbb{C}\, f$ is the map $x + D^{\perp\perp} \mapsto xf$. Moreover, the map restricts to an isometric isomorphism $C/D \cong C/D^{\perp\perp} \cong \mathbb{C}\, f$. This is because if the range of this restriction is not $\mathbb{C}\, f$ then it is (0), so that $C = D$, which implies the contradiction $e = 1$. By , there is an element $d \in \mathfrak{F}_C$ such that $df = 2f$. If $b = d/2 \in \frac{1}{2}\mathfrak{F}_C$, then $bf = f$. We see that $1 - b \in \frac{1}{2}\mathfrak{F}_{A^1}$, and $(1-b)e = 1 - b$, so $(1-b) \in \frac{1}{2}\mathfrak{F}_A$. Moreover, $(1-b)1 = q$, since $b \in q^{\perp}(A^1)^{**}q^{\perp}$. Choose $a = 1 - \frac{b+b^{\dagger}}{2}$, then we are done. $\square$

Corollary 7.34. *Let A be an approximately unital operator $*$-algebra. Then the infimum of any family of $\dagger$-compact projections in A^{**} is a $\dagger$-compact projection in A^{**}. Also, the supremum of two commuting $\dagger$-compact projections in A^{**} is a $\dagger$-compact projection in A^{**}.*

Proof. Note that the infimum and supremum of $\dagger$-projections are still $\dagger$-projections. Then the results follow immediately from [15, Corollary 2.3]. $\square$

Corollary 7.35. *Let A be an approximately unital operator $*$-algebra, with an approximately unital closed $\dagger$-subalgebra D. A projection $q \in D^{\perp\perp}$ is $\dagger$-compact in D^{**} if and only if q is $\dagger$-compact in A^{**}.*

Corollary 7.36. *Let A be an approximately unital operator $*$-algebra. If a $\dagger$-projection q in A^{**} is dominated by an open projection p in A^{**}, then q is $\dagger$-compact in $pA^{**}p$.*

In much of what follows we use the *peak projections* $u(a)$ defined and studied in e.g. [15, 21]. These may be defined to be projections in A^{**} which are the weak* limits of a^n for some $a \in \mathrm{Ball}(A)$, in the case such weak* limit exists. We will not take the time to review the properties of $u(a)$ here. We will however several times below use silently the following fact:

Lemma 7.37. *If $a \in \mathrm{Ball}(A)$ for an operator $*$-algebra A, and if $u(a)$ is a peak projection, with $a^n \to u(a)$ weak*, then $u((a + a^\dagger)/2) = u(a) \wedge u(a)^\dagger$ in A^{**} and this is a peak projection. Indeed $((a + a^\dagger)/2)^n \to u((a + a^\dagger)/2)$ weak*.*

Proof. Clearly $(a^\dagger)^n \to u(a)^\dagger$ weak*, so that $u(a^\dagger) = u(a)^\dagger$ is a peak projection. Then $u((a+a^\dagger)/2) = u(a) \wedge u(a)^\dagger$ by [15, Proposition 1.1], and since this is a projection it is by [15, Section 3] a peak projection, is †-selfadjoint, and $((a+a^\dagger)/2)^n \to u((a+a^\dagger)/2)$ weak*. $\square$

The following is the involutive variant of the version of the Urysohn lemma for approximately unital operator $*$-algebras in [15, Theorem 2.6].

Theorem 7.38. *Let A be an approximately unital operator $*$-algebra. If a †-compact projection q in A^{**} is dominated by a †-open projection p in A^{**}, then there exists $b \in \frac{1}{2}H(\mathfrak{F}_A)$ with $q = qb, b = pb$. Moreover, $q \leq u(b) \leq s(b) \leq p$, and b may also be chosen to be 'nearly positive' in the sense of the introduction to* [22]*: we can make it as close in norm as we like to an actual positive element.*

Proof. If $q \leq p$ as stated, then by the last corollary we know q is †-compact in $D^{**} = pA^{**}p$, where D is a †-HSA supported by p. By Theorem 7.33, there exists a †-selfadjoint $b \in \frac{1}{2}\mathfrak{F}_D$ such that $q = qb$ and $b = bp$. The rest follows as in [15, Theorem 2.6]. $\square$

Theorem 7.39. *Suppose that A is an operator $*$-algebra (not necessarily approximately unital), and that $q \in A^{**}$ is a projection. The following are equivalent:*

(1) *q is $\dagger$-compact with respect to A.*

(2) *q is $\dagger$-closed with respect to A^1 and there exists $a \in \mathrm{Ball}(A)_\dagger$ with $aq = qa = q$.*

(3) *q is a decreasing weak* limit of $u(a)$ for $\dagger$-selfadjoint element $a \in \mathrm{Ball}(A)$.*

Proof. (2) $\Rightarrow$ (3) Given (2) we certainly have q compact with respect to A by [21, Theorem 6.2]. By [15, Theorem 3.4], $q = \lim_t u(z_t)$, where $z_t \in \mathrm{Ball}(A)$ and $u(z_t)$ is decreasing. We have $q = q^\dagger = \lim_t u(z_t^\dagger)$. Moreover, $u(z_t) \wedge u(z_t^\dagger) = u(\frac{z_t + z_t^\dagger}{2})$. Hence, q is a decreasing weak* limit of $u(\frac{z_t + z_t^\dagger}{2})$ since the involution preserves order.

The rest follows from [21, Theorem 6.2]. $\square$

Corollary 7.40. *Let A be a (not necessarily approximately unital) operator $*$-algebra. If q is $\dagger$-compact then q is a weak* limit of a net of $\dagger$-selfadjoint elements (a_t) in $\mathrm{Ball}(A)$ with $a_t q = q$ for all t.*

7.4 Involutive peak projections

Definition 7.41. Let A be an operator $*$-algebra. A $\dagger$-projection $q \in A^{**}$ is called an *involutive peak projection* or a $\dagger$-*peak projection* if it is a peak projection.

Proposition 7.42. *Suppose A is a separable operator $*$-algebra (not necessarily approximately unital), then the $\dagger$-compact projections in A^{**} are precisely the peak projections $u(a)$, for some $\dagger$-selfadjoint $a \in \mathrm{Ball}(A)$.*

Proof. If A is separable then a projection in A^{**} is compact if and only if $q = u(a)$, for some $a \in \mathrm{Ball}(A)$ (see [21, Proposition 6.4]). If q is †-selfadjoint, then

$$q = u(a^\dagger) = u(a) \wedge u(a^\dagger) = u((a^\dagger + a)/2),$$

using e.g. Lemma 7.37. □

Proposition 7.43. *If $a \in \frac{1}{2}\mathfrak{F}_A$ with $a^\dagger = a$, then $u(a)$ is a †-peak projection and it is a peak for a.*

Proof. Since $u(a) = \lim a^n$ weak* in this case, we see that $u(a)$ is †-selfadjoint. From [15, Lemma 3.1, Corollary 3.3], we know that $u(a)$ is a peak projection and is a peak for a. □

Theorem 7.44. *If A is an approximately unital operator $*$-algebra, then*

(i) *A projection $q \in A^{**}$ is †-compact if only if it is a decreasing limit of †-peak projections.*

(ii) *If A is a separable approximately unital operator $*$-algebra, then the †-compact projections in A^{**} are precisely the †-peak projections.*

(iii) *A projection in A^{**} is a †-peak projection in A^{**} if and only if it is of form $u(a)$ for some $a \in \frac{1}{2}H(\mathfrak{F}_A)$.*

Proof. (ii) Follows from Proposition 7.42 and Proposition 7.43.

(i) One direction is obvious. Conversely, let $q \in A^{**}$ be a †-compact projection with $q = qx$ for some †-selfadjoint element $x \in \mathrm{Ball}(A)$. Then $q \leq u(x)$. Now $1 - q$

is an increasing limit of $s(x_t)$ for $\dagger$-selfadjoint elements $x_t \in \frac{1}{2}\mathfrak{F}_{A^1}$, by Theorem 7.27 and the remark after it. Therefore, q is a decreasing weak* limit of the $q_t = s(x_t)^\perp = u(1-x_t)$. Let $z_t = \frac{1-x_t+x}{2}$, then $u(z_t)$ is a projection. Since $q \leq q_t$ and $q \leq u(x)$, then $q \leq u(z_t)$. Note that z_t is $\dagger$-selfadjoint, so $u(z_t) = u(z_t)^\dagger$. Let $a_t = z_t x \in \mathrm{Ball}(A)$, then $u(a_t) = u(z_t)$ by the argument in [15, Lemma 3.1]. Thus, $u(a_t) = u(z_t) \searrow q$ as in that proof. Moreover, $u(a_t^\dagger) = u(a_t)^\dagger \searrow q$, which implies by an argument above that $u(\frac{a_t+a_t^\dagger}{2}) \searrow q$.

(iii) One direction is trivial. For the other, if q is a $\dagger$-peak projection, then by the operator algebra case there exists $a \in \frac{1}{2}\mathfrak{F}_A$ such that $q = u(a)$. Let $b = (a+a^\dagger)/2$, then $q = u(b)$ by e.g. Lemma 7.37. $\square$

Corollary 7.45. *Let A be an operator $*$-algebra. The supremum of two commuting $\dagger$-peak projections in A^{**} is a $\dagger$-peak projection in A^{**}.*

Lemma 7.46. *For any operator $*$-algebra A, the $\dagger$-peak projections for A are exactly the weak* limits of a^n for $\dagger$-selfadjoint element $a \in \mathrm{Ball}(A)$ if such limit exists.*

Proof. If q is a $\dagger$-peak projection, then there exists $a \in \mathrm{Ball}(A)$ such that $q = u(a)$ which is also the weak* limit of a^n. Since q is $\dagger$-selfadjoint, by Lemma 7.37 we have $q = u(a^\dagger) = u(\frac{a+a^\dagger}{2})$, which is the weak* limit of $((a+a^\dagger)/2)^n$. The converse follows from [21, Lemma 1.3]. $\square$

Remark 7.47. Similarly the theory of peak projections for operator $*$-algebras A which are not necessarily approximately unital follows the development in [21, Section 6], with appropriate tweaks in the proofs. Thus a projection is called a $\dagger$-$\mathfrak{F}$*-peak projection* for A if it is $\dagger$-selfadjoint and $\mathfrak{F}$-peak. A projection in A^{**} is $\dagger$-$\mathfrak{F}$*-compact* if it is a decreasing limit of $\dagger$-$\mathfrak{F}$-peak projections. We recall that A_H was discussed above Theorem 7.27. One may then prove:

(i) A projection q in A^{**} is $\dagger$-$\mathfrak{F}$-compact iff q is a $\dagger$-compact projection for A_H.

(ii) A projection in A^{**} is a $\dagger$-$\mathfrak{F}$-peak projection iff it is a $\dagger$-peak projection for A_H.

(iii) If A is separable then every $\dagger$-$\mathfrak{F}$-compact projection in A^{**} is a $\dagger$-$\mathfrak{F}$-peak projection.

7.5 Some interpolation results

Item (ii) in the following should be compared with Theorem 7.38 which gets a slightly better result in the case that A is approximately unital.

Theorem 7.48 (Noncommutative Urysohn lemma for operator $*$-algebras)**.** *Let A be a (not necessarily approximately unital) operator $\dagger$-subalgebra of C^*-algebra B with a second involution $\dagger$. Let q be a $\dagger$-compact projection in A^{**}.*

(i) *For any $\dagger$-open projection $p \in B^{**}$ with $p \geq q$, and any $\varepsilon > 0$, there exists an $a \in \mathrm{Ball}(A)_\dagger$ with $aq = q$ and $\|a(1-p)\| < \varepsilon$.*

(ii) *For any $\dagger$-open projection $p \in A^{**}$ with $p \geq q$, there exists a $\dagger$-selfadjoint element $a \in \mathrm{Ball}(A)$ with $q = qa$ and $a = pa$.*

Proof. (i) By [21, Theorem 6.6] there exists $b \in \mathrm{Ball}(A)$ such that

$$bq = q, \ \|b(1-p)\| < \varepsilon \text{ and } \|(1-p)b\| < \varepsilon.$$

Then $a = \frac{b+b^\dagger}{2} \in \mathrm{Ball}(A)_\dagger$ does the trick, since

$$\|(\frac{b+b^\dagger}{2})(1-p)\| \leq \frac{1}{2}\|b(1-p)\| + \frac{1}{2}\|((1-p)b)^\dagger\| < \varepsilon.$$

(ii) Apply Theorem 7.38 in A^1 to obtain a $\dagger$-selfadjoint element $a \in \mathrm{Ball}(A^1)$, $p \in A^{\perp\perp}$ and $ap = q$. Then $a \in A^{\perp\perp} \cap A^1 = A$. □

The following is an involutive variant of the noncommutative peak interpolation type result in [21, Theorem 5.1].

Theorem 7.49. *Suppose that A is an operator $*$-algebra and that q is a $\dagger$-closed projection in $(A^1)^{**}$. If $b = b^\dagger \in A$ with $bq = qb$, then b achieves its distance to the right ideal $J = \{a \in A : qa = 0\}$ (this is a r-$\dagger$-ideal if $1 - q \in A^{**}$), and also achieves its distance to $\{x \in A : xq = qx = 0\}$ (this is a $\dagger$-HSA if $1 - q \in A^{**}$). If further $\|bq\| \leq 1$, then there exists a $\dagger$-selfadjoint element $g \in \mathrm{Ball}(A)$ with $gq = qg = bq$.*

Proof. Proceed as in the proof of [21, Theorem 5.1]. The algebra $\tilde{D}$ is a $\dagger$-HSA in A^1. Thus if C is as in that proof, C is $\dagger$-selfadjoint and $\tilde{D}$ is a $\dagger$-ideal in C. Also $I = C \cap A$ and $D = I \cap \tilde{D}$ are $\dagger$-selfadjoint in C. Note that if $x \in A$ with $xq = qx = 0$ then $x \in \tilde{D} \cap A \subset C \cap A = I$, so $x \in \tilde{D} \cap A \subset \tilde{D} \cap I = D$. So $D = \{x \in A : xq = qx = 0\}$. By the proof we are following, there exists $y \in D \subset J$ such that

$$\|b - y\| = \|b - y^\dagger\| = d(b, D) = \|bq\| = d(b, J) \geq \|b - z\|,$$

where $z = (y + y^\dagger)/2$. Set $g = b - z$, then g is $\dagger$-selfadjoint with $gq = qg = bq$ (since D is $\dagger$-selfadjoint), and $\|g\| = \|bq\|$. □

Theorem 4.10 in [22] is the (noninvolutive) operator algebra version of the last result (and [21, Theorem 5.1]), but with the additional feature that the 'interpolating element' g in the last result is also in $\frac{1}{2}\mathfrak{F}_A$. Whence after replacing g by $g^{\frac{1}{n}}$, it is 'nearly positive' in the sense of the introduction to [22]: we can make it as close in norm as we like to an actual positive element. There seems to be a mistake in Theorem 4.10 in [22]. It is claimed there (and used at the end of the proof) that D is approximately unital. However this error disappears in what is perhaps the most important case, namely that q is the 'perp' of a (open) projection in A^{**}. Then D is certainly a HSA in A, and is approximately unital. Hence we have, also in the involutive case:

Theorem 7.50. *Suppose that A is an operator $*$-algebra p is a $\dagger$-open projection in A^{**}, and $b = b^\dagger \in A$ with $bp = pb$ and $\|b(1-p)\| \leq 1$ (where 1 is the identity of the unitization of A if A is nonunital). Suppose also that $\|(1-2b)(1-p)\| \leq 1$. Then there exists a $\dagger$-selfadjoint element $g \in \frac{1}{2}\mathfrak{F}_A \subset \mathrm{Ball}(A)$ with $g(1-p) = (1-p)g = b(1-p)$. Indeed such g may be chosen 'nearly positive' in the sense of the introduction to [22], indeed it may be chosen to be as close as we like to an actual positive element.*

Theorem 7.51. (A noncommutative Tietze theorem) *Suppose that A is an operator $*$-algebra (not necessarily approximately unital), and that p is a $\dagger$-open projection in A^{**}. Set $q = 1 - p \in (A^1)^{**}$. Suppose that $b = b^\dagger \in A$ with $bp = pb$ and $\|bq\| \leq 1$, and that the numerical range of bq (in $q(A^1)^{**}q$ or $(A^1)^{**}$) is contained in a compact convex set E in the plane satisfying $E = \bar{E}$. We also suppose, by fattening it slightly if necessary, that $E \not\subset \mathbb{R}$. Then there exists a $\dagger$-selfadjoint element $g \in \mathrm{Ball}(A)$ with $gq = qg = bq$, such that the numerical range of g with respect to A^1 is contained in*

E.

Proof. By [22, Theorem 4.12], there exists $a \in \text{Ball}(A)$ with $aq = qa = bq$, such that the numerical range of a with respect to A^1 is contained in E. Then $g = (a + a^\dagger)/2$ is $\dagger$-selfadjoint. Let B be a unital C^*-cover of A^1 with compatible involution $\sigma(b)^*$ as usual. If φ is a state of B then $\varphi \circ \sigma$ is a state too, and so

$$\varphi(a^\dagger) = \overline{\varphi(\sigma(a))} \in \bar{E} = E.$$

From this it is clear that $\varphi(g) \in E$. □

Corollary 7.52. *Suppose that A is an operator $*$-algebra (not necessarily approximately unital), and that J is an approximately unital closed $\dagger$-ideal in A. Suppose that $b = b^\dagger$ is an element in $\mathfrak{F}_{A/J}$ (resp. in $\mathfrak{r}_{A/J}$). Then there exists a $\dagger$-selfadjoint element a in $\mathfrak{F}_A$ (resp. in $\mathfrak{r}_A$) with $a + J = b$.*

Proof. Indeed the operator $*$-algebra variant of [20, Proposition 6.1] and [20, Corollary 6.1] hold. The $\mathfrak{r}_{A/J}$ lifting follows from the last theorem with $E = [0, K] \times [-K, K]$ and $K = \|b\|$ say. However both results also follow by the usual $(a + a^\dagger)/2$ trick. □

The following is the 'nearly positive' case of a simple noncommutative peak interpolation result which has implications for the unitization of an operator $*$-algebra.

Proposition 7.53. *Suppose that A is an approximately unital operator $*$-algebra, and B is a C^*-algebra generated by A with compatible involution $\dagger$. If $c = c^\dagger \in B_+$*

with $\|c\| < 1$ *then there exists a* $\dagger$*-selfadjoint* $a \in \frac{1}{2}\mathfrak{F}_A$ *with* $|1-a|^2 \leq 1-c$. *Indeed such* a *can be chosen to also be nearly positive.*

Proof. As in [22, Proposition 4.9], but using our Theorem 6.18 (2), there exists nearly positive $\dagger$-selfadjoint $a \in \frac{1}{2}\mathfrak{F}_A$ with

$$c \leq \mathrm{Re}(a) \leq 2\mathrm{Re}(a) - a^*a,$$

and $|1-a|^2 \leq 1-c$. □

We end with an involutive case of the best noncommutative peak interpolation result (from [9]), a noncommutative generalization of a famous interpolation result of Bishop. See [9] for more context and an explanation of the classical variant, and the significance of the noncommutative variant. Unfortunately we cannot prove this result for operator $*$-algebras without imposing a further strong condition (d commutes with b and $q^{\perp}(A^1)^{**}q^{\perp} \cap A^1$). This is a good example of a complicated result which is not clear in advance whether it has 'involutive variants'. In this case it is valid, without the strong condition just mentioned, for involutions of types (3) and (4) at the start of Section 6.1. We treat the type (3) case. For an operator algebra A, let $\bar{A}$ be $(A^{\star})^{\circ}$. In this case a conjugate linear completely isometric involution π on A of type (3) at the start of Section 6.1, gives rise after composition with the canonical map $- : A \to \bar{A}$, to a linear completely isometric isomorphism $A \to \bar{A}$. This map extends to a $*$-isomorphism $C^*_{\max}(A) \to C^*_{\max}(\bar{A}) = (C^*_{\max}(A)^{\circ})^{\star}$. Composing this with the canonical map $-$, we obtain a conjugate linear $*$-automorphism on $B = C^*_{\max}(A)$ which we will also write as π. This is the compatible conjugate linear involution on B.

Theorem 7.54. *Suppose that A is a operator algebra, with a conjugate linear completely isometric involution π of type (3) at the start of Section 6.1. Suppose that A is a subalgebra of a unital C^* -algebra B with compatible conjugate linear $*$-automorphism π on B. Suppose that q is a closed projection in B^{**} which lies in $(A^1)^{\perp\perp}$ and satisfies $\pi^{**}(q) = q$. If b is an element in A with $bq = qb$ and $b = \pi(b)$, and if $qb^*bq \leq qd$ for an invertible positive $d \in B$ with $d = \pi(d)$ which commutes with q, then there exists a $g \in \mathrm{Ball}(A)$ with $gq = qg = bq, g = \pi(g)$, and $g^*g \leq d$.*

Proof. By the proof of [9, Theorem 3.4], there exists $h \in A$ with $qh = hq = bq$, and $h^*h \leq d$. (We remark that $f = d^{-\frac{1}{2}}$ in that proof.) Thus also $\pi(h^*h) \leq \pi(d) = d$. Let $g = \frac{h+\pi(h)}{2}$. Then $g = \pi(g)$ and $qg = gq = bq$. Also

$$g^*g \leq (\frac{h+\pi(h)}{2})^*(\frac{h+\pi(h)}{2}) + (\frac{h-\pi(h)}{2})^*(\frac{h-\pi(h)}{2}).$$

Thus

$$g^*g \leq \frac{h^*h}{2} + \frac{\pi(h)^*\pi(h)}{2} = \frac{h^*h}{2} + \pi(\frac{h^*h}{2}) \leq d,$$

as desired. $\square$

Bibliography

[1] C. A. Akemann, *Left ideal structure of C^*-algebras,* J. Funct. Anal. **6** (1970), 305-317.

[2] J. Arazy and B. Solel, *Isometries of nonselfadjoint operator algebras,* J. Funct. Anal. **90** (1990), no. 2, 284–305.

[3] W. B. Arveson, *Analyticity in operator algebras,* Amer. J. Math. **89** (1967), 578–642.

[4] W. B. Arveson, *Subalgebras of C^*-algebras,* Acta Math. **123** (1969), 141–224.

[5] C. A. Bearden, D. P. Blecher and S. Sharma, *On positivity and roots in operator algebras,* Integral Equations Operator Theory **79** (2014), no. 4, 555–566.

[6] B. Blackadar, *Operator algebras. Theory of C*-algebras and von Neumann algebras,* Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[7] D. P. Blecher, *Commutativity in operator algebras,* Proc. Amer. Math. Soc. **109** (1990), 709–715.

[8] D. P. Blecher, *Multipliers, C^*-modules, and algebraic structure in spaces of Hilbert space operators,* pp. 85-128 in "Operator Algebras, Quantization, and Noncommutative Geometry: A Centennial Celebration Honoring John von Neumann and Marshall H. Stone" Contemp. Math. 365, Amer. Math. Soc., 2004.

[9] D. P. Blecher, *Noncommutative peak interpolation revisited*, Bull. London Math. Soc. **45** (2013), 1100-1106.

[10] D. P. Blecher, *Generalization of C*-algebra methods via real positivity for operator and Banach algebras,* pages 35–66 in "Operator algebras and their applications: A tribute to Richard V. Kadison", (ed. by R.S. Doran and E. Park), vol. 671, Contemporary Mathematics, American Mathematical Society, Providence, R.I. 2016.

[11] D. P. Blecher, D. M. Hay, and M. Neal, *Hereditary subalgebras of operator algebras,* J. Operator Theory **59** (2008), 333-357.

[12] D. P. Blecher, J. Kaad, and B. Mesland, *Operator $*$-correspondences in analysis and geometry,* Proc. London Math. Soc. **117** (2018), 303–344.

[13] D. P. Blecher, K. Kirkpatrick, M. Neal and W. Werner, *Ordered involutive operator spaces,* Positivity **11** (2007), 497–510.

[14] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach,* Oxford Univ. Press, Oxford (2004).

[15] D. P. Blecher and M. Neal, *Open projections in operator algebras II: Compact projections,* Studia Math. **209** (2012), 203–224.

[16] D. P. Blecher and M. Neal, *Completely contractive projections on operator algebras,* Pacific J. Math. **283-2** (2016), 289–324.

[17] D. P. Blecher and M. Neal, *Contractive projections on Jordan operator algebras (tentative title),* In preparation.

[18] D. P. Blecher and M. Neal, *Noncommutative topology and Jordan operator algebras,* Preprint, 2017.

[19] D. P. Blecher and N. Ozawa, *Real positivity and approximate identities in Banach algebras*, Pacific J. Math. **277** (2015), 1–59.

[20] D. P. Blecher and C. J. Read, *Operator algebras with contractive approximate identities,* J. Functional Analysis **261** (2011), 188–217.

[21] D. P. Blecher and C. J. Read, *Operator algebras with contractive approximate identities II,* J. Functional Analysis **264** (2013), 1049–1067.

[22] D. P. Blecher and C. J. Read, *Order theory and interpolation in operator algebras,* Studia Math. **225** (2014), 61–95.

[23] D. P. Blecher, Z-J. Ruan, and A. M. Sinclair, *A characterization of operator algebras*, J. Functional Analysis **89** (1990), 288–301.

[24] D. P. Blecher and B. Solel, *A double commutant theorem for operator algebras,* J. Operator Theory **51** (2004), 435–453.

[25] D. P. Blecher and Z. Wang, *Involutive operator algebras,* preprint (2018), to appear Positivity .

[26] D. P. Blecher and Z. Wang, *Jordan operator algebras: basic theory,* Mathematische Nachrichten **291** (2018), 1629–1654.

[27] D. P. Blecher and Z. Wang, *Jordan operator algebras revisited,* preprint (2018), to appear Mathematische Nachrichten.

[28] D. P. Blecher and V. Zarikian, *The calculus of one-sided M-ideals and multipliers in operator spaces,* Memoirs Amer. Math. Soc. **842** (2006).

[29] L. J. Bunce, B. Feely, amd R. M. Timoney, *Operator space structure of JC*-triples and TROs, I,* Math. Z. **270** (2012), 961–982.

[30] E. G. Effros and Z. Ruan, *Operator spaces,* Oxford Univ. Press, Oxford (2000).

[31] M. Cabrera Garcia and A. Rodriguez Palacios, *Non-associative normed algebras. Vol. 1,* The Vidav-Palmer and Gelfand-Naimark theorems. Encyclopedia of Mathematics and its Applications, 154. Cambridge University Press, Cambridge, 2014.

[32] S. R. Garcia, E. Prodan, and M. Putinar, *Mathematical and physical aspects of complex symmetric operators,* J. Phys. A **47** (2014), 353001.

[33] S. R. Garcia and W. Wogen, *Some new classes of complex symmetric operators,* Trans. Amer. Math. Soc. *362* (2010), 6065–6077.

[34] K. Guo, Y. Ji, and S. Zhu, *A C^*-algebra approach to complex symmetric operators,* Trans. Amer. Math. Soc. **367** (2015), 6903–6942.

[35] M. Hamana, *Triple envelopes and Silov boundaries of operator spaces,* Math. J. Toyama Univ. **22** (1999), 77–93.

[36] H. Hanche-Olsen and E. Stormer, *Jordan operator algebras,* Monographs and Studies in Mathematics, 21. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[37] L. A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces,* Lecture Notes in Math., **364**, Springer-Verlag, Berlin - New York (1973).

[38] L. A. Harris, *A generalization of C^*-algebras,* Proc. Lond. Math. Soc. **42** (1981), 331–361.

[39] D. M. Hay, *Closed projections and peak interpolation for operator algebras,* Integral Equations Operator Theory **57** (2007), 491–512.

[40] N. Jacobson, *General representation theory of Jordan algebras,* Trans. Amer. Math. Soc. **70**, (1951), 509–530.

[41] R. V. Kadison, *Isometries of operator algebras,* Ann. of Math. **54** (1951), 325–338.

[42] J. Kaad and M. Lesch, *Spectral flow and the unbounded Kasparov product,* Adv. Math. **248** (2013), 495–530.

[43] J. D. Maitland Wright, *Jordan C^*-algebras,* Michigan Math. J. **24** (1977), 291–302.

[44] K. McCrimmon, *A taste of Jordan algebras,* Universitext. Springer-Verlag, New York, 2004.

[45] B. Mesland, *Unbounded bivariant K-theory and correspondences in noncommutative geometry,* J. Reine Angew. Math. **691** (2014), 101–172.

[46] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic analysis of operators on Hilbert space,* Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010.

[47] M. Neal, *Inner ideals and facial structure of the quasi-state space of a JB-algebra,* J. Funct. Anal. **173** (2000), 284–307.

[48] M. Neal and B. Russo, *Operator space characterizations of C*-algebras and ternary rings*, Pacific J. Math. **209** (2003), 339–364.

[49] M. Neal and B. Russo, *Representation of contractively complemented Hilbertian operator spaces on the Fock space,* Proc. Amer. Math. Soc. **134**, No. 2, (2005), 475-485

[50] M. Neal and B. Russo, *A holomorphic characterization of operator algebras*, Math. Scand. **115** (2014), 229-268.

[51] V. I. Paulsen, *Completely bounded maps and operator algebras,* Cambridge Studies in Advanced Math., 78, Cambridge University Press, Cambridge, 2002.

[52] G. K. Pedersen, *C*-algebras and their automorphism groups,* Academic Press, London (1979).

[53] G. K. Pedersen, *Factorization in* C^**-algebra,* Exposition. Math. **16** (1988), 145–156.

[54] J. Peters and W. Wogen, *Commutative radical operator algebras,* J. Operator Theory **42** (1999), 405–424.

[55] G. Pisier, *Introduction to operator space theory,* London Math. Soc. Lecture Note Series, 294, Cambridge University Press, Cambridge, 2003.

[56] A. Rodriguez-Palacios, *Jordan structures in analysis,* In: Jordan Algebras (Oberwolfach, 1992), pp. 97–186 de Gruyter, Berlin (1994).

[57] B. Russo, *Structure of JB*-triples,* In: Jordan Algebras, Proc. of Oberwolfach Conference 1992, Ed. W. Kaup, et al., de Gruyter (1994), 209–280.

[58] S. Sharma, *Real operator algebras and real completely isometric theory,* Positivity **18** (2014), 95–118.

[59] J. Shen and S. Zhu, *Complex symmetric generators for operator algebras,* J. Operator Theory **77** (2017), 421–454.

[60] D. M. Topping, *Jordan algebras of self-adjoint operators,* Mem. Amer. Math. Soc. **53** (1965).

[61] H. Upmeier, *Symmetric Banach manifolds and Jordan* C^**-algebras,* North-Holland Mathematics Studies, 104. North-Holland Publishing Co., Amsterdam, 1985.

[62] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 67 (1987).

[63] S. Zhu and J. Zhao, *Complex symmetric generators of* C^**-algebras,* J. Math. Anal. Appl. **456** (2017), 796–822.