A NEW GAP THEOREM RESULT FOR PROPER HOLOMORPHIC MAPPINGS BETWEEN COMPLEX BALLS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

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May 2014

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Acknowledgements

I would like to thank everyone who helped me get to the end of my doctoral program, including my adviser Shanyu Ji, Xiaojun Huang, and Wanke Yin. The contributions of everyone on this team enabled us to find a solution to the tricky problem covered in this dissertation, and Dr. Ji's work as my adviser helped me improve my ability as a researcher and as a teacher. I would also like to thank Gordon Heier, Min Ru, and Yunjiao Wang for their work on my committee. Gordon Heier's complex geometry seminars gave me an opportunity to prepare for my defense ahead of time. I must also extend thanks to Rice University professor Stephen Semmes, who taught me to enjoy analysis and inspired me to pursue my education here.

I would also like to thank my wife, Kathleen Nguyen, for her support and love.

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Abstract

In this dissertation, we study how rigidity properties of proper holomorphic mappings from complex balls of dimension n to complex balls of dimension N allow us to classify all such maps for particular values of n and N, up to equivalence by automorphism on the boundary. For maps $F : \mathbb{B}^n \to \mathbb{B}^N$, when N takes values in certain intervals, all maps are equivalent to $(G, 0)$, where G is a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^M$, with $M < N$. These intervals are established by the First, Second, and Third Gap Theorems. The cases where N lies on the upper boundary of one of these gaps are more difficult than cases where N is smaller. We review the cases where $N < 3n-3$ and where $3n < N < 4n-6$, and then we prove a classification theorem for the case where $N = 3n-3$.

In Chapter 10, we preemptively simplify our calculation by examining the coefficients of the $(1, 1)$ terms of the Taylor expansion of the codimension components $\phi_{k\ell}$ of a normalized map. Given any p, our map F is linear fractional along some affine subspace through p, which we use to show that all but one of the coefficients we investigate in this chapter are 0.

We assume the geometric rank $\kappa_0 = 2$, because the smaller cases are covered in previous papers, and larger κ_0 has been shown to be impossible (cf. [HJX06, Corollary 1.3]). In Chapter 11, we perform a long calculation to show that $\deg(F) \leq 2$ along a certain Segre variety. This takes two steps. First, we show that $\deg(F) \leq 3$, and then we use an explicit statement of F as a rational map to show $\deg(F) \leq 2$. Our calculation allows us to apply two theorems, [HJX06,Theorem 5.4] and [Leb11, Theorem 1.5], to prove that F must be equivalent to the Generalized Whitney Map.

Contents

Part I

Introduction

Chapter 1

Overview

In complex analysis, a common way to characterize objects that are infinite in some sense is to take advantage of rigidity properties, which are properties that allow some larger collection of objects to be uniquely described by a smaller collection of properties than we might expect. The classic example of this phenomenon is seen in a holomorphic function. A holomorphic function on the unit disc is determined by its values on the boundary. It is also determined over its entire domain by its derivatives at any single point. A polynomial is determined by its values on any infinite set. Those of degree two provide an instructive case; there are infinitely many of them, but there are only three kinds: parabolas, hyperbolas, and ellipses, up to equivalence by rotations and translations.

We are especially interested in the complex unit ball, which is an important object both because it is our prototypical pseudoconvex shape that is itself biholomorphic to other useful spaces (e.g., by the Riemann mapping theorem, $\mathbb{B}^1$ is biholomorphic to any simply connected subset of $\mathbb{C}^1$, though this is not true in higher dimensions) and because many holomorphic maps between open subsets of $\mathbb{C}^n$ can be shown to be restrictions of automorphisms on $\mathbb{B}^n$. If we can classify maps up to automorphism on $\mathbb{B}^n$, we can classify holomorphic maps between a large number of open subsets of $\mathbb{C}^n$. It turns out that proper holomorphic maps from balls of dimension n to balls of dimension N, $n \leq N$, have very strong rigidity properties that allow us to classify them neatly, and much work has been done in this area already. The case we solve in this dissertation is the following. The work to prove this theorem was a joint undertaking by J. Andrews, X. Huang, S. Ji, and W. Yin.

1.1 Theorem (Main Theorem)**.** *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$, $n \geq 4$. Then F is equivalent to a linear embedding, the Whitney Map, the D'Angelo Map, or the Generalized Whitney Map.*

Understanding the context for this result requires some background. First, our equivalence relation is up to automorphism on the boundary of either the domain $\mathbb{B}^n$ or the range $\mathbb{B}^N$ (here, $N = 3n - 3$), so we have only four kinds of maps instead of infinitely many. A proper map is one such that the preimage of any compact set is compact. For definitions of the maps mentioned in the theorem, see pages 29 and 30 of this dissertation. We proceed by mentioning an important result from Poincaré. He showed in [Po07] that any biholomorphic map between two pieces of spheres in $\mathbb{C}^2$ is a restriction of an automorphism on $\mathbb{B}^2$ (see Theorem 2.7), and Tanaka, in [Ta62], extended this result to all higher dimensions. The next breakthrough was from Alexander, in [Al74] and [Al76], where he proved that a proper holomorphic self-map of $\mathbb{B}^n$, $n \geq 2$, is an automorphism. This is enough to understand proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^n$, but the question of what happens when we look at proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^N$, $N > n$, remains highly interesting. In subsequent years, several very interesting properties of these maps would come to light.

In [We97], Webster investigated proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$, $n > 2$, and he was able to show that if such a map F is C^3 smooth up to the boundary, then it is a linear embedding, up to equivalence. The next big result came from Faran in [Fa82], where he classified all such maps from $\mathbb{B}^2$ to $\mathbb{B}^3$, showing four different maps up to equivalence. Cima and Suffridge, in [CS83], eased a condition on F and showed that the results of Webster and Faran hold when F is only C^2.

In D'Angelo's classic text on CR geometry, he described a family of maps that are not linear embeddings in the case where $N = 2n - 1$. The open question at the time was what would happen for $N \in (n, 2n-1)$. Cima and Suffridge conjectured that the only mappings in these cases would be $(id, 0, \ldots, 0)$. Faran showed in [Fa86] that this is the case when f is real analytic up to the boundary. Forstneric, in [Fo89], then reduced Faran's regularity condition by showing that a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ is rational if the map is C^{N-n+1}. Later Mir, in [Mir03], would ease the restriction on Forstneric's theorem, replacing $\mathbb{B}^n$ with a real analytic hypersurface. Many sources, such as [Fo92], found a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ that was not C^2 at any boundary point.

The next really big result came in 1999, when Huang published a paper on what we will call the First Gap Theorem. If we think of N as falling somewhere along a number line, then the first gap is the open interval $(n, 2n-1)$, and everywhere on this gap, a twice continuously differentiable proper holomorphic map will be equivalent to a linear embedding. The proof involves showing that a certain integer κ_0 associated with the map, called the geometric rank of F, must be 0 and then proving that when $\kappa_0 = 0$, a map must be linear.

Several useful tools for treating further cases came from Huang's paper, including a lemma so important that we simply call it Huang's Lemma. The paper also first introduced a normal form for F that would be used and expanded on in all subsequent work.

The boundary case of the first gap, when $N = 2n - 1$, is a bit tricky, and it was not solved until [HJ01] by Huang and Ji. They demonstrated that there were two classes of proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$, $n > 2$, the linear map and the Whitney Map. (The case where $n = 2$ was covered already, as that is just mapping $\mathbb{B}^2$ to $\mathbb{B}^3$.) The main idea of the proof is still to show something about the geometric rank κ_0 of F, which can be 0 or 1 this time, and then to show that F is equivalent to the Whitney Map when

$\kappa_0 = 1$. The basic setup is the same as in the First Gap Theorem, and we can use the same normal form for F. Our starting point is the same linearity criterion as in [Hu99], namely the fact that $\kappa_0 = 0$ implies that F is linear, and then we assume that F is not linear.

The proof proceeds by detailed technical arguments looking for a direction in which F is linear, through analyzing normal forms and looking at various derivatives. In particular, we get another normal form F^{***}. From there, we can show that F has degree two when restricted to a certain special type of variety, called a Segre variety, and then an extremely important lemma [HJ01,Lemma 5.4] gives us the fact that F has degree two everywhere. From here, it suffices to write down F explicitly and get the Whitney Map.

The Second Gap Theorem was proved in 2002 in [HJX02]. When $N \in (2n, 3n-3)$, $n \geq 4$, and $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ (i.e., it is a proper holomorphic map and C^3), there is no new minimum map. That is, all maps for N in that interval are equivalent to a map $(G, 0, \ldots, 0)$ where G is a map from B^n to B^{2n}. The family of maps from $\mathbb{B}^n$ to $\mathbb{B}^{2n}$ that we did not have before was the D'Angelo family

$$F_\theta(z, w) = (z, w\cos\theta, z_1 w\sin\theta, \ldots, z_{n-1} w\sin\theta, w^2\sin\theta).$$

The proof uses the fact that the geometric rank is less than or equal to $n-2$ (which makes the case where $n = 2$ very complicated, though Catlin and D'Angelo discuss it in [CD96]). This is required to allow the F^{***} normalization. Results from [HJ03] and [HJX05] show that these maps are rational and $(n - \kappa_0)-$linear, meaning that for any point $p \in \mathbb{B}^n$, we have an affine subspace S_p^a along which F is linear fractional. The Second Gap Theorem follows from describing precisely what these spaces look like.

We skip ahead to the case where $N \in (3n, 4n-6)$, $n \geq 7$, in [HJY14]. For such large N, it is extremely difficult to classify maps simply by writing them all down, so the proof of the Third Gap Theorem requires a different tactic. If we take F such that $F(0) = 0$, we can look at the Taylor formula for F to get $F(z) = \sum_\alpha \frac{D^\alpha F}{\alpha!}(0) z^\alpha$, which tells us that the range of F is spanned by $\{D^\alpha F(0)\}$. The proof of the theorem is essentially a long search for a basis for this space, and it involves using Huang's Lemma to show that there is only one linearly independent element coming from any jet of higher order than two, and we use the fact that $3n < N < 4n - 6$ to apply the lemma.

Our main theorem is the boundary case of the Second Gap Theorem.

The setup for the proof uses the F^{***} normalization and the fact that $\kappa_0 \leq 2$, by the equation mentioned earlier. We assume $\kappa_0 = 2$, because the other two cases are covered by [Hu99] and [HJX06]. We need $n \geq 4$ so that we can use the fact that $\kappa_0 \leq n - 2$. Our method, though long in practice, can be summarized by saying that we must prove that $\deg(F) \leq 2$, after which we can apply a theorem from a recent paper by Lebl, [Le11, Theorem 1.5], to get that F is equivalent to the Generalized Whitney Map.

In order to understand the proof in detail, we first investigate some important background concepts in the remainder of Part I. Once thoroughly grounded in these, in Parts II and III, we examine the First, Second, and Third Gap Theorems, as well as the boundary case for the First Gap Theorem, in order to become acquainted with some more tools that will be useful for our case. A special case of our result, $F \in Rat(\mathbb{B}^4, \mathbb{B}^9)$, was shown in [JX04]. We prove our main result in Part IV. Of note is that our proof provides evidence for a conjecture of D'Angelo, which says that $\deg(F) \leq \frac{N-1}{n-1}$ for $n > 2$.

Chapter 2

Preliminary material

Various pseudoconvex spaces

We begin with some motivation for our study.

Let $\mathbb{B}^n$ be the usual n-dimensional unit ball, $\{z \in \mathbb{C}^n : |z| < 1\}$, where our norm is $|z|^2 = \sum_{j=1}^n |z_j|^2$, and $|z_j| = \sqrt{x_j^2 + y_j^2}$, where $z_j = x_j + iy_j$.

2.1 Definition. *A holomorphic map F from $\mathbb{B}^n$ to $\mathbb{B}^N$ is called proper if the preimages of compact subsets of $\mathbb{B}^N$ are compact subsets of $\mathbb{B}^n$.*

2.2 Definition (K92, p128)**.** *Let Ω be a domain with C^2 boundary. Let P be a point on the boundary of Ω.. Let ρ be a defining function for $\partial\Omega$ near P. We say that P is a point of Levi pseudoconvexity if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) w_j \overline{w}_k \geq 0$$

for any tangent vector w satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j = 0.$$

We have strong pseudoconvexity if this inequality is strict.

We care about pseudoconvexity, not convexity, because the former is conserved by biholomorphic mappings, whereas the latter is not. This gives us an analog to convexity when we can take complex tangent vectors. The concept shows up in our application of the Lewy Extension Theorem (later).

Now, we have some theorems involving pseudoconvexity. These will give scope to our ultimate work, because they allow us to use nice shapes (unit balls!) in place of more complicated ones, and we will often take advantage of the geometric simplicity of our setting.

2.3 Theorem (Fe74). *Let $D_1, D_2 \subset \mathbb{C}^n$ be smooth strongly pseudoconvex domains with C^∞ boundaries. Then the following are equivalent:*

$(i)\ There\ exists\ a\ biholomorphic\ map\ f : D_1 \to D_1.$

$(ii)\ There\ is\ a\ C^\infty\ CR\ isomorphism\ F : \partial D_1 \to \partial D_2.$

2.4 Theorem (CJ96). *(i) If Ω is a bounded simply connected domain in $\mathbb{C}^{n+1}$ with connected smooth spherical real analytic boundary, then Ω is globally biholomorphic to the unit ball $\mathbb{B}^{n+1}$.*

(ii) (HJ98) We can ignore the requirement that Ω is simply connected if the boundary is defined by a real polynomial.

We note that the boundary ∂D of a domain $D \subset \mathbb{C}^n$ is a real hypersurface in $\mathbb{C}^n$.

For the unit ball, the boundary is

$$\partial\mathbb{B}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = 1\}.$$

This boundary is the unit sphere.

We denote by $\mathbb{H}^n$ the Siegel upper-half space:

$$\mathbb{H}^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(w) > |z|^2\}.$$

The boundary of this is

$$\partial\mathbb{H}^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(w) = |z|^2\}.$$

We call the boundary the Heisenberg hypersurface.

In the slightly more general case, we have

$$\mathbb{H}^n_\ell = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(w) > |z|^2_\ell\},$$

where $|z|^2_\ell = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2$, with boundary

$$\partial\mathbb{H}^n_\ell = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(w) = |z|^2_\ell , z \in \mathbb{C}^{n-1}\}.$$

We call this boundary the Levi nondegenerate hyperquadric.

Similarly, we can define

$$\mathbb{B}^n_\ell = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z|^2_\ell + |w|^2 < 1\}$$

and

$$\partial\mathbb{B}^n_\ell = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z|^2_\ell + |w|^2 = 1\}.$$

This is relevant because we have:

$$\Psi_n : \mathbb{H}^n_\ell \to \mathbb{B}^n_\ell, \Psi_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right).$$

This biholomorphic map, called the Cayley transformation, allows us to identify $\mathbb{B}^n_\ell$ with $\mathbb{H}^n_\ell$ and $\partial\mathbb{B}^n_\ell$ with $\partial\mathbb{H}^n_\ell$.

Thus, any results we get for the Heisenberg hypersurface are applicable to a wide range of complex domains.

Another important concept that will show up in the proofs of the boundary cases of the First Gap Theorem and Second Gap Theorem is the idea of a Segre family and Segre variety.

Let $M \subset \mathbb{C}^n$ be a local real analytic hypersurface containing 0. Let U be a small neighborhood of 0, and let $M = \{z \in U : r(z,\overline{z}) = 0\}$, where dr never vanishes and where $r(z,\overline{z})$ is a real analytic function.

Then the Segre family of M is defined by

$$\mathcal{M} = \{(z, \xi) \in U \times U' : r(z, \xi) = 0\},$$

where $U' = \{\overline{z} : z \in U\}$. Essentially, $\mathcal{M}$ is the complexification of M. It is a complex manifold of real dimension $2n-1$ in $\mathbb{C}^n \times \mathbb{C}^n$. If we must, we shrink U so that the power series $r(z, \xi)$ converges.

We may also write

$$\mathcal{M} = \{(z, w) \in U \times U : r(z, \overline{w}) = 0\}.$$

Write r as a local power series near 0:

$$r(z, \overline{z}) = \sum_{I,J} r_{IJ} z^I \overline{z}^J.$$

Because r is real-valued, we have

$$r(z, \overline{z}) = \overline{r(z, \overline{z})} = \overline{r}(\overline{z}, z).$$

Thus, $r_{IJ} = \overline{r_{IJ}}$.

Then we have the following theorem.

2.5 Theorem (Ji10). *(i) $\mathcal{M}$ is independent of the choice of defining function r of M.*

(ii) A holomorphic function on $\mathcal{M}$ which vanishes on M also vanishes on any open subset of $\mathcal{M}$ that contains a point of M.

(iii) Let $f : U \to V$ be a biholomorphic map where $U, V \subset \mathbb{C}^{n+1}$ are open subsets. Suppose f maps M into another real hypersurface N with real analytic defining function r'. Denote by $\mathcal{M}$ and $\mathcal{N}$ the corresponding Segre families of M and N, respectively. Let $F(z, \overline{w}) = (f(z), \overline{f}(\overline{w}))$, the analytic continuation. Then $F(\mathcal{M}) \subseteq \mathcal{N}$. When $f(M) = N$, then $f(\mathcal{M}) = \mathcal{N}$.

Proof. (i) If r' is another defining function of M, then $r'(z, \overline{z}) = s(z, \overline{z}) r(z, \overline{z})$, where s is a real analytic function on U that never vanishes on U for small enough U.

(ii) Consider the power series of $r(z, \overline{z})$ and $r(z, \xi)$, and consider that these are real analytic functions.

(iii) We have

$$r'(f(z)\overline{f}(\overline{w})) = s(z, \overline{w})r(z, \overline{w}),$$

with $s \neq 0$. This implies (iii).

$\square$

We define the idea of a Segre variety as follows. Let $M \subset \mathbb{C}^n$ be a local smooth real hypersurface such that $M \cap U = \{z \in U : r(z, \overline{z}) = 0\}$, where r is a defining function. For $w \in U$, we define the Segre variety of w with respect to M by

$$Q_w = \{z \in U : r(z, \overline{w}) = 0\}.$$

For example, consider the Heisenberg hypersurface $M = \partial\mathbb{H}^n$ defined by

$$r(z, \overline{z}) = \frac{w - \overline{w}}{2i} - \sum_{j=1}^{n-1} |z_j|^2 .$$

Let $p = (z_0, w_0) \in \partial\mathbb{H}^n$. Then

$$Q_p = \left\{ (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \frac{w - \overline{w_0}}{2i} - \sum_{j=1}^{n-1} z_j \overline{z_{0j}} = 0 \right\}.$$

Q_p turns out to be a complex hyperplane, and it can be identified with the holomorphic tangent space to $\partial\mathbb{H}^n$ at p. When $p = 0$, $Q_0 = \{(z, 0) \in \mathbb{C}^{n-1} \times \mathbb{C}\}$. Locally, p determines Q_p, and Q_p determines p uniquely.

2.6 Proposition (Ji10)**.** *(i)* $r(z, \overline{w}) = \overline{r}(\overline{w}, z) = \overline{r(w, \overline{z})}$.

(ii) $z \in Q_w \Leftrightarrow w \in Q_z$

(iii) $z \in M \Leftrightarrow z \in Q_z$

(iv) Q_z *is invariant under local biholomorphisms, i.e., if* f *is a biholomorphic map such that* $f(M) = M'$*, then* $f(Q_w) = Q'_{f(w)}$.

Furthermore, we have an important result from Poincaré:

2.7 Theorem (P07). *Any non-constant holomorphic map* $f : U \to V$ *satisfying* $f(U \cap \partial\mathbb{B}^2) \subset V \cap \partial\mathbb{B}^2$ *is a map in* $Aut(\partial\mathbb{B}^2)$*, where* U *and* V *are open subsets of* $\mathbb{C}^2$*.*

CR structures and differential operators

Now, let M be a smooth real hypersurface in $\mathbb{C}^n$. For any point $p \in M$, define a complex vector space

$$\mathcal{V}_p = \mathbb{C}T_pM \cap T_p^{(0,1)}\mathbb{C}^n.$$

Here, $\mathbb{C}T_pM = T_pM \otimes \mathbb{C}$.

We see that $\dim_{\mathbb{C}} \mathcal{V}_p = n - 1$ for any $p \in M$.

Then $\mathcal{V} = \bigcup_{p \in M} \mathcal{V}_p$ defines a subbundle of $\mathbb{C}TM$ satisfying $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$, where $\overline{\mathcal{V}} = \mathbb{C}TM \cap T^{(1,0)}$.

We call M a CR manifold. The bundle $\mathcal{V}$ is its CR structure, and $\dim_{\mathbb{C}} V_p$, which is independent of p, is called the CR dimension. We call a section over $\mathcal{V}$ a CR vector field over M.

We now find a basis for CR vector fields over $\mathcal{V}$.

First, we consider a real case. Let M be a real hypersurface in $\mathbb{R}^n$ defined by $\rho(x) = 0$. Let $\gamma : [0, 1] \to M$, $t \mapsto \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ be a curve in M. Then $\rho(\gamma(t)) = 0$ for all $t \in [0, 1]$. By the chain rule,

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j} \frac{\partial \gamma_j}{\partial t} = 0, \forall t \in [0, 1].$$

Because we can pick any curve γ, and thus $\sum_{j=1}^{n} \frac{d\gamma_j}{dt}$ is any tangent vector, we have $\left(\frac{\partial \rho}{\partial x_1}, \ldots, \frac{\partial \rho}{\partial x_n} \right) \perp T(M)$.

Let $L = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$. Then $\sum_{j=1}^{n} b_j \frac{\partial \rho}{\partial x_j} = 0$ if and only if $(b_1, \ldots, b_j) \perp \left(\frac{\partial \rho}{\partial x_1}, \ldots, \frac{\partial \rho}{\partial x_j} \right)$, and this is true if and only if L is a tangent vector of M.

Consider a real hypersurface M in $\mathbb{C}$ defined by $\rho(z, \overline{z}) = 0$. We think of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ and $(z, \overline{z})$ as a basis of vectors over $\mathbb{R}$.

Let $L = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j} + \sum_{k=1}^n c_k \frac{\partial}{\partial \overline{z}_k}$. Then L is a tangent vector of M if and only if $\sum_{j=1}^n b_j \frac{\partial \rho}{\partial z_j} + \sum_{k=1}^n c_k \frac{\partial \rho}{\partial \overline{z}_k} = 0$.

Then, for a $(1,0)$-type vector $L_1 = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j}$, we have that L_1 is a tangent vector of M if and only if $\sum_{j=1}^n b_j \frac{\partial \rho}{\partial z_j} = 0$.

Similarly, for a $(0,1)$-type vector $L_2 = \sum_{k=1}^n c_k \frac{\partial}{\partial \overline{z}_k}$, L_2 is a tangent vector if and only if $\sum_{k=1}^n c_k \frac{\partial \rho}{\partial \overline{z}_k} = 0$.

Let M be locally defined by $\rho = v - \phi(z, \overline{z}, u) = 0$, where (z, w) are holomorphic coordinates and $w = u + iv$.

Define $\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} - wi \frac{\phi_{\overline{z_j}}}{1+i\phi_u} \frac{\partial}{\partial \overline{w}}, 1 \leq j \leq n$. Here, $\phi_{\overline{z_j}} = \frac{\partial \phi}{\partial \overline{z_j}}$ and $\phi_u = \frac{\partial \phi}{\partial u}$.

Noting that $\rho = \frac{w - \overline{w}}{2i} - \phi(z, \overline{z}, \frac{w+\overline{w}}{2})$, we only need to show that $\overline{L}_j(\rho) = 0$.

$$
\begin{aligned}
\overline{L}_j(\rho) &= \left(\frac{\partial}{\partial \overline{z}_j} - 2i \frac{\phi_{\overline{z_j}}}{1+i\phi_u} \frac{\partial}{\partial \overline{w}} \right) \left(\frac{w-\overline{w}}{2i} - \phi(z, \overline{z}, \frac{w+\overline{w}}{2}) \right) \\
&= -\phi_{\overline{z_j}} - 2i \frac{\phi_{\overline{z}_j}}{1+i\phi_u} \left(-\frac{1}{2i} - \phi_u \frac{1}{2} \right) = 0.
\end{aligned}
$$

Thus, we get that $\{\overline{L}_1, \ldots, \overline{L}_{n-1}\}$ form a basis for the CR bundle over $\mathcal{V}$.

We say that a CR manifold is a differentiable manifold together with a subbundle $\mathcal{V}$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes \mathbb{C}$ such that $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. We call $\mathcal{V}$ the CR structure on M. Then $\mathcal{V} \oplus \overline{\mathcal{V}}$ is the complex tangent bundle of M, and $\dim_{\mathbb{C}} \mathcal{V}_p$, which is independent of p, is the CR dimension. A section of $\mathcal{V}$ is called a CR vector field over M. A C^1-smooth function f is called a CR function if it is locally annihilated by any CR vector field. A CR mapping is a smooth mapping F between CR manifolds $(M, \mathcal{V}_M)$ and $(N, \mathcal{V}_N)$ such that $df(\mathcal{V}_M) \subseteq (\mathcal{V}_N)$.

Let us look at an example. Let $M = \partial \mathbb{H}_n \subset \mathbb{C}^n$ be the Heisenberg hypersurface. We let our defining function $\rho(z)$ be

$$
\rho(z) = Im(w) - |z|^2 = \frac{w - \overline{w}}{2i} - \sum_{j=1}^{n-1} |z_j|^2 .
$$

It is easy to see that this is a defining function.

Following the example where we found a basis for the CR bundle, we have

$$\phi(z, \overline{z}, \frac{w+\overline{w}}{2}) = |z|^2,$$

and then $\phi_{\overline{z}_j} = z_j$ and $\phi_u = 0$. Then

$$\overline{L}_j := \frac{\partial}{\partial \overline{z}_j} - 2iz_j \frac{\partial}{\partial \overline{w}}, 1 \leq j \leq n-1$$

is a basis for $\mathcal{V} = \mathbb{C}T^{(0,1)}(\partial \mathbb{H}^n)$, and

$$L_j := \frac{\partial}{\partial z_j} + 2i\overline{z}_j \frac{\partial}{\partial w}, 1 \leq j \leq n-1$$

is a basis for $\overline{\mathcal{V}} = \mathbb{C}T^{(1,0)}(\partial \mathbb{H}^n)$.

We also get that $T = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}$ is a tangent vector of M. We note that $T = \overline{T}$, so T is a real vector. Then the vector fields $\{L_j, \overline{L}_j, T\}_{1 \leq j \leq n-1}$ form a basis of the tangent vector bundle $T(M)$.

The vector field $T = \frac{\partial}{\partial Re(w)} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}$ is called the Reeb vector field.

We say that a real nonvanishing 1-form θ over M a contact form if $\theta \wedge (d\theta)^n \neq 0$. For example, if M is given by a defining function r, then $\theta = i\partial r$ is a contact form of M.

To make this concrete, let $M = \partial \mathbb{H}^n \subset \mathbb{C}^n$. Then $\rho(z) = -Im(w) + |z|^2 = -\frac{w-\overline{w}}{2i} + \sum_{j=1}^{n-1} |z_j|^2$. Our contact form is then

$$\theta = -i\partial\rho = \frac{1}{2}dw - i\sum_{j=1}^{n} \overline{z}_j dz_j.$$

We now go into a bit more detail about the geometry of our surfaces. In particular, we want to define the Levi form at a point p on a CR manifold $(M, \mathcal{V})$. This form is

$$\begin{aligned} h_p : \ \overline{\mathcal{V}}_p &\rightarrow \{T_p(M) \otimes \mathbb{C}\}/(\mathcal{V}_p \oplus \overline{\mathcal{V}}_p) \\ v_p &\mapsto \frac{1}{2i}\pi_p\{[v, \overline{v}]\} \end{aligned}$$

where v is any vector field in $\overline{\mathcal{V}}$ that is equal to v_p at p and $\pi_p : T_p(M) \otimes \mathbb{C} \to \{T_p(M) \otimes \mathbb{C}\}/(\mathcal{V} \oplus \overline{\mathcal{V}})$ is the natural projection. This definition does not depend on v.

If M is an embedded CR manifold, then we take $\overline{\mathcal{V}} = T^{(1,0)}(M)$ and identify the quotient space with X_p, the complexified totally real part of the tangent bundle:

$$\begin{array}{rccc} h_p : & H_p^{1,0}(M) & \to & X_p(M) \\ & v_p & \mapsto & \frac{1}{2i}\pi_p\{[v, \overline{v}]\} \end{array}.$$

We can also think of the Levi form of an embedded CR manifold $M \subset \mathbb{C}^n$ as

$$\begin{array}{rccc} \tilde{h}_p : & H_p^{(1,0)}(M) & \to & N_p(M) \\ & v_p & \mapsto & \frac{1}{2i}\tilde{\pi}_p(J[\overline{v}, v])_p \end{array},$$

where v is any $H^{1,0}(M)$-vector field extension of v_p, $N_p(M)$ is the normal space of M at p, and J is the complex structure map for $T_p(\mathbb{C}^n)$, and $\tilde{\pi}_p : T_p(\mathbb{C}^n) \mapsto N_p(M)$ is the orthogonal projection map.

Now, for a commonly used definition, let $M = \{\rho = 0\}$ be a smooth real hypersurface. Let $p \in M$, and then rescale so that $|\nabla\rho(p)| = 1$, a unit base for $N_p(M)$. Our Levi form is then

$$\tilde{h}_p = -\sum_{j,k=1}^{n} \frac{\partial^2\rho(p)}{\partial\zeta_j\partial\overline{\zeta}_k} w_j\overline{w}_k\nabla\rho(p), \forall W = \sum_{k=1}^{n} w_k\frac{\partial}{\partial\zeta_k} \in H_p^{1,0}(M).$$

We can write a Levi form in terms of some contact form θ (remembering that we can get a contact form from the defining function, in the right setting):

$$h_\theta(v, w) := -d\theta(v, \overline{w}) = \theta([v, \overline{w}]), \forall v, w \in \mathcal{V} \oplus \overline{\mathcal{V}}.$$

This uses the Cartan formula,

$$\langle d\theta, v \wedge \overline{w}\rangle = v\langle\theta, \overline{w}\rangle - \overline{w}\langle\theta, v\rangle - \langle\theta, [v, \overline{w}]\rangle,$$

and the fact that $\langle\theta, T\rangle = 0, \forall T \in \mathcal{V} \oplus \overline{\mathcal{V}}$ so that $\langle\theta, \overline{w}\rangle = \langle\theta, v\rangle = 0$.

We can think of the Levi form of M as a Hermitian 2-form, or a metric, on $\overline{\mathcal{V}} := T^{1,0}M$ defined by $h_\theta : T^{1,0}M \otimes T^{1,0}M \to \mathbb{C}$. Then (M, θ) is said to be Levi nondegenerate if h_θ is Levi nondegenerate at every point of M. We say that (M, θ) is strongly pseudoconvex if h_θ is positive definite or just pseudoconvex if h_θ is positive semidefinite.

Some extension theorems

2.8 Theorem (Bochner's Extension Theorem)**.** *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a bounded open subset, with C^∞ boundary $M := \partial\Omega$, and suppose that $\mathbb{C}^n - \overline{\Omega}$ is connected. If $f \in C^\infty(M)$ is a CR function, there is a unique function $F \in C^\infty(\overline{\Omega})$ such that $F|_M = f$ and F is holomorphic on Ω.*

There is a local version of this.

Let $M = \{z \in \mathbb{C}^n : \rho(z) = 0\}$ by a hypersurface where ρ is a C^k-smooth defining function with $d\rho \neq 0$ on M with $2 \leq k \leq \infty$. If ρ is scaled so that $|\nabla\rho(p)| = 1 \forall p \in M$, then the Levi form of M at p is the map

$$W \mapsto \left(-\sum_{j,k=1}^{n} \frac{\partial^2 \rho(p)}{\partial \zeta_j \partial \overline{\zeta}_k} w_j \overline{w}_k \right) \nabla\rho(p), \forall W = \sum_{j=1}^{n} w_j \frac{\partial}{\partial \zeta_j} \in H_p^{1,0}(M).$$

The eigenvalues of the Levi form of M at p are the ones of the matrix $\left(\frac{\partial^2 \rho}{\partial \zeta_j \partial \overline{\zeta}_k}\right)$. Now, we use the notation $\Omega^+ = \{\rho > 0\}$ and $\Omega^- = \{\rho < 0\}$.

2.9 Theorem (Lewy Extension Theorem)**.** *Let $M \subset \mathbb{C}^n$ be a C^k-smooth real hypersurface with $3 \leq k \leq \infty$ and $n \geq 2$. Let $p \in M$ be a point.*

1. If the Levi form of M at p has at least one positive eigenvalue, then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each C^1-smooth CR function f on ω, there is a unique function F which is holomorphic on $U \cap \Omega^+$ and continuous on $U \cap \overline{\Omega}^+$ such that $F|_{U \cap M} = f$.

2. If the Levi form of M at p has at least one negative eigenvalue, then the conclusion above holds with Ω^+ replaced by Ω^-.

3. If the Levi form of M at p has eigenvalues of opposite sign, then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each C^1-smooth CR function f on ω, there is a unique function F which is holomorphic on U with $F|_{U\cap M} = f$.

2.10 Theorem (Ji10). *Let $M \subset \mathbb{C}^{n+1}$ be a real analytic hypersurface, $p \in M$, and f a CR function in a neighborhood of p on M Then the following two statements are equivalent:*

1. f extends to a holomorphic function on a neighborhood of p in $\mathbb{C}^{n+1}$.

2. f is real analytic in a neighborhood of p in M.

Proof. Locally, assume that M is given by $v = \phi(z, \overline{z}, w)$, where $z = (z_1, \ldots, z_n)$, $w = u + iv$, and ϕ is real analytic with $\phi(0) = 0, d\phi(0) = 0$.

We know that the map

$$(z, \overline{z}, u) \mapsto (z, w) = (z, u + i\phi(z, \overline{z}, u))$$

is a parameterization of M with parameters $(z, \overline{z}, u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$.

We recall that a local basis of CR vector fields is given by

$$\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} - 2i\frac{\phi_{\overline{z}_j}}{1 + i\phi_u}\frac{\partial}{\partial \overline{w}}, 1 \leq j \leq n,$$

where $\phi_{\overline{z}_j} = \frac{\partial\phi}{\partial\overline{z}_j}$ and $\phi_u = \frac{\partial\phi}{\partial u}$.

Next, let

$$F(z, w) = f(z, \overline{z}, \zeta),$$

where ζ satisfies $\zeta + i\phi(z, \overline{z}, \zeta) = w$. Hence $\zeta = \zeta(z, \overline{z}, w)$ is uniquely determined by the equation according to the implicit function theorem. We note that even though ζ might not be real-valued in general, when $w \in M$, then $\zeta = Re(w)$ is real-valued. Differentiate both sides of $\zeta + i\phi(z, \overline{z}, \zeta) = w$ to get

$$\frac{\partial \zeta}{\partial \overline{z}_j} + i\phi_{\overline{z}_j} + i\phi_\zeta \frac{\partial \zeta}{\partial \overline{z}_j} = 0.$$

We note that $F|_M = f$, since $F(z, u + i\phi(z, \overline{z}, u)) = f(z, \overline{z}, u)$ for any $(z, w) \in M$.

It remains to be shown that F is holomorphic.

Since $\zeta = \zeta(z, \overline{z}, w)$ is real analytic without the $\overline{w}$ terms, F is holomorphic in w. Thus, we just need to show that F is holomorphic for each z_j, $1 \leq j \leq n$.

For any j,

$$\frac{\partial F}{\partial \overline{z}_j} = \frac{\partial f}{\partial \overline{z}_j} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial \overline{z}_j} = \frac{\partial f}{\partial \overline{z}_j} - \frac{i\phi_{\overline{z}_j}}{1 + i\phi_\zeta}\frac{\partial f}{\partial \zeta} = \overline{L}_j f = 0.$$

The second equality comes from rearranging $\frac{\partial \zeta}{\partial \overline{z}_j} + i\phi_{\overline{z}_j} + i\phi_\zeta \frac{\partial \zeta}{\partial \overline{z}_j} = 0$. The third comes from our definition of $\overline{L}_j$, and the fourth comes from the fact that f is a CR function. □

Now, let $F = (F_1, \ldots, F_n) : M \to N$ be a real analytic CR map between real analytic hypersurfaces $M, N \subset \mathbb{C}^{n+1}$. Since each F_j is a CR function, by the theorem we just proved, F extends holomorphically to a neighborhood of M.

2.11 Lemma (Hopf Lemma). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary, let $a \in \Omega$, and let $v(a)$ be the inward normal to $\partial\Omega$ at a. Then for any subharmonic function u on Ω with $u < 0$ on Ω, we get*

$$\overline{\lim}\frac{u(z)}{|z-a|} \leq -c$$

for some constant $c > 0$, where $\overline{\lim}$(the $\limsup$) is as $z \to a$ along $v(a)$.

Proof. Since $\partial\Omega$ is C^2 smooth, we can take a ball $B_R(z_0)$ with center z_0 and radius R in $\mathbb{C}^n$ such that it is tangent to $\partial\Omega$ at a and $B_R(z_0) \subset \Omega$. Such z_0 can be chosen in a fixed compact subset of Ω.

For $0 < r < R$, define a function on $B_R(z_0) - \overline{B_r(z_0)}$:

$$g(z) := e^{-\lambda|z-z_0|^2} - e^{-\lambda R^2}.$$

When λ is large compared to r, this function is subharmonic. In fact,

$$\frac{\partial^2}{\partial \overline{z}_j \partial z_j} g = \frac{\partial}{\partial \overline{z}_j} \left(-\lambda(\overline{z}_j - \overline{z}_0) e^{-\lambda |z - z_0|^2} \right) = \lambda(\lambda \left| z_j - z_0 \right|^2 - 1) e^{-\lambda |z - z_0|^2} > 0$$

holds on $B_R(z_0) - \overline{B_r z_0}$ as $\lambda >> 0$.

It is easy to see that $g = 0$ for $|z - z_0| = R$.

Since $u < 0$ on Ω, by taking a sufficiently small $\epsilon > 0$, we have $u + \epsilon g \leq 0$ on the boundary of $B_R(z_0) - \overline{B_r(z_0)}$. Thus, we can apply the maximum principle, concluding that $u(z) \leq -\epsilon g(z)$, i.e., $\frac{u(z)}{g(z)} \leq -\epsilon$ for $r < |z - z_0| < R$. We must then show that

$$\frac{u(z)}{|z - a|} \leq \ constant \cdot \frac{u(z)}{g(z)}$$

as $z \to a$ along $v(a)$. Since $u > 0$, it is enough to prove that $g(z) \leq \ constant \cdot |z - a|$, and we get this by taking the Taylor series expansion of f. □

2.12 Corollary. *Let Ω, a, and u be as above. If $u \leq 0$ on $\partial\Omega$ and $\overline{\lim}_{z \to a} \frac{u(z)}{z-a} = 0$, where $z \to a$ along the direction of the normal vector, then $u \equiv 0$.*

Proof. Assume for the sake of contradiction that u is not identically 0. By the maximum principle, $u < 0$ on Ω. By the Hopf lemma, $\overline{\lim} \frac{u(z)}{z-a} \leq -\epsilon < 0$, which is a contradiction. □

Our setting

We now have most of the preliminary results out of the way, so we are ready to describe some of the basics of our setting. We wish to cover some basic facts about proper holomorphic maps between balls.

2.13 Proposition. *Let $D, D' \subset \mathbb{C}^n$ be bounded domains, and let $f : D \to D'$ be a holomorphic map. Then f is proper if and only if for any sequence z_v that converges to a point in ∂D, the image sequence $\{f(z_v)\}$ tends to $\partial D'$.*

Proof. ($\Rightarrow$) Suppose that $\{f(z_v)\}$ does not tend to $\partial D'$. Then there is a subsequence $\{z_{v_k}\}$ such that $\{f(z_{v_k})\}$ is relatively compact in D'. This contradicts the fact that f is proper.

($\Leftarrow$) Suppose there is a compact subset $K \subset D'$ such that $f^{-1}(K)$ is not compact in D. Then there is a sequence $\{z_v\}$ converging to ∂D such that $\{f(z_v)\}$ does not tend to $\partial D'$. □

Now, we can see that we should focus on a subclass of the set of CR submanifolds of a sphere.

We will use certain notation for different classes of maps between these manifolds. These will be used throughout this dissertation.

$$\begin{aligned} Prop(\mathbb{B}^n, \mathbb{B}^N) &:= \{proper\ holomorphic\ maps\ F : \mathbb{B}^n \to \mathbb{B}^N\} \\ Prop_k(\mathbb{B}^n, \mathbb{B}^N) &:= Prop(\mathbb{B}^n, \mathbb{B}^N) \cap C^k(\overline{\mathbb{B}^n}) \\ Rat(\mathbb{B}^n, \mathbb{B}^N) &:= Prop(\mathbb{B}^n, \mathbb{B}^N) \cap \{rational\ maps\} \end{aligned}$$

We say that F and G are equivalent, i.e., $F \sim G$, if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$.

2.14 Theorem (Al77). *Any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^n$ must be an automorphism when $n \geq 2$.*

For the case where $n = 1$, we have the following.

2.15 Proposition (Ji10).

$$Prop(\mathbb{B}^1, \mathbb{B}^1) = \left\{ F(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a}_j z},\ with\ |a_j| < 1 \right\}$$

Proof. If f is proper, then $f^{-1}(0)$ is compact. In particular, $f^{-1}(0) = \sum_{j=1}^{N} m_j[a_j]$, where $a_j \in \mathbb{B}^1$ and $m_j \in \mathbb{Z}^+$. Let

$$g(z) = \prod_{j=1}^{N} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right)^{m_j}.$$

We want to show that $\frac{f}{g}$ is constant and $\left|\frac{f}{g}\right| \equiv 1$, which implies that $f \equiv e^{i\theta} g$.

Now, both $\frac{f}{g}$ and $\frac{g}{f}$ are meromorphic and have only removable singularities. Thus, $\frac{f}{g}$ and $\frac{g}{f}$ are holomorphic in $\mathbb{B}^1$. Applying the most recent proposition, we have that for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$1 - \epsilon \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1}{1-\epsilon}, \forall\, |z| > 1 - \delta.$$

We apply the maximum principle:

$$1 - \epsilon \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1}{1-\epsilon}, \forall\, |z| \leq 1 - \delta.$$

Hence, $\frac{f}{g} \equiv\ constant$, and if we let $\epsilon \to 0$, we get $\left|\frac{f(z)}{g(z)}\right| \equiv 1$. $\square$

Let $F = (f, \phi, g) = (\tilde{f}, g) : M_1 \cap \partial\mathbb{H}_n \to \mathbb{H}_N$ be a non-constant C^2-smooth CR map with $F(0) = 0$, where M_1 is an open subset of $\partial\mathbb{H}_n$. We write $f = (f_1, \ldots, f_{n-1})$, $\phi = (\phi_1, \ldots, \phi_{N-1})$, and $\tilde{f} = (f, \phi)$. Because of how we define our hypersurface, we have the following recurring basic equation:

$$Im(g) = \tilde{f} \cdot \overline{\tilde{f}} = \langle \tilde{f}, \overline{\tilde{f}} \rangle, \forall (z, w) \in M_1.$$

The same thing, written differently, looks like

$$\frac{g - \overline{g}}{2i} = \sum_{j=1}^{n-1} |f_j|^2 + \sum_{j=1}^{N-1} |\phi_j|^2, Im(w) = |z|^2.$$

The Lewy Extension Theorem applied here tells us that F extends holomorphically to a pseudoconvex side of M_1.

The F^* and F^{**} normalizations and the Chern-Moser operator

One of the primary tools for dealing with these spaces will be to apply our differential operators to the basic equation, often restricting ourselves to a certain Segre variety from which we can later generalize our findings. Keeping in mind that $L_j\overline{f} = 0$, we get

$$\frac{L_\ell g}{2i} = \sum_j L_\ell f_j \cdot \overline{f_j}^t + \sum_j L_\ell \phi_j \cdot \overline{\phi_j}^t = L_\ell \tilde{f} \cdot \overline{\tilde{f}}^t,$$

and

$$\frac{Tg - \overline{Tf}}{2i} = T\tilde{f} \cdot \overline{\tilde{f}}^t + \tilde{f}\overline{T\tilde{f}}^t.$$

Before applying higher degree differential operators, we mention a lemma, whose proof is straightforward.

2.16 Lemma (Ji10,2.3.1). *(i)* $TL_j = L_jT, T\overline{L_j} = \overline{L_j}T$, *and* $L_jL_k = L_kL_j$.

(ii) For any continuous CR function h *over an open subset* $M_1 \subset \partial\mathbb{H}_n$, Th *is a CR distribution over* M_1. *For* $1 \leq j, j \leq n-1$, $\overline{L_k}(L_jh)$ =[$L_j, \overline{L_k}$]$h = 2i\delta_k jTh$.

(iii) Let h *be a* C^2*-smooth CR function over* $\partial\mathbb{H}_n$, *and let* χ *be a* C^1 *function over* $\partial\mathbb{H}_n$. *Then for any* $k > 0$,

$$\overline{L_k}(L_k^2(h)\chi) = 4iL_k(T(h))\chi + L_k^2(h)\overline{L_k}(\chi)$$

and

$$\overline{L_k}(L_k(T(h))\chi) = 2iT^2(h)\chi + L_k(T(h))\overline{L_k}(\chi).$$

(iv) For k, ℓ, j *and a* C^2*-smooth CR function* h, *we have*

$$\overline{L_k}L_\ell L_jh = 2i\delta_k^\ell TL_jh + 2i\delta_k^jTL_\ell h.$$

We can now apply our differential operators to the geometric equation, as we said we wanted to do. Looking at second order differential operators L_kL_ℓ, TL_ℓ, T^2, and $\overline{L_k}L_\ell$, we get

$$\begin{array}{rcl}
\frac{L_kL_\ell g}{2i} & = & L_k(L_\ell\tilde{f}) \cdot \overline{\tilde{f}}^t \\
\frac{1}{2i}T(L_\ell g) & = & T(L_\ell\tilde{f}) \cdot \overline{\tilde{f}}^t + L_\ell(\tilde{f}) \cdot \overline{T\tilde{f}}^t \\
Im(T^2g) & = & 2Im(iT^2\tilde{f} \cdot \overline{\tilde{f}}^t) + 2\left|T\tilde{f}\right|^2 \\
\frac{1}{2i}\overline{L_k}L_\ell g & = & \overline{L_k}L_\ell\tilde{f} \cdot \overline{\tilde{f}}^t + L_\ell\tilde{f} \cdot \overline{L_k\tilde{f}}^t.
\end{array}$$

By a simple calculation, we confirm that $\overline{L_\ell} L_\ell = 2iT$, and thus, applying this,

$$Tg = 2i\langle T\tilde{f}, \overline{\tilde{f}}\rangle + \left| L_j \tilde{f} \right|^2.$$

Next, we apply $\overline{L_k} L_j L_\ell$:

$$\frac{1}{2}\overline{L_k}(L_j(L_\ell g)) = \overline{L_k}(L_j(L_\ell f)) \cdot \overline{f}^t + L_j(L_\ell f) \cdot \overline{L_k f}^t. \qquad (\star)$$

When $k = j = \ell$, we use part (iv) of Lemma 2.4, getting

$$L_j(L_\ell f) \cdot \overline{L_k f}^t = 0.$$

When $k = j = \ell$, again applying Lemma 2.4 to $(\star)$, we get

$$T(L_\ell g) = 2iT(L_\ell f) \cdot \overline{f}^t + L_j(L_\ell f) \cdot \overline{L_j f}^t.$$

Again,

$$T(L_j g) = 2iT(L_j f) \cdot \overline{f}^t + L_\ell(L_j f) \cdot \overline{L_\ell f}^t.$$

Again,

$$2T(L_k g) = 4iT(L_k f) \cdot \overline{f}^t + L_k(L_k f) \cdot \overline{L_k f}^t.$$

We now use a geometric fact: $F(0) = 0$. Apply this to our initial first and second order applications of the differential operators to the basic equation to obtain

$$\left.\frac{\partial}{\partial z_j}\right|_0 = \left.\frac{\partial^2 g}{\partial z_k \partial z_\ell}\right|_0 = 0.$$

This leads to the first two of our important normalizations for F. That is, we will exploit the fact that F is equivalent to a map F^* and a map F^{**}, with some particular useful properties. More details about these normalizations, including derivations of them, appear in the papers on the Gap Theorems, and thus they will show up in context in later chapters, but for now, we state some of the results.

First, we have F^* such that

$$F^*(0) = 0,\ \frac{\partial f_j^*}{\partial z_\ell}\bigg|_0 = \delta_j^\ell,\ \frac{\partial \phi_j^*}{\partial z_\ell}\bigg|_0 = 0,\ \frac{\partial g^*}{\partial z_\ell}\bigg|_0 = 0,\ \frac{\partial g^*}{\partial w}\bigg|_0 = 1.$$

We further find F^{**} with the following properties:

$$F^{**}(0) = 0,\ \frac{\partial f_\ell^{**}}{\partial z_j}\bigg|_0 = \delta_{\ell j},\ \frac{\partial f^{**}}{\partial w}\bigg|_0 = 0,\ \frac{\partial \phi_k^{**}}{\partial z_\ell}\bigg|_0 = 0,$$

$$\frac{\partial \phi_k^{**}}{\partial w}\bigg|_0 = 0,\ \frac{\partial g^{**}}{\partial z_\ell}\bigg|_0 = 0,\ \frac{\partial g^{**}}{\partial w}\bigg|_0 = 1,$$

and

$$\frac{\partial^2 g^{**}}{\partial z_j \partial z_k}\bigg|_0 = \frac{\partial^2 g^{**}}{\partial w^2}\bigg|_0 = 0.$$

The next important tool we will use in our approach to these problems is the Chern-Moser operator.

Let F satisfy our F^{**} normalization. Then

$$f = z + \hat{f}, g = w + \hat{g}, \hat{f}, \hat{h}, \phi \in \mathcal{O}(|(z,w)|^2),$$

$$\frac{\partial^2 \hat{g}}{\partial z_\ell \partial z_k}\bigg|_0 = \frac{\partial^2 \hat{g}}{\partial w^2}\bigg|_0 = 0.$$

From this,

$$Im(w + \hat{g}) = \sum_{j=1}^{n-1} \left| z_j + \hat{f}_j \right|^2 + \sum_{j=1}^{N-n} |\phi_j|^2. \qquad (\star\star)$$

The concept of a weighted degree will appear in many of the proofs to come. We will assign weights of 1 and 2 to z and w terms, respectively, so that we can define $o_{wt}(k)$ to be a class of function such that $f \in o_{wt}(k)$ if

$$\lim_{t \to 0^+} \frac{h(tz, t^2 w, t\overline{z}, t^2 \overline{w})}{t^k} = 0,$$

where this limit is uniform with respect to $(z,w) \approx (0',0) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

In other circumstances, we will refer directly to polynomials of weighted degree k. A homogeneous polynomial $z^s w^\ell$ has weighted degree k if $k = s + 2\ell$.

If we have a function f, we can refer to the part of f that is a homogeneous polynomial of weighted degree k by $f^{(k)}$.

That said, we can write

$$\begin{aligned} \hat{f}_j &= \textstyle\sum_{s=2}^{m-1} f_j^{(s)} + o_{wt}(m-1), \\ \hat{g} &= \textstyle\sum_{s=3}^{m} g^{(s)} + o_{wt}(m), \\ \phi - j &= \textstyle\sum_{s=\ell}^{m-\ell} \phi_j^{(s)} + o_{wt}(m-\ell), \ell \geq 2. \end{aligned}$$

Note the simple property of complex numbers that says that $a + \overline{a} = Im(2ia)$.

Now, we use our expressions for $\hat{f}_j, \hat{g}$ and ϕ_j with ($\star\star$) to get

$$Im(w) + Im(\hat{g}) = \sum_j (z_j + \hat{f}_j)\left(\overline{z_j} + \overline{\hat{f}_j}\right) + \sum_k \left(\sum_s \phi_k^{(s)}\right)\left(\sum_t \overline{\phi_k^{(t)}}\right)$$

$$= |z|^2 + \sum_j \left(z_j \overline{\hat{f}_j} + \hat{f}_j \overline{z_j} + \left|\hat{f}_j\right|^2\right) + \sum_k \left(\sum_s \phi_j^{(s)}\right)\left(\sum_j \overline{\phi_k^{(t)}}\right)$$

$$= |z|^2 + \sum_j Im(2i\langle \overline{z_j}, \hat{f}_j\rangle) + \sum_j \left|\hat{f}_j\right|^2 + \sum_k \left(\sum_s \phi_k^{(s)}\right)\left(\sum_t \overline{\phi_k^{(s)}}\right), \forall Im(w) = |z|^2.$$

Next,

$$Im(\hat{g}) = Im(2i\langle \overline{z}, \hat{f}\rangle) + \left|\hat{f}\right|^2 + \sum_k \left(\sum_s \phi_k^{(2)}\right)\left(\sum_t \overline{\phi_k^{(s)}}\right), \forall Im(w) = |z|^2.$$

We can separate the above equation by the weighted degree of its terms. Those of weighted degree s can be written thusly:

$$Im(g^{(s)} - 2i\langle \overline{z}, f^{(s-1)}\rangle) = \sum_{j=1}^{N-n} \sum_{p=\ell}^{s-\ell} \phi_j^{(s-p)} \overline{\phi_j^{(p)}} + G^{(s)}, \forall (z, w) \in \partial \mathbb{H}_n. \qquad (\star\star\star)$$

Here, $G^{(s)}$ is a homogeneous polynomial of weighted degree s that contributed to our expression by $f^{(\sigma-1)}$ and $g^{(\sigma)}$, where $\sigma \leq s - 1$.

Now, the Chern-Moser operator is defined as

$$\mathcal{L}(f,g) = Im(\hat{g} - 2i\langle \overline{z}, \hat{f}\rangle).$$

Clearly, $G^{(s)} \equiv 0$ if $f^{(\sigma)-1} \equiv g^{(\sigma)} \equiv 0$.

We have two cases to examine. First, we consider the case where $s = 2k$ for some integer k.

Now, if $f^{(\sigma-1)} \equiv g^{(\sigma)} \equiv 0$, we have

$$Im(g^{(2k)}(z,w) - 2i\langle \overline{z}, f^{(2k-1)}\rangle) = \sum_{j=1}^{N-n} \phi_j^{(k)}\overline{\phi_j^{(k)}}, \forall (z,w) \in M_1.$$

Note that the $\phi^{(s-p)}\overline{\phi^{(p)}}$ terms must be 0 when $s - p \neq p$.

For the second case, suppose $s = 2k+1 \leq m$. Now, if $f^{(\sigma-1)} \equiv g^{(\sigma)} \equiv 0$,

$$Im(g^{(2k+1)} - 2i\langle \overline{z}, f^{(2k)}\rangle) = 0.$$

Here, all of the $\phi^{(p)}$ terms vanish, since $s - p \neq p$ for all p.

2.17 Lemma (Ji10,2.7.1)**.** *Let* $F = F^{**} \in Prop_2(\mathbb{H}_n, \mathbb{H}_N)$. *Then*

(i) $f^{(2)} \equiv 0$, $f^{(3)} \equiv a^{(1)}(z)w, \phi^{(2)}(z,w) = \phi^{(2)}(z), g^{(3)} \equiv g^{(4)} \equiv 0.$

and

(ii) $-2i\langle a^{(1)}(z), \overline{z}\rangle \left|z\right|^2 = \sum_{j=1}^{N-n} \left|\phi_j^{(2)}(z)\right|^2.$

Proof. When $s = 2$, both sides of $(\star\star\star)$ are 0, so the equation is satisfied.

We consider

$$Im(g^{(2k+1)}(z,w) - 2i\langle \overline{z}, f^{(2k)}(z,w)\rangle) = 0,$$

with $m = s = 3$.

Then

$$Im(g^{(3)} - 2i\langle \overline{z}, f^{(2)}\rangle) = 0.$$

We rewrite $f^{(2)}(z,w)=a^{(2)}(z)$ and $g^{(3)}(z,w)=c^{(3)}(z)+c^{(1)}(z)w$. Then

$$Im(c^{(3)}(z)+c^{(1)}(z)w-2i\langle\overline{z},a(2)(z)\rangle)=0.$$

By the parameterization $u=i\left|z\right|^2$, we get that $c^{(1)}(z)$ and $a^{(2)}(z)\equiv 0$ and so $c^{(3)}(z)=0$.

Next, we look at $s=m=4$ in the even case:

$$Im(g^{(4)}-2i\langle\overline{z},f^{(3)}\rangle)=\sum_{j=1}^{N-n}\left|\phi_j^{(2)}\right|^2.$$

We claim that $g^{(4)}\equiv 0$, $\phi_j^{(2)}\equiv\phi_j^{(2)}(z)$, $f^{(3)}\equiv a^{(1)}(z)w$, and $-2i\langle a^{(1)}(z),\overline{z}\rangle\left|z\right|^2=\sum_{j=1}^{N-n}\left|\phi_j^{(2)}(z)\right|^2$.

Here, $a^{(1)}(z)$ is a homogeneous polynomial of degree 1.

We write

$$f^{(3)}(z,w)=a^{(1)}(z)w+a^{(3)}(z)$$

and

$$\phi_j^{(2)}(z)=b_j^{(2)}(z)$$

and

$$g^{(4)}(z,w)=c^{(4)}(z)+c^{(2)}(z)w.$$

The last of these comes from the fact that $\frac{\partial^2 g}{\partial w^2}=0$.

Now, we substitute these back into the main equation:

$$Im(c^{(4)}(z)+c^{(2)}(z)w-2i\langle\overline{z},a^{(1)}(z)\rangle\left|z\right|^2-2i\langle\overline{z},a^{(3)}(z)\rangle)=\sum_{j=1}^{N-n}\left|b_j^{(2)}(z)\right|^2.$$

Use the parameterization $w=u+i\left|z\right|^2$, and then look at the terms with a u factor and those without:

$$Im(c^{(4)}(z)+ic^{(2)}(z)\left|z\right|^2+2\langle\overline{z},a^{(1)}(z)\rangle\rangle w-2i\langle\overline{z},a^{(3)}(z)\rangle)=\sum_{j=1}^{N-n}\left|b_j^{(2)}(z)\right|^2$$

and

$$Im(c^{(2)}(z) - 2i\langle \overline{z}, a^{(1)}(z)\rangle)u = 0.$$

The second equation tells us that $c^{(2)}(z) \equiv 0$ and $Im(2i\langle \overline{z}, a^{(1)}(z)\rangle) \equiv 0$, and thus the first tells us that $c^{(4)}(z) \equiv 0$ and $a^{(3)}(z) \equiv 0$, proving our claim. □

The map F_p

The next tool we need to build is a map we will call F_p, since it might not be the case that $F(0) = 0$.

Let $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $M_1 \subset \partial\mathbb{H}_n$ into $M_2 \subset \partial\mathbb{H}_N$.

For any point $p \in M_1$, we have a CR map $F_p = \tau_p^F \circ F \circ \sigma_p^0$, equivalent to F, with $F_p(0) = 0$. Our map F_p maps a small neighborhood of $0 \subset \partial\mathbb{H}_n$ to $0 \in \partial\mathbb{H}_N$.

In particular, σ_p^0 takes $0 \in \partial\mathbb{H}_n$ to p, and then τ_p^F takes $F(p) \in \partial\mathbb{H}_N$ to 0.

Here, σ_p^0 and τ_p^F are automorphisms, and $p = (z_0, w_0)$, and

$$\sigma_p^0(z, w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle),$$

while

$$\tau_p^F(z^*, w^*) = (z^* - \tilde{(f)}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\tilde{f}(z_0, w_0)}\rangle).$$

Normalized properly, we have the following:

$$
\begin{aligned}
&\left(\tilde{f}_p\right)'_{z_\ell}\Big|_0 = L_\ell(\tilde{f})(p) = E_\ell(p) \\
&\left(\tilde{f}_p\right)'_{w}\Big|_0 = T(\tilde{f})(p) = E_w(p) \\
&\lambda(p) = \left|L_j\tilde{f}\right|^2(p), \forall j \in \{1,\dots,n-1\} \\
&(g_p)'_{z_\ell}\Big|_0 = L_\ell g(p) - 2iL_\ell\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t = 0 \\
&(g_p)'_{w}\Big|_0 = Tg(p) - 2iT\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t = \left|L_j\tilde{f}_p(0)\right|^2, 1\le j\le n-1 \\
&\left(\tilde{f}_p\right)''_{z_\ell z_k}\Big|_0 = L_\ell L_k(\tilde{f})(p) \\
&\left(\tilde{f}_p\right)''_{z_\ell w}\Big|_0 = TL_\ell(\tilde{f})(p) \\
&\left(\tilde{f}_p\right)''_{w^2}\Big|_0 = T^2(\tilde{f})(p) \\
&(g_p)''_{z_\ell z_k}\Big|_0 = L_\ell L_k g(p) - 2iL_\ell L_k\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t = 0 \\
&(g_p)''_{wz_\ell}\Big|_0 = L_\ell\left(Tg(p) - 2iT\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right) = 2iL_\ell\tilde{f}(p)\cdot\overline{T\tilde{f}(p)}^t \\
&(g_0)''_{w^2}\Big|_0 = T^2g(p) - 2iT^2\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t - 2iT\tilde{f}(p)\cdot T\overline{\tilde{f}(p)}^t.
\end{aligned}
$$

The fourth of these follows from

$$\frac{L_\ell g}{2i} = \sum_j L_\ell f_j\cdot\overline{f_j}^t + \sum_j L_\ell\phi_j\cdot\overline{\phi_j}^t = L_\ell\tilde{f}\cdot\overline{\tilde{f}}^t.$$

The ninth follows from

$$\frac{L_k L_\ell g}{2i} = L_k(L_\ell\tilde{f})\cdot\overline{\tilde{f}}^t.$$

The last part of the tenth follows from

$$g - \overline{g} = 2i\tilde{f}\cdot\overline{\tilde{f}}^t$$

and

$$TL_\ell g = 2iTL_p\tilde{f}\cdot\overline{\tilde{f}}^t + 2iL_\ell\tilde{f}\cdot T\tilde{f}^t.$$

We can now define $F_p^* = (\tilde{f}_p{}^*, g_p^*)$ by

$$F_p^* = (f_p^*, \phi_p^*, g_p^*) = \left(f_{p,\ell}^*, \phi_{p,k}^*, g_p^*\right)$$

with

$$\begin{aligned}
f_{p,\ell}^* &= \tfrac{1}{\lambda_p}\tilde{f}_p \cdot \overline{E_\ell(p)}^t \\
\phi_{p,k}^* &= \tfrac{1}{\sqrt{\lambda_p}}\tilde{f}_p \cdot \overline{C_k(p)}^t \ , \\
g_p^* &= \tfrac{1}{\lambda_p} g_p
\end{aligned}$$

where $1 \le \ell \le n-1$ and $1 \le k \le N-n$.

We now have F_p^*, which satisfies

$$\begin{aligned}
F_p^*(0) &= 0 \\
\left(f_{p,j}^*\right)'_{z_\ell}\Big|_0 &= \delta_j^\ell \\
\left(\phi_{p,j}^*\right)'_{z_\ell}\Big|_0 &= 0 \\
\left(g_p^*\right)'_{z_\ell}\Big|_0 &= 0 \\
\left(g_p^*\right)'_{w}\Big|_0 &= 1.
\end{aligned}$$

Next, we choose vectors $C_1(p), \ldots, C_{N-n}(p) \in \mathbb{C}^{N-1}$ such that

$$\left\{\frac{E_1(p)^t}{\sqrt{\lambda}}, \ldots, \frac{E_{n-1}(p)^t}{\sqrt{\lambda}}, C_1(p)^t, \ldots, C_{N-n}(p)^t\right\}$$

form an $(N-1)\times(N-1)$ unitary matrix.

Then

$$\begin{aligned}
\left(f_{p,\ell}^*\right)'_{z_k}\Big|_0 &= \tfrac{1}{\lambda(p)} L_k\tilde{f}(p)\cdot\overline{E_\ell(p)}^t = \tfrac{1}{\lambda(p)} L_k(\tilde{f})(p)\cdot\overline{L_\ell(\tilde{f})(p)}^t = \delta_\ell^k \\
\left(f_{p,\ell}^*\right)'_{w}\Big|_0 &= \tfrac{1}{\lambda(p)} E_w(p)\cdot\overline{E_\ell(p)}^t = \tfrac{1}{\lambda(p)} T(\tilde{f})(p)\cdot\overline{L_\ell(\tilde{f})(p)}^t \\
\left(\phi_{p,\ell}^*\right)'_{z_k}\Big|_0 &= \tfrac{1}{\sqrt{\lambda(p)}} L_k\tilde{f}(p)\cdot\overline{C_\ell(p)}^t = 0 \\
\left(\phi_{p,k}^*\right)'_{w}\Big|_0 &= \tfrac{1}{\sqrt{\lambda(p)}} E_w(p)\cdot\overline{C_k(p)}^t = \tfrac{1}{\sqrt{\lambda(p)}} T(\tilde{f})(p)\cdot\overline{C_k(p)}^t \\
\left(g_p^*\right)'_{z_\ell}\Big|_0 &= \tfrac{1}{\lambda(p)}\left(L_\ell g(p) - 2iL_\ell\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right) = 0 \\
\left(g_p^*\right)'_{w}\Big|_0 &= \tfrac{1}{\lambda(p)}\left(Tg(p) - 2iT\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right) = 1.
\end{aligned}$$

We get the fifth of these from

$$\frac{L_\ell g}{2i} = L_\ell \tilde{f} \cdot \overline{\tilde{f}}^t$$

and the sixth from

$$Tg = 2i\langle T\tilde{f}, \overline{\tilde{f}}\rangle + \left|L_j \tilde{f}\right|^2.$$

We also have

$$\begin{aligned}
\left(f^*_{p,j}\right)''_{z_k z_\ell}\Big|_0 &= \tfrac{1}{\lambda(p)} L_k L_\ell \tilde{f}(p) \cdot \overline{L_j \tilde{f}(p)}^t \\
\left(f^*_{p,\ell}\right)''_{z_j w}\Big|_0 &= \tfrac{1}{\lambda(p)} L_j T(\tilde{f})(p) \cdot \overline{L_\ell(\tilde{f})(p)}^t \\
\left(f^*_{p,j}\right)''_{w^2}\Big|_0 &= \tfrac{1}{\lambda(p)} T^2 \tilde{f}(p) \cdot \overline{L_j \tilde{f}(p)}^t
\end{aligned}$$

and

$$\begin{aligned}
\left(\phi^*_{p,j}\right)''_{z_k z_\ell}\Big|_0 &= \tfrac{1}{\lambda(p)} L_k L_\ell \tilde{f}(p) \cdot \overline{C_j(p)}^t \\
\left(\phi^*_{p,j}\right)''_{z_k w}\Big|_0 &= \tfrac{1}{\sqrt{\lambda(p)}} T L_k \tilde{f}(p) \cdot \overline{C_j(p)}^t \\
\left(\phi^*_{p,j}\right)''_{w^2}\Big|_0 &= \tfrac{1}{\sqrt{\lambda(p)}} T^2 \tilde{f}(p) \cdot \overline{C_j(p)}^t
\end{aligned}$$

and

$$\begin{aligned}
\left(g^*_p\right)''_{z_\ell z_k}\Big|_0 &= \tfrac{1}{\lambda(p)} \left(L_\ell L_k g(p) - 2i L_\ell L_k \tilde{f}(p) \cdot \overline{\tilde{f}(p)}^t\right) = 0 \\
\left(g^*_p\right)''_{z_\ell w}\Big|_0 &= \tfrac{1}{\lambda(p)} L_\ell \left(Tg(p) - 2iT\tilde{f}(p) \cdot \overline{\tilde{f}(p)}^t\right) = \tfrac{2i}{\lambda(p)} L_\ell \tilde{f}(p) \cdot \overline{T\tilde{f}(p)}^t \\
\left(g^*_p\right)''_{w^2}\Big|_0 &= \tfrac{1}{\lambda(p)} \left(T^2 g(p) - 2iT^2 \tilde{f}(p) \cdot \overline{\tilde{f}(p)}^t - 2iT\tilde{f}(p) \cdot \overline{T\tilde{f}(p)}^t\right).
\end{aligned}$$

We can now define

$$G_p = \frac{(z^* - a(p)w^*, w^*)}{1 + 2i\langle z^*, \overline{a(p)}\rangle + (r(p) - i\left|a(p)\right|^2)w^*},$$

with

$$a(p) = \left(\tilde{f_p}^*\right)'_w\Big|_0 = (a(p), b(p)) = (a_1(p), \ldots, a_{n-1}(p), b_1(p), \ldots, b_{N-n}(p)) =$$

$$= \left(\ldots, \frac{T\tilde{f}(p) \cdot \overline{L_j \tilde{f}(p)}}{\lambda(p)}, \ldots; \ldots, \frac{T\tilde{f}(p) \cdot \overline{C_j(p)}^t}{\sqrt{\lambda(p)}}, \ldots \right)$$

and

$$r(p) = \frac{1}{2} Re \left(g_p^* \right)''_{w^2} \bigg|_0 = \frac{1}{2\lambda(p)} Re \left(T^2 g(p) - 2iT^2 \tilde{f}(p) \cdot \overline{\tilde{f}(p)}^t \right).$$

Because $A = \left(\frac{E_j}{\sqrt{\lambda}}, C_k \right)$ is unitary,

$$|a(p)|^2 = \frac{1}{\lambda(p)} |E_w(p)| 2 = \frac{1}{\lambda(p)} \left| T\tilde{f}(p) \right|^2.$$

From here, we have the following normalization:

$$\begin{aligned}
F_p^{**} &= (\tilde{f}_p^{\ **}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) = G_p \circ F_p^* \\
f_{p,j}^{**} &= \tfrac{f_{p,j}^* - a_j(p) g_p^*}{1 + 2i \langle \tilde{f}_p^{\ *}, \overline{a(p)} \rangle - (-r(p) + i|a(p)|^2) g_p^*} \\
\phi_{p,j}^{**} &= \tfrac{\phi_{p,j}^* - b_j(p) g_p^*}{1 + 2i \langle \tilde{f}_p^{\ *}, \overline{a(p)} \rangle - (-r(p) + i|a(p)|^2) g_p^*} \\
g_p^{**} &= \tfrac{g_p^*}{1 + 2i \langle \tilde{f}_p^{\ *}, \overline{a(p)} \rangle - (-r(p) + i|a(p)|^2) g_p^*}.
\end{aligned}$$

We use this normalization because each of the following vanish at $(z, w) = 0$:

$$F_p^{**}, \left(f_p^{**} - z \right)'_{z_\ell}, \left(f_p^{**} \right)'_w, \left(\phi_p^{**} \right)'_{z_\ell}, \left(\phi_p^{**} \right)'_w$$

$$(g_p)'_{z_\ell}, \left(g_p^{**} - w \right)'_w, \left(g_p^{**} \right)''_{z_\ell z_k}, \left(g_p^{**} \right)''_{w^2}.$$

By our the normalization and the fact that $\frac{\partial^2 g_p^*}{\partial z_j \partial z_k} = 0$, we have

$$\begin{aligned}
(f_{p,j}^{**})'_{z_\ell} \bigg|_0 &= \delta_j^\ell \\
(f_{p,j}^{**})'_w \bigg|_0 &= (f_{p,j}^*)'_w \bigg|_0 - a_j(p) = 0 \\
(\phi_{p,j}^{**})'_{z_\ell} \bigg|_0 &= 0 \\
(\phi_{p,j}^{**})'_w \bigg|_0 &= (\phi_{p,j}^*)'_w - b_j(p) = 0 \\
(g_p^*)'_{z_\ell} \bigg|_0 &= 0 \\
(g_p^*)'_w \bigg|_0 &= 0
\end{aligned}$$

and

$$\left(f_{p,j}^{**}\right)''_{z_k z_\ell}\Big|_0 = \left(f_{p,j}^{*}\right)''_{z_k z_\ell}\Big|_0 - 2i\delta_j^k\overline{a_\ell(p)} - 2i\delta_j^\ell\overline{a_k(p)}$$

$$= \frac{1}{\lambda(p)}L_k L_\ell \tilde{f}(p)\cdot\overline{L_j\tilde{f}(p)}^t - \frac{2i\delta_j^k}{\lambda(p)}\overline{T\tilde{f}(p)}\cdot L_\ell\tilde{f}(p)^t - \frac{2i\delta_j^\ell}{\lambda(p)}\overline{T\tilde{f}(p)}\cdot L_k\tilde{f}(p)^t = 0.$$

Then,

$$\left(f_{p,\ell}^{**}\right)''_{z_j w}\Big|_0 = \left(f_{p,\ell}^{*}\right)''_{z_j w}\Big|_0 - a_\ell(p)\left(g_p^*\right)''_{z_j w}\Big|_0 - \delta_j^\ell\left(2i(\tilde{f_p}^*)'_w\Big|_0\cdot\overline{a} + (r(p) - i\left|a(p)\right|^2)\right)$$

$$= \left(f_{p,\ell}^{**}\right)''_{z_j w}\Big|_0 - a_\ell(p)\left(g_p^*\right)''_{z_j w}\Big|_0 - \delta_j^\ell\left(i\left|a(p)\right|^2 + r(p)\right)$$

$$= \frac{1}{\lambda(p)}L_j T\tilde{f}(p)\cdot\overline{L_\ell\tilde{f}(p)}^t - \frac{2i}{\lambda(p)^2}\left(T\tilde{f}(p)\cdot\overline{L_\ell\tilde{f}(p)}^t\right)\left(L - j\tilde{f}(p)\cdot\overline{T\tilde{f}(p)}^t\right)$$

$$-\frac{i\delta_j^\ell}{\lambda(p)}\left|T\tilde{f}(p)\right|^2 - \frac{\delta_j^\ell}{2\lambda(p)}Re\left(T^2 g(p) - 2iT^2\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right).$$

Next, we look at what happens when we apply T^2 to both sides of $Im(g) = \left|\tilde{f}\right|^2$:

$$0 = 2iIm(iT^2\tilde{f}\cdot\overline{\tilde{f}}^t) + 2i\left|T\tilde{f}\right|^2 - iIm(T^2 g).$$

Here, we use the fact that

$$Im(T^2 g) = 2Im(iT^2\tilde{f}\cdot\overline{\tilde{f}}^t) + 2\left|T\tilde{f}\right|^2.$$

We combine this with the previous equation, getting

$$\left(f_{p,\ell}^{**}\right)''_{z_j w}\Big|_0 = \frac{1}{\lambda(p)}L_j T\tilde{f}(p)\cdot\overline{L_\ell\tilde{f}(p)}^t - \frac{2i}{\lambda(p)^2}\left(T\tilde{f}(p)\cdot\overline{L_\ell\tilde{f}(p)}^t\right)\left(L_j\tilde{f}(p)\cdot\overline{T\tilde{f}(p)}^t\right)$$

$$-\frac{\delta_j^\ell}{2\lambda(p)}\left(T^2 g(p) - 2iT^2\tilde{f}(p)\cdot\overline{\tilde{f}(p)}\right).$$

$$\left(f^{**}_{p,\ell}\right)''_{w^2}\Big|_0 = \left(f^{*}_{p,\ell}\right)''_{w^2}\Big|_0 - a_\ell(p)\left(g^*_0\right)''_{w^2}\Big|_0$$

$$= \frac{1}{\lambda(p)}T^2\tilde f(p)\cdot\overline{L_\ell\tilde f(p)}^t - \frac{1}{\lambda(p)^2}\left(T\tilde f\cdot\overline{L_\ell\tilde f}^t\right)\left(T^2 g - 2iT^2\tilde f\cdot\overline{\tilde f}^t - 2i\left|T\tilde f\right|^2\right)(p).$$

$$\left(\phi^{**}_{p,\ell}\right)''_{z_jz_k}\Big|_0 = \left(\phi^{*}_{p,\ell}\right)''_{z_jz_k}\Big|_0 - b_\ell(g^*_p)''_{z_jz_k} = \left(\phi^{*}_{p,\ell}\right)''_{z_jz_k}\Big|_0 = \frac{1}{\sqrt{\lambda(p)}}L_jL_k\tilde f(p)\cdot\overline{C_\ell(p)}^t.$$

We use the fact that $(g^{**}_p)''_{z_jz_k}\Big|_0 = 0.$

Next,

$$\begin{aligned}
\left(\phi^{**}_{p,\ell}\right)''_{z_jw}\Big|_0 &= \left(\phi^{*}_{p,\ell}\right)''_{z_jw}\Big|_0 - b_\ell(p)(g^*_p)''_{z_jw}\Big|_0 \\
&= \tfrac{1}{\sqrt{\lambda(p)}}TL_j\tilde f(p)\cdot\overline{C_\ell(p)} \\
&\quad -\tfrac{1}{(\lambda(p))^{(3/2)}}\left(T\tilde f(p)\cdot\overline{C_\ell(p)}^t\right)L_j\left(Tg(p)-2iT\tilde f(p)\cdot\overline{\tilde f(p)}^t\right) \\
&= \tfrac{1}{\sqrt{\lambda(p)}}TL_j\tilde f(p)\cdot\overline{C_\ell(p)}t \\
&\quad -\tfrac{2i}{(\lambda(p))^{(3/2)}}\left(T\tilde f(p)\cdot\overline{C_\ell(p)}^t\right)\left(L_j\tilde f(p)\cdot T\overline{\tilde f(p)}^t\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\phi^{**}_{p,\ell}\right)''_{w^2}\Big|_0 &= \left(\phi^{*}_{p,\ell}\right)''_{w^2}\Big|_0 - b_j(p)\left(g^*_p\right)''_{w^2}\Big|_0 \\
&= \tfrac{1}{\sqrt{\lambda(p)}}T^2\tilde f(p)\cdot\overline{C_\ell(p)}^t - \\
&\tfrac{1}{(\lambda(p))^{(3/2)}}\left(T\tilde f(p)\cdot\overline{C_\ell(p)}^t\right)\left(T^2g(p)-2iT^2\tilde f(p)\cdot\overline{\tilde f(p)}^t - 2i\left|T\tilde f(p)\right|^2\right)
\end{aligned}$$

and

$$\left(g^{**}_p\right)''_{z_jz_k}\Big|_0 = 0$$

and

$$\left(g^{**}_p\right)''_{z_jw}\Big|_0 = \left(g^{*}_p\right)''_{z_jw}\Big|_0 - 2i\overline{a_j(p)} = \frac{2i}{\lambda(p)}L_j\tilde f(p)\cdot\overline{T\tilde f(p)}^t - \frac{2i}{\lambda(p)}\overline{T\tilde f(p)}\cdot\overline{L-j\tilde f(p)}^t = 0$$

and

$$\left(g_p^{**}\right)''_{w^2}\bigg|_0 = \left(g_p^{*}\right)''_{w^2}\bigg|_0 - 2\left(i\left|a_j(p)\right|^2 + r(p)\right)$$

$$= \frac{1}{\lambda(p)}\left(T^2 g(p) - 2iT^2\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right)$$

$$-\frac{2}{\lambda(p)}\left(i\left|T\tilde{f}(p)\right|^2 + \frac{1}{2}Re\left(T^2 g(p) - 2iT^2\tilde{f}(p)\cdot\overline{\tilde{f}(p)}^t\right)\right) = 0.$$

Then, we have another normalization theorem.

2.18 Theorem (Hu99,5.3). *Let $F \in Prop_2(\mathbb{H}_n, \mathbb{H}_N)$, $2 \le n \le N$, with $F(0) = 0$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such that $F_p^{**} = \tau_p^{**} \circ F_p = (f_p^{**}, \phi_p^{**}, g_p^{**})$ satisfies the following normalization:*

$$\begin{aligned}
&f_p^{**} = z + \tfrac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3)\\
&\phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2)\\
&g_p^{**} = w + o_{wt}(4)\\
&\langle \overline{z}, a_p^{**(1)}(z)\rangle \left|z\right|^2 = \left|\phi_p^{**(2)}(z)\right|^2.
\end{aligned}$$

Geometric rank

We are often interested in classifying the maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ according to their geometric rank. Some major theorems can be proved by showing that maps of a particular geometric rank must be of a certain form and then showing that every map in a certain context must have a certain geometric rank. Here, we briefly define the concept and state a few facts about it.

First, we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$, where

$$\mathcal{A}(p) = -2i\left(\frac{\partial^2 f_{p,\ell}^{**}}{\partial z_j \partial w}\bigg|_0\right)_{1\le j,\ell\le n-1}$$

is an $(n-1)\times(n-1)$ matrix. We note that $\mathcal{A}(p)$ is Hermitian. To show this, we write, from our normalization theorem,

$$z\mathcal{A}(p)\overline{z}^t\left|z\right|^2 = \left|\phi^{**(2)}(z)\right|^2 = z\overline{\mathcal{A}(p)}^t\overline{z}^t\left|z\right|^2.$$

Then,

$$z(\mathcal{A}(p)-\overline{\mathcal{A}(p)}^t)\overline{z}^t = 0,$$

and thus $\mathcal{A}(p)=\overline{\mathcal{A}(p)}^t$.

We also have that $\mathcal{A}(p)$ is semi-positive.

We define the $Rk_F(p)$ of F at p to be the rank of the matrix $\mathcal{A}(p)$, and the geometric rank of F is defined as

$$\kappa_0 = \kappa_0(F) = \max_{p\in\partial\mathbb{H}_n}(Rk_F(p)).$$

We note several things.

First, $\kappa_0(F)$ is an invariant.

Second, $0\le\kappa_0(F)\le n-1$, since $\mathcal{A}(p)$ is an $(n-1)\times(n-1)$ matrix.

Third, we have that $\kappa_0(F)=Rk_F(p)$ if and only if F is equivalent to F_p^{**}, where F_p^{**} satisfies

$$\begin{cases} f_{j,p}^{**} = z_j + \frac{i\mu_j(p)}{2}z_jw + o_{wt}(3), 1\le j\le\kappa_0, \mu_j(0)>0 \\ f_{j,p}^{**} = z_j + o_{wt}(3), \kappa_0+1\le j\le n-1 \\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2) \\ g_p^{**} = w + o_{wt}(4). \end{cases}$$

Finally, when $\kappa_0(F)=n-1$, classification problems are harder because of how much space $F(\partial\mathbb{H}_n)$ takes up. When $\kappa_0(F)\le n-2$, F is much simpler.

In the simplest case, κ_0, F turns out to be a linear map. The proof of this appears in the chapter on the First Gap Theorem, as it is in [Hu99].

Minimum maps and the Gap Theorems

Suppose $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$. We say that F is minimum if it is not equivalent to a map $G \in Prop(\mathbb{B}^n, \mathbb{B}^m)$, where $m < N$.

This allows us to reformulate the Gap Theorems in terms of minimum maps.

2.19 Theorem (First Gap Theorem). *There is no minimum map in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ if $N \in \mathcal{I}_1 = \{m \in \mathbb{Z}^+ : n < m < 2n - 1\}$.*

2.20 Theorem (Second Gap Theorem). *There is no minimum map in $Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ for $n \geq 4$ and $N \in \mathcal{I}_2 = \{m \in \mathbb{Z}^+ : 2n < m < 3n - 3\}$.*

2.21 Theorem (Third Gap Theorem). *There is no minimum map in $Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ for $n \geq 7$ and $N \in \mathcal{I}_3 = \{m \in \mathbb{Z}^+ : 3n < m < 4n - 6\}$.*

In general, let

$$K(n) = \max\{t \in \mathbb{Z}^+ : \frac{t(t+1)}{2} < n\}.$$

Then, for $1 \leq k \leq K(n)$,

$$\mathcal{I}_k = \left\{ m \in \mathbb{Z}^+ : kn < m < (k+1)n - \frac{k(k+1)}{2} \right\}.$$

We have a related theorem from [HJY09]:

2.22 Theorem. *For $n > 2$, let $K(n)$ be as above. For each k with $1 \leq k \leq K(n)$, let $\mathcal{I}_k$ be as above. Then, for each $N > n$ with*

$$N \notin \bigcap_{n=1}^{K(n)} \mathcal{I}_k,$$

there exists a minimum monomial map in $Rat(\mathbb{B}^n, \mathbb{B}^N)$.

2.23 Conjecture (Ji10,p64). For $n > 2$, let $K(n)$ be as above. For each k with $1 \leq k \leq K(n)$, let $\mathcal{I}_k$ be as above. Then, for each $N > n$, the following two statements are equivalent.

(i) There are no minimum maps in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$.

(ii) $N \in \mathcal{I}_k$ for some k with $1 \leq k \leq K(n)$.

When $N = n \leq 2$, we have from [A77] that any map in $Prop_2(\mathbb{B}^n, \mathbb{B}^N) = Aut(\mathbb{B}^n)$ is equivalent to the identity map.

According to [Hu99], for any map in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$, $n < N < 2n - 1$, $n \geq 2$, we have that our map is equivalent to a linear map $F(z, w) = (z, 0, w)$.

By [HJ01], for $N - 2n - 1$ and $n \geq 3$, F is equivalent to the linear map $F(z, w) = (z, 0, w)$ or the Whitney Map

$$W_{n,1} = (z', wz),\ where\ z = (z', w) \in \mathbb{C}^{n-1} \times \mathbb{C}.$$

By [Fa82], for $n - 2$ and $N - 2n - 1 = 3$, we have four maps:

$$(z, w, 0)$$
$$(z, zw, w^2)$$
$$(z^2, \sqrt{2}zw, w^2)$$
$$(z^3, \sqrt{3}zw, w^3).$$

By [DA88], for $N = 2n$, we have the D'Angelo Map

$$F_\theta = (z, w\cos\theta, z_1 w \sin\theta, \ldots, z_{n-1}, w\sin\theta, w^2\sin\theta),\ with\ 0 \leq \theta \leq \frac{\pi}{2}.$$

Let

$$W_{n,1}(z; z, \lambda) = (z', \lambda z_n, \sqrt{1 - \lambda^2} z_n h(z)).$$

Here, $z = (z', w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $\lambda \in [0, 1]$, and h is a homomorphic map from $\overline{\mathbb{B}}^n$ to $\mathbb{B}^m$, $m < N$. When $h(z) = z$, then this is the D'Angelo Map

$$W_{n,1}(z; z, \lambda) = (z', \lambda z_n, \sqrt{1 - \lambda} z_n z).$$

When $2n < N < 3n - 3$ and $n \geq 4$, and $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$, we have that F is equivalent to $(W_{n,q}(z; z, \lambda), 0)$.

We also have an example of a Generalized Whitney Map.

Let

$$\begin{cases} \psi_1 = (z_1, \sqrt{2}z_2, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n) \\ \psi_2 = (z_2, \sqrt{2}z_3, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n) \\ \ldots \\ \psi_{k-1} = (z_{k-1}, \sqrt{2}z_k, z_{k+1}, \ldots, z_n) \\ \psi_k = (z_k, z_{k+1}, \ldots, z_n) \\ \psi_{k+1} = (z_{k+1}, \ldots, z_n). \end{cases}$$

Then the Generalized Whitney Map is

$$W_{n,k}(z) = W_{n,k}(z_1, \ldots, z_n) = (z_1\psi_1, \ldots, z_k\psi_k, \psi_{k+1}).$$

This is a map from $\mathbb{B}^n$ to $\mathbb{B}^N$, where $N = (k+1)n - \frac{k(k+1)}{2}$.

2.24 Lemma (HJY09). *Let $F : \mathbb{B}^n \to \mathbb{B}^{n(k-k_0)}$ be a minimum proper polynomial map with $k > k_0 > 0$, with $F(0) = 0$. Then a new map*

$$W_{n,k_0}(z; F, \lambda_1, \ldots, \lambda_\tau) : \mathbb{B}^n \to \mathbb{B}^N,$$

with

$$N = (k+1)n - \frac{k_0(k_0+1)}{2} \text{ and } 0 \le \tau \le k_0 \le n$$

is a proper polynomial minimum map.

We can now prove the previous theorem by constructing a minimum proper monomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$, where either

$$(k+1)n - k(k+1)/2 \le N \le (k+1)n, \text{ where } k \le K(n)$$

or

$$N \ge (K(n)+1)n - K(n)(K(n)+1).$$

We notice that $K(n) \leq \sqrt{2}n$, and we let $k \leq n$.

From [Ji10,3.2 Example C], we have a minimum proper monomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$ in the case where $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n - k(k-1)/2$.

When $k-1 > 0$, we can apply our Lemma 2.12 with $\kappa_0 = k-1$ and $\tau = 0, \ldots, k-1$, and that gives us proper monomial maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $(k+1)n - k(k-1)/2 \leq N \leq (k+1)n - (k-1)(k-2)/2 - 1$.

We apply the same lemma with $\kappa_0 = k-2$ and $\tau = 0, 1, \ldots, k-2$ to get minimum proper monomial maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ when $(k+1)n - (k-1)(k-2)/2 - 1 \leq N \leq (k+1)n - (k-2)(k-3)/2 - 1$.

By induction, we solve the case where $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n$ for $k \leq n$.

For $k = n+1$, we apply the lemma inductively with $\kappa_0 = n, n-1, \ldots,$ and we get the maps we need when $(n+1)n - n(n+1)/2 - 1 \leq N \leq (n+2)n$, and this gives us what we want when $(n+1)n \leq N \leq (n+2)n$. By induction, then, we have any case where $N \geq (n+1)n$.

The degree of a map

We define the degree of a rational map $\frac{(P_1,\ldots,P_N)}{q}$ by

$$\deg(F) = \max\{\deg(P_1), \ldots, \deg(P_N), \deg(q)\}.$$

We shall make brief use of the concept of complex projective space. The space $\mathbb{CP}^n$ is defined as the quotient space $\{\mathbb{C}^{n+1} - 0\}/\sim$, where $[z_0 : z_1 : \cdots : z_n] \sim \lambda[z_0 : z_1 : \cdots : z_n] = [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n]$, $\lambda \in \mathbb{C}, \lambda \neq 0$.

Now, if $F = \frac{(P_1,\ldots,P_N)}{q}$, then F induces a rational map from $\mathbb{CP}^n$ to $\mathbb{CP}^N$:

$$\hat{F}([z_1 : \cdots : z_n : t]) = \left[t^k P\left(\frac{z}{t}\right) : t^k q\left(\frac{z}{t}\right)\right].$$

Here, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and $\deg(F) = k > 0$.

Since $\hat{F}$ might not be holomorphic, we let $Sing(\hat{F})$ be the singular set of F, i.e., the points where $\hat{F}$ is not holomorphic and cannot be extended holomorphically.

We note that $Sing(\hat{F})$ is an algebraic subvariety with codimension at least 2 in $\mathbb{CP}^n$. We let

$$\mathbb{B}_1^n = \{[z_1 : \cdots : z_n : t] \in \mathbb{CP}^n : \sum_{j=1}^{n} |z_j|^2 < |t|^2\}.$$

The following theorem comes from [FHJZ10].

2.25 Theorem. *Let F be a non-constant rational holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$, $N \geq n \geq 1$. Then F is equivalent to a holomorphic polynomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$, namely, there are $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma$ is a holomorphic polynomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$, if and only if there exist complex hyperplanes $H \subset \mathbb{CP}^n$ and $H' \subset \mathbb{CP}^N$ such that $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, $H' \cap \overline{\mathbb{B}_1^N} = \emptyset$, and $\hat{F}(H/Sing(\hat{F})) \subset H'$, $\hat{F}\left(\mathbb{CP}^n/(H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^N/H'$.*

Let F be a non-constant polynomial map. Then $\hat{F} = [t^k F(\frac{z}{t}), t^k]$, where $\deg(F) = k > 0$. Let $H = H_\infty$, and let $H' = H'_\infty$.

If F is equivalent to a polynomial map G, then there exist $\sigma^* \in U(n+1,1)$ and $\tau^* \in U(n+1,1)$ such that $\hat{F} = \tau^* \circ \hat{F} \circ \sigma^*$. For the forward implication in the theorem, let $H = \sigma^{*-1}(H_\infty)$. and $H' = \tau^*(H'_\infty)$.

For the converse, let $\hat{F}$, H, and H' be as stated in the theorem. By a lemma that will follow this proof, there exist $\sigma^* \in U(n+1,1)$ and $\tau^* \in U(n+1,1)$ so that $\sigma^*(H) = H_\infty$ and $\tau^*(H') = H'_\infty$.

Now, let $\hat{Q} = \tau^* \circ \hat{F} \circ \sigma^{*-1}$. Now $\hat{Q}$ gives us a rational holomorphic map Q from $\mathbb{B}^n$ to $\mathbb{B}^N$. If $Q = \frac{P}{q}$ where $(P, q) = 1$ and $\deg(Q) = k > 0$, we have

$$\hat{Q} = [t^k P\left(\frac{z}{t}\right) : t^k q\left(\frac{z}{t}\right)].$$

Suppose that q is not constant. Let $z_0 \in \mathbb{C}^n$ be a root of q where $P(z_0) \neq 0$. Then $[z_0 : 1] \notin Sing(\hat{Q}) \cup H_\infty$, and $\hat{Q}([z_0 : 1]) \in H'_\infty$. Now $\hat{Q}(H_\infty/Sing(\hat{Q})) \subset H'_\infty$, and

$\hat{Q}\left(\mathbb{CP}^n/(H_\infty \cup Sing(\hat{Q}))\right) \subset \mathbb{CP}^N/H'_\infty$, which is a contradiction. Thus, q is identically constant, and so Q is a polynomial.

Now, we state the lemma we just used.

2.26 Lemma (Ji10,3.3.2)**.** *For any hyperplane $H \subset \mathbb{CP}^n$ with $H \cap \overline{B_1^n} = \emptyset$, there is a $\sigma \in U(n+1,1)$ such that $\sigma(H) = H_\infty = \{[z_1 : \cdots : z_n : 0] \in \mathbb{CP}^n\}$.*

Let $H : \sum_{j=1}^n a_j z_j - a_{n+1} t = 0$, where $a = (a_1, \ldots, a_{n+1}) \neq 0$. Assuming $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, it follows that $a_{n+1} \neq 0$, so we can let that coordinate (in projective space) be 1.

Let U be a unitary $n \times n$ matrix so that

$$(a_1, \ldots, a_n)\overline{U} = (\lambda, 0, \ldots, 0)$$

for some complex λ. Now let $\sigma = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$.

We have that $\sigma(H) = \{[z : t] \in \mathbb{CP}^n : \lambda z_1 - t = 0\}$, where $|\lambda| < 1$. Now, define $T \in Aut(\mathbb{B}^n)$:

$$T(z_1, z') = \left(\frac{z_1 - \overline{\lambda}}{1 - \lambda z_1}, \frac{\sqrt{1 - |\lambda|^2} z'}{1 - \lambda z_1} \right).$$

Here, $z' = (z_2, \ldots, z_n)$. Now $\hat{T} \in U(n+1,1)$ is such that

$$\hat{T}([z_1 : z' : t]) = [z_1 - \overline{\lambda} t : \sqrt{1 - |\lambda|^2} z' : t - \lambda z_1].$$

Thus, $\hat{T} \circ \sigma$ maps H to H_∞.

We have another theorem from [FHJZ10].

2.27 Theorem. *A map $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ of degree 2 is equivalent to a polynomial proper holomorphic map in $Poly(\mathbb{B}^2, \mathbb{B}^N)$.*

Also, we have , from [Le11]:

2.28 Theorem. *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $n \geq 3$ and $\deg(F) = 2$. Then F is equivalent to a monomial map.*

Next, we shall discuss the degree of rational maps between balls. This will be important when we prove our main theorem at the end of this dissertation, one of the major objectives during the proof will be to find a bound for the degree of our map.

2.29 Definition. *For any rational map H that is not 0, where $H = \frac{(P_1,\ldots,P_m)}{Q}$, with P_i and Q polynomials whose* $\gcd$ *is 1, we define*

$$\deg(H) = \max(\deg(P_i), \deg(Q)).$$

By convention, when $H \equiv 0$, we say that $\deg(H) = -\infty$.

When we deal with $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ for $n = 2$ and $n = 3$, D'Angelo (see [DA93]) conjectured that

$$\deg(F) \leq \begin{cases} 2N-3, & when\ n = 2 \\ \frac{N-1}{n-1}, & when\ n \geq 3 \end{cases}.$$

D'Angelo also classified all monomial maps $F \in Rat(\mathbb{B}^2, \mathbb{B}^5)$.

$$\begin{cases} degree\ 3: & 31\ isolated\ maps\ or\ continuous\ families \\ degree\ 4: & 47\ isolated\ maps\ or\ continuous\ families \\ degree\ 5: & 24\ isolated\ maps\ or\ continuous\ families \\ degree\ 6: & 5\ isolated\ maps\ or\ continuous\ families \\ degree\ 7: & 3\ isolated\ maps \end{cases}$$

The following lemma will be essential to the proof of our main theorem. Our main trick will be to look at maps on a particular Segre variety, and we will be able to show that all of our maps there have degree less than or equal to a certain k, and from there we can generalize to our entire space.

2.30 Lemma (HJ01,Lemma 5.4)**.** *Let $H = \frac{(P_1,\ldots,P_m)}{R}$ be a rational map from $\mathbb{C}^n$ into $\mathbb{C}^m$ where P_j, and R are holomorphic polynomials whose* $\gcd$ *is 1. Assume that for each*

$p \in \partial\mathbb{H}^n$ *near 0,*

$$\deg\left(H\Big|_{Q_p}\right) \leq k.$$

Here, $k > 0$ *is some fixed integer, and* $Q_{(\zeta,\eta)} = \{(z,w) : \frac{w-\overline{\eta}}{2i} = \sum_{j=1}^{n-1} z_j\overline{\eta_j}\}$ *is the Segre variety of* $\partial\mathbb{H}^n$*. Then* $\deg(H) \leq k$*.*

We also have this theorem.

2.31 Theorem. *Let* $F \in Rat(\mathbb{B}^2, \mathbb{B}^3)$*. Then* $\deg(F) \leq 3$*.*

Proof. By the lemma we just mentioned, we need only look at $F \in Rat(\mathbb{H}^2, \mathbb{H}^3)$ along some Segre variety Q_{p_0} for some $p_0 \in \partial\mathbb{H}^n$. (We have $\mathbb{H}$ instead of $\mathbb{B}$ by Cayley transformation.)

It is equivalent to show that, for every $p \in \partial\mathbb{H}_2$, we have

$$\deg\left(F_p^{**}\Big|_{Q_0}\right) \leq 3.$$

Recall that $Q_0 = \{w = 0\}$, and

$$\begin{aligned}\deg\left(F\Big|_{Q_p}\right) &= \deg\left(F\Big|_{\sigma_p(Q_0)}\right) = \deg\left((F\circ\sigma_0)\Big|_{Q_0}\right)\\ &= \deg\left((\sigma\circ(F_p^{**})\circ\tau)\Big|_{Q_0}\right) = \deg\left((F_p^{**})\Big|_{Q_0}\right).\end{aligned}$$

We write

$$\begin{aligned}&F_p^{**} = (f,\phi,g)\\ &f = z + \textstyle\sum_{j+k\geq 2} a_{jk}z^jw^k\\ &\phi = \textstyle\sum_{j+k\geq 2} b_{jk}z^jw^k\\ &g = w + \textstyle\sum_{j+k\geq 4} c_{jk}z^jw^k.\end{aligned}$$

Our basic equation is

$$\frac{g-\overline{g}}{2} = f\bar{\cdot}f + \phi\cdot\overline{\phi}.$$

Recall the differential operator $L = \frac{\partial}{\partial z} + 2i\overline{z}\frac{\partial}{\partial w}$.

We apply L and L^2 to the basic equation.

$$\begin{cases} \frac{1}{2i}Lg = Lf \cdot \overline{f} + L\phi \cdot \overline{\phi} \\ \frac{1}{2i}L^2 g = L^2 f \cdot \overline{f} + L^2\phi \cdot \overline{\phi}. \end{cases}$$

We rewrite this using matrix notation.

$$\frac{1}{2i}\begin{bmatrix} Lg \\ L^2 g \end{bmatrix} = \begin{bmatrix} Lf & L\phi \\ L^2 f & L^2\phi \end{bmatrix}\begin{bmatrix} \overline{f} \\ \overline{\phi} \end{bmatrix}.$$

The complexified version of L is $\mathcal{L} = \frac{\partial}{\partial z} + 2i\zeta\frac{\partial}{\partial w}$, and $\mathcal{H}^2 = \{(z, w, \zeta, \eta) \in \mathbb{C}^4 : \frac{w-\eta}{2i} = z\zeta\}$. ($\partial\mathcal{H}^2$ is the Segre family of $\partial\mathbb{H}^2$.)

We now complexify this, getting

$$\frac{1}{2i}\begin{bmatrix} \mathcal{L}g(z,w) \\ \mathcal{L}^2 g(z,w) \end{bmatrix} = \begin{bmatrix} \mathcal{L}f(z,w) & \mathcal{L}\phi(z,w) \\ \mathcal{L}^2 f(z,w) & \mathcal{L}^2\phi(z,w) \end{bmatrix}\begin{bmatrix} \overline{f}(\zeta,\eta) \\ \overline{\phi}(\zeta,\eta) \end{bmatrix}.$$

This will hold for any $(z, w, \zeta, \eta) \in \partial\mathcal{H}^2$.

We see that the point $(0, 0, \zeta, 0) \in \partial\mathcal{H}^2$, so

$$\begin{cases} \mathcal{L}f\Big|_{(0,0,\zeta,0)} = 1 \\ \mathcal{L}\phi\Big|_{(0,0,\zeta,0)} = 0 \\ \mathcal{L}g\Big|_{(0,0,\zeta,0)} = 2i\zeta \\ \mathcal{L}^2 f\Big|_{(0,0,\zeta,0)} = -8a_{02}\zeta^2 + 4ia_{11}\zeta \\ \mathcal{L}^2\phi\Big|_{(0,0,\zeta,0)} = -8b_{02}\zeta^2 + 4ib_{11}\zeta + 2b_{20} \\ \mathcal{L}^2 g\Big|_{(0,0,\zeta,0)} = 0. \end{cases}$$

Now, we see

$$\det\begin{bmatrix} \mathcal{L}f & \mathcal{L}\phi \\ \mathcal{L}^2 f & \mathcal{L}^2\phi \end{bmatrix}\Bigg|_{(0,0,0,0)} = \det\begin{bmatrix} 1 & 0 \\ 0 & 2b_{20} \end{bmatrix} = 2b_{20} \neq 0.$$

It follows that

$$\begin{bmatrix} \overline{f}(\zeta,0) \\ \overline{\phi}(\zeta,0) \end{bmatrix} = \frac{1}{2i} \begin{bmatrix} \mathcal{L}f & \mathcal{L}\phi \\ \mathcal{L}^2 f & \mathcal{L}^2\phi \end{bmatrix}^{-1} \Bigg|_{(0,0,\zeta,0)} \cdot \begin{bmatrix} \mathcal{L}g \\ \mathcal{L}^2 g \end{bmatrix} \Bigg|_{(0,0,\zeta,0)}$$

$$= \frac{1}{2i} \begin{bmatrix} 2i\zeta \\ \frac{2i\zeta(8a_{02}\zeta^2 - 4ia_{11}\zeta)}{-8b_{02}\zeta^2+4ib_{11}\zeta+2b_{20}} \end{bmatrix} = \begin{bmatrix} \zeta \\ \frac{\zeta(8a_{02}\zeta^2 - 4ia_{11}\zeta)}{-8b_{02}\zeta^2+4ib_{11}\zeta+2b_{20}} \end{bmatrix}.$$

Thus,

$$f(z,0) = z$$

and

$$\phi(z,0) = \frac{4\overline{a_{02}}z^3 + 2i\overline{a_1 1}z^2}{-4\overline{b_{02}}z^2 - 2i\overline{b_{11}}z + \overline{b_{20}}}.$$

Evaluating

$$\frac{g(z,w) - \overline{g}(\zeta,\eta)}{2i} = f(z,w)\overline{f}(\zeta,\eta) + \phi(z,w)\overline{\phi}(\zeta,\eta)$$

at $(0,0,\zeta,0)$, we get $g(z,0) = 0$. This proves that $\deg\left(F_p^{**}\big|_{Q_0}\right) \leq 3$, which proves our theorem. □

We use a similar method for the next theorem.

2.32 Theorem (HJ01, lemma 5.2)**.** *Let* $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^{2n-1})$*,* $n \geq 3$*. Then* F *is rational, and* $\deg(F) \leq 2$*.*

Proof. Using a Cayley transformation, we consider $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^{2n-1})$. By our lemma, we need only consider that $\deg(F_{Q_{p_0}}) \leq 2$ for any $p_0 \in \partial\mathbb{H}^n$.

This is equivalent to showing that for any $p \in \partial\mathbb{H}_n$, we have

$$\deg\left(F_p^{***}\Big|_{Q_0}\right) \leq 2,$$

where $Q_0 = \{w = 0\}$.

We use a normalization $F_0^{***} = (f, \phi, g)$ satisfying

$$\begin{cases} F_p^{***}(0,w) = (0,w) \\ f_1 = z_1 + \frac{i}{2} z_1 w + z_1 \tilde{a^{(1)}}(z) w + o_{wt}(4) \\ f_\ell = z_\ell + o_{wt}(4), 2 \leq \ell \leq n-1 \\ \phi_j = z_1 z_j + b_j z_1 w + b_j^{(3)}(z) + o_{wt}(3), 1 \leq j \leq n-1 \\ g = w + o(|(z,w)|^3). \end{cases}.$$

Now, our basic equation, as usual, is

$$\frac{g(z,w) - \overline{g(\zeta,\eta)}}{2i} = \sum_{\ell=1}^{n-1} f_\ell(z,w)\overline{f_\ell(\zeta,\eta)} + \sum_{\ell=1}^{n-1} \phi_\ell(z,w)\overline{\phi(\zeta,\eta)}.$$

We apply our differential operators $\mathcal{L}_j$ and $\mathcal{L}_1\mathcal{L}_j$, and we work on the Segre variety $z = w = \eta = 0$:

$$\begin{pmatrix} \overline{\zeta_1} \\ \dots \\ \overline{\zeta_{n-1}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ A_{(n-1)\times(n-1)} & B_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} \overline{f(,\zeta,0)} \\ \overline{\phi(\zeta,0)} \end{pmatrix}.$$

We write $I_{(n-1)\times(n-1)}$ for the $(n-1)$-dimensional identity matrix.

Now,

$$A_{(n-1)\times(n-1)} = A = \begin{pmatrix} -2\overline{\zeta_1} & 0 & \dots & 0 \\ -\overline{\zeta_2} & 0 & \dots & 0 \\ \dots & 0 & \dots & 0 \\ -\overline{\zeta_{n-1}} & 0 & \dots & 0 \end{pmatrix}$$

and

$$B_{(n-1)\times(n-1)} = B = \begin{pmatrix} 2 + 4ib_1\overline{\zeta_1} & 4ib_2\overline{\zeta_1} & \dots & 4ib_{n-1}\overline{\zeta_1} \\ 2ib_1\overline{\zeta_2} & 1 + 2ib_2\overline{\zeta_2} & \dots & 2ib_{n-1}\overline{\zeta_2} \\ \dots & \dots & \dots & \dots \\ 2ib_1\overline{\zeta_{n-1}} & 2ib_2\overline{\zeta_{n-1}} & \dots & 1 + 2ib_{n-1}\overline{\zeta_{n-1}} \end{pmatrix}.$$

After a detailed calculation similar to one we will do later, this yields

$$\tilde{f}(z,0) = \left(z, \frac{z_1 z}{1 - 2i\sum_{j\geq 1}\overline{b_j}z_j}\right).$$

Then, by applying $z = w = \eta = 0$, we get $\overline{g}(\zeta, 0) = 0$, and so we have that $F(z,0)$ can be written as the quotient of a degree 2 polynomial and a linear function, and thus its degree is less than or equal to 2. □

Other proofs using similar methods include:

2.33 Theorem (JX04)**.** *Let* $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ *with geometric rank* $\kappa_0, 1 \leq \kappa_0 \leq n-2$ *and* $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. *Then* $\deg(F) \leq \kappa_0 + 2$.

2.34 Theorem (HJX05)**.** *Let* $F \in Rat(\mathbb{B}^3, \mathbb{B}^6)$ *with geometric rank* $\kappa_0(F) = 2$. *Then* $\deg(F) \leq 4$.

In [HJ01], we have the following theorem:

2.35 Theorem. *Let* $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^{2n-1})$. *Then* F *is equivalent to a linear map or the Whitney Map* $W_{n,1}(z,w) = (z, w(z,w))$, *where* $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

The full proof is found elsewhere in this dissertation, but there are five key components to it.

First, regarding $F^{***} = (f, \phi, g)$,

$$f_1 = z_1 + \tfrac{i}{2}z_1 w + o_{wt}(3).$$

$$f_j = z_j + o_{wt}(3), 2 \leq j \leq n-1.$$

$$\phi_j = z_1 z_j + o_{wt}(2), 2 \leq j \leq n-1.$$

$$g = w + o_{wt}(4).$$

Second, $\kappa_0 = 1$.

Third,

$$f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3).$$
$$f_j = z_j, 2 \le j \le n-1.$$
$$\phi_j = z_1 z_j + o_{wt}(2), 2 \le j \le n-1.$$
$$g = w.$$

Fourth, F is equivalent to a map satisfying

$$F = (z_1 \tilde{f}_1, z_1, \ldots, z_{n-1}, z_1 \tilde{\phi}_1, \ldots, z_1 \tilde{\phi}_{n-1}, w),$$

where $\Phi = (\tilde{f}_1, \tilde{\phi}_1, \ldots, \tilde{\phi}_{n-1})$ is a biholomorphic map from $\mathbb{H}^n$ to $\mathbb{B}^n$.

Fifth, $F\Big|_{z_1=0}$ is linear fractional.

These five points can be generalized into useful theorems.

First:

2.36 Theorem (Hu03). *Let $F \in Prop_3(\mathbb{H}^n, \mathbb{H}^N)$. Then F is equivalent to a map $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$ such that*

$$\begin{cases} f_{\ell,p}^{***} = \sum_{j=1}^{\kappa_0} z_j f_{\ell j}^*(z,w), f_{\ell j}^*(z,w) = \delta_\ell^j + \frac{i\delta_\ell^j \mu_\ell}{2} w + \mathcal{O}(|(z,w)|^2), \ell \le \kappa_0, \\ f_{j,p}^{***} = z_j + o_{wt}(3), \kappa_0 + 1 \le j \le n-1, \\ \phi_{\ell k,p}^{***} = \mu_{\ell k} z_\ell z_k + o_{wt}(2), (\ell,k) \in \mathcal{S} \\ g = w + o_{wt}(4). \end{cases}$$

Here, $\mathcal{S}_0 = \{(j,\ell) : 1 \le j \le \kappa_0, j \le \ell, 1 \le \ell \le n-1\}$. This is a set of indices for $\phi_{\ell k,p}$ whose $z_\ell z_k$ terms have non-zero coefficients.

Also,

$$\mathcal{S} = \mathcal{S}_0 \cup \left\{(j,\ell) : j = \kappa_0 + 1, \kappa_0 + 1 \le \ell \le N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\right\}$$

indexes all of $\phi_{\ell k,p}$.

Also,

$$\mu_{j\ell} = \begin{cases} \sqrt{\mu_j + \mu_\ell}, & j,\ell \le \kappa_0, j \ne \ell, \\ \sqrt{\mu_j}, & j \le \kappa_0, \ell > \kappa_0, \ or j = \ell \le \kappa_0. \end{cases}$$

Second:

2.37 Corollary (H03). *Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank κ_0. Then*

$$N \geq n + \frac{\kappa_0(2n - \kappa_0 - 1)}{2}.$$

Third:

2.38 Theorem (HJX06,Theorem 3.1). *Let $F \in Prop_3(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a map $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$ as follows:*

$$\begin{cases} f_{\ell,p}^{***} = \sum_{j=1}^{\kappa_0} z_j f_{\ell j}^*(z,w), f_{\ell j}^*(z,w) = \delta_\ell^j + \frac{i\delta_\ell^j \mu_\ell}{2} w + \mathcal{O}(|(z,w)|^2), \ell \leq \kappa_0 \\ f_{j,p}^{***} = z_j, \kappa_0 + 1 \leq j \leq n-1 \\ \phi_{\ell k,p}^{***} = \mu_{\ell k} z_\ell z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{\ell k j,p}^*(z,w) = o_{wt}(2), (\ell,k) \in \mathcal{S}_0 \\ \phi_{\ell j,p}^{***} = \sum_{j=1}^{\kappa_0} z_j \phi_{\ell k j,p}^* = \mathcal{O}(|(z,w)|^3), (\ell,k) \notin \mathcal{S}_0 \\ g_p^{***} w \end{cases}$$

Fourth:

2.39 Theorem (HJX06). *Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $3 \leq n \leq N$ and geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a proper holomorphic map of the form*

$$H = (z_1, \ldots, z_{n-\kappa_0}, H_1, \ldots, H_{N-n+\kappa_0}),$$

with $H_j = \sum_{\ell=n-\kappa_0+1}^{n} z_\ell H_{j,\ell}$ and $H_{j,\ell}$ holomorphic on $\overline{\mathbb{B}^n}$. When $\kappa_0 = 1$, then F is equivalent to a map (z, wh), where $h \in Rat(\mathbb{B}^n, B^{N-n+1})$.

Fifth:

2.40 Theorem (H03). *Let $F \in Prop_3(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank $\kappa_0 \leq n-2$. For any $p \in \mathbb{B}^n$, there exists an affine $(n-\kappa_0)$-dimensional complex subspace S_p^a containing p such that $F|_{S_p^a}$ is linear fractional.*

Constructing F^{***}

Finally, we have almost all of the tools we will need to look at the more complicated proofs. Our final task in this chapter is to construct our normalization F^{***}.

First, recall that we already have the normalization $F^{**} = (f^{**}, \phi^{**}, g^{**})$, with

$$f^{**} = z + \frac{i}{2}a^{**(1)}(z)w + o_{wt}(3)$$
$$\phi^{**} = \phi^{**(2)}(z) + o_{wt}(2)$$
$$g^{**} = w + o_{wt}(4).$$

Also,

$$\langle \overline{z}, a^{**(1)}(z)\rangle \left|z\right|^2 = \left|\phi^{**}(z)\right|^2.$$

Now, we can look at $\sigma \in Aut_0(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that

$$\sigma = \frac{(\lambda(z+aw)\cdot U, \lambda^2 w)}{1 - 2i\langle \overline{a}, z\rangle + (r - i\left|a\right|^2)w}$$

with $\lambda > 0, r \in \mathbb{R}$, and a an $(n-1)$-tuple and U an $(n-1)\times(n-1)$ unitary matrix. Now, let

$$\tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*)\cdot U^*, \lambda^{*2} w^*)}{1 - 2i\langle \overline{a}^*, z^*\rangle + (r^* - i\left|a^*\right|^2)w^*}$$

with $\lambda^* > 0$, $R^* \in \mathbb{R}$, a^* an $(N-1)$-tuple, and U^* an $(N-1)\times(N-1)$ unitary matrix.

We now have a theorem.

2.41 Theorem (H03)**.** *(A) Let $F = (f, \phi, g)$ and $F^* = (f^*, \phi^*, g^*)$ be C^2-smooth CR maps from a neighborhood of 0 in $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ $(N \geq n \geq 1)$, where these satisfy our F^{**} normalization. Suppose $F^* = \tau^* \circ F \circ \sigma$, with σ and τ^* as we just defined them. Then*

$$\lambda^* = \lambda^{-1}, a_1^* = -\lambda^{01} a \cdot U, a_2^* = 0, r^* = -\lambda^{-2} r, U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}.$$

Here $a^* = (a_1^*, a_2^*)$ with a_1^* the first $(n-1)$ components, and U_{22}^* is an $(N-n)\times(N-n)$ unitary matrix.

Conversely, if τ^* and σ are as given, then suppose F satisfies the F^{**} normalization. Then $F^* = \tau^* \circ \sigma$ also satisfies it.

(B) Let F and $F^* = \tau^* \circ F \circ \sigma$ satisfy the F^{**} normalization. Write

$$f(z,w) = z + \frac{i}{2} z\mathcal{A}w + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\bigg|_0 w^2 + o(|(z,w)|^2)$$

and

$$f^*(z,w) = z + \frac{i}{2} z\mathcal{A}^*w + \frac{1}{2}\frac{\partial^2 f^*}{\partial w^2}\bigg|_0 w^2 + o(|(z,w)|^2)$$

and

$$\phi(z,w) = \frac{1}{2} z(B^1, \ldots, B^{N-n})z^t + z\mathcal{B}w + \frac{1}{2}\frac{\partial^2 \phi}{\partial w^2}\bigg|_0 w^2 + o(|(z,w)|^2)$$

and

$$\phi^*(z,w) = \frac{1}{2} z(B^{*1}, \ldots, B^{*N-n})z^t + z\mathcal{B}^* + \frac{1}{2}\frac{\partial^2 \phi^*}{\partial w^2}\bigg|_0 w^2 + o(|(z,w)|^2).$$

Here,

$$\mathcal{A} = -2i \begin{pmatrix} \frac{\partial^2 f_1}{\partial z_1 \partial w} & \cdots & \frac{\partial^2 f_{n-1}}{\partial z_1 \partial w} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 f_1}{\partial z_{n-1} \partial w} & \cdots & \frac{\partial^2 f_{n-1}}{\partial z_{n-1} \partial w} \end{pmatrix}\Bigg|_0$$

and

$$B^k = \begin{pmatrix} \frac{\partial^2 \phi_{(k)}}{\partial z_1^2} & \frac{\partial^2 \phi_{(k)}}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_1 \partial z_{n-1}} \\ \frac{\partial^2 \phi_{(k)}}{\partial z_2 \partial z_1} & \frac{\partial^2 \phi_{(k)}}{\partial z_2^2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_2 \partial z_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1} \partial z_1} & \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1} \partial z_2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1}^2} \end{pmatrix}\Bigg|_0, 1 \le k \le N-n.$$

Also,

$$\mathcal{B} = \left(\begin{array}{ccc} \frac{\partial^2 \phi_{(1)}}{\partial z_1 \partial w} & \dots & \frac{\partial \phi_{(N-n)}}{\partial z_1 \partial w} \\ \dots & \dots & \dots \\ \frac{\partial^2 \phi_{(1)}}{\partial z_{n-1} \partial w} & \dots & \frac{\partial^2 \phi_{(N-n)}}{\partial z_{n-1} \partial w} \end{array} \right) \Bigg|_0 .$$

We define $\mathcal{A}^*, B^{*k}$, and $\mathcal{B}$ in the same way.

Now,

$$\mathcal{A}^* = \lambda^2 U \mathcal{A} U^{-1}$$

and

$$\frac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 a U \mathcal{A} U^{-1} + \lambda^2 \frac{\partial^2 f}{\partial w^2}(0) U^{-1}$$

and

$$z(B^{*1}, \dots, B^{*N-n}) z^t = \lambda z U(B^1, \dots, B^{N-n}) U^t z^t U_{22}^*$$

and

$$\mathcal{B}^* = \lambda U(B^1, \dots, B^{N-n}) U^t a^t U_{22}^* + \lambda^2 U \mathcal{B} U_{22}$$

and

$$\frac{1}{2} \frac{\partial^2 \phi^*}{\partial w^2}\Bigg|_0 = \frac{1}{2} \lambda a U(B^1, \dots, U^{N-n}) U^t a^t U_{22}^* + \lambda^2 a U \mathcal{B} U_{22}^* + \frac{1}{2} \lambda^3 \frac{\partial^2 \phi}{\partial w^2}\Bigg|_0 U_{22}^*.$$

(C) Let F_1 be a non-constant C^2 CR map from $M \subset \partial \mathbb{H}_n$ into $\partial \mathbb{H}_N$. Assume $F_2 = \tau \circ F_1 \circ \sigma$ with $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$. Then

$$Rk_{F_2}(p) = Rk_{F_1(\sigma(p))}.$$

With this, we can construct F^{***} by choosing an appropriate σ and τ.

A conjecture and some maps

Finally, we present a conjecture and a theorem related to our main result, though we preface these with some other results. Forstneric showed in [Fo89] that if $F \in Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N)$, then F must be rational, and $\deg(F) \leq N^2(N-n+1)$. A related result is the following.

2.42 Theorem (HJX05, Corollary 1.3). *If $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$, with either $\kappa_0 \leq n-1$ or $N \leq \frac{n(n+1)}{2}$, then F must be rational.*

2.43 Conjecture (D'Angelo). If $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$, with $n > 2$, then $\deg(F) \leq \frac{N-1}{n-1}$.

This conjecture is relevant to the current work because our method for proving our main theorem (classifying maps from $\mathbb{B}^n$ to $\mathbb{B}^{3n-3}$) will involve proving that the degree of those maps is less than or equal to 2. In fact, our result follows almost directly from this.

D'Angelo's conjecture, if true, gives us immediately that

$$\deg(F) \leq \frac{(3n-3)-1}{n-1} < 3.$$

Thus, our result would follow from D'Angelo's conjecture.

More importantly, our result, along with some others obtained elsewhere, provides evidence that the conjecture is true. However, it remains an open question.

The theorem related to our main theorem is the following.

2.44 Theorem (Le11, Theorem 1.5). *Let $f : \mathbb{S}^{2n-1} \to \mathbb{S}^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map. In particular, f is spherically equivalent to a map taking $(z_1, \ldots, z_n)$ to*

$$\Big(\sqrt{t_1}z_1, \sqrt{t_2}z_2, \ldots, \sqrt{t_n}z_n, \sqrt{1-t_1}z_1^2, \sqrt{1-t_2}z_2^2, \ldots, \sqrt{1-t_n}z_n^2,$$

$$\sqrt{2-t_1-t_2}z_1z_2, \sqrt{2-t_1-t_3}z_1z_3, \ldots, \sqrt{2-t_{n-1}-t_n}z_{n-1}z_n\Big),$$

with $0 \leq t_1 \leq \cdots \leq t_n \leq 1$*,* $(t_1, t_2, \ldots, t_n) \neq (1, 1, \ldots, 1)$*. Furthermore, all such maps are mutually spherically inequivalent for different parameters* $(t_1, \ldots, t_n)$*.*

The statement of this theorem in Lebl's paper has a 1 where there should be a 2. Also note that the dimension listed here is the real dimension of a CR manifold, which is why we have $2n-1$ (it must be odd).

Now, we can count the components in Lebl's map. There are n components $\sqrt{t_i}z_i$. There are n components $\sqrt{1-t_i}z_i^2$. There are $n-1$ components $\sqrt{2-t_1-t_i}z_1z_i$. There are $n-2$ components $\sqrt{2-t_2-t_i}z_2z_i$, and so on. Thus, we have

$$n + \sum_{j=1}^{n} j = 2n + \frac{n(n-1)}{2}.$$

Thus, $N \geq 2n + \frac{n(n-1)}{2}$ unless some components of f are mapped to 0.

We can consider several cases now.

First, if $n = 4$, we get that $2n + \frac{n(n-1)}{2} = 8 + \frac{12}{2} = 14$.

Now, if $t_1 = t_2 = t_3 = t_4 = 0$, our map is

$$(z_1^2, z_2^2, z_3^2, z_4^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, \sqrt{2}z_1z_4, \sqrt{2}z_2z_3, \sqrt{2}z_2z_4, \sqrt{2}z_3z_4).$$

Here, $f \in Rat(\mathbb{B}^4, \mathbb{B}^{10})$.

If $t_1 = t_2 = t_3 = 0$ and $t_4 = 1$, then we get

$$(z_4, z_1^2, z_2^2, z_3^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, z_1z_4, \sqrt{2}z_2z_3, z_2z_4, z_3z_4),$$

and then $f \in Rat(\mathbb{B}^4, \mathbb{B}^{10})$.

If $t_1 = t_2 = 0$ and $t_3 = t_4 = 1$, then we get

$$(z_3, z_4, z_1^2, z_2^2, \sqrt{2}z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4).$$

Here, $f \in Rat(\mathbb{B}^4, \mathbb{B}^9)$. Note that $N = 3n - 3$. This case was covered in [JX04].

If $t_1 = 0$ and $t_2 = t_3 = t_4 = 1$, then we get

$$(z_2, z_3, z_4, z_1^2, z_1z_2, z_1z_3, z_1z_4).$$

Here, $f \in Rat(\mathbb{B}^4, \mathbb{B}^7)$.

If $t_1 = t_2 = t_3 = t_4 = 1$, then we get

$$(z_1, z_2, z_3, z_4),$$

and $f \in Rat(\mathbb{B}^4, \mathbb{B}^4)$.

We now consider the general case for n.

If $t_1 = t_2 = \cdots = t_n = 1$, then we get

$$(z_1, z_2, \ldots, z_n),$$

and $f \in Rat(\mathbb{B}^n, \mathbb{B}^n)$.

If $t_1 = 0$ and $t_2 = t_3 = \cdots = t_n$, then we get

$$(z_2, z_3, \ldots, z_n, z_1^2, z_1 z_2, \ldots, z_1 z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-1})$. This map is the Whitney Map, and $2n-1$ is notable as the boundary point for the First Gap Theorem, as describe d in great detail in the next chapter.

If $0 < t_1 < 1$ and $t_2 = t_3 = \cdots = t_n = 1$, then we get

$$(\sqrt{t_1} z_1, z_2, z_3, \ldots, z_n, \sqrt{1-t_1} z_1^2, \sqrt{1-t_1} z_1 z_2, \ldots, \sqrt{1-t_1} z_1 z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$. This is the D'Angelo Map, and $2n$ is the lower boundary term of the second gap, also described in great detail later.

If $t_1 = t_2 = 0$ and $t_3 = t_4 = \cdots = t_n = 1$, then we get

$$(z_3, z_4, z_1^2, z_2^2, \sqrt{2} z_2 z_2, z_1 z_3, \ldots, z_1 z_n, z_3 z_3, z_2 z_4, \ldots, z_2 z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$. This case is our primary focus, as it is the upper boundary case of the second gap.

If $t_1 = 0$, $0 < t_2 < 1$, and $t_3 = t_4 = \cdots = t_n = 1$, then we get

$$(\sqrt{t_2} z_2, z_3, z_4, \ldots, z_n, z_1^2, \sqrt{t_2} z_2^2, \sqrt{2-t_2} z_1 z_2, z_1 z_3, \ldots,$$

$$z_1z_n, \sqrt{1-t_2}z_2z_3, \sqrt{1-t_2}z_2z_4, \ldots, \sqrt{1-t_2}z_2z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-2})$. This case falls between the second gap and the third gap.

If $0 < t_1 \leq t_2 < 1$ and $t_3 = t_4 = \cdots = t_n = 1$, then we get

$$(\sqrt{t_1}z_1, \sqrt{t_2}z_2, z_3, z_4, \ldots, z_n, \sqrt{t_1}z_1^2, \sqrt{t_2}z_2^2, \sqrt{2-t_1-t_2}z_1z_2, \sqrt{1-t_1}z_1z_3,$$

$$\ldots, \sqrt{1-t_1}z_1z_n, \sqrt{1-t_2}z_2z_3, \ldots, \sqrt{1-t_2}z_2z_n),$$

with $f \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-1})$.

One might expect otherwise, but there is no minimal map of degree 2 in $Rat(\mathbb{B}^n, \mathbb{B}^{3n})$. As demonstrated in the paper on the Third Gap Theorem and in a later chapter, we only have one of degree 3 here.

If $t_1 = t_2 = t_3 = 0$ and $t_4 = t_5 = \cdots = t_n = 1$, then we get

$$(z_4, z_5, \ldots, z_n, z_1^2, z_2^2, z_3^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, z_1z_4,$$

$$\ldots, z_1z_n, \sqrt{2}z_2z_3, z_2z_4, \ldots, z_2z_n, z_3z_4, \ldots, z_3z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{4n-6})$, which is the upper boundary point of the third gap.

If $t_1 = t_2 = 0, 0 < t_3 < 1$, and $t_4 = t_5 = \cdots = t_n = 1$, then we get

$$(\sqrt{t_3}z_3, z_4, z_5, \ldots, z_n, z_1^2, z_2^2, \sqrt{1-t_3}z_3^2, \sqrt{2}z_1z_2, \sqrt{2-t_3}z_1z_3, z_1z_4,$$

$$\ldots, z_1z_n, \sqrt{2-t_3}z_2z_3, z_2z_4, \ldots, z_2z_n, \sqrt{1-t_3}z_3z_4, \ldots, \sqrt{1-t_3}z_3z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{4n-5})$, with $4n-5$ between the third and fourth gaps.

If $t_1 = 0, 0 < t_2 \leq t_3 < 1$, and $t_4 = t_5 = \cdots = t_n$, then we get

$$(\sqrt{t_2}z_2, \sqrt{t_3}z_3, z_4, \ldots, z_n, z_1^2, \sqrt{1-t_2}z_2^2, \sqrt{1-t_3}z_3^2, \sqrt{2-t_2}z_1z_2, \sqrt{2-t_3}z_1z_3, z_1z_4,$$

$$\ldots, z_1z_n, \sqrt{2-t_2-t_3}z_2z_3, \sqrt{1-t_2}z_2z_4, \ldots,$$

$$\sqrt{1-t_2}z_2z_n, \sqrt{1-t_3}z_3z_4, \ldots, \sqrt{1-t_3}z_3z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{4n-4})$, with $4n-4$ between the third and fourth gaps.

If $0 < t_1 \leq t_2 \leq t_3 < 1$ and $t_4 = t_5 = \cdots = t_n = 1$, then we get

$$(\sqrt{t_1}z_2, \sqrt{t_2}z_2, \sqrt{t_3}z_3z_4, z_5, \ldots, z_n\sqrt{1-t_1}z_1^2, \sqrt{1-t_2}z_2^2, \sqrt{1-t_3}z_3^2,$$

$$\sqrt{2-t_1-t_2}z_1z_2, \sqrt{2-t_1-t_3}z_1z_3, \sqrt{1-t_1}z_1z_4, \ldots,$$

$$\sqrt{1-t_1}z_1z_n, \sqrt{2-t_2-t_3}z_2z_3, \sqrt{1-t_2}z_2z_4, \ldots,$$

$$\sqrt{1-t_2}z_2z_n, \sqrt{1-t_3}z_3z_4, \ldots, \sqrt{1-t_3}z_3z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{4n-3})$, with $4n-3$ between the third and fourth gaps.

If $t_1 = t_2 = t_3 = t_4 = 0$ and $t_5 = \cdots = t_n = 1$, then we get

$$(z_5, z_6, \ldots, z_n, z_1^2, z_2^2, z_3^2, z_4^2, \sqrt{2}z_1z_2, \sqrt{2}z_1z_3, \sqrt{2}z_1z_4, z_1z_5,$$

$$\ldots, z_1z_n, \sqrt{2}z_2z_3, \sqrt{2}z_2z_4, z_2z_5, \ldots, z_2z_n, \sqrt{2}z_3z_4, z_3z_5, \ldots, z_3z_n, z_4z_5, \ldots, z_4z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{4n-6})$, with $4n-6$ the boundary of the third gap.

If $t_1 = t_2 = \cdots = t_{n-1} = 0$ and $t_n = 1$, then we get

$$(z_n, z_1^2, z_2^2, \ldots, z_{n-1}^2, \sqrt{2}z_1z_2, \ldots, \sqrt{2}z_1z_{n-1}, z_1z_n,$$

$$\sqrt{2}z_2z_3, \ldots, \sqrt{2}z_2z_{n-1}, z_2z_n, \ldots, z_{n-1}z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{n+\frac{n(n+1)}{2}})$.

If $t_1 = t_2 = \cdots = t_n = 0$, then we get

$$(z_1^2, z_2^2, \ldots, z_n^2, \sqrt{2}z_1z_2, \ldots, \sqrt{2}z_1z_n, \ldots, \sqrt{2}z_{n-1}z_n),$$

where $f \in Rat(\mathbb{B}^n, \mathbb{B}^{n+\frac{n(n+1)}{2}})$.

By exhausting all of these cases, we conclude that there is only one map in $Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})/Rat(\mathbb{B}^n, \mathbb{B}^{3n-4})$.

Here, of course, by a map $f \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-4})$, we mean a map equivalent to $f = (g, 0)$, where $g \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-4})$.

Future work

Our main result remains the boundary case of the Second Gap Theorem. Recall that we will show that $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$ is one of four different maps, and we will do so by showing that $\deg(F) \leq 2$. Recall also that this is evidence for the conjecture of D'Angelo that says that for $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $n > 2$, $\deg(F) \leq \frac{N-1}{n-1}$.

Related to this work, we have questions about a geometric condition for semi-linearity. Since [Hu03], we have known that what $\kappa_0 \leq n-2$, we have that there exists a complex variety E such that, for any point Z not on E, we have a hyperplane $S_Z + Z$, where S_Z is a vector subspace of codimension κ_0, on which the restriction of F is linear fractional. We can assume $0 \notin E$.

For a point $\epsilon \approx 0$, we can define the hyperplane L_ϵ by

$$z_\ell = \sum_{j=\kappa_0+1}^{n-1} a_{j\ell}(\epsilon) z_j + a_{n\ell}(\epsilon) w + \epsilon_\ell, 1 \leq \ell \leq \kappa_0.$$

Here, $a_{j\ell}$ and $a_{n\ell}$ are holomorphic functions that are 0 at 0. From [HJX06], when $\kappa_0 = 1$, we have only one equation, $z_1 = \epsilon \left(\sum_{j=2}^{n-1} a_j z_j + a_n w + 1 \right)$, and that a_j and a_n are constant. Our concern now is the case where $\kappa_0 = 2$. An upcoming paper will address this case.

Another direction for future work involves the proof or disproof of Conjecture 2.23, as studied by Ebenfelt.

We need a definition. We say that f is (m,s)-degenerate at p if the codimension of $E_m(p) = s$, where

$$E_m(p) = span_{\mathbb{C}} \left\langle ({\omega_{\alpha_1}}^a{}_{\alpha_2;\alpha_3\ldots\alpha_j})_{a=1}^{N-n} : \alpha_1, \ldots, \alpha_j \in \{1, \ldots, n\}, j \leq m \right\rangle.$$

2.45 Theorem (Eb13). *Let $f : M^n \to \S^N$, and assume there are integers*

$0 \leq k_2, k_3, \ldots, k_m \leq n-1$, *with* m *the degeneracy of* f, *such that, for* $p \in M^n$,

$$d_j(p) - d_{j-1}(p) < \sum_{i=0}^{k_j}(n-i), j = 2, \ldots, m, d_1(p) = 0.$$

If $k < n$, *where* $k = \sum_{j=1}^{m} k_j$ *and* $d = \max_{p \in M} d(p)$, *then* $f(M)$ *is contained in a complex plane* P^{N_0+1} *with* $N_0 - n = d + k$.

This relies on a lemma.

2.46 Lemma (Eb13). *Let* $z = (z_1, \ldots, z_n)$, *and let* $p_1(z), \ldots, p_t(z)$ *be homogeneous polynomials of degree* m *linearly independent in* $\mathbb{C}[z]$. *Let* S *be the vector space of homogeneous polynomials* $r(z)$ *such that there are linear polynomials* $q_1(\overline{z}), \ldots, q_m(\overline{z})$ *(depending on* $r(z)$*) satisfying*

$$\sum_{j=1}^{t} p_j q_j(\overline{z}) = r(z)\|z\|^2.$$

If for some $0 \leq k \leq n-1$ *we have* $t < \sum_{j=1}^{k}(n-j)$, *then* $\dim S < k$.

Then Conjecture 2.23 follows from Theorem 2.45 and the following conjecture.

2.47 Conjecture (Eb14). Given a Hermitian polynomial $A(z, \overline{z})$ such that for some holomorphic polynomials $p_j(z)$ of degree m in n variables, with $p_j(0) = 0$,

$$A(z, \overline{z})\|z\|^2 = \sum_{j=1}^{p} |p_j(z)|^2, p = N - n.$$

Then the minimum number q such that $A(z, \overline{z})\|z\|^2$ can be written

$$A(z, \overline{z}) = \sum_{j=1}^{q} |q_j(z)|^2$$

can only take certain values:

$$
\begin{aligned}
&0,\\
&n,\\
&2n-1, 2n,\\
&3n-3, 3n-2, 3n-1, 3n\\
&\vdots\\
&n+(n-1)+\cdots+(n-k+1), \ldots, kn,\\
&\tfrac{n^2}{2}.
\end{aligned}
$$

This last conjecture is related to Huang's Lemma, and it raises deep questions about the nature of rigidity properties of proper holomorphic maps. Its resolution may be difficult to achieve, but it would be an extraordinary result.

Part II

On the First and Second Gap Theorems

Chapter 3

The First Gap Theorem

Our main result of this chapter is that of Huang [Hu99], which showed that any mapping in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$, $N \leq 2n-2$, is equivalent to a linear map. Some of the ideas used when studying other cases are useful to prove our main theorem in the final chapters. Some of these cases involve smaller N than that of our main theorem, where $N = 3n-3$, and some involve larger N.

The theorem proved by Huang in 1999 can be stated as follows:

3.1 Theorem (First Gap Theorem)**.** *Let M_1 and M_2 be two connected open pieces of the boundaries of $\mathbb{B}^n \subset \mathbb{C}^n$ and $\mathbb{B}^N \subset \mathbb{C}^N$, respectively. Let f be a non-constant twice continuously differentiable CR mapping from M_1 into M_2. Suppose that $n > 1$ and $N < 2n-1$. Then f is the restriction of a certain totally geodesic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$. More precisely, there exist $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ f \circ \sigma(z_1, \ldots, z_n) \equiv (z_1, \ldots, z_n, 0, \ldots, 0)$. In particular, f is real analytic over M_1.*

Before presenting a corollary, we recall the outline of the proof mentioned in the first chapter of this dissertation. There are two steps to the proof in Huang's paper. The first is to show that the F^{**} normalization gives us enough information to conclude that the geometric rank κ_0 of F is 0. Then, we can show that all maps with $\kappa_0 = 0$ are linear.

3.2 Corollary. *Let f be a proper holomorphic mapping from $\mathbb{B}^n$ into $\mathbb{B}^N$ that is twice continuously differentiable up to the boundary. Suppose that $n > 1$ and $N < 2n-1$. Then there exist $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ f \circ \sigma(z_1, \ldots, z_n) \equiv (z_1, \ldots, z_n, 0, \ldots, 0)$.*

Let M_1 and M_2 be pieces of $\partial\mathbb{B}^n$ and $\mathbb{B}^N$. We can assume that these are represented by:

$$Im\ w = \sum_{j=1}^{n-1} |z_j|^2, (z,w) \in M_1,\ and\ Im\ w^* = \sum_{j=1}^{N-1} \left|z_j^*\right|^2, (z^*, w^*) \in M_2.$$

We use certain familiar differential operators, a basis for $T^{1,0}M_1$:

$$L_j = 2i\overline{z_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}, 1 \leq j \leq n-1.$$

We then call our map $F = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g) = (f, \phi, g)$. In particular, this F is a non-constant, C^2-smooth CR map with $F(0) = 0$, mapping M_1 to M_2, and because it is a CR map, we have that $\overline{L_j}F \equiv 0$.

We now apply several theorems from Chapter 2. First, by the Lewy extension theorem, we get that F extends holomorphically to a pseudoconvex side of M_1, which we call Ω.

We assume that Ω is filled in by holomorphic discs attached to M_1, and then we apply the maximum principle and the Hopf lemma to a certain subharmonic function:

$$-Im\ g + \sum_{j=1}^{n-1} |f_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2.$$

We conclude that

$$F(\Omega) \subset \mathbb{H}_N = \{(z^*, w^*) : Im(w^*) > |z^*|^2\}.$$

Also, $\left.\frac{\partial(Im\ g)}{\partial(Im\ w)}\right|_0 = \lambda > 0$. Then, because $\left.\frac{\partial(Im\ g)}{\partial(Re_w)}\right|_0 = 0$, we get that $\left.\frac{\partial g}{\partial w}\right|_0 = \lambda$.

We now write:

$$\frac{g - \overline{g}}{2i} = \sum_{j=1}^{n-1} |f_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2,$$

and we apply our differential operators L_ℓ, $L_k L_\ell$, and $\overline{L_k} L_\ell$ to both sides of this, yielding

$$\frac{L_\ell g}{2i} = \sum_j L_\ell f_j \overline{f}_j + \sum_j L_\ell \phi_j \overline{\phi}_j$$

and

$$\frac{L_k L_\ell g}{2i} = \sum_j L_k(L_\ell f_j)\overline{f}_j + \sum_j L_k(L_\ell \phi_j)\overline{\phi}_j$$

and

$$\frac{\overline{L}_k L_\ell g}{2i} = \sum_j (\overline{L}_k L_\ell f_j \cdot \overline{f}_j + L_\ell f_j \cdot \overline{L}_k \overline{f}_j) + \sum_j (\overline{L}_k L_\ell \phi_j \cdot \overline{\phi}_j + L_\ell \phi_j \cdot \overline{L}_k \overline{\phi}_j).$$

We let $(z, w) = (0, 0)$ in the first two of these in order to get

$$\left.\frac{\partial g}{\partial z_\ell}\right|_0 = \left.\frac{\partial^2 g}{\partial z_k \partial z_\ell}\right|_0 = 0.$$

Now, we use the notation $\tilde{f} = (f, \phi)$ (basically, everything except for g), and we also write

$$E_\ell = \left.\left(\frac{\partial \tilde{f}}{\partial z_\ell}\right)\right|_0 = \left.\left(\frac{\partial f_1}{\partial z_\ell}, \ldots, \frac{\partial f_{n-1}}{\partial z_\ell}, \frac{\partial \phi_1}{\partial z_\ell}, \ldots, \frac{\partial \phi_{N-n}}{\partial z_\ell}\right)\right|_0$$

and, similarly,

$$E_w = \left.\left(\frac{\partial \tilde{f}}{\partial w}\right)\right|_0 = \left.\left(\frac{\partial f_1}{\partial w}, \ldots, \frac{\partial f_{n-1}}{\partial w}, \frac{\partial \phi_1}{\partial w}, \ldots, \frac{\partial \phi_{N-n}}{\partial w}\right)\right|_0.$$

Note that

$$\frac{\overline{L}_k L_\ell g}{2i} = \left(-2iz_k \frac{\partial}{\partial \overline{w}} + \frac{\partial}{\partial \overline{z}_k}\right)\left(2i\overline{z}_\ell \frac{\partial}{\partial w} + \frac{\partial}{\partial z_\ell}\right)\frac{g}{2i}$$

$$= \left(-2iz_k \frac{\partial}{\partial \overline{w}} + \frac{\partial}{\partial \overline{z_k}}\right)\left(\overline{z}_\ell \frac{\partial g}{\partial w}\right) = \frac{\partial \overline{z_\ell}}{\partial \overline{z}_k}\frac{\partial g}{\partial w} = \delta_\ell^k \frac{\partial g}{\partial w},$$

and, at 0, this is equal to $\delta_\ell^k \cdot \lambda$.

Thus, $E_\ell \overline{E}_k^t = \delta_\ell^k \lambda$.

Look at $\{E_1/\sqrt{\lambda}, \ldots, E_{n-1}/\sqrt{\lambda}\}$, and extend this to an orthonormal basis for $\mathbb{C}^{N-1}$:

$$\{E_1/\sqrt{\lambda}, \ldots, E_{n-1}/\sqrt{\lambda}, C_1, \ldots, C_{N-n}\}.$$

Then

$$A := \begin{pmatrix} E_1/\sqrt{\lambda} \\ \vdots \\ E_{n-1}/\sqrt{\lambda} \\ C_1 \\ \vdots \\ C_{N-1} \end{pmatrix}$$

is a unitary matrix.

We now write

$$F^* = (\tilde{f}^*, g^*) = (f_1^*, \ldots, f_{n-1}^*, \phi_1^*, \ldots, \phi_{N-n}^*, g^*) = \frac{1}{\sqrt{\lambda}} F \cdot \begin{pmatrix} \overline{A^t} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}.$$

We see that $F^*(M_1) \subset \partial \mathbb{H}_N$, and

$$F^{*\prime}(0) = \begin{pmatrix} (id)_{(n-1)\times(n-1)} & (0)_{(n-1)\times(n-1)} & 0 \\ * & * & 1 \end{pmatrix}.$$

$$= \frac{1}{\lambda} \left(\sum_j \frac{\partial \overline{F}_j}{\partial \bar{z}_1} F_j, \quad \ldots, \quad \sum_j \frac{\partial \overline{F}_j}{\partial \bar{z}_{n-1}} F_j, \quad \sum_j C_{1,j} F_j, \quad \ldots, \quad \sum_j C_{N-n}, j F_j, \quad g \right),$$

from which we get that $g = \lambda g^*$.

Thus,

$$f_j^* = z_j + a_j w + \sum_{k,\ell=1}^{n-1} e_{k\ell}^j z_k z_\ell + \mathcal{O}(|w|^2) + o(|(z,w)|^2),$$

and

$$\phi_j^* = b_j w + \sum_{k,\ell=1}^{n-1} q_{k\ell}^j z_k z_\ell + \mathcal{O}(\left|w^2\right| + |zw|) + o(|(z,w)|^2),$$

and

$$g^* = w + dw^2 + \mathcal{O}(|zw|) + o(|z,w|^2).$$

Here, $a_j, b_j, e^j_{k\ell}$, and $q^j_{k\ell}$ are constants with $e^j_{k\ell} = e^j_{\ell k}$ and $q^j_{k\ell} = q^j_{\ell k}$.

Now, we write

$$a = (a_1, \ldots, a_{n-1}, b_1, \ldots, b_{N-1},)$$

and

$$r = \frac{1}{2} Re \left\{ \frac{\partial^2 g^*}{\partial w^2}\bigg|_0 \right\},$$

and we define $G \in Aut(\mathbb{H}_N)$ by

$$G = \left(\frac{z^* - aw^*}{1 + 2i\langle z^*, \overline{a}\rangle - (-r + i\langle a, \overline{a}\rangle)w^*}, \frac{w^*}{1 + 2i\langle z^*, \overline{a}\rangle - (-r + i\langle a, \overline{a}\rangle)w^*} \right).$$

Here, $\langle x, y\rangle$ is the standard inner product $\sum_j x_j y_j$.

We get that $G \circ F^*$ takes M_1 into $\partial\mathbb{H}_N$, and we write $F^{**} := (\tilde{f}^{**}, g^{**}) = G \circ F^*$.

Then F^{**} satisfies the following normalization condition:

$$F^{**}, \frac{\partial(f^{**} - z)}{\partial z_\ell}, \frac{\partial f^{**}}{\partial w}, \frac{\partial \phi^{**}}{\partial z_\ell}, \frac{\partial \phi^{**}}{\partial w}, \frac{\partial g^{**}}{\partial z_\ell}, \frac{\partial (g^{**} - w)}{\partial w}, \frac{\partial^2 g^{**}}{\partial z_\ell \partial z_k}, Re \frac{\partial^2 g^{**}}{\partial w^2} = 0$$

at $(z, w) = 0$.

Now, we observe that

$$a_\ell = \frac{1}{\lambda} E_w \cdot \overline{E_\ell}^t, b_\ell = \frac{1}{\sqrt{\lambda}} E_w \cdot \overline{C_\ell}^t, e^j_{k\ell} = \frac{1}{2\lambda} \tilde{f}''_{z_k z_\ell}(0) \cdot \overline{E_j}^t, q^j_{k\ell} = \frac{1}{2\sqrt{\lambda}} \tilde{f}''_{z_k z_\ell}(0) \cdot \overline{C_j}^t.$$

We must now state a lemma.

3.3 Lemma (Hu99, Lemma 2.1)**.** *Suppose that*

$$f^{**} = z + \mathcal{O}(|w|^2 + |zw|) + o(|(z,w)|^2)$$

and

$$\phi^{**} = \mathcal{O}(|w|^2 + |zw|) + o(|(z,w)|^2).$$

In particular, this means that $\frac{\partial^2 f^{**}}{\partial z_k \partial z_j}\Big|_0 = \frac{\partial^2 \phi^{**}}{\partial z_k \partial z_j}\Big|_0 = 0$ *for any* k, j*. Then we have the following:*

$$\tilde{f}''_{z_k z_\ell}(0) = \frac{2i}{\lambda}(\overline{E_w} \cdot E^t_\ell)E_k + \frac{2i}{\lambda}(\overline{E_w} \cdot E^t_k)E_\ell.$$

Proof. We see that

$$\phi^{**}_j = \frac{\phi^*_j - b_j g^*}{1 + 2i\langle \tilde{f}^*, \overline{a}\rangle - (-r + i\langle a, \overline{a}\rangle)g^*}.$$

We compare the coefficients of the terms with $z_\ell z_k$ in the Taylor expansion of this, and we get that $q^j_{k\ell} = 0$ for any k, ℓ and j. Since $q^j_{k\ell} = \frac{1}{2\sqrt{\lambda}} f''_{z_\ell z_k}(0) \cdot \overline{C_j}^t$, we get that $\overline{C_j}^t$ is orthogonal to $\tilde{f}''_{z_\ell z_k}(0)$, and since C_j were vectors that, together with E_j, form an orthogonal basis of $\mathbb{C}^{N-1}$, we have $\{\lambda^j_{k\ell}\}$ such that

$$\tilde{f}''_{z_k z_\ell}(0) = \sum_{j=1}^{n-1} \lambda^j_{k\ell} E_j.$$

We next consider

$$f^{**}_j = \frac{f^*_j - a_j g^*}{1 + 2i\langle \tilde{f}^*, \overline{a}\rangle - (-r + i\langle a, \overline{a}\rangle)g^*},$$

and we use a similar method to get $\sum e^j_{k\ell} z_k z_\ell - 2i \sum_\ell \overline{a}_\ell z_j z_\ell \equiv 0$. Thus, $e^j_{k\ell} = i(\delta^j_k \overline{a}_\ell + \delta^j_\ell \overline{a}_k)$, and so, from our earlier expression for $e^j_{k\ell}$,

$$\frac{1}{2\lambda}\tilde{f}''_{z_k z_\ell}(0)\overline{E_j}^t = \frac{i\delta^j_k}{\lambda}\overline{E_w} \cdot E^t_\ell + \frac{i\delta^j_\ell}{\lambda}\overline{E_w} \cdot E^t_k.$$

Using this with $\tilde{f}''_{z_k z_\ell}(0) = \sum_{j=1}^{n-1} \lambda^j_{k\ell} E_j$ and $E_\ell \cdot \overline{E_j}^t = \lambda \delta^j_\ell$, we get

$$\lambda^j_{k\ell} = \frac{2i}{\lambda}(\delta^j_k \overline{E_w} \cdot E^t_\ell + \delta^j_\ell \overline{E_w} \cdot E^t_k).$$

□

Chern-Moser Lie-derivate and a semi-linear equation

The previous section allows us to assume the following normalization of $F = (f, \phi, g)$.

$$f = z + \hat{f}, g = w + \hat{g},\ with\ \hat{f}, \hat{g}, \phi = \mathcal{O}(|(z,w)|^2),\ and\ \left.\frac{\partial^2 \hat{g}}{\partial z_\ell \partial z_k}\right|_0, Re\left.\frac{\partial^2 \hat{g}}{\partial w^2}\right|_0 = 0.$$

The general idea is two-fold. First, in this section, we extend a lemma from Chern-Moser theory to our situation, and then we use this to set up the hypothesis of the lemma from the end of the previous section. Then we compose f with some automorphisms to get a family of mappings from which we will construct a differential equation, which will be enough to get $F = (z, 0, w)$.

Recall the definition of weighted degree mentioned in Chapter 2.

Now, using our normalization for f together with our functional equation for F, we get

$$Im(w + \hat{g}) = \sum_{j=1}^{n-1} \left| z_j + \hat{f}_j \right|^2 + \sum_{j=1}^{N-n} |\phi_j|^2, (z,w) \in M_1. \qquad (\star)$$

We use the notation $f^{(s)}$ denote the part of f that is a homogeneous polynomial of weighted degree s.

Suppose we can decompose $\hat{f}_j = \sum_{s=2}^{m-1} \hat{f}_j^{(s)} + o_{wt}(m-1)$ and $\hat{g} = \sum_{s=2}^{m} \hat{g}^{(s)} + o_{wt}(m)$ and $\phi_j = \sum_{s=1}^{m-\ell} \phi_j^{(s)} + o_{wt}(m-\ell), \ell \geq 2$.

If we collect the terms of $(\star)$ of weighted degree k, we get the following k^{th} semi-linearization:

$$Im\left(\hat{g}^{(k)}(z,w) - 2i\langle \overline{z}, \hat{f}^{(k-1)}(z,w)\rangle\right) = \sum_{j=1}^{N-n} \sum_{p=\ell}^{k-1} \phi_j^{(k-p)} \overline{\phi}_j^{(p)} + N^{(k)}, (z,w) \in M_1,$$

where $m \geq k \geq \ell$ and $N^{(k)}$ is a homogeneous polynomial of weighted degree k contributed by $\hat{f}^{(\sigma-1)}$ with $\sigma \leq k-1$. Also, $N^{(k)} \equiv 0$ if $\hat{f}^{(\sigma-1)} \equiv 0$ for $\sigma \leq k-1$. Thus, when $\hat{g}^{(\sigma_1)}, \hat{f}^{(\sigma_1 - 1)}, \phi^{(\sigma_2)} \equiv 0$ for $\sigma_1 < 2k$ and $\sigma_2 < k, 2k \leq m$, we can further simplify the $2k^{th}$ linearization of $(\star)$ as follows:

$$Im\left(\hat{g}^{(2k)}(z,w) - 2i\langle \overline{z}, \hat{f}^{(2k-1)}(z,w)\rangle\right) = \sum_{j=1}^{N-n} \phi_j^{(k)} \overline{\phi}_j^{(k)}, (z,w) \in M_1.$$

Similarly , when $\hat{g}^{(\sigma_1)}, \hat{f}^{(\sigma_1 - 1)}, \phi^{(\sigma_2)} \equiv 0$ for $\sigma_1 \leq 2k$ and $\sigma_2 \leq k, 2k+1 \leq m$, we can simplify the $(2k+1)^{th}$ linearization of $(\star)$ as follows:

$$Im\left(\hat{g}^{(2k+1)}(z,w) - 2i\langle \overline{z}, \hat{f}^{(2k)}(z,w)\rangle\right) = 0, (z,w) \in M_1.$$

We now use the notation $\mathcal{L}(\hat{f}^{(\sigma-1)}, \hat{g}^{(\sigma)}) = Im(\hat{g}^{(\sigma)} - 2i\langle \overline{z}, \hat{f}^{(\sigma-1)}\rangle)\Big|_{M_1}$, the Lie derivative of $\rho_1 = Im\ w - \sum_{j=1}^{n-1} |z_j|^2$ along the vector field $X = 2Re(\hat{g}^{(\sigma)}\frac{\partial}{\partial w} + \sum \hat{f}_j^{(\sigma-1)}\frac{\partial}{\partial z_j})$ restricted to M_1. In fact, the operator $\mathcal{L}$ is the basic tool used to construct the local form of strongly pseudoconvex hypersurfaces, as seen in [CM74] and [Vi90].

Our case involves the following lemma from Chern-Moser.

3.4 Lemma (CM74, Lemma 2.1)**.** *Consider the linear equation* $\mathcal{L}(f,g) = Im(g = 2i\langle \overline{z}, f\rangle) = 0$ *with* f, g *holomorphic near the origin. Then there is a unique solution* $(f,g) \equiv (0,0)$ *if the following normalization condition holds:*

$$f\Big|_0, g\Big|_0, \frac{\partial f}{\partial w}\Big|_0, \frac{\partial f}{\partial z_\ell}\Big|_0, \frac{\partial g}{\partial z_\ell}\Big|_0, \frac{\partial g}{\partial w}\Big|_0, \frac{\partial^2 g}{\partial z_\ell \partial z_k}\Big|_0, Re\left(\frac{\partial^2 g}{\partial w^2}\right)\Big|_0 = 0.$$

We shall also rely on another key proposition involving $\mathcal{L}$:

3.5 Proposition (Hu99)**.** *Consider the following semi-linear equation of weighted degree 4 with respect to the polynomials* $P^{(4)}$, $Q^{(3)} = (Q_1^{(3)}, \ldots, Q_{n-1}^{(3)})$, *and* $\Phi_j^{(2)}, j = 1, \ldots, k$ *of weighted degree 4, 3, and 2, respectively:*

$$\mathcal{L}(Q^{(3)}, P^{(4)}) = \sum_{j=1}^{k} \left|\Phi_j^{(2)}(z,w)\right|^2, (z,w) \in M_1.$$

Assume the following normalization condition holds:

$$\frac{\partial \Phi_j^{(2)}}{\partial w} = Re\left(\frac{\partial^2 P^{(4)}}{\partial w^2}\right) = 0.$$

Then $P^{(4)} \equiv 0, Q^{(3)} \equiv a^{(1)}(z)w$ *with*

$$-2i\langle a^{(1)}(z), \overline{z}\rangle\,|z|^2 = \sum_{j=1}^{k} \left|\Phi_j^{(2)}(z)\right|^2,$$

where $a^{(1)}(z)$ is a vector-valued polynomial of degree 1. Moreover, if $k \leq n-2$, then the first equation here has only the trivial solution $Q_j^{(3)} = P^{(4)} = \Phi_j^{(2)} \equiv 0$.

We note that when all $\Phi_j^{(2)}$ are assumed to be 0, the proposition is reduced to the 4^{th} order case of the lemma. Also, when $k \geq n-1$, the last statement in the proposition fails.

The geometric interpretation of the proposition is that $\sum_{j=1}^{k} \left|\Phi_j^{(2)}\right|^2$ stays in the Moser normal space $\mathcal{N}^4$ (see [CM74]) when $k \leq n-2$.

We prove the proposition.

By the normaliztion condition of the proposition, we can write

$$Q^{(3)}(z,w) = a^{(1)}(z)w + b^{(3)}(z),$$

$$P^{(4)}(z,w) = A^{(4)}(z) + B^{(2)}(z)w + D_0 w^2$$

with $Re(D_0) = 0$ and $\Phi_j^{(2)}(z,w) = h_j^{(2)}(z)$.

Then our semi-linear equation becomes

$$Im\left(A^{(4)} + B^{(2)}w + D_0 w^2 - 2i\langle \overline{z}, a^{(1)}\rangle w - 2i\langle \overline{z}, b^{(3)}\rangle\right) = \sum_{j=1}^{k} \left|h_j^{(2)}(z)\right|^2, (z,w) \in M_1.$$

Let $w = u + i\left|z\right|^2$. Then we can rewrite this as two separate equations:

$$Im\left(A^{(4)}(z) + iB^{(2)}(z)\left|z\right|^2 - D_0\left|z\right|^4 + 2\langle \overline{z}, a(1)(z)\left|z\right|^2 - ib^{(3)}(z)\rangle\right) = \sum_{j=1}^{k} \left|h_j^{(2)}(z)\right|^2$$

and

$$Im\left(B^{(2)}(z) + 2iD_0\left|z\right|^2 - 2i\langle \overline{z}, a^{(1)}(z)\rangle\right) u = 0,$$

and thus

$$Im\left(D_0 u^2\right) = 0.$$

Apply the normalization condition to this last equation to get $D_0 = 0$. Also, if we note that u is real, and then we collect the $z_k z_\ell$ terms, we get that $B^{(2)} \equiv 0$, and so

$$Im\left(i\langle \overline{z}, a^{(1)}(z)\rangle\right) \equiv 0.$$

Looking at the first of the split equations, we can do the same thing to get $A^{(4)} \equiv 0$ and $b^{(3)} = 0$, and thus we get $P^{(4)} \equiv 0$ and $Q^{(3)} = a^{(1)}(z)w$. Put all this together with the same equation we just used to get

$$-2i\langle \overline{z}, a^{(1)}(z)\rangle\, |z|^2 = \sum_{j=1}^{k} \left|h_j^{(2)}(z)\right|^2.$$

For convenience, we now restate Huang's Lemma.

3.6 Lemma (Huang's Lemma)**.** *Let* $\{\psi_j\}_{j=1}^{k}$ *and* $\{\chi_j\}_{j=1}^{k}$ *be holomorphic function in* $z \in \mathbb{C}^n$ *near 0. Assume that* $\psi_j(0) = \chi_j(0) = 0$. *Let* $H(z, \overline{z})$ *be a real analytic function for* $z \approx 0$ *such that*

$$\sum_{j=1}^{k} \psi_j(z)\overline{\chi_j(z)} = |z|^2 H(z, \overline{z}).$$

Then when $k \leq n-1$, $H(z, \overline{z}) \equiv 0$.

Proof. To prove this, we first complexify the above equation.

$$\sum_{j=1}^{k} \psi_j(z)\overline{\chi_j(\xi)} = \langle z, \overline{\xi}\rangle H(z, \overline{\xi})$$

with z, ξ independent variables. We can assume that ψ_j is not identically 0 for each j. Thus, there is a point z_0 that is close to 0 such that $\psi_j(z_0) - \epsilon_j \neq 0$ for each j¿

By the assumption that $k \leq n-1$, we get that

$$V_{z_0} = \{z : \psi_j(z) = \psi_j(z_0), j = 1, \ldots, k\}$$

defines a complex analytic variety of dimension at least 1 near z_0. Because of how we chose z_0 and because $\psi(0) = 0$, we get that V_{z_0} does not contain a complex line passing through 0. Thus, there is a point $z^* \in V_{z_0}$ such that V_{z_0} contains a complex curve C^* near z^* parameterized by an equation of the form

$$z(t) = z^* + vt + o(t),$$

with $\{z^*, v\}$ independent vectors and $|t| < 1$. For each $z \in C^*$ and ξ with $\langle z, \overline{\xi} \rangle - 0$ and ξ close to 0, then, we get $\sum \overline{\epsilon}_j \chi_j(\xi) = 0$.

By $z(t) = z^* + vt + o(t)$, we get that all such ξ fill an open subset of $\mathbb{C}^n$. Thus, $\sum_j \overline{\epsilon}_j \chi_j(z) \equiv 0$. Thus,

$$\sum_{j=1}^{k=1} (\psi_j(z) = \frac{\epsilon_j}{\epsilon_j} \psi_k(z)) \overline{(\chi_j(z))} - \langle z, \overline{z} \rangle H(z, \overline{z}).$$

It follows from induction that $\sum \psi_j \overline{\chi}_j \equiv 0$, and $H \equiv 0$. $\square$

Criterion for linearity

Next, we give a criterion for linearity for C^2-smooth CR mappings between spheres, and we note that this is the case for any $N \geq n$. In particular, this is the section where we examine Huang's proof that $\kappa_0 = 0$ implies that a map is linear.

First, we take $T = \frac{\partial}{\partial u}$ to be a real tangent vector field along $\partial \mathbb{H}_n$ transversal to $T^{(1,0)} \partial \mathbb{H}_n + T^{(0,1)} \partial \mathbb{H}_n$. Then

$$\{L_1, \ldots, L_{n-1}, \overline{L_1}, \ldots, \overline{L_{n-1}}, T\}$$

is a basis for $\mathbb{C}T\partial \mathbb{H}_n$.

Because of how we define M_1 as a hypersurface, we can parameterize it by $\mathbb{R}^{2n-1}$:

$$(z, u) \mapsto (z, w) = (z, u + i\,|z|^2).$$

We use the standard inner product $(\cdot, \cdot)$, and for any differential operator Y with smooth coefficients on M_1, we have the standard adjoint according to this product:

$$(Y(\chi), \rho) = (\chi, Y^*(\rho)).$$

Thus, for any continuous h, $Y(h)$ can be identified as a distribution acting on the testing function space $C_0^\infty(M_1)$ in the standard way:

$$(Y(h), \rho) = (h, Y^*(\rho)).$$

Then h is called a CR function if $\overline{L}_j(h) \equiv 0$ for each j, i.e., it is locally annihilated by any CR vector field.

3.7 Lemma (Hu99)**.** *(a) T commutes with L_j for each j. Hence, for any continuous CR function h over M_1, Th is a CR distribution over M_1. Moreover, $Th = \frac{1}{2i}[L_j, \overline{L}_j]h = \frac{1}{2i}\overline{L}_j(L_j h)$.*

(b) Let h be a C^2-smooth CR function over M_1, and let χ be a C^1-smooth function over M_1. Then for any k, we have

$$\overline{L_k}(L_k^2(h)\chi) = 4iL_k(T(h))\chi + L_k^2(h)\overline{L_k}(\chi),$$

and

$$\overline{L_k}(L_k(T(h))\chi) = 2iT^2(h)\chi + K_k(T(h))\overline{L_k}(\chi).$$

Now, let F be a twice continuously differentiable CR mapping from M_1 into M_2 such that $F(0) = 0$, satisfying the normalization from the previous section. Then for each $(z_0, w_0) \in M_1$, we write $\sigma_{(z_0,w_0)} \in Aut(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$. Define $\tau_{(z_0,w_0)} \in Aut(\mathbb{H}_n)$ by

$$\tau(z^*, w^*) = (z^* - \tilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - wi\langle z^*, \overline{\tilde{f}(z_0, w_0)}\rangle),$$

with $\tilde{f} = (f, \phi)$.

Then

$$F_{(z_0,w_0)}(z, w) = (f_{(z_0,w_0)}, \phi_{(z_0,w_0)}, g_{(z_0,w_0)}) = \tau_{(z_0,w_0)} \circ F \circ \sigma_{(z_0,w_0)}$$

is a twice continuously differentiable CR mapping from a small open subset of $\partial\mathbb{H}_n$ containing 0 to $\partial\mathbb{H}_n$, with $F_{(z_0,w_0)}(0) = 0$. As in the second section of this chapter, we have

$$\lambda_{(z_0,w_0)}, (E_\ell)_{(z_0,w_0)}, (E_w)_{(z_0,w_0)}, (C_\ell)_{(z_0,w_0)}, A_{(z_0,w_0)}, F^*_{(z_0,w_0)},$$
$$a_{(z_0,w_0)}, r_{(z_0,w_0)}, (e^j_{k\ell})_{(z_0,w_0)}, (q_{k\ell})^j_{(z_0,w_0)}, \text{ and } G_{(z_0,w_0)},$$

which now depend on our choice of (z_0, w_0).

As before, we can normalize $F^*_{(z_0,w_0)}$ by composing with $G_{(z_0,w_0)}$, yielding $F^{**}_{(z_0,w_0)} = (f^{**}_{(z_0,w_0)}, \phi^{**}_{(z_0,w_0)}, g^{**}_{(z_0,w_0)})$, which still depends on our choice of (z_0, w_0).

Now, we derive the linearity criterion.

3.8 Theorem (Linearity Criterion)**.** *Let F be a twice continuously differentiable CR mapping from M_1 into M_2 such that $F(0) = 0$ which satisfies our normalization. Assume $F^{**}_{(z_0,w_0)}$ is defined as above. Suppose that for each $(z_0, w_0) \approx 0 \in M_1$, we always have that*

$$f^{**}_{(z_0,w_0)} = z + \mathcal{O}(|w|^2 + |z|\,|w|) + o(|(z,w)|^2)$$

and

$$\phi^{**}_{(z_0,w_0)} = \mathcal{O}(|w|^2 + |z|\,|w|) + o(|(z,w)|^2).$$

Then $F(z,w) \equiv (z, 0, w)$.

We note that the assumption here is equivalent to

$$\left.\frac{\partial^2(\phi^{**}_{(z_0,w_0)})}{\partial z_k \partial z_\ell}\right|_0 = 0, \text{ or } \left.\frac{\partial^2(f^{**}_{(z_0,w_0)})}{\partial z_k \partial z_\ell}\right|_0 = 0, \ \forall k, \ell, \text{ with } (z_0, w_0) \approx 0.$$

We put our Lemma 3.3 (which is [Hu99,2.1]) in a form that we can apply more easily. First, we write

$$z_0 = (z_{1,0}, \ldots, z_{n-1,0})$$

and

$$\tilde{f}_{(z_0,w_0)} = \left((f_1)_{(z_0,w_0)}, \ldots, (f_{n-1})_{(z_0,w_0)}, (\phi_1)_{(z_0,w_0)}, \ldots, (\phi_{N-n})_{(z_0,w_0)}\right).$$

Now, we gather certain terms together:

$$\begin{aligned}
(E_w)_{(z_0,w_0)} &= (\tilde{f}_{(z_0,w_0)})'(0) = \tilde{f}'_w(z_0,w_0) \\
(E_\ell)_{(z_0,w_0)} &= (\tilde{f}_{(z_0,w_0)})'_{z_\ell}(0) = \tilde{f}'_{z_\ell}(z_0,w_0) + 2i\overline{z_{\ell,0}}\tilde{f}'_w(z_0,w_0) = L_\ell(\tilde{f})(z_0,w_0) \\
(g_{(z_0,w_0)})'_{z_\ell}(0) &= L_\ell(g - 2i\langle < \tilde{f}, \overline{\tilde{f}}\rangle)(z_0,w_0) \\
(g_{(z_0,w_0)})'_w &= g'_w(z_0,w_0) - 2i\langle \tilde{f}'_w(z_0,w_0), \overline{\tilde{f}(z_0,w_0)}\rangle \\
\lambda(z_0,w_0) &= \lambda_{(z_0,w_0)} = g'_w(z_0,w_0) - 2i\langle \tilde{f}'_w(z_0,w_0), \overline{\tilde{f}(z_0,w_0)}\rangle \\
\left(\tilde{f}_{(z_0,w_0)}\right)''_{z_\ell z_k}(0) &= (\tilde{f})''_{z_k z_\ell}(z_0,w_0) + 2i\overline{z_{0,\ell}}(\tilde{f})''_{z_k w}(z_0,w_0) + 2i\overline{z_{0,k}}(\tilde{f})''_{z_\ell w}(z_0,w_0) - \\
&\quad 4\overline{z_{0,\ell} z_{0,k}}(\tilde{f})''_{ww}(z_0,w_0) \\
&= L_\ell(L_k(\tilde{f}))(z_0,w_0)
\end{aligned}$$

If we notice that (z_0,w_0) is arbitrary, and we apply Lemma 3.3 to $F^{**}_{(z_0,w_0)}$, then these equations allow us to restate the lemma thusly:

3.9 Lemma. *Assume the above notation, and assume the hypothesis in the this section's theorem. For each j and k at each point $(z,w) \approx 0 \in M_1$, we have the following:*

$$L_j(L_k(\tilde{f})) = \frac{2i}{\lambda(z,w)}(\overline{\tilde{f}'_w} \cdot (L_j\tilde{f})^t)L_k(\tilde{f}) + \frac{2i}{\lambda(z,w)}(\overline{\tilde{f}'_w} \cdot (L_k\tilde{f})^t)L_j(\tilde{f}).$$

We now prove our normalization criterion theorem.

Proof. Write $A_k(z,w) = \frac{2i\overline{\tilde{f}'_w} \cdot L_k(\tilde{f})^t}{\lambda(z,w)}$, which is obviously C^1-smooth over M_1. We may assume that $L_k(f_k) \neq 0$ over M_1. Then, Lemma 3.9 gives us

$$A_k(z,w) = \frac{L_k^2(f_k)}{2L_k(f_k)}$$

and

$$L_j(L_k(\phi)) = A_j(z,w)L_k(\phi) + A_k(z,w)L_j(\phi).$$

In particular,

$$L_k^2(\phi) = 2A_k(z,w)L_k(\phi).$$

Next, we will use the fact that two continuous functions are the same if they are also the same in the sense of distributions.

Applying $\overline{L_k}$ to the previous equation and using Lemma 3.7(b), we get

$$4iL_k(T(\phi)) = B_{k,1}(z,w)L_k(\phi) + B_{k,2}T(\phi), \qquad (\star)$$

where $B_{k,2} = 4iA_k(z,w) \in C^1(M_1)$ and $B_{k,1} = \overline{L_k}(2A_k(z,w)) \in C^0(M_1)$. Note, now, that there are functions $b_{k,j}$ such that

$$B_{k,1} = b_{k,1}(z,w)L_k(T(f_k)) + b_{k,2}(z,w)L_k^2(f_k), \qquad (\star\star)$$

with $b_{k,j}(z,w) \in C^1(M_1)$ for $j = 1, 2$. By the same lemma, we have

$$\overline{L_k}(A_k) = \overline{L_k}\left(\frac{1}{2L_k(f_k)}\right)L_k^2(f_k) + \frac{2i}{L_k(f_k)}L_k(T(f_k)).$$

Thus, to get $(\star\star)$, we only need $b_{k,2}(z,w) = \overline{L_k}(\frac{1}{L_k(f_k)})$ and $b_{k,1} = \frac{2i}{L_k(f_k)}$.

We apply $\overline{L_k}$ to $(\star)$, getting, as a distribution,

$$-8T^2(\phi) = h + \overline{L_k}(B_{k,2}(T(\phi)),$$

with $h = \overline{L_k}((b_{k,1}L_k(T(f_k)) + b_{k,2}L_k^2(f_k))L_k(\phi))$. By Lemma 3.7 again, we identify h with

$$(c_{k,1}L_k(T(f_k)) + c_{k,2}T^2(f_k) + c_{k,3}L_k^2(f_k)))L_k(\phi) + (c_{k,4}L_k(T(f_k)) + c_{k,5}L_k^2(f_k))T(\phi)$$

with $c_{k,j}$ continuous over M_1. Hence, h is equivalent to a continuous function, and we obtain, pointwise,

$$T^2(\phi) = C_{k,1}(z,w)L_k(\phi) + C_{(k,2)}(z,w)T(\phi),$$

with $C_{k,j}$ continuous functions over M_1.

Because ϕ are CR mappings over M_1, we have that $\overline{L_k}T(\phi) \equiv 0$ and $\overline{L_j}(\overline{L_k}(\phi)) \equiv 0$. Identify M_1 with a neighborhood of 0 in $\mathbb{R}^{2n-1}$ via

$$(x, y, u) \mapsto (z = (x = iy), w = u + i\,|z|^2),$$

and write X for the vector $(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial u})$. Combining several of our results from this section gives us that there is a matrix $\Lambda(x, y, u)$ with entries continuous functions over M_1 such that X satisfies

$$DX = \Lambda(x, y, u)X^t, X(0) = 0.$$

If we think of this as a differential equation in X, by uniqueness theory, it follows that $X \equiv 0$.

In particular, for any $p = (x_0, y_0, u_0) \approx 0$, let $X_p(t) = X(tp)$ for $0 \le t \le 1$. Then $\frac{dX_p}{dt} = \Lambda_p(t)X_p(t)^t$ for a continuous matrix function $\lambda_p(t)$ with $\|\lambda_p(t)\| < \|p\|$, where $\|\cdot\|$ is the super norm.

Since $X_p(0) = 0$, $X_p(t) = \int_0^t \Lambda_p(p\tau)X_p(\tau)^t d\tau$, and thus, $\|X_p\| \le C\|p\|\|X_p\|$ for some constant $C > 0$, with C independent of p. It follows that $X_p \equiv 0$ when $\|p\| < \frac{1}{C}$. Since $\phi(0) = 0$, we also get $\phi \equiv 0$ over M_1.

Since $\phi \equiv 0$, we get that (f, g) is a C^2 CR diffeomorphism between open subsets of $\partial\mathbb{H}_n$. By a theorem of Alexander, we get that (f, g) extends to an automorphism of $\mathbb{H}_n$. Since (f, g) satisfies our normalization condition, we get that $(f, g) = (z, w)$ by Lemma 3.4. □

Completion of the proof of the First Gap Theorem

3.10 Proposition (Hu99, 5.1). *Let $F = (f, \phi, g)$ be any twice differentiable CR mapping from M_1 to M_2 satisfying our normalization. Assume $N \le 2n - 2$. Then $f = z + o_{wt}(3)$, $g = w + o_{wt}(4)$, and $\phi = o_{wt}(2)$.*

We specify some notation. Let P_1 and P_2 be polynomials that could be different in different contexts. We say that $wt(P_1) \geq k$ if $|P_1(tz, t^2w, t\overline{z}, t^2\overline{w})| \leq Ct^k$.

We need two more lemmas.

3.11 Lemma (Hu99, 5.2). *(a) For any function $\chi \in C^k(M_1 \cap o_{wt}(k)$ with $k \geq 2$, $L_j\chi, \overline{L_j}\chi \in o_{wt}(k-1)$ and $T\chi \in o_{wt}(k-2)$.*

(b) If $\chi \in C^1(M_1)$ is such that $\overline{L_j}(\chi)$, $L_j(\chi) \in \mathcal{P} + o_{wt}(k-1)$ for each j, and $T(\chi) \in \mathcal{P} + o_{wt}(k-2)$, then $\chi \in \mathcal{P} + o_{wt}(k)$. Moreover, write $\chi = h_1 + h_2$, where h_1 is a polynomial with weighted degree larger than k and $h_2 \in o_{wt}(k)$. Then in case χ is CR, h_1 must be a polynomial in (z, w).

3.12 Lemma (Hu99, 5.3). *Let $F = (f, \phi, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g)$ be a twice differentiable CR map from M_1 into M_2 satisfying our normalization condition. Then $f = Q + o_{wt}(3)$ and $g = P + o_{wt}(4)$, with P and Q certain polynomials. In fact, we can choose P and Q such that $P = w$ and $Q = z + \frac{iw}{2}a^{(1)}(z)$, where $a^{(1)}(z)$ is a certain polynomial of degree 1 with $\langle a^{(1)}(z), \overline{z}\rangle = \sum_{j=1}^{k} \left|\phi_j^{(2)}(z)\right|^2$.*

Proof. We prove Lemma 3.12.

First,

$$\frac{1}{2i}L_\ell g = \sum L_\ell(f_j)\overline{f_j} + \sum L_\ell(\phi_j)\overline{\phi_j}. \qquad (\star)$$

We can see that

$$L_\ell f_j\overline{f_j} = (P_1 + o_{wt}(1))(P_2 + o_{wt}(2)) \in \mathcal{P} + o_{wt}(2),$$

where $wt(P_2) \geq 1$, and

$$L_\ell \phi_j\overline{\phi_j} = (P_1 + o_{wt}(1))(P_2 + o_{wt}(2)) \in \mathcal{P} + o_{wt}(3),$$

where $wt(P_1) \geq 1$ and $wt(P_2) \geq 2$. Thus, $L_\ell g \in \mathcal{P} + o_{wt}(2)$.

However, since $g \in C^2(M_1), T(g) \in \mathcal{P} + o_{wt}(1)$, then by Lemma 3.11(b), we have $g \in \mathcal{P} + o_{wt}(3)$.

We see that $g = w + o_{wt}(2)$. Write $\hat{g} = \hat{g}_3 + o_{wt}(3)$, with $\hat{g}_3$ a weighted homogeneous polynomial of degree 3. By Lemma 3.11(b), $\hat{g}_3$ is holomorphic. To see that $\hat{g}^{(3)} = \hat{g}_3 = 0$ and $\hat{f}^{(2)} = 1$, we now just need to apply Lemma 3.4 with $k = 1$.

Now, apply L_j to $(\star)$, yielding $\frac{1}{2i}L_j(L_\ell g) = L_j(L_\ell(\tilde{f})) \cdot \overline{\tilde{f}}$. Apply $\overline{L_k}$ to this, and use Lemma 3.7, and we get $L_j(L_\ell(\tilde{f})) \cdot \overline{L_k \tilde{f}} = 0$ for $j, \ell \neq k$, and

$$L_j(L_\ell(\tilde{f})) \cdot \overline{L_k \tilde{f}} + cT(L_v(\tilde{f})) \cdot \overline{\tilde{f}} = \frac{c}{2i} T L_v(g),$$

where $k = j$ or $k = \ell$.

When $l \neq j$, we have $c = 2i, v = \{\ell, j\} \backslash \{k\}$. When $\ell = j$, we have $c = 4i$ and $v = k$. Since $L_k(f_\mu) = \delta^k_\mu + o_{wt}(1)$, we can solve for $L_j L_\ell$ to see that

$$L_j L_\ell(f_k) - \frac{cTL_\ell(g)}{2iL_k(f_k)} \in \mathcal{P} + o_{wt}(1).$$

Because

$$2iT(L_\ell(g)) = L_\ell(T(\tilde{f})) \cdot \overline{\tilde{f}} + (L_\ell(\tilde{f})) \cdot T(\tilde{f}),$$

we get that $T(L_\ell(g)) \in \mathcal{P} + o_{wt}(1)$. Thus,

$$L_j(L_\ell(f_k)) \in \mathcal{P} + o_{wt}(1) \forall j, \ell, k.$$

Since F is twice differentiable, $\overline{L_j}(L_\ell f_k) = \delta^j_\ell T(f_k) \in \mathcal{P} + o_{wt}(1)$, and $T(L_\ell(f_k)) \in \mathcal{P} + o_{wt}(0)$. Using Lemma 3.11, we get that $L_\ell(f_k) \in \mathcal{P} + o_{wt}(2)$ for all ℓ, k. Now, since $T(f_k) \in \mathcal{P} + o_{wt}(1)$, $f_k \in \mathcal{P} + o_{wt}(3)$ for any k. By Lemma 3.11(b), we have $f = Q + o_{wt}(3)$ for some polynomial Q.

With this, we can verify that $L_\ell f_j \overline{f_j}$ and $\|L_\ell f_j\|^2$ stay in $\mathcal{P} + o_{wt}(3)$ and $\mathcal{P} + o_{wt}(2)$, respectively. By $(\star)$, we have $2iT(g) = \langle 2iTf, \overline{f} \rangle + \|L_\ell f\|^2 + \langle 2iT\phi, \overline{\phi} \rangle + \|L_\ell \phi\|^2$, and we get that $L_j(g) \in \mathcal{P} + o_{wt}(3)$ and $T(g) \in \mathcal{P} + o_{wt}(2)$.

Similarly, $g = P + o_{wt}(4)$ for some polynomial P. Then, we apply the first part of Proposition 3.5 to

$$Im\left(\hat{g}^{(2k)}(z, w) - 2i\langle \overline{z}, \hat{f}^{(2k-1)}(z, w) \rangle\right) = \sum_{j=1}^{N-n} \phi^{(k)}_j \overline{\phi}^{(k)}_j, (z, w) \in M_1,$$

which we will need again, so we will call this $(\star)$. $\square$

Next, we prove Proposition 3.10.

Proof. By Lemma 3.12, we have $f = z + \hat{f}^{(3)} + o_{wt}(3)$ and $g = w + o_{wt}(4)$, where $\hat{f}^{(3)}$ is a polynomial of weighted degree 3. We now need only apply Proposition 3.5 to $(\star)$. $\square$

Lastly, we finish the proof of our main theorem for this chapter.

Proof. Let F be as in the theorem, and assume that $N \leq 2n-2$. After the normalization, we may assume that F satisfies our normalization condition. Now, as in the fourth section, using the Heisenberg group structures of $\partial\mathbb{H}_n$ and $\partial\mathbb{H}_N$, for any $(z_0, w_0) \approx 0$ and $(C_\ell)_{(z_0,w_0)}$ as obtained before, we have a new twice differentiable map $F^{**}_{(z_0,w_0)}$ from M_1 into M_2. This map still satisfies our normalization. Applying Proposition 3.10, we get that $f^{**}_{(z_0,w_0)} = z + o_{wt}(3), g^{**}_{(z_0,w_0)} = w + o_{wt}(4)$ and $\phi^{**}_{(z_0,w_0)} = o_{wt}(2)$. Thus, we get from Theorem 4.2 that $F = (z, 0, w)$ $\square$

This completes the proof of the First Gap Theorem. We shall consider other cases in subsequent chapters.

Chapter 4

The boundary case of the First Gap Theorem

Recall that, in [DA88], D'Angelo demonstrated a family of mutually inequivalent polynomial proper embeddings in the case where $N = 2n$, which first showed that any general description of $\tilde{R}(n, N)$ would be considerably more complicated than the case where $N < 2n - 1$.

We note that Faran showed in [Fa82] that there four elements of $\tilde{R}(2, 3)$, so the case where $N = 2n - 1$, $n = 2$ is an old problem. We concern ourselves in this chapter with the $N = 2n - 1, n > 2$ case.

Our primary theorem is this:

4.1 Theorem (Theorem 1). *When $n \geq 3$, $Rat(n, 2n - 1)$ has exactly two equivalence classes. One is generated by the standard linear embedding*

$$L(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, 0, \ldots, 0, z_n),$$

and the other is generated by the Whitney Map

$$W(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_n z_1, z_n z_2, \ldots, z_n z_n).$$

More precisely, any rational proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ with $n > 2$ is equivalent to either the standard linear map $L(z)$ or the Whitney Map $L(z)$.

As shown in [HS], [Low], [Fo92], and [Hu01], we note that a major discovery in the 1980s was that the class $Prop(\mathbb{B}^n, \mathbb{B}^N)$ is much larger than the class $Rat(\mathbb{B}^n, \mathbb{B}^N)$. Thus, we now often find ourselves asking about the conditions under which an element of $Prop(\mathbb{B}^n, \mathbb{B}^N)$ is linear or rational. Applications of this principle show up in classical dynamics and geometry, as we can see in [Yue] or [LN]. We also see many studies, such as [Fo2], [Yue], [Hu01], and [Hu94], in which the interaction of the boundary regularity or the dynamical property of the maps with the rationality and the linearity was investigated. For example, we have a theorem of Forstneric [Fo89]: If $f : \mathbb{B}^n \to \mathbb{B}^N$ is a proper holomorphic map that is $(N - n + 1)$ times continuously differentiable up to the boundary, then f is rational. Work by Huang established that linearity and rationality hold for maps only assumed to be twice differentiable up to the boundary for $N < 2n - 1$. This allows us to reformulate Theorem 1 for proper maps only twice differentiable up to the boundary.

4.2 Theorem (Theorem 2)**.** *Let F be a proper holomorphic embedding from $\mathbb{B}^n$ to $\mathbb{B}^N$, $N = 2n - 1$, $n > 1$, twice differentiable up to the boundary. Then F is equivalent to the standard linear embedding $L(z)$.*

Note that the conclusion here is different from the conclusion of Theorem 1, and the hypothesis is that we are dealing with a proper holomorphic embedding in Theorem 2.

The proof of Theorem 1 takes some cues from the proof of the First Gap Theorem. First, we establish the Chern-Moser formal theory for the normalized map, and this gives us a sort of jet relation. In particular, we will prove the rank of a certain matrix of second derivatives of the map to be generically 1 if the map is not linear. This condition holds for $N = 2n - 1$, but it only gives a non-trivial and intrinsic restriction to the map for $n > 2$. Using the automorphism group of the balls, we use the rank to get a partial differential equation, which, unlike in the previous chapter, we cannot easily solve directly. However, we can use certain formal arguments to get some relations between the third derivatives of the map, and that relation will prove the existence of a "characteristic direction" along which the map is linear and the solutions must take a specific normal form. We then show that the degree of the map as in the theorem is at most 2, and this simplifies the normal form. From there, we can prove Theorem 1.

The formal consideration

First, we note that the notation in this chapter will be the same as in the previous chapter, particularly for $p_0 = (z_0, w_0)$, that of $F^{**}_{p_0}$, as well as $f^{(k)}$ for a homogeneous polynomial of weighted degree k, where we assign a weight of 1 to z and 2 to w.

We also assume that F is not linear.

Now, let $F = (f, \phi, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g) : \partial\mathbb{H}_n \to \partial\mathbb{H}_{2n-1}$ be a non-constant rational map that satisfies our normalization condition. We will show that F is equivalent to the Whitney Map.

We wish to normalize F to make it even simpler than before, and to this end, we first look at the functional equation $Im(g) = \left|\tilde{f}\right|^2$ from $w = Im(z)^2$. Because F^{**}_p is equivalent to F, we can, after replacing F by $F^{**}_{p_0}$ for a certain p_0, assume that $a^{(1)}, \phi^{(2)}$ are not identically 0. We can also assume that the $a^{(1)}$ and $\phi^{(2)}$ from $F^{**}_{p_0}$ do not vanish identically for any p_0 sufficiently close to 0. We write $a^{(1)}(z) = z \cdot A$. Then $\langle \overline{z}, a^{(1)}(z)\rangle = zA\overline{z}^t$ and $\overline{A}^t = A$.

4.3 Lemma (HJ01,Lemma 3.1)**.** *Assume that F is as above such that $a^{(1)}$ and $\phi^{(2)}$ are not identically 0. Then:*

(i) $Rank(A) = 1$.

(ii) F is equivalent to a map with satisfied our normalization condition with $a^{(1)}(z) = (z_1, 0, \ldots, 0)$ and $\phi^{(2)}_j = z_1 z_j, 1 \le j \le n-1$. Namely, F is equivalent to a map of the

form:

$$\begin{cases} f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \\ f_j = z_j + o_{wt}(3), 2 \le j \le n-1 \\ \phi_j = z_1 z_j + o_{wt}(2), 1 \le j \le n-1, \\ g = w + o_{wt}(4) \ near \ 0 \in \partial\mathbb{H}_n. \end{cases} \qquad (4.1.1)$$

Proof. To prove this lemma, we note that A is a semi-positive Hermitian matrix. Since A is not identically 0, there is a unitary matrix U such that

$$U^{-1}AU = diag(\kappa_1, \ldots, \kappa_k, 0, \ldots, 0)$$

with $\kappa_v > 0$ for $1 \le v \le k$. Replacing $F(z,w)$ by

$$(f(zU^{-1}, w) \cdot U, \phi(zU^{-1}, w), g(zU^{-1}, w)),$$

we can assume without loss of generality that A takes such a form.

Now, we claim that $k = 1$. In fact, write $\phi_j^{(2)} = \sum_{\alpha \le \beta} b_{\alpha\beta}^{(j)} z_\alpha z_\beta$. Then

$$\left|\phi_j^{(2)}(z)\right|^2 = \sum_{\alpha \le \beta} \left|b_{\alpha\beta}^{(j)}\right|^2 |z_\alpha|^2 |z_\beta|^2 + \sum_{(\alpha,\beta) \ne (\gamma,\ell), \alpha \le \beta, \gamma \le \ell} \left\{ b_{\alpha\beta}^{(j)} \overline{b_{\alpha\beta}^{(j)}} z_\alpha z_\beta \overline{z_\gamma z_\ell} \right\}.$$

Write $B_{\alpha\beta} = (b_{\alpha\beta}^{(1)}, \ldots, b_{\alpha\beta}^{(n-1)})$. Now we have

$$\left(\sum_{j=1}^{k} \kappa_j |z_j|^2 \right) |z|^2 \equiv \sum_{j=1}^{n-1} \left|\phi_j^{(2)}(z)\right|^2. \qquad (4.1.2)$$

We compare the coefficients of the $|z_1|^2 z_\ell \overline{z_\beta}$ terms, and we get $|B_{1\beta}|^2 > 0$ and $B_{1\ell} \cdot \overline{B_{1\beta}}^t = 0$ for all $\ell \ne \beta$. Thus, $\{B_{11}, \ldots, B_{1(n-1)}\}$ forms an orthogonal basis for $\mathbb{C}^{n-1}$. For $B_{s\ell}$ with $s \ne \ell$, we get from (4.1.2) that $B_{1\beta} \cdot \overline{B_{s\ell}}^t = 0$ for all β, and so $B_{s\ell} = 0$ for $s \ne 1$. Thus, $k = 1$, since z_1 appears on the right side of (4.1.2).

From all of this, we get that the left side of (4.1.2) becomes $\kappa_1 |z_1|^2 |z|^2$ and $\phi_j^{(2)} = z_1 \sum_\beta b_{1\beta}^{(j)} z_\beta = z_1 \tilde{B}_j \cdot z^t$, where $\tilde{B}_j = (b_{11}^{(j)}, \ldots, b_{1(n-1)}^{(j)})$.

Write $\begin{pmatrix} \phi_1^{(2)} \\ \dots \\ \phi_{n-1}^{(2)} \end{pmatrix} = z_1\tilde{B}\cdot z^t$, where $\tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \dots \\ \tilde{B}_{n-1} \end{pmatrix}$. Applying (4.1.2), we get $\tilde{B}\overline{\tilde{B}}^t = \kappa_1(id)$.

Replace (f,ϕ,g) with $(f,\phi\frac{\overline{\tilde{B}}}{\sqrt{\kappa_1}},g)$, and the new map takes the form

$$f_1 = z_1 + \frac{i}{2}\kappa_1 z_1 w + \dots, \phi_j = \sqrt{\kappa_1}z_1 z_j + \dots.$$

Next, replace it with

$$(z,w) \mapsto \left(\sqrt{\kappa_1}\tilde{f}\left(\frac{z}{\sqrt{\kappa_1}},\frac{w}{\sqrt{\kappa_1}}\right), \kappa_1 g\left(\frac{z}{\sqrt{\kappa_1}},\frac{w}{\sqrt{\kappa_1}}\right)\right).$$

Now, we have a new map equivalent to F with the form we want.

4.4 Remark (HJ01, Remark $3.1'$). Suppose that F is an arbitrary non-linear holomorphic map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ satisfying our normalization condition with $a^{(1)}$ and $\phi^{(2)}$ not identically 0. Then Lemma 4.3(i) gives us, for $N = 2n-1$,

$$Rank\left.\begin{pmatrix} (f_1)_{z_1 w} & \dots & (f_1)_{z_{n-1}w} \\ \vdots & \ddots & \vdots \\ (f_{n-1})_{z_1 w} & \dots & (f_{n-1})_{z_{n-1}w} \end{pmatrix}\right|_0 = 1.$$

Then, when $N < 2n-1$, the rank of the above matrix becomes 0, as is shown in [Hu1], and this led to the proof of the First Gap Theorem. For the case we consider now, this rank is 1, and this is the key fact that leads to our proof of Theorem 1. (Observe: $n = 2$ and the rank $= 1$ condition say that $f''_{zw}(0) \neq 0$, which holds for any non-linear map. That is, the rank condition only gives a crucial restriction for the map when $n > 2$.)

Now, assume F satisfies the normal form in (4.1.1).

We look at the functional equation

$$Im(g) = \left|\tilde{f}\right|^2$$

over $Im(w) = |z|^2$, as we did in some of the cases for smaller N and as we shall do again for higher cases. We collect terms of weighted degree 5.

$$Im\{g^{(5)} - 2i\overline{z}f^{(4)}\} = 2Re\sum_{j=1}^{n-1}\phi_j^{(2)}\overline{\phi_j^{(3)}}, \forall(z,w) \in \partial\mathbb{H}_n.$$

As we see whenever this technique appears, the sum of real parts comes because $(a+bi)(c-di)+(a-bi)(c+di) = 2ac+2bd$.

Let $z_1 = 0$, and then use (4.1.1) to get

$$Re\sum_{j=1}^{n-1}\phi_j^{(2)}\overline{\phi_j^{(3)}} \equiv 0. \qquad (4.1.3)$$

Using the Chern-Moser lemma (see [CM, Lemma 2.1] or [Hu99, Lemma 3.0]), we have $g^{(5)}(0,z',w) \equiv 0$, and $f_j^{(4)}(0,z',w) \equiv 0$ for $j > 1$, where $z = (z_1, z')$. Thus:

$$\begin{cases} f_1^{(4)} = a^{(4)}(z) + a^{(2)}(z)w + a_{02}w^2, \\ f_j^{(4)} = z_1(a_j^{(1)}(z)w + a_j^{(3)}(z)), 2 \le j \le n-1 \\ \phi_j^{(3)} = b_j^{(3)}(z) + b_j^{(1)}(z)w, 1 \le j \le n-1 \\ g^{(5)}z_1[c^{(2)}(z)w + c_{12}w^2 + c^{(4)}(z)]. \end{cases} \qquad (4.1.3')$$

4.5 Lemma (HJ01, Lemma 3.2)**.** *With the above notation, the following holds:*

$$a^{(2)}(z) = z_1\tilde{a^{(1)}}(z), a_j^{(1)}(z) = \frac{\mu}{2}z_j, b_1^{(1)}(z) = b_1z_1, b_j^{(1)} = b_jz_1 + \frac{\overline{\mu}}{2}z_j,$$

$$c^{(2)}(z) = 0, c^{(4)}(z) = 0, a^{(4)}(z) = 0, a_j^{(3)}(z) = 0,$$

where μ, b_1 and $b_j, j > 1$ are complex numbers and $\tilde{a^{(1)}}(z)$ is a linear function in z. Moreover,

$$\mu = 2c_{12} = 4i\overline{a_{02}} \qquad (4.1.4).$$

Proof. Substituting $(4.1.3')$ into (4.1.3), we get

$$Im\left\{g^{(5)} - 2i\overline{z}\cdot f^{(4)} - 2i\sum\phi_j^{(2)}\overline{\phi_j^{(3)}}\right\} \equiv 0,$$

or

$$Im\Big\{z_1[c^{(2)}(z)w+c_{12}w^2+c^{(4)}(z)]-2i\overline{z_1}[a^{(2)}(z)w+a_{02}w^2+a^{(4)}(z)]$$

$$-2i\overline{z'}\cdot[z_1(a^{(1)'}(z)w+a^{(3)'}(z))]-2iz_1^2\left(\overline{b_1^{(3)}(z)}+\overline{b_1^{(1)}(z)w}\right)-$$

$$-2iz_1\sum_{j\geq 2}z_j\left(\overline{b_j^{(3)}(z)}+\overline{b_J^{(1)}(z)}\right)\Big\}\equiv 0. \qquad (4.1.5)$$

Here, $a^{(k)'}=(a_2^{(k)},\dots,a_{n-1}^{(k)})$.

We parameterize w by $w=u+i\,|z|^2$ (another technique that is used for other Gap Theorem proofs), and we substitute this in the previous equation. We can then look at the coefficients of the u^2 terms to get

$$Im\,\{c_{12}z_1-2i\overline{z_1}a_{02}\}=0,\forall z_1,$$

which gives us $c_{12}=2i\overline{a_{02}}$. Collect the harmonic terms to get

$$c^{(4)}(z)=0.$$

Collect the coefficients of the u terms to get

$$Im\Big\{z_1^{(2)}(z)+wic_{12}z_1\,|z|^2-2i\overline{z_1}a^{(2)}(z)-2i\overline{z_1}a_{02}(2i\,|z|^2)-$$

$$-2iz_1\sum_{j\geq 2}a_j^{(1)}(z)\overline{z_j}-2iz_1\sum_{j\geq 1}z_j\overline{b_j^{(1)}(z)}\Big\}\equiv 0.$$

Thus,

$$c^{(2)}(z)\equiv 0,$$

and

$$\begin{aligned}2ia_1\textstyle\sum_{j\geq 1}z_j\overline{b_j^{(1)}(z)}+2iz_1\sum_{j\geq 2}a_j^{(1)}(z)\overline{z_j}+2ia^{(2)}(z)\overline{z_1}\\ \equiv 2ic_{12}z_1\,|z|^2-4\overline{a_{02}}z_1\,|z|^2.\end{aligned} \qquad (4.1.5').$$

Next, we collect the terms that look like $z^\alpha \overline{z}^\beta$, with $|\alpha| = 4$ and $|\beta| = 1$. Then we get $a^{(4)}(z) = 0$ and $a^{(3)'} = 0$.

Because the left-hand side divides z_1, we have $a^{(2)}(z) - z_1 a^{\tilde{(1)}}(z)$ for some $a^{\tilde{(1)}}(z)$. Write $\mu = 2c_{12} = 4ia_{02}$. Now

$$z_1 \overline{b_1^{(1)}(z)} + \sum_{j\geq 2} z_j \overline{b_j^{(1)}(z)} + \sum_{j\geq 2} a_j^{(1)} \overline{z_j} + a^{\tilde{(1)}}(z) \equiv \mu |z|^2 . \qquad (4.1.6)$$

Next, consider the $|z_1|^2, z_1\overline{z_i}(i > 1)$, and $z_j\overline{z_\ell}(\ell, j > 1)$ terms.

$$\mu \equiv \overline{\frac{\partial b_1^{(2)}(z)}{\partial z_1}} + \frac{\partial a^{\tilde{(1)}}(z)}{\partial z_1}, 0 \equiv \overline{\frac{\partial b_1^{(1)}(z)}{\partial z_i}} + \frac{\partial a_i^{(1)}(z)}{\partial z_1},$$

$$\overline{\frac{\partial b_j^{(1)}}{\partial z_\ell}} + \frac{\partial a_\ell^{(1)}}{\partial z_j} = \mu\delta_j^\ell. \qquad (4.1.7)$$

Collect the terms with no factor of u:

$$Im\left\{z_1 c_{12}(-|z|^4) - 2i\overline{z_1}\left(a^{(2)}(z)i|z|^2 - a_{02}|z|^4\right) - 2i\overline{z'}\cdot\left[z_1 a^{(1)'}(z)(i|z|^2)\right] - \right.$$

$$\left. -2iz_1^2\left(\overline{b_1^{(2)}(z)} - i|z|^2\overline{b_1^{(1)}(z)}\right) - 2iz_1\sum_{j\geq 2} z_j\left(\overline{b_j^{(3)}(z)} - i|z|^2\overline{b_j^{(3)}(z)}\right)\right\} \equiv 0.$$

Next, collect the terms that look like $a^\alpha\overline{z}^\beta$, where $|\alpha| = 3$ and $|\beta| = 2$:

$$(-c_{12}z_1 + 2i\overline{a_{02}}z_1)|z|^4 + 2z_1\overline{z_1}a^{\tilde{(1)}}(z)|z|^2 + 2z_1\overline{z'}\cdot a^{(1)}(z)|z|^2 - 2z_1^2\overline{b_1^{(1)}(z)}|z|^2 -$$

$$-2z_1\left(\sum_{j\geq 2} z_j\overline{b_j^{(1)}(z)}\right)|z|^2 - 2i\overline{z_1^2}b_1^{(3)}(z) - 2i\overline{z_1}\sum_{j\geq 2}\overline{z_j}b_j^{(3)}(z) \equiv 0.$$

Since we have $c_{12} = 2i\overline{a_{02}}$, this gives us

$$z_1|z|^2\left(\overline{z_1}a^{\tilde{(1)}}(z) + \overline{z'}\cdot a^{(1)}(z) - z_1\overline{b_1^{(1)}(z)} - \sum_{j\geq 2} z_j\overline{b_j^{(1)}(z)}\right)$$

$$\equiv i\overline{z_1}^2 b_1^{(3)}(z) + i\overline{z_1}\sum_{j\geq 2}\overline{z_j}b_j^{(3)}(z). \qquad (4.1.8)$$

If we just look at the $\overline{z_j z_\ell}$ terms of (4.1.8), we get

$$z_1 a_j^{(1)}(z) + z_j a_\ell^{(1)}(z) - z_\ell z_1 \overline{\frac{\partial b_1^{(1)}}{\partial z_j}} - z_j z_1 \overline{\frac{\partial b_1^{(1)}}{\partial z_\ell}}$$

$$-z_j z_\ell \overline{\frac{\partial b_\ell^{(1)}}{\partial z_\ell}} - z_\ell z_1 \overline{\frac{\partial b_j^{(1)}}{\partial z_j}} \equiv 0. \qquad (4.1.9)$$

Looking at just the $z_1 z_\ell$ terms of (4.1.9), with $\ell \neq 1$, we get

$$\frac{\partial a_j^{(1)}(z)}{\partial z_1} - \overline{\frac{\partial b_z^{(1)}}{\partial z_j}} \equiv 0.$$

Combine this with the second equation of (4.1.7):

$$\frac{\partial b_1^{(1)}(z)}{\partial z_j} \equiv \frac{\partial a_j^{(1)}(z)}{\partial z_1} \equiv 0.$$

Thus,

$$b_1^{(1)}(z) \equiv b_1 z_1 \ for\ some\ b_1 \in \mathbb{C},$$

and

$$a_j^{(1)}(z)\ has\ no\ z_1\ terms\ for\ 2 \leq j \leq n-1. \qquad (4.1.10)$$

Now, we look at the $\overline{z_j}^2$ terms of (4.1.8), with $j \neq 1$, getting

$$a_j^{(1)}(z) - z_1 \overline{\frac{\partial b_1^{(1)}(z)}{\partial z_j}} - \sum_{k=1}^{n-1} z_k \overline{\frac{\partial b_k^{(1)}(z)}{\partial z_j}} \equiv 0.$$

From this and $b_z^{(1)}(z) = b_1 z_1$, we get

$$a_j^{(1)}(z) - \sum_{k=2}^{n-1} z_k \overline{\frac{\partial b_k^{(1)}(z)}{\partial z_j}} \equiv 0. \qquad (4.1.11)$$

Combine this with the third equation of (4.1.7) to get that $a_j^{(1)}(z)$ has only a z_j terms for $2 \leq j \leq n-1$. Also, $\frac{\partial a_j^{(1)}}{\partial z_j} = \overline{\frac{\partial b_j^{(1)}}{\partial z_j}}$, for $j \neq 1$.

Now, we use (4.1.11) and the third equation of (4.1.7) to get

$$a_j^{(1)}(z) = \frac{\mu}{2} z_j, 2 \leq j \leq n-1. \qquad (4.1.12)$$

Now we can combine (4.1.7), (4.1.12), and (4.1.11) to get

$$b_j^{(1)}(z) = b_j z_1 + \frac{\mu}{2} z_j, 2 \leq j \leq n-1,$$

where $b_j \in \mathbb{C}$, and this completes the proof. □

An application of the group structure of the Heisenberg Hypersurface

In this section, we wish to show that $c_{03} = |c_{12}|^2$, which we will use in the next section to show that we can compose F with certain automorphisms to make $g(0,w) \equiv w$. We do this by using several earlier results to get a system of differential equations, and then we compare the coefficients of the u and $|z_\ell|^2$ terms.

4.6 Lemma (Hu01, Lemma 4.1)**.** *Let F be as in (4.1.1). Write $c_{03} = \frac{1}{6}\frac{\partial^3 g}{\partial w^3}\Big|_0$ and $c_{12} = \frac{1}{2}\frac{\partial^3 g}{\partial z_1 \partial w^2}\Big|_0$. Then $c_{03} = |c_{12}|^2$.*

Proof. From earlier results (see [HJ01,(2.3)]), we have

$$\frac{\partial^2 (f_j^{**})_p}{\partial z_\ell \partial w}\Big|_0 = \frac{1}{\lambda_p} (\tilde{f}_p)''_{wz_\ell}\Big|_0 \cdot \overline{E_j}^t_p - \frac{1}{\lambda_p^2}((E_w)_p \cdot \overline{(E_\ell)_p}^t (g_p)''_{wz_j}\Big|_0 -$$

$$-\delta_\ell^j \left(\frac{i\,|(E_w)_p|^2}{\lambda_p} + \frac{1}{2\lambda_p} Re\left\{ (g_p)''_{ww}(0) \right\} \right).$$

We write $P_\ell^j = \frac{\partial^2 (f_j)^{(**)}_{z,w}}{\partial z_\ell \partial w}\Big|_0$. Using formulas from [Hu99], we have

$$2\lambda P_\ell^j = 2L_\ell(\tilde{f}_w) \cdot \overline{L_j(\tilde{f})}^t - \frac{2}{\lambda}\left(\tilde{f}'_w \cdot \overline{L_\ell(\tilde{f})}^t \right) \cdot L_j\left(g'_w - 2i\tilde{f}'_w \cdot \overline{\tilde{f}}^t \right) -$$

$$-2i\delta_\ell^j \left|\tilde{f}_w'\right|^2 - \delta_\ell^j Re\{g_{ww}''\} + 2\delta_\ell^j Re\{i\tilde{f_{ww}''} \cdot \overline{\tilde{f}}^t\}, \qquad (4.2.1)$$

where $p = (z, w) \in \partial\mathbb{H}_n$ and $\lambda = \lambda(p) = \lambda_p$. is a smooth positive function in p.

Now, we apply T^2 to our basic functional equation, where $T = \frac{\partial}{\partial u}$ is the real tangent vector field along $\partial\mathbb{H}_n$ and $w = u + i|z|^2$, yielding

$$0 = 2iIm\left\{i\tilde{f}_{ww}'' \cdot \overline{\tilde{f}}^t\right\} + 2i\left|\tilde{f}_w'\right|^2 - iIm\left\{g_{ww}''\right\}, \forall (z, w) \in \partial\mathbb{H}_n.$$

We multiply by δ_ℓ^j and then add this to (4.2.1):

$$\begin{aligned} 2\lambda P_\ell^j \;&=\; 2L_\ell(\tilde{f}_w{}') \cdot \overline{L_j(\tilde{f})}^t - \tfrac{2}{\lambda}\left(\tilde{f}_w' \cdot \overline{L_\ell(\tilde{f})}^t\right) \cdot L_j\left(g_w' - 2i\tilde{f}_w{}' \cdot \overline{\tilde{f}}^t\right) - \\ &\quad \delta_\ell^j\left(g_{ww}'' - 2i\tilde{f_{ww}''} \cdot \overline{\tilde{f}}^t\right), \end{aligned}$$

for $p = (z, w) \in \partial\mathbb{H}_n$. We now apply L_jT to $Im(g) = \left|\tilde{f}\right|^2$:

$$L_j\left(g_w' - 2i\tilde{f}_2{}' \cdot \overline{\tilde{f}}^t\right) = 2iL_j(\tilde{f}) \cdot \overline{\tilde{f}_w'}^t, \forall (z, w) \in \partial\mathbb{H}_n.$$

Next, we write

$$\lambda^* = 2i\tilde{f_{ww}''} \cdot \overline{\tilde{f}}^t - g_{ww}''. \qquad (4.2.1')$$

By both (4.1.1) and Lemma 4.3, $\lambda = 1 + |z_1|^2 + o_{wt}(2)$. Thus,

$$2\lambda P_\ell^j = 2L_\ell(\tilde{f}_w{}') \cdot \overline{L_j(\tilde{f})}^t - 4i\left(\tilde{f}_w' \cdot \overline{L_\ell(\tilde{f})}^t\right)\left(L_j(\tilde{f}) \cdot \overline{\tilde{f}_w'}^t\right) + \delta_\ell^j\lambda^* + o_{wt}(2). \qquad (4.2.2)$$

Now, we write $q_\ell^j = 2\lambda P_\ell^j - \delta_\ell^j\lambda^*$, and we have

$$q_\ell^j = 2L_\ell(\tilde{f}_w{}') \cdot \overline{L_j(\tilde{f})}^t - 4i\left(\tilde{f}_w' \cdot \overline{L_\ell(\tilde{f})}^t\right)\left(L_j(\tilde{f}) \cdot \overline{\tilde{f}_w'}^t\right) + o_{wt}(2) \qquad (4.2.3)$$

and

$$2\lambda P_\ell^j = q_\ell^j + \delta_\ell^j\lambda^*. \qquad (4.2.4)$$

By Lemma 4.3, applied to F_p^{**}, we have $Rank(P_\ell^j) \equiv 1$. Thus

$$\frac{P_1^1}{P\ell^1} = \frac{P_1^\ell}{P_\ell^\ell}, 2 \le \ell \le n-1,$$

or

$$q_\ell^\ell + \lambda^* = \frac{q_\ell^1}{q_1^1 + \lambda^*} q_1^\ell, 2 \le \ell \le n-1.$$

This is where we need $n \geq 3$.

This gives us the following second order partial differential equations:

$$(\lambda^*)^2 + \lambda^*(q_1^1 + q_\ell^\ell) + (q_1^1 q_\ell^\ell - q_\ell^1 q_1^\ell) \equiv 0, 2 \le \ell \le n-1, (z,w) \in \partial\mathbb{H}_n. \quad (4.2.5)$$

We analyze this equation by fixing ℓ and only considering the u and $|z_\ell|^2$ terms of the Taylor expansion of the left side of (4.2.5). This will be enough to prove our lemma.

We combine (4.1.1),(4.1.3′), and Lemma 4.5 to get

$$f_1 = z_1 + \frac{i}{2} z_1 w + a_{02} w^2 + z_1 \tilde{a^{(1)}}(z) w + \tilde{a_1^{(1)}}(z) w^2 + \tilde{a_1^{(3)}}(z) w + o_{wt}(5),$$

and

$$f_j = z_j + \frac{\mu}{2} z_1 z_j w + z_1 \tilde{a^{(1)}}(z) w + \tilde{a_j^{(3)}}(z) w + o_{wt}(5), j > 1.$$

An important technique that will show up again in the proof of the Third Gap Theorem is that we can see that f does not have any degree 5 terms in z by looking at the weighted 6^{th} degree terms of $Im(g) = \left|\tilde{f}\right|^2$:

$$Im\{g^{(6)} - 2i\overline{z}f^{(5)}\} = 2Re\left(\sum_j (\phi_j^{(2)}\overline{\phi_j^{(4)}} + \frac{1}{2}\phi_j^{(3)}\overline{\phi_j^{(3)}})\right).$$

We now include some convenient notation.

Let $A, B, C_1, \ldots, C_m$ be functions.

Then we will say that $A = B \mod (C_1, \ldots, C_m)$ if $A = B + \sum_{j=1}^m k_j C_j$ for constants k_j.

For real analytic functions $H(Z, \overline{Z})$ and $Q(Z, \overline{Z})$ near 0, we write

$$H = Q \mod (\textit{terms other than } Z^{\alpha_1}, \overline{Z^{\beta_1}}, \ldots, Z^{\alpha_k}\overline{Z^{\beta_k}})$$

if

$$\left.\frac{\partial^{|\alpha_m|+|\beta_m|}(H-Q)}{\partial Z^{\alpha_m}\partial \overline{Z}^{\beta_m}}\right|_0 = 0 \ for \ 1 \leq m \leq k.$$

We also note that we can parameterize $\partial\mathbb{H}_n$ by $z = x + iy$ and $w = u + i\,|z|^2$.

Combining (4.1.1),(4.1.3$'$), Lemma 4.3, and Lemma 4.5, we get:

$$\begin{aligned}
f_1 &= z_1 + \tfrac{i}{2} z_1 w + a_{02} w^2 + z_1 a^{\widetilde{(1)}}(z) w + a_1^{\widetilde{(1)}}(z) w^2 + a_1^{\widetilde{(3)}}(z) w + \\
&\quad o_{wt}(5) \cap o(|(z,w)|^4), \quad \mathrm{mod}\ (z_s z_t w^2, z_s w^3, w^3, w^4), \\
f_j &= z_j + \tfrac{\mu}{2} z_1 z_j w + a_j^{\widetilde{(1)}}(z) w^2 + a_j^{\widetilde{(3)}}(z) w + o_{wt}(5) \cap o(|(z,w)|^4), \\
&\qquad \mathrm{mod}\ (z_s z_t^2, z_s w^3, w^3, w^4), j \geq 2, \\
\phi_1 &= z_1^2 + b_1 z_1 w + b_z^{(0)} w^2 + b_z^{(3)}(z) + o(|(z,w)|^2) \cap o_{wt}(3), \\
\phi_j &= z_1 z_j + \left(b_j z_1 + \tfrac{\overline{\mu}}{2} z_j\right) w + b_j^{(0)} w^2 + b_j^{(2)}(z) w + b_j^{(3)}(z) \\
&\quad + o_{wt}(3) \cap o(|(z,w)|^2), j \geq 2, \\
g &= w + c_{12} z_1 w^2 + c_{03} w^3 + o_{wt}(5) \cap o(|(z,w)|^3).
\end{aligned}$$

As before, $X^{(k)}$ is a polynomial of degree k.

Thus,

$$
\begin{aligned}
(f_1)'_w &= \tfrac{i}{2}z_1 + 2a_{01}w + z_1\tilde{a^{(1)}} + 2\tilde{a_1^{(1)}}w + \tilde{a_1^{(3)}} + o(|(z,w)|^3), \\
&\qquad \mathrm{mod}\ (z_s z_t w, z_s w^2, w^2, w^3), \\
(f_1)''_{ww} &= 2a_{02} + 2\tilde{a_1^{(1)}}(z) + o(|(z,w)|^2), \\
&\qquad \mathrm{mod}\ (z_s z_t, z_s w, w, w^2), \\
(f_j)'_w &= \tfrac{\mu}{2}z_1 z_j + 2\tilde{a_j^{(1)}}(z)w + \tilde{a_j^{(3)}}(z) + o(|(z,w)|^3), \\
&\qquad \mathrm{mod}\ (z_s z_t w, z_s w^2, w^2, w^3), \\
(f_j)''_{ww} &= 2\tilde{a_j^{(1)}}(z) + o(|(z,w)|^2), \\
&\qquad \mathrm{mod}\ (z_s z_t, z_s w, w, w^2), \\
(\phi_1)'_w &= b_1 z_1 + 2b_1^{(0)}w + o(|(z,w)|), \\
(\phi_1)''_{ww} &= 2b_1^{(0)} + o(1), \\
(\phi_j)'_w &= b_j z_1 + \tfrac{\overline{\mu}}{2}z_j + 2b_j^{(0)}w + b_j^{(2)}(z) + o_{wt}(2), \\
(\phi_j)''_{ww} &= 2b_j^{(0)} + o(1), \\
(g_{ww})'' &= 2c_{12}z_1 + 6c_{03}w + o(|(z,w)|^2), \\
&\qquad \mathrm{mod}\ (z_s z_j, z_s w, w^2),
\end{aligned}
$$

with $2 \le j \le n-1$.

Thus,

$$
\begin{aligned}
\lambda^* &= 2i\tilde{f''_{ww}} \cdot \overline{\tilde{f}}^t - g''_{ww} = 4i\frac{\partial \tilde{a_\ell^{(1)}}}{\partial z_\ell}|z_\ell|^2 - 6c_{03}w, \\
&\qquad \mathrm{mod}\ (\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u).
\end{aligned} \tag{4.2.6}
$$

Thus,

$$
(\lambda^*)^2 = 0 \quad \mathrm{mod}\ (\textit{terms other than } |z_\ell|^2, u). \tag{4.2.7}
$$

We compute q_1^1 directly:

$$L_1(f_1) = 1 + \frac{i}{2}w$$
$$L_1((f_1)'_w) = \frac{i}{2} + \frac{\partial \tilde{a^{(1)}}(z)}{\partial z_\ell} z_\ell + 2\frac{\partial \tilde{a_1^{(1)}}(z)}{\partial z_1} w$$
$$L_1(f_j) = 0$$
$$L_1((f_j)'_w) = \frac{\mu}{2}\delta_\ell^j z_\ell + 2\frac{\partial \tilde{a_j^{(1)}}(z)}{\partial z_1} w, \quad \text{mod } (\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u),$$
$$L_1(\phi_1) = b_1 w$$
$$L_1((\phi_1)'_w) = b_1 + o(1)$$
$$L_1(\phi_j) = z_j + b_j w$$
$$L_1((\phi_j)'_w) = b_j + \frac{\partial b_j^{(2)}(z)}{\partial z_1} + o_{wt}(1),$$

with $2 \leq j \leq n-1$.

Thus,

$$\begin{aligned} q_1^1 &= 2L_1(\tilde{f}'_w) \cdot \overline{L_1(\tilde{f})}^t \\ &= 4i\left|\tilde{f}'_w \cdot \overline{L-1(\tilde{f})}^t\right|^2 \\ &= i + 2\frac{\partial \tilde{a^{(1)}}(z)}{\partial z_\ell} z_\ell + 4\frac{\partial \tilde{a_1^{(1)}}(z)}{\partial z_1} w + \frac{1}{2}\overline{2} + \\ &\quad 2\sum_{j=1}^{n-1} |b_j|^2 \overline{w} + 2b_\ell \overline{z_\ell} + 2\frac{\partial^2 b_\ell^{(2)}(z)}{\partial z_1 \partial z_\ell} |z_\ell|^2, \\ &\qquad \text{mod } (\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u). \end{aligned} \tag{4.2.8}$$

We calculate q_ℓ^ℓ directly:

$$\begin{aligned} L_\ell(f_j) &= \delta_\ell^j, 1 \leq j \leq n-1, \\ L_\ell((f_1)'_w) &= 4ia_{02}\overline{z_\ell} + 4i\frac{\partial \tilde{a_1^{(1)}}(z)}{\partial z_\ell} |z_\ell|^2 + 2\frac{\partial \tilde{a_1^{(1)}}}{\partial z_\ell} w, \\ L_\ell((f_j)'_w) &= 4i\frac{\partial \tilde{a_j^{(1)}}}{\partial z_\ell} |z_\ell|^2 + 2\frac{\partial \tilde{a_j^{(1)}}}{\partial z_\ell} w, j \neq 1, \\ &\qquad \text{mod } (\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u), \\ L_\ell(\phi_1) &= o_{wt}(1), \\ L_\ell(\phi_k) &= \delta_k^\ell i\overline{\mu} |z_\ell|^2 + \frac{\overline{\mu}}{2}\delta_k^\ell w, \\ L_\ell((\phi_1)'_w) &= o(1), \\ L_\ell((\phi_k)'_w) &= \frac{\overline{\mu}}{2}\delta_k^\ell + 4ib_k^{(0)}\overline{z_\ell} + \frac{\partial b_k^{(2)}(z)}{\partial z_\ell} + o_{wt}(1), \end{aligned}$$

with $2 \leq k \leq n-1$. We get

$$\begin{aligned}
q_\ell^\ell &= 2L-\ell(\tilde{f}'_w)\cdot\overline{L_\ell(\tilde{f})}^t - 4i\left|\tilde{f}'_w\cdot\overline{L_\ell(\tilde{f})}^t\right|^2 \\
&= 2\left(4i\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}|z_\ell|^2+2\frac{\partial a_\ell^{\tilde{(1)}}}{\partial z_\ell}\right)+ \\
&\quad 2\left(\frac{\mu}{2}+4ib_1^{(0)}\overline{z_\ell}+\frac{\partial b_\ell^{(2)}(z)}{\partial z_\ell}\right)\overline{\left(i\overline{\mu}\,|z_\ell|^2+\frac{\mu}{2}w\right)}+o_{wt}(2) \qquad (4.2.9)\\
&= 8i\frac{\partial a_\ell^{\tilde{(1)}}}{\partial z_\ell}|z_\ell|^2+4\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}w-i|\mu|^2|z_\ell|^2+\frac{|\mu|^2}{2}\overline{w}, \\
&\qquad \bmod (\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u).
\end{aligned}$$

Thus, from (4.2.8), we have

$$q_1^1 q_\ell^\ell = -8\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}|z_\ell|^2+4i\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}w+|\mu|^2|z_\ell|^2+\frac{i}{2}|\mu|^2\overline{w}, \qquad (4.2.10)$$

$\bmod$ ($\textit{terms other than } 1, z_\ell, \overline{z_\ell}, |z_\ell|^2, u$).

Next, we get

$$\begin{cases} q_1^\ell = \mu z_\ell, \quad \bmod (\textit{terms other than } 1, z_\ell, \overline{z_\ell}), \\ q_\ell^1 = 8ia_{02}\overline{z_\ell}+\overline{\mu z_\ell}, \quad \bmod (\textit{terms other than } 1, \overline{z_\ell}). \end{cases} \qquad (4.2.11)$$

By Lemma 4.5, $a_{02} = \frac{i\overline{\mu}}{4}$, so from this we get

$$q_1^\ell q_\ell^1 = -|\mu|^2|z_\ell|^2, \quad \bmod (\textit{terms other than } |z_\ell|^2, u). \qquad (4.2.12)$$

We combine (4.2.6), (4.2.8), and (4.2.9):

$$\begin{aligned}
\lambda^*(q_1^1+q_\ell^\ell) &= -i6c_{03}w-4\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}|z_\ell|^2, \qquad (4.2.13)\\
&\qquad \bmod (\textit{terms other than } |z_\ell|^2, u).
\end{aligned}$$

We combine (4.2.7),(4.2.10),(4.2.11),(4.2.12), and (4.2.13), and we write the u and $z_i\overline{z_j}$ terms in (4.2.5), yielding the following:

$$-i6c_{03}w-4\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}|z_\ell|^2-8\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}|z_\ell|^2+4i\frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_e ll}w+|\mu|^2|z_\ell|^2+$$

$$+\frac{i}{2}|\mu|^2\overline{w}-(-|\mu|^2|z_\ell|^2)\equiv 0, \quad \bmod (\textit{terms other than } |z_\ell|^2, u). \qquad (4.2.14)$$

Collect the u terms of (4.2.14) to get

$$-6ic_{03} + 4i\gamma + \frac{i}{2}|\mu|^2 = 0,$$

with $\gamma = \frac{\partial a_\ell^{\tilde{(1)}}(z)}{\partial z_\ell}$.

Thus,

$$12c_{03} = 8\gamma + |\mu|^2. \qquad (4.2.15)$$

Collect the $|z_\ell|^2$ terms of (4.2.14):

$$-6c_{03}(i) - 4\gamma - 8\gamma + 4i\gamma(i) + |\mu|^2 + \frac{i}{2}|\mu|^2(-i) + |\mu|^2 = 0. \qquad (4.2.16)$$

We combine (4.15), (4.16), and $\mu = 2c_{12}$ to get

$$c_{03} = \frac{1}{4}|\mu|^2 = |c_{12}|^2.$$

□

Linear direction and degree estimate

Our normalization (4.1.1) gives us $g = w + o_{wt}(4)$. We recall that if we also have $c_{03} = 0$, then we get $g(0, w) \equiv w$ through an argument from Burns-Krantz [BK]. In this section, we will show that replacing F by a certain $\hat{F}_c$ in the same equivalence class allows us to have something that still satisfies (4.1.1) but for which $c_{03} = 0$. In particular, we will show that $\hat{F}_c$ has degree 2 along Segre variety Q_0, and by looking at F_p^{**} instead of F, we will get that F has degree 2 when restricted to Q_p. Thus, $\deg(F) = 2$. We will apply some of this section in Lemma 4.11 later, and that will show that F is linear when restricted to a certain subspace, and this will help us prove Theorem 1.

First, then, we must define $\hat{F}_c$.

Let $p_1 = -\overline{c_{12}}, p = (p_1, 0') \in \mathbb{C}^{n-1}$, and $(p, 0') \in \mathbb{C}^{2n-2}$. Let $\hat{F}_c = \tau_c \circ F \circ \sigma_c$, where

$$\sigma_c(z, w) = \left(\frac{z + wp}{1 - 2i\overline{p_1}z_1 - i|p_1|^2 w}, \frac{w}{1 - 2i\overline{p_1}z_1 - i|p_1|^2 w}\right), \qquad (4.3.1)$$

and

$$\tau_c(z^*,w^*) = \left(\frac{z^* - w^*(p,0')}{1+2i\overline{p_1}z_1^* - i\left|p_1\right|^2 w^*}, \frac{w^*}{1+2i\overline{p_1}z_1^* - i\left|p_1\right|^2 w^*}\right). \qquad (4.3.2)$$

4.7 Lemma (HJ01, Lemma 5.1)**.** *Let F be as in (4.1.1) and $\hat{F}_c = \tau_c \circ F \circ \sigma_c = (\hat{f}_c, \hat{\phi}_c, \hat{g}_c) = (\tilde{\hat{f}}_c, \hat{g}_c) = (\hat{f}_{1,c}, \ldots, \hat{\phi}_{(n-1),c}, \hat{g}_c)$ be defined as above, with $n > 2$. Then $\hat{F}_c$ satisfies (4.1.1). Moreover, $\left.\frac{\partial^3 \hat{g}_c}{\partial w^3}\right|_0 = 0$.*

We can see directly that

$$\hat{g}_c = w + o(|(z,w)|), \hat{f}_c = z + o(|(z,w)|), \hat{\phi}_c = o(|(z,w)|).$$

Proof. By [Hu1,Lemma 5.3], to show that $\hat{F}_c$ satisfies our older normalization condition, we need only show that $\left.Re\left(\frac{\partial^2 \hat{g}_c}{\partial w^2}\right)\right|_0 = 0$, since $\left.\frac{\partial^2 \hat{g}_c}{\partial z_k \partial z_\ell}\right|_0 = 0$ already holds.

We calculate the coefficients of w^2 and w^3 in the expansion of $\hat{g}_c$. To do this, let $z=0$, then consider

$$\hat{g}_c(0,w) = \frac{g\left(\frac{p_1 w}{1-i|p_1|^2 w}, 0, \frac{w}{1-i|p_1|^2 w}\right)}{1+2i\overline{p_1}f_1\left(\frac{p_1 w}{1-i|p_1|^2 w}, 0, \frac{w}{1-i|p_1|^2 w}\right) - i\left|p_1\right|^2 g\left(\frac{p_1 w}{1-i|p_1|^2 w}, 0, \frac{w}{1-i|p_1|^2 w}\right)}.$$

Then, since $f_1(z_1,0,w) = z_1 + \frac{i}{2}z_1 w + a_{02}w^2 + z_1 a^{(1)}(\tilde{z}_1, 0)w + o_{wt}(5)$ and $g(z_1,0,w) = w + c_{12}z_1 w^2 + c_{03}w^3 + o(|(z,w)|^3)$, we get, after using the geometric series formula to find the Taylor expansion,

$$\hat{g}_c(0,w) = \left(\frac{w}{1-i\left|p_1\right|^2 w} + c_{12}p_1 w^3 + c_{03}w^3\right) \times$$

$$\times\left[1 - 2i\overline{p_1}\left(\frac{p_1 w}{1-i\left|p_1\right|^2 w} + \frac{i}{2}\frac{p_1 w^2}{1-i\left|p_1\right|^2 w} + a_{02}w^2\right) + \right.$$

$$\left. + i\left|p_1\right|^2 \frac{w}{1-i\left|p_1\right|^2 w} + \left(-2i\overline{p_1}\frac{p_1 w}{1-i\left|p_1\right|^2 w} + i\left|p_1\right|^2 w\right)^2\right] + o(w^3)$$

$$= (w + i\left|p_1\right|^2 w^2 + (-\left|p_1\right|^4 + c_{12}p_1 + c_{03})w^3)\times$$

$$\times\left\{1-2i\overline{p_1}\Big[p_1w(1+i\left|p_1\right|^2w)+\frac{i}{2}p_1w^2+a_{02}w^2\Big]+i\left|p_1\right|^2\Big[w(1+i\left|p_1\right|^2w)\Big]+\right.$$

$$\left.+\Big[-2i\left|p_1\right|^2w(1+i\left|p_1\right|^2w)+i\left|p_1\right|^2w\Big]^2\right\}+o(w^3)$$

$$=w+\left(i\left|p_1\right|^2-2i\left|p_1\right|^2+i\left|p_1\right|^2\right)w^2+\Big(-\left|p_1\right|^4+c_{12}p_1+c_{03}+$$

$$+2\left|p_1\right|^4-\left|p_1\right|^4+2\left|p_1\right|^4+\left|p_1\right|^2-2ia_{02}\overline{p_1}-\left|p_1\right|^4-\left|p_1\right|^4\Big)\times$$

$$\times w^3+o(w^3)=w+o(w^3).$$

We use

$$p_1=-\overline{c_{12}}=2ia_{02}$$

and

$$c_{03}=\left|c_{12}\right|^2=\left|p_1\right|^2.$$

This shows that our normalization holds.

Now, we notice that

$$\hat{\phi}_{c,j}(z,0)=\frac{\phi_j\left(\frac{z}{1-2i\overline{p_1}z_1},0\right)}{1+2i\overline{p_1}f_1\left(\frac{z}{1-2i\overline{p_1}z_1},0\right)-i\left|p_1\right|^2g\left(\frac{z}{1-2i\overline{p_1}z_1},0\right)}.$$

Then, it follows from (4.1.1) that

$$(\hat{\phi}_j)_c=z_1z_j+o_{wt}(2).$$

By a similar argument,

$$\hat{f}_{1,c}=z_1+\frac{i}{2}z_1w+o_{wt}(3)$$

and

$$\hat{f}_{1,j}=z_j+o_{wt}(3),j\geq 2.$$

This is enough to show that $\hat{F}_c$ satisfies (4.1.1) and $(\hat{g}_c)'''_{www}(0)=0$, which completes the proof of Lemma 4.7. □

Because $Im(\hat{g}_c(0,w)) \geq 0$ for $Im(w) \geq 0$ and $\hat{g}_c(0,w) = w + o(|w|^3)$, we use a result from Burns-Krantz (a version of the Hopf lemma, seen in [BK]) to get $\hat{g}_c(0,w) \equiv w$. In particular, consider the harmonic function

$$h(w) = Im\left(\frac{1}{w} - \frac{1}{\hat{g}_c(0',w)}\right)$$

over $\mathbb{H}^+ = \{w \in \mathbb{C} : Im(w) > 0\}$. We can see that $h(w) = o(|w|)$ as $w \to 0$, and $\liminf_{w \to x} h(w) \geq 0$, so 0 is the minimum value for $h(w)$. By the Hopf lemma, then, $h(w) \equiv 0$. Thus, $\hat{g}_c(0',w) \equiv w$, and because $Im(\hat{g}_c(0,w)) = \left|\tilde{\hat{f}}_c(0,w)\right|^2$ for $Im(w) = 0$, we get that $\tilde{\hat{f}}_c(0,w) \equiv 0$. Thus, for $\hat{F}_c$, we have the following:

$$\begin{cases} \hat{F}_c(0,w) = (0,w) \\ \hat{f}_{1,c} = z_1 + \frac{i}{2} z_1 w + z_1 \tilde{a}^{(1)}(z) w + o_{wt}(4) \\ \hat{f}_{\ell,c} = z_\ell + o_{wt}(4), 2 \leq \ell \leq n-1 \\ \hat{\phi}_{j,c} = z_1 z_j + b_j z_1 w + b_j^{(3)}(z) + o_{wt}(3), 1 \leq j \leq n-1 \\ \hat{g}_c = w + o(|(z,w)|^3). \end{cases} \tag{4.3.3}$$

This uses the fact that for $c_{03} = \frac{1}{6}(\hat{g}_c)'''_{www}(0)$, $c_{12} = (\hat{g}_c)'''_{wwz_1}(0)$, and $a_{02} = \frac{1}{2}(\hat{f}_{1,c})''_{ww}(0)$, we get $|c_{12}|^2 = c_{03} = 0$ and $2i\overline{a_{02}} = c_{12} = 0$.

The remainder of this section is an important technique we use in the proof of the boundary case of the Second Gap Theorem, and in fact, it was the inspiration for this later work and, indeed, this dissertation. For the rest of this section, we shall endeavor to show that the degree of F is 2, through the application of Segre family theory.

First, we write $\mathcal{L}_j$ for the complexification of L_j. In particular,

$$\mathcal{L}_j = 2i\overline{\xi_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}.$$

Then $\{\mathcal{L}_1, \ldots, \mathcal{L}_j\}$ are a basis for the holomorphic tangent space of the Segre variety

$$Q_{(\xi,\eta)} = \left\{(z,w) : \frac{w - \overline{\eta}}{2i} = \sum z_j \overline{\xi_j}\right\}, \forall (\xi,\eta).$$

4.8 Lemma (5.2). *When restricted to $Q_0 = \{w = 0\}$, the complex tangent space of $\partial\mathbb{H}_n$ at 0, $\hat{F}_c = (f_{j,c}, \phi_{j,c}, \hat{g}_c)$ has degree 2. In particular, $\hat{F}_c(z,0) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials in z with $\deg(P), \deg(Q) \le 2$.*

Proof. Along any $Q_{(\xi,\eta)}$, we have

$$\frac{\hat{g}_c(z,w) - \overline{\hat{g}_c(\xi,\eta)}}{2i} = \sum_{\ell=1}^{n-1} \hat{f}_{\ell,c}(z,w)\overline{\hat{f}_{\ell,c}(\xi,\eta)} + \sum_{\ell=1}^{n-1} \hat{\phi}_{\ell,c}(z,w)\overline{\phi_{\ell,c}(\xi,\eta)}. \quad (4.3.4)$$

We apply $\mathcal{L}_j$ and $\mathcal{L}_1\mathcal{L}_j$ to the above, using the facts we obtained in (4.3.3) and letting $(z,w) = 0, \eta = 0$. We get

$$\begin{pmatrix} \overline{\xi_1} \\ \dots \\ \overline{\xi_{n-1}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ A_{(n-1)\times(n-1)} & B_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} \overline{\hat{f}_c(\xi,0)} \\ \overline{\hat{\phi}_c(\xi,0)} \end{pmatrix}.$$

Here, I is the identity matrix, and

$$A_{(n-1)\times(n-1)} = \begin{pmatrix} -2i\overline{xi_1} & 0 & \dots & 0 \\ -\overline{\xi_2} & 0 & \dots & 0 \\ \dots & 0 & \dots & 0 \\ -\overline{\xi_{n-1}} & 0 & \dots & 0 \end{pmatrix},$$

and

$$B_{(n-1)\times(n-1)} = \begin{pmatrix} 2 + 4ib_1\overline{\xi_1} & 4ib_2\overline{\xi_1} & \dots & 4ib_{n-1}\overline{\xi_1} \\ 2ib_1\overline{\xi_2} & 1 + 2ib_2\overline{\xi}_2 & \dots & 2ib_{n-1}\overline{\xi_2} \\ \dots & \dots & \dots & \dots \\ 2ib_1\overline{\xi_{n-1}} & 2ib_2\overline{\xi_{n-1}} & \dots & 1 + 2ib_{n-1}\overline{\xi_{n-1}} \end{pmatrix}.$$

In condensed form, this is

$$\overline{\tilde{\hat{f}}_c(\xi,0)}^t = C^{-1} \begin{pmatrix} \overline{\xi}^t \\ 0 \end{pmatrix} \ where\ C = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}, \overline{\xi} = (\overline{\xi_1}, \dots, \overline{\xi_{n-1}}).$$

Since we know C, we get that

$$C^{-1} = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix}.$$

So,

$$\overline{\tilde{\hat{f}}_c(\xi,0)}^t = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} \overline{\xi}^t \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\xi}^t \\ -B^{-1}A\overline{\xi}^t \end{pmatrix}.$$

Thus, we must study $B^{-1}A\overline{\xi}^t$. First, we split B up, writing it as $B = D + \tilde{B}$, where

$$D = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \; and \; \tilde{B} = \begin{pmatrix} 4i\overline{\xi_1} \\ 2i\overline{\xi_2} \\ \dots \\ 2i\overline{\xi_{n-1}} \end{pmatrix} (b_1, b_2, \dots, b_{n-1}).$$

Now, $B^{-1} = (D+\tilde{B})^{-1} = (I+B^*)^{-1}D^{-1}$, where $B^* = 2i\overline{\xi}^t \cdot b$. We notice that $B^{*2} = (2i)\ell(\xi)B^*$, with $\ell(\xi) = \sum_{j\geq 1} b_j\overline{\xi_j}$, $B^{*3} = (2i)^3\ell^2(\xi)B^*$, and so on. Thus, by the geometric series formula,

$$B^{-1} = \left(\sum_{j=0}^{\infty}(-1)^j B^{*j}\right) D^{-1} = \left(I - \sum_{j=0}^{\infty}(-1)^j(2i)^j\ell(\xi)^j B^*\right) D^{-1}$$

$$= \left(I - \frac{1}{1+2i\ell(\xi)}\right) D^{-1}$$

$$-B^{-1}A\overline{\xi}^t = \left(I - \frac{2i\overline{\xi}^t \cdot b}{1+2i\ell(\xi)}\right) \begin{pmatrix} \overline{\xi_1}^2 \\ \overline{\xi_1\xi_2} \\ \dots \\ \overline{\xi_1\xi_{n-1}} \end{pmatrix},$$

and

$$\overline{\tilde{\hat{f}}_c(\xi,0)}^t = \begin{pmatrix} \overline{\xi}^t \\ \left(I - \frac{2i\overline{\xi}^t \cdot b}{1+2i\ell(\xi)}\right)\overline{\xi_1\xi}^t \end{pmatrix} = \begin{pmatrix} \overline{\xi}^t \\ \frac{\overline{\xi_1\xi}^t}{1+2i\ell(\xi)} \end{pmatrix}.$$

Finally,

$$\tilde{\hat{f}}_c(z,0) = \left(z, \frac{z_1 z}{1 - 2i\sum_{j\geq 1}\overline{b_j} z_j}\right). \qquad (4.3.5)$$

We put $z = w = \eta = 0$ in (4.3.4), and by (4.3.5), we get $\overline{\hat{g}_c(\xi, 0)} = 0$.

We see that $\hat{F}_c(z,0)$ can be written as the quotient of a vector-valued polynomial of degree 2 and a linear function, so its degree is 2, as desired. □

We say that the degree of a rational map $H = \frac{(P_1,\ldots,P_m)}{R}$, with $\gcd(P_1,\ldots,P_m,R) = 1$, is

$$\deg(H) = \max(\deg(P_j)_{1\leq j\leq m}, \deg R).$$

For completeness, we note that if $H \equiv 0$, we say that $\deg(H) = -\infty$.

Then, for some point $q = (q_1,\ldots,q_n) \approx 0$, we defined the degree of $H|_{Q_q}$ to be the degree of the rational mapping

$$H(z_1,\ldots,z_{n-1}, q_n + wi\sum_{j=1}^{n-1}\overline{q_j}z_j).$$

We say that H is linear if $\deg(H) = 1$. Also, any automorphism of H has degree 1.

4.9 Lemma (HJ01, Lemma 5.3)**.** *Assume that F is in the normal form of (4.1.1). Then* $\deg(F) = 2$.

Proof. For any $p_0(z_0, w_0) \in \partial\mathbb{H}_n$, near 0, given $F^{**}_{p_0}$, we can construct the map $(\hat{F^{**}_{p_0}})_c$ such that Lemma (5.2) applies to $(\hat{F^{**}_{p_0}})_c$. There exists $\tau \in Aut(\mathbb{H}_{2n-1})$ and $\sigma \in Aut(\mathbb{H}_n)$ such that $\tau\circ(F^{**}_{p_0})_c\circ\sigma = F\circ\sigma$, with $\sigma(0) = 0$ and $\sigma_{p_0}(z,w) = (z+z_0, w+w_0+2i\langle z, \overline{z_0}\rangle)$. (See section 2 of [HJ06].) Plus, we have that $\sigma_{p_0}(Q_0) = Q_{p_0}$, by an important property we noted in the section on Segre families and Segre varieties in Chapter 2.

Also, $\sigma_{p_0}|_{Q_0} = (z + z_0, w_0 + 2i\langle z, \overline{z_0}\rangle)$.

Now, because $\deg(\hat{F_{p_0}^{**}})_c|_{Q_0} = 2$ by Lemma 4.8, and since $\sigma(Q_0) = Q_0$, we get that $\deg((\hat{F_{p_0}^{**}})_c \circ \sigma)|_{Q_0} = 2$. Since τ is linear, we get that

$$\deg\left((\tau \circ (\hat{F_{p_0}^{**}})_c \circ \sigma)\Big|_{Q_0} \right) = \deg((\hat{F_{p_0}^{**}})_c)\Big|_{Q_0} = 2.$$

Thus, for any $p(\approx 0) \in \partial\mathbb{H}_n, \deg(F)|_{Q_p} = 2$. The remainder of the proof is a direct result of the next lemma. In fact, our proof in the boundary case of the Second Gap Theorem also relies on this lemma. $\square$

4.10 Lemma (HJ01, Lemma 5.4). *Let $H = \frac{(P_1,\ldots,P_m)}{R}$ be a rational map from $\mathbb{C}^n$ to $\mathbb{C}^m$, where P_j and R are polynomials and $\gcd(P_1, \ldots, P_m, R) = 1$ and $m > n > 1$. Assume that for $p \in \partial\mathbb{H}_n$ near 0, $\deg(\mathbb{H}_{Q_p}) \leq k$ with $k > 0$, with k some fixed integer. Then $\deg(H) \leq k$.*

Proof. Assume for the sake of contradiction that $\deg(H) \geq k + 1$. Then consider the irreducible decomposition of the following affine variety:

$$Zero(P, R) = \{P_1 = \cdots = P_m = R = 0\} = Z_1 \cup Z_2 \cup \cdots \cup Z_\ell.$$

Since $\gcd(P_1, \ldots, P_m, R) = 1$, we have that each Z_j is of codimension at least 2 in $\mathbb{C}^n$.

Now, for any polynomial $h(z, w) = \sum_{|\alpha|+j\leq s} a_{\alpha j} z^\alpha w^j$ of degree $s > 0$, we get that there is a proper real analytic subvariety S of $\partial\mathbb{H}^n$ such that for any $p \in \partial\mathbb{H}^n \backslash S$, $\deg(h|_{Q_p}) = s$. Note that $h|_{Q_p} = \sum_{|\alpha|+j=s} a_{\alpha j} z^\alpha (wi \sum_{j=1}^{n-1} \overline{p_j} z_j)^j + \textit{lower order terms}$. Thus, there is a proper analytic subvariety A of C^{n-1} such that

$$\sum_{|\alpha|+j=s} a_{\alpha j} z^\alpha (2i \sum_{j=1}^{n-1} \overline{p_j} z_j)^j$$

is not identically 0 for any $(p_1, \ldots, p_{n=1}) \notin A$. We can take S to be $(A \times \mathbb{C}) \cap \partial\mathbb{H}_n$.

We can find a real analytic subvariety S_0 inside $\partial\mathbb{H}_n$ such that $\deg(P_j) = \deg(P_j|_{Q_p})$ and $\deg(R) = \deg(R|_{Q_p})$ for any $p \in \mathbb{H}^n \backslash S_0$. For such p, from our hypothesis, we conclude that $\{P_1|_{Q_p}, \ldots, P_m|_{Q_p}, R|_{Q_p}\}$ has a non-constant polynomial factor H_p^* whose zero induces a complex analytic subvariety, which we denote by $Z(p)$, in Q_p. Since $Z(p) \subset Zero\left(P|_{Q_p}, R|_{Q_p}\right)$ and $Z(p)$ has codimension 1 in Q_p, there is an irreducible component Z_{j_0} with $Z_{j_0} \cap Q_p$ containing an irreducible component of maximum dimension of $Z(p)$. (Note that j_0 might depend on p.)

The irreducible variety Z_{j_0} has codimension at least 2 in $\mathbb{C}^n$, and $Z_{j_0} \cap Q_p$ has codimension 1 in Q_p, so we get that $Z_{j_0} \subset Q_p$.

Since there are only finitely many choices of Z_{j_0}, and since for any Z_j, the set $\{p : Z_j \subset Q_p\}$ is a closed subset in $\partial\mathbb{H}_n \backslash S$, we get that there is a non-empty open subset $U \subset \partial\mathbb{H}^n$ such that, for any $p \in U$, Q_p contains a certain fixed Z_j, which has codimension 2 in $\mathbb{C}^n$. This is a contradiction. If not, pick $Z^* \in Z_j$. Since $z \in Q_{z^*}$ if and only if $z^* \in Q_z$, we would have Q_{z^*} containing an open piece of $\partial\mathbb{H}^n$, which is not possible. □

Completion of the proof of Theorem 1

We now finish our proof of Theorem 1.

First, we F be normalized to $\hat{F}_c$, as in Lemma 5.1, but write it as $F = (f, \phi, g)$ as before. Recall that $\deg(F) = 2$, $F(0, w) = (0, w), f(z, 0) = z, \phi(z, 0) = \frac{z_1 z}{1-2iR(z)}$ with $R(z) = \sum_{j=1}^{n-1} \overline{b_j} z_j$, and $g = w + \mathcal{O}(|(z, w)|^4)$, from (5.5) and (5.3). First, we show that F is partially linear.

4.11 Lemma (HJ01, Lemma 6.1)**.** *We have that $f|ell = z_1$ for $\ell \geq 2$, $g = 2$, $\phi_j = z_1\tilde{\phi}_j$, and $f_1 = z_1\tilde{f}_1$, where $\Phi = (\tilde{f}_1, \tilde{\phi}_1, \ldots, \tilde{\phi}_{n-1})$ extends as a biholomorphic map from $\mathbb{H}_n$ into $\mathbb{B}^n$.*

Assume for the sake of contradiction that $g \neq q$. Write $\tilde{g} = g - w = \frac{P}{Q}$ with $\deg(P), \deg(Q) \leq 3$. Then $\tilde{g}Q = P$. Assume that P is not identically 0. Since $\deg(P) \leq 3$¡ there is some multi-index α with $|\alpha| \leq 3$ such that $D^\alpha P\Big|_0 \neq 0$. However, $D^\beta(\tilde{g})\Big|_0 \equiv 0$

for any $|\beta| \leq 3$ because $g = w + \mathcal{O}(|(z,w)|^4)$. This is a contradiction, and thus $g \equiv w$.

Now, by (4.3.5), (4.3.3), Lemma 4.9, and the fact that $F(0,w) = (0,w)$, ϕ_ℓ takes the following form:

$$\phi_\ell = \frac{z_1 z_\ell + b_\ell z_1 w}{1 - 2iR(z) + B^{(0)}w + \tilde{B}^{(1)}(z)w + \tilde{B}^{(0)}w^2}.$$

In particular, we have that $\phi_\ell(0, z', w) \equiv 0$, where $z' = (z_2, \ldots, z_{n-1})$.

Now, we look at $Im(g) = |f|^2 + |g|^2$.. Letting $z_1 = 0$, we get

$$Im(w) = |f(0,z',w)|^2, \ when \ Im(w) = |z'|^2. \qquad (4.4.1)$$

We claim that

$$f(0,z',w) \equiv (0,z'). \qquad (4.4.2)$$

When $n \geq 4$, by (4.4.1) and [We], the map $(f(0,z',w),w)$ is linear.

Since $f_1(0,z',w) = o(|(z',w)|)$, it follows that $f_1(0,z',w) \equiv 0$. Since $f_\ell(0,z',w) = z_\ell + o_{wt}(4)$, by (5.3), we also have that $f_\ell(0,z',w) \equiv z_\ell$ for $\ell > 1$.

When $n = 3$, by Lemma 4.9, the map $(f(0,z_2,w),w) : \mathbb{H}_2 \to \mathbb{H}_3$ can be written $\left(\frac{P_1}{Q}, \frac{P_2}{Q}\right)$, where P_1, P_2, and Q are polynomials with degree ≤ 2. Since $f(0,w) = 0$ and $f(z,0) = z$ by (4.3.5), we can write this map as:

$$\begin{cases} f_1(0,z_2,w) = \frac{A_1 z_2 w}{1+B_1z_2+B_2w+B_3z_2^2+B_4z_2w+B_5w^2} \\ f_2(0,z_2,w) = \frac{z_2+A_2z_2w+A_3z_2^2}{1+B_1z_2+B_2w+B_3z_2^2+B_4z_2w+B_5w^2} \end{cases}$$

Because $f_1(0,z_2,w) = o_{wt}(4)$, we get that $A_1 = 0$, so $f_1(0,z_2,w) \equiv 0$.

Since $|z_1|^2 = |f_2(0,z_2,w)|^2$ holds for $Im(w) = |z_2|^2$ by (4.4.1), we get that

$$f_2(0,z_2,w) \equiv z_2.$$

Thus, (6.1) also holds when $n - 2$, and so (4.4.2) holds for any $n > 2$.

Now, by Lemma 4.9, we can write

$$f_\ell = \frac{z_\ell + A_\ell^{(1)}(z)w + A_\ell^{(2)}(z) + A_\ell^{(0)}w + \tilde{A}_\ell^{(0)}w^2}{1 + \mathcal{A}^{(1)}(z)w + \tilde{\mathcal{A}}^{(1)}(z) + \mathcal{A}^{(2)}(z) + \mathcal{A}^{(0)}w + \tilde{\mathcal{A}}^{(0)}w^2}, \forall \ell \geq 2.$$

Note the difference between A and $\mathcal{A}$.

By (4.3.5), with $w=0$, we get $f_\ell(z,0)=\frac{z_\ell+A_\ell^{(2)}(z)}{1+\tilde{\mathcal{A}}^{(1)}(z)+\mathcal{A}^{(2)}(z)}=z_\ell$.

Thus, $A_\ell^{(2)}=z_\ell\tilde{\mathcal{A}}^{(1)}(z)$, and $\mathcal{A}^{(2)}\equiv 0$. Since $f_\ell(0,w)-\frac{A_\ell^{(0)}w+\tilde{A}^{(0)}w^2}{1+\mathcal{A}^{(0)}(z)w+\tilde{\mathcal{A}}^{(0)}w^2}\equiv 0$, we get that $A_\ell^{(0)}=\tilde{A_\ell}^{(0)}=0$, and thus

$$f_\ell=\frac{z_\ell+A_\ell^{(1)}(z)w+z_\ell\tilde{\mathcal{A}}^{(1)}(z)}{1+\mathcal{A}^{(1)}(z)w+\tilde{\mathcal{A}}^{(1)}(z)+\mathcal{A}^{(0)}w+\tilde{\mathcal{A}}^{(0)}w^2},\forall\ell\geq 2.$$

By (4.4.2), we get

$$z_\ell=\frac{z_\ell+A_\ell^{(1)}(0,z')w+z_\ell\tilde{\mathcal{A}}^{(1)}(0,z')}{1+\mathcal{A}^{(1)}(0,z')w+\tilde{\mathcal{A}}^{(1)}(0,z')+\mathcal{A}^{(0)}w+\tilde{\mathcal{A}}^{(0)}w^2}.$$

Thus, $\tilde{\mathcal{A}}^{(0)}=0,\mathcal{A}^{(1)}(z)=az_1$, and $A_\ell^{(1)}(z)=a_\ell^*z_1+z_\ell\mathcal{A}^{(0)}$. Thus,

$$f_\ell=\frac{z_\ell+(a_\ell^*z_1+\mathcal{A}^{(0)}z_\ell)w+z_\ell\tilde{\mathcal{A}}^{(1)}(z)}{1+az_1w+\tilde{\mathcal{A}}^{(1)}(z)+\mathcal{A}^{(0)}w},\forall\ell\geq 2.$$

Since $f_\ell=z_\ell+o_{wt}(4)$, by comparing the z_1w terms, we get $a_\ell^*=0$. Thus,

$$f_\ell=\frac{z_\ell+\mathcal{A}^{(0)}z_\ell w+z_\ell\tilde{\mathcal{A}}^{(1)}z}{1+az_1w+\tilde{\mathcal{A}}^{(1)}(z)+\mathcal{A}^{(0)}w}=z_\ell-az_1z_\ell w+o_{wt}(4),\forall\ell\geq 2.$$

Thus, $a=0$, and so

$$f_\ell=\frac{z_\ell+\mathcal{A}^{(0)}z_\ell w+z_\ell\tilde{\mathcal{A}}^{(1)}(z)}{1+\tilde{\mathcal{A}}(z)+\mathcal{A}^{(0)}w}=z_\ell,\forall\ell\geq 2.$$

Then, comparing (4.4.2) and the expression for ϕ, we get $f_1=z_1\tilde{f}_1$ and $\phi=z_1\tilde{\phi}$. Moreover, we conclude that $\left|\tilde{f}_1\right|^2+\left|\tilde{\phi}\right|^2\equiv 1$ over $\partial\mathbb{H}_n$. We note that $\tilde{\phi}_\ell=z_\ell+b_\ell w+o(|(z,w)|^2)$, and $\tilde{f}_1=1+\frac{i}{2}w+o(|(z,w)|^2)$. Then $\Phi=(\tilde{\phi}_1,\ldots,\tilde{\phi}_{n-1},\tilde{f}_1)$ is biholomorphic to $\mathbb{H}_n$ to $\mathbb{B}^n$, with $\Phi(0)=(0,\ldots,0,1)$, by the Poincaré-Tanaka-Alexander Theorem in [Po07],[Ta62].

Now, we look again at $H:\mathbb{B}^n\to\mathbb{B}^{2n-1}$, letting $H=$
$psi_{2n-1}\circ F\circ\psi_n^{-1}$, where

$$\psi_k(z,w)=\left(\frac{2z}{i+w},\frac{i-w}{i+w}\right)$$

is the Cayley transformation from $\mathbb{H}_k$ to $\mathbb{B}^k$. Then we see that

$$H = \left(z_1\tilde{h}_1, z_2, \ldots, z_{n-1}, z_1\tilde{h}_2, \ldots, z_1\overline{h}_{n-1}, w\right).$$

Here, $(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_{n-1}) \in Aut(\mathbb{B}^n)$.

Putting everything together, we get the following proposition.

4.12 Proposition (HJ01, Proposition 6.2)**.** *Any non-linear rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$ with $n > 2$ is equivalent to a map of the form $H(z) = (z_1, \ldots, z_{n-1}, z_n h(z))$, with $h \in Aut(\mathbb{B}^n)$.*

Now, we show the following lemma.

4.13 Lemma (HJ01, Lemma 6.3)**.** *Let $H = (z_1, z_2, \ldots, z_{n-1} z_h h_1, z_n h_2, \ldots, z_h h_n)$ be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$ with $(h_1, h_2, \ldots, h_n) \in Aut(\mathbb{B}^n)$. Then F is equivalent to the Whitney Map.*

To show this, we first remark that we need only assume that $n \geq 2$.

Let $h = (h_1, \ldots, h_n)$ be such that $h(0) = p_0$. Then there is a unitary matrix U such that $h(0)U = (0, 0, \ldots, 0, b)$, with $1 > b \geq 0$. Replacing H by $(z', z_n hU)$ where $z' = (z_1, \ldots, z_{n-1})$, we can assume that $h(0) = (0, 0, \ldots, 0, b)$ with $b \in [0, 1)$. Thus, $h = \varphi_b \cdot \tilde{U}$, where

$$\varphi_b(z', z_n) = \left(\frac{s_b z'}{1 - b z_n}, \frac{b - z_n}{1 - b z_n}\right). \qquad (4.4.3)$$

Here, $s_b = \sqrt{1 - b^2}$, and $\tilde{U}$ is a unitary matrix. Thus, we can assume that

$$H = (z', z_n \varphi_b) \; with \; b \in [0, 1). \qquad (4.4.4)$$

Now, starting from the Whitney Map $W(z', z_n) = (z', z' z_n, z_n^2)$, for any $a \in [0, 1)$,

$$W \circ \varphi_a = \left(\frac{s_a z'}{1 - a z_n}, \frac{s_a z'(a - z_n)}{(1 - a z_n)^2}, \frac{(a - z_n)^2}{(1 - a z_n)^2}\right).$$

Let $\tilde{\varphi}_{a^2} \in Aut(\mathbb{B}^{2n-1})$:

$$\tilde{\varphi}_{a^2}(z^{*\prime}, z^*_{2n-1}) - \left(\frac{s_{a^2} z^{*\prime}}{1 - a^2 z^*_{2n-1}}, \frac{a^2 - z^*_{2n-1}}{1 - a^2 a^*_{2n-1}}\right).$$

Here, $z^{*\prime} = (z^*_1, \ldots, z^*_{2n-1})$.

Then,

$$\tilde{\varphi}_{a^2} \circ W \circ \varphi_a = \left(\frac{s_{a^2} z'(1 - az_n)}{s_1(1 + a^2 - 2az_n)}, \frac{s_{a^2} z'(a - z_n)}{s_a(1 + a^2 - 2az_n)}, \frac{2az_n - (1 + a^2)z_n^2}{1 + a^2 - 2az_n}\right).$$

Let U_a be a $(2n-1) \times (2n-1)$ unitary matrix given by

$$U_a = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $A = \frac{1}{\sqrt{1+a^2}} I, B = -\frac{a}{\sqrt{1+a^2}} I, C = \frac{1}{\sqrt{1+a^2}} I$, and $D = \frac{1}{\sqrt{1+a^2}} I$, where I is the $(n-1) \times (n-1)$ identity matrix.

Define

$$\psi = U_a \circ \tilde{\varphi}_{a^2} \circ W \circ \varphi_a = (I, II, III),$$

where

$$\begin{cases} I = \frac{z'(1-a^2)}{1+a^2-2az_n} \\ II = \frac{z'[2a-(a^2+1)z_n]}{1+a^2-2az_n} \\ III = \frac{z_n[2a-(a^2+1)z_n]}{1+a^2-2az_n} \end{cases}.$$

Write $\beta = \frac{2a}{1+a^2}$. Then

$$\psi = \left(\frac{s_\beta z'}{1 - \beta z_n}, \frac{z'(\beta - z_n)}{1 - \beta z_n}, \frac{z_n(\beta - z_n)}{1 - \beta z_n}\right).$$

Let $(z'^*, z^*_n) = \varphi_\beta(z', z_n)$. Then,

$$\psi \circ \varphi_\beta^{-1} = (z'^*, z^*_n \varphi_\beta(z'^*, z^*_n)). \qquad (4.4.5)$$

Here, we use the fact that $\varphi_\beta^{-1} = \varphi_\beta$. If we choose a such that $b = \beta$, we get that the Whitney Map is equivalent to (4.4.5).

Now, Theorem 1 follows from Proposition 4.12 and Lemma 4.13, while Theorem 2 follows from Theorem 1 and Corollary 1.2 in [Hu01].

Chapter 5

The Second Gap Theorem

In [DA88], D'Angelo introduced the following family of mutually inequivalent proper polynomial embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$:

$$F_\theta(z', w) = (z', (\cos\theta)w, (\sin\theta)z_1 w, \ldots, (\sin\theta)z_{n-1}w, (sin\theta)w^2), 0 \le \theta \le \frac{\pi}{2},$$

with $z = (z', w) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Hamada, in [Ha05], showed that, for $n \ge 4$, any map in $Rat(\mathbb{B}^n, \mathbb{B}^{2n})$ is equivalent to F_θ for some $0 \le \theta \le \frac{\pi}{2}$. The following theorem is related.

5.1 Theorem (HJX06). *Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$, $n \ge 4$ and $n \le N \le 3n-4$. Then F is equivalent to*

$$F'_\theta = (F_\theta(z, w), 0 \ldots, 0) = (z, w\cos\theta, z_1 w \sin\theta, \ldots, z_{n-1}w\sin\theta, w^2\sin\theta, 0 \ \ldots, 0)$$

for some $0 \le \theta \le \frac{\pi}{2}$.

In particular, there are no new minimal maps in the Second Gap, $(2n, 3n-3)$.

Preliminaries

Note that the possible maps for $N < 3n-3$ all have geometric rank ≤ 1. The boundary case of the Second Gap Theorem will cover what happens when $N = 3n-3$, when we can add certain maps of geometric rank 2 (those equivalent to the generalized Whitney Map).

We Chapter 2 of this dissertation (or read ahead to Definition 5.5) for the definition of geometric rank, which is an integer associated with each map $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$. The paper on the First Gap Theorem, [Hu99], demonstrates that F has geometric rank 0 if and only if it is equivalent to a linear fractional map.

Now, for $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ and $\kappa_0 \leq n-2$, we say that F has degenerate geometric rank. The results of [Hu03] and [HJX05] show that these maps must be rational and $(n-\kappa_0)$-linear. By k-linear, we mean a map $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ such that, for any $p \in \mathbb{B}^n$, there is an affine complex subspace $\mathbb{S}_p^a$ containing p of complex dimension k so that the restriction of F to S_p^a is a linear fractional map.

Also, we note that Forstneric, in [Fo89], proved that

$$Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = Rat(\mathbb{B}^n, \mathbb{B}^N).$$

The proof of Theorem 5.1 comes about from considering the normalization problem for maps in $Rat(\mathbb{B}^n, \mathbb{B}^N)$. For $n \geq 3$ and for maps with geometric rank 1, we see precisely the hyperplanes along which the maps are linear fractional, and the proof of the theorem follows from this. For the more general case of degenerate geometric rank, we derive a normal form the same way. For a non-linear map $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ with $N \geq 3$, we get $\kappa_0 = 1 = n-1$, and this turns out to be a very complicated case, as detailed in [Fa82] and [CD96].

Our main theorem this chapter is the following.

5.2 Theorem (Second Gap Theorem)**.** *Let F be a non-linear proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$, with $N \geq n \geq 3$. Assume that F is C^3-smooth up to the boundary as has geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a proper holomorphic map of the form*

$$H = (z_1, \ldots, z_{k^0}, H_1, \ldots, H_{N-k^0}),$$

where $k^0 = n - \kappa_0$, and $H_j = \sum_{\ell=k^0-1}^{n} z_\ell H_{j,\ell}$, with $H_{j,\ell}$ holomorphic over $\overline{\mathbb{B}^n}$. Moreover, with $\kappa_0 = 1$, $(H_1, \ldots, H_{N-n+1}) = z_n \cdot h$, with h a rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N-n+1}$. Both H and h are affine linear maps along each hyperplane defined by $z_n = constant$.

For $N \leq 3n - 3$, we have $N - n + 1 \leq 2n - 2$. Then if F is given as in the Second Gap Theorem, with geometric rank 1, then the corresponding h must be linear fractional, according to the linearity theorem in [Fa86] and [Hu99]. Thus, H can be thought of as a proper map from $\mathbb{B}^n$ to $\mathbb{B}^{2n}$. We combine this with [Hu03, Theorem 1.1], [HJX05, Corollary 1.3], [HJ01, §6], and [Ha05, Theorem 1.1], and this proves Theorem 5.1.

We also get the following corollary of our main theorem.

5.3 Corollary (HJX06, Corollary 1.3)**.** *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with geometric rank 1. Assume $n \geq 3$. Then the degree of F is bounded by $\frac{N-1}{n-1}$.*

Notation

Write $\mathbb{H}_n$ for the Siegel upper-half space, $\{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}^n : Im(w) > |z|^2\}$. Let ρ_n be the Cayley transformation

$$\rho_n : \mathbb{H}_n \to \mathbb{B}^n, \rho_n(z, w) = \left(\frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw} \right).$$

This is a biholomorphic mapping from $\mathbb{H}_n$ to $\mathbb{B}^n$, so we can identify a map $F \in Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ or $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in $Prop_k(\mathbb{H}_n, \mathbb{H}_N)$ or $Rat(\mathbb{H}_n, \mathbb{H}_N)$. We have, by calculation, that

$$\rho_n^{-1}(z, w) = \left(\frac{z}{1 + w}, \frac{i - iw}{1 + w} \right).$$

As we have done before, we can parameterize $\partial\mathbb{H}_n$ by $(z, \overline{z}, u)$ using the map

$$(z, \overline{z}, u) \mapsto (z, u + i|z|^2).$$

We assign z a weight of 1 and u a weight of 2. For a non-negative integer m, a function $h(z,\overline{z},u)$ defined over a ball U of 0 in $\partial\mathbb{H}_n$ has weighted degree $o_{wt}(m)$ if $\lim_{t\to 0}\frac{h(tz,t\overline{z},t^2u)}{|t|^m} = 0$ uniformly for (z,u) on any compact subset of U. We use the notation $h^{(k)}$ for a polynomial of weighted degree k.

Let $F = (f,\phi,g) = (\tilde{f},g) = (f_1,\ldots,f_{n-1},\phi_1,\ldots,\phi_{N-n},g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$. For $p \in M$ near 0, we write $\sigma_p^0 \in Aut(\mathbb{H}_n)$ for the map sending (z,w) to $(z+z_0, w+w_0+2i\langle z,\overline{z_0}\rangle)$ and $\tau_p^F \in Aut(\mathbb{H}_N)$ for

$$\tau_p^F(z^*,w^*) = (z^* - \tilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\tilde{f}(z_0,w_0)}\rangle).$$

Then, F is equivalent to

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p,\phi_p,g_p).$$

Now $F_0 = F$ and $F_p(0) = 0$.

We restate [Hu99,Lemma 5.3].

5.4 Lemma. *Let F be a C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$, $2 \le n \le N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such that $F_p^{**} = \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$\begin{aligned} f_p^{**} &= z + \tfrac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3) \\ \phi_p^{**} &= \phi_p^{**(2)}(z) + o_{wt}(2) \\ g_p^{**} &= w + o_{wt}(4) \end{aligned}$$

with

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle\,|z|^2 = \left|\phi_p^{**(2)}(z)\right|^2.$$

5.5 Definition. *The geometric rank $Rk_F(p)$ of F at p is the rank of the matrix*

$$\mathcal{A}(p) = -2i\left(\left.\frac{\partial^2 (f_p)_\ell^{**}}{\partial z_j \partial w}\right|_0\right),$$

and we say that the geometric rank of F *is* $\kappa_0 = \max_{p\in\partial\mathbb{H}_n} Rk_F(p)$. *We will define the geometric rank of* $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ *to be the one associated with* $\rho_N^{-1} \circ F \circ \rho_n \in Prop_2(\mathbb{H}_n, \mathbb{H}_N)$.

It was shown in [Hu03] that κ_0 depends only on the equivalence class of F, and, as we mentioned before and as was shown in [Hu99], $\kappa_0 = 0$ if and only if F is linear fractional.

For this chapter, we may think of κ_0 as greater than or equal to 1.

Let $F = (f, \phi, g) \in Rat(\mathbb{H}_n, \mathbb{H}_N)$ satisfy Lemma 5.4. We find a basis for sections of the complex tangent bundle $T^{(1,0)}\partial\mathbb{H}_n$ by letting $L_j = 2i\overline{z}_j\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ for $1 \leq j \leq n-1$. The complexification of this will also be important, so we note that it looks like

$$\mathcal{L}_j = 2i\overline{\xi}_j\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}.$$

Now, we examine the complexified version of the basic geometric equation

$$\frac{g(z,w) - \overline{g(\xi,\eta)}}{2i} = \tilde{f}(z,w)\cdot\overline{\tilde{f}(\xi,\eta)}, w - \overline{\eta} = 2iz\cdot\overline{\xi}.$$

We consider the case where $z = w = \eta = 0$, and we get $g(z,0) \equiv 0$.

We apply $\mathcal{T}^j, j = 1,\ldots,k$, with $\mathcal{T} = \frac{1}{2}\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial\overline{\eta}}\right)$ to the basic equation with $z = w = \eta = 0$, and then we get $g(z,w) = w + o(|(z,w)|^k)$ if $g(0,w) = w$ and $\tilde{f}(0,w) = o(w^k)$.

Applying it once, we get

$$\frac{1}{2}\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial\overline{\eta}}\right)\frac{g(z,w) - \overline{g(\xi,\eta)}}{2i} = \frac{1}{2}\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial\overline{\eta}}\right)\tilde{f}(z,w)\cdot\overline{\tilde{f}(\xi,\eta)}$$

$$\frac{1}{4i}\left(\frac{\partial^k g(0,0)}{\partial w^k} - \frac{\partial^k\overline{g(\xi,0)}}{\partial\overline{\eta}^k}\right) = \frac{1}{2}\frac{\partial^k\tilde{f}(0,0)}{\partial w^k}\overline{\tilde{f}(\xi,0)} + \frac{1}{2}\tilde{f}(0,0)\frac{\partial^k\overline{\tilde{f}(\xi,0)}}{\partial\overline{\eta}^k}.$$

By our normalization, $g(0,w) = w + o_{wt}(4)$, so $g(0,w) - w = o_{wt}(4)$. Also, $\tilde{f}$ has no constant term, so $\tilde{f}(0,0) \equiv 0$.

Suppose $g(0,w) - w = o(w^k)$ and $\tilde{f}(0,w) = o(w^k)$.

Then the previous equation becomes

$$\frac{1}{4i}(A - A + o(2)) = B\overline{\tilde{f}(\xi,0)},$$

since the constant term of the k^{th} partial derivative of $g(z,w)$ is the same as the constant term of the k^{th} partial derivative of $\overline{g(\xi,\eta)}$, and it is non-zero by our assumption. Similarly, the constant term of the k^{th} derivative of $\tilde{f}(z,w)$ is non-zero. Essentially, the $\tilde{f}^{(0,k)}$ part of $\tilde{f}$ and the $g^{(0,k)}$ part of g are non-zero, where $h^{(a,b)}(z)$ is a homogeneous polynomial in z of degree a such that $h^{(a,b)}(z)w^b$ is a term of $h(z,w)$.

Applying $\mathcal{T}^j$ with $j < k$ yields 0 on the right-hand side (and thus also on the left).

We similarly apply $\mathcal{L}_j$ to the basic equation for each j and then let $\xi = w = \eta = 0$ to get $f(z,0) = z$.

These results give us a lemma.

5.6 Lemma (HJX06, Lemma 2.3). *Let $F = (f, \phi, g) \in Rat(\mathbb{H}_n, \mathbb{H}_N)$ satisfy the normalization in Lemma 5.1. Then*

(a) $g(z,0) = 0, \frac{\partial g}{\partial w}(z,0) \equiv 1$, and $f(z,0) = z$.

(b) If we further assume that $(\tilde{f}(0,w), g(0,w)) = (0,w)$, then $g \equiv w$.

We can find (a) in [BER99] and [HJ01]. We can find (b) in [HJ01] for the case where $N = 2n-1$, and (b) was shown for $N = 2n$ in [Ha05], and that argument can be used to show (b) for any $N \geq n$..

Now, for a rational holomorphic map $H = \frac{(P_1,\ldots,P_m)}{Q}$ over $\mathbb{C}^n$, with P_j and Q polynomials with $\gcd(P_1,\ldots,\mathbb{P}_m,Q) = 1$, we can define

$$\deg(H) = \max\{\deg(P_j), 1 \leq j \leq m, \deg(Q)\}.$$

If H is a rational map, and S is a complex affine subspace of dimension k, we have that H is linear fractional along S if S is not contained in the singular set of H, and for any linear parameterization $z_j = z_j^0 + \sum_{\ell=1}^k a_{j\ell}t_\ell$ with $j = 1,\ldots,n$, we have $H^*(t_1,\ldots,t_k) = H(z_1^0 + \sum_{\ell=1}^k a_{1\ell}t_\ell, \ldots, z_n^0 + \sum_{j=1}^k a_{nj}t_j)$ has degree 1 in $(t_1,\ldots,t_k)$.

Mappings with geometric rank bounded by $n-2$

Now, we let $F \in Prop_3(\mathbb{H}_n, \mathbb{H}_N)$ with $1 \leq \kappa_0 \leq n-2$. We can assume F is rational, according to Theorem 2.3 in [Hu03] and Corollary 1.3 in [HJX05]. Using several results (lemmas 3.2,3.3,4.1, and 4.3, equation (3.6.4), and Claim 4.4 from [Hu03], as well as our Lemma 5.6), we get that this map is equivalent to a map with the following normalization:

$$\begin{cases} f_\ell = z_\ell + \frac{i}{2}\mu_\ell z_\ell w + a_\ell^{(2)}(z)w + o_{wt}(4), \mu_\ell > 0, \ell \leq \kappa_0 \\ f_j = z_j + o_{wt}(4), \kappa + 1 \leq j \leq n-1, \\ \phi_{j\ell} = \mu_{j\ell} z_j z_\ell + o_{wt}(2), (j,\ell) \in \mathcal{S}_0, \\ \phi_{j\ell} = o_{wt}(2), (j,\ell) \notin \mathcal{S}_0, \\ g = w, \\ F(0,\ldots,0,z_{\kappa_0+1},\ldots,z_{n-1}w) = (0,\ldots,0,z_{\kappa_0+1},\ldots,z_{n-1},0,\ldots,0,w). \end{cases} \quad . \qquad (\star)$$

Here, $\mathcal{S}_0 = \{(j,\ell) : 1 \leq j \leq \kappa_0, 1 \leq \ell \leq (n-1), j \leq \ell\}$, and $\mu_{j\ell} = \sqrt{\mu_j + \mu_\ell}$ for $j < \ell \leq \kappa_0$ and $\mu_{j\ell} = \sqrt{\mu_j}$ for $j \leq \kappa_0$ and $\ell > \kappa_0$ or $j = \ell \leq \kappa_0$.

We let E_0 be the proper complex analytic variety consisting of the poles and the non-immersed points of our normalized F. Following [Hu03], we define $\mathcal{V}_F = \{(Z, S_Z) \in (\mathbb{C}^n \backslash E_0) \times G_{n,k^0}(\mathbb{C}), F$ is linear fractional when restricted to $S_Z + Z\}$. That is, we look at pairings of points not on E_0 and corresponding spaces along which F is linear fractional.

Still following [Hu03, Lemma 5.1], we get that $\mathcal{V}_F$ is a complex analytic variety with $\pi : \mathcal{V}_F \to \mathbb{C}^n \backslash E_0$ as its proper holomorphic projection. Furthermore, given any point $Z \in \mathbb{C}^n \backslash E_1$, for a certain proper complex analytic subvariety E_1 of $X = \mathbb{C}^n \backslash E_0$, we get that Z has a unique preimage in $\mathcal{V}_F$. What this means is that there is a unique complex subspace S_Z of dimension k^0 so that F is linear fractional along $Z + S_Z$.

Write $\bigcup_j \mathcal{V}^{(j)}$ for the irreducible decomposition of $\mathcal{V}_F$. Write $\mathcal{V}^{(1)}$ for the irreducible component whose projection to $\mathbb{C}^n \backslash E_0$ lies in $\mathbb{H}_n$ with a piece of $\partial\mathbb{H}_n$ containing 0 as part of its boundary, as in [Hu03, Lemma 5.3]. Write $(\mathcal{K}, \rho\mathcal{V}^{(1)})$ for the desingularization of $\mathcal{V}^{(1)}$, and let A be the singular set of $\pi \circ \rho$ (where $\pi \circ \rho$ is not biholomorphic). Let

$B = \pi \circ \rho(A) \cup \pi(Sing(\mathcal{V}^{(1)}))$. We get that E_1 is the locally finite union of B with projections of the other irreducible components. Thus, perhaps by choosing another point nearby, we may assume that $0 \notin E_1$, so π is biholomorphic near $(0, S_0) \in \mathcal{V}_F$. Then, for $\epsilon = (\epsilon_1, \dots, \epsilon_{\kappa_0}) \approx 0$, $S_{(\epsilon,0)} + (\epsilon, 0)$ can be defined by κ_0 equations

$$z_\ell = \sum_{j=\kappa_0+1}^{n} a_{j\ell}(\epsilon) z_j + \epsilon_\ell, \ell = 1, \dots, \kappa_0.$$

Here, $a_{j\ell}$ is holomorphic, and we identify z_n with w. We see that $a_{j\ell}(0) = 0$.

Consider

$$z_\ell = \sum_{j=\kappa_0+1}^{n} a_{j\ell}(z'_1, \dots, z'_{\kappa_0}) z_j + z'_\ell, \ell = 1, \dots, \kappa_0.$$

From this and $a_{j\ell}(0) = 0$, we get

$$z'_\ell = \psi_\ell(z_1, \dots, z_n) = z_\ell + \sum_{j=\kappa_0+1}^{n} \mathcal{O}(|z|) z_j, \ell = 1, \dots, \kappa_0.$$

Next, we show that $f_j \equiv z_j$ for $j \geq \kappa_0 + 1$. Fix such a j. Then, since $(\tilde{f}(z), w)$ has degree 1 along each complex affine subspace defined by $z'_\ell = \epsilon_\ell (\ell = 1, \dots, \kappa_0)$, we get

$$f_j(z, w) = b_0(z'_1, \dots, z'_{\kappa_0}) + \sum_{\ell=\kappa_0+1}^{n-1} b_\ell(z'_1, \dots, z'_{\kappa_0}) z_\ell + b_n(z'_1, \dots, z'_{\kappa_0}) w.$$

Recalling that $f(z, 0) = z$, we have that $f_j(z_1, \dots, z_{n-1}, 0) = z_j$, and we have

$$0 = f_j(z_1, \dots, z_{\kappa_0}, 0' = b_0(z'_1, \dots, z'_{\kappa_0}).$$

Thus, $b_0 \equiv 0$.

We write

$$b_\ell(z'_1, \dots, z'_{\kappa_0}) - \delta^j_\ell = b^{(k_\ell)}_\ell(z'_1, \dots, z'_{\kappa_0}) + \sum_{\ell=\kappa_0+1}^{n} o(\left|(z'_1, \dots, z'_{\kappa_0})\right|^{k_\ell}).$$

We can do this because of our normalization. Since $f_j = z_j + o_{wt}(4)$, and $f_j(z, 0) = \sum b_\ell(z') z_\ell + b_n(z') w$, we note that for $\ell = j$, the first term of that is going to be equal to

$z_j + o_{wt}(4)$, and thus it contains a linear term and higher order terms of weighted degree at least 4.

Here, $b_\ell^{(k_\ell)}$ is a homogeneous polynomial in $(z'_1, \ldots, z'_{\kappa_0})$ with degree k_ℓ. We assume $b_\ell^{(k_\ell)}$ is not identically 0 when b is not δ_ℓ^j. The formulas for ψ_ℓ then tell us that

$$b_\ell = \delta_\ell^j + b_\ell^{(k_\ell)}(z_1, \ldots, z_{\kappa_0}) + \sum_{\ell=\kappa_0+1}^{n} \mathcal{O}(|(z_1, \ldots, z_n)|^{k_\ell}) z_j.$$

Because $f_j(z_1, \ldots, z_{n-1}, 0) = z_j$, we have

$$\sum_{\ell=\kappa_0+1}^{n-1} b_\ell^{(k_\ell)}(z_1, \ldots, z_{\kappa_0}) z_\ell + \sum_{\ell=\kappa_0+1}^{n-1} o(|(z_1, \ldots, z_{n-1})|^{k_\ell}) z_\ell = 0.$$

We wish to show that this implies that $b_\ell \equiv 0$ for $\ell \neq n$.

Assume for the sake of contradiction that this is not the case.

Then $b_{\ell_0}^{(k_{\ell_0})} \neq 0$ has a minimum degree for a certain $\ell_0 < n-1$. We compare the coefficients of the homogeneous polynomials in $(z_1, \ldots, z_{n-1})$ of degree $k_{\ell_0}+1$, and we get $b_{\ell_0}^{(k_{\ell_0})} \equiv 0$. This is a contradiction, so $b_\ell \equiv 0$ for any $\ell < n$.

Next, we complexify the first equation in [Hu03, (3.4.5)], getting

$$\overline{T(\tilde{F})} \cdot L_j(\tilde{f})^t = \frac{1}{2i} L_j T(g) - (L_j T(\tilde{f})) \cdot \overline{\tilde{f}}.$$

Now, we let $z = w = \eta = 0$, and, from our normalization $(\star)$ and the fact that $\left.\frac{\partial^2 \phi_{j\ell}}{\partial z_j \partial w}\right|_0 = 0$ for $j \geq \kappa_0 + 1$ (see [Hu03, Lemma 3.3(c)]),

$$b_n(\phi_1(z_1, \ldots, z_{n-1}, 0), \ldots, \phi_{\kappa_0}(z_1, \ldots, z_{n-1}, 0)) \equiv 0.$$

This, in turn, forces $b_n \equiv 0$.

Since $b_\ell \equiv 0$ for $l < n$, we are able to get that $f_j \equiv z_j$.

Next, we look at $\phi \equiv 0$ and $f_\ell(0, \ldots, 0, z_{\kappa_0+1}, \ldots, z_{n-1}, w) \equiv 0$ for $\ell \leq \kappa_0$. We can write f_ℓ and ϕ as follows:

$$f_\ell = \sum_{\tau=1}^{\kappa_0} z_\tau f_{\ell\tau}^*$$

and

$$\phi = \sum_{\tau=1}^{\kappa_0} z_\tau \phi^*_{\ell\tau}.$$

Next, we assume $\kappa_0 = 1$, and we write our defining equation of $S_{(\epsilon,0)} + (\epsilon, 0)$, $\epsilon \approx 0$, as

$$z_1 = \sum_{j=2}^{n} a_j(\epsilon) z_j + \epsilon, \ where\ w\ is\ z_n.$$

We have three cases to consider.

Case (11): $a_j(\epsilon) = \epsilon a_j^0(\epsilon)$, where $a_{j_0}^0(\epsilon) \neq constant$ for a certain $j_0 \leq n$.

Case (22): $a_j(\epsilon) = \epsilon a_j^0$ with $a_j^0 = constant$ and $Im(a_n^0) = -\sum_{j=2}^{n-1} \left|\frac{a_j^0}{2}\right|^2$.

Case (33): $a_j(\epsilon) = \epsilon a_j^0$ with $a_j^0 = constant$ but the other condition of (22) is not met.

We examine Case (11) first.

In this case, we have that S_0 intersects $S_{(\epsilon,0)+(\epsilon,0)}$. We vary ϵ and take the union of the resulting sets, and we get that this union contains an open piece of S_0. Now $S_0 \backslash E_0$ is an irreducible complex analytic variety of $\mathbb{C}^n \backslash E_0$. By uniqueness of complex analytic varieties, we get that $S_0 \subset E_1 \cup E_0$. However, this contradicts our initial assumption that $0 \notin E_1 \cup E_0$, so Case (11) does not actually occur.

Next, we can consider Case (22), writing

$$z_1' = \frac{z_1}{1 + \sum_{j=2}^n a_j^0 z_j} = \frac{z_1}{1 - 2i\langle \overline{\alpha}, z\rangle + (r - i\,|\alpha|^2)w}$$

for some $r \in \mathbb{R}$ and $\alpha = (0, \alpha_2, \ldots, \alpha_{n-1}) \in \mathbb{C}^{n-1}$. We define

$$\sigma(z', w') = \frac{((z' - \alpha w'), w')}{q'(z', w')}$$

where $q'(z', w') = 1 + 2i\langle \overline{\alpha}, z'\rangle + (-r - i\,|\alpha|^2)w'$. We define

$$\tau^*(z^*, w^*) = \frac{((z^* + \alpha^* w^*), w^*)}{q^*(z^*, w^*)},$$

where $q^*(z^*, w^*) = 1 - 2i\langle\overline{\alpha}, z^*\rangle + (r - i|\alpha^*|^2)w^*$, with $\alpha^* = (\alpha, 0)$. We get that $\sigma \in Aut_0(\mathbb{H}_n)$ and $\tau^* \in Aut_0(\mathbb{H}_N)$, and $\tau^* \circ F \circ \sigma$ is still normalized. To avoid the confusion, we refer to the new map as F.

This new F is linear fractional along each hyperplane defined by $z'_1 = constant$. Because $g = w$, we have that F is affine linear along the hyperplanes defined by $z'_1 = constant$. We write (z, w) for (z', w'). Now $f_1 = z_1 f_z^*$ and $\phi_j = z_1\phi_j^*$.

We also get that $\Psi = (\phi_j^*, f_1^*)$ is a proper map from $\mathbb{H}_N$ into $\mathbb{B}^{N-n+1}$, and Ψ is affine linear along $z_1 = constant$. If we let $z_1 = 0$, we get that

$$\Psi(0, z_2, \ldots, z_{n-1}, w) = (b_1 w, z_2 + b_2 w, \ldots, z_{n-1} + b_{n-1}w, 0, \ldots, 0, 1 + \frac{i}{2}w)$$

maps $Im(w) = \sum_{j=2}^{n-1} |z_j|^2$ into the unit sphere. In particular,

$$|b_1 w|^2 + \sum_{j=2}^{n-1} |b_j w + z_j|^2 + \left|1 + \frac{i}{2}w\right|^2 = 1,$$

over $w = u + i\sum_{j=2}^{n-1} |z_j|^2$.

Now, looking at this last equation, we compare the terms with a u factor, and we get another contradiction, so we have that Case (22) does not occur.

In the final case, we can assume that F, possibly after a unitary transformation before and after, is linear fractional (affine linear, even) along hyperplanes defined by $z_1 = constant \cdot (1 + bz_2 + cw)$, with $b \geq 0$.

For $\alpha = (0, K, 0, \ldots, 0)$ with $K \in \mathbb{C}$, we see that the inverse of

$$\sigma(z', w') = \frac{((z' - \alpha w'), w')}{q(z', w')},$$

with $q(z', w') = 1 + 2i\langle\overline{\alpha}, z'\rangle + (-r - i|\alpha|^2)w'$ transforms hyperplanes defined by $z_1 = constant \cdot (1 + bz_2 + cw)$ into hyperplanes defined by $z'_1 = constant \cdot (q(z', w') + bz'_2 - bKw' + cw')$.

Now,

$$q(z', w') + bz'_2 - bKw' + cw' = 1 + 2i(\overline{K} - \frac{i}{2}b)z'_2 + (-r - i|K|^2 + c - bK)w'.$$

This follows from how we defined α and from our expression for $q(z', w')$.

Choose $K = -\frac{i}{2}b$ and $r = Re(c)$. Then, we can make $b = 0$ and c purely imaginary and nonzero if we compose F appropriately with this σ and the τ mentioned in our discussion of Case (22). Through stretching F on both sides, we can make $c = \pm i$, and we will later show that $c \neq i$. This leads to the following normalization:

5.7 Theorem (HJX06, Theorem 3.1)**.** *Let $F \in Prop_3(\mathbb{H}_n \mathbb{H}_N)$ with geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a map of the following form:*

$$\begin{cases} f_\ell = \sum_{j=1}^{\kappa_0} z_j f_{ij}^*(z), \ell \leq \kappa_0 \\ f_j = z_j, \kappa_0 + 1 \leq j \leq n-1 \\ \phi_{\ell k} = \mu_{\ell k} z_\ell z_k + \sum_{j=1}^{\kappa_0} z_J \phi_{\ell k j}^*, (\ell, k) \in \mathcal{S}_0 \\ g = w \\ f_{\ell j}^*(z,w) = \delta_\ell^j + \frac{i \delta_\ell^j \mu_\ell}{2} w + \mathcal{O}(|(z,w)|^2), \\ \phi_{\ell k j}^*(z,w) = \mathcal{O}_{wt}(2), (\ell, k) \in \mathcal{S}_0, \\ \phi_{\ell k} = \sum_{j=1}^{\kappa_0} z_j \phi_{\ell k j}^* = \mathcal{O}(|(z,w)|^3), (\ell, k) \notin \mathcal{S}_0. \end{cases} \qquad (\star\star)$$

Here, μ_j and $\mu_{j\ell}$ are as in $(\star)$, and when $\kappa_0 = 1$, F is an affine linear map along each hyperplane defined by $z_1 = constant \cdot (1 - iw)$.

Next, we finish proving Theorem 5.7 and Theorem 5.2.

First, when $\kappa_0 = 1$, then F from Theorem 5.7 is affine linear along hyperplanes defined by $z_1 = constant \cdot (1 - iw)$, since $\deg((\tilde{f}, w)) = 1$ along those hyperplanes.

Now, we consider the case where $\kappa_0 = 1$. We wish to show that, along hyperplanes defined by $z_1 = constant \cdot (1 + i2)$, F cannot be affine linear. As in [HJ01], using the part of Theorem 5.7 already proved, we can use the Cayley transformation to get a proper holomorphic mapping $H = \rho_N \circ F \circ \rho_n^{-1}$ from $\mathbb{B}^n$ into $\mathbb{B}^N$:

$$H = (H - 1, \ldots, H_{\kappa_0}, z_{\kappa_0 + 1}, \ldots, z_{n-1}, H_n, \ldots, H_{N-1}, w),$$

where $H_j = \sum_{\ell=1}^{\kappa_0} z_\ell H_{j,\ell}$ and $H_{j,\ell}$ holomorphic over $\overline{\mathbb{B}}^n$. We have that

$$\rho^{-1}(z', w') = \left(\frac{iz'}{i + iw'}, \frac{w' - 1}{i + iw'} \right).$$

Thus, H is affine linear along one of two families of hyperplanes.

(1) Those defined by $z_1 = constant$.

(2) Those defined by $z_1 = constant \cdot w$.

Assume for the sake of contradiction that F is affine linear along hyperplanes defined by $z_1 = constant \cdot w$.

This implies that $H^*(z_1, \ldots, z_n) = (H_{1,1}, \ldots, H_{N-1,1})$ must be a constant map along each $z_1 = constant \cdot w$. Because H^* is a proper map from $\mathbb{B}^n$ into $\mathbb{B}^{N-n+1}$, this is impossible, which is a contradiction. Thus, we have proved Theorem 5.7.

Changing coordinates, we can rewrite H as

$$H = (z_1, \ldots, z_{k^0}, H_1, \ldots, H_{N-k^0}).$$

Here, $k^0 = n - \kappa_0$, and $H_j = \sum_{\ell=k^0+1}^{n} z_\ell H_{j,\ell}$.

So, when $\kappa_0 = 1$, write $(H_1, \ldots, H_{N-n+1}) = z_n \cdot h = z_n(h_1, \ldots, h_{N-n+1})$. Then we have that h is a proper holomorphic mapping from $\mathbb{B}^n$ into $\mathbb{B}^{N-n+1}$, and h is affine linear along hyperplanes defined by $z_n = constant$. This proves Theorem 5.2.

Next, we prove Corollary 5.3.

Proof. For $N \geq n \geq 3$, there is a unique positive integer k so that $k(n-1)+1 \leq N \leq (k+1)(n-1)$. We will prove our corollary by induction on k.

When $k = 1$, $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-2})$, and so $\deg(f) = 1 \leq \frac{N-1}{n-1}$. Let $\deg(F) \leq \frac{N-1}{n-1}$ for $k = k_0$. Now, for $k = k_0 + 1$ and $N \leq (k_0 + 2)(n-1)$, we have from Theorem 1.1 that F is equivalent to (z, wh), where $h \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$. with geometric rank 0 or 1.

Since $N-(n-1) \leq (k_0+1)(n-1)$, by the induction hypothesis, $\deg(F) \leq 1 + \deg h \leq 1 + \frac{(N-n+1)-1}{n-1} = \frac{N-1}{n-1}$. □

We now prove Theorem 5.1.

Let F be as in Corollary 5.3. Assume that F is not linear. Then, by [Hu03], F must have geometric rank 1. By Theorem 5.2, we have that F is equivalent to $H = (z_1, \ldots, z_{n-1}, w \cdot h(z, w))$, where $h(z, w)$ is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N-n+1}$.

Since $N \leq 3n - 3$ and $N - n + 1 \leq 2n - 2$, we have that h is equivalent to a linear embedding, by the First Gap Theorem. Thus, there is a unitary transformation U mapping $h(\mathbb{B}^n)$ into the intersection of the ball with the affine complex subspace of $\mathbb{C}^{N-n+1}$ defined by $z_1 \equiv c_1, \ldots, z_{N-2n+1} \equiv c_{N-2n+1}$, where each c_j is a non-negative constant.

We write $|c| = \sqrt{c_1^2 + \cdots + c_{N-2n+1}^2}$.

Composing H with (Id, U) from the left reveals that H is equivalent to a map (which, to avoid confusion, we will also call H)

$$H = (z_1, \ldots, z_{n-1}, (c_1, \ldots, c_{N-2n+1}w), \sqrt{1 - |c|^2} w\tilde{h}(z, w)),$$

with $\tilde{h}(z, w) \in Aut(\mathbb{B}^n)$. This is equivalent to the following, after a unitary transformation:

$$H = (z_1, \ldots, z_{n-1}\, |c|\, w, 0, \ldots, 0, w\sqrt{1 - |c|^2}\tilde{h}).$$

From [Ha05,§4], we get that H is equivalent to $(F_\theta, 0)$, with $\theta \in (0, \frac{\pi}{2}]$, and from here, the theorem follows directly from applying [Ha05, Theorem 1.1]. □

From this, we get an even better result, which is relevant for when we later consider the boundary case of the Second Gap Theorem.

5.8 Theorem (HJX06, Theorem 3.2)**.** *Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$, with $4 \leq n \leq N = 3n - 3$. Suppose that F has geometric rank 1. Then F is equivalent to*

$$F'_\theta = (F_\theta(z, w), 0, \ldots, 0) = (z, \cos\theta, z_1 w \sin\theta, \ldots, z_{n-1} w \sin\theta, w^2 \sin\theta, 0, \ldots, 0)$$

for some $0 \leq \theta \leq \frac{pi}{2}$. In other words, it is equivalent to the D'Angelo Map.

To see that this is exact, we look at the following example, from [Hu03] and [HJX06].

5.9 Example (HJX06, Example 3.3). Let

$$\begin{aligned}
\psi_1 &= (z_1^2, \sqrt{2}z_1z_2, \ldots, \sqrt{2}z_1k_{k-1}, z_1z_k, \ldots, z_1z_n),\\
\psi_2 &= (z_2^2, \sqrt{2}z_2z_3, \ldots, \sqrt{2}z_2z_{k-1}, z_2z_k, \ldots, z_2z_n),\\
&\vdots\\
\psi_{k-1} &= (z_{k-1}^2, z_{k-1}z_k, \ldots, z_{k-1}z_n),\\
\psi_k &= (z_k, \ldots, z_n)
\end{aligned}$$

Note that ψ_j has $n-j+1$ components.

Let $W_{n,k} = (\psi_1, \ldots, \psi_k)$. We can see easily that $W_{n,1}$ is the identity map. Also, $W_{n,2}$ is the standard Whitney Map. For $k \leq n$, we see that $W_{n,k}$ is a proper polynomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $N = n + P(n, k-1)$. Here, $W_{n,k}$ is not $(n-(k-1)+1)$-linear for $P(n,k) = \frac{k(2n-k+1)}{2}$. Then, for $k = 3$, we get $W_{n,3}$, from $\mathbb{B}^n$ to $\mathbb{B}^{3n-3}$, is not $(n-1)$-linear, which tells us that $W_{n,3}$ is not equivalent to $(F_\theta, 0, \ldots, 0)$. Thus, we establish the right side of the Second Gap Theorem through constructing a map not equivalent to any of the earlier ones.

Note that there is a misprint in the formula for $P(n,k)$ given in [HJX06].

A normal form for $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2

The final part of [HJX06] covers a normalization problem for mappings with geometric rank 1. We have, from [Hu03] and [CD96], that these maps may be very abstract, and thus it is important to normalize them under the action of the isotropic group of the Heisenberg hypersurfaces, though even with these restrictions, we are forced to use maps with degree 2.

Let $F = (f, \phi, g)$ be a proper rational map of degree 2 from $\mathbb{H}_2$ into $\mathbb{H}_N$, with $F(0) = 0$ and 0 a generic point of F, meaning $\kappa_F(0) = 1$. We assume $N \geq 4$.

First, we apply [Hu03, Lemma 3.2], to show that F is equivalent to

$$\begin{cases} f = z + \frac{i}{2}zw + o_{wt}(3), \frac{\partial^2 f}{\partial w^2}(0) = 0 \\ g = w + o_{wt}(4) \\ \phi_1 = z^2 + A_1 zw + B_1 w^2 + E_1 z^3 + \dots \\ \phi_j = o_{wt}(2), j \geq 2. \end{cases} \qquad (\star\star\star)$$

For a certain unitary matrix U, we can replace $(\phi_2, \dots, \phi_{N-2})$ with $(\phi_2, \dots, \phi_{N-2}) \cdot U$, so that we can get

$$\phi_j = A_j zw + B_j w^2 + o(|(z,w)|^2), j \geq 2$$

and

$$A_j = 0, j \geq 3.$$

Similarly, we can assume that $B_j = 0$ for $j \geq 4$, as long as $N \geq 6$.

We use the assumption that F has degree 2, and this allows us to refine our normalization thusly:

$$\begin{cases} \phi_2 = A_2 zw + B_2 w^2 + o(|(z,w)|^2) \\ \phi_2 = B_3 w^2 + o(|(z,w)|^2) \\ \phi_j = 0, j \geq 4. \end{cases}$$

We now look at the terms of our basic geometric equation having weighted degree 5. First, our equation is

$$Im(g) = |f|^2 + |\phi|^2, Im(w) = |z|^2.$$

Considering the terms of weighted degree 5, we get

$$Im(g^{(5)} - 2iz\overline{f^{(4)}}) = 2Re(z^2\overline{A_1}\overline{zw} + z^2\overline{E_1 z^3}).$$

We get this by moving the terms we get from f to the left side.

It follows from this and our normalization that $g^{(5)} \equiv 0$ and $f^{(4)} = az^2w$ with $a = -\overline{A_1}$ and $E_1 = -2ia$.

$$\begin{aligned} Im(2iz\overline{f^{(4)}}) &= 2Re(z^2\overline{A_1 zw} + z^2\overline{E_1 z^3}) \\ 2Re(-z\overline{f^{(4)}}) &= 2Re(z^2\overline{A_1 zw} + z^2\overline{E_1 z^3}) \\ Re(-z\overline{az^2w}) &= Re(-z^2 a\overline{zw} - z^2\overline{2iaz^3}) \\ Re(\overline{azw}) &= Re(za\overline{w} + 2i\left|z\right|^2\overline{az}) \\ Re(\overline{azw}) &= Re(za\overline{w} + 2i\tfrac{w-\overline{w}}{2i}\overline{az}) \\ Re(\overline{azw}) &= Re(za\overline{w} - w\overline{az} + \overline{waz}) \\ Re(\overline{azw}) &= Re(za\overline{w} - za\overline{w} + \overline{azw}). \end{aligned}$$

We use the fact that $\left|z\right|^2 = \frac{w-\overline{w}}{2i}$ and $Re(w\overline{az}) = Re(za\overline{w})$.

Next, we look at the terms of the basic equation with weighted degree 6:

$$Im(g) = 2Re(-\frac{1}{4}zw\overline{zw} + z^2\overline{B_1 w^2} + A_1 zw\overline{E_1 z^3}).$$

We parameterize this by $w = (u + i\left|z\right|^2)$, getting

$$\begin{aligned} Im(g) \quad = \quad & 2Re(-\tfrac{1}{4}\left|z\right|^2(u + i\left|z\right|^2)(\overline{u + i\left|z\right|^2}) + \\ & z^2\overline{B_1(u + i\left|z\right|^2)^2} + A_1\left|z\right|^2(u + i\left|z\right|^2)\overline{E_1 z^2}). \end{aligned}$$

The coefficients of the u^3 terms give us this:

$$g = w + \mu w^3 + \mathcal{O}(\left|(z,w)\right|^4) \cap o_{wt}(5), \mu \in \mathbb{R}.$$

We take advantage of the fact that F has degree 2, and we write $f = \frac{P_0}{Q}, \phi_j = \frac{P_j}{Q}$, $j = 1, \ldots, N-2$, and $g = \frac{G}{Q}$, with all of these polynomials of degree less than or equal to 2. Also $Q = 1 + L(z,w) + Q_0(z,w)$, with L linear and Q_0 quadratic.

We combine our normalization $(\star\star\star)$ with our expression for g to get that

$$P_0 = z(1 + L(z,w) + iw/2)$$

and

$$G = w(1 + L(z,w))$$

and

$$Q_0 = -\mu w^2.$$

We can write our linear function L as $id_1 z + id_2 w$.

We choose

$$\sigma = \frac{(z,w)}{1+rw}, \tau = \frac{(z^*,w^*)}{1-rw^*}$$

with $r = -Re(id_2)$, and then we replace F with $\tau \circ F \circ \sigma$. Then we can assume that $d_2 \in \mathbb{R}$.

We find a new σ and τ to modify F again:

$$\begin{aligned}\sigma(z,w) &= (e^{i\theta}z, w), \tau(z_1^*, z_2^*, \ldots, z_{N-1}^* w^*) \\ &= (e^{i\theta}z_1^*, e^{-2i\theta}z_2^*, e^{(i\beta_3)}z_3^*, \ldots, e^{(i\beta_{N-1})}z_{N-1}^*, w^*).\end{aligned}$$

We can choose θ and β_j so that $A_1 \geq 0$, $A_2 \geq 0$, and $B_3 \geq 0$, and when $A_2 = 0$, then $B_2 \geq 0$.

We rewrite our basic equation as follows:

$$Im\left(w(1+L(z,w))(1+\overline{L(z,w)} - \mu\overline{w}^2)\right) =$$

$$|z(1+L(z,w)+i/2w)| + \left|z^2 + A_1 zw + B_1 w^2\right|^2 + \left|A_2 zw + B_2 w^2\right|^2.$$

Now we look only at the terms of weighted degree 8:

$$-Im(id_2\mu w^2\overline{w}^2) = \sum_{j=1}^{3} |b_j|^2 |w|^4.$$

Since $\mu \in \mathbb{R}$ and $|w|^4 \in \mathbb{R}$, we get

$$-\mu d_2 = \sum_{j=1}^{3} |B_j|^2.$$

Now, looking at the terms of weighted degree 7 rather than 8, we get

$$-Im(id_1\mu z\overline{w}\,|w|^2) = \sum_{j=1}^{2}(A_j\overline{B_j}z\overline{w}\,|w|^2 + B_j\overline{A_j}w\overline{z}\,|w|^2),$$

and, using reasoning similar to what we just used, we get

$$-\mu d_1 = 2\sum_{j=1}^{2} A_j \overline{B_j}.$$

Next, we look at the terms of weighted degree 6, and we see:

$$B_1 = 0, \mu = \frac{1}{4} + d_2 + \sum_{j=1}^{2} |A_j|^2.$$

We have $E_1 = id_1$ and $a = \frac{d_1}{2}$, so by our result for the degree 7 terms, we get $d_1 \in \mathbb{R}$ and $A_1 = -\frac{d_1}{2}$. It follows that $d_1 \leq 0$.

Using the degree 7 results again, we get $B_2 \in \mathbb{R}$. When $A_2 \neq 0$, we can see that $B_2 \geq 0$. If $A_2 = 0$, then $B_2 \geq 0$ and $B_3 = 0$, by applying a unitary matrix to the $\phi_2, \ldots, \phi_{N-2}$ components.

Now, when $\mu = 0$, we have $g = w$, $f = zf^*$, and $\phi = z\phi^*$, where (f^*, ϕ^*) is a linear fractional map from $\mathbb{H}_2$ into $\mathbb{B}^{N-1}$, and the pullback of F to the map from $\mathbb{B}^2$ to $\mathbb{B}^N$ is equivalent to the D'Angelo Map:

$$F_\theta(z, w) = (z, w\cos\theta, zw\sin\theta, w^2\sin\theta, 0, \ldots, 0).$$

This leads to a theorem.

5.10 Theorem (HJX06, Theorem 4.1)**.** *Let $F \in Rat(\mathbb{H}_2, \mathbb{H}_N)$ with degree 2, where $F(0) = 0$ and $\kappa_F(0) = 1$, with $N \geq 4$. Then there are $\sigma \in Aut_0(\partial\mathbb{H}_2)$ and $\tau \in Aut_0(\partial\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$, which we write as (f, ϕ, g), takes the following form:*

$$f(z,w) = \frac{z - 2id_1z^2 + (i/2 - id_2)zw}{1 - id_2w - \mu w^2 - 2id_1z}$$
$$\phi_1(z,w) = \frac{z^2 + d_1zw}{1 - id_2w - \mu w^2 - 2id_1z}$$
$$\phi_2(z,w) = \frac{c_1w^2 + \nu zw}{1 - id_2w - \mu w^2 - 2id_1z}$$
$$\phi_3(z,w) = \frac{c_2w^2}{1 - id_2w - \mu w^2 - 2id_1z}$$
$$\phi_j \equiv 0, j \geq 4$$
$$g(z,w) = \frac{w - id_2w^2 - 2id_zzw}{1 - id_2w - \mu w^2 - 2id_1z}.$$

When $N =$, ϕ has only two components, ϕ_1 and ϕ_2. Here, $\nu, \mu, d_1, d_2, c_1,$ and c_2 are non-negative real numbers.

Also,

$$\mu d_2 = c_1^2 + c_2^2$$

$$\mu + d_2 = 14 + d_1^2 + \nu^2$$

$$\mu d_1 = \nu c_1$$

$$c_2 = 0 \; if \; \nu = 0.$$

We have two more results.

(I) We have that ν, μ, d_1, d_2, c_1 and c_2 are uniquely determined by F. Converse, for any non-negative real ν, μ, d_1, d_2, c_1 and c_2 satisfying the relations above, the corresponding F is an element of $Rat(\mathbb{H}_2, \mathbb{H}_N)$ of degree 2 with $F(0) = 0$ and $\kappa_F(0) = 1$.

(II) If $\mu = 0$, then $\rho_N^{-1} \circ F \circ \rho_n$, where $\rho_n = \left(\frac{2z}{1-iw}, \frac{1+i2}{1-iw}\right)$, is equivalent to $(F_\theta, 0)$, where F_θ is the D'Angelo Map.

Note that [HJX06] lists ρ_2 here instead of ρ_n.

5.11 Remark (HJX06). Theorem 5.10 implies that any rational proper holomorphic map from $\mathbb{B}^2$ into $\mathbb{B}^N$ with $N \geq 5$ of degree 2 is equivalent to a rational proper holomorphic map from $\mathbb{B}^2$ into $\mathbb{B}^5$. Similarly, for any positive integers $k, n,$ and N with $N \geq n \geq 2$, we have an integer N_0 such that any rational proper map of degree bounded by k from $\mathbb{B}^n$ into $\mathbb{B}^N$ is equivalent to a rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N_0}$.

We end this chapter by proving the remainder of Theorem 5.10. We need only show the uniqueness of μ, ν, d_1, d_2, c_1 and c_2.

Suppose $F^* = \tau^* \circ F \circ \sigma = (f^*, \phi^*, g^*)$ with $\sigma \in Aut_0(\partial\mathbb{H}_2)$ and $\tau^* \in Aut_0(\partial\mathbb{H}_N)$. Suppose F and F^* satisfy the normalization in the theorem. We apply [Hu03, Lemma 2.2(A)],[Hu03,(2.4.1)], and [Hu03,(2.4.2)] to get

$$\sigma(z, w) = \frac{(\lambda(z + aw) \cdot U, \lambda^2 w)}{q(z,w)}$$

$$\tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*) \cdot U^*, \lambda^{*2} w^*)}{q^*(z^*, w^*)},$$

with

$$
\begin{aligned}
&q(z,w) = 1 - 2i\langle \overline{a}, z\rangle + (r - i\,|a|^2)w \\
&\lambda > 0 \\
&r \in \mathbb{R} \\
&a \in \mathbb{C} \\
&|U| = 1 \\
&q^*(z^*,w^*) = 1 - 2i\langle \overline{a^*}, z^*\rangle + (r^* - i\,|a^*|^2)w^* \\
&\lambda^* > 0 \\
&r^* \in \mathbb{R} \\
&a^* \in \mathbb{C}^{N-1} \\
&U^* \textit{ a unitary } (N-1)\times(N-1) \textit{ matrix.}
\end{aligned}
$$

By [Hu03,(2.5.2),(2.5.2)],

$$
\begin{aligned}
&\lambda^* = \lambda^{-1} \\
&a_1^* = -\lambda^{-1} a U \\
&a_2^* = 0 \\
&r^* = -\lambda^{-2} r
\end{aligned}
$$

and

$$
U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix},
$$

with $a^* = (a_1^*, a_2^*)$ and U_{22}^* a unitary $(N-2)\times(N-2)$ matrix. Now, we write

$$
\begin{aligned}
&A = -2i\tfrac{\partial^2 f}{\partial z \partial w}(0) \\
&B^i = \tfrac{\partial^2 \phi_i}{\partial z^2}(0) \\
&B^{*i} = \tfrac{\partial^2 \phi_i^*}{\partial z^2}(0) \\
&\mathcal{B} = \left(\tfrac{\partial^2 \phi_1}{\partial z \partial w}, \ldots, \tfrac{\partial^2 \phi_{N-2}}{\partial z \partial w}\right) \\
&\mathcal{B}^* = \left(\tfrac{\partial^2 \phi_1^*}{\partial z \partial w}, \ldots, \tfrac{\partial^2 \partial_{N-2}^2}{\partial z \partial w}\right),
\end{aligned}
$$

with $i = 1, \ldots, N-2$.

By [Hu03,Lemma 2.2(A)], we get

$$\frac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU \cdot A \cdot U^{-1} + \lambda^3 \frac{\partial^2 f}{\partial w^2}(0)U^{-1}$$
$$(B^{*1}, \ldots, B^{*N-2}) = \lambda U(B^1, \ldots, B^{N-2})U^t U_{22}^*$$
$$\mathcal{B}^* = \lambda U(B^1, \ldots, B^{N-2})U^t a^t U_{22}^* + \lambda^2 U\mathcal{B}U_{22}^*$$
$$\frac{\partial^2 \phi^*}{\partial w^2}(0) = \lambda aU(B^1, \ldots, B^{N-2})U^t a^t U_{22}^* + 2\lambda^2 aU\mathcal{B}U_{22}^* + \lambda^3 \frac{\partial^2 \phi}{\partial w^2}(0)U_{22}^*.$$

We have that both F and F^* satisfy the normalization, and $a = 0$ and $\lambda = 1$. We also have that $d_2, d_2^* \in \mathbb{R}$, and it follows directly that $r = 0$.

We can write U_{22}^* as $(\alpha_{k\ell})$. By the second equation above, we have $\alpha_{11}e^{2i\theta} = 1$, and $\alpha_{1j} = 0$ for $j \geq 2$.

Now, $d_1, d_1^*, nu, \nu^* \geq 0$. By this fact that the third equation above, we get $(d_1^*, \nu^*) = (d_1, \nu)$.

We see that $\mu = 0$ implies that $g = g^* = w$.

Thus, c_1, c_2, c_1^*, c_2^*, and $\mu^* = 0$, and since $f^* = f$, we have $d_1 = d_1^*$, $d_2 = d_2^*$, and $\nu = \nu^*$.

Suppose $\mu, \mu^* \neq 0$. The rest follows from $g = g^*$ if ν does not vanish. We get something similar if $\mu \neq 0$ and $\nu = 0$.

Part III

On the Third Gap Theorem

Chapter 6

Introduction

We give an overview of the Third Gap Theorem and its proof, which appeared in a recent paper, [HJY14]. As the main theorem of this dissertation occurs in a setting between the second and third gaps (at the boundary of the second one), understanding the third gap puts the second in context, and the proof of our main theorem uses a few results from the proof of the Third Gap Theorem.

The main goal of this discussion is to prove a certain fact about how many maps there are, up to automorphism on $\partial\mathbb{B}^n$ and $\partial\mathbb{B}^N$, from $\mathbb{B}^n$ to $\mathbb{B}^N, 3n+1 \leq N \leq 4n-7$ (with $n > 7$). This is, we recall, the third such gap, with the previous two having been discussed at length in earlier chapters. Recall, too, that we say that two maps F and G are equivalent if there exist $\tau \in Aut(\mathbb{B}^N)$ and $\sigma \in Aut(\mathbb{B}^n)$ such that $G = \tau \circ F \circ \sigma$.

We state the theorem.

6.1 Theorem (Third Gap Theorem)**.** *Let F be a proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^N$, with $n > 7$ and $3n+1 \leq N \leq 4n-7$. Then there is an automorphism $\tau \in Aut(\mathbb{B}^N$ such that $\tau \circ F = (G, 0') = (G, 0, 0, \ldots, 0)$, where G is a proper holomorphic rational map from $\mathbb{B}^n$ to $\mathbb{B}^{3n}$.*

We use an example to show that this is a sharp inequality.

6.2 Example. For $n \geq 2$, $\lambda, \mu \in (0,1)$, define the proper monomial map F from $\mathbb{B}^n$ to $\mathbb{B}^{3n}$ as follows:

$$F = (z_1, \ldots, z_{n-2}, \lambda z_{n-1}, z_n, \sqrt{1-\lambda^2} z_{n-1}(z_1, \ldots, z_{n-1}, \mu z_n, \sqrt{1-\mu_2} z_n z)).$$

There is no $\tau \in Aut(B^{3n})$ such that $\tau \circ F = (G, 0')$. Also, there are proper monomial maps F from B^n into B^{4n-6} such that for any $\tau \in Aut(\mathbb{B}^{4n-6})$, $\tau \circ F$ cannot be of the form $(G, 0')$.

Now, we can actually restate the Third Gap Theorem in simpler form, because of a rationality theorem proved in [Hu03] and [HJX05]. Any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $N \leq n(n+1)/2$ that is three times differentiable up to the boundary must be rational. Thus:

6.3 Theorem (Third Gap Theorem)**.** *Let F be a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $n > 7$ and $3n+1 \leq N \leq 4n-7$. Assume that F is C^3-smooth up to the boundary. Then there is an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F = (G, 0')$, where G is a proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{3n}$.*

Earlier results give us a complete picture of the linearity problem for proper holomorphic embeddings from $\mathbb{B}^n$ to $\mathbb{B}^N$ that are C^2-smooth up to the boundary, but we might wish to know about the linearity for mappings with a more complicated structure. For instance, the following remains an open question:

6.4 Conjecture (Siu,Mok). Let f be a proper holomorphic mapping from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $1 < n < N$. Write $M = F(\mathbb{B}^n)$. Suppose that there is a subgroup Γ of $Aut(\mathbb{B}^N)$ such that (1) for any $\sigma \in \Gamma, \sigma(M) = M$, and (2) M/Γ is compact. Then f is a linear embedding.

We denote by $\mathcal{I}_1$ the interval $(2n, 2n-1)$ and by $\mathcal{I}_2$ the interval $(2n, 3n-3)$, i.e., the first and second gaps. We denote by $\mathcal{I}_3 = [3n+1, 4n-7]$ the third gap.

Put generally, for any $n \geq 3$, write $K(n)$ for the largest positive integer m such that $m(m+2)/2 < n$. Then $K(n) = \frac{-1+\sqrt{1+8n}}{2}$ if $\frac{-1+\sqrt{1+8n}}{2} \notin \mathbb{Z}$ and $\frac{-1+\sqrt{1+8n}}{2} - 1$ otherwise. For each $1 \leq k \leq K(n)$, define $\mathcal{I}_k := [kn+1, (k+1)n - \frac{k(k+1)}{2} - 1]$. Then $\mathcal{I}_k$ is a closed interval containing positive integers if $n \geq 2 + \frac{k(k+1)}{2}$, and $\mathcal{I}_k \cap \mathcal{I}_\ell = \emptyset$ for $k \neq \ell$. We have that this defines the same $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ as the Gap Theorems. Write $\mathcal{I} = \bigcup_{k=1}^{K(n)} \mathcal{I}_k$. Then,

$$\begin{aligned} \max_{N \in \mathcal{I}} N &= (K(n)+1)n - \tfrac{K(n)(K(n)+1)}{2} - 1 \approx \tfrac{-1+\sqrt{1+8n}}{2} n - n - 1 \\ &\approx \sqrt{2} n^{3/2} - n - 1. \end{aligned}$$

In [HJY07], it was shown that for any $N \notin \mathcal{I}$, there are many monomial proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^N$ that are not equivalent to maps of the form $(G, 0')$.

Such maps have not been found for $N \notin \mathcal{I}$, and thus we have the following conjecture:

6.5 Conjecture (HYJ07). Let $n \geq 3$ be a positive integer, and let $\mathcal{I}_k (1 \leq k \leq K(n))$ be as defined above. Then any proper holomorphic rational map F from $\mathbb{B}^n$ into $\mathbb{B}^N$ is equivalent to a map of the form $(G, 0')$ if and only if $N \in \mathcal{I}_k$ for some $1 \leq k \leq K(n)$.

We now present an overview of the method used to prove the Third Gap Theorem.

When $N \in \mathcal{I}_3$, it is much more difficult to classify every map than it was in the earlier cases, so we use a different approach, starting from [HJ01], [Ha05], and [HJX06].

Consider the case of the Heisenberg hypersurface. Let F be a holomorphic map into $\mathbb{C}^n$ defined near 0 with $F(0) = 0$. By Taylor's formula, we have

$$F(z) = \sum_{\alpha} \frac{D^\alpha F}{\alpha!}(0) z^\alpha.$$

Because the constant term of this is zero, the image of $F(0)$ is spanned by $\{D^\alpha F(0)\}_\alpha$. That is, $F(0)$ is a linear combination of the derivatives of F evaluated at 0. Thus, if $\{D^\alpha F(0)\}_\alpha$ do not span $\mathbb{C}^N$, there is some value that F does not hit. As it is put in the Third Gap Theorem paper, F has a gap.

Our goal is, then, to find a basis for $span\{D^\alpha F(0)\}_\alpha$, and we will do this by finding a good normal form for F.

6.6 Definition (Jet). *Let M be an n-dimensional manifold, and let (E, π, X) be a fiber bundle over M. Let $p \in M$, and let $\Gamma(p)$ be the set of all local sections whose domains contain p. Let $I = I_1, I_2, \ldots, I_n$ be a multi-index. Then*

$$|I| := \sum_{i=1}^{n} I_i$$

and

$$\frac{\partial^{|I|}}{\partial z^I} := \prod_{i=1}^{n} \left(\frac{\partial}{\partial z^i} \right)^{I_i}.$$

We then say that two sections σ and τ have the same r-jet if

$$\left. \frac{\partial^{|I|} \sigma^\alpha}{\partial z^I} \right|_p = \left. \frac{\partial^{|I|} \tau^\alpha}{\partial z^I} \right|_p, 0 \leq |I| \leq r.$$

This defines an equivalence relation, and r is called the order of the equivalence class.

In our case, we find that it is easy to get a linearly independent set from the first two jets, but finding anything linearly independent for those of order three or higher is considerably more difficult. (Jets, here, are coordinate-independent versions of the Taylor expansion.)

To do this, we use Huang's Lemma, and it turns out that for $N \in \mathcal{I}_3$, there is only one more linearly independent element for the map from the higher order jets. This is fortunate, because we need to determine this result very precisely, and we can only do that because of the extra structure in our setting.

At many times in the proof of the Third Gap Theorem, we will need to use [Hu1,Lemma 3.2], but it so happens that $N \in \mathcal{I}_3$ is the very assumption we need to make in order to apply that lemma.

We review our notation and other basic concepts that we will need later.

The Siegel upper-half space is defined as $\mathbb{H}_n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(w) > |z|^2\}$. We want to define proper rational maps from $\mathbb{H}_n$ to $\mathbb{H}_N$. Recall the Cayley transformation

$$\rho_n : \mathbb{H}_n \to \mathbb{B}^n, \rho_n(z, w) = \left(\frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw} \right) \qquad (6.1.1)$$

Here, ρ_n is biholomorphic. We can thus identify a proper rational map F from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $\rho_N^{-1} \circ F \circ \rho_n$, and that is a proper rational map from $\mathbb{H}_n$ to $\mathbb{H}_N$. A theorem by Cima-Suffridge [CS2] says that F extends holomorphically across the boundary $\partial\mathbb{B}^n$.

We parameterize $\partial\mathbb{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \mapsto (z, u + i|z|^2)$, and we will assign weights of 1 and 2 to z and u, respectively.

6.7 Definition (Weighted Degree). *For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}_n$ is said to be of weighted degree $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2 w}{|t|^m} \to 0$ uniformly for (z, u) on any compact subset of U as $t \to 0$.*

Write use the notation $h^{(k)}$ for a polynomial h with weighted degree k.

If we have a holomorphic function or map $H(z, w)$, we may write

$$H(z, w) = \sum_{k,\ell=0}^{\infty} H^{(k,\ell)}(z) w^\ell$$

with $H^{(k,\ell)}(z)$ a polynomial of degree k in z. That is, $H^{(k,\ell)}$ is the coefficient of the w^ℓ term of $H(z, w)$ with degree k.

We write $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g)$ for a non-constant C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$.

Now, for any $p = (z_0, w_0) \in M$ near 0, we write $\sigma_p^0 \in Aut(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and $\tau_P^F \in Aut(\mathbb{H}_N)$ by

$$\tau_p^F(z^*, w^*) = (z^* - \tilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - wi\langle z^*, \overline{\tilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to F_p:

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p). \qquad (6.1.2)$$

We note that $F_0 = F$ and $F_p(0) = 0$.

6.8 Lemma (Hu99,§2, Lemma 5.3). *Let F be a C^2-smooth CR map from $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$, $2 \leq n \leq N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such* *that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3), \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2), g_p^{**} = w + o_{wt}(4),$$

with

$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = \left| \phi_p^{**(2)(z)} \right|^2.$$

6.9 Definition (Hu03)**.** *Write $\mathcal{A}(p) = -2i \left(\frac{\partial^2 (f_p^{**})_\ell}{\partial z \partial w} \Big|_0 \right)_{1 \le j, \ell \le (n-1)}$ in the above lemma. We call the rank of the $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the geometric rank of F at p.*

Define the geometric rank of F to be $\kappa_0(F) = \max_{p \in \partial \mathbb{H}_n} Rk_F(p)$. Define the geometric rank of a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ that is C^2-smooth up to the boundary to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n$. By [Hu2], $\kappa_0(F)$ depends only on the equivalence class of F.

When $N < \frac{n(n+1)}{2}$, $\kappa_0(F) \le n-2$.

In [HJX06], the following normalization theorem for maps with geometric rank bounded by $n-2$ was shown:

6.10 Theorem (HJX1)**.** *Suppose that F is a rational proper holomorphic map from $\mathbb{H}_n$ into $\mathbb{H}_N$ with geometric rank $1 \le \kappa_0 \le n-2$, with $F(0) = 0$. Then there exist $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which we shall still denote $F = (f, \phi, g) = (\tilde{f}, g)$ for convenience:*

$$\begin{cases} f_\ell = \sum_{j=1}^{\kappa_0} z_j f_{\ell j}^*(z,w), \ell \le \kappa_0 \\ f_j = z_j, \kappa_0 + 1 \le j \le n-1 \\ \phi_{\ell k} = \mu_{\ell k} z_\ell z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{\ell j k}^*, (\ell, k) \in \mathcal{S}_0 \\ \phi_{\ell k} = \sum_{j=1}^{\kappa_0} z_j \phi_{\ell j k}^* = \mathcal{O}_{wt}(3), (\ell, k) \in \mathcal{S}_1 \\ g = w \\ f_{\ell j}^*(z,w) = \delta_\ell^j + \frac{i \delta_\ell^j \mu_\ell}{2} w + b_{\ell j}^{(1)}(z) w + O_{wt}(4), 1 \le \ell \le \kappa_0, \mu_\ell > 0 \\ \phi_{\ell j k}^*(z,w) = \mathcal{O}_{wt}(2), (\ell, k) \in \mathcal{S}_1. \end{cases}$$

Here, for $1 \leq \kappa_0 \leq n-2$, we write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, the index set for all components of ϕ, where $\mathcal{S}_0 = \{(j,\ell) : 1 \leq j \leq \kappa_0, 1 \leq \ell \leq n-1, j \leq \ell\}$ and $\mathcal{S}_1 = \{(j,\ell) : j = \kappa_0+1, \kappa_0+1 \leq \ell \leq N-n-\frac{(2n-\kappa_0-1)\kappa_0}{2}\}$. Also, $\mu_{j\ell} = \sqrt{\mu_j + \mu_\ell}$ for $j < \ell \leq \kappa_0$, and $\mu_{j\ell} = \sqrt{\mu_j}$ if $j \leq \kappa_0 < \ell$ or if $j = l \leq \kappa_0$.

We need one more lemma, which will be fundamental in the proof later on.

6.11 Lemma (Hu99)**.** *Let k be a positive integer such that $1 \leq k \leq n-2$. Assume that $a_1, \ldots, a_k, b_1, \ldots, b_k$ are germs at $0 \in \mathbb{C}^{n-1}$ of holomorphic functions such that $a_j(0) = 0, b_j(0) = 0$, and*

$$\sum_{i=1}^{k} a_i(z)\overline{b_i(z)} = A(z,\overline{z})\,|z|^2 \qquad (6.1.4)$$

where $A(z,\overline{z})$ is a germ at $0 \in \mathbb{C}^{n-1}$ of a real analytic function. Then

$$A(z,\overline{z}) = \sum_{i=1}^{k} a_i(z)\overline{b_i(z)} = 0.$$

Chapter 7

Analysis of the Chern-Moser equation

Now, let $F = (f, \phi, g)$ be a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_N$ that satisfies the normalization from Theorem 6.10, with $1 \le \kappa_0 \le n-2$. Write the codimension part ϕ of the map F as $\phi := (\Phi_0, \Phi_1)$ with $\Phi_0 = (\phi_{\ell k})_{(\ell,k)\in\mathcal{S}_0}$ and $\Phi_1 = (\phi_{\ell k})_{(\ell,k)\in\mathcal{S}_1}$. Write

$$\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j, \quad \Phi_1^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} \hat{e}_j z_j,$$

with $e_j \in \mathbb{C}^{\#(\mathcal{S}_0)} = \mathbb{C}^{\kappa_0 n - \frac{\kappa_0(\kappa_0+1)}{2}}$, $\hat{e}_j \in \mathbb{C}^{\#(\mathcal{S}_1)}, \xi_j(z) = \overline{e}_j \cdot \Phi_0^{(2,0)}(z)$,
and $\xi = (\xi_1, \ldots, \xi_{\kappa_0})$. We also write

$$\phi^{(1,1)}(z)w = \sum e_j^* z_j w, \ with\ e_j^* = (e_j, \hat{e}_j),$$

$$H = \sum_{(i_1,\ldots,i_{n-1},i_n)} H^{(i_1,\ldots,i_n)} z_1^{i_1}\cdots z_{n-1}^{i_{n-1}} w^{i_n} = \sum_{k,j=0}^{\infty} H^{(k,j)}(z)w^j \ for\ H = f\ or\ \phi.$$

Here, $H^{(k,j)}(z)$ is a homogeneous polynomial of degree k in z.

7.1 Lemma (HJY14, Lemma 3.1). *Let* $\left(\Gamma_j^{[h]}(z)\right)_{1\le j\le\kappa_0, h=1,2}$ *be some holomorphic functions of* z. *Let* $\mu_{j\ell}$ *and* μ_j *be as in Theorem 6.10. Suppose that for* $h = 1, 2$, $\left(\Lambda_{j\ell}^{[h]}\right)_{(j,\ell)\in\mathcal{S}_0}$

are defined as follows:

1. $\mu_{j\ell}\Lambda_{j\ell}^{[h]}(z) = 2i(z_j\Gamma_\ell^{[h]} + z_\ell\Gamma_j^{[h]}), j < \ell \leq \kappa_0$
2. $\mu_{jj}\Lambda_{jj}^{[h]}(z) = 2iz_j\Gamma_j^{[h]}(z), j \leq \kappa_0$
3. $\mu_{j\ell}^{[h]}(z) = 2iz_\ell\Gamma_j^{[h]}(z), j \leq \kappa_0 < \ell$

Then we have (7.1.1)*:*

$$\sum_{(j,\ell)\in\mathcal{S}_0} \overline{\Lambda_{j\ell}^{[1]}}\Lambda_{j\ell}^{[2]} = 4|z|^2\left(\sum_{j\leq\kappa_0}\tfrac{1}{\mu_j}\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]}\right) - \sum_{j<\ell\leq\kappa_0}\tfrac{4}{\mu_j\mu_\ell(\mu_j+\mu_\ell)}\left(\mu_j\overline{z}_j\overline{\Gamma_\ell^{[1]}} - \mu_\ell\overline{z}_\ell\overline{\Gamma_j^{[1]}}\right)\cdot\left(\mu_j z_j\Gamma_\ell^{[2]} - \mu_\ell z_\ell\Gamma_j^{[2]}\right).$$

Proof. We use the formulas in Theorem 6.10 for $\mu_{j\ell}$, μ_j and μ_ℓ, and we compute the following:

$$\begin{aligned}\tfrac{1}{4}\sum_{(j,\ell)\in\mathcal{S}_0}\overline{\Lambda_{j\ell}^{[1]}}\Lambda_{j\ell}^{[2]} &= \sum_{1\leq j\leq\kappa_0}\tfrac{|z_j|^2}{\mu_j}\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]} + \sum_{j\leq\kappa_0<\ell}\tfrac{|z_\ell|^2}{\mu_j}\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]} \\ &\quad + \sum_{j<\ell\leq\kappa_0}\tfrac{1}{\mu_j+\mu_\ell}\overline{(z_j\Gamma_\ell^{[1]} + z_\ell\Gamma_j^{[1]})}\cdot(z_j\Gamma_\ell^{[2]} + z_\ell\Gamma_j^{[2]}) \\ &= \left(\sum_{j\leq\kappa_0}\tfrac{1}{\mu_j}\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]}\right)|z|^2 - \sum_{\ell\leq\kappa_0,\ell\neq j\leq\kappa_0}\tfrac{1}{\mu_j}|z_\ell|^2\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]} \\ &\quad + \sum_{j<\ell\leq\kappa_0}\tfrac{1}{\mu_j+\mu_\ell}\overline{(z_j\Gamma_\ell^{[1]} + z_\ell\Gamma_j^{[2]})}\cdot(z_j\Gamma_\ell^{[2]} + z_\ell\Gamma_j^{[2]}).\end{aligned}$$

We observe an elementary identity:

$$\frac{\mu_j}{\mu_\ell}|z_j|^2\overline{\Gamma_\ell^{[1]}}\Gamma_\ell^{[2]} + \frac{\mu_\ell}{\mu_j}|z_\ell|^2\overline{\Gamma_j^{[1]}}\Gamma_j^{[2]} - z_j\Gamma_\ell^{[2]}\overline{\Gamma_j^{[1]}}\overline{z}_\ell - \overline{z}_j\overline{\Gamma_\ell^{[1]}}\Gamma_j^{[2]}z_\ell$$

$$= \frac{1}{\mu_j\mu_\ell}(\mu_j\overline{z}_j\overline{\Gamma_\ell^{[1]}} - \mu_\ell\overline{z}_\ell\overline{\Gamma_\ell^{[1]}})\cdot(\mu_j z_j\Gamma_\ell^{[2]} - \mu_\ell z_\ell\Gamma_j^{[2]}).$$

The lemma follows from this. □

7.2 Lemma (HJY14, Lemma 3.2)**.**

$$\tfrac{1}{4}\left|\Phi_0^{(3,0)}(z)\right|^2 = \left(\sum_{j\leq\kappa_0}\tfrac{1}{\mu_j}|\xi_j(z)|^2\right)|z|^2 - \sum_{j<\ell\leq\kappa_0}\tfrac{1}{\mu_j+\mu_\ell}\left|\sqrt{\tfrac{\mu_j}{\mu_\ell}}z_j\xi_\ell - \sqrt{\tfrac{\mu_\ell}{\mu_j}}z_\ell\xi_j\right|^2. \qquad (7.1.2)$$

Proof. We know from our setting that

$$-Im(g(z,w)) + |f(z,w)|^2 + |\phi(z,w)|^2 = 0 \ over\ Im(w) = |z|^2, \qquad (7.1.3)$$

so we can look at the terms of weighted degree 5 to get, again over $Im(w) = |z|^2$ (noting that Φ has terms of degree 2 and 3 but not 1, Φ_1 has no applicable terms, while f has terms of degree 1 and 4 but not 3),

$$z\overline{f^{(4)}(z,w)} + \overline{z}f^{(4)}(z,w) + \Phi_0^{(2)}(z,w)\overline{\Phi_0^{(3)}(z,w)} + \Phi_0^{(3)}(z,w)\overline{\Phi_0^{(2)}(z,w)} = 0$$

or, because we can parameterize $\partial\mathbb{H}_n$ by $w = u + i\,|z|^2$ (paying attention to Theorem 6.10 to see what $f^{(4)}$ looks like),

$$\begin{gathered} z\overline{f^{(2,1)}(z)(u+i\,|z|^2)} + \overline{z}f^{(2,1)}(z)(u+i\,|z|^2) + \\ \Phi_0^{(2)}(z)\overline{\Big(\Phi_0^{(3,0)}(z) + (\textstyle\sum e_j z_j)w\Big)} + \Big(\Phi_0^{(3,0)}(z) + (\textstyle\sum e_j z_j)w\Big)\overline{\Phi_0^{(2)}(z)} \equiv 0. \end{gathered} \tag{7.1.4}$$

The second part of this comes from the fact that the terms of ϕ of weighted degree 3 are $\sum_{j=1}^{\kappa_0} z_j\phi^*_{\ell jk}$, where $\phi^*_{\ell jk} = \mathcal{O}_{wt}(2)$, so it has components $\phi^{(3,0)}(z)$ and $\phi^{(1,1)}(z)w$, and we noted at the beginning of this section that $\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j$.

We collect the terms $\overline{z}^\alpha z^\beta$, where $|\alpha| = 1$ and $|\beta| = 2$, yielding

$$\overline{z}f^{(2,1)}(z)w + \Phi_0^{(2)}(z)\overline{\sum e_j z_j w} = 0, \ \ or,$$

$$\overline{z}f^{(2,1)}(z) = -\overline{(z_1,\ldots,z_{\kappa_0})}\cdot\xi(z). \qquad (7.1.5)$$

The $z^\alpha\overline{z}^\beta$ terms, with $|\alpha| = 3$ and $|\beta| = 2$, give us

$$i\overline{z}f^{(2,1)}(z)\,|z|^2 + \overline{\Phi_0^{(2)}(z)}\Phi_0^{(3,0)}(z) + \Phi_0^{(2)}(z)\overline{\sum_{j=1}^{\kappa_0} e_j z_j (i\,|z|^2)} \equiv 0, \qquad (7.1.6)$$

with the last part coming from $w = u + i\,|z|^2$, and thus

$$\overline{\Phi_0^{(2)}(z)}\Phi_0^{(3,0)}(z) = 2i\overline{(z_1,\ldots,z_{\kappa_0})}\cdot\xi(z)\,|z|^2. \qquad (7.1.7)$$

We get (7.1.7) by replacing $\overline{z}f^{(2,1)}(z)$ in (7.1.6) with $-\overline{(z_1,\ldots,z_{\kappa_0})}\cdot\xi(z)$ and then replacing $\Phi_0^{(2)}(z)\overline{\sum_{j=1}^{\kappa_0} e_j z_j}$ with $\overline{(z_1,\ldots,z_{\kappa_0})}\cdot\xi(z)$, as per the equality mentioned near the beginning of this section.

We expand this:

$$\sum_{(j,\ell)\in\mathcal{S}_0} \mu_{j\ell}\overline{z}_j\overline{z}_\ell\phi^{(3,0)}_{j\ell}(z) = 2i\left(\sum_{k=1}^{\kappa_0}\overline{z}_k\cdot\xi_k\right)\left(\sum_{k=1}^{n-1}z_k\overline{z}_k\right).$$

Equivalently,

1. $\mu_{l\ell}\phi^{(3,0)}_{j\ell}(z) = 2i(z_j\xi_\ell(z) + z_\ell\xi_j(z)),\ j < \ell \leq \kappa_0$
2. $\mu_{jj}\phi^{(3,0)}_{jj}(z) = 2iz_j\xi_j(z),\ j \leq \kappa_0$
3. $\mu_{j\ell}\phi^{(3,0)}_{j\ell}(z) = 2iz_\ell\xi_j(z),\ j \leq \kappa_0 < \ell.$

We then apply Lemma 7.1, and we are done. □

7.3 Lemma (HJY14, Lemma 3.3)**.** $\left|\phi^{(3,0)}(z)\right|^2 = A(z,\overline{z})\,|z|^2$*, with* $A(z,\overline{z})$ *a real analytic polynomial in* $(z,\overline{z})$*.*

Proof. Collecting terms of weighted degree 6 in (7.1.3), we have

$$z\overline{f^{(5)}(z,w)} + \overline{z}f^{(5)}(z,w) + \Phi^{(2)}_0(z,w)\cdot\overline{\Phi^{(4)}_0(z,w)} + \overline{\Phi^{(2)}_0(z,w)}\cdot\Phi^{(4)}_0(z,w)$$

$$+\left|\phi^{(3)}(z,w)\right|^2 + \left|f^{(3)}(z,w)\right|^2 = 0,\ over\ Im(w) = |z|^2.$$

We note that we have a few terms of Φ_1 this time. The $z^3\overline{z}^3$ terms (see Theorem 6.10) give us the result immediately this time. □

We notice that $\left|\phi^{(3,0)}(z)\right|^2 = \left|\Phi^{(3,0)}_0(z)\right|^2 + \left|\Phi^{(3,0)}_1(z)\right|^2$.

Also, we look back at (7.1.2), and we count the number of negative terms, getting $\frac{\kappa_0(\kappa_0+1)}{2} - \kappa_0$. (We get $\sum_{k=1}^{\kappa_0-1}k$.) We then count the number of components of Φ_1, getting $N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2} - \kappa_0$.

7.4 Proposition (DA93,p.102)**.** *Suppose that* B *is an open ball about 0 in* $\mathbb{C}^q$ *and* F, G *are holomorphic mappings from* B *to* $\mathbb{C}^N$ *for which*

$$\|F\|^2 = \|G\|^2.$$

Then there exists a unitary matrix $U \in \mathbb{U}(N)$ *for which* $F = UG$*.*

Proof. Suppose $F(z) = \sum F_a z^a$ and $G(z) = \sum G_a z^a$ are convergent power series in B with complex coefficients. We get

$$\langle F_a, F_b \rangle = \langle G_a, G_b \rangle$$

for all choices of $(a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Choose a maximal linearly independent set among G_a, then define U on that set by $UG_a = F_a$. Extend this by linearity.

We need to check that U is well-defined on the span of G_a.

Assume for the sake of contradiction that we have

$$G_k = \sum_j c_k^j G_j.$$

Then F_k has two definitions. To check that U is well-defined, we need to check that the two formulas agree:

$$F_k = UG_k$$

and

$$F_k = U \sum_j c_k^j G_j = \sum_j c_k^j UG_j = \sum_j c_k^j F_j.$$

To check this, we compute

$$\|F_k - \sum c_k^j F_k\|^2 = \|G_k - \sum c_k^j G_j\|^2 = 0,$$

and this holds after expanding each squared norm because $\langle F_j, F_k \rangle = \langle G_j, G_k \rangle$. Thus, U is well-defined, and it preserves the inner product of every pair of vectors in its domain, so it is an isometry from $span\{G_a\}$ to $span\{F_a\}$. If we define U to be the identity on the complement of this span, we obtain a unitary transformation of $\mathbb{C}^N$. □

7.5 Corollary (HJY14, Corollary 3.4)**.** *Suppose that* $\kappa_0 \geq 2$ *and* $(\kappa_0 + 1)n - \kappa_0 \leq N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2$. *Then*

$$\Phi_1^{(3,0)}(z) = \left(\frac{2}{\sqrt{\mu_j + \mu_\ell}} \left(\sqrt{\frac{\mu_j}{\mu_\ell}} z_j \xi_\ell - \sqrt{\frac{\mu_\ell}{\mu_j}} z \ell \xi_j \right), 0' \right)_{1 \leq j < \ell \leq \kappa_0}$$

$$\left|\phi^{(3,0)}\right|^2 = 4\left(\sum_{j\leq\kappa_0}\frac{1}{\mu_j}\left|\xi_j(x)\right|^2\right)|z|^2.$$

Proof. This follows from Proposition 7.5 and Huang's Lemma, since $N \geq (\kappa_0+1)n-\kappa_0$ implies that $N > \#(\mathcal{S}_0)+n$ for $\kappa_0 > 1$. □

From here on, we can assume that the normalization required for Corollary 7.5 and Theorem 6.10 hold.

Chapter 8

Partial linearity and further applications of the Chern-Moser equation

This entire section is dedicated to proving the following theorem:

8.1 Theorem. *(HJY14, Theorem 4.1)*

Assume that F is as in Theorem 6.10 with $\kappa_0 = 2, n \geq 7$, and $3n-2 \leq N \leq 4n-6$. Also, assume that $\Phi_1^{(3,0)}(z)$ is normalized as in Corollary 7.5. Then the following holds:

$(1)\Phi_1^{(4,0)}(z) = (\phi_{33}^{(4,0)}(z), 0, \ldots, 0),\ where$
$\phi_{33}^{(4,0)}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_2^*\right), \eta_1^* = \phi^{(3,0)}(z)\cdot\overline{e_1^*}, \eta_2^* = \phi^{(3,0)}(z)\cdot\overline{e_2^*},$
$(2)D_z^\alpha\Phi_1^{(2,1)}(z) \in span\{1, 0, \ldots, 0, \hat{e}_1, \hat{e}_2\}\ for\ \ |\alpha| = 2,$
$(3)D_z^\alpha\Phi_1^{(1,2)}(z) \in span\{\hat{e}_1, \hat{e}_2\}\ for\ \ |\alpha| = 1.$

Here, $\hat{e}_1, \hat{e}_2, e_1^*, e_2^*$ are as defined at the beginning of the previous chapter, and D is the regular differential operator.

Notice that $g = w$. By the partial linearity theorem proved by Huang in [Hu01], we can let $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{C}^2$ near 0, and we can assume that there is a unique affine subspace

L_ϵ of codimension 2 defined by equations of the form

$$\begin{aligned} z_1 &= \textstyle\sum_{i=3}^{n-1} a_i(\epsilon) z_i + a_n(\epsilon) w + \epsilon_1 \\ z_2 &= \textstyle\sum_{i=3}^{n-1} b_i(\epsilon) z_i + b_n(\epsilon) w + \epsilon_2 \qquad (8.1.1) \\ a_i(0) &= b_i(0) = 0 \end{aligned}$$

with F a linear map on L_ϵ and a_j and b_j holomorphic functions in ϵ near 0.

Then,

$$\left.\frac{\partial^2 H}{\partial w^2}\right|_{L_\epsilon} = 0 \ for \ H = f \ or \ \phi.$$

By a calculation shown elsewhere (see, for instance, Chapter 10),

$$\begin{aligned} 0 &= \left.\tfrac{\partial^2 H(L_\epsilon)}{\partial w^2}\right|_{(\epsilon_1,\epsilon_2)} \\ &= \Big(\tfrac{\partial^2 H}{\partial z_1^2} a_n^2 + \tfrac{\partial^2 H}{\partial z_2^2} b_n^2 + 2\tfrac{\partial^2 H}{\partial z_1 \partial z_2} a_n b_n + \qquad (8.1.2) \\ &\quad \left. 2\tfrac{\partial^2 H}{\partial z_2 \partial w} a_n + 2\tfrac{\partial^2 H}{\partial z_2 \partial w} b_n + \tfrac{\partial^2 H}{\partial w^2}\Big)\right|_{(\epsilon_1,\epsilon_2,0,\ldots,0)}. \end{aligned}$$

We then let $a_n^{(1)}(\epsilon)$ and $b_n^{(1)}(\epsilon)$ be the linear parts of a_n and b_n. Letting $H = f_1, f_2$ and ϕ, in turn, in (8.1.2) yields

$$\begin{aligned} &\tfrac{i}{2}\mu_1 a_n^{(1)}(\epsilon) + f_1^{(1,2)}(\epsilon, 0, \ldots, 0) = 0 \\ &\tfrac{i}{2}\mu_2 b_n^{(1)}(\epsilon) + f_2^{(1,2)}(\epsilon, 0, \ldots, 0) = 0 \qquad (8.1.3) \\ &\phi^{(1,2)}(\epsilon, 0, \ldots, 0) + e_1^* a_n^{(1)}(\epsilon) + e_2^* b_n^{(1)}(\epsilon) = 0. \end{aligned}$$

By Theorem 6.10, $F^{(1,m)}(z)$ depends only on (z_1, z_2) for all m. Thus, we start with

$$\phi^{(1,2)}(\epsilon, 0, \ldots, 0) = -e_1^* a_n^{(1)}(\epsilon - e_2^* b_n^{(1)}(\epsilon)$$

and multiply the first part of (8.1.3) by $\frac{2i}{\mu_1}$ and the second part by $\frac{2i}{\mu_2}$ to get

$$a_n^{(1)} = \frac{2i}{\mu_1} f_1^{(1,2)}(\epsilon, 0, \ldots, 0)$$

and

$$b_n^{(1)} = \frac{2i}{\mu_2} f_2^{(1,2)}(\epsilon, 0, \ldots, 0),$$

which leads us to

$$\phi^{(1,2)}(\epsilon,0,\ldots,0) = -\frac{2i}{\mu_1} f_1^{(1,2)}(\epsilon,0,\ldots,0)e_1^* - \frac{2i}{\mu_2} f_2^{(1,2)}(\epsilon,0,\ldots,0)e_2^*. \qquad (8.1.4)$$

This is enough to prove the third part of Theorem 8.1.

We shall use the notation $I_j = (0,\ldots,1,\ldots,0)$, where the 1 is in the j^{th} position.

We now use the previous result and the fact that $\xi_j(z) = \overline{e}_j \cdot \Phi_0^{(2,0)}(z)$ to multiply $\overline{\Phi_0^{(1,2)}}$ and $\Phi_0^{(2,0)}(z)$:

$$\overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,0)}(z) = \frac{2i}{\mu_1}\overline{f_1^{(1,2)}(z)}\overline{e_1} \cdot \Phi_0^{(2,0)}(z) + \frac{2i}{\mu_2}\overline{f_2^{(1,2)}(z)}\overline{e_2} \cdot \Phi_0^{(2,0)}(z)$$

$$= \frac{2i}{\mu_1}(\overline{f_1^{(I_1+2I_n)}}\overline{z}_1 + \overline{f_1^{(I_2+2I_n)}}\overline{z}_2)\xi_1 + \frac{2i}{\mu_2}(\overline{f_2^{(I_1+2I_n)}}\overline{z}_1 + \overline{f_2^{(I_2+2I_n)}}\overline{z}_2)\xi_2. \qquad (8.1.5)$$

We rearrange the above and take out the $\overline{z}_1$ and $\overline{z}_2$ to get

$$\overline{\Phi_0^{(I_1+2I_n)}} \cdot \Phi_0^{(2,0)}(z) = 2i\left(\frac{\xi_1}{\mu_1}\overline{f_1^{(I_1+2I_n)}} + \frac{\xi_2}{\mu_2}\overline{f_2^{(I_1+2I_n)}}\right)$$

$$\overline{\Phi_0^{(I_2+2I_n)}} \cdot \Phi_0^{(2,0)}(z) = 2i\left(\frac{\xi_1}{\mu_1}\overline{f_1^{(I_2+2I_n)}} + \frac{\xi_2}{\mu_2}\overline{f_2^{(I_2+2I_n)}}\right). \qquad (8.1.6)$$

Also, applying (8.1.4),

$$2i\left(\frac{\overline{\xi}_1}{\mu_1}\overline{Phi_0^{(I_1+2I_n)}} \cdot \Phi_0^{(2,0)}(z) + \frac{\overline{\xi}_2}{\mu_2}\overline{\Phi_0^{(I_2+2I_n)}} \cdot \Phi_0^{(2,0)}(z)\right)$$

$$= \frac{-4\overline{\xi}_1}{\mu_1}\left(\frac{\xi_1}{\mu_1}\overline{f_1^{(I_1+2I_n)}} + \frac{\xi_2}{\mu_2}\overline{f_2^{(I_1+2I_n)}}\right) + \frac{-4\overline{\xi}_2}{\mu_2}\left(\frac{\xi_1}{\mu_1}\overline{f_1^{(I_2+2I_n)}} + \frac{\xi_2}{\mu_2}\overline{f_2^{(I_2+2I_n)}}\right)$$

$$= \frac{-4\xi_1}{\mu_1}\left(\overline{f_1^{(I_1+2I_n)}}\frac{\overline{\xi}_1}{\mu_1} + \overline{f_1^{(I_2+2I_n)}}\frac{\overline{\xi}_2}{\mu_2}\right) - \frac{4\xi_2}{\mu_2}\left(\overline{f_2^{(I_1+2I_n)}}\frac{\overline{\xi}_1}{\mu_1} + \overline{f_2^{(I_2+2I_n)}}\frac{\overline{\xi}_2}{\mu_2}\right). \qquad (8.1.7)$$

We look back at (7.1.3) and observe the terms of degree 6.

Note that

$$(a+bi)(c-di) + (a-bi)(c+di) = 2Re(a+bi)(c-di).$$

Also note that there will be some f terms of degree 1 and 5 and 3, and there will be Φ_0 terms of degree 2 and 4 and Φ_1 terms of degree 2, 4, and 3. In addition, $-Im(g(z,w)) = -w$, which does not have weighted degree 6.

We get

$$2Re\left\{\overline{z}f^{(5)}(z,w) + \overline{\Phi_0^{(2)}(z,w)} \cdot \Phi_0^{(4)}(z,w)\right\} + \left|f^{(3)}(z,w)\right|^2 + \left|\phi^{(3)}(z,w)\right|^2, \qquad (8.1.8)$$

and then, using our parameterization of w,

$$\begin{aligned} &2Re\left\{\overline{z}f^{(3,1)}(z)(u+i|z|^2) + f^{(1,2)}(z)(u+i|z|^2)^2\right\} \\ &+\overline{\Phi_0^{(2,0)}(z)}\left(\Phi_0^{(4,0)}(z) + \Phi_0^{(2,1)}(z)\cdot(u+i|z|^2)\right) \\ &+\left|f^{(1,1)}(z)(u+i|z|^2)\right|^2 + \left|\phi^{(3,0)}(z) + \phi^{(1,1)}(z)(u+i|z|^2)\right|^2 = 0. \end{aligned} \qquad (8.1.9)$$

See [HJX06, Lemma 2.3(A)] for why the term $f^{(5,0)}(z)$ does not appear. We collect the terms with w^2 for an important result:

$$2Re\left(\overline{z}f^{(1,2)}(z)\right) + \left|f^{(1,1)}(z)\right|^2 + \left|\phi^{(1,1)}(z)\right|^2 = 0. \qquad (8.1.10)$$

We now get the terms that look like $z^\alpha\overline{z}^\beta u$ with $|\alpha| = 3$ and $|\beta| = 1$:

$$\overline{z}f^{(3,1)}(z) + \phi^{(3,0)}(z)\cdot\overline{\phi^{(1,1)}(z)} = 0. \qquad (8.1.11)$$

Next, the terms with $|\alpha| = 2$ and $|\beta| = 2$:

$$2Re\left(2i\overline{z}f^{(1,2)}(z)|z|^2 + \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\right) = 0. \qquad (8.1.12)$$

Next, the terms that look like $z^\alpha\overline{z}^\beta$ with $|\alpha| = 4$ and $|\beta| = 2$ (no u this time):

$$i|z|^2\,\overline{z}f^{(3,1)}(z) + \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{4,0)}(z) - i|z|^2\,\overline{\phi^{(1,1)}(z)}\cdot\phi^{(3,0)}(z) = 0. \qquad (8.1.13)$$

Next, the terms with $|\alpha| = |\beta| = 3$:

$$2Re\left(-\overline{z}f^{(1,2)}(z)|z|^4 + i|z|^2\,\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\right)$$

$$+|z|^4\cdot\left|f^{(1,1)}(z)\right|^2 + \left|\phi^{(3,0)}(z)\right|^2 + |z|^4\left|\phi^{(1,1)}(z)\right|^2 = 0. \qquad (8.1.14)$$

Next, we rearrange (8.1.11) to read

$$\overline{z}f^{(3,1)}(z) = -\phi^{(3,0)}(z) \cdot \overline{\phi^{(1,1)}(z)},$$

we place this in (8.1.13) to get

$$-i\,|z|^2\,\phi^{(3,0)}(z) \cdot \overline{\phi^{(1,1)}(z)} + \overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(4,0)}(z) - i\,|z|^2\,\overline{\phi^{(1,1)}(z)} \cdot \phi^{(3,0)}(z) = 0,$$

and thus

$$\overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(4,0)}(z) = 2i\,|z|^2\,\overline{\phi^{(1,1)}(z)} \cdot \phi^{(3,0)}(z). \qquad (8.1.15)$$

Rearrange (8.1.10):

$$-2Re\left(\overline{z}f^{(1,2)}(z)\right) = \left|f^{(1,1)}(z)\right|^2 + \left|\phi^{(1,1)}(z)\right|^2.$$

Now, locate the right hand side of this in (8.1.14) and replace it with the left hand side, yielding

$$2Re\left(-\overline{z}f^{(1,2)}(z)\,|z|^2 + i\overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(2,1)}(z)\right)|z|^2 + \left|\phi^{(3,0)}\right|^2 - 2Re\left(\overline{z}f^{(1,2)}(z)\right) = 0$$

or

$$2Re\left(-2\overline{z}f^{(1,2)}(z)\,|z|^2 + i\overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(2,1)}(z)\right)|z|^2 + \left|\phi^{(3,0)}(z)\right|^2 = 0. \qquad (8.1.16)$$

By (8.1.12), $2Re\left(2i\overline{z}f^{(1,2)}(z)\,|z|^2 + \overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(2,1)}(z)\right) = 0$, meaning the term inside the parenthesis is entirely imaginary. The term in the parenthesis of (8.1.16) is equal to (8.1.14) multiplied by i, and thus it is entirely real, so

$$2\left(-2\overline{z}f^{(1,2)}(z)\,|z|^2 + i\overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(2,1)}(z)\right)|z|^2 + \left|\phi^{(3,0)}\right|^2 = 0 \qquad (8.1.17)$$

without any need to specify that we are looking at the real part of the term in parenthesis.

By Corollary 7.5, we have

$$\left|\phi^{(3,0)}\right|^2 = 4\,|z|^2\left(\frac{|\xi_1|^2}{\mu_1} + \frac{|\xi_2|}{\mu_2}\right). \qquad (8.1.18)$$

By (8.1.18) and (8.1.17):

$$2\left(-2\overline{z}f^{(1,2)}(z)\left|z\right|^2 + i\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\right)\left|z\right|^2 + 4\left|z\right|^2\left(\frac{\left|\xi_1\right|^2}{\mu_1}+\frac{\left|\xi_2\right|^2}{\mu_2}\right)$$

and thus

$$2\left(-2\overline{z}f^{(1,2)}(z)\left|z\right|^2 + i\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\right) + 4\left(\frac{\left|\xi_1\right|^2}{\mu_1}+\frac{\left|\xi_2\right|^2}{\mu_2}\right). \qquad (8.1.19)$$

By our definitions of our terms, we have $\left|\xi_j\right|^2 = \overline{\xi}_j\xi_j = \xi_j e_j \cdot \overline{\Phi_0^{(2,0)}(z)}$.

We will define some new terms:

$$\tilde{\phi}^{(2,1)}(z) = \phi^{(2,1)}(z) - 2i\sum_{j=1}^{2}\frac{\xi_j}{\mu_j}e_j^*, \quad \tilde{\Phi}_0^{(2,1)}(z) = \Phi_0^{(2,1)}(z) - 2i\sum_{j=1}^{2}\frac{\xi_j}{\mu_1}e_j. \qquad (8.1.20)$$

Now, we get to the key equations that we use to solve the boundary case of the Third Gap Theorem.

First, multiply (8.1.19) by $\frac{i}{2}$ and get

$$\overline{\Phi_0^{(2,0)}(z)}\cdot\tilde{\Phi}_0^{(2,1)}(z) = -2i\left|z\right|^2\overline{z}\cdot f^{(1,2)}(z). \qquad (8.1.21)$$

Because $f_j = z_j$ for $3 \leq j \leq n-1$, and thus $f_j^{(1,2)}(z) = 0$ for $j > 2$, we have $\overline{z}\cdot f^{(1,2)}(z) = (\overline{z}_1, \overline{z}_2)\cdot(f_1^{(1,2)}(z), f_2^{(1,2)}(z))$, and, recalling that $\Phi_{\ell k}^{(2,0)}(z) = \mu_{\ell k}z_\ell z_k$ for $(\ell, k) \in \mathcal{S}_0$, we get

$$\begin{aligned}
\tilde{\Phi}_{11}^{(2,1)}(z) = &\ \tfrac{-2i}{\sqrt{\mu_1}}z_1 f_1^{(1,2)}(z)\\
\tilde{\Phi}_{12}^{(2,1)}(z) = &\ \tfrac{-2i}{\sqrt{\mu_1+\mu_2}}\left(z_1 f_2^{(1,2)}(z) + z_2 f_1^{(1,2)}(z)\right)\\
\tilde{\Phi}_{22}^{(2,1)}(z) = &\ \tfrac{-2i}{\sqrt{\mu_2}}z_2 f_2^{(1,2)}(z) \qquad\qquad (8.1.22)\\
\tilde{\Phi}_{1j}^{(2,1)}(z) = &\ \tfrac{-2i}{\sqrt{\mu_1}}z_j f_1^{(1,2)}(z), j \geq 3\\
\tilde{\Phi}_{2j}^{(2,1)}(z) = &\ \tfrac{-2i}{\sqrt{\mu_2}}z_j f_2^{(1,2)}(z), j \geq 3.
\end{aligned}$$

The reason for this is not entirely obvious. If we expand (8.1.21), we have

$$\sum_{\ell,k}\mu_{\ell k}\overline{z}_\ell\overline{z}_k\tilde{\Phi}_{\ell k}^{(2,1)}(z) = -2i(\sum_j z_j\overline{z}_j)(\overline{z}_1 f_1^{(1,2)}(z) + \overline{z}_2 f_2^{(1,2)}(z)).$$

Note that $\ell < \kappa_0$. Now, we match the $\overline{z}_\ell \overline{z}_k$ terms on the left hand side with the $\overline{z}_\ell \overline{z}_k$ terms on the right.

Looking at Lemma 7.1 and letting $\Gamma_j^{[1]} = \Gamma_j^{[2]} = f_j^{(1,2)}$ and $\Lambda_{j\ell}^{[1]} = \Lambda_{j\ell}^{[2]} = \tilde{\Phi}_0^{(2,1)}(z)$, we have

$$\left|\tilde{\Phi}_0^{(2,1)}(z)\right|^2 = 4\left|z\right|^2 \sum_{j=1}^{2} \frac{1}{\mu_j} \left|f_j^{(1,2)}(z)\right|^2$$

$$-\frac{4}{\mu_1\mu_2(\mu_1+\mu_2)} \left|\mu_1 z_1 f_2^{(1,2)}(z) - \mu_2 z_2 f_1^{(1,2)}(z)\right|^2. \qquad (8.1.23)$$

Now, let $\Lambda_{j\ell}^{[1]} = \tilde{\Phi}_0^{(2,1)}$ and $\Lambda_{j\ell}^{[2]} = \Phi_0^{(3,0)}$ so $\Gamma_j^{[1]} = f_j^{(1,2)}$ and $\Gamma_j^{[2]} = \xi_j$. By Lemma 7.1, we have

$$\overline{\tilde{\Phi}_0^{(2,1)}(z)}\Phi_0^{(3,0)}(z) = -4\left|z\right|^2 \sum_{j=1}^{2} \frac{1}{\mu_j} \overline{f_j^{(1,2)}(z)}\xi_j$$

$$+\frac{4}{\mu_1\mu_2(\mu_1+\mu_2)} \overline{\left(\mu_1 z_2 f_2^{(1,2)}(z) - \mu_2 z_2 f_1^{(1,2)}(z)\right)} \cdot \left(\mu_1 z_1 \xi_2 - \mu_2 z_2 \xi_1\right). \qquad (8.1.24)$$

We look back at (8.1.10) and replace z_1 with $\frac{\xi_1}{\mu_1}$ and z_2 with $\frac{\xi_2}{\mu_2}$, getting

$$\begin{aligned} &2Re\left\{\tfrac{\overline{\xi_1}}{\mu_1}\left(f_1^{(I_1+2I_n)}\tfrac{\xi_1}{\mu_1} + f_1^{(I_2+2I_n)}\tfrac{\xi_2}{\mu_2}\right) + \tfrac{\overline{\xi_2}}{\mu_2}\left(f_2^{(I_1+2I_n)}\tfrac{\xi_1}{\mu_1} + f_2^{(I_2+2I_n)}\tfrac{\xi_2}{\mu_2}\right)\right\} + \\ &\tfrac{1}{4}(\left|\xi_1\right|^2 + \left|\xi_2\right|^2) + \left|\tfrac{\xi_1}{\mu_1}e_1^* + \tfrac{\xi_2}{\mu_2}e_2^*\right|^2 = 0. \end{aligned} \qquad (8.1.25)$$

For clarity, we remind ourselves that $\phi^{(1,1)}(z) = e_1^* z_1 + e_2^* z_2$, where $e_j^* = (e_j, \hat{e}_j)$, $e_j \in \mathbb{C}^{\#(\mathcal{S}_0)}$, and $\hat{e}_j \in \mathbb{C}^{\#(\mathcal{S}_1)}$, and, from Theorem 6.10, $f_j^{(1,1)}(z) = \frac{i}{2}\mu_j z_j$ for $j = 1, 2$.

Now, we multiply (8.1.25) by 4 and apply (8.1.7) to get

$$\begin{aligned} &-2Re\left\{2i\left(\tfrac{\overline{\xi}_1}{\mu_1}\overline{\Phi_0^{(I_1+2I_n)}} + \tfrac{\overline{\xi}_2}{\mu_2}\overline{\Phi_0^{(I_2+2I_n)}}\right) \cdot \Phi_0^{(2,0)}(z)\right\} \\ &+(\left|\xi_1\right|^2 + \left|\xi_2\right|^2) + 4\left|\tfrac{\xi_1}{\mu_1}e_1^* + \tfrac{\xi_2}{\mu_2}e_2^*\right|^2 = 0. \end{aligned} \qquad (8.1.26)$$

Next, we turn back to (7.1.3) and look at the terms of weighted degree 7. We note that there are no degree 2 terms of f, degree 1 terms of ϕ, or degree 2 terms of Φ_1.

$$\begin{aligned} &2Re\Big\{\overline{z}f^{(6)}(z,w) + \overline{f^{(3)}(z,w)}f^{(4)}(z,w) \\ &+\overline{\Phi_0^{(2)}}(z,w)\Phi_0^{(5)}(z,w) + \overline{\phi^{(3)}(z,w)}\phi^{(4)}(z,w)\Big\} = 0. \end{aligned} \qquad (8.1.27)$$

Filling in the details using Theorem 6.10 and our parameterization of w, we get

$$
\begin{aligned}
&2Re\left\{\overline{z}\left(f^{(4,1)}(z)(u+i|z|^2)+f^{(2,2)}(z)(u+i|z|^2)^2\right)+\right.\\
&\overline{f^{(1,1)}(z)(u+i|z|^2)}\cdot f^{(2,1)}(z)\cdot(u+i|z|^2)\\
&+\overline{\Phi_0^{(2,0)}(z)}\left(\Phi_0^{(5,0)}(z)+\Phi_0^{(3,1)}(z)(u+i|z|^2)+\Phi_0^{(1,2)}(z)(u+i|z|^2)^2\right)\\
&\left.+\overline{\left(\phi^{(3,0)}(z)+\phi^{(1,1)}(z)(u+i|z|^2)\right)}\cdot\left(\phi^{(4,0)}(z)+\phi^{(2,1)}(z)(u+i|z|^2)\right)\right\}.
\end{aligned}
$$

We note [HJX06, Lemma 2.3(A)]; no $F^{(6,0)}(z)$ appears here.

We now go through a familiar process, collecting the $z^\alpha\overline{z}^\beta u^2$ terms, with $|\alpha|=2$ and $|\beta|=1$:

$$
\begin{aligned}
&\overline{z}f^{(2,2)}(z)+\overline{f^{(1,1)}(z)}\cdot f^{(2,1)}(z)+\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)+\\
&\overline{\phi^{(1,1)}(z)}\cdot\phi^{(2,1)}(z)=0.
\end{aligned}
\tag{8.1.29}
$$

Next, $z^\alpha\overline{z}^\beta u$ with $|\alpha|=3$ and $|\beta|=2$.

$$
\begin{aligned}
&2i\overline{z}|z|^2 f^{(2,2)}(z)+\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)-2i|z|^2\,\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)\\
&+\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z)=0.
\end{aligned}
\tag{8.1.30}
$$

Next, $z^\alpha\overline{z}^\beta$ with $|\alpha|=4$ and $|\beta|=3$.

$$
\begin{aligned}
&-\overline{z}f^{(2,2)}(z)|z|^4+\overline{f^{(1,1)}}(z)\cdot f^{(2,1)}(z)|z|^4+i|z|^2\,\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)\\
&-|z|^4\,\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)+\overline{\Phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z)\\
&-i|z|^2\,\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z)+|z|^4\,\overline{\phi^{(1,1)}(z)}\cdot\phi^{(2,1)}(z)=0.
\end{aligned}
\tag{8.1.31}
$$

Now, we multiply (8.1.29) by $|z|^4$ and subtract the result from (8.1.31), yielding

$$
\begin{aligned}
&-2\overline{z}f^{(2,2)}(z)|z|^4+i|z|^2\,\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)-2|z|^4\,\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)\\
&+\overline{\phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z)-i|z|^2\,\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z)=0.
\end{aligned}
\tag{8.1.32}
$$

Next, we take $i|z|^2$ times (8.1.30), subtract that from (8.1.32), and rearrange the result to get

$$
\overline{\phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z)=4|z|^4\,\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)+2i|z|^2\,\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z).
\tag{8.1.33}
$$

Rearrange (8.1.30):

$$2i\overline{z}\,|z|^2\,f^{(2,2)}(z) + \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z) = 2i\,|z|^2\,\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z) - \overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z).$$

The right hand side of this appears in (8.1.33). Replace it with the left hand side:

$$\overline{\phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z) = -2i\,|z|^2\left(2i\,|z|^2\,\overline{z}f^{(2,2)}(z) + \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)\right).$$

Recall (8.1.15):

$$\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(4,0)}(z) = 2i\,|z|^2\,\overline{\phi^{(1,1)}(z)}\cdot\phi^{(3,0)}(z).$$

By Theorem 6.10, we get $\overline{\phi_{\ell k}^{(2,0)}(z)} = \mu_{\ell k}\overline{z}_\ell\overline{z}_k$,

Now, we expand (8.1.15):

$$\sum_{\ell,k}\mu_{\ell k}\overline{z}_\ell\overline{z}_k\Phi_{\ell k}^{(4,0)}(z) = 2i\left(\sum_j z_j\overline{z}_j\right)\left(\sum_{m,n}\overline{\phi_{mn}^{(1,1)}}\phi_{mn}^{(3,0)}(z)\right)$$

$$\sum_{\ell,k}\mu_{\ell k}\overline{z}_\ell\overline{z}_k\Phi_{\ell k}^{(4,0)}(z) = 2i\left(\sum_j z_j\overline{z}_j\right)\left(\overline{z}_1\overline{e}_1^* + \overline{z}_2\overline{e}_2^*\right)\cdot\left(\phi^{(3,0)}\right).$$

Now, by matching $\overline{z}_\ell\overline{z}_k$ terms on the left and right hand sides of this, we get:

$$\begin{aligned}
\mu_{11}\cdot\Phi_{11}^{(4,0)}(z) &= 2iz_1\phi^{(3,0)}(z)\cdot\overline{e}_1^*,\\
\mu_{12}\cdot\Phi_{12}^{(4,0)}(z) &= 2iz_1\phi^{(3,0)}(z)\cdot\overline{e}_2^* + 2iz_2\phi^{(3,0)}(z)\cdot\overline{e}_1^*,\\
\mu_{22}\cdot\Phi_{22}^{(4,0)}(z) &= 2iz_2\phi^{(3,0)}(z)\cdot\overline{e}_2^* \qquad (8.1.35)\\
\mu_{1j}\cdot\Phi_{1j}^{(4,0)}(z) &= 2iz_j\phi^{(3,0)}(z)\cdot\overline{e}_1^*, j\geq 3\\
\mu_{2j}\cdot\Phi_{2j}^{(4,0)}(z) &= 2iz_j\phi^{(3,0)}(z)\cdot\overline{e}_2^*, j\geq 3.
\end{aligned}$$

We now use the following new notation:

$$\begin{aligned}
\eta_1^* &= \phi^{(3,0)}(z)\cdot\overline{e}_1^*,\\
\eta_2^* &= \phi^{(3,0)}(z)\cdot\overline{e}_2^*,\\
\eta_1 &= \Phi_0^{(3,0)}(z)\cdot\overline{e}_1^*,\\
\eta_2 &= \Phi_0^{(3,0)}(z)\cdot\overline{e}_2^*.
\end{aligned}$$

Using Lemma 7.1 again, with $\Lambda^{[1]}_{j\ell} = \overline{\Phi^{(3,0)}_0(z)}$, $\Lambda^{[2]}_{j\ell} = \Phi^{(4,0)}_0(z)$, $\Gamma^{[1]}_j = \overline{\xi_j}$, and $\Gamma^{[2]}_j = \eta^*_j$, we get

$$\overline{\Phi^{(3,0)}_0(z)}\Phi^{(4,0)}_0(z) = 4\,|z|^2\left(\frac{\overline{\xi}_1\eta^*_1}{\mu_1} + \frac{\overline{\xi}_2\eta^*_2}{\mu_2}\right)$$

$$-\frac{4}{\mu_1+\mu_2}\left(\sqrt{\frac{\mu_2}{\mu_1}}\overline{z}_2\overline{\xi}_1 - \sqrt{\frac{\mu_1}{\mu_2}}\overline{z}_1\overline{\xi}_2\right)\cdot\left(\sqrt{\frac{\mu_2}{\mu_2}}z_2\eta^*_1 - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta^*_2\right). \qquad (8.1.36)$$

This is not quite in the same form as what we see in Lemma 7.1, but we note that we have multiplied the last two factors each by -1, and we note the following:

$$\tfrac{1}{AB(A+B)}\left(AX - BY\right)\left(AZ - BW\right) =$$
$$\tfrac{1}{AB(A+B)}\left(A^2XZ - BAYZ - BAXW + B^2YW\right)$$

and

$$\tfrac{1}{A+B}\left(\sqrt{\tfrac{A}{B}}X - \sqrt{\tfrac{B}{A}}Y\right)\left(\sqrt{\tfrac{A}{B}}Z - \sqrt{\tfrac{B}{A}}W\right) =$$
$$\tfrac{1}{A+B}\left(\tfrac{A}{B}XZ - YZ - XW + \tfrac{B}{A}YW\right),$$

and the second of those is obviously equal to the first.

We observe that (8.1.34) is $\overline{\phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z) = \overline{\Phi^{(3,0)}_0(z)}\cdot\Phi^{(4,0)}_0(z) + \overline{\Phi^{(3,0)}_1(z)}\cdot\Phi^{(4,0)}_1(z)$.

We note that the first term of (8.1.36) has $|z|^2$ as a factor, but the second term does not, since $\overline{z}_k$ shows up for all k, but z_k might not. All of (8.1.34) has $|z|^2$ as a factor.

Next, we combine (8.1.34) and (8.1.36) and use Huang's Lemma to get

$$\overline{\Phi^{(3,0)}_1(z)}\cdot\Phi^{(4,0)}_1(z) = \frac{4}{\mu_1+\mu_2}\left(\sqrt{\frac{\mu_2}{\mu_1}}\overline{z}_2\overline{\xi}_1 - \sqrt{\frac{\mu_1}{\mu_2}}\overline{z}_1\overline{\xi}_2\right)\cdot\left(\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta^*_1 - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta^*_2\right).$$

By Corollary 7.5,

$$\phi^{(3,0)}_{33}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_2}{\mu_1}}z_1\xi_2 - \sqrt{\frac{\mu_2}{\mu_1}}z_1\xi_2\right),$$

and $\phi^{(3,0)}_{\ell k}(z) = 0$ for $\ell \geq 3$ or $k > 3$.

It follows immediately that

$$\phi_{33}^{(4,0)}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_2^*\right). \qquad (8.1.37)$$

It also follows from the preceding, (8.1.36), and (8.1.34) that

$$2i\left(\frac{\overline{\xi}_1\eta_1^*}{\mu_1} + \frac{\overline{\xi}_2\eta_2^*}{\mu_2}\right) = 2i\,|z|^2\,\overline{z}f^{(2,2)}(z) + \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z). \qquad (8.1.38)$$

We now write:

$$\begin{aligned}\tilde{\phi}^{(3,1)}(z) = &\quad \phi^{(3,1)}(z) - 2i\left(\tfrac{\eta_1^*}{\mu_1}e_1^* + \tfrac{\eta_2^*}{\mu_2}e_2^*\right)\\ \tilde{\Phi}_0^{(3,1)}(z) = &\quad \Phi_0^{(3,1)}(z) - 2i\left(\tfrac{\eta_1^*}{\mu_1}e_1 + \tfrac{\eta_2^*}{\mu_2}e_2\right)\end{aligned}\;. \qquad (8.1.39)$$

Now,

$$\overline{\Phi_0^{(2,0)}(z)}\tilde{\Phi}_0^{(3,1)}(z) = \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z) - \overline{\Phi_0^{(2,0)}(z)}\cdot 2i\left(\frac{\eta_1^*}{\mu_1}e_1 + \frac{\eta_2^*}{\mu_2}e_2\right)$$

and, by definition of ξ_j,

$$= \overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z) - 2i\left(\frac{\overline{\xi}_1\eta_1^*}{\mu_1} + \frac{\overline{\xi}_2\eta_2^*}{\mu_2}\right) = -2i\,|z|^2\,\overline{z}f^{(2,2)}(z).$$

By Theorem 6.10, we get

$$\begin{aligned}&\mu_{11}\cdot\tilde{\Phi}_{11}^{(3,1)}(z) = -2iz_1f_1^{(2,2)}(z)\\ &\mu_{12}\cdot\tilde{\Phi}_{12}^{(3,1)}(z) = -2i\left(z_1f_2^{(2,2)}(z) + z_2f_1^{(2,2)}(z)\right)\\ &\mu_{22}\cdot\Phi_{22}^{(3,1)}(z) = -2iz_2f_2^{(2,2)}(z)\\ &\mu_{1j}\cdot\Phi_{1j}^{(3,1)}(z) = -2iz_jf_1^{(2,2)}(z), j\geq 3\\ &\mu_{2j}\cdot\tilde{\Phi}_{2j}^{(3,1)}(z) = -2iz_jf_2^{(2,2)}(z), j\geq 3.\end{aligned} \qquad (8.1.40)$$

Our reasoning here is the same that led us to (8.1.22).

We apply Lemma 7.1 again, this time getting:

$$\begin{aligned}&\overline{\Phi_0^{(3,0)}(z)}\cdot\Phi_0^{(3,1)}(z) = -4\,|z|^2\left(\tfrac{\bar{\xi}_1}{\mu_1}f_1^{(2,2)}(z) + \tfrac{\xi_2}{\mu_2}f_2^{(2,2)}(z)\right) + \\ &\tfrac{4}{\mu_1+\mu_2}\left(\sqrt{\tfrac{\mu_1}{\mu_2}}\overline{z}_1\xi_2 - \sqrt{\tfrac{\mu_2}{\mu_2}}\overline{z}_2\overline{\xi}_1\right)\cdot\left(\sqrt{\tfrac{\mu_1}{\mu_2}}z_1f_2^{(2,2)}(z) - \sqrt{\tfrac{\mu_2}{\mu_1}}z_2f_1^{(2,2)}(z)\right).\end{aligned} \qquad (8.1.41)$$

Since $\overline{\eta}_1 = \overline{\Phi_0^{(3,0)}(z)} \cdot e_1$ and $\overline{\eta}_2 = \overline{\Phi_0^{(3,0)}(z)} \cdot e_2$, we see that

$$\overline{\Phi_0^{(3,0)}}(z) \cdot 2i \left(\frac{\eta_1^*}{\mu_1} e_1 + \frac{\eta_2^*}{\mu_2} e_2 \right) = 2i \left(\frac{\eta_1^*}{\mu_1} \overline{\eta}_1 + \frac{\eta_2^*}{\mu_2} \overline{\eta}_1 \right). \qquad (8.1.42)$$

Add this to (8.1.41) to get

$$\overline{\Phi_0^{(3,0)}(z)} \cdot \Phi_0^{(3,1)}(z) =$$

$$2i \left(\frac{\eta_1^*}{\mu_1} \overline{\eta}_1 + \frac{\eta_2^*}{\mu_2} \overline{\eta}_2 \right) - 4 |z|^2 \left(\frac{\overline{\xi}_1}{\mu_1} f_1^{(2,2)}(z) + \frac{\overline{\xi}_2}{\mu_2} f_2^{(2,2)}(z) \right)$$

$$+ \frac{4}{\mu_1 + \mu_2} \left(\sqrt{\frac{\mu_1}{\mu_2}} \overline{z}_1 \overline{\xi}_2 - \sqrt{\frac{\mu_2}{\mu_1}} \overline{z}_2 \overline{\xi}_1 \right) \cdot \left(\sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(2,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} f_2^{(2,2)}(z) \right). \qquad (8.1.43)$$

Since we know that $\overline{\phi^{(3,0)}(z)} \cdot \phi^{(4,0)}(z) = 4 |z|^2 \left(\frac{\overline{\xi}_1 \eta_1^*}{\mu_1} + \frac{\overline{\xi}_2 \eta_2^*}{\mu_2} \right)$, we equate this with (8.1.33), getting

$$\begin{array}{l} 4 |z|^2 \overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,1)}(z) + 2i \overline{\phi^{(2,1)}(z)} \cdot \phi(3,0)(z) = \\ \overline{\phi^{(3,0)}(z)} \cdot \phi^{(4,0)}(z) = 4 |z|^2 \left(\frac{\overline{\xi}_1 \eta_1^*}{\mu_1} + \frac{\overline{\xi}_2 \eta_2^*}{\mu_2} \right) \end{array}. \qquad (8.1.44)$$

So,

$$|z|^2 \overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,0)}(z) + 2i \overline{\phi^{(2,1)}(z)} \cdot \phi^{(3,0)}(z) = 0,$$

or, because $4 |z|^2 \left(\frac{\overline{\xi}_1 \eta_1^*}{\mu_1} + \frac{\overline{\xi}_2 \eta_2^*}{\mu_2} \right) = \phi^{(3,0)}(z) \cdot 4 \left(\frac{\overline{\xi}_1 \eta_1^*}{\mu_1} + \frac{\overline{\xi}_2 \eta_2^*}{\mu_2} \right)$,

$$|z|^2 A(z, \overline{z}) + 2i \overline{\tilde{\phi}^{(2,1)}(z)} \cdot \phi^{(3,0)}(z) = 0. \qquad (8.1.45)$$

Here, $A(z, \overline{z})$ is some function, as in the notation for Huang's Lemma.

Now, we combine this with Corollary 7.5 and (8.1.24). In particular, we notice that the sum needs to be 0. Looking only at the parts that don't have a factor of $|z|^2$ (the second term of (8.1.24)), we have

$$\begin{array}{l} \frac{4}{\mu_1 \mu_2 (\mu_1 + \mu_2)} \overline{\left(\mu_1 z_1 f_2^{(1,2)}(z) - \mu_2 z_2 f_1^{(1,2)}(z) \right)} \cdot (\mu_1 z_1 \xi_2 - \mu_2 z_2 \xi_1) \\ + \overline{\tilde{\phi}_{33}^{(2,1)}}(z) \cdot \phi_{33}^{(3,0)}(z) = 0. \end{array}$$

Since $\phi_{33}^{(3,0)}(z) = \frac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\xi_2 - \sqrt{\frac{\mu_2}{\mu_1}}z_2\xi_2\right)$, we divide by $\phi_{33}^{(3,0)}(z)$ and take the complex conjugate to get

$$\tilde{\phi}_{33}^{(2,1)}(z) = \frac{-2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\frac{\mu_1}{\mu_2}}z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}}z_2 f_1^{(1,2)}(z)\right). \qquad (8.1.46)$$

Our next goal is to show:

$$\phi_{3j}^{(4,0)}(z) = 0$$
$$\tilde{\phi}_{3j}^{(2,1)}(z) = 0$$

for $j = 4, \ldots, K$, where $K = N - n - (n-1) - (n-2)$.

We begin by looking at the terms of (7.1.3) with weighted degree 8:

$$2Re\left\{\overline{z}f^{(7)}(z,w) + \overline{f^{(3)}(z,w)}f^{(5)}(z,w) + \overline{\Phi_0^{(2)}(z,w)}\Phi_0^{(6)}(z,w) + \overline{\phi^{(3)}(z,w)}\phi^{(5)}(z,w)\right\} + \left|f^{(4)}(z,w)\right|^2 + \left|\phi^{(4)}(z,w)\right|^2 = 0 \qquad (8.1.47)$$

over, as usual, $Im(w) = |z|^2$.

We break this down further, using our parameterization of w:

$$\begin{aligned}
&2Re\left\{\overline{z}\cdot\left(f^{(5,1)}(z)(u+i|z|^2) + f^{(3,2)}(z)(u+i|z|^2)^2 + f^{(1,3)}(z)(u+i|z|^2)^3\right)\right.\\
&+\overline{f^{(1,1)}(z)(u+i|z|^2)}\cdot\left(f^{(3,1)}(z)(u+i|z|^2) + f^{(1,2)}(z)(u+i|z|^2)^2\right)\\
&+\overline{\Phi_0^{(2,0)}(z)}\cdot\left(\Phi_0^{(6,0)}(z) + \Phi_0^(4,1)(z)(u+i|z|^2) + \Phi_0^{(2,2)}(u+i|z|^2)^2\right)\\
&+\left(\overline{\phi^{(3,0)}(z)} + \overline{\phi^{(1,1)}(z)(u+i|z|^2)}\right)\cdot\left(\phi^{(5,0)}(z) + \phi^{(3,1)}(z)(u+i|z|^2)+\right.\\
&\left.\left.\phi^{(1,2)}(z)(u+i|z|^2)^2\right)\right\}\\
&+\left|f^{(2,1)}(z)(u+i|z|^2)\right|^2 + \left|\phi^{(4,0)}(z) + \phi^{(2,1)}(z)(u+i|z|^2)\right|^2 = 0.
\end{aligned} \qquad (8.1.48)$$

Note that there is no $f^{(7,0)}(z)$ term, as per [HJX06, Lemma 2.3(A)].

As before, we collect terms of the form $z^\alpha\overline{z}^\beta$, this time with $|\alpha| = 4$ and $|\beta| = 4$:

$$\begin{aligned}
&2Re\left\{-i\overline{z}f^{(1,3)}(z)|z|^6 - \overline{f^{(1,1)}(z)}(-i|z|^2)f^{(1,2)}|z|^4\right.\\
&+\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)(-|z|^4)\\
&\left.+\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z)i|z|^2 + \overline{\phi^{(1,1)}(z)}(-i|z|^2)\phi^{(1,2)}(z)(-|z|^4)\right\}\\
&+\left|\phi^{(4,0)}(z)\right|^2 + \left(\left|f^{(2,1)}(z)\right|^2 + \left|\phi^{(2,1)}(z)\right|^2\right)|z|^4 = 0.
\end{aligned} \qquad (8.1.49)$$

Next, we collect the terms of the form $z^\alpha \overline{z}^\beta u^2$ with $|\alpha| = 2$ and $|\beta| = 2$, getting:

$$\begin{aligned}&2Re\left\{\overline{z}f^{(1,3)}(z)3i\,|z|^2 + \overline{f^{(1,1)}(z)}f^{(1,2)}(z)i\,|z|^2 + \overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)\right.\\&\left.+\overline{\phi^{(1,1)}(z)}\phi^{(1,2)}(z)i\,|z|^2\right\} + \left(\left|f^{(2,1)}(z)\right|^2 + \left|\phi^{(2,1)}(z)\right|^2\right) = 0.\end{aligned} \tag{8.1.50}$$

Next, the terms of the form $z^\alpha \overline{z}^\beta u$ with $|\alpha| = 3$ and $|\beta| = e$:

$$\begin{aligned}&2Re\left\{\overline{z}f^{(1,3)}(z)3(-|z|^4) + \overline{f^{(1,1)}(z)}f^{(1,2)}(z)\,|z|^4 + \overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)2i\,|z|^2\right.\\&\left.+\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z) + \overline{\phi^{(1,1)}(z)}\phi^{(1,2)}(z)\,|z|^4\right\} = 0.\end{aligned} \tag{8.1.51}$$

Multiply (8.1.50) by $|z|^4$ and subtract the result from (8.1.49) to get

$$\begin{aligned}&|z|^2 \cdot 2Re\left\{-4i\overline{z}f^{(1,3)}(z)\,|z|^4 - 2\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)(|z|^2)\right.\\&\left.+i\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z)\right\} + \left|\phi^{(4,0)}(z)\right|^2 = 0.\end{aligned} \tag{8.1.52}$$

Multiply (8.1.51) by $i\,|z|^2$ and combine with (8.1.52):

$$\begin{aligned}&|z|^6\,A(z,\overline{z}) + 2\,|z|^2 \cdot \left(-2\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)(|z|^2) + i\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z)\right) +\\&\left|\phi^{(4,0)}\right|^2 = 0.\end{aligned} \tag{8.1.53}$$

Using (8.1.35) and Lemma 7.1, with $\Lambda^{[1]} = \Lambda^{[2]} = \Phi_0^{(4,0)}(z)$ and $\Gamma_j^{[1]} = \Gamma_j^{[2]} = \eta_j^*$ and doing a bit of algebra:

$$\begin{aligned}&\tfrac{1}{4}\left|\Phi_0^{(4,0)}(z)\right|^2 = |z|^2\left(\tfrac{1}{\mu_1}|\eta_1^*|^2 + \tfrac{1}{\mu_2}|\eta_2^*|^2\right)\\&-\tfrac{1}{\mu_1+\mu_2}\left|\sqrt{\tfrac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\tfrac{\mu_1}{\mu_2}}z_1\eta_2^*\right|^2\end{aligned}\quad. \tag{8.1.54}$$

Replace $\left|\phi^{(4,0)}(z)\right|^2$ in (8.1.53) with $4\cdot$(8.1.54), and then notice that almost every term has a factor of $|z|^2$.

We have:

$$\begin{aligned}&|z|^4\,A(z,\overline{z}) - 4\,|z|^2\,\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) + 2i\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z) +\\&\left|\Phi_0^{(4,0)}(z)\right|^2 + \left|\Phi_1^{(4,0)}(z)\right|^2 = 0\end{aligned}\quad.$$

Expanding $\left|\phi^{(4,0)}(z)\right|^2$ and splitting it into its $\mathcal{S}_0$ and $\mathcal{S}_1$ parts:

$$|z|^4\,A(z,\overline{z}) - 4\,|z|^2\,\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) + 2i\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z) + \frac{4}{\mu_1}|\eta_1^*|^2 + \frac{4}{\mu_2}|\eta_2^*|^2$$

$$-\frac{1}{\mu_1+\mu_2}\left|\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\right|^2 + \left|\Phi_1^{(4,0)}(z)\right|^2 = 0.$$

Apply Huang's Lemma to the $|z|^2$ terms:

$$\begin{aligned}&|z|^4 A(z,\overline{z}) - 4|z|^2\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) + 2i\overline{\phi^{(3,0)}(z)}\phi^{(3,1)}(z)\\&+\tfrac{4}{\mu_1}|\eta_1^*|^2 + \tfrac{4}{\mu_2}|\eta_2^*|^2 = 0.\end{aligned} \tag{8.1.55}$$

It then follows that

$$\frac{1}{4}\left|\Phi_1^{(4,0)}(z)\right|^2 = \frac{1}{\mu_1+\mu_2}\left|\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\right|^2. \tag{8.1.56}$$

Now, we recall (4.37), and we see that

$$\begin{aligned}&\phi_{33}^{(4,0)}(z) = \tfrac{2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\tfrac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\tfrac{\mu_1}{\mu_2}}z_1\eta_2^*\right)\\&\phi_{3j}^{(4,0)}(z) = 0, j > 3.\end{aligned} \tag{8.1.57}$$

This completes the proof of part 1 of Theorem 8.1.

We begin the last part by substituting (8.1.43) directly into (8.1.55) (and adding the Φ_1 term that does not appear in (8.1.43)):

$$\begin{aligned}&|z|^4 A(z,\overline{z}) - 4|z|^2\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) + 2i\overline{\Phi_1^{(3,0)}(z)}\Phi_1^{(3,1)}(z)\\&-8i|z|^2\left(\tfrac{\overline{\xi}_1}{\mu_1}f_1^{(2,2)}(z) + \tfrac{\overline{\xi}_2}{\mu_2}f_2^{(2,2)}(z)\right)\\&+\tfrac{8i}{\mu_1+\mu_2}\left(\sqrt{\tfrac{\mu_1}{\mu_2}}\overline{z}_1\overline{\xi}_2 - \sqrt{\tfrac{\mu_2}{\mu_1}}\overline{z}_2\overline{\xi}_1\right)\cdot\left(\sqrt{\tfrac{\mu_1}{\mu_2}}z_1f_2^{(2,2)}(z) - \sqrt{\tfrac{\mu_2}{\mu_1}}z_2f_1^{(2,2)}(z)\right)\\&+\tfrac{4}{\mu_1}\eta_1^*\overline{(\eta_1^*-\eta_1)} + \tfrac{4}{\mu_2}\eta_2^*\overline{(\eta_2^*-\eta_2)} = 0.\end{aligned} \tag{8.1.58}$$

Looking at the terms of this with a factor of $|z|^2$, we divide everything by 4 and apply Huang's Lemma:

$$\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z) = -2i\left(\frac{\overline{\xi}_1}{\mu_1}f_1^{(2,2)}(z) + \frac{\overline{\xi}_2}{\mu_2}f_2^{(2,2)}(z)\right) + |z|^2 A(z,\overline{z}). \tag{8.1.59}$$

Looking at (8.1.29) and Theorem 6.10, we get

$$\begin{aligned}&f_1^{(2,2)}(z) = \tfrac{i}{2}\mu_1 f_1^{(2,1)}(z) - \overline{\Phi_0^{(I_1+2I_n)}}\Phi_0^{(2,0)}(z) - \overline{e}_1^*\phi^{(2,1)}(z)\\&f_2^{(2,2)}(z) = \tfrac{i}{2}\mu_2 f_2^{(2,1)}(z) - \overline{\Phi_0^{(I-2+2I_n)}}\Phi_0^{(2,0)}(z) - \overline{e}_2^*\phi^{(2,1)}(z).\end{aligned} \tag{8.1.60}$$

We look next at the following, which we will expand:

$$2Re\left\{-2i\left(\frac{\overline{\xi}_1}{\mu_1}f_1^{(2,2)}(z)+\frac{\overline{\xi}_2}{\mu_2}f_2^{(2,2)}(z)\right)\right\}=I+II+III \qquad (8.1.61)$$

$$\begin{aligned}
I \quad &= 2Re\left\{-2i\left(\tfrac{\bar{\xi}_1}{\mu_1}\tfrac{i}{2}\mu_1(-\xi_1)+\tfrac{\bar{\xi}_2}{\mu_2}\tfrac{i}{2}(-\xi_2)\right)\right\}\\
&= -2(|\xi_1|^2+|\xi_2|^2)\\
II \quad &= 2Re\left(2i\tfrac{\bar{\xi}_1}{\mu_1}\overline{\Phi_0^{(I_1+2I_n)}}\Phi_0^{(2,0)}(z)+2i\tfrac{\bar{\xi}_2}{\mu_2}\overline{\Phi_0^{(I_2+2I_n)}}\Phi_0^{(2,0)}(z)\right) \qquad (8.1.62)\\
&= (|\xi_1|^2+|\xi_2|^2)+4\left|\tfrac{\xi_1}{\mu_1}e_1^*+\tfrac{\xi_2}{\mu_2}e_2^*\right|^2\\
III \quad &= 2Re\left(2i\tfrac{\bar{\xi}_1}{\mu_1}\overline{e}_1^*\phi^{(2,1)}(z)+2i\tfrac{\bar{\xi}_2}{\mu_2}\overline{e}_2^*\phi^{(2,1)}(z)\right).
\end{aligned}$$

Here, the first part comes from (7.1.5). The second part follows immediately from (8.1.26).

Rearranging (8.1.50) using the notation from Huang's Lemma, we have

$$|z|^2 A(z,\overline{z})+2Re\left(\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(2,2)}(z)\right)+\left(\left|f^{(2,1)}(z)\right|^2+\left|\phi^{(2,1)}(z)\right|^2\right)=0. \qquad (8.1.63)$$

We now replace the second term of (8.1.63) using (8.1.59), (8.1.61), and (8.1.62), and we replace the third term using (7.1.5), yielding

$$\begin{aligned}
&|z|^2 A(z,\overline{z})-2(|\xi_1|^2+|\xi_2|^2)+(|\xi_1|^2+|\xi_2|^2)+4\left|\tfrac{\xi_1}{\mu_1}e_1^*+\tfrac{\xi_2}{\mu_2}e_2^*\right|^2\\
&+2Re\left(2i\tfrac{\bar{\xi}_1}{\mu_1}\overline{e}_1^*\phi^{(2,1)}(z)+2i\tfrac{\bar{\xi}_2}{\mu_2}\overline{e}_2^*\phi^{(2,1)}(z)\right) \qquad (8.1.64)\\
&+(|\xi_1|^2+|\xi_2|^2)+\left|\phi^{(2,1)}(z)\right|^2=0.
\end{aligned}$$

Several terms sum to zero. The remaining terms are $|z|^2 A(z,\overline{z})$ and a square, so we can rewrite this as

$$|z|^2 A(z,\overline{z})+\left|\phi^{(2,1)}(z)-2i\left(\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\right)\right|^2=0. \qquad (8.1.65)$$

This is really

$$|z|^2+\left|\tilde{\phi}^{(2,1)}(z)\right|^2=0.$$

Thus, we can replace the second term by (8.1.23) (while remembering to add the $\mathcal{S}_1$ terms):

$$|z|^2 A(z,\overline{z}) + \left|\tilde{\Phi}_1^{(2,1)}(z)\right|^2 - \frac{4}{\mu_1+\mu_2}\left|\sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(1,2)}(z)\right|^2 = 0. \quad (8.1.66)$$

By Huang's Lemma, the first term is zero, and by (8.1.46), $\left|\tilde{\phi}_{33}^{(2,1)}(z)\right|$ is equal to -1 times the third term. Thus,

$$\begin{aligned} &\tilde{\phi}_{33}^{(2,1)}(z) = \tfrac{-2}{\sqrt{\mu_1+\mu_2}}\left(\sqrt{\tfrac{\mu_1}{\mu_2}} z_1 f_2^{(1,2)}(z) - \sqrt{\tfrac{\mu_2}{\mu_1}} z_2 f_1^{(1,2)}(z)\right) \\ &\tilde{\phi}_{3j}^{(2,1)}(z) = 0, j > 3. \end{aligned} \quad (8.1.67)$$

This and (8.1.20) are all we need to complete our proof of the second part of Theorem 8.1.

Chapter 9

Proof of Theorem 6.1

In this section, we prove the Third Gap Theorem. We do this in two steps. First, we cover the case where $\kappa_0 = 1$, and then we cover the case where $\kappa_0 = 2$. By [Hu03, Lemma 2.3], when $N \le 4n - 7$, $F \in Rat(\mathbb{B}^n, B^N)$ cannot have geometric rank greater than 2. We do not need to treat the case where $\kappa_0 = 0$, because then F is linear.

Step I:An application of a normal form in [HJX06] for maps with geometric rank 1

First, we use recall the Second Gap Theorem, [HJX06, Theorem 1.2].

9.1 Theorem (HJX06, Theorem 1.2)**.** *Let F be a non-linear proper holomorphic map from $\mathbb{B}^n$ into B^N with $N \ge n \ge 3$. Assume that F is C^3-smooth up to the boundary and has geometric rank $\kappa_0 \le n - 2$. Then F is equivalent to a proper holomorphic map of the form*

$$H := (z_1, \ldots, z_{k^0}, H_1, \ldots, H_{N-k^0})$$

where $k^0 = n - \kappa_0$ and $H_j = \sum_{\ell=k^0+1}^{n} z_\ell J_{j,\ell}$, with $H_{j,\ell}$ holomorphic over $\overline{\mathbb{B}^n}$. Moreover, when $\kappa_0 = 1$, $(H_1, \ldots, H_{N-n+1}) = z_n \cdot h$, with h a rational proper holomorphic map from

$\mathbb{B}^n$ *into* $\mathbb{B}^{N-n+1}$. *Both* H *and* h *are affine linear maps along each hyperplane defined by* $z_n =$ *constant.*

We then consider $F \in Rat(\mathbb{B}^n, B^N)$, $\kappa_0 = 1$. By this theorem, F is equivalent to a map of the form

$$\Phi = (z_1, \ldots, z_{n-1}, z_n \cdot H(z)) = (\phi_1, \ldots, \phi_N),$$

where $H \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$ also has geometric rank 1.

9.2 Lemma (HJY14, Lemma 5.1). *If* $H(\mathbb{B}^n)$ *is contained in an affine subspace of dimension* m *in* $\mathbb{C}^{N-n+1}$, *then* $F(\mathbb{B}^n)$ *is contained in an affine subspace of dimension* $m + n$ *in* $\mathbb{C}^N$.

Proof. Linear transformations map affine linear subspaces to affine linear subspaces. We have that $F(\mathbb{B}^n)$ is contained in an affine linear subspace of dimension m if and only if F is equivalent to a map of the form $(G, 0)$ with G having m components. Supposing the image of $H = (h_1, \ldots, h_{N-n+1})$ is contained in an affine subspace of dimension $m \leq N - n$, there are N - n - m + 1 linearly independent vectors $\mu_j = (a_{j1}, \ldots, a_{jk})$, $k = N - n + 1$, such that $\sum_{\ell=1}^{k} a_{j\ell} h_\ell \equiv c_j$ for some $c_j \in \mathbb{C}$. If $c_j = 0$ for all j, then $\sum_{\ell=1}^{k} a_{j\ell} \phi_{n-1+\ell}(z) \equiv 0$, and so $\Phi(\mathbb{B}^n)$ is contained in an affine linear subspace of dimension $n + m - 1$. If at least one c_j is nonzero, we assume $c_1 = 1$, and then we look at $\sum_{\ell=1}^{k} (a_{j\ell} - c_j a_{1\ell}) \phi_{n-1}(z) \equiv 0$.

Specifically,

$$\sum_{\ell=1}^{k} (a_{j\ell} - c_j a_{1\ell}) \phi_{n-1}(z) = \sum a_{j\ell} \phi_{n-1+\ell}(z) + \sum c_j a_{1\ell} \phi_{n-1+\ell}(z) = c_j - (c_j)(1) \equiv 0.$$

We note that $\{\mu_2 - c_2 \mu_1, \ldots, \mu_{N-n-m+1} - c_{N-n-m+1} \mu_1\}$ is linearly independent, and thus $\Phi(\mathbb{B}^n)$ is contained in an affine subspace of dimension $m + n$.

Essentially, for an m-dimensional affine subspace, we've found $k - m$ "orthogonal" vectors. □

If we apply the main theorem from [HJX06] to H and use Lemma 9.2 (since $N \leq 4n-5$ implies $H \in Rat(\mathbb{B}^n, B^N)$, with $N \leq 3n-4$), we get that when $3n+1 \leq N \leq 4n-5$ and $n \geq 6$, $F(\mathbb{B}^n)$ is contained in an affine linear subspace of dimension $3n$, which proves Theorem 6.1 for the case where $\kappa_0 = 1$.

Step II: The case where $\kappa_0 = 2$

When $\kappa_0 = 2$, our theorem is a special case of the following:

9.3 Theorem (HJY14, Theorem 5.2)**.** *Let F be a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_N$ with geometric rank $\kappa_0 = 2$. Assume that $n \geq 7$ and $3n \leq N \leq 4n-6$. Then F is equivalent to a map of the form $(G, 0')$ where G is a proper rational map from $\mathbb{H}_n$ into $\mathbb{H}_{3n}$.*

Then, since F is rational, by a result of Cima-Suffridge, it extends holomorphically across $\partial\mathbb{H}_n$.

Let $N \leq 4n-6$, and let F be a proper rational holomorphic map from $\mathbb{H}_n$ into $\mathbb{H}_N$ with geometric rank $\kappa_0 = 2$ and $F(0) = 0$. We may assume that F satisfies the normalization in Theorem 6.10.

Write $\mathcal{L}_j = \frac{\partial}{\partial z_j} - 2i\overline{z}_j \frac{\partial}{\partial w}$ for $1 \leq j \leq n-1$. This is the usual basis of (1,0)-type tangent vectors along ∂H_n. Let $\mathcal{L}^\alpha$, where α is a multi-index, be the standard higher-order version of $\mathcal{L}$.

We notice that

$$\mathcal{L}^\alpha h\Big|_0 = \frac{\partial^{|\alpha|}}{\partial z^\alpha} h\Big|_0 - 2i(0)\frac{\partial^{|\alpha|}}{\partial w^{|\alpha|}} h.$$

We assume the normalization from of F found in Corollary 7.5, and we assume $\phi_{33}^{(3,0)}(z)$ is not identically 0.

We have that

$$span_{|\beta| \leq 3} \left\{ \mathcal{L}^\beta f \Big|_0 \right\} =$$

$$span\left\{(0,\ldots,0,1^{j^{th}},0,\ldots,0), 1\leq j\leq n+\#\mathcal{S}_0 = n+(n-1)+n-2\right\}. \qquad (9.1.1)$$

Because we are taking the result at 0, we are essentially only looking at components with a nonzero term of degree 3 or lower, which means we lose most of Φ_1. By Theorem 6.10, there are no lower (weighted) degree terms, and by Corollary 7.5, all but one of the degree 3 terms is 0. (We do not need the (1,2) or (2,1) terms, since $\mathcal{L}^\alpha|_0$, $|\alpha|=3$, takes those to 0, too.)

Now we begin to use Theorem 8.1, which we went through so much trouble to prove. Because of the first part of the theorem, $\Phi_1^{(4,0)} = (\phi_{33}^{(4,0)}, 0)$, so

$$span_{|\beta|\leq 4}\left\{\mathcal{L}^\beta F\Big|_0\right\} = span\left\{(0,\ldots,0,1^{j^{th}},0,\ldots,0), 1\leq j\leq n+\#\mathcal{S}_0\right\}. \qquad (9.1.2)$$

Thus,

$$span_{|\beta|\leq 4}\left\{\mathcal{L}^\beta F\Big|_0\right\} = span_{|\beta|\leq 3}\left\{\mathcal{L}^\beta F\Big|_0\right\}. \qquad (9.1.3)$$

For any $p\in\mathbb{H}_n$ near 0, there are automorphisms $\tau_p\in Aut_0(\mathbb{H}_N)$ and $\sigma_p\in Aut_0(\mathbb{H}_n)$ such that $G_p = \tau_0\circ F_p\circ\sigma_p$ satisfies the normalization in Theorem 6.10, and we can have $\Phi_1^{(3,0)}(z)$ not identically 0, since we can choose τ_p and σ_p to depend smoothly on p, so we can apply some unitary matrix transformation U_p to normalize the Φ_1 part, giving us what we have in Corollary 7.5. Now,

$$span_{|\beta|\leq 4}\left\{\mathcal{L}^\beta G_0\Big|_0\right\} = span_{|\beta|\leq 3}\left\{\mathcal{L}^\beta G_0\Big|_0\right\},$$

or

$$span_{|\beta|\leq 4}\left\{D_z^\beta G_0\Big|_0\right\} = span_{|\beta|\leq 3}\left\{D_z^\beta G_0\Big|_0\right\} \qquad (9.1.4),$$

and the dimension of this span is $n+\#\mathcal{S}_0$.

Because we can always do this normalization, we will assume it, and we will write τ_p instead of $U_p\circ\tau_p$.

Then $F_p = \tau_p^{-1}\circ G_p\circ\sigma_p^{-1}$.

Next, we make a claim: For $|\alpha| =$,

$$D_z^\alpha(\tau_p^{-1} \circ G_0 \circ \sigma_p^{-1})\Big|_0 \in span_{|\beta|\le 3}\left\{D_z^\beta(\tau_p^{-1} \circ G_p \circ \sigma_p^{-1})\right\},$$

or

$$\mathcal{L}^\alpha F_p\Big|_0 \in span_{|\beta|\le 3}\left\{\mathcal{L}^\beta F_p\Big|_0\right\}. \qquad (9.1.5)$$

To show this, we write

$$\sigma_p^{-1} = \left(\mu\frac{z-aw}{q(z,w)}A, \mu^2\frac{w}{q(z,w)}\right), \tau_p^{-1} = \left(\tilde{\mu}\frac{\tilde{z}-\tilde{a}\tilde{w}}{\tilde{q}(\tilde{z},\tilde{w})}\tilde{A}, \tilde{\mu}^2\frac{\tilde{w}}{\tilde{q}(\tilde{z},\tilde{w})}\right),$$

where $\mu, \tilde{\mu} \neq 0$, A,$\tilde{A}$ are unitary matrices, and $q(0) = \tilde{q}(0) = 1$.

We write $G_p = (h(z,w), w)$. Then

$$F_p(z,0) = \left(\frac{\tilde{\mu}}{q^*(z)}h(\frac{\mu z}{q(z,0)}A, 0)\tilde{A}, 0\right). \qquad (9.1.6)$$

Here, $q^*(z)$ is a holomorphic function with $q^*(0) = 1$.

Now, we prove our claim by showing that, for $|\alpha| = 4$,

$$D_z^\alpha h\left(\frac{\mu z}{q(z,0)}A, 0\right)\Big|_0 \in span_{|\beta|\le 3}\left\{D_z^\beta\left(\frac{\mu z}{q(z,0)}A, 0\right)\Big|_0\right\}.$$

We note something that is not quite obvious:

$$span_{|\alpha|\le k}\left\{D_z^\alpha h\left(\frac{\mu z}{q(z,0)}A, 0\right)\Big|_0\right\} = span_{|\alpha|\le k}\left\{D_z^\alpha h(z,0)\Big|_0\right\}.$$

For example,

$$\begin{aligned}&\tfrac{\partial}{\partial z_j}h\left(\tfrac{\mu z_1 a_{\ell 1}+\mu z_2 a_{\ell 2}+\cdots+\mu z_k a_{\ell k}}{q(z,0)}\right) = \\ &h^{(j)}\left(\tfrac{\mu a_{\ell 1}z_1+\cdots+\mu a_{\ell k}z_k}{q(z,0)}\right)\times \\ &\left(-\tfrac{\mu z_1}{(q(z,0)^2)} + \cdots + \left(\tfrac{\mu a_\ell j}{q(z,0)} - \tfrac{\mu z_j a_{\ell j}}{(q(z,0))^2}\right) + \cdots - \tfrac{\mu z_k}{(q(z,0))^2}\right)\end{aligned}$$

and, when $z = 0$, this is

$$h^{(j)}(0,0)\cdot(\mu a_{\ell j}).$$

Then, by (9.1.4),

$$span_{|\alpha|\le 4}\left\{D_z^\alpha h(z,0)\Big|_0\right\} = span_{|\alpha|\le 3}\left\{D_z^\alpha h(z,0)\Big|_0\right\}.$$

Therefore,

$$span_{|\alpha|\le 4}\left\{D - z^\alpha F_p(z,0)\Big|_0\right\} = span_{|\alpha|\le 3}\left\{D_z^\alpha h(z,0)\Big|_0\right\}.$$

This proves the claim. We also get from (9.1.6) that

$$\dim\left(span_{|\alpha|\le 3}\left\{D_z^\alpha F_p(z,0)\Big|_0\right\}\right) = \dim\left(span_{|\alpha|\le 3}\left\{D_z^\alpha G_p(z,0)\Big|_0\right\}\right) = n + \#\mathcal{S}_0.$$

Putting several results together, since $\mathcal{L}^\alpha(F_p)\big|_0 = \mathcal{L}^\alpha(F)(p)$, we get that, for $|\alpha| = 4$, $\mathcal{L}^\alpha F(p) \in span_{|\beta|\le 3}\left\{\mathcal{L}^\beta F(p)\right\}$.

This has a fixed dimension, so we can now take any α and express $\mathcal{L}^\alpha F(p)$ as a smooth linear combination of basis elements from $span_{|\beta|\le 3}\left\{\mathcal{L}^\beta f(p)\right\}$. Because of this, we can express a total differential of any order $|\alpha|$ using this basis, so by the multi-variable version of Taylor's theorem,

$$F(z,w) \in span_{|\beta|\le 3}\left\{D^\beta F(0)\right\}$$

for (z,w) close to 0.

Next, we remind ourselves that $\phi^{(1,1)}(z)w = (e_1^* z_1 + e_2^* z_2)w$. Theorem 8.1, parts 2 and 3, gives us the rest of the span of $D_z^\alpha \Phi_1^{(1,2)}(z)$ and $D_z^\alpha \Phi_1^{(2,1)}(z)$. This lets us say that $span_{|\beta|\le 3}\left\{D^\beta f(0)\right\}$ is in the span of

$$\left\{(0,\dots,0,1^{j^{th}},0,\dots,0),(0,\dots,0,1),(0,\dots,0,\hat{e}_1,0),(0,\dots,0,\hat{e}_2,0)\right\}$$

with $1 \le j \le ((n-1)+(n-1)+(n-2))+1$, which is the f terms, the Φ_0 terms, and ϕ_{33}. The total here comes to $3n$. Thus, in this setting, we have our result from Theorem 6.1.

If, for a certain point p_0 near 0, the $\phi_{33}^{(3,0)}(z)$ associated with F_{p_0} is not a zero polynomial, we can look at F_{p_0} instead of F and apply the same argument to get the proof of Theorem 9.2.

If, after the normalization of Corollary 7.5, we have $\Phi_1^{(3,0)} \equiv 0$, then we end up with $\mathcal{L}^\alpha F(p) \in span_{|\beta|\leq 2}\left\{\mathcal{L}^\beta F(p)\right\}$ with $\dim\left(span_{|\beta|\leq 2}\left\{\mathcal{L}^\beta F(p)\right\}\right) = n + \#\mathcal{S}_0 - 1$, and so $F(\mathbb{B}^n)$ is contained in a subspace of dimension $3n-1$, spanned by

$$\left\{(0,\ldots,0,1^{j^{th}},0,\ldots,0),(0,\ldots,0,1),(0,\ldots,0,\hat{e}_1,0),(0,\ldots,0,\hat{e}_2,0)\right\}.$$

Here, $1 \leq j \leq (n-1)+(n-1)+(n-2)$, which corresponds to the f part and the Φ_0 part.

This proves the Third Gap Theorem.

Part IV

The Proof of the Main Theorem

Chapter 10

Proving several parameters are zero

We now restate our main theorem and attempt to prove it.

10.1 Theorem (Main Theorem)**.** *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$, $n \geq 4$. Then F is equivalent to a linear embedding, the Whitney Map, the D'Angelo Map, or the Generalized Whitney Map.*

The first three of these maps show up in cases with $N < 3n-3$. Recall that all maps for $n \leq N < 2n-1$ are linear, then we can find the Whitney Map when $N = 2n-1$, and then for $2n \leq N < 3n-3$, we also have the family of maps described by D'Angelo, which we sometimes call the D'Angelo Map. In our current case, we will assume that our map has geometric rank $\kappa_0 = 2$, because lower geometric rank puts us in one of the other cases mentioned, and higher geometric rank is impossible, since Huang showed in [Hu03] that $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Since $\kappa_0 \leq n-2$, we have degenerate geometric rank.

The final version of our calculation is much less complex than the final version because we were able to show that many of the relevant terms of ϕ are actually 0. It is possible to prove our theorem without doing this, but the results from this chapter make it much easier to write our proof down; the original version was about three times as long. We shall do the simplification in this chapter.

We recall that we have a parameter $e = \begin{pmatrix} e_{1,11} & e_{1,12} & e_{1,1j} & e_{1,22} & e_{1,2j} \\ e_{2,11} & e_{2,12} & e_{2,1j} & e_{2,22} & e_{2,2j} \end{pmatrix}$. In particular, $e_{j,k\ell}$ is the coefficient of the $z_j w$ term in the Taylor series for one of the codimensional components of $\phi_{k\ell}$.

By [Hu03] and [HJX05] and the fact that F has degenerate geometric rank, we have that F is $(n-\kappa_0)$-linear, meaning for any $p \in \mathbb{B}^n$, there is an affine subspace S_p^a of codimension κ_0 containing p such that F is linear fractional along S_p^a.

Proving that $e_{1,1j} = 0$

For the first step, we take L_ε to be given by

$$\begin{cases} z_1 = \sum_{j=3}^{n-1} a_j(\varepsilon) z_j + a_n(\varepsilon) w + \varepsilon_1 \\ z_2 = \sum_{j=3}^{n-1} b_j(\varepsilon) z_j + b_n(\varepsilon) w + \varepsilon_2 \end{cases}. \qquad (10.1.1)$$

Now we look at an automorphism on this space, $\hat{\sigma}_{\overline{c}}(L_\varepsilon)$, given by

$$\begin{cases} Z_1 = \sum_{j=3}^{n-1} A_j(\varepsilon) Z_j + A_n(\varepsilon) W + \rho_1(\varepsilon) \\ Z_2 = \sum_{j=3}^{n-1} A_j(\varepsilon) Z_j + A_n(\varepsilon) W + \rho_2(\varepsilon) \end{cases} \qquad (10.1.2)$$

where

$$\begin{aligned} \hat{\sigma}_{\overline{c}}^{-1}(Z,W) &:= \tfrac{(Z_1, Z_2, Z_3 + c_3 W, \ldots, Z_{n-1} + c_{n-1} W, W)}{q_c} \\ &= (z_1, z_2, z_3, \ldots, z_{n-1}, w) \end{aligned} \qquad (10.1.3)$$

and $q_c := 1 - 2i\overrightarrow{c} \cdot Z - i\,|\overrightarrow{c}|^2\, W$, with $\overrightarrow{c} = (0, \ldots, 0, c_3, \ldots, c_{n-1})$.

$$\begin{aligned} & q_c = 1 - i(2\textstyle\sum_{j=3}^{n=1} c_j Z_j + \sum_{j=3}^{n-1} |c_j|^2\, W) \\ & z_1 = \tfrac{Z_1}{q_c} = \tfrac{Z_1}{1 - i(2\sum_{j=3}^{n=1} c_j Z_j + \sum_{j=3}^{n-1} |c_j|^2 W)} \\ & z_1 = \textstyle\sum_{j=3}^{n-1} a_j(\varepsilon) z_j + a_n(\varepsilon) w + \varepsilon_1 = \tfrac{\sum_{j=3}^{n=1} A_j(\varepsilon) Z_j + A_n(\varepsilon) W + \rho_1(\varepsilon)}{1 - i(2\sum_{j=3}^{n=1} c_j Z_j + \sum_{j=3}^{n-1} |c_j|^2 W)} \\ & z_1 = \textstyle\sum_{j=3}^{n-1} a_j(\varepsilon) z_j + a_n(\varepsilon) w + \varepsilon_1 = \tfrac{\sum_{j=3}^{n=1} A_j(\varepsilon) Z_j + A_n(\varepsilon) W + \rho_1(\varepsilon)}{q_c} \\ & \tfrac{Z_1}{q_c} = \textstyle\sum_{j=3}^{n-1} a_j(\varepsilon) z_j + a_n(\varepsilon) w + \varepsilon_1 = \sum_{j=3}^{n-1} a_j(\varepsilon) \tfrac{Z_j + c_j W}{q_c} + a_n(\varepsilon)(W) + \varepsilon_1 \\ & \Rightarrow Z_1 = \textstyle\sum_{j=3}^{n-1} a_j(\varepsilon)(Z_j + c_j W) + a_n(\varepsilon) W + \varepsilon_1 q_c. \end{aligned}$$

By the same argument,

$$Z_2 = \sum_{j=3}^{n-1} b_j(\varepsilon)(Z_j + c_j W) + b_n(\varepsilon)W + \varepsilon_2 q_c.$$

We set these equal to our definitions for Z_1 and Z_2:

$$\begin{cases} \sum_{j=3}^{n-1} A_j(\varepsilon)Z_j + A_n(\varepsilon)W + \rho_1(\varepsilon) = \\ \sum_{j=3}^{n=1} a_j(\varepsilon)(Z_j + c_j W) + a_n(\varepsilon)W + \varepsilon_1(1 - 2i\overrightarrow{c} \cdot Z - i\left|c\right|^2 W) \\ \sum_{j=3}^{n-1} B_j(\varepsilon)Z_j + B_n(\varepsilon)W + \rho_2(\varepsilon) = \\ \sum_{j=3}^{n=1} b_j(\varepsilon)(Z_j + c_j W) + b_n(\varepsilon)W + \varepsilon_2(1 - 2i\overrightarrow{c} \cdot Z - i\left|c\right|^2 W). \end{cases}$$

Looking at Z_j for $3 \leq j \leq n-1$, we see that the coefficients on the left hand side and right hand side are:

$$\begin{cases} A_j(\varepsilon) = a_j(\varepsilon) - 2i\varepsilon_1 c_j \\ B_j(\varepsilon) = b_j(\varepsilon) - 2i\varepsilon_2 c_j. \end{cases}$$

This means that we can choose c so that $a_j^{(I_1)} = 0$. In other words, no matter where we start, we can choose an automorphism so that the result of applying which makes $a_j^{(I_1)} = 0$.

Moving along, we note that for maps from $\mathbb{H}^n$ to $\mathbb{H}^{3n-3}$, we have $\phi_{33} \equiv 0$. In particular, we have $\phi_{33}^{(2,1)} \equiv 0$.

$$0 = \tilde{\phi}_{33}^{(2,1)} = \frac{-2}{\sqrt{\mu_1 + \mu_2}} \left(\sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(1,2)}(z) \right).$$

We apply this to [HYJ14,(4.3)]:

$$\frac{i}{2}\mu_1 a_n^{(1)}(\varepsilon) + f_1^{(1,2)}(\varepsilon, 0, \ldots, 0) = 0, \frac{i}{2}\mu_2 b_n^{(1)}(\varepsilon) + f_2^{(1,2)}(\varepsilon, 0, \ldots, 0) = 0.$$

Multiply the first equation by $\mu_2 z_2$ and the second equation by $\mu_1 z_1$, getting

$$\frac{i}{2} z_2 \mu_1 \mu_2 a_n^{(1)}(\varepsilon) + \mu_2 z_2 f_1^{(1,2)}(\varepsilon, 0, \ldots, 0) = 0, \frac{i}{2} z_1 \mu_1 \mu_2 b_n^{(1)}(\varepsilon) + \mu_1 z_1 f_2^{(1,2)}(\varepsilon, 0, \ldots, 0)$$

$$-\frac{i}{2} z_2 \mu_1 \mu_2 a_n^{(1)}(\varepsilon) = -\mu_2 z_2 f_1^{(1,2)}(\varepsilon, 0, \ldots, 0) = -\mu_1 z_1 f_2^{(1,2)}(\varepsilon, 0, \ldots, 0) = \frac{i}{2} z_1 \mu_1 \mu_2 b_n^{(1)}(\varepsilon)$$

$$z_2 a_n^{(1)}(\varepsilon) = z_1 b_n^{(1)}(\varepsilon).$$

Since $a_n^{(1)}(z)$ and $b_n^{(1)}(z)$ are linear functions of z, we get that

$$a_n^{(1)}(z) = \zeta z_1, b_n^{(1)}(z) = \zeta z_2 \qquad (10.1.4)$$

for some $\zeta \in \mathbb{C}$.

10.2 Lemma. *Let $F \in Rat(\mathbb{H}^n, \mathbb{H}^{3n-3})$ with geometric rank $\kappa_0 = 2$ and $n \geq 4$. Then*

$$\begin{cases} e_{1,22} = 0, e_{1,2j} = 0, 3 \leq j \leq n-1 \\ e_{2,11} = 0, e_{2,1j} = 0, 3 \leq j \leq n-1, \\ e_{2,12} = \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11}, \\ e_{2,22} = \frac{\sqrt{\mu_2(\mu_1+\mu_2)}}{\mu_1} e_{1,12}, \\ e_{2,2j} = \sqrt{\frac{\mu_2}{\mu_2}} e_{1,1j}, 3 \leq j \leq n-1, \end{cases}$$

$$\xi_1 = \sqrt{\mu_1}\overline{e_{1,11}} z_1^2 + \sqrt{\mu_1 + \mu_2}\overline{e_{1,12}} z_1 z_2 + \sum_{j=1}^{n-1} \sqrt{\mu_1}\overline{e_{1,1j}} z_1 z_j,$$

$$(10.1.5)$$

$$\xi_2 = \sqrt{\mu_1}\overline{e_{2,12}} z_1 z_2 + \sqrt{\mu_1 + \mu_2}\overline{e_{2,22}} z_2^2 + \sum_{j=3}^{n-1} \sqrt{\mu_1}\overline{e_{2,2j}} z_2 z_j.$$

Recall that $\xi_j(z)$ is defined in [HJY14] by $\xi_j(z) = \overline{e_j} \cdot \Phi_0^{(2,0)}(z)$.

Next, we let H be an affine linear function along L_ε. Then $\frac{\partial^2 H|_{L_\varepsilon}}{\partial z_j \partial w} \equiv 0$. We write

$$H|_L = H\left(\sum_{k=3}^{n} a_k z_k + \varepsilon_1, \sum_{k=3}^{n} b_k z_k + \varepsilon_2, z_3, \ldots, z_{n-1}, w\right).$$

Next, for $1 \leq j, k \leq n-1$,

$$\frac{\partial^2 H|_L}{\partial z_j \partial w} =$$
$$\frac{\partial^2 H}{\partial z_1^2} a_j a_n + \frac{\partial^2 H}{\partial z_1 \partial z_2}(a_n b_j + a_j b_n) + \frac{\partial^2 H}{\partial z_1 \partial z_j} a_n + \frac{\partial^2 H}{\partial z_1 \partial w} a_j + \cdot \qquad (10.1.6)$$
$$\frac{\partial^2 H}{\partial z_2^2} b_j b_n + \frac{\partial^2 H}{\partial z_2 \partial z_j} b_n + \frac{\partial^2 H}{\partial z_2 \partial w} b_j + \frac{\partial^2 H}{\partial z_j \partial w} = 0.$$

Note that H is only affine linear along L, so the partial derivatives not restricted to L are not all zero. To see more easily where the coefficients come from, think of what happens when you take the partial derivative with respect to w by itself, since w appears in the first, second, and n^{th} terms of H. Because of this fact and the chain rule, the resulting terms are $\frac{\partial H}{\partial w} + \frac{\partial H}{\partial z_1} a_n + \frac{\partial H}{\partial z_2} b_n$.

Looking only at $\frac{\partial H}{\partial w}$, let us apply $\frac{\partial}{\partial z_j}$. We get $\frac{\partial^2 H}{\partial z_1 \partial w} a_j + \frac{\partial^2 H}{\partial z_j \partial w} + \frac{\partial^2 H}{\partial z_2 \partial w} b_j$, since z_j appears in the first, second, and j^{th} terms of H.

Looking only at $\frac{\partial H}{\partial z_1} a_n$, we do the same thing, getting $\frac{\partial^2 H}{\partial z_1^2} a_j a_j + \frac{\partial^2 H}{\partial z_2 \partial z_1} b_j a_n + \frac{\partial^2 H}{\partial z_j \partial z_1} a_n$.

Looking only at $\frac{\partial H}{\partial z_2} b_n$, we do the same thing, getting $\frac{\partial^2 H}{\partial z_2^2} b_j b_n + \frac{\partial^2 H}{\partial z_1 \partial z_2} a_j b_n + \frac{\partial^2 H}{\partial z_j \partial z_2} b_n$.

Next, we let H be f_1.

$$\begin{aligned} f_1 &= \textstyle\sum_{j=1}^{2} z_j f_{1j}^*(z,w) \\ &= z_1 + \tfrac{i\mu_1}{2} z_1 w + b_{11}^{(1)}(z) z_1 w + b_{12}^{(1)}(z) z_2 w + \mathcal{O}_{wt}(4) \\ &= z_1 + \tfrac{i\mu_1}{2} z_1 w + b_{11}^{(1)}(z)(\textstyle\sum a_j z_j + \epsilon_1) w + b_{12}^{(1)}(z)(\textstyle\sum b_j z_j + \epsilon_2) w + \mathcal{O}_{wt}(4). \end{aligned}$$

We collect the ϵ_1 and ϵ_2 terms.

$$\begin{aligned} & \textstyle\sum a_j z_j + \epsilon_1 + \tfrac{i\mu_1}{2} z_1 w \\ & + b_{11,1} z_1 (\textstyle\sum a_j z_j + \epsilon_1) w + b_{11,2} z_2 (\textstyle\sum a_j z_j + \epsilon_1) w + \dots \\ & + b_{11,j} z_k (\textstyle\sum a_j z_j + \epsilon_1) w + \dots + b_{11,n} (\textstyle\sum a_j z_j + \epsilon_1) w^2 \\ & + b_{12,1} z_1 (\textstyle\sum b_j z_j + \epsilon_2) w + b_{12,2} z_2 (\textstyle\sum b_j z_j + \epsilon_2 w) + \dots \\ & + b_{12,n} (\textstyle\sum b_j z_j + \epsilon_2) w^2 + \mathcal{O}_{wt}(4) \\ = \; & \textstyle\sum a_j z_j + \epsilon_1 + \tfrac{i\mu_1}{2} z_1 w \\ & + b_{11,1} (\textstyle\sum a_j z_j + \epsilon_1)(\textstyle\sum a_j z_j + \epsilon_1) w + b_{11,2} (\textstyle\sum b_j z_j + \epsilon_2)(\textstyle\sum a_j z_j + \epsilon_1) w \\ & + \dots + b_{11,k} (\textstyle\sum a_j z_j + \epsilon_1) z_k w + \dots + b_{11,n} (\textstyle\sum a_j z_j + \epsilon_1) w^2 \\ & + b_{12,1} (\textstyle\sum a_j z_j + \epsilon_1)(\textstyle\sum b_j z_j + \epsilon_2) w + b_{12,2} (\textstyle\sum b_j z_j + \epsilon_2)(\textstyle\sum b_j z_j + \epsilon_2) w \\ & + \dots + b_{12,k} (\textstyle\sum b_j z_j + \epsilon_2) z_k w + \dots + b_{12,n} (\textstyle\sum b_j z_j + \epsilon_2) w^2 + \mathcal{O}_{wt}(4). \end{aligned}$$

We find that whenever ϵ_1 appears, it is with w, z_1, and either $z_1, z_2,$, or z_j, but if it's z_1 or z_2, we can replace that with $\sum a_k z_k + \epsilon_1$ or $\sum b_k z_k + \epsilon_2$. Then, we can apply $\frac{\partial^2 f_1}{\partial z_j \partial w}$ to remove the z_1 and w.

We are left with

$$\frac{i}{2}\mu_1 a_j^{(1)} + f_1^{(I_1+I_j+I_n)}\epsilon_1 + f_1^{(I_2+I_j+I_n)}\epsilon_2 = 0.$$

We check [HJY14,(3.5)], and we get $f_1^{(I_1+I_j+I_n)} = -\xi_1$, and by $(10.1.5)$, we have $f_1^{(I_1+I_j+I_n)} = -\sqrt{\mu_1}\overline{e_{1,1j}}$.

We conclude that

$$e_{1,1j} = 0.$$

Proving that $e_{1,12} = 0$

Next, we use the following facts:

$$\begin{aligned}
f_1^{(1,2)} &= -\tfrac{i}{2}\mu_1\zeta_1 z_1 \\
f_2^{(1,2)} &= -\tfrac{i}{2}\mu_2\zeta_2 z_2 \\
f_1^{(1,1)} &= \tfrac{i}{2}\mu_1 z_1 \\
f_2^{(1,1)} &= \tfrac{i}{2}\mu_2 z_2 \\
\phi^{(1,1)} &= (e_{1,11}z_1, e_{1,12}z_1 + e_{2,12}z_2, e_{2,22z_2}, 0, \ldots, 0).
\end{aligned}$$

The first two come from [HYJ14,(4.3)], and $\zeta z_1 = a_n^{(1)}(\epsilon)$, and $\zeta z_2 = b_n^{(1)}(\epsilon)$.

The second two are from [HYJ14,(2.3)].

The last comes from the first part of section 3 of [HYJ14]:

$$\phi^{(1,1)}(z)w = \sum e_j^* z_j w.$$

Now, [HYJ14,(4.10)] says

$$2Re(\overline{z}f^{(1,2)}z) + \left|f^{(1,1)}(z)\right|^2 + \left|\phi^{(1,1)}(z)\right|^2 = 0.$$

Combining these facts:

$$2Re\left\{\overline{z}_1 \cdot (-\frac{i}{2}\mu_1\zeta z_1) + \overline{z}_2 \cdot (-\frac{i}{2}\mu_2\zeta z_2)\right\} +$$

$$+\left|\frac{i}{2}\mu_1 z_1\right|^2+\left|\frac{i}{2}\mu_2 z_2\right|^2+|e_{1,11}z_1|^2+|e_{1,12}z_1+e_{2,12}z_2|^2+|e_{2,22}z_2|^2=0.$$

We break this apart, separating the $z_1\overline{z}_1$, $z_1\overline{z}_2$, and $z_1\overline{z}_2$ parts, respectively:

$$2Re\left\{-\tfrac{i}{2}\zeta_n\right\}\mu_1+\tfrac{\mu_1^2}{4}+|e_{1,11}|^2+|e_{1,12}|^2=0 \quad (10.1.7)$$

$$2Re\left\{-\tfrac{i}{2}\zeta_n\right\}\mu_2+\tfrac{\mu_2^2}{4}+|e_{2,22}|^2+|e_{2,12}|^2=0 \quad (10.1.8)$$

$$e_{1,12}\overline{e_{2,12}}=0. \quad (10.1.9)$$

We get rid of some of the terms in (10.1.7) and (10.1.8) by taking $\mu_2(10.1.7)-\mu_1(10.1.8)$:

$$\frac{\mu_1^2\mu_2}{4}-\frac{\mu_2^2\mu_1}{4}+\mu_2|e_{1,11}|^2-\mu_1 e_{1,12}{}^2+\mu_2 e_{1,12}{}^2-\mu_1 e_{2,22}{}^2=0.$$

Since $e_{2,12}$ is related to $e_{1,11}$ and $e_{1,12}$ is related to $e_{2,22}$ by

$$e_{2,12}=\frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2}}e_{1,11}, e_{1,12}=\frac{\mu_1}{\sqrt{\mu_2(\mu_1+\mu_2)}}e_{2,22},$$

we have

$$\frac{1}{4}\mu_1\mu_2(\mu_1-\mu_2)=\frac{\mu_1\mu_2}{\mu_1+\mu_2}|e_{1,11}|^2-\frac{\mu_1\mu_2}{\mu_1+\mu_2}|e_{2,22}|^2=0.$$

Rearranging some terms, we get

$$|e_{1,11}|^2=|e_{2,22}|^2+\frac{1}{4}(\mu_1+\mu_2)(\mu_2-\mu_1).$$

According to (10.1.9), we have that either $e_{1,11}$ or $e_{2,22}=0$. Suppose $\mu_2\geq\mu_1$. Then $e_{2,22}=0$, and thus $e_{1,12}=0$, and

$$|e_{1,11}|^2=\frac{1}{4}(\mu_1+\mu_2)(\mu_2-\mu_1).$$

Combining this with (10.1.7), we get

$$Im(\zeta)=-\frac{1}{4\mu_1}(\mu_1+\mu_2)(\mu_2-\mu_1)-\frac{1}{4}\mu_1=-\frac{\mu_2^2}{4\mu_1}.$$

Summarizing all of our conclusions here about e, we have that

$$e_{1,12}=e_{1,1j}=e_{1,22}=e_{1,2j}=e_{2,11}=e_{2,12}=e_{2,1j}=e_{2,2j}=0,$$

whereas $e_{1,11}$ is not, in general, 0.

Also,

$$e_{2,22} = \frac{\mu_2}{\sqrt{\mu_1(\mu_1 + \mu_2)}} e_{1,11}$$

and

$$|e_{1,11}|^2 = \frac{1}{4}(\mu_1 + \mu_2)(\mu_2 - \mu_1).$$

Chapter 11

Calculating $\deg(F)$

We note that it was shown in [JX04] that $\deg F \leq 4$ in our current case. In particular, when a map has degenerate geometric rank ($\kappa_0 \leq n-2$) and $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$, then $\deg(F) \leq \kappa_0 + 2$. This is relevant because we wish to show, first, that $\deg(F) \leq 3$, and then we will use that result to show $\deg(F) \leq 2$. Once we establish that, our Theorem 2.44 (Lebl's theorem) gives us the form of our map, which is the Generalized Whitney Map.

Demonstrating the degree of F seems to be a difficult task, but if we recall [HJX06, Theorem 5.4], it becomes easier. We need only show that the degree is less than or equal to some k along a certain Segre variety, and then we get that it is less than or equal to k everywhere.

Proving that $\deg(F) \leq 3$

In order to show that $\deg(F) \leq 3$ everywhere, we show that $\deg(F) \leq 3$ on a certain Segre variety $z = w = \eta = 0$.

We start by applying our differential operators $\mathcal{L}_1\mathcal{L}_2$ to both sides of the geometric equation

$$Im(w) = \sum_j |f_j|^2 + \sum_k |\phi_k|^2 .$$

Then we will see that

$$\begin{pmatrix} \overline{f(\xi,0)}^t \\ \overline{\phi(\xi,0)}^t \end{pmatrix}$$

has degree less than or equal to three. That is, we want to calculate it explicitly and show that each component of it has a numerator and denominator that is a polynomial whose degree is at most three.

We recall that, throughout, $o_{wt}(s)$ is a class of function such that

$$h \in o_{wt}(s) \Leftrightarrow \lim_{t\to 0+} \frac{f(tz, t^2 w, t\overline{z}, t^2\overline{w})}{t^s} = 0.$$

At times, we will use the notation $b_{ij}^{(d)}$ to refer to a polynomial of degree d. When we write $b^{(n,m)}(z)w^m$, that is the (n,m) part of a polynomial, where $b^{(n)}(z)$ is a polynomial in z of degree n.

From Lemma 3.2 from [JX01], we have $b_1^{(1)}(z) = b_z z_1$ and $b_j^{(1)} = b_j z_1 + \frac{\overline{\mu}}{2} z_j$.

Let

$$A = \left.\begin{pmatrix} \mathcal{L}_1\mathcal{L}_1 f_1 & \mathcal{L}_1\mathcal{L}_1 f_2 & \dots & \mathcal{L}_1\mathcal{L}_1 f_{n-1} \\ \mathcal{L}_1\mathcal{L}_2 f_1 & \mathcal{L}_1\mathcal{L}_2 f_2 & \dots & \mathcal{L}_1\mathcal{L}_2 f_{n-1} \\ \mathcal{L}_1\mathcal{L}_j f_1 & \mathcal{L}_1\mathcal{L}_j f_2 & \dots & \mathcal{L}_1\mathcal{L}_j f_{n-1} \\ \mathcal{L}_2\mathcal{L}_2 f_1 & \mathcal{L}_2\mathcal{L}_2 f_2 & \dots & \mathcal{L}_2\mathcal{L}_2 f_{n=1} \\ \mathcal{L}_2\mathcal{L}_j f_1 & \mathcal{L}_2\mathcal{L}_j f_1 & \dots & \mathcal{L}_2\mathcal{L}_j f_{n-1} \end{pmatrix}\right|_{(0,0,\xi,0)} , 3 \le j \le n-1.$$

We would like to write A explicitly.

We recall that $\mathcal{L}_j = \frac{\partial}{\partial z_j} + 2i\overline{\xi_j}\frac{\partial}{\partial w}$.

We also have:

$$f_1 = z_1\left(1+\tfrac{i\mu_1}{2}w + b_{11}^{(1)}(z)w + O_{wt}(4)\right) + z_2\left(b_{12}^{(1)}(z)w + O_{wt}(4)\right)$$
$$= z_1 + \tfrac{i\mu_1}{2}z_1w + b_{11}^{(1)}(z)z_1w + b_{12}^{(1)}(z)z_2w + z_1O_{wt}(4) + z_2O_{wt}(4)$$
$$= z_1 + \tfrac{i\mu_1}{2}z_1w + b_{11,1}z_1^2w + b_{11,2}z_1z_2w + \cdots +$$
$$b_{11,n-1}z_1z_{n-1}w + b_{12,1}z_1z_2w + b_{12,2}z_2^2w + \cdots + b_{12,n-1}z_2z_{n-1}$$
$$+z_1O_{wt}(4) + z_2O_{wt}(4).$$

And,

$$\mathcal{L}_1 = \frac{\partial}{\partial z_1} + 2i\overline{\xi_1}\frac{\partial}{\partial w}.$$

So, we look at $\mathcal{L}_1 f_1$:

$$\mathcal{L}_1f_1 = 1 + \tfrac{i\mu_1}{2}w + 2b_{11,1}z_1w + b_{11,2}z_2w + \cdots + b_{11,n-1}z_{n-1}w + b_{12,1}z_2w + 2i\overline{\xi_1}\tfrac{i\mu_1}{2}z_1$$
$$+2i\overline{\xi}_1b_{11,1}z_1^2 + 2i\overline{\xi}_1b_{11,2}z_1z_2 + \ldots$$
$$+2i\overline{\xi}_1b_{11,n-1}z_1z_{n-1} + 2i\overline{\xi}_1b_{12,1}z_1z_2 + 2i\overline{\xi}_1b_{12,2}z_2^2 + \cdots + 2i\overline{\xi}_1b_{12,n-1}z_2z_{n-1}$$
$$+\mathcal{L}_1z_1O_{wt}(4) + \mathcal{L}_1z_2O_{wt}(4).$$

This prepares us to apply $\mathcal{L}_1$ again, yielding:

$$0+0+2b_{11,1}w + 0 - \overline{\xi}_1\mu_1 + 4i\overline{\xi}_1b_{11,1}z_1 + 2i\overline{\xi}_1b_{11,1}z_2 + 2i\overline{\xi}_1b_{12,1}z_2 + 0 + 0$$
$$-\overline{\xi}_1\mu_1 + 4i\overline{\xi}_1b_{11,1}z_1 + 2i\overline{\xi}_1b_{11,2}z_2 + \cdots + \qquad .$$
$$2i\overline{\xi_1}b_{11,n-1}z_{n-1} - 4\overline{\xi_1}^2(0) + \mathcal{L}_1\mathcal{L}_1\left(z_1O_{wt}(4) + z_2O_{wt}(4)\right).$$

If we evaluate this at $z = w = \eta = 0$, we get $-2\overline{\xi}_1\mu_1$.

Thus, $A_{11} = -2\overline{\xi}_1\mu_1$.

Next, we compute $\mathcal{L}_1f_2$.

We note that

$$f_2 = b_{21,1}z_1^2w + b_{21,2}z_1z_2w + \cdots + b_{21,n-1}z_1z_{n-1}w$$
$$+z_2 + \tfrac{i\mu_2}{2}z_2w + b_{22,1}z_1z_2w + b_{22,2}z_2w + \cdots +$$
$$b_{22,n-1}z_2z_{n-1}w + z_1O_{wt}(4) + z_2O_{wt}(4).$$

Applying $\mathcal{L}_1$, we get:

$$\begin{aligned}
&2b_{21,1}z_1w + b_{21,2}z_2w + \cdots + b_{21,n-1}z_{n-1}w + 0 + 0 + b_{22,1}z_2w + 0+\\
&2i\bar{\xi}_1\left(b_{21,1}z_1^2 + b_{21,2}z_1z_2 + \cdots + b_{21,n-1}z_1z_{n-1} + 0+\right.\\
&\left.\tfrac{i\mu_2}{2}z_2 + b_{22,1}z_1z_2 + b_{22,2}z_2^2 + \cdots + b_{22,n-1}z_2z_{n-1}\right) +\\
&\mathcal{L}_1\left(z_1O_{wt}(4) + z_2O_{wt}(4)\right).
\end{aligned}$$

Applying $\mathcal{L}_1$ again, we get

$$\begin{aligned}
&2b_{21,1}w + 0 + 0 + 0 + 0 + 0+\\
&2i\bar{\xi}_1\left(2b_{21,1}z_1 + b_{21,2}z_2 + 0 + 0 + b_{22,1}z_2 + 0\right)\\
&+2i\bar{\xi}_1\left(2b_{21,1}z_1 + b_{21,2}z_2 + 0 + 0 + b_{22,1}z_2 + 0\right) +\\
&-4\bar{\xi}_1^2\left(0 + 0 + 0 + 0 + 0 + 0\right)\\
&+\mathcal{L}_1\mathcal{L}_1(z_1O_{wt}(4) + z_2O_{wt}(4)).
\end{aligned}$$

When evaluated at $z = w = \eta = 0$, this is 0, so $A_{12} = 0$.

For $3 \le j \le n-1$, $f_j = z_j$, and it is clear that $\mathcal{L}_1\mathcal{L}_1 z_j = 0$, so the entire rest of the first row of A is also 0, and thus we have calculated one row of A.

We now look again at f_1 and apply $\mathcal{L}_2$:

$$\begin{aligned}
&0 + 0 + b_{11,2}z_1w + \cdots + 0\\
&+b_{12,1}z_1w + 2b_{12,2}z_2w + \cdots + b_{12,n-1}z_{n-1}w\\
&+2i\bar{\xi}_2\left(\tfrac{i\mu_1}{2}z_1 + b_{11,1}z_1^2 + \ldots b_{11,n-1}z_1z_{n-1} + b_{12,1}z_1z_2 + \cdots + b_{12,n-1}z_2z_{n-1}\right)\\
&+\mathcal{L}_2(z_1O_{wt}(4) + z_2O_{wt}(4)).
\end{aligned}$$

Next, we apply $\mathcal{L}_1$ to this:

$$\begin{aligned}
&b_{12,2}w + b_{12,1}w + 0 + \cdots + 0\\
&+2i\bar{\xi}_2\left(\tfrac{i\mu_1}{2} + 2b_{11,1}z_1 + \cdots + b_{11,n-1}z_{n-1} + b_{12,1}z_2 + 0\right)\\
&+2i\bar{\xi}_1\left(b_{11,2}z_1 + b_{12,1}z_1 + 2b_{12,2}z_2 + \cdots + b_{12,n-1}z_1\right)\\
&-4\bar{\xi}_1\bar{\xi}_2\left(0\right) + \mathcal{L}_1\mathcal{L}_2(z_2O_{wt}(4) + z_2O_{wt}(4)).
\end{aligned}$$

This, when evaluated at $z = w = \eta = 0$, is $-\bar{\xi}_2\mu_1$, and thus $A_{21} = -\bar{\xi}_2\mu_1$.

We now check $\mathcal{L}_2 f_2$:

$$b_{21,2}z_1w + 1 + \tfrac{i\mu_2}{2}w + b_{22,1}z_1w + 2b_{22,2}z_2w + \cdots + b_{22,n-1}z_{n-1}w$$
$$+2i\overline{\xi}_2\,(b_{21,1}z_1^2 + \cdots + b_{21,n-1}z_1z_{n-1} + 0+$$
$$\tfrac{i\mu_2}{2}z_2 + b_{22,1}z_1z_2 + b_{22,2}z_2^2 + \cdots + b_{22,n-1}z_2z_{n-1}\big)$$
$$+\mathcal{L}_2(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

We apply $\mathcal{L}_1$ to this:

$$b_{21,2}w + 0 + 0 + b_{22,1}w + 0 + \cdots + 0$$
$$+2i\overline{\xi}_1\left(2b_{21,1}z_1 + \tfrac{i\mu_2}{2} + \cdots + b_{21,n-1}z_{n-1} + 0 + 0 + b_{22,1}z_2 + \cdots + 0\right)$$
$$+2i\overline{\xi}_2\,(b_{21,2}z_1 + 0 + b_{22,1}z_1 + 2b_{22,2}z_2 + \cdots + b_{22,n-1}z_{n-1})$$
$$-4\overline{\xi}_1\overline{\xi}_2\,(0) + \mathcal{L}_1\mathcal{L}_2(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

When we evaluate this at $z = w = \eta = 0$, we get $-\overline{\xi_2}\mu_2$, so $A_{22} = -\overline{\xi_1}\mu_2$.

Since $f_j = z_j$ for $3 \leq j \leq n-1$, and $\mathcal{L}_1\mathcal{L}_2 z_j = 0$, we get that the rest of the second row is also 0, so we have now calculated the entire second row of A.

We next evaluate $\mathcal{L}_j f_1$:

$$b_{11,j}z_1w + b_{12,j}z_2w$$
$$+2i\overline{\xi}_j\left(\tfrac{i\mu_1}{2}z_1 + b_{11,1}z_1^2 + \cdots + b_{11,n-1}z_1z_{n-1} + b_{12,1}z_1z_2 + \cdots + b_{12,n-1}z_2z_{n-1}\right)$$
$$+\mathcal{L}_j(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

We then apply $\mathcal{L}_1$ to this result:

$$b_{11,j}w + 0$$
$$+2i\overline{\xi}_j\left(\tfrac{i\mu_1}{2} + 2b_{11,1}z_1 + \cdots + b_{11,n-1}z_{n-1} + b_{12,1}z_2\right)$$
$$+2i\overline{\xi}_1\,(b_{11,j}z_1 + b_{12,j}z_2)$$
$$-4i\overline{\xi}_1\overline{\xi}_j\,(0)$$
$$+\mathcal{L}_1\mathcal{L}_j(z_1O_{wt}(4) + z_2O_{wt}(4)),$$

and when $z = w = \eta = 0$, this is $-\overline{\xi}_j\mu_1$. Thus, $A_{j1} = -\overline{\xi}_j$ for $3 \leq j \leq n-1$.

We now check $\mathcal{L}_j f_2$:

$$b_{21,j}z_1w + b_{22,j}z_2w$$
$$+2i\overline{\xi}_j\left(b_{21,1}z_1^2 + \cdots + b_{21,n-1}z_1z_{n-1} + \tfrac{i\mu_2}{2}z_2 + b_{22,1}z_1z_2 + \cdots + b_{22,n-1}z_2z_{n-1}\right)$$
$$+\mathcal{L}_j(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

We then apply $\mathcal{L}_1$ to this, and we get:

$$b_{21,j}w + 0$$
$$+2i\overline{\xi}_j\left(2b_{21,1}z_1 + b_{21,2}z_2 + \cdots + b_{21,n-1}z_{n-1} + 0 + b_{22,1}z_2 + 0\right)$$
$$+2i\overline{\xi}_1\left(b_{21,j}z_1 + b_{22,j}z_2\right)$$
$$-4\overline{\xi}_1\overline{\xi}_j\left(0\right) + \mathcal{L}_1\mathcal{L}_j(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

Evaluated at $z = w = \eta = 0$, this is 0, so $A_{j2} = 0$ for $j \leq 3 \leq n-1$.

As $f_j = z_j$ for $3 \leq j \leq n-1$, we get that $\mathcal{L}_1\mathcal{L}_j f_j = 0$ for $3 \leq j \leq n-1$, and so $A_{j\ell} = 0$ for $3 \leq j, \ell \leq n-1$.

Next, we look again at $\mathcal{L}_2 f_1$:

$$0 + 0 + b_{12,2}z_1w + \cdots + 0$$
$$+b_{12,1}z_1w + 2b_{12,2}z_2w + \cdots + b_{12,n-1}z_{n-1}w$$
$$+2i\overline{\xi}_2\left(\tfrac{i\mu_1}{2}z_1 + b_{11,1}z_1^2 + \ldots b_{11,n-1}z_1z_{n-1} + b_{12,1}z_1z_2 + \cdots + b_{12,n-1}z_2z_{n-1}\right)$$
$$+\mathcal{L}_2(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

We apply $\mathcal{L}_2$ to this again:

$$2b_{12,2}w + 2i\overline{\xi_2}\left(b_{11,2}z_1 + b_{12,1}z_1 + \cdots + b_{12,n-1}z_{n-1}\right)$$
$$2i\overline{\xi_2}\left(b_{11,2}z_1 + b_{12,1}z_1 + 2b_{12,2}z_2 + \cdots + b_{12,n-1}z_{n-1}\right)$$
$$-4\overline{\xi}\left(0\right)$$
$$+\mathcal{L}_2\mathcal{L}_2\left(z_1O_{wt}(4) + z_2O_{wt}(4)\right).$$

When $z = w = \eta = 0$, this is 0, so $A_{n1} = 0$.

Now, we look at $\mathcal{L}_2 f_2$:

$$b_{21,2}z_1w + 1 + \tfrac{i\mu_2}{2}w + b_{22,1}z_1w + 2b_{22,2}z_2w + \cdots + b_{22,n-1}z_{n-1}w$$
$$+2i\overline{\xi}_2\left(b_{21,1}z_1^2 + \cdots + b_{21,n-1}z_1z_{n-1} + 0 + \tfrac{i\mu_2}{2}z_2 + b_{22,1}z_2^2 + \cdots + b_{22,n-1}z_2n_{n-1}\right)$$
$$+\mathcal{L}_2(z_1O_{wt}(4) + z_2O_{wt}(4)).$$

We apply $\mathcal{L}_2$ to this:

$$
\begin{aligned}
&0+0+0+2b_{22,2}w+0+\\
&+2i\overline{\xi_2}\left(b_{21,2}z_1+\tfrac{i\mu_2}{2}+b_{22,1}z_1+2b_{22,2}z_2+\cdots+b_{22,n-1}z_{n-1}\right)\\
&+2i\overline{\xi_2}\left(b_{21,2}z_1+\tfrac{i\mu_2}{2}+b_{22,1}z_1+2b_{22,2}z_2+\cdots+b_{22,n-1}z_{n-1}\right)\\
&-4\overline{\xi_2}^2(0)+\mathcal{L}_2\mathcal{L}_2(z_1O_{wt}(4)+z_2O_{wt}(4)).
\end{aligned}
$$

When $z=w=\eta=0$, this is $-2\overline{\xi_2}\mu_2$, so $A_{n2}=-2\overline{\xi}_2\mu_2$.

For $3\le j\le n-1$, we have $f_j=z_j$, so $\mathcal{L}_2\mathcal{L}_2 f_j=0$.

Next, we look at $\mathcal{L}_j f_1$:

$$
b_{11,j}z_1w+b_{12,j}z_2w
$$

$$
+2i\overline{\xi}_j\left(\frac{i\mu_1}{2}z_1+b_{11,1}z_1^2+\ldots b_{11,n-1}z_1z_{n-1}+b_{12,1}z_1z_2+\cdots+b_{12,n-1}z_2z_{n-1}\right).
$$

We apply $\mathcal{L}_2$ to this:

$$
\begin{aligned}
&0+b_{12,j}w\\
&+2i\overline{\xi}_j\left(0+b_{11,2}z_1+b_{12,1}z_1+2b_{12,2}z_2+\cdots+b_{12,n-1}z_{n-1}\right)\\
&+2i\overline{\xi}_2\left(b_{11,j}z_1+b_{12,j}z_2\right)\\
&-4i\overline{\xi}_2\overline{\xi}_j(0)+\mathcal{L}_2\mathcal{L}_j(z_1O_{wt}(4)+z_2O_{wt}(4)).
\end{aligned}
$$

When $z=w=\eta=0$, this is 0, so $A_{(n+j-1)1}=0$.

Next, we look at $\mathcal{L}_j f_2$:

$$
\begin{aligned}
&b_{21,j}z_1w+b_{22,j}z_2w\\
&+2i\overline{\xi}_j\left(b_{21,1}z_1^2+b_{21,2}z_1z_2+\cdots+b_{21,n-1}z_1z_{n-1}+\tfrac{i\mu_2}{2}z_2+\right.\\
&\left.b_{22,1}z_1z_2+b_{22,2}z_2^2+\cdots+b_{22,n-1}z_2z_{n-1}\right).
\end{aligned}
$$

We apply $\mathcal{L}_2$ to this:

$$
\begin{aligned}
&b_{22,j}w+2i\overline{\xi}_j\left(b_{21,2}z_1+\tfrac{i\mu_2}{2}+b_{21,1}z_1+2b_{21,2}z_2+\cdots+b_{21,n-1}z_n-1\right)\\
&+2i\overline{\xi}_2\left(b_{21,j}z_1+b_{22,j}z_2-4i\overline{\xi}_2\overline{\xi}_j(0)+\mathcal{L}_2\mathcal{L}_j(z_1O_{wt}(4)+z_2O_{wt}(4))\right).
\end{aligned}
$$

When $z = w = \eta = 0$, this is $-\overline{\xi}_j\mu_2$, so $A_{(n+j-1)2} = -\overline{\xi}_j\mu_2$.

Thus,

$$A = \begin{pmatrix} -2\overline{\xi}_1\mu_1 & 0 & 0 & \dots & 0 \\ -\overline{\xi}_2\mu_1 & -\overline{\xi}_1\mu_2 & 0 & \dots & 0 \\ -\overline{\xi}_j\mu_1 & 0 & 0 & \dots & 0 \\ 0 & -2\overline{\xi}_2\mu_2 & 0 & \dots & 0 \\ 0 & -\overline{\xi}_j\mu_2 & 0 & \dots & 0 \end{pmatrix}.$$

We now turn our attention to the matrix B:

$$B = \left.\begin{pmatrix} \mathcal{L}_1\mathcal{L}_1\phi_{11} & \mathcal{L}_1\mathcal{L}_1\phi_{12} & \mathcal{L}_1\mathcal{L}_1\phi_{1j} & \mathcal{L}_1\mathcal{L}_1\phi_{22} & \mathcal{L}_1\mathcal{L}_1\phi_{2j} \\ \mathcal{L}_1\mathcal{L}_2\phi_{11} & \mathcal{L}_1\mathcal{L}_2\phi_{12} & \mathcal{L}_1\mathcal{L}_2\phi_{1j} & \mathcal{L}_1\mathcal{L}_2\phi_{22} & \mathcal{L}_1\mathcal{L}_2\phi_{2j} \\ \mathcal{L}_1\mathcal{L}_k\phi_{11} & \mathcal{L}_1\mathcal{L}_k\phi_{12} & \mathcal{L}_1\mathcal{L}_k\phi_{1j} & \mathcal{L}_1\mathcal{L}_k\phi_{22} & \mathcal{L}_1\mathcal{L}_k\phi_{2j} \\ \mathcal{L}_2\mathcal{L}_2\phi_{11} & \mathcal{L}_2\mathcal{L}_2\phi_{12} & \mathcal{L}_2\mathcal{L}_2\phi_{1j} & \mathcal{L}_2\mathcal{L}_2\phi_{22} & \mathcal{L}_2\mathcal{L}_2\phi_{22} \\ \mathcal{L}_2\mathcal{L}_k\phi_{11} & \mathcal{L}_2\mathcal{L}_k\phi_{12} & \mathcal{L}_2\mathcal{L}_k\phi_{1j} & \mathcal{L}_2\mathcal{L}_k\phi_{22} & \mathcal{L}_2\mathcal{L}_k\phi_{2j} \end{pmatrix}\right|_{(0,0,\xi,0)}$$

where

$$\begin{aligned} \phi_{11} &= \sqrt{\mu_1}z_1^2 + z_1(O_{wt}(2)) + z_2(O_{wt}(2)) \\ \phi_{12} &= \sqrt{\mu_1+\mu_2}z_1z_2 + z_1(O_{wt}(2)) + z_2(O_{wt}(2)) \\ \phi_{1j} &= \sqrt{\mu_1}z_1z_j + z_1(O_{wt}(2)) + z_2(O_{wt}(2)), 3 \leq j \leq n-1 \\ \phi_{22} &= \sqrt{\mu_2}z_2^2 + z_2(O_{wt}(2)) + z_2(O_{wt}(2)) \\ \phi_{2j} &= \sqrt{\mu_2}z_2z_j + z_1(O_{wt}(2)) + z_2(O_{wt}(2)), 3 \leq j \leq n-1. \end{aligned}$$

Most of these calculations are short. They follow from the Taylor series for $\phi_{k\ell}$, with the $e_{j,k\ell}$ the coefficient of the z_jw term of $\phi_{k\ell}$. Thus,

$$\frac{B}{2i} = \begin{pmatrix} \frac{\sqrt{\mu_1}}{i} + 2\overline{\xi}_1e_{1,11} & \overline{\xi}_1e_{1,12} & \overline{\xi}_1e_{1,1j} & \overline{\xi}_1e_{1,22} & \overline{\xi}_1e_{1,2j} \\ \overline{\xi}_2e_{1,11} + \overline{\xi}_1e_{2,11} & \beta_{12,12} & \overline{\xi}_2e_{1,1j} + \overline{\xi}_1e_{2,1j} & \overline{\xi}_2e_{1,22} + \overline{\xi}_1e_{2,22} & \beta_{12,2j} \\ \overline{\xi}_ke_{1,11} & \overline{\xi}_ke_{1,12} & \frac{\sqrt{\mu_1}}{2i} + \overline{\xi}_ke_{1,1j} & \overline{\xi}_ke_{1,22} & \overline{\xi}_ke_{1,2j} \\ \overline{\xi}_2e_{2,11} & \overline{\xi}_2e_{2,12} & \overline{\xi}_2e_{2,1j} & \frac{\sqrt{\mu_2}}{i} + 2\overline{\xi}_2e_{2,22} & \overline{\xi}_2e_{2,2j} \\ \overline{\xi}_ke_{2,11} & \overline{\xi}_ke_{2,12} & \overline{\xi}_ke_{2,1j} & \overline{\xi}_ke_{2,22} & \beta_{2j,2j} \end{pmatrix}$$

where $\beta_{22} = \frac{\sqrt{\mu_1+\mu_2}}{2i} + \overline{\xi}_2 e_{1,12} + \overline{\xi}_1 e_{2,12}$, $\beta_{12,2j} = \overline{\xi}_2 e_{1,2j} + \overline{\xi}_1 e_{2,2j}$, and $\beta_{2j,2j} = \frac{\sqrt{\mu_2}}{2i} + \overline{\xi}_k e_{2,2j}$.

From our calculations in the previous chapter, we have that

$$e_1 = \begin{pmatrix} e_{1,11} & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$
$$e_2 = \begin{pmatrix} 0 & \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

so

$$B = 2i \begin{pmatrix} \frac{\sqrt{\mu_1}}{i} + \overline{\xi}_1 e_{1,11} & 0 & 0 & 0 & 0 \\ \overline{\xi}_2 e_{1,11} & \frac{\sqrt{\mu_1+\mu_2}}{2i} + \beta_{22} & 0 & 0 & 0 \\ \overline{\xi}_k e_{1,11} & 0 & \frac{\sqrt{\mu_1}}{2i} & 0 & 0 \\ 0 & \overline{\xi}_2 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & \frac{\sqrt{\mu_2}}{i} & 0 \\ 0 & \overline{\xi}_k \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & 0 & \frac{\sqrt{\mu_2}}{2i} \end{pmatrix},$$

with $\beta_{22} = \overline{\xi}_1 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11}$.

We define a matrix $\tilde{B}$ where $B = (D + \tilde{B})$, with D a diagonal matrix:

$$\tilde{B} = 2i \begin{pmatrix} 2\overline{\xi}_1 e_{1,11} & 0 & 0 & 0 & 0 \\ \overline{\xi}_2 e_{1,11} & \overline{\xi}_1 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & 0 & 0 \\ \overline{\xi}_j e_{1,11} & 0 & 0 & 0 & 0 \\ 0 & 2\overline{\xi}_2 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & 0 & 0 \\ 0 & \overline{\xi}_j \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & 0 & 0 & 0 \end{pmatrix}.$$

Then, if we let

$$f = \begin{pmatrix} 4i\overline{\xi}_1 & 0 \\ 2i\overline{\xi}_2 & 2i\overline{\xi}_1 \\ 2i\overline{\xi}_j & 0 \\ 0 & 4i\overline{\xi}_2 \\ 0 & 2i\overline{\xi}_j \end{pmatrix}$$

we get $\tilde{B} = fe$.

Now, $B = \tilde{B} + D$ and $B^{-1} = (I + B^*)^{-1} D^{-1}$, where D^{-1} is easy to compute. Choose $k_1, \ldots, k_5$ so that

$$D = \begin{pmatrix} 2\sqrt{\mu_1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\mu_1 + \mu_2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\mu_1} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{\mu_2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\mu_2} \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} \frac{1}{2\sqrt{\mu_1}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\mu_1+\mu_2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mu_1}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{\mu_2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu_2}} \end{pmatrix}.$$

Next, we say that

$$b = 2i \begin{pmatrix} \frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 & 0 \\ \frac{1}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_2 & \frac{1}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_1 \\ \frac{1}{\sqrt{\mu_1}}\overline{\xi}_j & 0 \\ 0 & \frac{1}{\sqrt{\mu_1}}\overline{\xi}_2 \\ 0 & \frac{1}{\sqrt{\mu_1}}\overline{\xi}_j \end{pmatrix}.$$

Now, $B = \tilde{B} + D$, and $B^{-1} = (I + B^*)^{-1} D^{-1}$, where $B^* = b \cdot e$.

We compute B^{-1}.

First, we note that $(I + B^*)^{-1} = (-B^*)^0 + (-B^*)^1 + (-B^*)^2 + \ldots$.

This is the same as:

$$I - b(e \cdot b)^0 e + b(e \cdot b)^1 e - b(e \cdot b)^2 e + \ldots$$
$$I - b(\textstyle\sum_{j=0}^{\infty} (-1)^j (e \cdot b)^j) e$$
$$I - b(I + e \cdot b)^{-1} e.$$

Here, $I + e \cdot b$ is a 2×2 matrix, so we can compute its inverse. First, we write

$$e \cdot b = \begin{pmatrix} 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} & 0 \\ \frac{2i}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_2 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} & \frac{2i}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_1 \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} \end{pmatrix}.$$

$$e \cdot b = \begin{pmatrix} 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} & 0 \\ 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2 e_{1,11} & 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} \end{pmatrix}.$$

So, $I + e \cdot b$ is

$$I + e \cdot b = \begin{pmatrix} 1 + 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} & 0 \\ 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2 e_{1,11} & 1 + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} \end{pmatrix}.$$

We need the determinant of $I + e \cdot b$:

$$\begin{aligned} \mathcal{D} &= \left(1 + 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11}\right)\left(1 + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11}\right) \\ &= 1 + 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2 (e_{1,11})^2. \end{aligned}$$

Now,

$$(I + e \cdot b)^{-1} = \frac{1}{\mathcal{D}} \begin{pmatrix} 1 + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} & 0 \\ -2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2 e_{1,11} & 1 + 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} \end{pmatrix}.$$

Now, if we multiply this on the right by e, we get a matrix we will call M.

We observe that $B^{-1} = I - \frac{1}{\mathcal{D}} b \cdot M$, where

$$\begin{pmatrix} e_{1,11} + \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 (e_{1,11})^2 & 0 & 0 & 0 & 0 \\ -\frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2 (e_{1,11})^2 & \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11} + \frac{2i\mu_2}{\mu_1\sqrt{\mu_1+\mu_2}}\overline{\xi}_1 (e_{1,11})^2 & 0 & 0 & 0 \end{pmatrix}.$$

We now multiply this on the left by b to get the matrix N. This matrix is very large, so we only show the first two columns N'. All other elements of N are 0.

$$N' = \begin{pmatrix} \frac{2i}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2 (e_{1,11})^2 & 0 \\ \frac{2i}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_2 e_{1,11} & \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - \frac{4\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2 (e_{1,11})^2 \\ \frac{2i}{\sqrt{\mu_1}}\overline{\xi}_j e_{1,11} - \frac{4\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_j (e_{1,11})^2 & 0 \\ \frac{4\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2^2 (e_{1,11})^2 & \frac{2i\sqrt{\mu_2}}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_2 e_{1,11} - \frac{4\sqrt{\mu_2}}{\mu_1\sqrt{\mu_1+\mu_2}}\overline{\xi}_1\overline{\xi}_2 (e_{1,11})^2 \\ \frac{4\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_2\overline{\xi}_j (e_{1,11})^2 & \frac{2i\sqrt{\mu_2}}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_j e_{1,11} - \frac{4\sqrt{\mu_2}}{\mu_1\sqrt{\mu_1+\mu_1}}\overline{\xi}_1\overline{\xi}_j (e_{1,11})^2 \end{pmatrix}$$

Next, we want to write $P = D^{-1}A\overline{\xi}^t$.

$$P = D^{-1}A\overline{\xi}^t = \begin{pmatrix} -\sqrt{\mu_1}\overline{\xi}_1^2 \\ -\sqrt{\mu_1+\mu_2}\overline{\xi}_1\overline{\xi}_2 \\ -\sqrt{\mu_1}\overline{\xi}_1\overline{\xi}_j \\ -\sqrt{\mu_2}\overline{\xi}_2^2 \\ -\sqrt{\mu_2}\overline{\xi}_2\overline{\xi}_j \end{pmatrix}.$$

Now, for the finale, we look at $\overline{\phi(\xi,0)}^t$, one row at a time. It should be noted that the original version of this calculation was extremely long and complicated, and our result from Chapter 10 greatly simplifies things.

The first row is $-\frac{1}{\mathcal{D}}(-\sqrt{\mu_1}\overline{\xi}_1^2\mathcal{D} - N_{1,\cdot}\cdot P)$. We look at $\mathcal{D}\cdot(-\sqrt{\mu_1}\overline{\xi}_1^2)$ first.

$$\mathcal{D}\cdot(-\sqrt{\mu_1}\overline{\xi}_1^2) = -\sqrt{\mu_1}\overline{\xi}_1^2 - 2i\overline{\xi}_1^3 e_{1,11} - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^3 e_{1,11} + 4\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1^4(e_{1,11})^2.$$

Next, we look at $N_{1,\cdot}\cdot P$. The first term is:

$$N_{11}\cdot P_1 = -2i\overline{\xi}_1^3 e_{1,11} + 4\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}k_2\overline{\xi}_1^4(e_{1,11})^2.$$

The second term is $N_{12}\cdot P_2 = 0$.

Next, $N_{13}\cdot P_3 = 0$.

Next, $N_{14}\cdot P_4 = 0$.

Next, $N_{15}\cdot P_5 = 0$.

We get that

$$\mathcal{D}P_1 - N_{(1,\cdot)}\cdot P_1 = -\sqrt{\mu_1}\overline{\xi}_1^2 - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^3 e_{1,11}.$$

Our next task is to do the same thing for the second row.

First, we need $\mathcal{D}\cdot P_2$:

$$\begin{aligned}\mathcal{D}\cdot P_2 = \quad & -\sqrt{\mu_1+\mu_2}\overline{\xi}_1\overline{\xi}_2 - 2i\tfrac{\sqrt{\mu_1}}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_1^2\overline{\xi}_2 e_{1,11} - 2i\tfrac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_1^2\overline{\xi}_2 \\ & -2i\tfrac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_1^2\overline{\xi}_2 e_{1,11} + \tfrac{4\mu_2}{\mu_1\sqrt{\mu_1+\mu_2}}\overline{\xi}_1^3\overline{\xi}_2(e_{1,11})^2.\end{aligned}$$

Next, we want $N_{(2,\cdot)} \cdot P$.

We start with $N_{21} \cdot P_1$:

$$-2i\frac{\sqrt{\mu_1}}{\sqrt{\mu_1+\mu_2}}\overline{\xi}_1^2\overline{\xi}_2 e_{1,11}.$$

Now, $N_{22} \cdot P_2$:

$$-2i\frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_1^2\xi_2 e_{1,11} + \frac{4\mu_2}{\mu_1\sqrt{\mu_1+\mu_2}}\overline{\xi}_1^3\xi_2(e_{1,11})^2.$$

Now, $N_{23} \cdot P_3 = N_{24} \cdot P_4 = N_{25} \cdot P_5 = 0$.

We get that

$$\mathcal{D}P_2 - N_{(2,\cdot)} \cdot P = -\sqrt{\mu_1+\mu_2}\overline{\xi}_1\overline{\xi}_2 - \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_1^2\overline{\xi}_2 e_{1,11}.$$

We move on to the third row.

First, we look at $\mathcal{D} \cdot P_3$:

$$\mathcal{D} \cdot P_3 = -\sqrt{\mu_1}\overline{\xi}_1\overline{\xi}_j - 2i\overline{\xi}_1^2\overline{\xi}_j e_{1,11} - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^2\overline{\xi}_j e_{1,11} + \frac{4\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1^3\overline{\xi}_j(e_{1,11})^2.$$

Now, we look at $N_{(3,\cdot)} \cdot P$.

First, $N_{31} \cdot P_1$:

$$-2i\overline{\xi}_1^2\overline{\xi}_j e_{1,11} + \frac{4\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1^3\overline{\xi}_j(e_{1,11})^2.$$

Then, $N_{32} \cdot P_2 = N_{33} \cdot P_3 = N_{34} \cdot P_4 = N_{35} \cdot P_5 = 0$.

We get that

$$\mathcal{D}P_3 - N_{(3,\cdot)} \cdot P = -\sqrt{\mu_1}\overline{\xi}_1\overline{\xi}_j - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^2\overline{\xi}_j e_{1,11}.$$

Next, we look at $\mathcal{D} \cdot P_4$:

$$\mathcal{D} \cdot P_4 = -\sqrt{\mu_2}\overline{\xi}_2^2 - 2i\tfrac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\overline{\xi}_1\overline{\xi}_2^2 e_{1,11} - 2i\tfrac{\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2^2 e_{1,11} +$$
$$\tfrac{4\mu_2\sqrt{\mu_2}}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2\overline{\xi}_2^2(e_{1,11})^2.$$

We compute $N_{(4,\cdot)} \cdot P$.

First, $N_{41} \cdot P_1$:

$$-\frac{4\sqrt{\mu_2}}{\mu_1+\mu_2}\overline{\xi}_1^2\overline{\xi}_2^2(e_{1,11})^2.$$

Next, $N_{42} \cdot P_2$:

$$-\frac{2i\sqrt{\mu_2}}{\sqrt{\mu_1}}\overline{\xi}_1\overline{\xi}_2^2 e_{1,11} + \frac{4\sqrt{\mu_2}}{\mu_1+\mu_2}\overline{\xi}_1^2\overline{\xi}_2^2(e_{1,11})^2 + \frac{4\mu_2\sqrt{\mu_2}}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2\overline{\xi}_2^2(e_{1,11})^2.$$

Next, $N_{43} \cdot P_3 = N_{44} \cdot P_4 = N_{45} \cdot P_5 = 0$.

We get that

$$\mathcal{D}P_4 - N_{(4,\cdot)} \cdot P = -\sqrt{\mu_2}\overline{\xi}_2^2 - \frac{2i\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2^2 e_{1,11}.$$

Finally, we want to calculate the fifth row.

We list $\mathcal{D} \cdot P_5$:

$$\mathcal{D} \cdot P_5 = -\sqrt{\mu_2}\overline{\xi}_2\overline{\xi}_j - \tfrac{2i\sqrt{\mu_2}}{\sqrt{\mu_1}}\overline{\xi}_1\overline{\xi}_2\overline{\xi}_j e_{1,11} - \\ \tfrac{2i\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2\overline{\xi}_j e_{1,11} + \tfrac{4\mu_2\sqrt{\mu_2}}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2\overline{\xi}_2\overline{\xi}_j(e_{1,11})^2.$$

Next, $N_{51} \cdot P_1$:

$$-\frac{4\sqrt{\mu_2}}{\mu_1+\mu_2}\overline{\xi}_2^2\overline{\xi}_i\overline{\xi}_j(e_{1,11})^2.$$

Next, $N_{52} \cdot P_2$:

$$-2i\frac{\sqrt{\mu_2}}{\sqrt{\mu_2}}\overline{\xi}_1\overline{\xi}_2\overline{\xi}_j e_{1,11} + \frac{4\sqrt{\mu_2}}{\mu_1+\mu_2}\overline{\xi}_2^2\overline{\xi}_i\overline{\xi}_j(e_{1,11})^2 + \frac{4\mu_2\sqrt{\mu_2}}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_2^2\overline{\xi}_i\overline{\xi}_j(e_{1,11})^2.$$

Next, $N_{53} \cdot P_3 = N_{54} \cdot P_4 = N_{55} \cdot P_5 = 0$.

We get that

$$\mathcal{D}P_5 - N_{(5,\cdot)} \cdot P = -\sqrt{\mu_2}\overline{\xi}_2\overline{\xi}_j - \frac{2i\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2\overline{\xi}_j e_{1,11}.$$

Finally, we can write our expressions for ϕ.

$$-\overline{\phi_{11}(\xi,0)} = \frac{-\sqrt{\mu_1}\overline{\xi}_1^2 - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^3 e_{1,11}}{1 + 2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2(e_{1,11})^2}.$$

$$-\overline{\phi_{12}(\xi,0)} = \frac{-\sqrt{\mu_1+\mu_2}\overline{\xi}_1\overline{\xi}_2 - \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}}\overline{\xi}_1^2\overline{\xi}_2 e_{1,11}}{1+2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2(e_{1,11})^2}.$$

$$-\overline{\phi_{1j}(\xi,0)} = \frac{-\sqrt{\mu_1}\overline{\xi}_1\overline{\xi}_j - 2i\frac{\mu_2}{\mu_1+\mu_2}\overline{\xi}_1^2\overline{\xi}_j e_{1,11}}{1+2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2(e_{1,11})^2}.$$

$$-\overline{\phi_{22}(\xi,0)} = \frac{-\sqrt{\mu_2}\overline{\xi}_2^2 - \frac{2i\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2^2 e_{1,11}}{1+2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2(e_{1,11})^2}.$$

$$-\overline{\phi_{2j}(\xi,0)} = \frac{-\sqrt{\mu_2}\overline{\xi}_2\overline{\xi}_j - \frac{2i\mu_2\sqrt{\mu_2}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1\overline{\xi}_2\overline{\xi}_j e_{1,11}}{1+2i\frac{1}{\sqrt{\mu_1}}\overline{\xi}_1 e_{1,11} + 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{\xi}_1 e_{1,11} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}\overline{\xi}_1^2(e_{1,11})^2}.$$

Thus,

$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1}z_1^2\left(1 - \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}}z_1\right)}{1-2i\frac{1}{\sqrt{\mu_1}}z_1\overline{e_{1,11}} - 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\overline{e_{1,11}} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}z_1^2(\overline{e_{1,11}})^2}.$$

$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2}z_1z_2\left(1 - \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}}z_1\right)}{1-2i\frac{1}{\sqrt{\mu_1}}z_1\overline{e_{1,11}} - 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\overline{e_{1,11}} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}z_1^2(\overline{e_{1,11}})^2}.$$

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1}z_1z_j\left(1 - \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}}z_1\right)}{1-2i\frac{1}{\sqrt{\mu_1}}z_1\overline{e_{1,11}} - 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\overline{e_{1,11}} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}z_1^2(\overline{e_{1,11}})^2}.$$

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2}z_2^2\left(1 - \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}}z_1\right)}{1-2i\frac{1}{\sqrt{\mu_1}}z_1\overline{e_{1,11}} - 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\overline{e_{1,11}} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}z_1^2(\overline{e_{1,11}})^2}.$$

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j\left(1 - \frac{2i\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}}z_1\right)}{1-2i\frac{1}{\sqrt{\mu_1}}z_1\overline{e_{1,11}} - 2i\frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\overline{e_{1,11}} - 4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}z_1^2(\overline{e_{1,11}})^2}.$$

Because

$$\begin{pmatrix} f(z,0) \\ \phi(z,0) \end{pmatrix} = \begin{bmatrix} z^t \\ -\overline{B^{-1}A}z^t \end{bmatrix},$$

it is clear from this that $\deg(F) \le 3$. To show that $\deg(F) \le 2$, we need to look more closely at ϕ.

Proving that $\deg(F) \leq 2$

We look at one component at a time, starting with ϕ_{11}.

Now,

$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2 \left(1 - 2i \frac{z_1 \mu_2 \overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\right)}{1 - 2i\frac{1}{\sqrt{\mu_1}} z_1 \overline{e_{1,11}} - 2i \frac{\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)} z_1 \overline{e_{1,11}} - 4 \frac{\mu_2}{\mu_1(\mu_1+\mu_2)} z_1^2 (\overline{e_{1,11}})^2}.$$

Let's factor the denominator, using the quadratic formula $r_i = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Here,

$$\begin{aligned} a = &\quad -4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2 \\ b = &\quad -2i\frac{\mu_1+2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}} \\ c = &\quad 1. \end{aligned}$$

So,

$$\begin{aligned}
r = &\quad \frac{2i\frac{\mu_1+2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}\overline{e_{1,11}} \pm \sqrt{-4\frac{\mu_1^2+4\mu_1\mu_2+4\mu_2^2}{\mu_1(\mu_1^2+2\mu_1\mu_2+\mu_2^2)}(\overline{e_{1,11}})^2 + 16\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2}}{-8\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2}. \\
\overline{e_{1,11}} r &= \frac{2i\frac{\mu_1+2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)} \pm \sqrt{\frac{-4\mu_1^2-16\mu_1\mu_2-16\mu_2^2+16\mu_1\mu_2+16\mu_2^2}{\mu_1(\mu_1+\mu_2)^2}}}{-8\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}}. \\
\overline{e_{1,11}} r = &\quad \frac{2i\frac{\mu_1+2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)} \pm \sqrt{\frac{-4\mu_1^2}{\mu_1(\mu_1+\mu_2)^2}}}{-8\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}}. \\
\overline{e_{1,11}} r_1 = &\quad 2i\frac{\frac{2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}}{-8\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}} \\
\overline{e_{1,11}} r_1 = &\quad \frac{-i}{2}\sqrt{\mu_1} \\
\overline{e_{1,11}} r_2 = &\quad \frac{2i\frac{2\mu_1+2\mu_2}{\sqrt{\mu_1}(\mu_1+\mu_2)}}{-8\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}} \\
\overline{e_{1,11}} r_2 = &\quad \frac{-i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2} \\
r_1 = &\quad \frac{-i\sqrt{\mu_1}}{2\overline{e_{1,11}}} \\
r_2 = &\quad \frac{-i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}.
\end{aligned}$$

We rewrite $\phi_{11}(z,0)$:

$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2 \left(1-2i\frac{z_1\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2 \left(1-2i\frac{z_1\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2 \left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}+z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}.$$

We notice the common factor in the numerator and denominator, so we can reduce this expression to

$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{11}(z,0) = \frac{\sqrt{\mu_1} z_1^2}{-\frac{2e_{1,11}}{i\sqrt{\mu_1}}z_1+1}.$$

Thus, $\phi_{11}(z,0)$ has degree 2.

Next, we look at $\phi_{12}(z,0)$.

$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2} z_1 z_2 \left(1-\frac{2i\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2} z_1 z_2 \left(1-\frac{2i\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2} z_1 z_2 \left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}+z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2} z_1 z_2}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$
$$\phi_{12}(z,0) = \frac{\sqrt{\mu_1+\mu_2} z_1 z_2}{-\frac{2e_{1,11}}{i\sqrt{\mu_1}}z_1+1}.$$

Thus, the ϕ_{12} component has degree 2.

Next, we look at $\phi_{1j}(z,0)$.

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1} z_1 z_j \left(1 - 2i \frac{\mu_2 \overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)} z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)}$$

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1} z_1 z_j \left(1 - 2i \frac{\mu_2 \overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)} z_1\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}$$

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1} z_1 z_j \left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}} + z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}$$

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1} z_1 z_j}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e}_{1,11}}\right)}$$

$$\phi_{1j}(z,0) = \frac{\sqrt{\mu_1} z_1 z_j}{-\frac{2e_{1,11}}{i\sqrt{\mu_1}} z_1 + 1}.$$

Thus, $\phi_{1j}(z,0)$ has degree 2.

Next, we look at $\phi_{22}(z,0)$.

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2} z_2^2 \left(1 - \frac{2i\mu_2 \overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)} z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)}$$

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2} z_2^2 \left(1 - \frac{2i\mu_2 \overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)} z_1\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}$$

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2} z_2^2 \left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}} + z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(z_1 + \frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e_{1,11}}}\right)}$$

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2} z_2^2}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1 + \frac{i\sqrt{\mu_1}}{2\overline{e}_{1,11}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2 \overline{e}_{1,11}}\right)}$$

$$\phi_{22}(z,0) = \frac{\sqrt{\mu_2} z_2^2}{-\frac{2e_{1,11}}{i\sqrt{\mu_1}} z_1 + 1}.$$

Thus, the ϕ_{22} component has degree 2.

Finally, we look at $\phi_{2j}(z,0)$.

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j\left(1-\frac{2i\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)}$$

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j\left(1-\frac{2i\mu_2\overline{e_{1,11}}}{\sqrt{\mu_1}(\mu_1+\mu_2)}z_1\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}+z_1\right)}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(z_1+\frac{i\sqrt{\mu_1}(\mu_1+\mu_2)}{2\mu_2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j}{\left(-4\frac{\mu_2}{\mu_1(\mu_1+\mu_2)}(\overline{e_{1,11}})^2\right)\left(z_1+\frac{i\sqrt{\mu_1}}{2\overline{e_{1,11}}}\right)\left(\frac{i}{2}\frac{\sqrt{\mu_1}(\mu_1+\mu_2)}{\mu_2\overline{e_{1,11}}}\right)}$$

$$\phi_{2j}(z,0) = \frac{\sqrt{\mu_2}z_2z_j}{-\frac{2e_{1,11}}{i\sqrt{\mu_1}}z_1+1}.$$

Thus, the $\phi_{2,j}$ component is of degree 2.

This means that $\phi(z,0)$ has degree 2, and so $\phi(z,w)$ has degree 2, as well. Because $\deg(F) \leq 2$, we have that F is linear, the Whitney Map, the D'Angelo Map, or the Generalized Whitney Map, by [Le11, Theorem 1.5].

Part V

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