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NAIMARK'S PROBLEM *for* GRAPH C^* -ALGEBRAS

A Dissertation Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
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May 2017

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To my parents, who gave me their love, and their genes.

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Abstract

Naimark's problem asks if a C^* -algebra with exactly one irreducible representation up to unitary equivalence is isomorphic to $K(\mathcal{H})$, the algebra of compact operators on some Hilbert space \mathcal{H} . A C^* -algebra that satisfies the premise of this question but not its conclusion is a *counterexample to Naimark's problem*. It is known that neither separable C^* -algebras nor Type I C^* -algebras can be counterexamples to Naimark's problem. In 2004, Akemann and Weaver constructed an \aleph_1 -generated counterexample using Jensen's \diamond axiom (pronounced "diamond axiom"), which is known to be independent of ZFC. In fact, they showed that the existence of an \aleph_1 -generated counterexample is independent of ZFC. The general problem remains open. In this thesis we focus on Naimark's problem for a subclass of C^* -algebras called *graph C^* -algebras*. We show that *approximately finite-dimensional* (denoted AF) graph C^* -algebras cannot be counterexamples to Naimark's problem. We also show that, as a consequence, C^* -algebras of row-countable graphs cannot be counterexamples to Naimark's problem. Since C^* -algebras with unique irreducible representations up to unitary equivalence must be simple, and since simple graph C^* -algebras are either AF or purely infinite, a complete answer to Naimark's problem for all graph C^* -algebras now hinges on an examination of the class of purely infinite graph C^* -algebras.

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CHAPTER 1

Introduction and Historical Background

1.1 C^* -algebras

At the beginning of the 20th century, the mathematician Fredholm began an investigation of integral equations of the type

$$u(y) - \int_0^1 k(y, x)u(x)dx = f(y),$$

where the problem was to find the function u , given the functions k and f [5]. Such equations arose in connection with theoretical physics, in particular the Dirichlet problem, and today find applications in signal processing. Fredholm realized that since the integral can be thought of as a limit of finite sums, such an integral equation could itself be viewed as an infinite-dimensional counterpart to a finite-dimensional system of linear algebraic equations represented by the matrix equation

$$\vec{u} - K\vec{u} = \vec{f},$$

with finite dimensional vectors \vec{u} and \vec{f} being replaced by functions and the matrix K being replaced by the *integral operator*

$$T_k = \int_0^1 k(y, x) \cdot dx \quad ; \quad u \mapsto T_k(u) = \int_0^1 k(y, x)u(x)dx.$$

It was soon recognized that such integral operators are best viewed as continuous (bounded) linear operators on the *normed vector space* of *square integrable functions* on the compact interval $[0, 1]$ (i.e., functions u on $[0, 1]$ such that $\int_0^1 |u(x)|^2 dx < \infty$), with addition and scalar multiplication of functions given by point-wise addition and point-wise scalar multiplication respectively, and the norm of a square integrable function u given by

$$\|u\| = \left(\int_0^1 |u(x)|^2 dx \right)^{1/2}.$$

This norm is *complete* (in the sense of Cauchy), and is induced by an inner product

$$\langle u, v \rangle = \int_0^1 u(x)\overline{v(x)}dx.$$

The square integrable functions on $[0, 1]$ therefore constitute a *Hilbert space*, denoted $L^2([0, 1])$.

Definition 1.1.1 (Hilbert Space). *A Hilbert space \mathcal{H} is a real or complex vector space with an inner product, such that \mathcal{H} is complete (in the sense of Cauchy) with respect to the norm induced by the inner product.*

This shift in viewpoint marked the birth of the study of operator algebras. Fredholm set forth on a systematic exploration of the relationship between integral equations and matrix equations, and found, for instance, that the kernel and cokernel of an integral operator both have finite, and equal, dimension. His work sparked the interest of Hilbert, who made a detailed study of Fredholm's integral operators in

the special case where the function k is symmetric in the variables x and y . These symmetric integral operators are infinite-dimensional counterparts of real symmetric matrices, and Hilbert showed that they are diagonalizable, just as real symmetric matrices are [8]. In other words, given a symmetric integral operator T_k , there exist infinitely many functions u_1, u_2, u_3, \dots which satisfy

$$T_k(u_n) = \int_0^1 k(y, x)u_n(x)dx = \lambda_n u_n(y)$$

for some scalars λ_n , such that any function $f(y)$ in $L^2([0, 1])$ may be written as an infinite linear combination

$$f(y) = \sum a_n u_n(y)$$

of them, with the coefficient a_n recoverable from the inner product of f and u_n , and the sum converging in the L^2 -norm. The functions u_n are called *eigenfunctions* of the operator, and they are mutually orthogonal. Hilbert's result may be restated more succinctly as the fact that eigenfunctions of a symmetric integral operator form an *orthogonal basis* of $L^2([0, 1])$. This result is known as the *spectral theorem for symmetric integral operators*, so called because the *spectrum* of an operator T on a Hilbert space \mathcal{H} is the collection of all scalars λ such that the operator $(T - \lambda I)$ does not have a bounded inverse. This generalizes the notion of the set of eigenvalues of a matrix. The theorem ignited a flurry of activity in the field. For instance, Koopman observed [11] that a measure-preserving automorphism ϕ on a measure space X induces a *unitary* (i.e. bounded, surjective and inner-product preserving) operator U on $L^2(X)$ given by

$$U(f)(x) = f(\phi(x)).$$

Building on this observation, von Neumann discovered a spectral theorem for unitary operators that mirrored Hilbert's result for symmetric ones, and used it to show that

if no non-constant function in $L^2(X)$ is fixed by U (which is one way of saying that the transformation which induces U is “sufficiently complicated”, or *ergodic*), then, for every function $f \in L^2(X)$, the sequence

$$\frac{1}{n} \sum_{k=1}^n U^k f$$

of averages converges in the strong operator topology to the constant function whose value everywhere is the average value of f [22]. This result is known as the *mean ergodic theorem*, and its discovery laid the foundation of the subject now known as *ergodic theory*.

It was in the 1920s, however, that operator theory underwent a renaissance triggered by the development of quantum mechanics. The key realization that revitalized the field was that observables of a physical system such as energy and momentum can be viewed as *self-adjoint* operators on a complex Hilbert space corresponding to that system, i.e., as operators that coincide with their *adjoints*. (The adjoint of an operator T on a Hilbert space \mathcal{H} is the unique operator T^* on \mathcal{H} such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for every x, y in \mathcal{H} .) The set of all observed values of the observable is then precisely the spectrum of the associated self-adjoint operator. Building on the ideas of Heisenberg, Schrödinger, Dirac, Weyl, Jordan, and others, von Neumann began the axiomatization of quantum theory in order to provide the mathematical setting for modeling the algebra of observables. This led to a deep structural analysis of the operator algebras we now call *von Neumann algebras*. In 1943, Gelfand and Naimark introduced the notion of a *C*-algebra* (pronounced “C star algebra”) as an abstraction of those subalgebras of the bounded linear operators on a complex

Hilbert space \mathcal{H} (denoted $B(\mathcal{H})$), which are topologically closed under the operator norm and algebraically closed under taking adjoints [6]. Such subalgebras of $B(\mathcal{H})$ are sometimes called *concrete C^* -algebras*.

Definition 1.1.2 (C^* -algebra). *A C^* -algebra is a complete normed complex algebra closed under a unary operation called an involution, denoted $*$ and pronounced “star”, satisfying the following conditions for all elements x and y of the algebra and all complex scalars λ :*

$$(1) (x^*)^* = x$$

$$(4) (xy)^* = y^*x^*$$

$$(2) (\lambda x)^* = \bar{\lambda}x^*$$

$$(5) \|x^*\| = \|x\|$$

$$(3) (x + y)^* = x^* + y^*$$

$$(6) \|xx^*\| = \|x\|^2.$$

The involution is precisely the adjoint in a concrete C^* -algebra. A *homomorphism of C^* -algebras* is a map between C^* -algebras that preserves all this structure. It turns out that it is enough for a map to be a $*$ -algebra homomorphism, often simply called a $*$ -homomorphism, in order for it to be a homomorphism of C^* -algebras.

Definition 1.1.3 ($*$ -homomorphism). *A map $\phi : A \rightarrow B$ between two C^* -algebras is a $*$ -homomorphism if it preserves the $*$ -algebra structure, i.e., if it is linear and satisfies*

$$\phi(xy) = \phi(x)\phi(y) \quad \text{and} \quad \phi(x^*) = \phi(x)^*$$

for all elements x and y in A .

Condition (6) in the definition of a C^* -algebra, called the *C^* -identity*, is a particularly restrictive constraint and has the consequence that $*$ -homomorphisms are automatically contractive, and therefore continuous. It also implies that an injective

-homomorphism is automatically isometric. C^ -algebras with *-homomorphisms constitute a category, with the bijective *-homomorphisms, called *-isomorphisms, being the isomorphisms in the category.

Gelfand showed that a commutative C^* -algebra is isomorphic to $C_0(\Omega)$, the algebra of continuous complex-valued functions on a locally-compact Hausdorff topological space Ω that vanish at infinity. This correspondence is known as the *Gelfand representation*. In particular, a commutative C^* -algebra is *unital* if and only if the corresponding topological space Ω in its Gelfand representation is compact.

The prototypical example of a C^* -algebra is $M_n(\mathbb{C})$, the algebra of $n \times n$ square matrices with complex entries. This is unital, and noncommutative for $n \geq 2$. In the special case when $n = 1$, it yields the algebra \mathbb{C} of complex numbers. Operators on \mathbb{C}^n written as matrices generalize to bounded operators on infinite-dimensional Hilbert spaces. Since C^* -algebras are defined as abstractions of closed and self-adjoint subalgebras of $B(\mathcal{H})$, it is no surprise that $B(\mathcal{H})$ is itself a C^* -algebra for any Hilbert space \mathcal{H} .

The smallest non-zero closed ideal in $B(\mathcal{H})$ is the closure of operators of finite rank, and is denoted $K(\mathcal{H})$. Operators in $K(\mathcal{H})$ are called *compact operators*.

Definition 1.1.4 (Compact Operator). *A compact operator on a Hilbert space \mathcal{H} is one that can be approximated in the norm topology by operators of finite rank. A compact operator takes bounded subsets to precompact ones. The algebra of all compact operators on \mathcal{H} is denoted $K(\mathcal{H})$.*

The algebra $K(\mathcal{H})$ is also noncommutative whenever $\dim(\mathcal{H}) \geq 2$, and it is unital if and only if $\dim(\mathcal{H}) < \infty$.

1.2 Representations of C^* -algebras

In the study of abstract groups, it is useful to *represent* group elements as linear transformations of a vector space, and study these concrete objects instead. Indeed, one can learn a lot about an abstract group by studying all of the various ways in which it can be represented on vector spaces. This approach reduces abstract group-theoretic problems to problems in linear algebra, a subject that is well-understood and is more amenable to computations. In a similar manner, one often wishes to study an abstract C^* -algebra by representing its elements as bounded linear operators on a Hilbert space, i.e., by representing it as a concrete C^* -algebra. Studying all possible such representations illuminates the structure of the C^* -algebra.

Definition 1.2.1 (C^* -algebra Representation). *A representation of a C^* -algebra A is a $*$ -homomorphism $\pi : A \rightarrow B(\mathcal{H})$ from A into the bounded linear operators on some Hilbert space \mathcal{H} . In this case, A is said to be represented on \mathcal{H} .*

An injective (and consequently isometric) representation is also called *faithful*.

Definition 1.2.2 (Unitary Equivalence). *Two representations*

$$\pi : A \rightarrow B(\mathcal{H}_\pi) \quad \text{and} \quad \rho : A \rightarrow B(\mathcal{H}_\rho)$$

of A are unitarily equivalent, denoted $\pi \sim_u \rho$, if there is a unitary operator $U : \mathcal{H}_\pi \rightarrow \mathcal{H}_\rho$ such that $\pi(a) = U^ \rho(a) U$ for every $a \in A$.*

It is easy to check that \sim_u is an equivalence relation on representations of A . The set of equivalence classes is called the *spectrum of A* and is denoted $\text{Spec } A$. Unitarily equivalent representations are geometrically indistinguishable — the unitary operator simply being a change of coordinates. Unsurprisingly, then, unitarily equivalent

representations share the same kernel.

A nonzero vector ξ in \mathcal{H} is *cyclic* for a representation π of A on \mathcal{H} if

$$\pi(A)(\xi) := \{\pi(a)(\xi) : a \in A\}$$

is norm-dense in \mathcal{H} . A representation which has a cyclic vector is itself called a *cyclic representation*. The *Gelfand-Naimark-Segal construction* (*GNS construction* for short) establishes a correspondence between equivalence classes of cyclic representations of a C^* -algebra A on the one hand and positive linear functionals of unit norm on A on the other. These functionals are called *states* on A , because in the C^* -algebraic formulation of quantum mechanics they correspond to states of a physical system, i.e., mappings which take physical observables, modeled as self-adjoint elements of the C^* -algebra, to their expected measurement outcomes.

The set of all states on a C^* -algebra A , called the *state space* of A and denoted $S(A)$, is a convex and weak*-compact subset of the Banach dual A^* of A . Consequently it has extreme points, and by the *Krein-Milman theorem*, it is the weak*-closure of their convex-hull. These extreme points are called *pure states*. Pure states are “atomic” in the sense that they cannot be approximated (in the weak*-topology) by convex-linear combinations of *other* states. Recall that the Gelfand representation shows that every commutative C^* -algebra A is of the form $C_0(\Omega)$, the algebra of complex-valued continuous functions on a locally compact Hausdorff space Ω that vanish at infinity. In this special case, the states on A correspond to positive Radon measures on Ω , with the pure states on A corresponding to the evaluation functionals

$$\xi_x : C_0(\Omega) \rightarrow \mathbb{C} ; f \mapsto f(x)$$

for every point x in Ω . States may therefore be viewed as non-commutative generalizations of positive Radon measures. Furthermore, the correspondence between states on a C^* -algebra A and cyclic representations on A established by the GNS construction restricts to a correspondence between pure states on A and special cyclic representations of A called *irreducible representations*. These representations are themselves “atomic” in an important sense which is made clear in the following discussion.

A *subrepresentation* of π is a representation $\pi_W : A \rightarrow B(W)$ of A on a closed, π -invariant subspace W of \mathcal{H} . (A subspace W is π -invariant if $\pi(a)(W) \subseteq W$ for every a in A .) It is easy to see that if W is a non-trivial, proper, π -invariant closed subspace of \mathcal{H} , then so is its *orthogonal compliment*

$$W^\perp := \{\xi \in \mathcal{H} : \text{there exists } w \in W \text{ such that } \langle \xi, w \rangle = 0\}.$$

Consequently, given a proper, non-trivial subrepresentation π_W of π , we may decompose π as a *direct sum of subrepresentations* $\pi = \pi_W \oplus \pi_{W^\perp}$ defined by

$$x \mapsto \pi_W(p_W(x)) + \pi_{W^\perp}(p_{W^\perp}(x)),$$

where p_W is the projection onto W and p_{W^\perp} is the projection onto W^\perp .

Definition 1.2.3 (Irreducible Representation). *A representation π is irreducible if it has no proper, non-trivial subrepresentations, and can therefore not be decomposed non-trivially as a direct sum of subrepresentations. In other words, $\pi : A \rightarrow B(\mathcal{H})$ is irreducible if the only closed proper subspace of \mathcal{H} that is invariant under the A -action π is $\{0\}$.*

An ideal I in A is called a *primitive ideal* if it is the kernel of some irreducible representation of A . Since unitarily equivalent representations share the same kernel,

we have a correspondence between irreducible representations of a C^* -algebra A and primitive ideals in A . In particular, a C^* -algebra with a unique irreducible representation up to unitary equivalence has a unique primitive ideal. The following lemma shows that in this case the primitive ideal is zero, and moreover that any such C^* -algebra is simple.

Lemma 1.3. *If A is a C^* -algebra with a unique irreducible representation up to unitary equivalence, then A is simple.*

Proof. It is a standard result that every closed proper ideal in a C^* -algebra is the intersection of the primitive ideals containing it. (See [19, Proposition A.17, p.212] for a statement and proof.) Since A has only one irreducible representation up to unitary equivalence, A has a unique primitive ideal I . Thus, every closed proper ideal of A must equal I . Since $\{0\}$ is a closed proper ideal of A , it follows that any closed proper ideal of A is equal to $\{0\}$. In particular, A is simple. \square

The set of all primitive ideals of A is called the *primitive spectrum of A* and is denoted $\text{Prim } A$.

After proposing their definition of a C^* -algebra, Gelfand and Naimark used the GNS construction to show that every C^* -algebra can be isometrically embedded in some $B(\mathcal{H})$ as a concrete C^* -algebra. (See [19, Theorem A.11, p.207] for a statement and proof.) This parallels Cayley's theorem from group theory and Whitney's embedding theorem from manifold theory, in that it confirms that a "nice" abstraction of a concrete C^* -algebra exists, and that Gelfand and Naimark's definition captures it. In particular, this means that the primitive spectrum (and therefore also the spectrum) of a C^* -algebra is never empty.

1.3 Naimark's Problem

In 1948, Naimark observed that every irreducible representation of $K(\mathcal{H})$ is unitarily equivalent to the inclusion representation $i : K(\mathcal{H}) \hookrightarrow B(\mathcal{H}) ; T \mapsto T$ [14]. Said differently, for any Hilbert space \mathcal{H} , the C^* -algebra $K(\mathcal{H})$ of compact operators on \mathcal{H} has a unique irreducible representation up to unitary equivalence. In 1951, he asked whether this property characterized the C^* -algebras $K(\mathcal{H})$ [15]. In other words, if A is a C^* -algebra with only one irreducible representation up to unitary equivalence, is A isomorphic to $K(\mathcal{H})$ for some Hilbert space \mathcal{H} ? This question is known as *Naimark's problem*. A C^* -algebra which satisfies the premise of this question but not its conclusion is a *counterexample to Naimark's problem*. In the years following its proposal, various results were obtained which provided partial answers to Naimark's problem.

In 1951, Kaplansky introduced the *GCR C^* -algebras* and began developing their representation theory [9]. Today these are more commonly known as *Type I C^* -algebras*.

Definition 1.3.1 (Type I C^* -algebras). *A C^* -algebra is Type I if the image of each of its irreducible representations contains the compact operators.*

It is known that no Type I C^* -algebra can be a counterexample to Naimark's problem. Indeed, this is not hard to see.

Theorem 1.3.1. *If A is a Type I C^* -algebra with $T : A \rightarrow B(\mathcal{H})$ its only irreducible representation up to unitary equivalence, then $A \cong K(\mathcal{H})$ via T .*

Proof. Lemma 1.3 shows that A is simple. Together with the fact that $K(\mathcal{H}) \subseteq \text{im } T$, this forces $\ker T = \{0\}$. In other words, $A \cong \text{im } T$ via T . Since $K(\mathcal{H})$ is an

ideal in $B(\mathcal{H})$ and therefore an ideal in $\text{im} T$, the simplicity of A in turn forces $\text{im} T = K(\mathcal{H})$. \square

Naimark's problem was quickly settled for separable C^* -algebras, when in 1953 A. Rosenberg proved the following theorem.

Theorem 1.3.2 (A. Rosenberg). [20, Theorem 4] *No separable C^* -algebra is a counterexample to Naimark's problem. Explicitly, if A is a separable C^* -algebra with a unique irreducible representation up to unitary equivalence, then A is isomorphic to $K(\ell^2(\mathbb{N}))$.*

Building on Kaplansky's ideas, Fell showed in 1960 that any two irreducible representations of a Type I C^* -algebra that have equal kernels must be unitarily equivalent [4]. That same year, Dixmier proved a partial converse: a separable C^* -algebra that is not Type I necessarily has unitarily *inequivalent* representations whose kernels are equal [3]. In fact, in 1961 Glimm showed that a separable C^* -algebra that is not Type I has *uncountably many* inequivalent irreducible representations [7]. These two results partially recovered Rosenberg's 1953 result of Theorem 1.3.2.

In 2004, however, Akemann and Weaver used Jensen's \diamond axiom (pronounced "diamond axiom"), which is a combinatorial principle known to be independent of ZFC, to construct a counterexample to Naimark's problem that is generated by \aleph_1 elements [1]. In fact, they showed that the existence of an \aleph_1 -generated counterexample is independent of ZFC. Akemann and Weaver's result suggests that there are set-theoretic obstructions to obtaining a general answer to Naimark's problem. In light of this, it is reasonable to consider restrictions of the problem to special subclasses of C^* -algebras. In this thesis we focus attention on *graph C^* -algebras*.

2.1 Directed Graphs

Definition 2.1.1 (Directed Graph). *A directed graph (E^0, E^1, r, s) consists of a set E^0 of vertices, a set E^1 of edges between them, and range and source maps*

$$r : E^1 \rightarrow E^0 \quad \text{and} \quad s : E^1 \rightarrow E^0$$

identifying, respectively, the range and source of each edge.

From here on, we shall dispense with the qualifier *directed* in an understanding that all graphs under discussion are directed.

A vertex v in E^0 is a *sink* if no edges come out of it (i.e., $s^{-1}(v)$ is empty), and it is an *infinite emitter* if infinitely many do (i.e., $|s^{-1}(v)| = \infty$). Sinks and infinite emitters are classified as *singular vertices*. Vertices that are neither sinks

nor infinite emitters are called *regular vertices*. A graph is called *row-finite* if every vertex v emits finitely many edges (i.e., $s^{-1}(v)$ is a finite, possibly empty, set), and it is called *row-countable* if every vertex v emits countably many edges (i.e., $s^{-1}(v)$ is a countable, possibly empty, set). A graph E is *finite* if both E^0 and E^1 are finite sets, and it is *countable* if both E^0 and E^1 are countable sets.

A *path* $\alpha = e_1 \dots e_n$ in a graph E is a finite succession of edges in which the range vertex of any edge coincides with the source vertex of the next edge in the succession (i.e., $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$), and we say that such a path has *length* $|\alpha| = n$. We consider vertices to be paths of length zero (also called empty paths) and edges to be paths of length one. We also let E^n denote the set of all paths of length n in E , and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of all paths in E . We extend the range and source maps to E^* in the obvious way: If $\alpha = e_1 \dots e_n$ then $r(\alpha) := r(e_n)$ and $s(\alpha) := s(e_1)$. An *infinite path* $e_1 e_2 \dots$ is an infinite succession of edges with $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. We let E^∞ denote the set of all infinite paths in E , and we extend the source map s to E^∞ in the obvious way: If $\alpha = e_1 e_2 \dots$, then $s(\alpha) := s(e_1)$. A *cycle* is a path $\alpha \in E^*$ such that $s(\alpha) = r(\alpha)$. If $\alpha = e_1 \dots e_n$ is a cycle, an *exit* for α is an edge $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$ for some $1 \leq i \leq n$. A graph is said to satisfy *Condition (L)* if every cycle in E has an exit.

If $v, w \in E^0$, we say v can reach w , written $v \geq w$, if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. A graph is called *cofinal* if whenever $v \in E^0$ and $\alpha := e_1 e_2 \dots \in E^\infty$, then $v \geq s(e_i)$ for some $i \in \mathbb{N}$. A subset $H \subseteq E^0$ is called *hereditary* if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. A hereditary subset H is called *saturated* if whenever v is a regular vertex and $r(s^{-1}(v)) \subseteq H$, then $v \in H$.

2.2 Graph C^* -algebras

As a locally compact Hausdorff space Ω contains enough topological information to produce the commutative C^* -algebra $C(\Omega)$, and a discrete group G contains enough algebraic information to produce its group C^* -algebra $C^*(G)$, so too a graph E contains enough combinatorial information to produce its *graph C^* -algebra* $C^*(E)$. Explicitly, there is a natural recipe for building a C^* -algebra from a graph. We may describe this recipe from several viewpoints. One viewpoint that helps make it concrete and transparent is the following.

Given a graph $E = (E^0, E^1, r, s)$, recall that E^* is the set of all (finite, possibly empty) paths in E and E^∞ is the set of all infinite paths in E . For this section, we shall denote the set $E^\infty \cup (E^* \setminus E^0)$ of *all* non-empty paths, finite or infinite, by Π . Each $\delta \in E^*$ (partially) acts on the set Π by pre-concatenation with its elements whenever possible:

$$\delta(\pi) = \delta\pi, \quad \pi \in \Pi.$$

We also define the (partial) inverse of this action as pre-truncation of paths beginning with δ :

$$\delta^*(\delta\pi) = \pi, \quad \pi \in \Pi.$$

Consequently we have $\delta\delta^*\delta = \delta$ and $\delta^*\delta\delta^* = \delta^*$. This action makes δ a partial bijection on Π . Consequently E^* can be viewed as an inverse semigroup. Taking this viewpoint, every (possibly empty) finite path *becomes* a bijection from one subset of Π to another. We denote the range of the *partial bijection* $\delta \in E^*$ by $\text{range}(\delta)$. (Not to be confused with the *range vertex* of the *path* δ , which is denoted by $r(\delta)$.) If the path δ is empty (i.e., it is just a vertex, say v), then it is, in fact, a partial identity on Π , and we have $v = v^* = v^2$. Also, different vertices have

disjoint ranges in Π . On the other hand, if δ is a non-empty finite path $e_1e_2e_3 \dots e_n$, where the e_i 's are edges, then every path of the form $\delta(\pi) = \delta\pi$ may be written as $e_1e_2e_3 \dots e_n\pi$, and the partial inverse δ^* may be written as $e_n^* \dots e_3^*e_2^*e_1^*$, so that the action of δ is the composition of the actions of its edges. Again, different edges have disjoint ranges in Π . Therefore the set $E^0 \cup E^1 = \{v, e : v \in E^0, e \in E^1\}$ of vertices and edges of E , subject to the relations

$$\begin{aligned}
 (1) \quad v &= v^* = v^2 & (4) \quad \text{range}(ee^*) &\subseteq \text{range}(s(e)) \\
 (2) \quad ee^*e &= e & (5) \quad \text{range}(v) &= \coprod_{s(e)=v} \text{range}(ee^*) \\
 (3) \quad e^*e &= r(e)
 \end{aligned}$$

generates the inverse semigroup E^* .

To move from an inverse semigroup to a C^* -algebra, we simply let E^* act in a similar manner, via bounded linear operators, on the Hilbert space $\ell^2(\Pi)$ with orthonormal basis $\{\xi_\pi\}$ indexed by the set of all (possibly infinite) non-empty paths $\pi \in \Pi$. Concretely, for every vertex v in E and every edge e in E , define the bounded linear operators p_v and s_e on $\ell^2(\Pi)$ by

$$p_v(\xi_\pi) := \begin{cases} \xi_\pi & \text{if } \pi \text{ begins at } v \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s_e(\xi_\pi) := \begin{cases} \xi_{e\pi} & \text{if } e\pi \text{ is a path} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the relations on the vertices and edges that determined the inverse semigroup E^* are carried over appropriately as relations on the p_v 's and s_e 's in this new setting:

$$\begin{aligned}
 (1) \quad p_v &= p_v^* = p_v^2 & (4) \quad s_e s_e^* &\leq p_{s(e)} \\
 (2) \quad s_e s_e^* s_e &= s_e & (5) \quad p_v &= \sum_{s(e)=v} s_e s_e^* \text{ whenever } v \in E^0 \\
 (3) \quad s_e^* s_e &= p_{r(e)} & & \text{is a regular vertex.}
 \end{aligned}$$

In particular, relation (1) says that the p_v 's are projections in $B(\ell^2(\Pi))$ and relation (2) says that the s_e 's are partial isometries in $B(\ell^2(\Pi))$. It follows from their definitions that the p_v 's are mutually orthogonal, and that the s_e 's have mutually orthogonal ranges. Also, for a path $\delta = e_1 e_2 \dots e_n$, we denote the composition $s_{e_1} s_{e_2} \dots s_{e_n}$ of partial isometries (which is itself a partial isometry) by s_δ . The set $\{p_v, s_e : v \in E^0, e \in E^1\}$, together with the relations above, generates a C^* -subalgebra of $B(\ell^2(\Pi))$. This C^* -algebra encodes much of the combinatorial information contained in the graph, and the recipe described above can serve as a bridge between the study of graphs on the one hand and of their associated C^* -algebras on the other. However, we would like to define the graph C^* -algebra of a graph E in a manner that is independent of the underlying Hilbert space $\ell^2(\Pi)$. This motivates the following.

Definition 2.2.1 (Graph C^* -algebra). *The graph C^* -algebra $C^*(E)$ of a graph $E := (E^0, E^1, r, s)$ is the universal C^* -algebra generated by mutually orthogonal projections $\{p_v : v \text{ is a vertex in } E\}$ together with partial isometries with mutually orthogonal ranges $\{s_e : e \text{ is an edge in } E\}$ satisfying*

$$(CK1) \quad s_e^* s_e = p_{r(e)} \text{ for all } e \in E^1,$$

$$(CK2) \quad s_e s_e^* \leq p_{s(e)} \text{ for all } e \in E^1, \text{ and}$$

$$(CK3) \quad p_v = \sum_{s(e)=v} s_e s_e^* \text{ whenever } v \in E^0 \text{ is a regular vertex.}$$

Such a universal C^* -algebra does indeed exist, and its uniqueness is an immediate consequence of universality. The three relations above are called the *Cuntz-Krieger relations*. Given a graph E , a family $\{p_v, s_e : v \in E^0, s \in E^1\}$ of mutually orthogonal projections p_v and partial isometries s_e with mutually orthogonal ranges that additionally satisfies the Cuntz-Krieger relations is called a *Cuntz-Krieger E -family*. One

can use the Cuntz-Krieger relations to show that $C^*(E) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}$. Moreover, $C^*(E)$ is separable if and only if E is a countable graph.

The graph C^* -algebras are a rich and tractable class. They include, for instance, the finite-dimensional C^* -algebras, $K(\mathcal{H})$ for \mathcal{H} not necessarily separable, $n \times n$ matrices over continuous functions on the circle, the Toeplitz algebras, the Cuntz algebras, and the Cuntz-Krieger algebras. Also, all separable AF algebras, as well as all Kirchberg algebras with free K_1 -group, are Morita equivalent to graph C^* -algebras. Furthermore, the class of graph C^* -algebras is closed under direct sums, stabilization, and tensoring by $M_n(\mathbb{C})$.

2.3 Standard Results for Graph C^* -algebras

There are numerous results relating the structure of a graph to the structure of its graph C^* -algebra. In this section, we state (with reference) those that we shall need.

Remark 2.3.1. Many of these results, as stated in the cited literature, rely on additional assumptions such as row-finiteness of the graph or separability of the graph C^* -algebra. Nevertheless, the proofs extend to the more general cases stated.

Theorem 2.3.1. [21, Theorem 12] [16, Theorem 4] *Let E be a directed graph.*

Then the following three conditions are equivalent:

- (1) $C^*(E)$ is simple.
- (2) E satisfies Condition (L), E is cofinal, and every vertex of E can reach every singular vertex of E .
- (3) E satisfies Condition (L) and the only saturated hereditary subsets of E are E^0 and \emptyset .

In this thesis, we call a C^* -algebra an *AF algebra*, or say the C^* -algebra is *AF* (short for approximately finite-dimensional), if it is the (generalized) direct limit of finite-dimensional C^* -algebras. Note that this differs from the standard usage of the term, which typically requires a *sequential* direct limit, and therefore implies AF algebras are separable. Our notion of AF coincides with the usual definition for separable C^* -algebras, but also allows for nonseparable AF algebras, which are direct limits of *directed sets* of finite-dimensional C^* -algebras. The following result gives a very nice characterization of AF graph C^* -algebras.

Theorem 2.3.2. [18, §5.4] *If E is a graph, then $C^*(E)$ is AF if and only if E has no cycles.*

If $E = (E^0, E^1, r_E, s_E)$ is a graph, a *subgraph* of E is a graph $F = (F^0, F^1, r_F, s_F)$ such that $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, and r_F and s_F are restrictions of the range and source maps that r_E and s_E induce on E^* such that $\text{im } r_F \subseteq F^0$ and $\text{im } s_F \subseteq F^0$.

If H is a hereditary subset of the graph $E = (E^0, E^1, r, s)$, the *restriction of E to H* is the graph $E_H := (E_H^0, E_H^1, r_H, s_H)$ with vertex set $E_H^0 := H$, edge set $E_H^1 := s^{-1}(H)$, and range and source maps $r_H := r|_{E_H^1}$ and $s_H := s|_{E_H^1}$. In addition, we let I_H denote the closed two-sided ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$.

Morita equivalence is an equivalence relation on the class of all C^* -algebras that is weaker than isomorphism but strong enough to preserve the ideal structure and representation theory. Explicitly, if C^* -algebras A and B are Morita equivalent then their ideal-lattices (of closed, two-sided ideals) are lattice isomorphic, and this isomorphism takes primitive ideals to primitive ideals. Moreover, there is a correspondence between $\text{Spec } A$ and $\text{Spec } B$ that restricts to a correspondence between $\text{Prim } A$ and $\text{Prim } B$. This is known as the *Rieffel correspondence*. (See [19, §3.3] for a detailed discussion of the Rieffel correspondence.)

Theorem 2.3.3. [2, Proposition 3.4] *If E is a graph and H is a hereditary subset of E , then $C^*(E_H)$ is Morita equivalent to I_H .*

Definition 2.3.1 (Relative Graph C^* -algebra). [13, §3] *If $E := (E^0, E^1, r, s)$ is a graph, let E_{reg}^0 denote the regular vertices of E , and let $S \subseteq E_{\text{reg}}^0$. A Cuntz-Krieger (E, S) -family is a collection of elements $\{s_e, p_v : e \in E^1, v \in E^0\}$ in a C^* -algebra such that $\{p_v : v \in E^0\}$ is a collection of mutually orthogonal projections and $\{s_e : e \in E^1\}$ is a collection of partial isometries with mutually orthogonal ranges satisfying the Cuntz-Krieger relations:*

$$(CK1) \quad s_e^* s_e = p_{r(e)} \text{ for all } e \in E^1,$$

$$(CK2) \quad s_e s_e^* \leq p_{s(e)} \text{ for all } e \in E^1, \text{ and}$$

$$(CK3) \quad p_v = \sum_{s(e)=v} s_e s_e^* \text{ whenever } v \in S.$$

The relative graph C^ -algebra $C^*(E, S)$ is the C^* -algebra generated by a universal Cuntz-Krieger (E, S) -family.*

Observe that if $S = E_{\text{reg}}^0$, then $C^*(E, E_{\text{reg}}^0)$ is simply the graph C^* -algebra $C^*(E)$. If $S = \emptyset$, then $C^*(E, \emptyset)$ is called the Toeplitz algebra of E , and often denoted $TC^*(E)$.

If $C^*(E, S)$ is a relative graph C^* -algebra and $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger (E, S) -family in $C^*(E, S)$, then for any $v \in E_{\text{reg}}^0 \setminus S$, we call $q_v := p_v - \sum_{s(e)=v} s_e s_e^*$ the *gap projection* at v .

If I is the ideal generated by $\{q_v : v \in E_{\text{reg}}^0 \setminus S\}$, then $C^*(E, S)/I \cong C^*(E)$, and hence the graph C^* -algebra is a quotient of the relative graph C^* -algebra $C^*(E, S)$.

In addition, whenever E is a graph and $S \subseteq E_{\text{reg}}^0$ there exists a graph E_S such

that $C^*(E_S)$ is isomorphic to $C^*(E, S)$. Thus every relative graph C^* -algebra is isomorphic to a graph C^* -algebra (of a possibly different graph).

Furthermore, we have the following Cuntz-Krieger Uniqueness Theorem for relative graph C^* -algebras.

Theorem 2.3.4. [13, Theorem 3.11] *Let E be a graph, let $S \subseteq E_{\text{reg}}^0$, and let $\phi : C^*(E, S) \rightarrow A$ be a homomorphism from $C^*(E, S)$ into a C^* -algebra A . If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger (E, S) -family in $C^*(E, S)$ and the following three conditions hold*

- (1) E satisfies Condition (L),
- (2) $\phi(p_v) \neq 0$ for all $v \in E^0$, and
- (3) $\phi(q_v) \neq 0$ for all $v \in E_{\text{reg}}^0 \setminus S$,

then ϕ is injective.

CHAPTER 3

Structure Results for certain Graph C^* -algebras

In this chapter we show that if $C^*(E)$ is AF and has a unique irreducible representation, then the graph E necessarily has one of two specific forms.

Remark 3.0.1. All results proved from this point forward are ours.

3.1 Preliminary Lemmas

Lemma 3.2. *If F is the graph*

$$w_1 \begin{array}{c} \xrightarrow{e_1} \\ \xleftarrow{f_1} \end{array} w_2 \begin{array}{c} \xrightarrow{e_2} \\ \xleftarrow{f_2} \end{array} w_3 \begin{array}{c} \xrightarrow{e_3} \\ \xleftarrow{f_3} \end{array} \cdots \quad (3.2.1)$$

with distinct vertices w_1, w_2, \dots and distinct edges $e_1, f_1, e_2, f_2, \dots$, then $C^(F)$ contains a full corner isomorphic to the UHF-algebra M_{2^∞} , and $C^*(F)$ is not a Type I C^* -algebra.*

Proof. Consider the corner $p_{w_1}C^*(F)p_{w_1}$. Since F has no cycles, the ideals of $C^*(F)$ are in bijective correspondence with the saturated hereditary subsets of F . Since $p_{w_1} \in p_{w_1}C^*(F)p_{w_1}$, and any hereditary subset containing w_1 must equal F^0 , we conclude that any ideal containing $p_{w_1}C^*(F)p_{w_1}$ is equal to $C^*(F)$. Thus $p_{w_1}C^*(F)p_{w_1}$ is a full corner of $C^*(F)$.

If $\{s_e, p_v : e \in F^1, v \in F^0\}$ is a generating Cuntz-Krieger F -family, then

$$p_{w_1}C^*(F)p_{w_1} = \overline{\text{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta) = w_1 \text{ and } r(\alpha) = r(\beta)\}.$$

For each $n \in \mathbb{N} \cup \{0\}$, define

$$E_n := \{\alpha \in F^* : s(\alpha) = w_1 \text{ and } r(\alpha) = w_n\}$$

and let

$$A_n := \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E_n\}.$$

Then we see that each A_n is a C^* -subalgebra of $C^*(F)$, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$, and

$$C^*(F) = \overline{\bigcup_{n=0}^{\infty} A_n}.$$

For each $n \in \mathbb{N} \cup \{0\}$, we see that $\{s_\alpha s_\beta^* : \alpha, \beta \in E_n\}$ is a set of matrix units, and since $|E_n| = 2^n$ it follows that $A_n \cong M_{2^n}(\mathbb{C})$. Furthermore, for each $s_\alpha s_\beta^*$ with $\alpha, \beta \in E_n$, we see that

$$s_\alpha s_\beta^* = s_\alpha p_{w_n} s_\beta^* = s_\alpha (s_{e_n} s_{e_n}^* + s_{f_n} s_{f_n}^*) s_\beta^* = s_{\alpha e_n} s_{\beta e_n}^* + s_{\alpha f_n} s_{\beta f_n}^*.$$

Hence if for all $n \in \mathbb{N} \cup \{0\}$ we identify A_n with $M_{2^n}(\mathbb{C})$ via an isomorphism, then for each n the inclusion map $A_n \hookrightarrow A_{n+1}$ may be identified with the map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$.

Thus $C^*(F) = \overline{\bigcup_{n=0}^{\infty} A_n}$ is isomorphic to the UHF-algebra M_{2^∞} .

Finally, it follows from [17, Theorem 6.5.7, p.211] that M_{2^∞} is not Type I. Since any C^* -subalgebra of a Type I C^* -algebra is Type I [17, Theorem 6.2.9, p.199], we conclude that $C^*(F)$ is not Type I. \square

Proposition 3.3. *Let E be a row-countable graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. If $v \in E^0$ and we define $H(v) := \{w \in E^0 : v \geq w\}$, then $E_{H(v)}$ is a countable graph, $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$, and $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for some separable Hilbert space \mathcal{H} .*

Proof. We see that $H(v) := \{w \in E^0 : v \geq w\}$ is a hereditary subset of E . In addition, if we let $H_0 = \{v\}$ and inductively define $H_{n+1} := r(s^{-1}(H_n))$, then one can easily verify that $H(v) = \bigcup_{n=0}^{\infty} H_n$. Since H_0 is finite and E is row-countable, an inductive argument shows that H_n is countable for all $n \in \mathbb{N}$. Hence $E_{H(v)}^0 := H(v) = \bigcup_{n=0}^{\infty} H_n$ is countable, and $E_{H(v)}^1 := s^{-1}(H(v))$ is countable. Thus, $E_{H(v)} := (E_{H(v)}^0, E_{H(v)}^1, r|_{H(v)}, s|_{H(v)})$ is a countable graph.

It follows from Theorem 2.3.3 that $C^*(E_{H(v)})$ is Morita equivalent to $I_{H(v)}$. Because $C^*(E)$ has a unique irreducible representation up to unitary equivalence, it follows from Lemma 1.3 that $C^*(E)$ is simple. Because $I_{H(v)}$ is a nonzero ideal of $C^*(E)$, it follows that $I_{H(v)} = C^*(E)$. Thus $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$.

Finally, since $C^*(E)$ has a unique irreducible representation up to unitary equivalence and $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$, we conclude that $C^*(E_{H(v)})$ has a unique irreducible representation up to unitary equivalence. (This is because the Rieffel correspondence establishes a bijection between representations of Morita equivalent C^* -algebras that preserves irreducibility and unitary equivalence.) Moreover, the fact that $E_{H(v)} := (E_{H(v)}^0, E_{H(v)}^1, r|_{H(v)}, s|_{H(v)})$ is a countable graph implies $C^*(E_{H(v)})$ is a separable C^* -algebra. Hence by Rosenberg's result (Theorem 1.3.2) we have $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for a separable Hilbert space \mathcal{H} . \square

3.2 A Structure Result for AF Graph C^* -algebras

Proposition 3.3. *Let E be a directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. Then the following are equivalent:*

- (1) E is row-countable.
- (2) E is row-finite.
- (3) $C^*(E)$ is AF.

Proof. (2) \implies (1): This is immediate from the definitions.

(1) \implies (3): Let $v \in E^0$, and set $H(v) := \{w \in E^0 : v \geq w\}$. Then $H(v)$ is a hereditary subset of E , and by Proposition 3.3 $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for some separable Hilbert space \mathcal{H} . Consequently $C^*(E_{H(v)})$ is AF, and Theorem 2.3.2 implies the graph $E_{H(v)}$ has no cycles. Hence E has no cycles with vertices in $H(v)$. Furthermore, $C^*(E)$ is simple by Lemma 1.3, and thus Theorem 2.3.1 implies that E is cofinal. Since vertices in the hereditary set $H(v)$ cannot reach cycles containing vertices in $E^0 \setminus H(v)$, we may conclude that E has no cycles. Thus Theorem 2.3.2 implies $C^*(E)$ is AF.

(3) \implies (2): Since $C^*(E)$ is AF, Theorem 2.3.2 implies the graph E has no cycles. In addition, since $C^*(E)$ has a unique irreducible representation up to unitary equivalence, Lemma 1.3 implies that $C^*(E)$ is simple. It then follows from Theorem 2.3.1 that every vertex of E can reach every singular vertex of E . Let $v \in E^0$, and suppose v is not a sink. Then there exists $e \in s^{-1}(v)$. Since E has no cycles, it follows that $r(e)$ cannot reach v . But this implies that v is not a singular vertex, and hence v emits a finite number of edges. Since every vertex of E that is not a sink emits a finite number of edges, E is row-finite. \square

3.3 A Forbidden Subgraph

Proposition 3.4. *Let E be a directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. If $C^*(E)$ is AF, then E does not contain a subgraph of the form*

$$v_1 \begin{array}{c} \xrightarrow{\beta_1} \\ \xleftarrow{\alpha_1} \end{array} v_2 \begin{array}{c} \xrightarrow{\beta_2} \\ \xleftarrow{\alpha_2} \end{array} v_3 \begin{array}{c} \xrightarrow{\beta_3} \\ \xleftarrow{\alpha_3} \end{array} \cdots \quad (3.4.1)$$

where v_1, v_2, \dots are distinct vertices and $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots$ are distinct paths.

Proof. For the sake of contradiction, suppose that E has a subgraph of the form in (3.4.1), and use the labeling of vertices and paths listed in (3.4.1). Since $C^*(E)$ is AF, Proposition 3.3 implies E is row-finite. If $H(v_1) := \{w \in E^0 : v_1 \geq w\}$, then $H(v_1)$ is a hereditary subset of E and Proposition 3.3 implies $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for a separable Hilbert space \mathcal{H} . Thus $C^*(E_{H(v)})$ is a Type I C^* -algebra. Furthermore, since v_1 can reach every vertex on each path α_i and each path β_i for all $i \in \mathbb{N}$, we conclude that the graph $E_{H(v)}$ has a subgraph of the form in (3.4.1).

Let $\{s_e, p_v : e \in E_{H(v)}^1, v \in E_{H(v)}^0\}$ be a generating Cuntz-Krieger $E_{H(v)}$ -family in $C^*(E_{H(v)})$, and consider the set $\{p_{v_i}, s_{\alpha_i}, s_{\beta_i}\}_{i=1}^\infty$. Also let F be the graph

$$w_1 \begin{array}{c} \xrightarrow{e_1} \\ \xleftarrow{f_1} \end{array} w_2 \begin{array}{c} \xrightarrow{e_2} \\ \xleftarrow{f_2} \end{array} w_3 \begin{array}{c} \xrightarrow{e_3} \\ \xleftarrow{f_3} \end{array} \cdots \quad (3.4.2)$$

and let

$$S := \{w_i : i \in \mathbb{N} \text{ and } p_{v_i} = s_{\alpha_i} s_{\alpha_i}^* + s_{\beta_i} s_{\beta_i}^*\}.$$

Then $\{p_{v_i}, s_{\alpha_i}, s_{\beta_i}\}_{i=1}^\infty$ is an (F, S) -family in $C^*(E_{H(v)})$, and there exists a homomorphism $\phi : C^*(F, S) \rightarrow C^*(E_{H(v)})$ (where $C^*(F, S)$ denotes the relative graph C^* -algebra of F with the (CK3) relation imposed at the vertices in S). We observe that F has no cycles, and whenever $i \in \mathbb{N}$ with $w_i \notin S$, then the gap projection

$p_{v_i} - s_{\alpha_i} s_{\alpha_i}^* - s_{\beta_i} s_{\beta_i}^* \neq 0$ whenever $v_i \notin S$, it follows from the Cuntz-Krieger Uniqueness Theorem for relative graph C^* -algebras that ϕ is injective. Hence $\text{im } \phi$ is a C^* -subalgebra of $C^*(E_{H(v)})$ isomorphic to $C^*(F, S)$.

It follows from Proposition 3.2 that $C^*(F)$ is not a Type I C^* -algebra. Because $C^*(F)$ is a quotient of $C^*(F, S)$, and all quotients of Type I C^* -algebras are Type I [17, Theorem 6.2.9, p.199], it follows that $C^*(F, S)$ is not a Type I C^* -algebra. Thus $\text{im } \phi \cong C^*(F, S)$ is a C^* -subalgebra of $C^*(E_{H(v)})$ that is not Type I, and since all C^* -subalgebras of Type I C^* -algebras are Type I [17, Theorem 6.2.9, p.199], it follows that $C^*(E_{H(v)})$ is not a Type I C^* -algebra. But this contradicts the fact that $C^*(E_{H(v)}) \cong K(\mathcal{H})$. \square

3.4 The Graph of a Simple AF Algebra

We are now in a position to prove the result promised at the beginning of this chapter.

Theorem 3.4.1. *Let E be a directed graph such that $C^*(E)$ is AF and has a unique irreducible representation up to unitary equivalence. Then one of two distinct possibilities must occur: Either*

- (1) *E has exactly one sink and no infinite paths; or*
- (2) *E has no sinks and E contains an infinite path $\alpha := e_1 e_2 e_3 \dots$ in which every vertex emits exactly one edge, i.e.,*

$$s^{-1}(s(e_i)) = \{e_i\}$$

for all $i \in \mathbb{N}$.

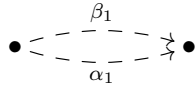
Proof. Because $C^*(E)$ has a unique irreducible representation up to unitary equivalence, it follows from Lemma 1.3 that $C^*(E)$ is simple, and it follows from Theorem 2.3.1 that E is cofinal, satisfies Condition (L), and every vertex of E can reach every singular vertex of E .

The fact that every vertex of E can reach every singular vertex of E implies that E has at most one sink. If E has one sink, then the cofinality of E implies that E has no infinite paths (because a sink cannot reach a vertex on the infinite path), and hence we are in situation (1) of the proposition.

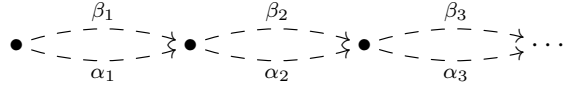
If E has no sinks, then E must contain an infinite path $f_1 f_2 \dots$. To show that we are in situation (2) it suffices to show that there exists $N \in \mathbb{N}$ such that $s^{-1}(s(f_i)) = \{f_i\}$ for all $i \geq N$. (For then we can take $e_i := f_{N+i}$, and $e_1 e_2 \dots$ is the desired path.)

Suppose to the contrary that the infinite path $f_1 f_2 \dots$ does not have our desired property. Then for each $k \in \mathbb{N}$ there exists $n \geq k + 1$ such that $s^{-1}(s(f_n))$ contains an element different from $e = f_n$. Choose a natural number n_0 such that $s^{-1}(s(f_{n_0}))$ contains an element g different from f_{n_0} , and define $w_0 := s(f_{n_0})$. By cofinality there exists a path μ with $s(\mu) = r(g)$ and $r(\mu) = s(f_{m_0})$ for some $m_0 \in \mathbb{N}$.

Since $C^*(E)$ is AF, it follows from Theorem 2.3.2 that E has no cycles, and hence $m_0 > n_0$. Using our hypothesis, we may choose $n_1 > m_0$ such that $s^{-1}(s(f_{n_1}))$ contains an element distinct from f_{n_1} . If we let $\alpha_1 := f_{n_0} f_{n_0+1} \dots f_{m_0} \dots f_{n_1-1}$ and $\beta_1 := g \mu f_{m_0} f_{m_0+1} \dots f_{n_1-1}$, then α_1 and β_1 are distinct paths in E with $s(\alpha_1) = s(\beta_1)$ and $r(\alpha_1) = r(\beta_1)$ and such that $r(\alpha_1) = r(\beta_1) = s(f_{n_1})$ with $s^{-1}(s(f_{n_1}))$ containing an element distinct from f_{n_1} .



Proceeding inductively, we are able to construct a subgraph of E of the following form.



Since $C^*(E)$ has a unique irreducible representation up to unitary equivalence, Proposition 3.4 implies that E does not have such a subgraph. Hence, we have a contradiction. \square

Naimark's Problem for certain Graph C^* -Algebras

4.1 Main Results

In this chapter we prove our two main results: (1) Naimark's Problem has an affirmative answer for the class of AF graph C^* -algebras, and (2) Naimark's Problem has an affirmative answer for the class of C^* -algebras of row-countable graphs.

If \mathcal{H} is a Hilbert space, then for any $x, y \in \mathcal{H}$, we let $\Theta_{x,y} : \mathcal{H} \rightarrow \mathcal{H}$ denote the rank-one operator given by

$$\Theta_{x,y}(z) := \langle y, z \rangle x.$$

Since $K(\mathcal{H})$ is the closure of the finite-rank operators, we see that if β is a basis for \mathcal{H} , then $K(\mathcal{H}) = \overline{\text{span}}\{\Theta_{x,y} : x, y \in \beta\}$.

If $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is an isometry between Hilbert spaces, we let $\text{Ad}_V : K(\mathcal{H}_1) \rightarrow K(\mathcal{H}_2)$ denote the homomorphism given by $\text{Ad}_V(T) := VTV^*$. It is straightforward

to verify that Ad_V is injective and for any $x, y \in \mathcal{H}_1$ we have $\text{Ad}_V(\Theta_{x,y}) = \Theta_{Vx, Vy}$.

Theorem 4.1.1. *Let E be a graph such that $C^*(E)$ is AF. If $C^*(E)$ has a unique irreducible representation up to unitary equivalence, then $C^*(E) \cong K(\mathcal{H})$ for some Hilbert space \mathcal{H} .*

Proof. Throughout, let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a generating Cuntz-Krieger E -family. By Proposition 3.4.1 there are two cases to consider.

CASE I: E has exactly one sink and no infinite paths.

Let v_0 denote the sink of E , and let $E^*(v_0) := \{\alpha \in E^* : r(\alpha) = v_0\}$. Define $I_{v_0} := \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_0)\}$. Since no path ending at the sink v_0 can be extended, for any $\alpha, \beta, \gamma, \delta \in E^*(v_0)$ we have

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) := \begin{cases} s_\alpha s_\delta^* & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma, \end{cases}$$

and $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_0)\}$ is a set of matrix units indexed by $E^*(v_0)$. Hence $I_{v_0} \cong K(\mathcal{H})$, where $\mathcal{H} := \ell^2(E^*(v_0))$. Furthermore, since v_0 is a sink, I_{v_0} is a nonzero ideal in $C^*(E)$. By Lemma 1.3 $C^*(E)$ is simple, and hence $I_{v_0} = C^*(E)$. Thus the result holds in this case.

CASE II: E contains an infinite path $\alpha := e_1 e_2 \dots$ with $s^{-1}(s(e_i)) = \{e_i\}$ for all $i \in \mathbb{N}$.

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \dots$$

For convenience of notation, let $v_i := s(e_i)$, and for each $n \in \mathbb{N}$ define $E^*(v_n) := \{\alpha \in E^* : r(\alpha) = v_n\}$. Let $\mathcal{H}_n := \ell^2(E^*(v_n))$, and for each $\alpha \in E^*(v_n)$ let $\delta_\alpha \in \mathcal{H}_n$ denote the point mass function at α , so that $\{\delta_\alpha : \alpha \in E^*(v_n)\}$ forms an orthonormal basis for \mathcal{H}_n .

For each $n \in \mathbb{N}$ define $V_n : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ to be the isometry with

$$V_n(\delta_\alpha) := \delta_{\alpha e_n}$$

for each $\alpha \in \ell^2(E^*(v_n))$. Also define $\text{Ad}_{V_n} : K(\mathcal{H}_n) \rightarrow K(\mathcal{H}_{n+1})$ by $\text{Ad}_{V_n}(T) := V_n T V_n^*$.

For each $n \in \mathbb{N}$ define $A_n := \overline{\text{span}} \{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)\}$. If we consider the generating set $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)\}$, then for any $\beta, \gamma \in E^*(v_n)$, we have $r(\beta) = r(\gamma) = v_n$, and since E has no cycles the only way for one of β and γ to extend the other is if $\beta = \gamma$. Hence for any $\alpha, \beta, \gamma, \delta \in E^*(v_n)$, we have

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) := \begin{cases} s_\alpha s_\delta^* & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma \end{cases}$$

and $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)\}$ is a set of matrix units indexed by $E^*(v_n)$. Hence there exists an isomorphism $\phi_n : K(\mathcal{H}_n) \rightarrow A_n$ satisfying $\phi_n(\Theta_{\alpha, \beta}) = s_\alpha s_\beta^*$.

Let $\iota_n : A_n \hookrightarrow A_{n+1}$ denote the embedding $s_\alpha s_\beta^* \mapsto s_{\alpha e_n} s_{\beta e_n}^*$. For each $n \in \mathbb{N}$ and for all $\alpha, \beta \in E^*(v_n)$ we have

$$\begin{aligned} \phi_{n+1} \circ \text{Ad}_{V_n}(\Theta_{\delta_\alpha, \delta_\beta}) &= \phi_{n+1}(V_n \Theta_{\delta_\alpha, \delta_\beta} V_n^*) = \phi_{n+1}(\Theta_{V_n \delta_\alpha, V_n \delta_\beta}) \\ &= \phi_n(\Theta_{\delta_{\alpha e_n}, \delta_{\beta e_n}}) = s_{\alpha e_n} s_{\beta e_n}^* = s_\alpha s_{e_n} s_{e_n}^* s_\beta^* = s_\alpha p_{s(e_n)} s_\beta^* = s_\alpha p_{s(e_n)} s_\beta^* \\ &= s_\alpha s_\beta^* = \iota_n \circ \phi_n(\Theta_{\delta_\alpha, \delta_\beta}). \end{aligned}$$

Thus for each $n \in \mathbb{N}$ we have $\phi_{n+1} \circ \text{Ad}_{V_n} = \iota_n \circ \phi_n$ and the diagram

$$\begin{array}{ccc} K(\mathcal{H}_n) & \xrightarrow{\text{Ad}_{V_n}} & K(\mathcal{H}_{n+1}) \\ \phi_n \downarrow & & \downarrow \phi_{n+1} \\ A_n & \xrightarrow{\iota_n} & A_{n+1} \end{array} \quad (4.1.1)$$

commutes. Since the direct limit of the sequence

$$A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} A_3 \xrightarrow{\iota_3} \dots$$

is equal to $\overline{\bigcup_{n=1}^{\infty} A_n}$, and since for all $n \in \mathbb{N}$ the map $\phi_n : A_n \rightarrow A_{n+1}$ is an isomorphism and the diagram in (4.1.1) commutes, we may conclude that

$$\varinjlim K(\mathcal{H}_n) \cong \overline{\bigcup_{n=1}^{\infty} A_n}, \quad (4.1.2)$$

where $\varinjlim K(\mathcal{H}_n)$ is the direct limit of the sequence

$$K(\mathcal{H}_1) \xrightarrow{\text{Ad}_{V_1}} K(\mathcal{H}_2) \xrightarrow{\text{Ad}_{V_2}} K(\mathcal{H}_3) \xrightarrow{\text{Ad}_{V_3}} \dots$$

Next we consider the set of infinite paths E^∞ . For any infinite path $\mu \in E^\infty$, we must have $\mu = \alpha e_i e_{i+1} e_{i+1} \dots$ for some $\alpha \in E^*$ and some $i \in \mathbb{N}$, for otherwise the vertex v_1 could not reach a vertex on μ , contradicting the cofinality of E .

Define $\mathcal{H}_\infty := \ell^2(E^\infty)$ and for $\mu \in E^\infty$ let δ_μ denote the point mass function at μ . Then $\{\delta_\mu : \mu \in E^\infty\}$ is an orthonormal basis for \mathcal{H}_∞ . For each $n \in \mathbb{N}$ define an isometry $W_n : \mathcal{H}_n \rightarrow \mathcal{H}_\infty$ by

$$W_n(\delta_\alpha) = \delta_{\alpha e_n e_{n+1} \dots}.$$

For each $n \in \mathbb{N}$ and for any $\alpha \in E^*(v_n)$ we have

$$W_{n+1}(V_n(\delta_\alpha)) = W_{n+1}(\delta_{\alpha e_n}) = \delta_{\alpha e_n e_{n+1} e_{n+2} \dots} = W_n(\delta_\alpha)$$

and hence $W_{n+1} \circ V_n = W_n$ for all $n \in \mathbb{N}$.

In addition, for any $n \in \mathbb{N}$ we define $\text{Ad}_{W_n} : K(\mathcal{H}_n) \rightarrow K(\mathcal{H}_\infty)$ by $\text{Ad}_{W_n}(T) := W_n T W_n^*$. For any $T \in K(\mathcal{H}_n)$ we have

$$\begin{aligned} \text{Ad}_{W_{n+1}} \circ \text{Ad}_{V_n}(T) &= \text{Ad}_{W_{n+1}}(V_n T V_n^*) = W_{n+1} V_n T V_n^* W_{n+1}^* \\ &= (W_{n+1} V_n) T (W_{n+1} V_n)^* = W_n T W_n^* = \text{Ad}_{W_n}(T) \end{aligned}$$

so that

$$\text{Ad}_{W_{n+1}} \circ \text{Ad}_{V_n} = \text{Ad}_{W_n}$$

for all $n \in \mathbb{N}$. By the universal property of the direct limit there exists a homomorphism $\psi : \varinjlim K(\mathcal{H}_n) \rightarrow K(\mathcal{H}_\infty)$ with $\text{im Ad}_{W_n} \subseteq \text{im } \psi$ for all $n \in \mathbb{N}$, and furthermore, since each Ad_{W_n} is injective for all $n \in \mathbb{N}$, we may conclude that $\psi : \varinjlim K(\mathcal{H}_n) \rightarrow K(\mathcal{H}_\infty)$ is injective.

Moreover, for any $\mu, \nu \in E^\infty$, we may write $\mu = \alpha e_n e_{n+1} \dots$ and $\nu = \beta e_n e_{n+1} \dots$ for some $n \in \mathbb{N}$ and some $\alpha, \beta \in E^*(v_n)$, from which it follows that

$$\begin{aligned} \Theta_{\delta_\mu, \delta_\nu} &= \Theta_{\delta_{\alpha e_n e_{n+1} \dots}, \delta_{\beta e_n e_{n+1} \dots}} = \Theta_{W_n(\delta_\alpha), W_n(\delta_\beta)} = W_n \Theta_{\alpha, \beta} W_n^* \\ &= \text{Ad}_{W_n}(\Theta_{\delta_\alpha, \delta_\beta}) \in \text{im Ad}_{W_n} \subseteq \text{im } \psi. \end{aligned}$$

Hence $\{\Theta_{\delta_\mu, \delta_\nu} : \mu, \nu \in E^\infty\} \subseteq \text{im } \psi$, so that $\text{im } \psi = K(\mathcal{H}_\infty)$, and ψ is surjective.

Hence $\psi : \varinjlim K(\mathcal{H}_n) \rightarrow K(\mathcal{H}_\infty)$ is an isomorphism, and

$$\varinjlim K(\mathcal{H}_n) \cong K(\mathcal{H}_\infty). \quad (4.1.3)$$

Next we let

$$\begin{aligned} H := \{v \in E^0 : p_v = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^* \text{ for some } \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n) \\ \text{satisfying } s(\alpha_i) = s(\beta_i) = v \text{ for all } 1 \leq i \leq k\} \end{aligned}$$

We shall show that H is a saturated and hereditary subset of E .

To show that H is hereditary, we first observe that for each $i \in \mathbb{N}$ we have $p_{v_i} = s_{e_i} s_{e_i}^*$ and that $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$, implying that $v_i \in H$. Thus $\{v_1, v_2, \dots\} \subseteq H$. Next, suppose that $e \in E^1$ and $s(e) \in H$. If $s(e) = v_i$ for some $i \in \mathbb{N}$, then from the previous sentence we have that $r(e) = v_{e+i} \in H$. If $s(e) \neq v_i$ for all $i \in \mathbb{N}$, we use the fact that $s(e) \in H$ to write

$$p_{s(e)} = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^*$$

for some $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n)$ with $s(\alpha_i) = s(\beta_i) = s(e)$ for all $1 \leq i \leq k$, and moreover, the fact that $s(e) \neq v_i$ for all $i \in \mathbb{N}$ implies that α_i and β_i are paths of length at least 1 for each $1 \leq i \leq k$. Consequently,

$$p_{r(e)} = s_e^* s_e = s_e^* p_{s(e)} s_e = s_e^* \left(\sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^* \right) s_e = \sum_{i=1}^k s_e^* s_{\alpha_i} s_{\beta_i}^* s_e. \quad (4.1.4)$$

For each $1 \leq i \leq k$, we may use the fact that α_i and β_i have lengths at least 1 to write $\alpha_i = f_1 \dots f_l$ and $\beta_i = g_1 \dots g_m$ for edges $f_1, \dots, f_l, g_1, \dots, g_m \in E^1$, and then we have

$$s_e^* s_{\alpha_i} s_{\beta_i}^* s_e = \begin{cases} s_{f_2 \dots f_l} s_{g_2 \dots g_m}^* & \text{if } f_1 = e \text{ and } g_1 = e \\ 0 & \text{otherwise.} \end{cases}$$

For the nonzero case above, we see that $s(f_2) = r(e)$ and $s(g_2) = r(e)$, and also $r(f_l) = r(g_m) = r(\alpha_i) = r(\beta_i)$ so that $s_e^* s_{\alpha_i} s_{\beta_i}^* s_e = s_{f_2 \dots f_l} s_{g_2 \dots g_m}^*$ has the properties given in defining the set H . Consequently, (4.1.4) shows that $r(e) \in H$. Hence H is hereditary.

To see that H is saturated, suppose that $v \in E^0$ is a regular vertex with $r(s^{-1}(v)) \subseteq H$. For each $e \in s^{-1}(v)$, the fact that $r(e) \in H$ allows us to write

$$p_{r(e)} = \sum_{i=1}^{k_e} s_{\alpha_i^e} s_{\beta_i^e}^*$$

for some $\alpha_1^e, \dots, \alpha_{k_e}^e, \beta_1^e, \dots, \beta_{k_e}^e \in \bigcup_{n=1}^{\infty} E^*(v_n)$ with $s(\alpha_i^e) = s(\beta_i^e) = v$ for all $1 \leq i \leq k_e$. Hence

$$\begin{aligned} p_v &= \sum_{s(e)=v} s_e s_e^* = \sum_{s(e)=v} s_e p_{r(e)} s_e^* = \sum_{s(e)=v} s_e \left(\sum_{i=1}^{k_e} s_{\alpha_i^e} s_{\beta_i^e}^* \right) s_e^* \\ &= \sum_{s(e)=v} \sum_{i=1}^{k_e} s_e s_{\alpha_i^e} s_{\beta_i^e}^* s_e^* = \sum_{s(e)=v} \sum_{i=1}^{k_e} s_{e\alpha_i^e} s_{e\beta_i^e}^* \end{aligned}$$

and since $s(e\alpha_i^e) = s(e\beta_i^e) = v$ and $r(e\alpha_i^e) = r(e\beta_i^e) \in \bigcup_{n=1}^{\infty} E^*(v_n)$, it follows that $v \in H$. Thus, H is saturated.

Because H is a nonempty saturated hereditary subset, and since $C^*(E)$ is simple by Lemma 1.3, it follows from Theorem 2.3.1 that $H = E^0$. Consequently, for any $v \in E^0$ we have that $p_v = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^*$ for some paths $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n)$ satisfying $s(\alpha_i) = s(\beta_i) = v$ for all $1 \leq i \leq k$. Thus $p_v \in \overline{\bigcup_{n=1}^{\infty} A_n}$.

Likewise, for any $e \in E^1$, we have $r(e) \in H$ and $p_{r(e)} = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^*$ for some $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n)$ satisfying $s(\alpha_i) = s(\beta_i) = r(e)$ for all $1 \leq i \leq k$. Thus $s_e = s_e p_{r(e)} = \sum_{i=1}^k s_e s_{\alpha_i} s_{\beta_i}^* = \sum_{i=1}^k s_e s_{\alpha_i} s_{\beta_i}^* \in \overline{\bigcup_{n=1}^{\infty} A_n}$.

Hence $\{p_v, s_e : v \in E^0, e \in E^1\} \subseteq \overline{\bigcup_{n=1}^{\infty} A_n}$, and it follows that

$$C^*(E) = \overline{\bigcup_{n=1}^{\infty} A_n}. \quad (4.1.5)$$

Combining (4.1.2), (4.1.3), and (4.1.5) gives the desired result. \square

Theorem 4.1.2. *If E is a row-countable directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence, then $C^*(E) \cong K(\mathcal{H})$ for some Hilbert space \mathcal{H} .*

Proof. It follows from Lemma 1.3 that $C^*(E)$ is simple, and since E is row-countable, Proposition 3.3 then implies that $C^*(E)$ is AF. The result then follows from Theorem 4.1.1. \square

4.2 Conclusion and Future Directions

We know that a graph C^* -algebra with a unique irreducible representation is simple. We have shown that if such a graph C^* -algebra is AF, then it is isomorphic to $K(\mathcal{H})$ for some Hilbert space \mathcal{H} . Furthermore, it is known that simple graph C^* -algebras exhibit the following dichotomy.

Theorem 4.2.1. [12, Corollary 3.10] *Let E be a graph. If $C^*(E)$ is simple, then either*

- (1) $C^*(E)$ is an AF algebra if E contains no cycle, or
- (2) $C^*(E)$ is purely infinite if E contains a cycle.

Remark 4.2.1. The result, as stated in the cited article, relies on additional assumptions. However, the proof extends to the general case stated above.

Definition 4.2.1 (Purely Infinite C^* -algebra). [10, Definition 4.1] *A C^* -algebra A is purely infinite if there are no characters on A and if for every pair of positive elements a, b in A , $a \leq b$ if and only if a belongs to the closed two-sided ideal \overline{AbA} generated by b .*

Consequently, obtaining a complete answer to Naimark's problem for all graph C^* -algebras now requires an examination of purely infinite graph C^* -algebras.

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