CHARACTERIZATIONS OF THE ADJUGATE MAP

A Thesis

Presented to

The Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Bachelor of Science

By

Timor Sever

August, 1988

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ABSTRACT

In this thesis, we discuss properties and characterizations of the adjugate map; that is the function that assigns to a complex matrix A the matrix consisting of the cofactors of A. The characterizations we develop arise from basic properties of the adjugate map. One important property of the adjugate function is that it is anti-product preserving. We present certain sets of abstract conditions and we show that these conditions define the adjugate function. Each nonsingular matrix Z can be expressed as a finite product of elementary matrices. Hence, using the anti-commutativity of the adjugate map we can express the image of an arbitrary nonsingular matrix in terms of the images of elementary matrices. We use this fact to verify whether the sets of conditions considered define the adjugate function. We attempt to minimize the number of conditions necessary to characterize the adjugate map.

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§1. INTRODUCTION

We require some preliminary notation and definitions. The field of complex numbers is denoted by $\mathbb{C}$. The field of real numbers is denoted by $\Re$. The algebra of all n x n complex (real) matrices is denoted by $M_n(\mathbb{C})$ ($M_n(\Re)$). The sets of natural numbers, integers, and rational numbers are denoted by **N**, **Z**, and **Q** respectively. The symbol ♦ will stand for Q.E.D. and denote the end of a proof. Throughout this paper, we suppose $n \in N$ with $n \geq 2$, and should the situation arise $0^0 = \frac{0}{0} = 1$.

__Definition 1.1__ : If $A \in M_n(\mathbb{C})$, the adjugate of A, denoted adj A, is (A_{ij}) where $A_{ij} = (-1)^{i+j}\det A(j|i)$ $i,j = 1,2,\ldots,n$ and $A(j|i)$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the j^{th} row and the i^{th} column of A. For n = 1, adj A = (1). We observe that some linear algebra texts refer to the adjugate matrix as the classical adjoint.

__Definition 1.2__ : If $A \in M_n(\mathbb{C})$ and $B \in M_p(\mathbb{C})$, then the direct sum of A and B, denoted by $A \oplus B$, is the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{n+p}(\mathbb{C})$. Throughout this paper, should the situation occur, we define $A \oplus I_0 = A \oplus 0_0 = A + 0_n = A$.

__Definition 1.3__ : If $A \in M_n(\mathbb{C})$, then an elementary row operation on A is one of the following three types :

I. The interchange of two rows of A ;

II. The multiplication of one row of A by a nonzero scalar ;

III. The replacement of the r^{th} row of A by row r plus c times row s, where $c \in \mathbb{C}$, and $r \neq s$.

Definition 1.4 : A matrix $E \in M_n(\mathbb{C})$ is said to be an elementary matrix if E can be obtained from I_n by means of a single elementary row operation.

Definition 1.5 : Let F be a field. We will denote by F[x], the ring of polynomials over F in a single indeterminant x.

Definition 1.6 : Let $A \in M_n(\mathbb{C})$. A characteristic value of A in $\mathbb{C}$ is a scalar $c \in \mathbb{C}$ such that the matrix $(c \cdot I_n - A)$ is singular. The polynomial $f = \det(x \cdot I_n - A)$ is said to the characteristic polynomial of A.

Definition 1.7 : An ideal in $\mathbb{C}[x]$ is a vector subspace M of $\mathbb{C}[x]$ such that $f \cdot g \in M$ whenever $f \in \mathbb{C}[x]$ and $g \in M$.

Definition 1.8 : Let $A \in M_n(\mathbb{C})$. The minimal polynomial for A is the unique monic generator of the ideal of all polynomials over F that annihilate A.

Definition 1.9 : Let X,Y be subsets of $\mathbb{C}$. The sets X and Y are said to be mutually separated if $X \cap Y = \varnothing$ and neither contains a limit point of the other. A set X is said to connected if it is not the union of two mutually separated sets. A set X is said to be dense in $\mathbb{C}$ if $p \in \mathbb{C}$, implies that $p \in X$ or p is a limit point of X.

Definition 1.10 : If $A \in M_n(\mathbb{C})$, then we define the norm of A, denoted $| A |$, by

$$| A | = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

Definition 1.11 : A function $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is said to be continuous on $M_n(\mathbb{C})$ if for any $A \in M_n(\mathbb{C})$ and $\varepsilon > 0$ in $\mathfrak{R}$, there exists a $\delta > 0$ in $\mathfrak{R}$ such that if $B \in M_n(\mathbb{C})$ and $| A-B | < \delta$, then $| \psi(A) - \psi(B) | < \varepsilon$.

Defintion 1.12 : Let $\psi : S \to \mathbb{C}$, where S is an open subset of $\mathbb{C}$. Then ψ is said to be analytic on S if ψ is complex differentiable at each $z \in S$.

Definition 1.13 : Let $f \in \mathbb{C}[x]$, where $f = c_0 + c_1x + c_2x^2 + \ldots + c_kx^k$ and $k \in N$. If $A \in M_n(\mathbb{C})$, then $f(A) \in M_n(\mathbb{C})$, is defined by the relation

$$f(A) = c_0I_n + c_1A + c_2A^2 + \ldots + c_kA^k .$$

Such a matrix f(A) is said to be a matrix poylnomial or a polynomial in A.

Theorem 1.1 : If A is nonsingular in $M_n(\mathbb{C})$, then A is a product of elementary matrices.

Theorem 1.2 : If A is type I elementary in $M_n(\mathbb{C})$, then A is the product of type II and type III elementary matrices. The factorization is not unique.

Proof : It is sufficient to observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The general case follows using direct sums. ♦

The following is a well known theorem concerning adjugate matrices :

Theorem 1.3 : If $A \in M_n(\mathbb{C})$, then $A(\text{adj } A) = (\text{adj } A)A = (\det A)I_n$.

We observe that if A is nonsingular in $M_n(\mathbb{C})$, then we obtain (from Theorem 1.3) $\text{adj } A = (\det A)A^{-1}$. Now we have an explicit formula for the adjugate of an arbitrary nonsingular matrix in terms of its inverse. However, we observe that if A is elementary in $M_n(\mathbb{C})$, then we may obtain a simpler expression for adj A. Hence, we have

Theorem 1.4 : If E is an elementary matrix in $M_n(\mathbb{C})$, then

$$\text{adj } E = (1 + \det E)I_n - E.$$

Proof : Suppose that $\lambda \in \mathbb{C}\backslash\{0\}$, and $E \in M_n(\mathbb{C})$ is elementary. Suppose first that E is type I elementary. Hence,

$$\text{adj } E = (\det E)E^{-1} = 0 \cdot I_n - E = (1 - 1)I_n - E = (1 + \det E)I_n - E.$$

Next, suppose that E is type II elementary. Without loss of generality we can suppose that $E = \text{diag}(\lambda,1,\ldots,1)$. Hence,

$$\begin{aligned}\text{adj } E = (\det E)E^{-1} &= \lambda \cdot \text{diag}(\lambda^{-1},1,\ldots,1) = \text{diag}(1,\lambda,\ldots,\lambda) \\ &= \text{diag}(1 + \lambda - \lambda, 1 + \lambda - 1,\ldots, 1 + \lambda - 1) \\ &= (1 + \lambda)I_n - \text{diag}(\lambda,1,\ldots,1) = (1 + \lambda)I_n - E \\ &= (1 + \det E)I_n - E.\end{aligned}$$

Suppose that E is type III elementary. Without loss of generality we can assume that $E = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. Whence, $E^{-1} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$ and

$$\operatorname{adj} E = (\det E)E^{-1} = E^{-1} = \begin{pmatrix} 2-1 & -\lambda \\ 0 & 2-1 \end{pmatrix} \oplus (2-1)I_{n-2}$$
$$= \operatorname{diag}(2,\ldots,2) - E = (1 + \det E)I_n - E.$$

Therefore if E is elementary in $M_n(\mathbb{C})$, then $\operatorname{adj} E = (1 + \det E)I_n - E$. ♦

Theorem 1.5 : (Cayley - Hamilton) Let $A \in M_n(\mathbb{C})$. If f is the characteristic polynomial for A, then $f(A) = 0_n$; in other words, the minimal polynomial divides the characterstic polynomial for A.

Theorem 1.6 : If $f, d \in \mathbb{C}[x]$ and d is different from 0, then there exist $q,r \in \mathbb{C}[x]$ such that $f = dq + r$ where either $r = 0$ or $\deg r < \deg d$.
The polynomials q, r are unique.

Theorem 1.7: If A is elementary in $M_n(\mathbb{C})\backslash\{I_n\}$, then A has minimum polynomial $m_A(x) = (x - 1)(x - \det A)$.

Proof : Suppose A is elementary in $M_n(\mathbb{C})\backslash\{I_n\}$. Suppose first that A is type I elementary. Without loss of generality we may suppose that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}$.
Hence, the characteristic polynomial $f_A(x) = -(x-1)^{n-2}(-x^2+1) = (x-1)^{n-2}(x^2-1)$ $= (x-1)^{n-1}(x+1)$. Whence, $m_A(x) = (x-1)(x+1)$ since

$$(A - I_n)(A + I_n) = A^2 - I_n = I_n - I_n = 0_n \text{ and } (A - I_n) \neq 0_n,\ (A + I_n) \neq 0_n.$$

Thus, $m_A(x) = (x - 1)(x - \det A)$. Next, suppose that A is type II elementary, then without loss of generality we suppose that $A = \mathrm{diag}(k,1,\ldots,1)$, where $k \in \mathbb{C}\setminus\{0,1\}$. Whence, $f_A(x) = (x-1)^{n-1}(x-k) = (x-1)^{n-1}(x - \det A)$. We observe that

$$(A - I_n)(A - kI_n) = \mathrm{diag}(k-1,0,\ldots,0)\cdot\mathrm{diag}(0,1-k,\ldots 1-k) = 0_n.$$

Thus, $m_A(x) = (x-1)(x - \det A)$. Suppose that A is type III elementary. Without loss of generality we suppose that $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$, where $k \in \mathbb{C}\setminus\{0\}$. Hence, $f_A(x) = (x-1)^n$. Moreover, $m_A(x) = (x-1)(x - \det A)$ since $\det A = 1$ and

$$A - I_n \neq 0_n \text{ and } (A - I_n)^2 = [\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2}]^2 = 0_n.$$

Therefore, if A is elementary in $M_n(\mathbb{C})\setminus\{I_n\}$,then A has minimum polynomial $m_A(x) = (x-1)(x - \det A)$. ♦

The following corollary follows immediately from Theorem 1.7.

<u>Corollary 1.7.1</u> : If E is an elementary matrix in $M_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$ is a polynomial in E, then there exist $\alpha,\beta \in \mathbb{C}$ such that $A = \alpha I_n + \beta E$.

The following theorem is a well-known fact concerning nonsingular matrices.

<u>Theorem 1.8</u> : The set of nonsingular matrices is dense in $M_n(\mathbb{C})$.

§2. PROPERTIES OF THE ADJUGATE MAP

The following theorem lists well-known properties of the adjugate map and can be found in most any linear algebra text.

Theorem 2.1 : In the formulas which follow, should the situation arise, $0^0 = \frac{0}{0} = 1$. Here $A \in M_n(\mathbb{C})$.

a) $\det(\operatorname{adj} A) = (\det A)^{n-1}$;

b) $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2} A$;

c) if P is nonsingular in $M_n(\mathbb{C})$, then $\operatorname{adj}(P^{-1}AP) = P^{-1}(\operatorname{adj} A)P$;

d) $\lambda \in \mathbb{C}$, $\operatorname{adj}(\lambda A) = \lambda^{n-1} \operatorname{adj} A$;

e) adj A is a polynomial in A ;

f) if A is real, then adj A is real ;

g) adj A is an analytic function of the entries of A ;

h) $\operatorname{adj}(AB) = (\operatorname{adj} B)(\operatorname{adj} A)$ for $B \in M_n(\mathbb{C})$.

The following theorem is a well-known result in linear algebra, in fact it appears as an exercise in some linear algebra texts. In [4], Sinkhorn extends this result to show that the adjugate map is onto the n x n matrices of rank n,1, and 0.

Theorem 2.2 : a) $\operatorname{rank}(A) = n \Rightarrow \operatorname{rank}(\operatorname{adj} A) = n$;

b) $\operatorname{rank}(A) = n-1 \Rightarrow \operatorname{rank}(\operatorname{adj} A) = 1$;

c) $\operatorname{rank}(A) < n-1 \Rightarrow \operatorname{rank}(\operatorname{adj} A) = 0$.

§3. CHARACTERIZATION OF THE ADJUGATE FUNCTION

As seen in Theorem 1.4, adj E = (1 + det E)$\cdot I_n$ - E when E is an elementary matrix in $M_n(\mathbb{C})$. It is then possible to calculate the adjugate of a nonsingular matrix A by calculating the adjugate of the elementary matrices which compose a factorization of A. The calculation of the adjugate of a nonsingular matrix in this manner (depending on the matrix) is not necessarily efficient. However, by observing some fundamental properties of the adjugate function, the notion of factoring a nonsingular matrix into a product of elementary matrices becomes important. In the following theorem, we investigate a continuous function from $M_n(\mathbb{C})$ to $M_n(\mathbb{C})$ which is a multiplicative anti-homomorphism and is defined only in terms of the images of elementary matrices.

<u>Theorem 3.1</u> : If $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is such that

a) $\varphi(AB) = \varphi(B)\varphi(A), \ \forall \ A,B \in M_n(\mathbb{C})$;

b) $\varphi(E) = (1 + \det E)I_n - E$, when $E \in M_n(\mathbb{C})$ is elementary ;

c) φ is a continuous function of A,

then $\varphi(A) = \operatorname{adj} A, \ \forall A \in M_n(\mathbb{C})$.

<u>Proof</u> : Suppose that $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is such that φ is continuous, $\varphi(AB) = \varphi(B)\varphi(A), \ \forall \ A,B \in M_n(\mathbb{C})$, and if E is elementary in $M_n(\mathbb{C})$, then $\varphi(E) = (1 + \det E)I_n - E$. Suppose A is nonsingular in $M_n(\mathbb{C})$. Hence, $A = E_1E_2\cdots E_k$ for some $k \in \mathbb{N}$, where $E_i \in M_n(\mathbb{C})$ is elementary for $i = 1,2,\ldots,k$. Whence,

$$\varphi(A) = \varphi(E_1E_2\cdots E_k) = \varphi(E_k)\varphi(E_{k-1})\cdots\varphi(E_1)$$

$$= [(1 + \det E_k)I_n - E_k][(1 + \det E_{k-1})I_n - E_{k-1}]\cdots[(1 + \det E_1)I_n - E_1].$$

Since $(1 + \det E_i)I_n - E_i = \operatorname{adj} E_i$ for $1 \le i \le k$,

$$\varphi(A) = (\operatorname{adj} E_k)(\operatorname{adj} E_{k-1})\cdots(\operatorname{adj} E_1) = \operatorname{adj}(E_1E_2\cdots E_k) = \operatorname{adj} A.$$

Since the nonsingular matrices are dense in $M_n(\mathbb{C})$ and φ is continuous, we observe that, $\forall A \in M_n(\mathbb{C})$, $\varphi(A) = \operatorname{adj} A$. ♦

In Theorem 3.1, we obtain a characterization of the adjugate function that explicitly relies on the images of elementary matrices under the adjugate function. However, one may note that this characterization gives little insight as to which properties of the adjugate map are essential in obtaining a characterization. We endeavor to find a set of conditions, independent of Theorem 1.4, which defines the adjugate function. Throughout the remainder of this paper, we suppose $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is continuous and satisfies the following properties :

1) $\operatorname{rank}(A) = n \Rightarrow \operatorname{rank}(\varphi(A)) = n$, where $A \in M_n(\mathbb{C})$;
2) $\varphi(AB) = \varphi(B)\varphi(A)$, $\forall A,B \in M_n(\mathbb{C})$;
3) $\varphi(\lambda A) = \lambda^{n-1}\varphi(A)$, $\forall A \in M_n(\mathbb{C})$, $\lambda \in \mathbb{C}$;
4) $\varphi(\varphi(A)) = \mu_A \cdot A$, where $\mu_A \in \mathbb{C}$ depends continuously on $A \in M_n(\mathbb{C})$.

From these conditions, we immediately obtain certain properties concerning φ.

<u>Theorem 3.2</u> : a) $\varphi(I_n) = I_n$;

b) $\mu_{\lambda I_n} = \lambda^{n(n-2)}$, $\lambda \in \mathbb{C}\backslash\{0\}$;

c) $\varphi(0_n) = 0_n$;

d) $\mu_{AB} = \mu_A \cdot \mu_B$, $\forall$ A,B $\in M_n(\mathbb{C})$;

e) if P is nonsingular in $M_n(\mathbb{C})$, then $\mu_{P^{-1}} = \mu_P^{-1}$;

f) if E is type I elementary in $M_n(\mathbb{C})$, then $(\mu_E)^2 = 1$.

__Proof__: Suppose that A,B,P $\in M_n(\mathbb{C})$ with P nonsingular, $\lambda \in \mathbb{C}\backslash\{0\}$.

a) $\varphi(I_n) = \varphi(I_n^2) = \varphi(I_n)^2$; since $\varphi(I_n)$ is nonsingular, $\varphi(I_n) = I_n$.

b) $\mu_{\lambda I_n} \cdot (\lambda I_n) = \varphi(\varphi(\lambda I_n)) = \varphi(\lambda^{n-1} I_n) = \lambda^{(n-1)^2} I_n$.

Hence, if $\lambda \neq 0$, then $\mu_{\lambda I_n} = \lambda^{n(n-2)}$.

c) $\varphi(0_n) = \varphi(0 \cdot I_n) = 0^{n-1} I_n = 0_n$.

d) $\mu_{AB} \cdot AB = \varphi(\varphi(AB)) = \varphi(\varphi(B)\varphi(A)) = \mu_A \mu_B \cdot AB$. Thus, if each of A,B is nonsingular, then $\mu_{AB} = \mu_A \mu_B$. By the continuity of μ,

$\mu_{AB} = \mu_A \mu_B$, $\forall$ A,B $\in M_n(\mathbb{C})$.

e) $1 = 1^{n(n-2)} = \mu_{I_n} = \mu_{PP^{-1}} = \mu_P \mu_{P^{-1}}$. Hence, $\mu_{P^{-1}} = \mu_P^{-1}$.

f) Suppose that E is type I elementary in $M_n(\mathbb{C})$. Hence, $(\mu_E)^2 = \mu_{E^2} = \mu_{I_n} = 1$. ♦

From Theorem 2.1, we observe that adj(adj A) = (det A $)^{n-2}$A. In addition, we note that from property (4) $\varphi(\varphi(A)) = \mu_A \cdot A$, where $\mu_A \in \mathbb{C}$ depends continuously on A. Since we intend φ = adj, we note that μ_A should equal $(\det A)^{n-2}$.

__Theorem 3.3__: $\mu_A = (\det A)^{n-2}$, $\forall A \in M_n(\mathbb{C})$.

Proof: Consider $D = \mathrm{diag}(\lambda_1\lambda_2\cdots\lambda_n, 1, \ldots, 1) \in M_n(\mathbb{C})$, where $\lambda_i \in \mathbb{C}$ for $i = 1,2,\ldots,n$. There exist type I elementary matrices $E_i \in M_n(\mathbb{C})$ $i = 1,2,\ldots,n-1$ and $D_j = \mathrm{diag}(1,\ldots,1,\lambda_j,1,\ldots,1) \in M_n(\mathbb{C})$ $j = 1,2,\ldots,n$ such that

$$D = D_1E_1D_2E_1E_2D_3E_2E_3\cdots E_{n-2}E_{n-1}D_nE_{n-1}.$$

Thus $\mu_D = \mu_{D_1}(\mu_{E_1})^2\mu_{D_2}(\mu_{E_2})^2\cdots(\mu_{E_{n-1}})^2\mu_{D_n} = \mu_{D_1D_2\cdots D_n} = \mu_{D_0}$, where $D_0 = D_1D_2\cdots D_n = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$. In particular, if D_0 is replaced by $D_0' = \lambda I_n$, then we know that

$$\mu_{D_0'} = \mu_{\lambda I_n} = \lambda^{n(n-2)} \quad (\text{ which equals 1, if } n = 2).$$

For any set $\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$ with $\lambda_i \in \mathbb{C}$, for $i = 1,2,\ldots,n$ by using $\lambda = (\lambda_1\lambda_2\cdots\lambda_n)^{1/n}$ in D_0',

$$\mu_{D_0} = \mu_D = \mu_{D_0'} = (\lambda_1\lambda_2\cdots\lambda_n)^{n-2} = (\det D_0)^{n-2}.$$

If $A \in M_n(\mathbb{C})$ has distinct eigenvalues, then there is a nonsingular $P \in M_n(\mathbb{C})$ such that $P^{-1}AP$ is diagonal. Thus

$$\mu_A = \mu_{P^{-1}}\cdot\mu_A\cdot\mu_P = \mu_{P^{-1}AP} = (\det P^{-1}AP)^{n-2} = (\det A)^{n-2}.$$

According to [1, p. 199] the set of matrices in $M_n(\mathbb{C})$ with distinct eigenvalues is dense in $M_n(\mathbb{C})$.

Therefore, by continuity of μ, if $A \in M_n(\mathbb{C})$, then $\mu_A = (\det A)^{n-2}$. ♦

Now, using the results in Theorem 3.2 and Theorem 3.3 we observe

Theorem 3.4: If $A \in M_n(\mathbb{C})$, then $\det \varphi(A) = (\det A)^{n-1}$.

Proof: Suppose that A is nonsingular in $M_n(\mathbb{C})$.

Let $B = \alpha\cdot(\det A)^{\frac{-(n-2)}{n-1}}\,\varphi(A)$, where $\alpha \in \mathbb{C}$ is such that $\alpha^{n-1} = 1$. Hence,

$$(3.1) \qquad \varphi(B) = (\det A)^{-(n-2)}\varphi(\varphi(A)) = (\det A)^{-(n-2)}(\det A)^{n-2}A = A.$$

Thus, $\varphi(A) = \varphi(\varphi(B)) = (\det B)^{n-2}B$ and we observe that

$$(\det A)^{n-2}A = (\det B)^{(n-2)(n-1)}\varphi(B) = (\det B)^{(n-2)(n-1)}A.$$

Whence, $(\det A)^{n-2} = (\det B)^{(n-2)(n-1)}$. For some $\alpha_A \in \mathbb{C}$ with $\alpha_A{}^{n-2} = 1$, $\alpha_A\cdot\det A = (\det B)^{n-1}$. Moreover,

$$\begin{aligned}\det \varphi(A) &= (\det B)^{n(n-2)}\cdot(\det B) = (\det B)^{(n-1)^2}\\ &= \alpha_A{}^{n-1}\cdot(\det A)^{n-1} = \alpha_A\cdot(\det A)^{n-1}.\end{aligned}$$

We observe that $\alpha_A = \dfrac{\det \varphi(A)}{(\det A)^{n-1}}$ is a continuous function of A. Since the possible values of α_A are discrete and the set of all nonsingular $A \in M_n(\mathbb{C})$ is connected, α_A is a constant β such that $\beta^{n-2} = 1$. Hence,

$$\beta = \frac{\det \varphi(I_n)}{(\det I_n)^{n-1}} = \frac{\det I_n}{1} = 1.$$

We see that in equation (3.1) that the set of nonsingular matrices are contained in $\varphi(M_n(\mathbb{C}))$. Hence, $\{\varphi(A) \mid A \text{ is nonsingular in } M_n(\mathbb{C})\}$ is a dense subset of $M_n(\mathbb{C})$. Therefore, by continuity of det, $\det \varphi(A) = (\det A)^{n-1}$, $\forall A \in M_n(\mathbb{C})$. ♦

The following lemmas are well-known theorems in complex analysis and are supplied for the readers convenience.

<u>Lemma 3.5</u> : If $\psi : \Re \to \Re$ is continuous and $\psi(x + y) = \psi(x) + \psi(y)$, $\forall\, x,y \in \Re$, then for some $C \in \Re$, $\psi(x) = Cx$.

<u>Proof</u> : Suppose $\psi : \Re \to \Re$ is continuous and $\psi(x + y) = \psi(x) + \psi(y)$, $\forall\, x,y \in \Re$. Hence, $\psi(0 + 0) = \psi(0) + \psi(0) \Rightarrow \psi(0) = 0$. Thus if $x \in \Re$, $\psi(0) = \psi(x - x) = \psi(x) + \psi(-x)$. Whence, $\psi(-x) = -\psi(x)$. If $b \in \mathbf{Z}$ with $b > 0$, then $\psi(bx) = \psi(x) + \ldots + \psi(x)$ (b times) $= b\cdot\psi(x)$. If $s \in \mathbf{Z}$ with $s > 0$, then $\psi(x) = \psi(s\frac{x}{s}) = s\cdot\psi(\frac{x}{s})$ and thus $\psi(\frac{x}{s}) = \frac{1}{s}\psi(x)$. Whence, $\psi(\frac{b}{s}x) = b\cdot\psi(\frac{x}{s}) = \frac{b}{s}\cdot\psi(x)$. Moreover, $\psi(-\frac{b}{s}x) = -\psi(\frac{b}{s}x) = -\frac{b}{s}\cdot\psi(x)$. Thus, if $r \in \mathbf{Q}$, then $\psi(rx) = r\cdot\psi(x)$. Since $\mathbf{Q}$ is dense in $\Re$, $\psi(rx) = r\cdot\psi(x)$ $\forall\, r \in \Re$. Therefore, $\forall\, r \in \Re$, $\psi(x) = \psi(x\cdot 1) = x\cdot\psi(1) = \psi(1)\cdot x$, so take $C = \psi(1)$. ♦

<u>Lemma 3.6</u> : If $\psi : \mathbb{C} \to \mathbb{C}$ is such that

a) $\psi(x + y) = \psi(x) + \psi(y)$ $\forall\, x,y \in \mathbb{C}$;

b) if $z \in \Re$, then $\psi(z) \in \Re$;

c) ψ is analytic ,

then there exists a $C \in \Re$ such that $\psi(z) = Cz$, $\forall\, z \in \mathbb{C}$.

<u>Proof</u> : Suppose that $\psi : \mathbb{C} \to \mathbb{C}$ is analytic such that if $x,y \in \mathbb{C}$, then $\psi(x + y) =$

$\psi(x) + \psi(y)$, and if $z \in \Re$, then $\psi(z) \in \Re$. Let $z = x + iy \in \mathbb{C}$, where $x, y \in \Re$ and $i^2 = -1$. Thus, $\psi(x + iy) = \psi(x) + \psi(iy) = \mathrm{Re}\,\psi(x) + i \cdot \mathrm{Im}\,\psi(x) + \mathrm{Re}\,\psi(iy) + i \cdot \mathrm{Im}\,\psi(iy)$. From the additivity and continuity of each of $\mathrm{Re}\,\psi(x)$, $\mathrm{Re}\,\psi(iy)$, $\mathrm{Im}\,\psi(x)$, and $\mathrm{Im}\,\psi(iy)$ there are real constants C, H ,G ,K such that $\mathrm{Re}\,\psi(x) = Cx$, $\mathrm{Re}\,\psi(iy) = Hy$, $\mathrm{Im}\,\psi(x) = Gx$, and $\mathrm{Im}\,\psi(iy) = Ky$. Whence $\psi(x + iy) = Cx + iGx + Hy + iKy$. Let $y = 0$. Since $x \in \Re$, $\psi(x) \in \Re$. Thus $G = 0$ and $\psi(x + iy) = (Cx + Hy) + iKy$. Since ψ is analytic, $\frac{\partial}{\partial x}(Cx + Hy) = \frac{\partial}{\partial y}(Ky)$ and $\frac{\partial}{\partial x}(Ky) = -\frac{\partial}{\partial y}(Cx + Hy)$, i.e. $C = K$ and $H = 0$. Therefore $\psi(z) = \psi(x + iy) = Cx + iCy = Cz$, $\forall z \in \mathbb{C}$. ♦

In the following example, we observe that the assumptions made concerning φ do not imply that $\varphi(A) = \mathrm{adj}\,A$, $\forall A \in M_n(\mathbb{C})$.

<u>Example 3.1</u> : Suppose, in addition to properties (1) - (4), that

$$\varphi(A) = (\det A)^{\frac{n-2}{n}} A^T,$$

where A^T denotes the transpose of A.

We observe that φ is continuous and each assumption holds :

(1) If $\mathrm{rank}(A) = n$, then $\mathrm{rank}(\varphi(A)) = n$.

(2) $\varphi(AB) = (\det AB)^{\frac{n-2}{n}} (AB)^T = (\det B)^{\frac{n-2}{n}} B^T \cdot (\det A)^{\frac{n-2}{n}} A^T = \varphi(B)\varphi(A)$.

(3) $\varphi(\lambda A) = (\det \lambda A)^{\frac{n-2}{n}} (\lambda A)^T = \lambda^{n-2} (\det A)^{\frac{n-2}{n}} \lambda A^T = \lambda^{n-1} (\det A)^{\frac{n-2}{n}} A^T$
$= \lambda^{n-1} \varphi(A)$.

(4) $\varphi(\varphi(A)) = \varphi((\det A)^{\frac{n-2}{n}} A^T) = (\det A^T)^{\frac{(n-2)(n-1)}{n}} \varphi(A^T)$

$$= (\det A)^{\frac{(n-2)(n-1)}{n}} (\det A)^{\frac{n-2}{n}} (A^T)^T = (\det A)^{n-2} A.$$

In addition, we observe that

$$\det \varphi(A) = \det (\varphi(A) = (\det A)^{\frac{n-2}{n}} A^T) = (\det A)^{n-2} (\det A^T) = (\det A)^{n-1}.$$

The function φ is not the adj map since if $A \in M_n(\mathbb{C})$ with rank(A) = n-1 and n > 2, then rank(φ(A)) = 0 $\neq$ 1 = rank(adj A). If n = 2, $\varphi(A) = A^T \neq \operatorname{adj} A$ unless A has the form $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Example 3.1 indicates that properties (1) - (4) do not imply that φ is the adjugate map. Hence, for the remainder of this paper we make the following additional assumptions concerning φ :

5) φ(A) is a polynomial in A ;
6) φ(A) is real when A is real ;
7) φ(A) is an analytic function of the entries of A (this replaces the continuity assumption).

In Theorem 3.1, we noted that the images of elementary matrices under the function φ played a major role in the characterization of the adjugate map. Hence, if we can determine that φ(E) = adj E when E is elementary then we may obtain another characterization of the adjugate map. Moreover, our task becomes simpler when we observe that type I elementary matrices are the product of type II and type III elementary matrices (Theorem 1.2). Thus, we have

Theorem 3.7 : If E is type II elementary in $M_n(\mathbb{C})$, then φ(E) = adj E.

<u>**Proof**</u> : Suppose E is type II elementary in $M_n(\mathbb{C})$. We suppose without loss of generality that $E = E_\lambda = (\lambda) \oplus I_{n-1}$, where $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\varphi(E_\lambda) = a_\lambda I_n + b_\lambda E_\lambda = (a_\lambda + b_\lambda \lambda) \oplus (a_\lambda + b_\lambda)I_{n-1}$ for some $a_\lambda, b_\lambda \in \mathbb{C}$ which may depend on λ. Since $\det \varphi(E_\lambda) = (\det E_\lambda)^{n-1}$, $(a_\lambda + b_\lambda)^{n-1} \cdot (a_\lambda + b_\lambda \lambda) = \lambda^{n-1}$. We observe that $a_\lambda + b_\lambda \neq 0$ since $\varphi(E_\lambda)$ is nonsingular and hence, we define $f(\lambda) = \dfrac{\lambda}{a_\lambda + b_\lambda}$.

Then $a_\lambda + b_\lambda \lambda = f(\lambda)^{n-1}$ and $a_\lambda + b_\lambda = \dfrac{\lambda}{f(\lambda)}$. Whence,

$$\varphi(E_\lambda) = \begin{pmatrix} f(\lambda)^{n-1} & 0 \\ 0 & \frac{\lambda}{f(\lambda)} I_{n-1} \end{pmatrix}, \ \lambda \neq 0.$$

For $\lambda, \mu \in \mathbb{C}\backslash\{0\}$, since $\varphi(E_{\mu\lambda}) = \varphi(E_\lambda E_\mu) = \varphi(E_\mu)\varphi(E_\lambda)$, we see that in particular $f(\mu\lambda) = f(\mu)f(\lambda)$. Let $\mu = e^w$, $\lambda = e^z$. Hence, $\ln f(e^{w+z}) = \ln f(e^w) + \ln f(e^z)$. Then, for some $C \in \mathbb{C}$, $\ln f(e^w) = Cw$, i.e. $f(e^w) = e^{Cw}$. Whence,

$$\varphi(E_{e^w}) = \begin{pmatrix} e^{C(n-1)w} & 0 \\ 0 & e^{(1-C)w} I_{n-1} \end{pmatrix}$$

$$= e^{(1-C)w} \cdot \begin{pmatrix} e^{(Cn-1)w} & 0 \\ 0 & I_{n-1} \end{pmatrix}; \text{ and}$$

$$\varphi(\varphi(E_\mu)) = e^{(1-C)(n-1)w} \cdot \begin{pmatrix} e^{C(Cn-1)(n-1)w} & 0 \\ 0 & e^{(1-C)(Cn-1)w} I_{n-1} \end{pmatrix}.$$

Moreover, $\varphi(\varphi(E_\mu)) = (\det E_\mu)^{n-2} E_\mu = \mu^{n-2} \cdot \begin{pmatrix} \mu & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Whence,

$$C(Cn-1)(n-1) + (1-C)(n-1) = n - 1 \text{ and}$$
$$(1-C)(n-1) + (1-C)(Cn-1) = n - 2.$$

Thus, $C^2 - 2C = 0$ and $nC^2 - 2C = 0$, respectively. The common solution is $C = 0$. Thus,

$$\varphi(E_{e^w}) = \begin{pmatrix} 1 & 0 \\ 0 & e^w I_{n-1} \end{pmatrix} \text{ or since } \mu = e^w, \quad \varphi(E_\mu) = \begin{pmatrix} 1 & 0 \\ 0 & \mu I_{n-1} \end{pmatrix}.$$

Thus, if $\lambda \in \mathbb{C}\backslash\{0\}$, then $\varphi(E_\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda I_{n-1} \end{pmatrix} = \operatorname{adj} E_\lambda$. Therefore, if E is type II elementary in $M_n(\mathbb{C})$, then $\varphi(E) = \operatorname{adj} E$. ♦

Theorem 3.8 : If $n > 2$ and E is type III elementary in $M_n(\mathbb{C})$, then $\varphi(E) = \operatorname{adj} E$.

Proof : Suppose that E is type III elementary in $M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. We suppose without loss of generality that $E = E_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. There exist $c_\lambda, d_\lambda \in \mathbb{C}$ which may depend on λ such that

$$\varphi(E_\lambda) = c_\lambda I_n + d_\lambda E_\lambda = \begin{pmatrix} c_\lambda + d_\lambda & d_\lambda \lambda \\ 0 & c_\lambda + d_\lambda \end{pmatrix} \oplus (c_\lambda + d_\lambda) I_{n-2}$$

$$= (c_\lambda + d_\lambda) \left(\begin{pmatrix} 1 & \dfrac{d_\lambda \lambda}{c_\lambda + d_\lambda} \\ 0 & 1 \end{pmatrix} \oplus I_{n-2} \right).$$

We observe that $c_\lambda + d_\lambda \neq 0$ since $\varphi(E_\lambda)$ is nonsingular. Thus, if $\mu = \dfrac{d_\lambda \lambda}{c_\lambda + d_\lambda}$, then

$$\varphi(\varphi(E_\lambda)) = (c_\lambda + d_\lambda)^{n-1}\left(\begin{pmatrix} c_\mu + d_\mu & d_\mu \mu \\ 0 & c_\mu + d_\mu \end{pmatrix} \oplus (c_\mu + d_\mu)I_{n-2}\right).$$

Moreover, $\varphi(\varphi(E_\lambda)) = (\det E_\lambda)^{n-2}E_\lambda = E_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. It follows that

$$(c_\lambda + d_\lambda)^{n-1}(d_\mu \mu) = \lambda \quad \text{and} \quad (c_\lambda + d_\lambda)^{n-1}(c_\mu + d_\mu) = 1.$$

Since $\det \varphi(E_\lambda) = (\det E_\lambda)^{n-1} = 1$, $(c_\lambda + d_\lambda)^n = 1$. Thus, $c_\lambda + d_\lambda = \alpha_\lambda$, where $\alpha_\lambda{}^n = 1$. By continuity and connectedness $\alpha_\lambda = \alpha$, a constant such that $\alpha^n = 1$. Since $\varphi(E_0) = \varphi(I_n) = I_n$ and this would equal $(c_0 + d_0)I_n = \alpha I_n$, $\alpha = 1$. Thus,

$$c_\lambda + d_\lambda = 1, \text{ and } \varphi(E_\lambda) = \begin{pmatrix} 1 & d_\lambda \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}.$$

Since $(c_\lambda + d_\lambda)^{n-1}(d_\mu \cdot \mu) = \lambda$, $d_\mu \cdot (d_\lambda \lambda) = \lambda$, i.e. $d_\mu = \frac{1}{d_\lambda}$ if $\lambda \neq 0$. From property (7), $d_\lambda \lambda$ is necessarily analytic and so d_λ itself is analytic. Since $\mu \to 0$ as $\lambda \to 0$, $d_0 = \frac{1}{d_0}$ as well. Moreover, $\varphi(E_{\mu+\lambda}) =$ $\varphi(E_\lambda E_\mu) = \varphi(E_\mu)\varphi(E_\lambda)$ and $E_{\mu+\lambda} = \begin{pmatrix} 1 & \mu+\lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. Whence,

$$\begin{pmatrix} 1 & d_{\mu+\lambda}(\mu+\lambda) \\ 0 & 1 \end{pmatrix} \oplus I_{n-2} = \begin{pmatrix} 1 & d_\mu \mu + d_\lambda \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}.$$

It follows that $d_{\mu+\lambda}(\mu+\lambda) = d_\mu \mu + d_\lambda \lambda$. There is a real constant D so that $d_\lambda \lambda = D\lambda$ and so $d_\lambda = D$. Moreover, $D = d_\mu = \frac{1}{d_\lambda}$ and so $d_\lambda = \frac{1}{d_\lambda}$, i.e. $d_\lambda{}^2 = 1$ or $D^2 = 1$. Consider the possibility that $D = 1$. Then $\varphi(E_\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. Since

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} XY, \text{ where}$$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \text{ and}$$

$$Y = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix},$$

it would appear that $\varphi\left(\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\right) = \begin{pmatrix} -3 & -2 & -10 \\ -2 & -3 & -10 \\ -22 & -18 & -75 \end{pmatrix}$.

Moreover, $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} AB$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$

we would have $\varphi\left(\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\right) = \begin{pmatrix} -3 & -4 & -2 \\ 2 & 1 & -2 \\ -2 & -6 & -3 \end{pmatrix}$. Thus, if $D = 1$ then φ is not well-defined if $n > 2$. Whence if $n > 2$, then $D = -1$ and

$$\varphi(E_\lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \oplus I_{n-2} = \operatorname{adj} E_\lambda.$$

Therefore, if $n > 2$ and E is type III elementary in $M_n(\mathbb{C})$, then $\varphi(E) = \operatorname{adj} E$. ♦

<u>Theorem 3.9</u>: If $n > 2$ and $A \in M_n(\mathbb{C})$, then $\varphi(A) = \operatorname{adj} A$.

Proof : Suppose that $n > 2$ in $\mathbb{N}$ and A is nonsingular in $M_n(\mathbb{C})$. Hence, $A = E_1E_2\cdots E_k$ for some $k \in \mathbb{N}$, where E_i is a type II or type III elementary matrix, for $1 \le i \le k$. Thus,

$$\begin{aligned}\varphi(A) &= \varphi(E_1E_2\cdots E_k)\\ &= \varphi(E_k)\varphi(E_{k-1})\cdots\varphi(E_1)\\ &= \operatorname{adj}(E_k)\operatorname{adj}(E_{k-1})\cdots\operatorname{adj}(E_1)\\ &= \operatorname{adj}(E_1E_2\cdots E_k)\\ &= \operatorname{adj} A.\end{aligned}$$

By continuity of φ, if $n > 2$, then $\varphi(A) = \operatorname{adj} A$, $\forall A \in M_n(\mathbb{C})$. ♦

In the case $n = 2$, we observe that there is a function that allows $D = 1$ and in fact satisfies all the assumptions concerning φ except for property (5) and $\varphi \neq \operatorname{adj}$.

Example 3.2 : We define $\varphi_0 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, by

$$(3.2) \qquad \varphi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

For this function φ_0, $D = 1$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{C})$.

Whence, $AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$ and so

$$\varphi_0(B)\varphi_0(A) = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} d\delta + c\beta & b\delta + a\beta \\ d\gamma + c\alpha & b\gamma + a\alpha \end{pmatrix} = \varphi_0(AB).$$

Moreover, we observe that $\varphi_0(\lambda A) = \lambda\varphi_0(A)$, $\varphi_0(\varphi_0(A)) = (\det A)^{2-2}A = A$, $\det \varphi_0(A) = \det A$, $\varphi_0(E) = \operatorname{adj} E$ if E is type II elementary, and $\varphi_0(E) = E$ if E is type III elementary. Hence, $\varphi_0(A) \neq \operatorname{adj} A$ unless $b = c = 0$. In general, this $\varphi_0(A)$ is not a polynomial in A. One would have $\varphi_0(A) = sI_2 + tA$, so if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} s + at & tb \\ tc & s + dt \end{pmatrix}.$$

This is not possible unless $b = c = 0$, or if $a = d$.

In Example 3.2, we note that φ_0 is not a counter example for the case $n = 2$. However, we prove that $\varphi = \varphi_0$ is the only continuous anti-homomorphism from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ that satisfies $\varphi(E) = \operatorname{adj} E$ when E is type II elementary and $\varphi(E) = E$ when E is type III elementary. Thus, we have

<u>**Theorem 3.10**</u> : If $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a continuous multiplicative anti-homomorphism such that

1) if E is type II elementary in $M_2(\mathbb{C})$, then $\varphi(E) = \operatorname{adj} E$ and

2) if E is type III elementary in $M_2(\mathbb{C})$, then $\varphi(E) = E$,

then $\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.

<u>**Proof**</u> : First suppose that $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a continuous multiplicative anti-homomorphism such that if $E \in M_2(\mathbb{C})$ is type II elementary, then $\varphi(E) = \operatorname{adj} E$ and, if $E \in M_2(\mathbb{C})$ is type III elementary, then $\varphi(E) = E$. Next, suppose that $\varphi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$, and $A \in M_2(\mathbb{C})$ is nonsingular. Hence, $A = E_1E_2 \cdots E_m$

for some $m \in N$ and where E_i is type II or type III elementary for $1 \le i \le m$. We show by induction that whenever A has m such factors then $\varphi(A) = \varphi_0(A)$. Since $\varphi(E) = \varphi_0(E)$ when E is type II or type III elementary the result holds for m = 1. Suppose it holds for m = p and consider A with factorization $E_1E_2 \cdots E_{p+1}$. If $B = E_1E_2 \cdots E_p$, then we obtain

$$\varphi(A) = \varphi(BE_{p+1}) = \varphi(E_{p+1})\varphi(B)$$
$$= \varphi_0(E_{p+1})\varphi_0(B) = \varphi_0(BE_{p+1}) = \varphi_0(A),$$

and the induction is finished.
Therefore, by continuity of φ, $\varphi(A) = \varphi_0(A)$ for all $A \in M_2(\mathbb{C})$. ♦

Theorem 3.10 along with the fact that φ_0 does not in general satisfy property (5) shows that for any $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ satisfying properties (1) - (7) must necessarily be the adjugate map. Thus (1) - (7) characterize the adjugate map for all $n \ge 2$, and we have
Corollary 3.10.1 : If $A \in M_n(\mathbb{C})$, then $\varphi(A) = \text{adj } A$.

We have that Theorem 3.1 and a continuous function φ with properties (1) - (7) yield two characterizations of the adjugate map. They are summarized, respectively as follows :

(3.3) If $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is such that

a) $\varphi(AB) = \varphi(B)\varphi(A)$, $\forall\ A,B \in M_n(\mathbb{C})$;

b) $\varphi(E) = (1 + \det E)I_n - E$, when $E \in M_n(\mathbb{C})$ is elementary ;

c) φ is a continuous function of A, then $\varphi(A) = \text{adj } A, \ \forall A \in M_n(\mathbb{C})$.

(3.4) If $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies :

1) $\text{rank}(A) = n \Rightarrow \text{rank}(\varphi(A)) = n$, where $A \in M_n(\mathbb{C})$;
2) $\varphi(AB) = \varphi(B)\varphi(A), \ \forall A,B \in M_n(\mathbb{C})$;
3) $\varphi(\lambda A) = \lambda^{n-1}\varphi(A), \ \forall A \in M_n(\mathbb{C}), \lambda \in \mathbb{C}$;
4) $\varphi(\varphi(A)) = \mu_A \cdot A$, where $\mu_A \in \mathbb{C}$ depends continuously on $A \in M_n(\mathbb{C})$.
5) $\varphi(A)$ is a polynomial in A ;
6) $\varphi(A)$ is real when A is real ;
7) $\varphi(A)$ is an analytic function of A of the entries of A,

then $\varphi(A) = \text{adj } A, \ \forall A \in M_n(\mathbb{C})$.

However, questions still remain concerning the minimality and independence of the assumptions made in (3.4). It may be the case that one assumption implies another and hence an assumption could then be removed from the list. As to the uniqueness of each assumption we observe that, assumption (5) may be replaced with an assumption (5') as follows : $A\varphi(A) = \varphi(A)A, \ \forall A \in M_n(\mathbb{C})$.

Gantmacher [2, p. 222] investigates the connections between commutativity and polynomials in a matrix. Hence, one can show that assumption (5') implies assumption (5) and we may then show that if $n > 2$, then $\varphi(A) = \text{adj } A, \ \forall A \in M_n(\mathbb{C})$. In (3.3) it is clear that each of the three assumptions are independent and that the list is indeed minimal.

BIBLIOGRAPHY

1. Bellman, R., *Introduction to Matrix analysis*, McGraw-Hill Book Company, Inc., 1960.

2. Gantmacher, F.R., *The Theory of Matrices*, Vol. I, Chelsea Publishing Company, 1959.

3. Hoffman, K. and Kunze, R., *Linear Algebra*, Prentice-Hall, Inc., 1971.

4. Sinkhorn, R., *The Range of the Adjugate Map*. To be published.