ENDOMORPHISMS OF LEAVITT PATH ALGEBRAS

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Mozahid Haque

May 2019

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Abstract

Directed graphs have played a prominent role as a tool for encoding information for certain classes of C^*-algebras, particularly AF-algebras and Cuntz-Krieger algebras. These constructions have been generalized to a class of C^*-algebras known as *graph C^*-algebras*, which have found applications to several areas of C^*-algebra theory. One prominent area of investigation has been the application of Elliott's classification program to the class of graph C^*-algebras. Rørdam was able to prove that K-theory invariants classify certain simple Cuntz-Krieger algebras, and this classification has been extended to broader classes of graph C^*-algebras, including even certain non-simple cases. Another avenue for extending these classification results is to consider *Leavitt path algebras*, algebraic analogues of the graph C^*-algebras, and ask to what extent K-theory groups can be used to classify them. This dissertation explores a specific, but important, aspect of the classification of Leavitt path algebras. In particular, we investigate the question of whether $L(E_2)$ and $L(E_2^-)$ are $*$-isomorphic. We do this by examining the diagonal of a Leavitt path algebra, and producing methods to construct endomorphisms of Leavitt path algebras that take a given maximal abelian subalgebra (MASA) to the diagonal.

Contents

Chapter 1

Introduction

1.1 History

Consider an infinite dimensional Hilbert space H with basis $\{\delta_i\}_{i=1}^{\infty}$. We can consider dividing up the Hilbert space into two other infinite dimensional subspaces by considering two maps:

$$T_1 : H \to H \text{ given by } T_1(\delta_i) = \delta_{2i-1} \text{ and}$$

$$T_2 : H \to H \text{ given by } T_2(\delta_i) = \delta_{2i}.$$

Notice $T_1^*T_1 = I_H$, $T_2^*T_2 = I_H$, and $T_1T_1^* + T_2T_2^* = I_H$. These maps are isometries, and one can consider the C^*-algebra $\mathcal{O}_2$ generated by $\{T_1, T_2\}$.

More generally, one can consider isometries $T_1, \ldots, T_n$ satisfying $T_i^*T_i = I$ for all $1 \leq i \leq n$ and $\sum_{i=1}^{n} T_iT_i^* = I$, and define $\mathcal{O}_n$ to be the C^*-algebra generated by $\{T_1, \ldots, T_n\}$. It turns out that the isomorphism class of this C^*-algebra is independent of the choice of the isometries $T_1, \ldots, T_n$. The C^*-algebras $\mathcal{O}_n$ for $n \geq 2$ are called the *Cuntz algebras*, after Joachim Cuntz who introduced them [5]. These C^*-algebras turned out to be pervasive in many aspects of C^*-algebra theory, and they have been generalized in several ways. One generalization was accomplished by Cuntz and Krieger in which they take a square matrix A with entries in $\{0,1\}$, and define a C^*-algebra $\mathcal{O}_A$, known as the *Cuntz-Krieger algebra*, which is generated by partial isometries satisfying relations determined by A. The Cuntz algebra $\mathcal{O}_n$ is equal to the Cuntz-Krieger algebra of the $n \times n$ matrix of all 1s. Cuntz and Krieger pointed out that $\mathcal{O}_A$ may be viewed as a C^*-algebra of the directed graph whose adjacency matrix is A, and that the structure of $\mathcal{O}_A$ is related to the dynamics of the directed graph or "topological Markov chain" corresponding to A [7].

The Cuntz-Krieger algebras were later generalized, by several authors, to construct C^*-algebra of any directed graph. In particular, the graph is allowed to have infinitely many vertices or edges, multiple edges between vertices, sink and sources, or vertices that emit or receive infinitely many edges. For a (directed) graph E the *graph C^*-algebra* $C^*(E)$ is define to be the universal C^*-algebra generated by the collection of projections $\{p_v : v \text{ is a vertex in } E\}$ and a collection of partial isometries with mutually orthogonal ranges $\{s_e : e \text{ is an edge in } E\}$ satisfying the Cuntz-Krieger relations

(1) $s_e^* s_e = p_{r(e)}$ for each edge e.

(2) $p_v = \displaystyle\sum_{s(e)=v} s_e s_e^*$ when the vertex v emits a finite and nonzero number of edges.

(3) $s_e s_e^* \leq p_{s(e)}$ for each edge e.

The graph C^*-algebras have been studied extensively and found applications in many areas of C^*-algebra theory. A good introduction to graph C^*-algebras can be found in Raeburn's book [11], which is the product of a CBMS conference on the subject. Graph C^*-algebra encompass a wide variety of C^*-algebras (e.g., the Cuntz and Cuntz-Krieger algebras, the Toeplitz algebra, all finite-dimensional C^*-algebras, all AF-algebras up to Morita equivalence, all Kirchberg algebras with free K_1-group, and various quantum spaces). In addition, many operator algebra properties of $C^*(E)$ correspond to graph properties of E, resulting in a theory in which the graph can be used to "visualize" C^*-algebraic properties.

Graph C^*-algebras have also been used as tools in the classification program for C^*-algebras, especially classification of non-simple C^*-algebras. Moreover, graph C^*-algebras have provided a bridge strengthening the already prevalent connection between C^*-algebra theory and dynamics. Specifically, C^*-algebraic properties of $C^*(E)$ are closely related to dynamical properties of the shift space of the graph E, and results from dynamics have had useful applications in classification of graph C^*-algebras. Graph C^*-algebras have also been generalized in myriad ways, to produce more general classes with diverse applications all motivated by the theory and results developed for graph C^*-algebras.

1.2 Classification of Graph C^*-algebras

It is possible for different graphs to have isomorphic C^*-algebras. In fact, one nice facet of graph C^*-algebra theory is that graph operations have been developed that change the graph without changing the associated C^*-algebra. This allows one to change the graph to a nicer form without altering the associated C^*-algebra, and it also allows one to gain multiple perspectives on a single C^*-algebra by using different graphs to represent it.

Since it is possible for different graphs to have isomorphic C^*-algebras, an important and natural problem is to determine when two given graphs have the same C^*-algebra. As mathematicians know, the phrase "the same" could have various meanings. The most natural version of the problem is to determine necessary and sufficient conditions for two graphs to have isomorphic C^*-algebras. However, besides isomorphism, one could also ask variants of the question for other notions of equivalence, which may be stronger or weaker than isomorphism. For example, one could ask for necessary and sufficient conditions for two graphs to have Morita equivalent C^*-algebras, or one could ask for necessary and sufficient conditions for two graphs to have C^*-algebras that are isomorphic via an isomorphism preserving a canonical subalgebra of the graph C^*-algebra. We state these classification questions here.

THE CLASSIFICATION QUESTION: Given two directed graphs E and F, what are necessary and sufficient conditions for $C^*(E)$ and $C^*(F)$ to be isomorphic?

VARIANTS OF THE CLASSIFICATION QUESTION: Given two directed graphs E and F, what are necessary and sufficient conditions for $C^*(E)$ and $C^*(F)$ to be "equivalent"? (Here "equivalent" can have various meanings, but Morita equivalent is probably one of the most important.)

While the isomorphism version of this question is perhaps the most important, variants for other notions of equivalence are also useful and can give important insights into the structure of the associated C^*-algebra.

(1) For instance, in certain situations the answer to the isomorphism version of the question may be unknown, while results for other notions of equivalence are available.

(2) Sometimes a variant of the question has conditions that are easier to verify or refute, and may be sufficient for certain purposes — for examples, a questions

about simplicity of ideal structure may only require one to know that a given graph C^*-algebra is Morita equivalent to another, and the sufficient conditions for Morita equivalence may be easier to check than the conditions for isomorphism.

(3) In situations where necessary and sufficient conditions for isomorphism are unknown, comparing results for notions of equivalence that are stronger than isomorphism and with results for notions of equivalence that are weaker than isomorphism allows one to hone in on conditions that are both necessary and sufficient for isomorphism. This can be useful for formulating conjectures as well as producing counterexamples.

A variety of partial answers to special cases of the classification questions have been found and, under certain hypotheses, necessary and sufficient conditions have even been obtained. One important achievement has been the classification of simple C^*-algebras of finite graphs up to Morita equivalence. We describe this result here:

There are conditions on a graph, which are both easy to describe and easy to verify, that characterize when the associated C^*-algebra is simple. We shall refer to these graphs as *simple graphs*. In addition, for any graph E, the vertex (or adjacency) matrix is defined to a square matrix A_E indexed by the vertices with entries

$$A(v, w) := \text{number of edges in } E \text{ from } v \text{ to } w.$$

Among many other applications, the vertex matrix can be used to calculate the K-theory of $C^*(E)$. If E has n vertices, and we consider the map $I - A_E^t : \mathbb{Z}^n \to \mathbb{Z}^n$, then the $K_0(C^*(E)) \cong \operatorname{coker}(I - A_E^t)$. (Recall that $\operatorname{coker}(I - A_E^t) := \mathbb{Z}^n / \operatorname{im}(I - A_E^t)$.) For a finite graph, this cokernel can easily be computed using some elementary Linear Algebra. Also note that the transpose appearing in A_E^t makes no difference, since $(I - A_E)^t = I - A_E^t$ so for a finite matrix $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_E)$; however, the transpose makes a difference for infinite graphs/matrices, so we use it to be consistent with the general formula.

It turns out that the K_0-group of $C^*(E)$ is equal to the Bowen-Franks group of the shift space associated to E. Using result from symbolic dynamics one can prove that if E and F are simple graphs with $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_F^t)$ and $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$, then the shift spaces of E and F are conjugate (i.e. isomorphic in the category of shift spaces) and there exist four graph moves, called (S), (O), (I), and (R), such that E can be transformed into F via a finite sequence of these moves. With a little work, one can prove that these moves preserve Morita equivalence of the C^*-algebra of the graph as well as the sign of $\det(I - A_E^t)$. Thus we have the following:

Theorem 1.2.1. *For finite simple graphs E and F, the following two conditions are equivalent:*

(1) $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_F^t)$ *and* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$*, and*

(2) E can be transformed into F via a finite sequence of (S), (O), (I), and (R) moves.

Moreover, each of these conditions implies $C^*(E)$ is Morita equivalent to $C^*(F)$.

While these conditions are sufficient for Morita equivalence, they are not necessary. In particular, the sign of the determinant condition is not needed for Morita equivalence of the graph C^*-algebras. To get around the sign of the determinant condition, another move on graphs needed to be developed — this move is called the *Cuntz splice*. If E is a simple graph and v is a vertex of E that is the base of a cycle, the Cuntz splice is the move that attaches a piece of the form

$$\begin{tikzcd} \cdots v \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] \arrow[loop above] & \bullet \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

to the graph E at v.

The effect of the Cuntz splice is that it leaves the cokernel of $I - A_E^t$ unchanged while flipping the sign of the determinant; in other words if E is a graph and $\widetilde{E}$ is the graph formed by attaching a Cuntz splice to E, then $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_{\widetilde{E}}^t)$ and $\operatorname{sign}\det(I - A_E^t) = -\operatorname{sign}\det(I - A_{\widetilde{E}}^t)$. Rørdam proved that performing a Cuntz splice to a finite simple graph preserves the Morita equivalence class of the associated C^*-algebra [12]. This allows for the following classification theorem, which gives necessary and sufficient conditions for Morita equivalence, both in terms of an algebraic invariant and in terms of moves performed on the graph.

Theorem 1.2.2. *For finite simple graphs E and F, the following conditions are equivalent:*

(1) $K_0(C^*(E)) \cong K_0(C^*(F))$,

(2) $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_F^t)$,

(3) E can be transformed into F via a finite sequence of (S), (O), (I), and (R) moves and the Cuntz splice, and

(4) $C^*(E)$ is Morita equivalent to $C^*(F)$.

Moreover, if $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$ *then only the moves (S), (O), (I), and (R) are needed in Condition (2), and if* $\operatorname{sign}\det(I - A_E^t) = -\operatorname{sign}\det(I - A_F^t)$, *then one application of the Cuntz splice folowed by a finite sequence of the moves (S), (O), (I), and (R) are needed in Condition (2). (In particular, at most one application of the Cuntz splice is necessary.)*

1.3 Leavitt Path Algebras and Motivation

Inspired by the successes of graph C^*-algebras, researches considered algebras (over an arbitrary field) analogous to graph C^*-algebras. In particular, if K is a fixed field, then for any graph E one defines the *Leavitt path algebra* $L_K(E)$ to be the universal algebra generated by elements $\{p_v, s_e, s_e^*\}$ satisfying the Cuntz-Krieger relations. When the field is the complex numbers, $L_{\mathbb{C}}(E)$ is isomorphic to a dense $*$-subalgebra of $C^*(E)$. Surprisingly, it has been found that many theorems for graph C^*-algebras have parallel — often very similar — theorems for Leavitt path algebras, although the proofs of these results typically have to obtained by very different methods.

Classification of Leavitt path algebras has made impressive strides, and Leavitt path algebras have provided some of the first examples of algebras (besides operator algebras) that can be classified by K-theory. It has been the case the classification for Leavitt path algebras has had more success for infinite graphs (where there is no sign of the determinant obstruction), but for finite graphs the sign of the determinant has been a major stumbling block.

As with graph C^*-algebras, one can show that the moves (S), (O), (I), (R) preserves the Morita equivalence class of the Leavitt path algebra of a graph. This allows one to obtain a version of Theorem 1.2.1 for Leavitt path algebras.

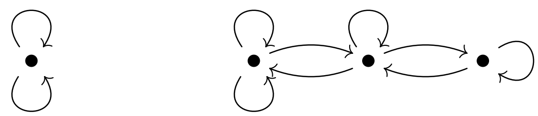
Theorem 1.3.1. *Let K be a field. If E and F are finite simple graphs, the following two conditions are equivalent:*

(1) $\operatorname{coker}(I - A_E^t) \cong \operatorname{coker}(I - A_F^t)$ *and* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$*, and*

(2) E can be transformed into F via a finite sequence of (S), (O), (I), and (R) moves.

Moreover, each of these conditions implies $L_K(E)$ is Morita equivalent to $L_K(F)$.

However, it is currently unknown whether the Cuntz splice preserves Morita equivalence of the associated Leavitt path algebra. In fact, this is unknown even in the easiest situation: the case of the graph E_2 with a single vertex and two edges. We let E_2^- denote the graph obtained by adding a Cuntz splice to E_2.



E_2 E_2^-

For graph C^*-algebras, $C^*(E_2) \cong \mathcal{O}_2$ and the C^*-algebra $C^*(E_2^-)$ is typically denoted $\mathcal{O}_2^-$. Rørdam's result shows that $\mathcal{O}_2$ is Morita equivalent to $\mathcal{O}_2^-$; in fact, it can be shown that $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2^-$. However, the corresponding result for Leavitt path algebras in currently open.

For a fixed field K, the associated Leavitt path algebras are denoted $L_2 := L_K(E_2)$ and $L_2^- := L_K(E_2^-)$. It is known that $K_0(L_2) = K_0(L_2^-) = 0$, and that $\det(I - A_{E_2}^t) = 1$ while $\det(I - A_{E_2^-}^t) = -1$. Two fundamental questions in the classification of Leavitt path algebras are the following:

QUESTION 1: For a given field, are L_2 and L_2^- isomorphic?

QUESTION 2: For a given field, are L_2 and L_2^- Morita equivalent?

These questions are often collectively referred to as the L_2-question or the L_2-problem for Leavitt path algebras. The lack of an answer to the L_2-problem is currently a major obstruction to the classification of simple unital Leavitt path algebras. While several researchers have worked on these problems, very little progress has been made.

Tomforde introduced the notion of a Leavitt path algebra over a ring [14], and in a very interesting result, Johansen and Sørensen have shown that $L_{\mathbb{Z}}(E_2)$ is not isomorphic to $L_{\mathbb{Z}}(E_2^-)$ [9]. It is unclear what this means for Leavitt path algebras over fields or what it means when the field is $\mathbb{C}$ (arguably, the most important case for C^*-algebraists). It remains unknown what the relationship is between L_2 and L_2^-.

1.4 An Approach to the L_2-problem

The purpose of this dissertation is to explore particular aspects of the structure of Leavitt path algebras motivated by the L_2-question(s). In particular, we consider endomorphisms on Leavitt path algebras and properties of the diagonal subalgebra. If $\alpha := e_1 \dots e_n$ is a (directed) path in E, we define $s_\alpha := s_{e_1} \dots s_{e_n}$. For a graph E the diagonal subalgebra of $L(E)$ is the subalgebra $D_{L(E)} := \text{span}\{s_\alpha s_\alpha^* : \alpha \text{ is a path in } E\}$ and the diagonal subalgebra of $C^*(E)$ is the closed subalgebra $D_{L(E)} := \overline{\text{span}}\{s_\alpha s_\alpha^* : \alpha \text{ is a path in } E\}$. Note that $D_{C^*(E)} = \overline{D_{L(E)}}$ when the underlying field of $L(E)$ is the complex numbers $\mathbb{C}$.

An important piece of Johansen and Sørensen's proof that $L_{\mathbb{Z}}(E_2)$ is not isomorphic to $L_{\mathbb{Z}}(E_2^-)$ relied on applying a theorem of Matsumoto and Matui involving the diagonal. The following result is a consequence of [10, Theorem 3.6] obtained by Matsumoto and Matui.

Theorem 1.4.1 (Matsumoto and Matui)**.** *Let E and F be finite simple graphs. Let A_E and A_F denote the vertex matrices of E and F, respectively, and let u_E and u_F denote the vectors of all 1s in the domains of A_E and A_F, respectively. The following are equivalent.*

(1) $C^*(E) \cong C^*(F)$ *and* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

(2) There exists an isomorphism $\Phi : \operatorname{coker}(I - A_E^t) \to \operatorname{coker}(I - A_F^t)$ *with* $\Phi([u_E]) = [u_F]$, *and* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

(3) There exists an isomorphism $\Phi : K_0(C^*(E)) \to K_0(C^*(F))$ *with* $\Phi([1_{C^*(E)}]_0) = [1_{C^*(F)}]_0$, *and* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

(4) There exists an isomorphism $\phi : C^*(E) \to C^*(F)$ *with* $\phi(D_{C^*(E)}) = D_{C^*(F)}$.

Inspired by Johansen and Sørensen's work, we describe an approach to the L_2-question that motivates the work done in this dissertation.

Suppose that there is an isomorphism $\psi : L_2 \to L_2^-$. We can show $D_{L_2^-}$ is a MASA in L_2^-, and hence we may conclude $M := \psi^{-1}(D_{L_2^-})$ is a MASA in L_2. If there exists an automorphism $\phi : L_2 \to L_2$ with $\phi(D_{L_2^-}) = M$, then $\psi \circ \phi : L_2 \to L_2^-$ is a diagonal-preserving isomorphism. One can argue that this extends to an isomorphism $\overline{\phi} : C^*(E_2) \to C^*(E_2^-)$ with $\phi(D_{C^*(E_2)}) = D_{C^*(E_2^-)}$, and because $\operatorname{sign}\det(I - A_{E_2}^t) = 1$ and $\operatorname{sign}\det(I - A_{E_2^-}^t) = -1$, this contradicts the equivalence of (1) and (4) in the above result of Matsumoto and Matui.

Therefore, if we can find an automorphism of L_2 that takes the diagonal D_{L_2} onto the MASA $M := \psi^{-1}(D_{L_2^-})$, we may conclude that L_2 is not isomorphic to L_2^-.

Thus, we wish to verify some of the claims asserted in the argument above, and ask about the existence of an automorphism of a Leavitt path algebra that carries the diagonal onto a specified MASA. Specifically, after some preliminaries in Chapter 2 we address the following.

(1) In Chapter 3, we examine the structure of the diagonal $D_{L(E)}$ for a general

Leavitt path algebra $L(E)$, compute the commutant of $D_{L(E)}$, and determine necessary and sufficient conditions for $D_{L(E)}$ to be a MASA.

(2) In Chapter 4, we consider the problem of lifting a morphism between Leavitt path algebras $\phi : L_{\mathbb{C}}(E) \to L_{\mathbb{C}}(F)$ to the enveloping C^*-algebras $\overline{\phi} : C^*(E) \to C^*(F)$ and make rigorous our claims regarding our approach to the L_2-problem.

(3) In Chapter 5, we study endormorphisms of $L(E)$. We prove there is a bijective correspondence between endomorphisms fixing vertex projections and unitaries that commute with vertex projections. We deduce results in which we describe properties of the endormorphism in terms of the property of the associated unitary.

(4) In Chapter 6, we study endomorphisms of $L(E)$ with the goal of constructing endormorphisms (and automorphisms) that map the diagonal $D_{L(E)}$ onto a given MASA in $L(E)$.

Although we are unable to answer the L_2-problem at this point, we hope that the insights we produce will help with its solution in the future. We are also optimistic that further investigations along these avenues will produce results useful for examining other questions related to the structure of Leavitt path algebras. Even though our motivating example is L_2, because of our desire to create a theory with many potential applications, we strive to obtain results for Leavitt path algebras in as much generality as possible.

Chapter 2

Preliminaries

In this chapter we establish notation and terminology that we shall use throughout future chapters.

2.1 Graphs

When we refer to a graph, we shall always mean a directed graph $E := (E^0, E^1, r, s)$ consisting of a countable set of vertices E^0, a countable set of edges E^1, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge.

Definition 2.1.1. Let $E := (E^0, E^1, r, s)$ be a graph. We say that a vertex $v \in E^0$ is a *sink* if $s^{-1}(v) = \emptyset$, and we say that a vertex $v \in E^0$ is an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular vertex* is a vertex that is either a sink or an infinite emitter, and we denote the set of singular vertices by E^0_{sing}. We also let $E^0_{\text{reg}} := E^0 \setminus E^0_{\text{sing}}$, and refer to the elements of E^0_{reg} as *regular vertices*; i.e., a vertex $v \in E^0$ is a regular vertex if and only if $0 < |s^{-1}(v)| < \infty$. We say a graph is *row-finite* if no vertex emits infinitely many edges.

Definition 2.1.2. If E is a graph, a *path* is a sequence $\alpha := e_1 e_2 \dots e_n$ of edges with $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. We say the path α has *length* $|\alpha| := n$, and we let E^n denote the set of paths of length n. We consider the vertices in E^0 to be paths of length zero. We also let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the paths of finite length, and we extend the maps r and s to E^* as follows: For $\alpha := e_1 e_2 \dots e_n \in E^n$, we set $r(\alpha) = r(e_n)$ and $s(\alpha) = s(e_1)$.

Definition 2.1.3. If E is a graph, a *cycle* is a path $\alpha \in E^*$ with $r(\alpha) = s(\alpha)$. A cycle $\alpha = e_1 \dots e_n$ is called a *simple cycle* if $s(e_i) \neq s(e_j)$ for $i \neq j$. Furthermore, if $\alpha := e_1 \dots e_n$ is a cycle, we say that an edge $f \in E^1$ is an *exit* for the cycle α if there exists $i \in \{1, \dots, n\}$ such that $s(f) = s(e_i)$ and $f \neq e_i$. We say a graph satisfies Condition (L) if every cycle in the graph has an exit. Note that in any graph every cycle has an exit if and only if every simple cycle has an exit.

2.2 Graph C^*-algebras and Leavitt Path Algebras

If E is a graph, a *Cuntz-Krieger E-family* in a C^*algebra A is a collection $\{p_v, s_e : v \in E^0, e \in E^1\}$ such that the elements of $\{p_v : v \in E^0\}$ are mutually orthogonal projections, the elements of $\{s_e : e \in E^1\}$ are partial isometries with pairwise orthogonal range projections (i.e., $s_e^* s_f = 0$ when $e \neq f$) and the collection satisfies the Cuntz-Krieger conditions:

(CK1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,

(CK2) $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in E^0_{\text{reg}}$, and

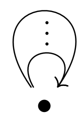
(CK3) $p_{s(e)} s_e = s_e$ for all $e \in E^1$.

The graph C^*-algebra $C^*(E)$ is a defined to be the C^*-algebra generated by a

Cuntz-Krieger E-family $\{p_v, s_e : v \in E^0, e \in E^1\}$ with the following universal property: If A is a C^*-algebra and $\{q_v, t_e :: v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in A, then there exists a $*$-homomorphism $\phi : C^*(E) \to A$ with $\phi(p_v) = q_v$ for all $v \in E^0$ and $\phi(s_e) = t_e$ for all $e \in E^1$. It is a consequence of the universal property that $C^*(E)$ is unique up to isomorphism. A proof of the existence of $C^*(E)$ and basic facts about graph C^*-algebras can be found in [11].

Inspired by the usefulness of graph C^*-algebras in functional analysis, algebraists created analogues known as Leavitt path algebras. If E is a graph, and A is an algebra over a field K, a Cuntz-Krieger E-family is a collection $\{p_v, s_e, s_e^* : v \in E^0, e \in E^1\}$ in A such that the elements of $\{p_v : v \in E^0\}$ are mutually orthogonal idempotents, the elements of $\{s_e : e \in E^1\}$ satisfy $s_e^* s_f = 0$ when $e \neq f$, and the collection satisfies the Cuntz-Krieger conditions (CK1)–(CK3) above. The Leavitt path algebra is defined to be the K-algebra $L_K(E)$ generated by a Cuntz-Krieger E-family $\{p_v, s_e, s_e : v \in E^0, e \in E^1\}$ with the following universal property: If A is a K-algebra and $\{q_v, t_e, t_e^* :: v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in A, then there exists a K-algebra homomorphism $\phi : L_K(E) \to A$ with $\phi(p_v) = q_v$ for all $v \in E^0$ and with $\phi(s_e) = t_e$ and $\phi(s_e^*) = t_e^*$ for all $e \in E^1$. It is a consequence of the universal property that $L_K(E)$ is unique up to isomorphism. A proof of the existence of $L_K(E)$ and basic facts about Leavitt path algebras can be found in [1]. When the field is fixed, it is common to simplify notation and to simply write $L(E) := L_K(E)$ for the Leavitt path algebra of E.

The following table lists some common graphs and the associated graph C^*-algebras and Leavitt path algebras.

Graph	C^*-algebra	Leavitt path algebra
$\bullet$	$\mathbb{C}$	K
	$C(\mathbb{T})$	$K[z, z^{-1}]$
 (n edges)	$\mathcal{O}_n$	L_n
$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_n$	$M_n(\mathbb{C})$	$M_n(K)$

For a path $\alpha = e_1 \dots e_n$, we define $s_\alpha := s_{e_1} \dots s_{e_n}$. If a path α has length zero; i.e., $\alpha = v$ for some vertex v, we define $s_v := p_v$. Note that $s_\alpha^* = (s_{e_1} \dots s_{e_n})^* = s_{e_n}^* \dots s_{e_1}^*$. For paths α and β with $r(\alpha) = s(\beta)$, we have $s_\alpha s_\beta = s_{\alpha\beta}$ and $s_\beta^* s_\alpha^* = (s_\alpha s_\beta)^* = s_{\alpha\beta}^*$. Using the Cuntz-Krieger relations, it is straightforward to show that the following multiplication rule holds: For any $\alpha, \beta, \gamma, \delta \in E^*$, we have

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) = \begin{cases} s_{\alpha\gamma'} s_\delta^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha s_\delta^* & \text{if } \beta = \gamma \\ s_\alpha s_{\delta\beta'}^* & \text{if } \beta = \gamma\beta' \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, any nonzero word in the symbols s_e and s_f^* may be rewritten as $s_\alpha s_\beta^*$ for $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$. Consequently,

$$C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ with } r(\alpha) = r(\beta)\}$$

and for any field K

$$L_K(E) = \operatorname{span}_K\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ with } r(\alpha) = r(\beta)\}.$$

One can show that if $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger family for $C^*(E)$, then the subalgebra $\operatorname{span}_{\mathbb{C}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ with } r(\alpha) = r(\beta)\}$ contained in $C^*(E)$ has the universal property for $L_{\mathbb{C}}(E)$ and hence is isomorphic to $L_{\mathbb{C}}(E)$. Thus $L_{\mathbb{C}}(E)$ is (isomorphic to) a dense subalgebra of $C^*(E)$.

Furthermore, suppose K is given an involution $z \mapsto \overline{z}$. For the complex numbers $\mathbb{C}$ the standard choice for $\overline{z}$ is complex conjugation. For any field, we are always free to choose the involution to be the identity $\overline{z} = z$, so every field has such an involution. We may then define a $*$-operation on $L_K(E)$ as follows: For a typical element $x = \sum_{i=1}^n z_i s_{\alpha_i} s_{\beta_i}^* \in L_K(E)$ we set

$$x^* := \sum_{i=1}^n \overline{z}_i s_{\beta_i}^* s_{\alpha_i}^* \in L_K(E).$$

This $*$-operation is conjugate-linear $(zx)^* = \overline{z}x^*)$, involutive $(\overline{\overline{x}} = x)$, and anti-multiplicative $(\overline{xy} = \overline{y}\,\overline{x})$. It is therefore a $*$-operation making $L_K(E)$ into a $*$-algebra. Furthermore, for $L_{\mathbb{C}}(E)$ this $*$-operation agrees with the adjoint operation on the C^*-algebra $C^*(E)$. This $*$-operation allows us to discuss projections (elements with $p = p^2 = p^*$), isometries (elements with $v^*v = 1$), and unitaries (elements with $uu^* = u^*u = 1$) in Leavitt path algebras.

Definition 2.2.1. If R is a ring, we say R is $\mathbb{Z}$-graded if there is a a collection of additive subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R with the following two properties.

1. $R = \bigoplus_{n\in\mathbb{Z}} R_n$.

2. $R_m R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}$.

The subgroup R_n is called the *homogeneous component of R of degree n*.

All Leavitt path algebras have a natural $\mathbb{Z}$-grading. If E is a graph, then we may define a $\mathbb{Z}$-grading on the associated Leavitt path algebra $L_K(E)$ by setting

$$L_K(E)_n := \left\{ \sum_{i=1}^{N} \lambda_i s_{\alpha_i} s_{\beta_i}^* : \alpha_i, \beta_i \in E^* \text{ and } |\alpha_i| - |\beta_i| = n \text{ for } 1 \leq k \leq N \right\}.$$

Note that, in fact, each $L_K(E)_n$ is closed under scalar multiplication by elements of K. Hence $L_K(E)$ is actually a graded algebra.

2.3 The Diagonal Subalgebra of a Leavitt Path Algebras

If E is a graph and K is field, let $L(E) := L_K(E)$ be the associated Leavitt path algebra, and let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a generating Cuntz-Krieger E-family. For each $\alpha \in E^*$ define

$$Q_\alpha := s_\alpha s_\alpha^*.$$

Each Q_α is a projection, and the elements of $\{Q_\alpha : \alpha \in E^*\}$ satisfy the following multiplication:

$$Q_\alpha Q_\beta = Q_\beta Q_\alpha = \begin{cases} Q_\alpha & \text{if } \alpha = \beta\alpha' \\ Q_\beta & \text{if } \beta = \alpha\beta' \\ 0 & \text{otherwise.} \end{cases}$$

In particular, Q_α and Q_β commute and the product is nonzero if and only if either α or β extends the other, and in this case, the product is the Q-projection corresponding to the longer path.

The *diagonal subalgebra* of $L(E)$ is defined to be

$$D_{L(E)} := \operatorname{span}_K\{Q_\alpha : \alpha \in E^*\}.$$

One may observe that $D_{L(E)}$ is an abelian $*$-subalgebra of $L(E)$ that resides inside of $L_K(E)_0$, the zero grade of $L(E)$.

Chapter 3

MASAs and the Diagonal

In this chapter we introduce the notion of a MASA (Maximal Abelian SubAlgebra) and establish basic results concerning them. We compute the commutant of the diagonal $D_{L(E)}$ in a general Leavitt path algebra $L(E)$, and we prove that the diagonal is a MASA if and only if the graph E satisfies Condition (L) (i.e., every cycle in E has an exit).

3.1 Definition and Basic Results for MASAs

If A is an algebra, a subalgebra $B \subseteq A$ is a *maximal abelian subalgebra* (or MASA for short) if B is abelian and whenever C is an abelian subalgebra of A with $B \subseteq C$, then $B = C$.

If A is an algebra and $S \subseteq A$, the *commutant of S* is the collection

$$S' := \{a \in A : as = sa \text{ for all } s \in S\}$$

of all elements of A that commute with each element of S.

Proposition 3.1.1. *Let A and B be algebras.*

(1) If S and T are subsets of A, then $S \subseteq T$ implies $T' \subseteq S'$.

(2) If $S \subseteq A$, then S' is a subalgebra of A.

(3) If A is a $*$-algebra and S is a selfadjoint subset of A (i.e., $x \in S$ implies $x^* \in S$), then S' is selfadjoint.

(4) If $\phi : A \to B$ is an algebra homomorphism and $S \subseteq A$, then $\phi(S') \subseteq \phi(S)'$.

(5) If $\phi : A \to B$ is a surjective algebra homomorphism and $S \subseteq B$, then $\phi^{-1}(S)' \subseteq \phi^{-1}(S')$.

Proof. (1) If $S \subseteq T$, then whenever $x \in A$ commutes with every element of T, then x commutes with every element of S. Hence $T' \subseteq S'$.

(2) If $x, y \in S'$, then for any $s \in S$ we have $(x+y)s = xs+ys = sx+sy = s(x+y)$ so $x + y \in S'$, and $(xy)s = x(ys) = x(sy) = (xs)y = (sx)y = s(xy)$ so $xy \in S'$. Also, if $x \in S'$ and λ is a scalar, then for any $s \in S$ we have $(\lambda x)s = \lambda(xs) = \lambda(sx) = s(\lambda x)$ so $\lambda x \in S'$. Hence S' is an algebra.

(3) Let $x \in S'$. For any $s \in S$, the fact S is selfadjoint implies $s^* \in S$, and we have $x^*s = (s^*x)^* = (xs^*)^* = sx^*$ so that $x^* \in S'$.

(4) If $y \in \phi(S')$, then $y = \phi(x)$ for some $x \in S'$. Hence for any $z \in \phi(S)$, we may write $z = \phi(s)$ for $s \in S$, and then $yz = \phi(x)\phi(s) = \phi(xs) = \phi(sx) = \phi(s)\phi(x) = zy$, so $y \in \phi(S)'$.

(5) Since ϕ is surjective, we have $\phi(\phi^{-1}(S)) = S$. Therefore, if $x \in \phi^{-1}(S)'$, then applying the result of (4) we conclude that $\phi(x) \in \phi(\phi^{-1}(S)') \subseteq \phi(\phi^{-1}(S))' = S'$, so that $x \in \phi^{-1}(S')$. $\square$

Proposition 3.1.2. *Let A be an algebra, and let $B \subseteq A$ be a subalgebra of A. Then the following are equivalent:*

(1) B is a MASA.

(2) $B' = B$.

Proof. Suppose (1) holds. Since B is abelian, it follows that $B \subseteq B'$. If $x \in B'$, then x commutes with every element of B, and if we let C denote the subalgebra of A generated by $B \cup \{x\}$, then C is abelian. Since $B \subseteq C$, the maximality of B implies that $B = C$, so that $x \in B$ and $B' \subseteq B$. Thus $B' = B$, and (2) holds.

Conversely, suppose that (2) holds. Since $B' = B$, we conclude that B is abelian. In addition, if C is any abelian subalgebra of A with $B \subseteq C$, then every element of C commutes with each element of B, so that $C \subseteq B'$. Hence $B \subseteq C \subseteq B'$, and the fact that $B' = B$ implies $B = C$. Thus B is a MASA. □

Proposition 3.1.3. *Let A and B be algebras, and let $\phi : A \to B$ be an injective algebra homomorphism. If C is a MASA in B with $C \subseteq \operatorname{im}\phi$, then $\phi^{-1}(C)$ is a MASA in A.*

Proof. Since ϕ is an algebra homomorphism, $\phi^{-1}(C)$ is a subalgebra of A. In addition, if $a, b \in \phi^{-1}(C)$, then $\phi(a), \phi(b) \in C$, and since C is abelian, $\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba)$. By the injectivity of ϕ, we conclude that $ab = ba$. Hence $\phi^{-1}(C)$ is an abelian subalgebra.

If D is an abelian subalgebra of A with $\phi^{-1}(C) \subseteq D$, then $\phi(\phi^{-1}(C)) \subseteq \phi(D)$, and using the fact that $C \subseteq \operatorname{im}\phi$, we obtain $C \subseteq \phi(D)$. Since D is an abelian subalgebra of A and ϕ is an algebra homomorphism, $\phi(D)$ is an abelian subalgebra of B. By the maximality of C we conclude $C = \phi(D)$, and hence by the injectivity of ϕ we have $\phi^{-1}(C) = \phi^{-1}(\phi(D)) = D$. $\square$

3.2 The Commutant of $D_{L(E)}$

In this section we calculate the commutant of the diagonal subalgebra $D_{L(E)}$ and give necessary and sufficient conditions for $D_{L(E)}$ to be a MASA.

Lemma 3.2.1. *Let E be a graph and let K be a field. If $\alpha_1, \ldots, \alpha_m$ are distinct paths in E, and $\beta_1, \ldots, \beta_n$ are distinct paths in E (with each β_j not necessarily distinct from the α_is), then $\{s_{\alpha_1}, \ldots, s_{\beta_m}, s_{\beta_1}^*, \ldots, , s_{\beta_n}^*\}$ is a linearly independent subset of $L_K(E)$.*

Proof. Suppose $a_1, \ldots, a_m, b_1, \ldots, b_m \in K$ with $\sum_{i=1}^n a_i s_{\alpha_i} + \sum_{j=1}^m b_j s_{\beta_j}^* = 0$. Using the grading on $L_K(E)$, we may conclude that each graded component of this linear combination is equal to zero. Hence it suffices to prove the result in the following two cases: when the paths are all α_i of the same length (and there are no β_j terms), and when the paths are all β_j of the same length (and there are no α_i terms). For the first case, suppose that $\sum_{i=1}^n a_i s_{\alpha_i} = 0$ and $|\alpha_i| = N$ for all $1 \leq i \leq n$. Because these paths have the same length, no α_i can extend α_j for $i \neq j$, and hence for any k we have

$$a_k p_{r(\alpha_k)} = a_k s_{\alpha_k}^* s_{\alpha_k} = \sum_{i=1}^n a_i s_{\alpha_k}^* s_{\alpha_i} = s_{\alpha_k}^* \sum_{i=1}^n a_i s_{\alpha_i} = 0$$

and since $p_{r(\alpha_k)}$ is a nonzero projection, $a_k = 0$.

The second case is similar: suppose $\sum_{j=1}^m b_j s_{\beta_j}^* = 0$ and $|\beta_j| = N$ for all $1 \leq j \leq m$. For each i we multiply each side of the equation on the right by s_{β_i} to obtain $\left(\sum_{j=1}^m b_i s_{\beta_j}^*\right) s_{\beta_i} = 0$, and $b_i p_{r(\beta_j)} = 0$. Hence $b_i = 0$. $\square$

Lemma 3.2.2. *Let E be a graph, K be a field, and $L(E) := L_K(E)$. If $x \in D'_{L(E)}$, then*

(i) $xQ_\beta = Q_\beta x \in D'_{L(E)}$ *for all* $\beta \in E^*$, *and*

(ii) $s_\beta^* x s_\beta \in D'_{L(E)}$ *for all* $\beta \in E^*$.

Proof. The result in (i) follows from the fact that $Q_\beta \in D_{L(E)}$ and $D_{L(E)}$ is commutative. To obtain the result in (ii), it suffices to show that $s_\beta^* x s_\beta$ commutes with $Q_\alpha := s_\alpha s_\alpha^*$ for all $\alpha \in E^*$. If $\alpha \in E^*$ with $s(\alpha) \neq r(\beta)$, then $(s_\beta^* x s_\beta) s_\alpha s_\alpha^* = 0 = s_\alpha s_\alpha^* (s_\beta^* x s_\beta)$. If $\alpha \in E^*$ with $s(\alpha) = r(\beta)$, then $(s_\beta^* x s_\beta) s_\alpha s_\alpha^* = s_\beta^* x s_{\beta\alpha} s_\alpha^* = s_\beta^* x s_{\beta\alpha} s_{\beta\alpha}^* s_\beta = s_\beta^* s_{\beta\alpha} s_{\beta\alpha}^* x s_\beta = s_\alpha s_{\beta\alpha}^* x s_\beta = s_\alpha s_\alpha^* (s_\beta^* x s_\beta)$. □

The next two lemmas give a decomposition of elements in the commutant of $D_{L(E)}$ that will give specific characteristics that we may look for in the graph E itself to determine the maximality of $D_{L(E)}$.

Lemma 3.2.3. *Let E be a graph, K be a field, and $L(E) := L_K(E)$. Suppose $x \in D'_{L(E)}$. Then x may be written in the form*

$$x = \sum_{i=1}^{n} a_i s_{\mu_i} s_{\nu_i}^* + \sum_{j=1}^{m} b_j s_{\gamma_j} s_{\gamma_j}^*$$

for $a_1, \ldots, a_n, b_1, \ldots, b_m \in K \setminus \{0\}$ and paths $\mu_1, \ldots, \mu_n, \nu_1, \ldots \nu_n, \gamma_1, \ldots, \gamma_m$ with the property that there exists $N \in \mathbb{N}$ for which $|\nu_i| = N$ for all $1 \leq i \leq n$.

Proof. Suppose that $x \in D'_{L(E)}$. As with any element of $L(E)$ we may write $x = \sum_{i \in I} c_i s_{\alpha_i} s_{\beta_i}^*$ for a finite set I. Whenever $\alpha, \beta \in E^*$ are paths with $v := r(\alpha) = r(\beta) \in E^0_{\text{reg}}$, then (CK3) implies $s_\alpha s_\beta^* = s_\alpha p_v s_\beta^* = s_\alpha \left(\sum_{s(e)=v} s_e s_e^* \right) s_\beta^* = \sum_{s(e)=v} s_{\alpha e} s_{\beta e}^*$. If we let $N := \max\{|\beta_i| : i \in I\}$, we may use repeated applications of (CK3) to rewrite

x in the form

$$x = \sum_{j \in J} d_j s_{\alpha_j} s_{\beta_j}^* + \sum_{k \in K} e_k s_{\gamma_k} s_{\delta_k}^*$$

with $r(\alpha_j) = r(\beta_j)$ and $|\beta_j| = N$ for all $j \in J$, and with $r(\gamma_k) = r(\delta_k) \in E^0_{\text{sing}}$ and $|\delta_k| \leq N - 1$ for all $k \in K$.

Since $s_\alpha s_\alpha^* \in D'_{L(E)}$ for any $\alpha \in E^*$, we may subtract any such terms off of x (note that if $x, y \in D'_{L(E)}$, then $x - y \in D'_{L(E)}$), and thus

$$x' := x - y = \sum_{j \in J'} d_j s_{\alpha_j} s_{\beta_j}^* + \sum_{k \in K'} e_k s_{\gamma_k} s_{\delta_k}^* \in D'_{L(E)}$$

for some $y \in D_{L(E)}$ and some subsets $J' \subseteq J$ and $K' \subseteq K$ satisfying $\alpha_j \neq \beta_j$ for all $j \in J'$ and $\gamma_k \neq \delta_k$ for all $k \in K'$.

To obtain the result, it suffices to show that $K = \emptyset$. For the sake of contradiction, suppose $K' \neq \emptyset$. Choose δ to be a path of minimal length from the set $\{\delta_k : k \in K'\}$. Likewise, choose $k_0 \in K'$ so that γ_{k_0} has minimal length among the set $\{\gamma_k : \delta_k = \delta\}$ (and observe this implies $\delta_{k_0} = \delta$) and define $\gamma := \gamma_{k_0}$. Note that the paths $\delta = \delta_{k_0}$ and $\gamma = \gamma_{k_0}$ have the the property that $|\delta| \leq |\delta_k|$ for all $k \in K'$ and $|\gamma| \leq |\gamma_k|$ for all $k \in K'$ with $\delta_k = \delta$.

Since $r(\delta) \in E_{\text{sing}}$, either $r(\delta) \in E^0_{\text{sink}}$ or $r(\delta) \in E^0_{\text{inf}}$. We shall now define paths η and θ based on each of these two cases: If $r(\delta) \in E^0_{\text{sink}}$ define $\eta := \delta$ and $\theta := \gamma$. If $r(\delta) \in E^0_{\text{inf}}$ we may choose an edge $e \in E^1$ with $s(e) = r(\delta)$ such that e does not appear as an edge in any of the paths of $\{\alpha_j, \beta_j : j \in J'\} \cup \{\gamma_k, \delta_k : k \in K'\}$, and in this case we define $\eta := \delta e$ and $\theta := \gamma e$.

Let $j \in J'$. Since η is either equal to δ with $r(\delta)$ a sink or equal to δe with e not in any β_j, and since $|\delta| \leq N - 1 < N = |\beta_j|$, we conclude that neither η nor β_j extends the other. Hence

$$Q_{\beta_j} Q_\eta = 0 \qquad \text{for all } j \in J'. \tag{3.2.3.1}$$

In addition, since $\gamma \neq \delta$ and since either η and θ have ranges equal to a sink or contain e as their last edge, neither η nor θ extends the other, and

$$Q_\theta Q_\eta = 0 \tag{3.2.3.2}$$

Similarly, by the minimality in the choice of δ, for any $k \in K'$ we see that either $\delta_k = \delta$ or neither of η and δ extend each other. Additionally, if $k \in K'$ with $\delta_k = \delta$, then by the minimality in the choice of γ either $\gamma_k = \gamma$ or neither of θ and γ extend each other. Consequently, for any $k \in K'$ we have

$$\begin{aligned} Q_\theta s_{\gamma_k} s_{\delta_k}^* Q_\eta &= \begin{cases} Q_\theta s_{\gamma_k} s_\delta^* Q_\eta & \text{if } \delta_k = \delta \\ 0 & \text{otherwise} \end{cases} \qquad (3.2.3.3) \\ &= \begin{cases} Q_\theta s_\gamma s_\delta^* Q_\eta & \text{if } \delta_k = \delta \text{ and } \gamma_k = \gamma \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} s_\theta s_\eta^* & \text{if } \delta_k = \delta \text{ and } \gamma_k = \gamma \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} s_\theta s_\eta^* & \text{if } k = k_0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Putting these equations together, we obtain

$$\begin{aligned} 0 &= Q_\theta Q_\eta x' \qquad \text{(by (3.2.3.2))} \\ &= Q_\theta x' Q_\eta \qquad (\text{since } x' \in D'_{L(E)}) \end{aligned}$$

$$= Q_\theta \left(\sum_{j \in J'} d_j s_{\alpha_j} s^*_{\beta_j} Q_\eta \right) + \sum_{k \in K'} e_k Q_\theta s_{\gamma_k} s^*_{\delta_k} Q_\eta$$

$$= 0 + \sum_{k \in K'} e_k Q_{\gamma_e} s_{\gamma_k} s^*_{\delta_k} Q_{\delta e} \qquad \text{(by (3.2.3.1))}$$

$$= e_{k_0} s_\theta s^*_\eta \qquad \text{(by (3.2.3.3))}$$

which contradicts the fact that $e_{k_0} \neq 0$.

Thus $K' = \emptyset$, $x' := x - y = \sum_{j \in J'} d_j s_{\alpha_j} s^*_{\beta_j}$ with $|\beta_j| = N$ for all $j \in J'$, and x has the claimed form. $\square$

Lemma 3.2.4. *Let E be a graph, K be a field, and $L(E) := L_K(E)$. Suppose $x \in D'_{L(E)}$ and*

$$x = \sum_{i=1}^{n} a_i s_{\mu_i} s^*_{\nu_i}$$

for $a_1, \ldots, a_n \in K$ and paths $\mu_1, \ldots, \mu_n, \nu_1, \ldots \nu_n \in E^*$ with the property that there exists $N \in \mathbb{N}$ for which $|\nu_i| = N$ for all $1 \leq i \leq n$. Then x has the form

$$x = \sum_{i=1}^{k} b_i s_{\alpha_i} s^*_{\alpha_i} + \sum_{i=1}^{l} c_i s_{\beta_i \sigma_i} s^*_{\beta_i} + \sum_{i=1}^{m} d_i s_{\gamma_i} s^*_{\gamma_i \theta_i}$$

where $b_1, \ldots, b_k, c_1, \ldots, c_l, d_1, \ldots, d_m \in K$ are elements of K, $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_l$, $\gamma_1, \ldots, \gamma_m$ are paths in E, and $\sigma_1, \ldots, \sigma_l$, $\theta_1, \ldots, \theta_m$ are cycles with no exits in E.

Proof. For each $\nu \in \{\nu_1, \ldots, \nu_n\}$. Let $x_\nu := xQ_\nu$. Note that since all the ν_i have the same length, no ν_i can extend a ν_j unless $\mu_i = \mu_j$. Hence

$$x_\nu = \sum_{\{i : \nu_i = \nu\}} b_i s_{\mu_i} s^*_\nu$$

and $x = \sum_\nu x_\nu$ where the sum ranges over the distinct values of $\{\nu_i\}_{i=1}^n$.

For the remainder of the proof, fix $\nu \in \{\nu_1, \ldots, \nu_n\}$. By Lemma 3.2.2(i), we have that $x_\nu := xQ_\nu \in D'_{L(E)}$. Moreover, we see that $s^*_\nu s_{\mu_i}$ is zero unless one of ν and μ_i

extends the other by the following computation:

$$x_\nu = xQ_\nu = xQ_\nu^2 = Q_\nu x Q_\nu = s_\nu s_\nu^* \left(\sum_{i=1}^{n} a_i s_{\mu_i} s_{\nu_i}^* \right) s_\nu s_\nu^* = \sum_{\{i:\mu_i=\nu_i\}} a_i s_\nu (s_\nu^* s_{\mu_i}) s_\nu^*.$$

Any zero terms may be discarded, and for the remaining terms we partition them into three sets: We let S_1 be the set of indices for which $\mu_i = \nu$, we let S_2 be the set of indices for which μ_i strictly extends ν, in which case we write $\mu_i = \nu\sigma_i$ for a nontrivial path σ_i, and we let S_3 be the set of indices for which ν strictly extends μ_i, in which case we write $\nu = \mu_i\theta_i$ for a nontrivial path θ_i. Thus

$$\begin{aligned} x_\nu &= \sum_{i\in S_1} a_i s_\nu (s_\nu^* s_\nu) s_\nu^* + \sum_{i\in S_2} a_i s_\nu (s_\nu^* s_{\nu\sigma_i}) s_\nu^* + \sum_{i\in S_3} a_i s_\nu (s_{\mu_i\theta_i}^* s_{\mu_i}) s_\nu^* \\ &= \sum_{i\in S_1} a_i s_\nu s_\nu^* + \sum_{i\in S_2} a_i s_{\nu\sigma_i} s_\nu^* + \sum_{i\in S_3} a_i s_\nu s_{\nu\theta_i}^*. \end{aligned} \tag{3.2.3.4}$$

Since $\mu_i = \nu\sigma_i$, we see that $s(\sigma_i) = r(\nu)$ and $r(\sigma_i) = r(\mu_i) = r(\nu)$. Hence θ_i is a cycle. Likewise, since $\nu = \mu_i\theta_i$, we have $r(\theta_i) = r(\nu)$, and $s(\theta_i) = r(\mu_i) = r(\nu)$, so that θ_i is a cycle. We shall show that the cycles σ_i and θ_i are all cycles with no exits.

Since $x_\nu \in D'_{L(E)}$, it follows from Lemma 3.2.2 that $s_\nu^* x_\nu s_\nu \in D'_{L(E)}$. Also, since $D_{L(E)}$ is commutative, $\sum_{i\in S_1} a_i s_\nu s_\nu^* \in D_{L(E)} \subseteq D'_{L(E)}$. Since $D'_{L(E)}$ is closed under differences, we conclude

$$\begin{aligned} &\sum_{i\in S_2} a_i s_{\sigma_i} + \sum_{i\in S_3} a_i s_{\theta_i}^* \\ &= s_\nu^* \left(\sum_{i\in S_2} a_i s_{\nu\sigma_i} s_\nu^* + \sum_{i\in S_3} a_i s_\nu s_{\nu\theta_i}^* \right) s_\nu \\ &= s_\nu^* \left(\sum_{i\in S_1} a_i s_\nu s_\nu^* + \sum_{i\in S_2} a_i s_{\nu\sigma_i} s_\nu^* + \sum_{i\in S_3} a_i s_\nu s_{\nu\theta_i}^* \right) s_\nu - \sum_{i\in S_1} a_i s_\nu s_\nu^* \\ &= s_\nu^* x_\nu s_\nu - \sum_{i\in S_1} a_i s_\nu s_\nu^* \\ &\in D'_{L(E)}. \end{aligned}$$

Fix $j \in S_2$ and consider the cycle σ_j. For the sake of contradiction, suppose that σ_j has an exit. Write $\sigma_i = e_1 \dots e_r \eta$ where $e_1, \dots, e_r$ are the edges of a simple cycle and $\eta \in E^*$ is a path. Then there exists an exit $f \in E^1$ with $s(f) = e_k$ for some $1 \leq k \leq r$ and $f \neq e_k$. Let $\xi := e_1 \dots e_r$ and let $\lambda := e_1 \dots e_{k-1} f$. Since neither of ξ and λ extends the other, we have $Q_\xi Q_\lambda = 0$. Using the fact that $\sum_{i \in S_2} a_i s_{\sigma_i} + \sum_{i \in S_3} a_i s^*_{\theta_i} \in D'_{L(E)}$ we obtain

$$\begin{aligned}
&\sum_{i \in S_2} a_i s^*_\xi s_{\sigma_i \lambda} + \sum_{i \in S_3} a_i s^*_{\theta_i \xi} s_\lambda \\
&= s^*_\xi \left(\sum_{i \in S_2} a_i s_{\sigma_i} + \sum_{i \in S_3} a_i s^*_{\theta_i} \right) s_\lambda = s^*_\xi Q_\xi \left(\sum_{i \in S_2} a_i s_{\sigma_i} + \sum_{i \in S_3} a_i s^*_{\theta_i} \right) Q_\lambda s_\lambda \\
&= s^*_\xi \left(\sum_{i \in S_2} a_i s_{\sigma_i} + \sum_{i \in S_3} a_i s^*_{\theta_i} \right) Q_\xi Q_\lambda s_\lambda = 0 \qquad (3.2.3.5)
\end{aligned}$$

If we consider $\sum_{i \in S_2} a_i s^*_\xi s_{\sigma_i \lambda}$, we observe that for each $i \in S_2$, ξ cannot strictly extend the cycle σ_i because ξ is a simple cycle. Thus the only nonzero terms in this sum occur when σ_i extends ξ in which case we write $\sigma_i = \xi \sigma'_i$.

Furthermore, since $|\lambda| \leq |\xi|$, for each $i \in S_3$ the term $s^*_{\theta_i \xi} s_\lambda$ is either equal to zero or equal to $s^*_{\rho_i}$ for some path $\rho_i \in E^*$. Hence (3.2.3.5) becomes

$$\sum_{\{i : \sigma_i = \xi \sigma'_i\}} a_i s_{\sigma'_i} + \sum_i a_i s^*_{\rho_i} = 0.$$

The fact that the σ_i are distinct implies that the σ'_i are distinct. Furthermore, from our prior choice of j we know that σ_j extends ξ, and hence j is one of the indices appearing in the left sum in the above equation. It follows from Lemma 3.2.1 that $a_j = 0$, which is a contradiction. (As stated earlier, if $a_j = 0$, we could simply discard that term.) Hence σ_j is a cycle without exits for all $j \in S_2$.

A very similar argument shows that the cycle θ_i has no exits for all $i \in S_3$. Since $x = \sum_\nu x_\nu$ and each x_ν has the form shown in (3.2.3.4), the claim holds. $\square$

Theorem 3.2.5. *Let E be a graph, let K be a field, and write $L(E) := L_K(E)$. The commutant of $D_{L(E)}$ is equal to the set of all elements of $L(E)$ of the form*

$$x = \sum_{i=1}^{k} b_i s_{\alpha_i} s_{\alpha_i}^* + \sum_{i=1}^{l} c_i s_{\beta_i \sigma_i} s_{\beta_i}^* + \sum_{i=1}^{m} d_i s_{\gamma_i} s_{\gamma_i \theta_i}^*$$

where $b_1, \ldots, b_k, c_1, \ldots, c_l, d_1, \ldots, d_m \in K$ are elements of K, $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_l$, $\gamma_1, \ldots, \gamma_m \in E^*$ are paths in E, and $\sigma_1, \ldots, \sigma_l$, $\theta_1, \ldots, \theta_m \in E^*$ are cycles with no exits in E.

Proof. It is straightforward to verify that elements of the above form commute with each element of $D_{L(E)}$, so that these elements are in $D'_{L(E)}$. Conversely given an element in $D'_{L(E)}$ Lemma 3.2.3 combined with Lemma 3.2.4 shows that the element has the above form. □

Corollary 3.2.6. *Let E be a graph, let K be a field, and write $L(E) := L_K(E)$. Then $D_{L(E)}$ is a MASA in $L(E)$ if and only if E satisfies Condition (L).*

Chapter 4

Lifting Morphisms and the L_2-problem

In this chapter, we consider how morphisms between Leavitt path algebras can be lifted to morphisms between graph C^*-algebras, and we verify claims that we have described in our approach to the L_2-problem.

4.1 Lifting $*$-homomorphisms from $L(E)$ to $C^*(E)$

If $C^*(E)$ is a graph C^*-algebra with generating Cuntz-Krieger E-family $\{p_v, s_e, v \in E^0, e \in E^1\}$, then the elements $\{p_v, s_e, s_e^* : v \in E^0, E^1\}$ satisfy the defining relations for the Leavitt path algebra $L(E)$ and hence by the universal property there exists a $*$-homomorphism $i_E : L(E) \to C^*(E)$ mapping the generators of $L(E)$ onto the generators of $C^*(E)$. A straightforward application of the graded uniqueness theorem [13, Theorem 4.8] shows that i_E is injective. Hence $L(E)$ is isomorphic to $\operatorname{im} i_E = \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}$. Rather than repeatedly referencing the map i_E, we shall simply identify $L(E)$ with the dense $*$-subalgebra $\operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}$ sitting inside of the graph C^*-algebra $C^*(E)$ so that

$$C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}.$$

Proposition 4.1.1. *Let E and F be graphs, and let $L(E) := L_{\mathbb{C}}(E)$ and $L(F) := L_{\mathbb{C}}(F)$. If $\phi : L(E) \to L(F)$ is a $*$-homomorphism, then there exists a $*$-homomorphism $\Phi : C^*(E) \to C^*(F)$ such that $\Phi|_{L(E)} = \phi$. Furthermore, if ϕ is injective, then Φ is injective; and if ϕ is surjective, then Φ is surjective.*

Proof. Since ϕ is a $*$-homomorphism, the set $\{\phi(p_v), \phi(s_e) : v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in $C^*(F)$, and hence by the universal property of $C^*(E)$ there exists a $*$-homomorphism $\Phi : C^*(E) \to C^*(E)$ with $\Phi(p_v) = \phi(p_v)$ for all $v \in E^0$ and $\Phi(s_e) = s_e$ for all $e \in E^1$. Since $\Phi|_{L(E)}$ and ϕ are $*$-homomorphisms on $L(E)$ that agree on generators, we conclude $\Phi|_{L(E)} = \phi$.

If ϕ is injective, then $\phi : L(E) \to \operatorname{im}\phi$ is a $*$-isomorphism. Let $\phi^{-1} : \operatorname{im}\phi \to L(E)$ be the inverse of this $*$-isomorphism. Let $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ denote the standard gauge action on $C^*(E)$. For each $z \in \mathbb{T}$, we have $\gamma_z(s_e) = zs_e$ and $\gamma(p_v) = p_v$, so we conclude that $\gamma_z(L(E)) \subseteq L(E)$. For each $z \in \mathbb{T}$ define $\beta_z := \phi \circ \gamma_z \circ \phi^{-1} : \operatorname{im}\phi \to \operatorname{im}\phi$. Since ϕ is the restriction of the $*$-homormorphism Φ and ϕ is inejctive, it follows that ϕ is isometric. Likewise γ_z is isometric. Hence β_z is bounded and extends to a $*$-homomorphism $\overline{\beta}_z : \overline{\operatorname{im}\phi} \to \overline{\operatorname{im}\phi}$. It is easy to verify that $\overline{\beta}_{\overline{z}}$ is an inverse of $\overline{\beta}_z$, and thus $\overline{\beta}_z \in \operatorname{Aut}\overline{\operatorname{im}\phi}$. Hence $z \mapsto \overline{\beta}_z$ gives a gauge action on $\overline{\operatorname{im}\phi}$. Moreover, we see that $\phi \circ \gamma_z = \phi \circ \gamma_z \circ \phi^{-1} \circ \phi = \overline{\beta}_z \circ \phi$. Thus $\Phi \circ \gamma_z = \overline{\beta}_z \circ \Phi$, since the maps on each side of the equals sign agree on generators. Since ϕ is injective, $\Phi(p_v) = \phi(p_v) \neq 0$. The gauge-invariant uniqueness theorem implies that Φ is injective.

If ϕ is surjective, then $\operatorname{im}\Phi \supseteq \Phi(L(E)) = \phi(L(E)) = L(F)$. Since $\operatorname{im}\Phi$ is closed,

$\operatorname{im}\Phi$ contains $\overline{L(F)} = C^*(F)$, and hence Φ is surjective. $\square$

Definition 4.1.2. Let E and F be graphs, let K be a field, and define $L(E) := L_K(E)$ and $L(F) := L_K(F)$. We say that a subset $M \subseteq L(E)$ is an $L(F)$-*diagonal* if there exists a $*$-isomorphism $\psi : L(F) \to L(E)$ with $\psi(D_{L(F)}) = M$.

Note that Corollary 3.2.6 implies that when E satisfies Condition (L), any $L(F)$-diagonal is a MASA.

Proposition 4.1.3. *Let E and F be finite graphs with the property that $L(E) := L_{\mathbb{C}}(E)$ and $L(F) := L_{\mathbb{C}}(F)$ are simple algebras. If there exists an $L(F)$-diagonal $M \subseteq L(E)$ and a $*$-automorphism $\phi \in \operatorname{Aut} L(E)$ with $\phi(D_{L(E)}) = M$, then* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

Proof. Since M is an $L(F)$-diagonal, there exists a $*$-isomorphism $\psi : L(F) \to L(E)$ with $\psi(D_{L(F)}) = M$. Thus $\lambda := \phi \circ \psi^{-1} : L(E) \to L(F)$ is a $*$-isomorphism with $\rho(D_{L(E)}) = D_{L(F)}$. By Proposition 4.1.1 there exists a $*$-isomorphism $\Lambda : C^*(E) \to C^*(F)$ with $\Lambda|_{L(E)} = \lambda$. Consequently, $\Lambda : C^*(E) \to C^*(F)$ is a $*$-isomorphism with

$$\Lambda(D_{C^*(E)}) = \Lambda(\overline{D_{L(E)}}) = \overline{\Lambda(D_{L(E)})} = \overline{\lambda(D_{L(E)})} = \overline{D_{L(F)}} = D_{C^*(E)}.$$

It follows from Theorem 1.4.1 that $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$. $\square$

Corollary 4.1.4. *Let E be a finite graph for which $L(E) := L_{\mathbb{C}}(E)$ is a simple algebra. If $L(E)$ has the property such that whenever M is a MASA in $L(E)$ there exists a $*$-automorphism $\phi \in \operatorname{Aut} L(E)$ with $\phi(D_{L(E)}) = M$, then whenever F is a graph and $L(E)$ is $*$-isomorphic to $L(F)$, we have* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

In fact, one may impose a weaker condition on the above corrollary due to simplicity implying injectivity for $*$-homomorphisms between Leavitt path algebras.

Corollary 4.1.5. *Let E be a finite graph for which $L(E) := L_{\mathbb{C}}(E)$ is a simple algebra. If $L(E)$ has the property such that whenever M is a MASA in $L(E)$ there exists a $*$-automorphism $\phi \in \operatorname{Aut} L(E)$ with $\phi(D_{L(E)}) = M$, then whenever F is a graph and there exists a surjective $*$-homomorphism from $L(E)$ to $L(F)$, we have* $\operatorname{sign}\det(I - A_E^t) = \operatorname{sign}\det(I - A_F^t)$.

Chapter 5

Automorphisms

In this chapter, we consider methods of constructing endormorphisms and automorphisms for a Leavitt path algebra.

5.1 Endormophisms and Unitaries in Leavitt Path Algebras

If $L(E)$ is a unital Leavitt path algebra, an element $U \in L(E)$ is called a *unitary* if $U^*U = U^*U = 1$. We let $\mathcal{U}(L(E)) := \{U \in L(E) : UU^* = U^*U = 1\}$ denote the set of unitaries in $L(E)$, and we let

$$\mathcal{U}_v(L(E)) := \{U \in \mathcal{U}(L(E)) : Up_v = p_vU \text{ for all } v \in E^0 \}$$

denote the unitaries in $L(E)$ that commute with each vertex projection. Note that if $U \in \mathcal{U}_v(L(E))$, then $Up_vU^* = U^*p_vU = p_v$ for all $v \in E^0$.

Let $\operatorname{End} L(E) := \{\phi : L(E) \to L(E) : \phi \text{ is a } *\text{-homomorphism}\}$ denote the endomorphisms on $L(E)$, and let

$$\operatorname{End}_v L(E) := \{\phi \in \operatorname{End} L(E) : \phi(p_v) = p_v \text{ for all } v \in E^0\}$$

denote the endomorphism that fix vertex projections. Note that if $\phi \in \operatorname{End}_v L(E)$, then $\phi(1) = \phi(\sum_{v\in E^0} p_v) = \sum_{v\in E^0} \phi(p_v) = \sum_{v\in E^0} p_v = 1$, so any $\phi \in \operatorname{End}_v L(E)$ is a unital endomorphism.

Proposition 5.1.1. *Let E be a finite graph with no sinks, let K be a field, and let $L(E) := L_K(E)$. If $U \in \mathcal{U}_v(L(E))$, then there exists a unique endomorphism $\alpha_U \in \operatorname{End}_v L(E)$ with $\alpha_U(s_e) = U^* s_e$ for all $e \in E^1$. If $\alpha \in \operatorname{End}_v L(E)$, then there exists a unique $U_\alpha \in \mathcal{U}_v(L(E))$ such that $\alpha(s_e) = U_\alpha^* s_e$ for all $e \in E^1$.*

Proof. Let $U \in \mathcal{U}_v(L(E))$. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger E-family for $L(E)$, then $\{U^* s_e, p_v : e \in E^1, v \in E^0\}$ is also a Cuntz-Krieger E-family. All the relations are straightforward to verify, and the fact that U commutes with vertex projections gives (CK2): $\sum_{s(e)=v}(U^*s_e)(U^*s_e)^* = \sum_{s(e)=v} U^* s_e s_e^* U = U^*(\sum_{s(e)=v} s_e s_e^*)U = U^* p_{r(e)} U = p_{r(e)}$. By the universal property of $L(E)$, there exists a $*$-homomorphism $\alpha_U : L(E) \to L(E)$ with $\alpha_U(s_e) = U^* s_e$ for all $e \in E^1$ and $\alpha_U(p_v) = p_v$ for all $v \in E^0$. Thus $\alpha_U \in \operatorname{End}_v(L(E))$ with $\alpha_U(s_e) = U^* s_e$ for all $e \in E^1$. Moreover, α is unique because its values on the generators $\{s_e, p_v : e \in E^1, v \in E^0\}$ is prescribed.

If $\alpha \in \operatorname{End}_v L(E)$, we define $U_\alpha := \sum_{e\in E^1} s_e \alpha(s_e)^*$. Then

$$\begin{aligned} U_\alpha U_\alpha^* &= \sum_{e\in E^1} s_e \alpha(s_e)^* \sum_{f\in E^1} \alpha(s_f) s_f^* = \sum_{e,f\in E^1} s_e \alpha(s_e^* s_f) s_f^* = \sum_{e\in E^1} s_e \alpha(p_{r(e)}) s_e^* \\ &= \sum_{e\in E^1} s_e p_{r(e)} s_e^* = \sum_{e\in E^1} s_e s_e^* = \sum_{v\in E^0} p_v = 1 \end{aligned}$$

and

$$U_\alpha^* U_\alpha = \sum_{e \in E^1} \alpha(s_e) s_e^* \sum_{f \in E^1} s_f \alpha(s_f)^* = \sum_{e,f \in E^1} \alpha(s_e) s_e^* s_f \alpha(s_f)^* = \sum_{e \in E^1} \alpha(s_e) p_{r(e)} \alpha(s_e)^*$$
$$= \sum_{e \in E^1} \alpha(s_e) \alpha(p_{r(e)}) \alpha(s_e)^* = \sum_{e \in E^1} \alpha(s_e p_{r(e)} s_e)^* = \alpha(\sum_{e \in E^1} s_e s_e^*) = \alpha(1) = 1.$$

Hence U_α is a unitary. Moreover,

$$U_\alpha^* p_v U_\alpha = (\sum_{e \in E^1} \alpha(s_e) s_e^*) p_v (\sum_{f \in E^1} s_f \alpha(s_f)^* = \sum_{s(e)=v} \alpha(s_e) s_e^* s_e \alpha(s_e)^*$$
$$= \sum_{s(e)=v} \alpha(s_e) p_{r(e)} \alpha(s_e)^* = \sum_{s(e)=v} \alpha(s_e) \alpha(p_{r(e)}) \alpha(s_e)^* = \sum_{s(e)=v} \alpha(s_e p_{r(e)} s_e^*)$$
$$= \sum_{s(e)=v} \alpha(s_e s_e^*) = \alpha(\sum_{s(e)=v} s_e s_e^*) = \alpha(p_v) = p_v.$$

Thus $U_\alpha \in U_e(L(E))$. Moreover, if V is another unitary with $V s_e = s_e$ for all $e \in E^1$, then $U_\alpha s_e = V s_e$ for all $e \in E^1$, and $U_\alpha = U_\alpha 1 = U_\alpha \sum_{e \in E^1} s_e s_e^* = \sum_{e \in E^1} U_\alpha s_e s_e^* = \sum_{e \in E^1} V s_e s_e^* = V \sum_{e \in E^1} s_e s_e^* = V1 = V$. Hence the unitary U_α is unique. $\square$

Proposition 5.1.1 shows that there is a bijective correspondence between the $*$-endomorphisms of $L(E)$ that fix vertex projections and unitaries of $L(E)$ that commute with vertex projections. The map $\alpha \mapsto U_\alpha$ is a bijection from $\operatorname{End}_v L(E)$ onto $\mathcal{U}_v(L(E))$ with inverse given by $U \mapsto \alpha_U$.

Notice that when a graph E has a single vertex (such as the graph E_2 for the Leavitt path algebra L_2), then the only vertex projection is equal to the multiplicative identity 1, and Proposition 5.1.1 gives a bijective correspondence between unital endomorphisms of $L(E)$ and unitaries in $L(E)$.

5.2 Properties of Endomorphisms and Automorphisms

Although Proposition 5.1.1 shows that the map $\alpha \mapsto U_\alpha$ is a bijective correspondence between $\operatorname{End}_v L(E)$ and $U_v(L(E))$, this map does not preserve the standard operations; i.e., composition on $\operatorname{End}_v L(E)$ does not correspond to multiplication in $\mathcal{U}_v(L(E))$. In particular, if $U, V \in \mathcal{U}_v(L(V))$, then $\alpha_U \circ \alpha_V$ is not equal to α_{UV}. Instead, we see that for each $e \in E^1$

$$\alpha_U \circ \alpha_V(s_e) = \alpha_U(V^* s_e) = \alpha_U(V^*)\alpha_U(s_e) = \alpha_U(V^*)U^* s_e = (U\alpha_U(V))^* s_e$$

so that

$$\alpha_U \circ \alpha_V = \alpha_{U\alpha_U(V)}.$$

Using this relation, we may define an operation $\star$ on $\mathcal{U}_v(L(E))$ by

$$U \star V := U\alpha_U(V).$$

This may be viewed as a multiplication satisfying a "cocycle condition". With this operation on $\mathcal{U}_v(L(E))$, the map $\alpha \mapsto \alpha_U$ becomes a bijective, operation-preserving morphism from $\operatorname{End}_v L(E)$ onto $\mathcal{U}_v(L(E))$. In particular, since $\operatorname{End}_v L(E)$ is a monoid under the operation of composition, $\mathcal{U}_v(L(E))$ is also a monoid under the $\star$ operation. Moreover, invertible elements (i.e., automorphisms) in $\operatorname{End}_v L(E)$ correspond to unitaries in $U_v(L(E))$ that are invertible with respect to the $\star$ operation.

Proposition 5.2.1. *Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$.*

(1) If $U \in \mathcal{U}_v(L(E))$, then α_U is surjective if and only if $U \in \operatorname{im} \alpha_U$.

(2) If E satisfies Condition (L), then for every $U \in \mathcal{U}_v(L(E))$ the endomorphism α_U is injective.

Proof. If α_U is surjective, then we trivially have $U^* \in \operatorname{im} \alpha_U$. Conversely, if $U^* \in \operatorname{im} \alpha_U$, then there exists $V \in L(E)$ with $\alpha_U(V) = U^*$. For each $e \in E^1$, we have

$$\alpha_U(V s_e) = \alpha_U(V)\alpha_U(s_e) = U^* U s_e = s_e$$

so that $s_e \in \operatorname{im} \alpha_U$. Since the set $\{s_e : e \in E^1\}$ generates $L(E)$, it follows that $\operatorname{im} \alpha_U$ contains $L(E)$, and hence α_U is surjective.

If E satisfies Condition (L), then for each $U \in \mathcal{U}_v(L(E))$ the endomorphism α_U fixes vertex projections, so that $\alpha_U(p_v) = p_v \neq 0$ for all $v \in E^0$. The Cuntz-Krieger uniqueness theorem then implies that α_U is injective. $\square$

Chapter 6

Endomorphisms and Subalgebras

In this chapter we study the image of the diagonal under an endomorphism α_U.

6.1 Inner Automorphisms

Definition 6.1.1. Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$. Define $\Psi : L(E) \to L(E)$ by

$$\Psi(x) := \sum_{e \in E^1} s_e x s_e^*.$$

For $k \in \mathbb{N} \cup \{0\}$, we let Ψ^k denote the k-fold composition of Ψ; i.e., $\Psi^0 := \mathrm{id}$, and $\Psi^k := \Psi \circ \Psi^{k-1}$ for $k \in \mathbb{N}$.

Lemma 6.1.2. *The map $\Psi : L(E) \to L(E)$ is linear and $*$-preserving. In addition, if $x, y \in L(E)$, and at least one of x and y is in the commutant of $\{p_v : v \in E^0\}$, then $\Psi(xy) = \Psi(x)\Psi(y)$. (In particular, if $x \in \mathcal{U}_v(L(E))$ or $y \in \mathcal{U}_v(L(E))$, then $\Psi(xy) = \Psi(x)\Psi(y)$.)*

Proof. It is straightforward to verify Ψ is linear and $*$-preserving. If either x or y is in the commutant of $\{p_v : v \in E^0\}$, then $\Psi(x)\Psi(y) = \sum_{e\in E^1} s_e x s_e^* \sum_{f\in E^1} s_f y s_f^* = \sum_{e,f\in E^1} s_e x s_e^* s_f y s_f^* = \sum_{e\in E^1} s_e x s_e^* s_e y s_e^* = \sum_{e\in E^1} s_e x p_{r(e)} y s_e^* = \sum_{e\in E^1} s_e x y s_e^* = \Psi(xy)$. □

Lemma 6.1.3. *Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$. If $U \in \mathcal{U}_v(L(E))$, then $\Psi(U) \in \mathcal{U}_v(L(E))$,*

$$\Psi(U^*)s_e = s_e U^* \qquad \text{for all } e \in E^1,$$

and for $k \in \mathbb{N}$

$$\Psi^k(U^*)s_e = s_e \Psi^{k-1}(U^*) \qquad \text{for all } e \in E^1.$$

Proof. It follows from Lemma 6.1.2 that $\Psi(U)\Psi(U)^* = \Psi(UU^*) = \Psi(1) = 1$ and $\Psi(U)^*\Psi(U) = \Psi(U^*U) = \Psi(1) = 1$, so that $\Psi(U)$ is a unitary. Furthermore, for each $v \in E^0$ we have

$$\Psi(U)p_v = \sum_{e\in E^1} s_e U s_e^* p_v = \sum_{s(e)=v} s_e U s_e^* = \sum_{e\in E^1} p_v s_e U s_e^* = p_v \sum_{e\in E^1} s_e U s_e^* = p_v \Psi(U)$$

so that $\Psi(U) \in \mathcal{U}_v(L(E))$. In addition,

$$\Psi(U^*)s_e = \sum_{f\in E^1} s_f U^* s_f^* s_e = s_e U^* s_e^* s_e = s_e U^* p_{r(e)} = s_e p_{r(e)} U^* = s_e U^*.$$

For the final claim, let $k \in \mathbb{N}$. Then $V := \Psi^{k-1}(U)$ is a unitary in $\mathcal{U}_v(L(E))$, and $\Psi(U^*)s_e = s_e U^*$ implies $\Psi^k(U^*)s_e = s_e \Psi^{k-1}(U^*)$. □

Definition 6.1.4. An endomorphism $\alpha : L(E) \to L(E)$ is *inner* if there exists a unitary $V \in L(E)$ such that $\alpha(x) = VxV^*$. For a unitary $V \in L(E)$ we define $\mathrm{Ad}(V) : L(E) \to L(E)$ by $\mathrm{Ad}(V)(x) := VxV^*$.

Theorem 6.1.5. *Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$. An endomorphism $\alpha_U \in \mathrm{End}_v(L(E))$ is inner if and only if there exists $V \in \mathcal{U}_v(L(E))$ such that $U = \Psi(V)V^*$. Moreover, in this situation $\alpha_U = \mathrm{Ad}(V)$.*

Proof. If $U = \Psi(V)V^*$ with $V \in \mathcal{U}_v(L(E))$, then for all $e \in E^1$ we apply Lemma 6.1.3 to obtain $\alpha_U(s_e) = U^* s_e = V\Psi(V)^* s_e = V\Psi(V^*)s_e = V s_e \Psi^0(V^*) = V s_e V^*$. Therefore, if $\alpha = e_1 \dots e_n$ and $\beta = f_1 \dots f_m$, we have

$$\begin{aligned}
\alpha_U(s_\alpha s_\beta^*) &= \alpha_U(s_{e_1}) \dots \alpha_U(s_{e_n}) \alpha_U(s_{f_m})^* \dots \alpha_U(s_{f_1})^* \\
&= V s_{e_1} V^* \dots V s_{e_n} V^* V s_{f_m}^* V^* \dots V s_{f_1}^* V^* \\
&= V s_{e_1} \dots s_{e_n} s_{f_m}^* \dots s_{f_1}^* V^* \\
&= V s_\alpha s_\beta^* V^*.
\end{aligned}$$

Since $L(E) = \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}$, by linearity we obtain $\alpha_U(x) = VxV^*$ for all $x \in L(E)$. Thus α_U is inner.

If α_U is inner, there exists a unitary $V \in L(E)$ such that $\alpha_U(x) = VxV^*$. For any $v \in E^0$, we have $V^* p_v V = V^* \alpha_U(p_v) V = V^* V p_v V^* V = p_v$, so that $V \in \mathcal{U}_v(L(E))$. Moreover, for every $e \in E^1$, we apply Lemma 6.1.3 to obtain

$$U^* s_e = \alpha_U(s_e) = V s_e V^* = V\Psi(V^*) s_e.$$

Thus

$$\begin{aligned}
U^* = U^* 1 = U^* \sum_{e \in E^1} s_e s_e^* &= \sum_{e \in E^1} U^* s_e s_e^* = \sum_{e \in E^1} V\Psi(V) s_e s_e^* \\
&= V\Psi(V) \sum_{e \in E^1} s_e s_e^* = V\Psi(V) 1 = V\Psi(V^*)
\end{aligned}$$

which implies $U = \Psi(V)V^*$. □

6.2 The Fixed-Point Algebra

Definition 6.2.1. Let E be a graph with no sinks. We define the *fixed-point algebra* of $L(E)$ to be the algebra

$$\mathcal{F}_{L(E)} := \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } |\alpha| = |\beta|\}.$$

For each $k \in \mathbb{N} \cup \{0\}$ we define

$$\mathcal{F}_{L(E),k} := \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } |\alpha| = |\beta| = k\}.$$

When E is row-finite with no sinks, an application of (CK2) shows

$$s_\alpha s_\beta^* = s_\alpha p_{r(\alpha)} s_\beta^* = s_\alpha \left(\sum_{s(e)=r(\alpha)} s_e s_e^* \right) s_\beta^* = \sum_{s(e)=r(\alpha)} s_{\alpha e} s_{\beta e}^*$$

and hence

$$\mathcal{F}_{L(E),0} \subseteq \mathcal{F}_{L(E),1} \subseteq \mathcal{F}_{L(E),2} \subseteq \dots$$

We also see that

$$\mathcal{F}_{L(E)} = \bigcup_{k=0}^{\infty} \mathcal{F}_{L(E),k}.$$

When the graph E is understood, we shall often write $\mathcal{F} := \mathcal{F}_{L(E)}$ and $\mathcal{F}_k := \mathcal{F}_{L(E),k}$ to make the notation easier to use.

Lemma 6.2.2. *Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$. Let $x \in \mathcal{F}$. Then there exists $k \in \mathbb{N}$ such that $x \in \mathcal{F}_k$, and*

$$\alpha_U(x) = \operatorname{Ad}(U^*\Psi(U^*) \dots \Psi^k(U^*))(x).$$

Proof. By linearity, it suffices to prove the result when x has the form $s_\alpha s_\beta^*$ for $\alpha, \beta \in E^k$. We shall prove this by induction on $k \in \mathbb{N}$. If $k = 1$, then for any $e, f \in E^1$ we have $\alpha_U(s_e s_f^*) = \alpha_U(s_e)(\alpha_U(s_f))^* = U^* s_e (U^* s_f)^* = U^* s_e s_f^* U = \operatorname{Ad}(U^*)(s_e s_f^*)$ and (1) holds. Assuming that the result holds for $k-1$, let $\alpha, \beta \in E^k$. Then $\alpha = e\alpha'$ and $\beta = f\beta'$ for $e, f \in E^1$ and $\alpha', \beta' \in E^{k-1}$. Hence, using repeated applications of Lemma 6.1.3, we have

$$\begin{aligned}\alpha_U(s_\alpha s_\beta^*) &= \alpha_U(s_e)\alpha_U(s_{\alpha'} s_{\beta'}^*)(\alpha_U(s_f))^* \\ &= U^* s_e U^* \Psi(U^*) \dots \Psi^{k-1}(U^*) s_{\alpha'} s_{\beta'}^* \Psi^{k-1}(U) \dots \Psi(U) U s_f^* U\end{aligned}$$

$$= U^*\Psi(U^*)s_e\Psi(U^*)\dots\Psi^{k-1}(U^*)s_{\alpha'}s^*_{\beta'}\Psi^{k-1}(U)\dots\Psi(U)s^*_f\Psi(U)U$$

$$\vdots$$

$$= U^*\Psi(U^*)\Psi^2(U^*)\dots\Psi^k(U^*)s_es_{\alpha'}s^*_{\beta'}s^*_f\Psi^k(U)\dots\Psi(U)U$$

$$= \mathrm{Ad}(U^*\Psi(U^*)\dots\Psi^k(U^*))(s_\alpha s^*_\beta).$$

□

Definition 6.2.3. Let $S \subseteq L(E)$. We say that an element $a \in L(E)$ *conjugates* S if $a^*Sa \subseteq S$, and we say a *normalizes* S if $a^*Sa \subseteq S$ and $a^*Sa \subseteq S$.

Lemma 6.2.4. *Let E be a finite graph with no sinks, let K be a field, and let $L(E) := L_K(E)$.*

(1) If a conjugates S and b conjugates S, then ab conjugates S.

(2) For any $e \in E^1$, s_e normalizes $D_{L(E)}$; i.e., $s^*_eD_{L(E)}s_e \subseteq D_{L(E)}$ and $s_eD_{L(E)}s^*_e \subseteq D_{L(E)}$.

(3) If a conjugates $D_{L(E)}$, then $\Psi(a)$ conjugates $D_{L(E)}$.

(4) If a conjugates $\mathcal{F}_{L(E)}$, then $\Psi(a)$ conjugates $\mathcal{F}_{L(E)}$.

Proof. (1) Suppose a and b both conjugate S. Then $(ab)^*S(ab) = b^*(a^*Sa)b \subseteq b^*Sb \subseteq S$. Thus ab conjugates S.

(2) For any $\alpha \in E^*$ write $\alpha = e_1\dots e_n$. Then $s_es_\alpha s^*_\alpha s^*_e = s_{\alpha e}s^*_{\alpha e} \in D_{L(E)}$. In addition, $s^*_es_\alpha s^*_\alpha s_e$ is nonzero if and only if $e = e_1$, in which case $s^*_es_\alpha s^*_\alpha s_e = s_{e_2\dots e_n}s^*_{e_2\dots e_n} \in D_{L(E)}$. Since $D_{L(E)} = \mathrm{span}\{s_\alpha s^*_\alpha : \alpha \in E^*\}$, we have by linearity that $s_eD_{L(E)}s^*_e \subseteq D_{L(E)}$ and $s^*_eD_{L(E)}s_e \subseteq D_{L(E)}$.

(3) Suppose a conjugates $D_{L(E)}$. Let $x = s_\alpha s_\alpha^*$ with $\alpha \in E^*$, and write $\alpha = \alpha_1 \dots \alpha_n$ with $\alpha_i \in E^1$ for $1 \le i \le n$. Then (2) and the fact that a conjugates $D_{L(E)}$ implies $s_{\alpha_1} a^* s_{\alpha_2 \dots \alpha_n} s_{\alpha_2 \dots \alpha_n}^* a s_{\alpha_1}^* \in D_{L(E)}$. Thus

$$\begin{aligned}\Psi(a)^* x \Psi(a) &= \left(\sum_{e \in E^1} s_e a^* s_e^*\right) s_\alpha s_\alpha^* \left(\sum_{f \in E^1} s_f a s_f^*\right) = s_{\alpha_1} a^* s_{\alpha_1}^* s_\alpha s_\alpha^* s_{\alpha_1} a s_{\alpha_1}^* \\ &= s_{\alpha_1} a^* s_{\alpha_2 \dots \alpha_n} s_{\alpha_2 \dots \alpha_n}^* a s_{\alpha_1}^* \in D_{L(E)}\end{aligned}$$

By linearity, $\Psi(a)^* D_{L(E)} \Psi(a) \subseteq D_{L(E)}$.

(4) Suppose a conjugates $\mathcal{F}_{L(E)}$. Let $x = s_\alpha s_\beta^*$ with $\alpha, \beta \in E^*$ and $|\alpha| = |\beta|$. Write $\alpha = \alpha_1 \dots \alpha_n$ and $\beta = \beta_1 \dots \beta_n$ with $\alpha_i, \beta_i \in E^1$ for $1 \le i \le n$. Since a conjugates $\mathcal{F}_{L(E)}$, we have $a^* s_{\alpha_2 \dots \alpha_n} s_{\beta_2 \dots \beta_n}^* a \in \mathcal{F}_{L(E)}$. Thus $s_{\alpha_1} a^* s_{\alpha_2 \dots \alpha_n} s_{\beta_2 \dots \beta_n}^* a s_{\beta_1}^* \in \mathcal{F}_{L(E)}$. Consequently,

$$\begin{aligned}\Psi(a)^* x \Psi(a) &= \left(\sum_{e \in E^1} s_e a^* s_e^*\right) s_\alpha s_\beta^* \left(\sum_{f \in E^1} s_f a s_f^*\right) = s_{\alpha_1} a^* s_{\alpha_1}^* s_\alpha s_\beta^* s_{\beta_1} a s_{\beta_1}^* \\ &= s_{\alpha_1} a^* s_{\alpha_2 \dots \alpha_n} s_{\beta_2 \dots \beta_n}^* a s_{\beta_1}^* \in \mathcal{F}_{L(E)}\end{aligned}$$

By linearity, $\Psi(a)^* \mathcal{F}_{L(E)} \Psi(a) \subseteq \mathcal{F}_{L(E)}$. $\square$

Definition 6.2.5. Let E be a finite graph, let K be a field, and let $L(E) := L_K(E)$. Define $\Upsilon : L(E) \to L(E)$ by

$$\Upsilon(x) := \sum_{e,f \in E^1} s_e x s_f^*.$$

Lemma 6.2.6. *Let E be a finite graph with no sinks, let K be a field, and let $L(E) := L_K(E)$.*

(1) $D_{L(E)}$ is generated as an algebra by the set $\{s_e s_e^* : e \in E^1\} \cup \Psi(D_{L(E)})$.

(2) If $U \in \mathcal{U}_v(L(E))$, then $\mathrm{Ad}(U) \circ \Psi \circ \alpha_U = \alpha_U \circ \Psi$.

(3) $\mathcal{F}_{L(E)}$ *is generated as an algebra by the set* $\{s_e s_f^* : e, f \in E^1\} \cup \Upsilon(\mathcal{F}_{L(E)})$.

(4) If $U \in \mathcal{U}_v(L(E))$*, then* $\mathrm{Ad}(U) \circ \Upsilon \circ \alpha_U = \alpha_U \circ \Upsilon$.

Proof. (1) It suffices to prove that for each $k \in \{2, 3, \dots\}$ and $\alpha \in E^k$ we have $s_\alpha s_\alpha^*$ is in the algebra generated by $\{s_e s_e^* : e \in E^1\} \cup \Psi(D_{L(E)})$. Given $\alpha \in E^k$ with $k \geq 2$, write $\alpha = f\alpha'$ with $f \in E^1$ and $\alpha' \in E^{k-1}$. Then $\Psi(s_{\alpha'} s_{\alpha'}^*) = \sum_{e \in E^1} s_e s_{\alpha'} s_{\alpha'}^* s_e^*$. Hence

$$s_f s_f^* \Psi(s_{\alpha'} s_{\alpha'}^*) s_f s_f^* = \sum_{e \in E^1} s_f s_f^* s_e s_{\alpha'} s_{\alpha'}^* s_e^* s_f s_f^* = s_f s_{\alpha'} s_{\alpha'}^* s_f^* = s_\alpha s_\alpha^*.$$

Thus $s_\alpha s_\alpha^*$ is in the algebra generated by $\{s_e s_e^* : e \in E^1\} \cup \Psi(D_{L(E)})$.

(2) For any $x \in L(E)$ we have

$$\begin{aligned} U^* \Psi(\alpha_U(x)) U &= U^* \left(\sum_{e \in E^1} s_e \alpha_U(x) s_e^* \right) U = \sum_{e \in E^1} U^* s_e \alpha_U(x) s_e^* U \\ &= \sum_{e \in E^1} \alpha_U(s_e) \alpha_U(x) \alpha_U(s_e^*) = \alpha_U \left(\sum_{e \in E^1} s_e x s_e^* \right) = \alpha_U(\Psi(x)). \end{aligned}$$

(3) It suffices to prove that for each $k \in \{2, 3, \dots\}$ and $\alpha, \beta \in E^k$ we have $s_\alpha s_\beta^*$ is in the algebra generated by $\{s_e s_f^* : e, f \in E^1\} \cup \Upsilon(\mathcal{F}_{L(E)})$. Given $\alpha, \beta \in E^k$ with $k \geq 2$, write $\alpha = g\alpha'$ and $\beta = h\beta'$ with $g, h \in E^1$ and $\alpha', \beta' \in E^{k-1}$. Then $\Upsilon(s_{\alpha'} s_{\beta'}^*) = \sum_{e,f \in E^1} s_e s_{\alpha'} s_{\beta'}^* s_f^*$. Hence

$$s_g s_g^* \Psi(s_{\alpha'} s_{\beta'}^*) s_h s_h^* = \sum_{e \in E^1} s_g s_g^* s_e s_{\alpha'} s_{\alpha'}^* s_f^* s_h s_h^* = s_g s_{\alpha'} s_{\alpha'}^* s_h^* = s_\alpha s_\beta^*.$$

Thus $s_\alpha s_\beta^*$ is in the algebra generated by $\{s_e s_f^* : e \in E^1\} \cup \Upsilon(\mathcal{F}_{L(E)})$.

(4) For any $x \in L(E)$ we have

$$U^* \Upsilon(\alpha_U(x)) U = U^* \left(\sum_{e,f \in E^1} s_e \alpha_U(x) s_f^* \right) U = \sum_{e,f \in E^1} U^* s_e \alpha_U(x) s_f^* U$$

$$= \sum_{e,f\in E^1} \alpha_U(s_e)\alpha_U(x)\alpha_U(s_e^*) = \alpha_U\left(\sum_{e,f\in E^1} s_e x s_f^*\right) = \alpha_U(\Upsilon(x)).$$

□

Theorem 6.2.7. *Let E be a finite graph with no sinks, let K be a field, and let $L(E) := L_K(E)$. If $U \in \mathcal{U}_v(L(E))$, we have the following.*

(1) If $U^* D_{L(E)} U \subseteq D_{L(E)}$, then $\alpha_U(D_{L(E)}) \subseteq D_{L(E)}$.

(2) If $\alpha_U(D_{L(E)}) = D_{L(E)}$, then $U^* D_{L(E)} U \subseteq D_{L(E)}$.

Proof. (1) Suppose $U^* D_{L(E)} U \subseteq D_{L(E)}$. To establish $\alpha_U(D_{L(E)}) \subseteq D_{L(E)}$, it suffices to show $\alpha_U(s_\mu s_\mu^*) \in D_{L(E)}$ for any $\mu \in E^*$. Let $x = s_\mu s_\mu^*$ with $\mu \in E^*$ and $|\mu| = k$. Then $x = s_\mu s_\mu^* \in \mathcal{F}_k$. Since U conjugates $D_{L(E)}$, it follows from Lemma 6.2.4(3), that each of $U, \Psi(U), \Psi^2(U), \ldots, \Psi^k(U)$ conjugates $D_{L(E)}$. By Lemma 6.2.4(1), the unitary $V := U\Psi(U)\Psi^2(U)\ldots, \Psi^k(U)$ conjugates $D_{L(E)}$. Therefore, applying Lemma 6.2.2 we have $\alpha_U(x) = \mathrm{Ad}(V)(x) = VxV^* \in D_{L(E)}$. Hence $\alpha_U(D_{L(E)}) \subseteq D_{L(E)}$.

(2) Suppose $\alpha_U(D_{L(E)}) = D_{L(E)}$. By Lemma 6.2.6(1) $D_{L(E)}$ is generated as an algebra by the set $\{s_e s_e^* : e \in E^1\} \cup \Psi(D_{L(E)})$. Since U is a unitary, to show U conjugates $D_{L(E)}$, it suffices to prove that U conjugates both $S := \{s_e s_e^* : e \in E^1\}$ and $\Psi(D_{L(E)})$. If $e \in E^1$, then $U^* s_e s_e^* U = \alpha_U(s_e)\alpha_U(s_e^*) = \alpha_U(s_e s_e^*) \in D_{L(E)}$, so $U^* S U \subseteq S$. Furthermore, using Lemma 6.2.6(2) and the fact that $\Psi(D_{L(E)}) \subseteq D_{L(E)}$, we have $U^*\Psi(D_{L(E)})U = U^*\Psi(\alpha_U(D_{L(E)}))U = \alpha_U(\Psi(D_{L(E)})) \subseteq \alpha_U(D_{L(E)}) = D_{L(E)}$. □

Theorem 6.2.8. *Let E be a finite graph with no sinks, let K be a field, and let $L(E) := L_K(E)$. If $U \in \mathcal{U}_v(L(E))$, then we have the following.*

(1) If $U^* \mathcal{F}_{L(E)} U \subseteq \mathcal{F}_{L(E)}$, then $\alpha_U(\mathcal{F}_{L(E)}) \subseteq \mathcal{F}_{L(E)}$.

(2) If $\alpha_U(\mathcal{F}_{L(E)}) = \mathcal{F}_{L(E)}$, then $U^*\mathcal{F}_{L(E)}U \subseteq \mathcal{F}_{L(E)}$.

Proof. (1) Suppose $U^*\mathcal{F}_{L(E)}U \subseteq \mathcal{F}_{L(E)}$. To establish $\alpha_U(\mathcal{F}_{L(E)}) \subseteq \mathcal{F}_{L(E)}$, it suffices to show $\alpha_U(s_\mu s_\nu^*) \in \mathcal{F}_{L(E)}$ for any $\mu, \nu \in E^*$ with $|\mu| = |\nu|$. Let $x = s_\mu s_\nu^*$ with $\mu, \nu \in E^*$ and $|\mu| = |\nu| = k$. Then $x = s_\mu s_\nu^* \in \mathcal{F}_k$. Since U conjugates $\mathcal{F}_{L(E)}$, it follows from Lemma 6.2.4(4), that each of $U, \Psi(U), \Psi^2(U), \dots, \Psi^k(U)$ conjugates $\mathcal{F}_{L(E)}$. By Lemma 6.2.4(1), the unitary $V := U\Psi(U)\Psi^2(U)\dots, \Psi^k(U)$ conjugates $\mathcal{F}_{L(E)}$. Therefore, applying Lemma 6.2.2 we have $\alpha_U(x) = \mathrm{Ad}(V)(x) = VxV^* \in \mathcal{F}_{L(E)}$. Hence $\alpha_U(\mathcal{F}_{L(E)}) \subseteq \mathcal{F}_{L(E)}$.

(2) Suppose $\alpha_U(\mathcal{F}_{L(E)}) = \mathcal{F}_{L(E)}$. By Lemma 6.2.6(3) $\mathcal{F}_{L(E)}$ is generated as an algebra by the set $\{s_e s_f^* : e, f \in E^1\} \cup \Psi(\mathcal{F}_{L(E)})$. Since U is a unitary, to show U conjugates $\mathcal{F}_{L(E)}$, it suffices to prove that U conjugates both $S := \{s_e s_f^* : e, f \in E^1\}$ and $\Upsilon(\mathcal{F}_{L(E)})$. If $e, f \in E^1$, then $U^* s_e s_f^* U = \alpha_U(s_e)\alpha_U(s_f^*) = \alpha_U(s_e s_f^*) \in \mathcal{F}_{L(E)}$, so $U^*SU \subseteq S$. Furthermore, using Lemma 6.2.6(4) and that $\Upsilon(\mathcal{F}_{L(E)}) \subseteq \mathcal{F}_{L(E)}$, we have $U^*\Upsilon(\mathcal{F}_{L(E)})U = U^*\Upsilon(\alpha_U(\mathcal{F}_{L(E)}))U = \alpha_U(\Upsilon(\mathcal{F}_{L(E)})) \subseteq \alpha_U(\mathcal{F}_{L(E)}) = \mathcal{F}_{L(E)}$. □

Theorem 6.2.9. *Let E be a graph that satisfies Condition (L), and let $U \in \mathcal{U}_v(L(E))$. The following are equivalent.*

(1) U conjugates $D_{L(E)}$.

(2) U normalizes $D_{L(E)}$.

(3) $U^*D_{L(E)}U = D_{L(E)}$.

Moreover, each of these conditions implies $\alpha_U(D_{L(E)}) \subseteq D_{L(E)}$.

Proof. (3) $\implies$ (2): If $U^*D_{L(E)}U = D_{L(E)}$, then $UD_{L(E)}U^* = U(U^*D_{L(E)}U)U^* = D_{L(E)}$, so U normalizes $D_{L(E)}$.

(2) $\implies$ (1): This follows trivially from the definitions.

(1) $\implies$ (3): Since E satisfies Condition (L), it follows from Corollary 3.2.6 that $D_{L(E)}$ is a MASA. Since U conjugates $D_{L(E)}$ se have $U^*D_{L(E)}U \subseteq D_{L(E)}$. Hence $D_{L(E)} = UU^*D_{L(E)}UU^* \subseteq UD_{L(E)}U^*$. Since U is a unitary and $D_{L(E)}$ is an abelian subalgebra, $UD_{L(E)}U^*$ is an abelian subalgebra. Because $UD_{L(E)}U^*$ contains $D_{L(E)}$, the fact $D_{L(E)}$ is a MASA implies $UD_{L(E)}U^* = D_{L(E)}$. Thus $U^*D_{L(E)}U = U^*(UD_{L(E)}U^*)U = D_{L(E)}$.

If any of conditions (1), (2), or (3) holds, then (3) holds, and $\alpha_U(D_{L(E)}) \subseteq D_{L(E)}$ by Theorem 6.2.7.

$\square$

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