Phase Retrieval for Finitely-Supported Complex Measures via the Fourier Transform

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ABSTRACT

We study the recovery of a finitely-supported complex measure $\mu = \sum_{j=1}^{s} c_j \delta_{t_j}$ from the magnitudes of linear measurements. The distribution μ is completely determined by the amplitude vector $c \in \mathbb{C}^s$ and the support set $\{t_1, t_2, \ldots, t_s\} \subset [0, \Lambda]$, where $\Lambda > 0$ is assumed to be known. We show that by using magnitudes of point evaluations of the Fourier transform $\widehat{\mu}$ of μ at $\{v_1, v_2, \ldots, v_n\} \subset [-\Omega, \Omega]$, along with magnitudes of differences of modulated point evaluations of $\widehat{\mu}$, we can construct injective maps over the space of all such measures of support length at most s. We follow a measurement design by Alexeev et al. [2] whereby point evaluations of $|\widehat{\mu}|^2$ are encoded as vertices of a graph, and edges of this graph correspond to interference measurements. In particular, if $\Lambda\Omega < s$, $d \geq 3$ and there is a Ramanujan graph, Γ, that is d-regular on $n > \frac{6(1+6/\ln(s/\Lambda\Omega))s}{1-2\sqrt{d-1}/d}$ vertices, then a set of $M = (d+1)n$ magnitude measurements associated with Γ is sufficient for identifying μ up to an overall unimodular multiplicative constant.

Under some additional assumptions, we provide two recovery algorithms. The first algorithm is based on phase propagation and the Prony method. We show that the reconstruction problem can be reduced to applying linear inverses and finding roots of a polynomial in the case of exact measurements for almost every signal of the above form. In the second algorithm, at the cost of introducing a truncation error, we follow the technique presented by Candès and Fernandez-Granda in [22] to show that the solution to a total-variation norm minimization problem defined by the given intensity measurements yields an approximation of μ. We give explicit error bounds for recovery using this method depending on the number of given samples, and discuss the effect of noise in this approach.

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Introduction

1.1 Background

Phase retrieval is an inverse problem in signal processing with applications in many fields such as wireless communications [34, 47], speech recognition [55, 6], medical imaging [67], and quantum mechanics [29, 40, 41, 33]. In this problem, we aim to recover a real or complex-valued signal of interest from the magnitude of linear measurements, also referred to as intensity measurements in the literature, assuming all phase information is lost. A number of interesting questions arise based on this limitation. As a first line of inquiry, we are interested in deriving necessary and sufficient conditions for the injectivity of the measurement map. In this direction, the minimal number of measurements that allow for recovery is important as it provides a condition for the feasibility of reconstruction strategies [30]. Second, we would like to obtain a reconstruction algorithm that allows the signal to be recovered, potentially up to an acceptable residual ambiguity, from as few intensity measurements as possible. Finally, this algorithm should be stable in the sense that the error of reconstruction is proportional to the the measurement error.

In pursuit of injectivity, the two main strategies are to impose additional assumptions either on the measurements, by increasing their number or selecting a particular design, or on the signal space, by restricting the domain to a particular subclass of signals. It is worth exploring these techniques in some variations of the phase retrieval problem that are fairly understood. For example, in the discrete (finite-dimensional) case, the signal is typically modeled as a vector in a Hilbert space, $\mathbb{H}^N$, where often $\mathbb{H} = \mathbb{R}$ or $\mathbb{H} = \mathbb{C}$. The measurements are then represented as magnitudes of inner products with a set of measurement vectors $\Phi = \{\varphi_n\}_{n=1}^M \subseteq \mathbb{H}^N$. It is easy to check that both x and cx give the same intensity measurements, where c is any complex number with $|c| = 1$. Therefore, it is natural to restrict the signal domain to the set of equivalence classes $\mathbb{H}/\mathbb{T} = \{[x] \,|\, x \in \mathbb{H}\}$, where $\mathbb{T}$ represents the unit circle in the complex plane in the case of $\mathbb{H} = \mathbb{C}$, $\mathbb{T} = \{-1, 1\}$ if $\mathbb{H} = \mathbb{R}$, and $[x]$ is defined by letting $x \sim y$ if and only if $y = cx$ for some $c \in \mathbb{C}$ with $|c| = 1$. Other common assumptions for inverse problems in the discrete case include sparsity [48, 22, 10], non-negativity [46, 11, 53] of the measured signals, and enforcing efficient design properties on the measurement vectors in Φ such as the restricted isometry property [21]. In addition, the linear map underlying the intensity measurements is often represented by point evaluations of the Fourier transform of the signal. The problem in this case is known as standard phase retrieval.

If the signal is known to be real-valued, a classification for when Φ allows for phase retrieval of every vector in $\mathbb{R}^N$ up to a unimodular constant was given by Balan, Casazza and Edidin in [6], using Frame theory. Recall the following definition:

Definition 1.1.1. *A frame on a Hilbert space, $\mathbb{H}$, is a set $\mathcal{X} = \{x_n\}_{n\in S} \subset \mathbb{H}$, such that there exist constants A and B with $0 < A \leq B < \infty$, and for all $x \in \mathbb{H}$*

$$A\|x\| \leq \sum_{n\in S} |\langle x, x_n\rangle| \leq B\|x\|.$$

A direct consequence of this definition is that $\mathcal{X}$ spans $\mathbb{H}$ (otherwise, there would be at least one vector $y \in \mathbb{H}$ that is orthogonal to all $x_n \in \mathcal{X}$, which would imply that $A \leq 0$, a contradiction). The following property turns out to be critical for phase retrieval in $\mathbb{R}^N$:

Definition 1.1.2. *A frame $\Phi = \{\varphi_n\}_{n=1}^M$ in $\mathbb{H}^N$ is said to have the complement property if for all subsets $S \subseteq \{1, ..., M\}$, either span$(\{\varphi_n\}_{n\in S}) = \mathbb{H}^N$ or span$(\{\varphi_n\}_{n\in S^c}) = \mathbb{H}^N$.*

Denote by $\mathcal{M} : \mathbb{R}^N \to \mathbb{R}^M$, the measurement map defined by $\mathcal{M}(x) = (|\langle x, \varphi_n\rangle|)_{n=1}^M$. The authors of [6] give the following characterization of injective measurement maps associated with frames in $\mathbb{R}^N$:

Theorem 1.1.3 (Theorem 2.8 from [6])**.** *A frame Φ gives phase retrieval if and only if it has the complement property.*

The same authors proceed to show that a specific choice of frames consisting of $2N-1$ vectors is both necessary and sufficient for phase retrieval up to a unimodular constant in $\mathbb{R}^N$, thereby providing a full classification of the real discrete case [6].

The first analogous results in the complex case were presented by Bandeira, Cahill, Mixon, and Nelson [9]. They give the following classification:

Theorem 1.1.4 (Theorem 4 from [9])**.** *Consider $\Phi = \{\varphi_n\}_{n=1}^M \subset \mathbb{C}^N$ and the mapping $A : \mathbb{C}^N/\mathbb{T} \to \mathbb{R}^M$ defined by $(A(x))(n) := |\langle x, \varphi_n\rangle|$ for $n \in \{1, ..., M\}$. Denote $S(u) := span_{\mathbb{R}}\{\varphi_n\varphi_n^* u\}_{n=1}^M \subseteq \mathbb{R}^{2N}$. Then the following are equivalent:*

(1) A is injective.

(2) $dim S(u) \geq 2N-1$ for every $u \in \mathbb{C}^N \setminus \{0\}$.

(3) $S(u) = span_{\mathbb{R}}\{iu\}^\perp$ for every $u \in \mathbb{C}^N \setminus \{0\}$.

The authors of [9] also conjecture that $4N-4$ vectors are both necessary and sufficient for recovery of every vector in $\mathbb{C}^N$ up to a unimodular constant. In [17], Bodmann and Hammen gave a frame of $4N-4$ vectors such that the associated measurement map is injective, thereby proving that the conjectured number is indeed sufficient. However, in [65] Cynthia Vinzant provided a choice of $4N-5$ vectors that allow for the recovery of every vector when $N = 4$, showing that this number is not necessary. The question regarding the minimal sufficient number of measurements therefore remains open in the case of $\mathbb{C}^N$.

Aside from characterizing injectivity, many recovery algorithms have been proposed in solving the discrete case. One property that is useful in analyzing the stability of these algorithms is the bilipschitz property of the measurement map [7, 5]. This property resembles a restricted isometry property for quadratic measurements [9], and is known to be necessary for recovery in both the real case [28], and the complex case [5].

The standard algorithms in the real case use alternating projections by choosing a feasible solution that fits the measured data in the Fourier domain and some a priori information in the signal domain. These include the Gerchberg-Saxton algorithm [58] and the Fienup input-output algorithm [31]. However, there are no theoretical guarantees for convergence of these algorithms even in the noiseless case [47]. In the complex case, the PhaseLift algorithm uses an embedding of the signal in a space of positive semidefinite matrices to transform the reconstruction problem into a convex optimization problem [23, 24]. See [66] for theoretical guarantees for stability of this method under the sparsity assumption. An alternative approach re-encodes the signal entries as coefficients of a complex polynomial, and uses an embedding of the space of complex polynomial spaces in a corresponding space of trigonometric polynomials [17, 18, 54]. For a review of reconstruction algorithms used in discrete case, see [47, 31]. The stability and robustness of these algorithms in the presence of noise have also been discussed [28, 59, 39]. Unfortunately, it has been shown that no stable recovery strategies exist in the case of phase retrieval for signals in infinite dimensional Hilbert spaces [19]. However, one may hope that additional support restrictions and sparsity assumptions for the signal may help.

In this dissertation, we consider phase retrieval for finitely-supported complex measures. Let $S_{[0,\Lambda],s}$ denote the set of complex measures having support of maximal size $s \in \mathbb{N}$ contained in the interval $[0,\Lambda]$, formally defined as

$$S_{[0,\Lambda],s} = \Big\{\mu = \sum_{j=1}^{s} c_j \delta_{t_j} \,:\, c \in \mathbb{C}^s \text{ and } 0 \le t_1 < t_2 < ... < t_s \le \Lambda\Big\},$$

let the union of all such sets be $S_{[0,\Lambda]} = \cup_{s=1}^{\infty} S_{[0,\Lambda],s}$, and for any measure $\mu = \sum_{j=1}^{s} c_j \delta_{t_j} \in S_{[0,\Lambda]}$, let the Fourier transform $\mathcal{F}(\mu) = \widehat{\mu} : \mathbb{R} \to \mathbb{C}$ be chosen according to the convention so that

$$\widehat{\mu}(\omega) = \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j}.$$

The first simplification of the problem is our assumption that μ has finite support. This can be interpreted as a continuous analog for sparsity. We will also focus on the standard case where the intensity measurements are taken of the Fourier transform of μ. In terms of ambiguities, we observe that for any linear functional ϕ on $S_{[0,\Lambda]}$ and any $c \in \mathbb{C}$ with $|c| = 1$, we have that $|\phi(c\mu)|^2 = |\phi(\mu)|^2$. Thus, our procedure will aim at recovering the equivalence class $[\mu] \in S_{[0,\Lambda]}/\sim$ where $\sim$ is the equivalence relation defined by $\mu \sim \nu$ if and only if $\mu = c\nu$, with $c \in \mathbb{C}$, $|c| = 1$.

In [14], the authors apply the Prony method to this problem to provide a sufficient condition for recovery using a number of measurements that is $O(s^2)$, where s represents the length of the signal. Uniqueness conditions exploiting the sparsity assumption were discussed in [56, 4] where it is shown that a certain condition on the distances between the nodes of the signal, *no collision*, is sufficient for recovery of almost every signal $\mu \in S_{[0,\Lambda],s}$ in the general case of phase retrieval using projections in the Fourier domain. The main procedure in both methods involves attempting to recover the Fourier transform of the auto-correlation function associated with μ, before recovering the support of the original signal along with the amplitudes in a second step. Recall that the auto-correlation function $a_\mu : \mathbb{R} \to \mathbb{C}$ associated with $\mu \in S_{[0,\Lambda],s}$ is defined by $a_\mu(t) := \int_{\mathbb{R}} \mu(x)\overline{\mu(x+t)}dx$. The auto-correlation function is relevant in our analysis since

$$a_\mu(t) = (\mu(x) * \overline{\mu}(-x))(t) = \mathcal{F}^{-1}(\widehat{\mu}(\omega).\overline{\widehat{\mu}}(\omega))(t) = \mathcal{F}^{-1}(|\widehat{\mu}|^2)(t)$$

by properties of the Fourier transform, where $*$ is the convolution operation. The auto-correlation function is therefore determined the by a number of parameters that is quadratic in s (the values of $t_j - t_k$ and $c_j\overline{c_k}$ for $j, k \in \{1, 2, \ldots, s\}$). However, we observe that the actual number of unknowns needed to determine the signal is only twice the support size and not quadratic in s, which motivates the results presented here. We wish to determine μ with a number of intensity measurements that is linear in s. Our contributions in this dissertation are as follows:

(1) **Injectivity of measurements.** We construct injective measurement maps over $S_{\Delta,[0,\Lambda],s}/\sim$ using a total of $O(s)$ magnitudes of the Fourier transform, along with magnitudes of differences

after applying a modulation to the Fourier transform. We apply phase propagation which requires ensuring the existence of non-zero magnitude samples, which can be accomplished using spectral properties of expander graphs, following the strategy presented by Alexeev et al. in [2].

(2) **Recovery algorithms.** We present two recovery algorithms that use $O(s)$ intensity measurements. The first algorithm is based on oversampling, polarization, and the Prony method. We show that if $n > \frac{Cds}{d-2\sqrt{d-1}}$ and n is prime, $d \geq 3$, and Γ is a degree d Ramanujan graph with $n+1$ vertices, then there is a measurement design associated with Γ that allows for the recovery of almost every $[\mu] \in S_{[0,\Lambda]}/\sim$. The second approach considers the application of the result by Candès and Fernandez-Granda in [22] which shows that it is possible to reconstruct μ from a specific set of linear measurements by solving a TV-norm minimization problem. We give guarantees for exact recovery using the second algorithm under some restrictions on the signal parameters, see Theorems 3.2.4 and 3.2.5.

(3) **Error bounds for signal recovery in the presence of noise.** The Prony method, as mentioned, involves finding the roots of a polynomial from its coefficients. The common iterative methods for used to find these roots are known to be highly sensitive to noise in the coefficients [42]. Thus, we will focus our error analysis on the second algorithm. We provide error bounds for recovery from any (not necessarily sufficient for injectivity) number of measurements, and consider the effect of noise on these error bounds. Formally, we show that a sufficiently large choice n exact intensity measurements allows for reconstruction with an error bound that is proportional to $\frac{1}{n^2}$, where n represents the number of measurements. In the presence of noise, the error bound we derive is inversely proportional to the signal-to-noise ratio, $\frac{\|\mu\|_{TV}}{\|\epsilon\|_\infty}$ (see Theorem 3.4.3).

Finally, we point out that the model we adopt in this study is widely applicable. For example, a common model used in X-ray crystallography is the three-dimensional version of our set-up (where the support for μ is taken to be contained $[0,\Lambda]^3 \in \mathbb{R}^3$ for some $\Lambda > 0$) [56, 47]. In this case, the goal is to determine the molecular structure of some sample using intensity measurements modeled as point evaluations of the Fourier transform of the density distribution function associated with the given material. Another example comes from speckle imaging in optical astronomy, where the objective is to obtain high resolution images of astronomical objects from ones that have been distorted by atmospheric turbulence [32, 56]. The problem also has connections to quantum state tomography, where it is desired to recover pure quantum states from measurements that are modeled as non-negative operator valued measures. In fact, the discrete phase retrieval problem can be seen as a special case of this set-up, so that analysis of the case of quantum states has direct implications for the solution in the case of finite-dimensional Hilbert spaces [45, 49]. In addition, despite the difficulty introduced by restriction to the case of Fourier measurements, it has been shown that this choice of measurements fits the models used in many applications [15].

1.2 Outline of the dissertation

All the work contained in this dissertation is based on a joint effort with Bernhard G. Bodmann. In Section 1.3, we will fix some notation and present some elementary results that will be useful in our analysis. In Chapter 2, we give an elementary characterization of injectivity that is analogous to one described in the discrete case in Lemma 9 of [9]. Then, we present a measurement design that ensures injectivity of the associated measurement map based on the Fourier transform of the signal. In particular, we show that a choice of $O(s)$ intensity measurements determines every signal in $S_{[0,\Lambda],s}$ up to a global unimodular constant. We utilize the technique of phase propagation presented in [18] to construct linear measurements of $[\mu]$ from the given quadratic measurements. However, this requires the selection of non-zero sample values in order for relative phase to be well-defined. To guarantee the existence of such samples, we need to obtain an upper bound on the number of roots of the Fourier transform of the signal in a bounded interval. Then, we rely on the connectivity and spectral properties of Ramanujan graphs to choose a measurement design that guarantees the existence of interference measurements associated with each non-zero sample value, following the technique described in [2].

In Chapter 3, we present two frameworks for recovery from a finite number of exact measurements. The key tools we use in this chapter are polarization, the Prony method, and interpolation. The first algorithm is based on the generalized Prony method and uses the injectivity result to provide a guarantee for recovery in the noiseless case. Our second method can be described as a two step procedure, wherein first, linear measurements of μ are constructed using a polarization-like identity similar to [18], then we perform super-resolution to extrapolate the spectrum outside of the sampling interval. We give several sufficient conditions for reconstruction from $O(s)$ exact measurements by imposing certain restrictions on the signal and the measurements. This recovery method requires solving an optimization problem that involves minimizing a a family of quadratic forms constructed using the given intensity measurements. We also provide error bounds for recovery from noisy measurements using the second method.

Chapter 4 explores the possibility of extending our framework for recovery to a larger class of signals. In particular, we show that the same techniques from Chapter 3 can be applied to spline functions with analogous error bounds arising in the presence of noise. Finally, we summarize our results and discuss future work related to this problem.

1.3 Preliminaries

In this section, we define elements of our results in later chapters. We use the following standard notation:

Definition 1.3.1. *For a function $g : \mathbb{R} \to \mathbb{R}$, we denote by $O(g(s))$ the set of all functions, $f(s)$, such that there exists constants $k > 0$ and $a \in \mathbb{R}$ satisfying $0 \leq f(s) \leq kg(s)$ for all $s > a$. We denote by $o(g(s))$ the set of all functions, $f(s)$, such that for every constant $k > 0$, there exists some $a \in \mathbb{R}$ satisfying $0 \leq f(s) \leq kg(s)$ for all $s > a$.*

It is easy to check that $o(g(s)) \subset O(g(s))$ for every function g. Next, we recall the definition of a useful tool for our second recovery algorithm in Chapter 3. The total-variation norm for the space of complex measures over $[-\Lambda, \Lambda]$ with bounded variation is defined as

$$\|.\|_{TV} = |.|([-\Lambda, \Lambda]) = \sup_{\cup_{i=1}^{n} E_i = [-\Lambda,\Lambda]} \sum_{i=1}^{n} |.(E_i)|.$$

It can be thought of as a continuous analog of the usual ℓ_1 norm. In fact, for any $\mu \in S_{[0,\Lambda],s}$ satisfying a minimum spacing requirement, $\min_{j \neq k} |t_j - t_k| \geq \Delta$ for some $\Delta > 0$, we can compute

$$\begin{aligned}
\|\mu\|_{TV} &= \sup_{\cup_{i=1}^{n} E_i = [-\Lambda,\Lambda]} \sum_{i=1}^{n} |\mu(E_i)| \\
&= \sup_{\cup_{i=1}^{n} E_i = [-\Lambda,\Lambda]} \sum_{i=1}^{n} |\sum_{j=1}^{s} c_j \delta_{t_j}(E_i)| \\
&= \sum_{j=1}^{s} |c_j \delta_{t_j}(E_i)| \\
&= \sum_{j=1}^{s} |c_j| = \|c\|_{\ell_1}.
\end{aligned}$$

The remainder of this section is divided into two parts; introducing notation and basic results for distributions and graphs that will be used throughout the remaining chapters.

1.3.1 Essentials of distributions

Let us begin by justifying the formula we give for the Fourier transform of μ. In this case, we think of μ as a tempered distribution. Recall the following definitions:

Definition 1.3.2. *The Schwartz space over* $\mathbb{R}$ *is defined as*

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \,|\, \forall c > 0, n \in \mathbb{N} \cup \{0\}, |f^{(n)}(x)| = o(|x|^c))\}$$

Note that the Schwartz space is not a Banach space with respect to any p-norm (in fact, it is not normable), however, it is dense in $L^2(\mathbb{R})$. The Fourier transform is a linear automorphism over $S(\mathbb{R})$. It is exactly this property that allows us to define the Fourier transform for tempered distributions over $S(\mathbb{R})$.

Definition 1.3.3. *A tempered distribution is a continuous linear functional over* $S(\mathbb{R})$*. The space of all such functionals is denoted by* $S^*(\mathbb{R})$*.*

The dirac delta is defined as the element of $S^*(\mathbb{R})$ with the unique property that

$$\delta_0(f) = \int_{\mathbb{R}} \delta_0 f dt = f(0)$$

for all $f \in S(\mathbb{R})$. We may defined the Fourier transform of any tempered distribution by

Definition 1.3.4. *The Fourier transform of a tempered distribution,* $\phi \in S^*(\mathbb{R})$*, is given by* $\widehat{\phi} : S(\mathbb{R}) \to \mathbb{C}$ *defined as* $\widehat{\phi}(f) = \langle \phi, \widehat{f} \rangle$*.*

Hence, we get for $\phi = \delta_{t_j}$ that $\widehat{\delta_{t_j}}(f) = \int_{\mathbb{R}} \delta_{t_j} \widehat{\mu} dt = \widehat{\mu}(t_j) = \int_{\mathbb{R}} f(x) e^{-2\pi x i t_j} dx$ so that $\mathcal{F}(\delta_{t_j})$ can be identified with $e^{-2\pi i t_j}$ as elements of $S^*(\mathbb{R})$.

Another advantage of the definition of $S(\mathbb{R})$ is that it allows us to derive the famous Poisson summation formula [37, 38] in a fairly straightforward way. We rely on the following proposition [61] for one of the sampling theorems used in Chapter 3.

Proposition 1.3.5 (Poisson Summation Formula)**.** *Let $f \in S(\mathbb{R})$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m).$$

Proof. Let $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Note that the summation on the RHS converges by the decay property of $f \in S(\mathbb{R})$. Thus, $F(x)$ is 1-periodic so it has a Fourier series expansion with coefficients

$$\begin{aligned}
\widehat{\mu}_k &= \int_{\mathbb{R}} F(x)e^{-2\pi ixk}dx \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi ixk}dx \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi ixk}dx \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x+n)e^{-2\pi ixk}dx \\
&= \int_{\mathbb{R}} f(x)e^{-2\pi ixk}dx \\
&= \widehat{f}(k)
\end{aligned}$$

where exchanging the integral and summation is allowed by absolute convergence since $f \in S(\mathbb{R})$. By definition of the Fourier series, we get that $F(x) = \sum_{m\in\mathbb{Z}} \widehat{\mu}(m)e^{ikx}$ and the result follows by setting $x = 0$. □

To derive the generalization of the Poisson summation formula, observe that the Fourier transform for $D(x) = \sum_{n\in\mathbb{Z}} \delta_n(x)$ is $D(x)$ itself. Also, since $D(x)$ is supported on $\mathbb{Z}$, it has a Fourier series expansion given by $D(x) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi inx}$, where $a_n = 1$ for all $n \in \mathbb{Z}$.

Next, in order to perform phase propagation, recall that the relative phase formula for a non-zero complex number $z_2 = |z_2|e^{i\theta_2}$ with respect to $z_1 = |z_1|e^{i\theta_1} \neq 0$ is given by

$$\rho_{12} = \frac{e^{i\theta_2}}{e^{i\theta_1}} = \frac{|z_1|}{z_1}\frac{z_2}{|z_2|}.$$

Thus, the relative phase for a sample value $\widehat{\mu}(v_k)$ with respect to $\widehat{\mu}(v_j)$, assuming both are non-zero, is given by

$$\begin{aligned}\rho_{jk} := \frac{|\widehat{\mu}(v_j)|}{\widehat{\mu}(v_j)} . \frac{\widehat{\mu}(v_k)}{|\widehat{\mu}(v_k)|} &= |\widehat{\mu}(v_j)| . \frac{\overline{\widehat{\mu}(v_j)}}{|\widehat{\mu}(v_j)|^2} . \frac{\widehat{\mu}(v_k)}{|\widehat{\mu}(v_k)|} \\ &= \frac{\overline{\widehat{\mu}(v_j)}\widehat{\mu}(v_k)}{|\widehat{\mu}(v_j)| . |\widehat{\mu}(v_k)|}.\end{aligned}$$

1.3.2 Essentials of graphs

Our measurement design in Chapter 2 follows the idea utilized in algorithm A of [2], where certain properties of a class of graphs called *Ramanujan graphs* are used to guarantee the success of phase propagation. The following definitions and results from graph theory can be found in greater detail in [51, Chapters 1-4]. Our goal is to construct arbitrarily large graphs which are sparse, robust and connected. These notions are made rigorous below, and families of graphs satisfying the relevant properties are termed *expander graphs*. Let us fix some notation.

Definition 1.3.6. *An unoriented graph, Γ, is a triple (V, E, p) such that V and E are any sets, and $p : E \to V^{(2)}$ is a map into $V^{(2)} = \{e \subseteq V \,|\, |e| = 1 \, or 2\}$.*

We will start using graph and unoriented graph interchangeably as all the graphs we will refer to are unoriented. In addition, we will use a pair (V, E) to represent Γ when $E \subseteq V^{(2)}$, since in this case p is simply the inclusion map into $V^{(2)}$. There is a one-to-one correspondence between finite unoriented graphs and symmetric matrices. The following definition gives a (unique) way of constructing such a matrix for each finite graph.

Definition 1.3.7. *For a finite graph Γ, the adjacency matrix, $A_\Gamma = (a(x, y))$, is the matrix with rows and columns indexed by V such that*

$$a(x, y) = |\{\alpha \in E \,|\, p(\alpha) = \{x, y\}\}|.$$

Definition 1.3.8. *We use $val(x)$ to denote the valency of a vertex $x \in V$, which is defined as the* *number of edges in E having x as an extremity. Formally,*

$$val(x) = |\{\alpha \in E \,\big|\, x \in p(\alpha)\}|$$

for each $x \in V$.

The following definition allows us to determine how "close" two vertices are in a graph.

Definition 1.3.9. *A path between two vertices v_j and v_k in $\Gamma = (V, E)$ is a finite sequence $P_{jk} \subseteq V^{|P_{jk}|}$, such that $(P_{jk})_1 = v_j$, $(P_{jk})_{|P_{jk}|} = v_k$ and $\{(P_{jk})_i, (P_{jk})_{i+1}\} \in E$ for all $i \in \{1, ..., |P_{jk}| - 1\}$.*

The distance between two vertices in a graph is then given by the length of the shortest path between them.

Definition 1.3.10. *The distance function on Γ is given by $d_\Gamma : V \times V \to \{0, 1, 2, ...\} \cup \{\infty\}$ where*

$$d_\Gamma(x, y) = \min\{|P_{xy}| \,\big|\, P \textit{ is any path between} x \textit{ and} y\}$$

and $d_\Gamma(x, y) = \infty$ if there is no path between x and y.

Graphs are naturally equipped with a measure space structure recognized in the following way

Definition 1.3.11. *The graph measure, ν_Γ, on Γ is the measure defined on V by*

$$\nu(\{x\}) = val(x)$$

for all $x \in V$.

The normalized graph measure, μ_Γ, on a finite graph Γ is the probability measure on V defined by

$$\mu_\Gamma(\{x\}) = \frac{val(x)}{\sum\limits_{y \in V} val(y)}$$

for all $x \in V$.

In fact, the space of functions on Γ, $L^2[\Gamma, \nu_\Gamma]$, defined as the vector space of all functions $\phi : \Gamma \to \mathbb{C}$ such that the series

$$\sum_{x \in V} val(x)|\phi(x)|^2$$

converges, has a Hilbert space structure when equipped with the inner product defined by

$$\langle \phi_1, \phi_2 \rangle_\Gamma = \sum_{x \in V} val(x)\phi_1(x)\overline{\phi_2(x)}$$

for all $\phi_1, \phi_2 \in L^2[\Gamma, \nu_\Gamma]$. The norm induced by this inner product may be expressed in terms of the adjacency matrix as

$$\|\phi\|^2 = \sum_{x \in V} val(x)|\phi(x)|^2 = \sum_{x,y \in V} a(x,y)|\phi(x)|^2.$$

The following operators defined on $L^2[\Gamma, \nu_\Gamma]$ will help us quantify spectral properties associated with certain classes of graphs.

Definition 1.3.12. *Let Γ be a countable graph with finite valencies and no isolated vertex. The Markov averaging operator on $L^2(\Gamma, \nu_\Gamma)$ is the linear map*

$$K_\Gamma : \begin{cases} L^2(,\Gamma, \nu_\Gamma) \to L^2(\Gamma, \nu_\Gamma) \\ \varphi \mapsto K_\Gamma \varphi \end{cases}$$

where

$$(K_\Gamma \varphi)(x) = \frac{1}{val(x)} \sum_{\substack{\alpha \in E \\ p(\alpha) = \{x,y\}}} \varphi(y) = \frac{1}{val(x)} \sum_{\substack{y \in V \\ d_\Gamma(x,y) \leq 1}} a(x,y)\varphi(y)$$

for each $x \in V$.

Definition 1.3.13. *Let Γ be a finite graph. The normalized Laplace operator of Γ, L_Γ, is the linear* *operator*

$$L_\Gamma : \begin{cases} L^2(,\Gamma,\nu_\Gamma) \to L^2(,\Gamma,\nu_\Gamma) \\ \varphi \mapsto (Id - K_\Gamma)\varphi \end{cases}$$

Now, we give concrete definitions for the desirable properties we would like the graph underlying our measurement design to possess.

Definition 1.3.14. *Let $d \geq 0$. A graph, Γ, is d-regular if $val(x) = d$ for all $x \in V$. It is simple if it has no loops and no multiple edges (or equivalently, if $p : V \to \widehat{E}^{(2)}$ is an injection, where $\widehat{V}^{(2)} = \{e \subseteq V \,\big|\, |e| = 2\} \subset V^{(2)}$). It is connected if $d_\Gamma(x,y) < \infty$ for all $x, y \in V$.*

In addition, a graph Γ is *robust* if it cannot be disconnected easily (meaning that removing a "small" number of edges does not disconnect large subsets of V from the remainder of the graph). It is *sparse* if $\frac{|E|}{|V|}$ is "small". These notions are quantified below.

Definition 1.3.15. *Let $\Gamma = (V, E, p)$ be a finite graph. Denote by $\mathcal{E}(W)$ the set of edges with one extremity in $W \subset V$ and one extremity in the complement of W in V. Formally,*

$$\mathcal{E}(W) := \{\alpha \in E \big| p(\alpha) \cap W \neq \emptyset \, and p(\alpha) \cap W^c \neq \emptyset\}.$$

The Cheeger (expansion) constant $h(\Gamma)$ is then defined by

$$h(\Gamma) := \min\{\frac{|\mathcal{E}(W)|}{|W|} \in [0,\infty) \big| \emptyset \neq W \subset V \, and |W| \leq \frac{1}{2}|V|\}$$

such that $h(\Gamma) = \infty$ if Γ has at most one vertex.

The Cheeger constant is a measure of the robustness of the graph in the sense that large values of $h(\Gamma)$ correspond to more difficulty in disconnecting large subsets of V from the rest of the graph. The following lemma makes this relationship rigorous:

Lemma 1.3.16 (Proposition 3.1.2 of [51])**.** *Let Γ be a finite graph with at least two vertices. Then,* *(i)* $h(\Gamma) > 0$ *if and only if* Γ *is connected, and*

(ii) if $W \subset V$ *such that* $|W| = \delta|V|$ *for some* $\delta \in (0, \frac{1}{2}]$*, then removing fewer than* $\delta h(\Gamma)|V|$ *edges from the graph does not disconnect* W *from the rest of the graph.*

Proof. (i) If Γ is not connected, then there are at least two components of Γ that are connected with at least one of them, W, being non-empty and satisfying $|W| \leq \frac{1}{2}|V|$. Since $\mathcal{E}(W) = \emptyset$, we get that $h(\Gamma) = 0$. Conversely, if $h(\Gamma) = 0$, then there exists at least one non-empty $W \subset V$ such that $|W| \leq \frac{1}{2}|V|$ and $\mathcal{E}(W) = \emptyset$. This means that for each $\alpha \in E$, either $p(\alpha) \subset W$, or $p(\alpha) \subset V \setminus W$, which implies that there is no path from any vertex $x \in W$ to any vertex $y \in V \setminus W$.
(ii) In order to disconnect $W \subset V$ from the rest of the graph, we must remove a set of edges, C, such that $\mathcal{E}(W) \subseteq C$. However, by definition of $h(\Gamma)$, we have that

$$|\mathcal{E}(W)| \geq h(\Gamma)|W| = \delta h(\Gamma)|V|,$$

which means that removing a set $C \subset E$ with $|C| = \delta h(\Gamma)|V| \leq |\mathcal{E}(W)|$ does not disconnect W. □

Another important quantity that is used to describe the robustness of a graph is the following:

Definition 1.3.17. *Let* Γ *be a connected non-empty finite graph without isolated vertices. The equidistribution radius of* Γ*,* ϱ_Γ*, is the maximum of the absolute value for an eigenvalue* λ *of* K_Γ *which is different from* ± 1*.*

Equivalently, ϱ_Γ is the spectral radius of the restriction of K_Γ to the subspace $L^2_0(\Gamma, \mu_\Gamma) := (ker(K_\Gamma - Id) \oplus ker(K_\Gamma + Id)^\perp)$. It can be easily checked that function $\varphi : V \to \mathbb{C}$, defined by $\varphi(x) = 1$ for all $x \in V$, is an eigenfunction of L_Γ corresponding to the eigenvalue $\lambda_0 = 0$. The *normalized spectral gap*, $\lambda_1(\Gamma)$, is the smallest non-zero eigenvalue of L_Γ.The relationship between the expansion constant the spectral gap of the Laplacian for a graph is of great importance. In many cases, it easier to derive a lower bound for $\lambda_1(\Gamma)$ than it is for $h(\Gamma)$ directly. To establish this relationship, we need the following lemma:

Lemma 1.3.18 (Lemma 3.3.5 from [51]). *Let Γ be a finite non-empty graph with no isolated vertices. Let $W \subset V$ and set $\varphi_0 = Id_W - \mu_\Gamma(W)$. Then,*

$$\langle (Id - K_\Gamma)\varphi_0, \varphi_0 \rangle = \langle (Id - K_\Gamma) Id_W, Id_W \rangle = \frac{|\mathcal{E}(W)|}{\sum_{x \in V} val(x)}$$

and

$$\|\varphi_0\|^2 = \mu_\Gamma(W)\mu_\Gamma(V \setminus W).$$

Proof. We have that

$$\begin{aligned}\langle (Id - K_\Gamma)\varphi_0, \varphi_0 \rangle &= \frac{1}{2\sum_{x \in V} val(x)} \sum_{x,y \in V} a(x,y)(\varphi_0(x) - \varphi_0(y))^2 \\ &= \frac{1}{2\sum_{x \in V} val(x)} \sum_{x,y \in V} a(x,y)(Id_W(x) - Id_W(y))^2 \\ &= \frac{1}{\sum_{x \in V} val(x)} \sum_{\substack{x \in W \\ y \in V \setminus W}} a(x,y) = \frac{|\mathcal{E}(W)|}{\sum_{x \in V} val(x)}\end{aligned}$$

Next, since φ_0 is orthogonal to constants, we get that

$$\|\varphi_0\|^2 = \|Id_W\|^2 - \mu_\Gamma(W)^2 = \mu_\Gamma(W) - \mu_\Gamma(W)^2 = \mu_\Gamma(W)\mu_\Gamma(V \setminus W).$$

□

Thus, we obtain the following relationship:

Theorem 1.3.19 (Discrete Cheeger Inequality - Proposition 3.1.2 from [51]). *Let Γ be a connected, non-empty, finite graph with no isolated vertices. Then,*

$$1 - \varrho_\Gamma \le \lambda_1(\Gamma) \le \left(\frac{2v_+}{v_-}\right) h(\Gamma)$$

where

$$v_- = \min_{x \in V} val(x), \quad \textit{and} \quad v_+ = \max_{x \in V} val(x).$$

In particular, if Γ *is* d*-regular, then we have*

$$1 - \varrho_\Gamma \leq \lambda_1(\Gamma) \leq \left(\frac{2}{d}\right) h(\Gamma).$$

Proof. The first inequality follows from the definition of ϱ_Γ and $\Lambda_1(\Gamma)$, so we will focus on the second inequality. First, observe that

$$\begin{aligned}
\frac{v_-|W|}{v_+|V|} &= \frac{|W| \min\limits_{x \in V} val(x)}{|V| \max\limits_{x \in V} val(x)} \\
&\leq \frac{\sum\limits_{x \in W} val(x)}{\sum\limits_{x \in V} val(x)} \\
&= \mu_\Gamma(W) \\
&\leq \frac{|W| \max\limits_{x \in V} val(x)}{|V| \min\limits_{x \in V} val(x)} \\
&= \frac{v_+|W|}{v_-|V|}.
\end{aligned}$$

Second, we have that

$$\lambda_1(\Gamma) = \min_{0 \neq \varphi \perp 1} \frac{\langle (Id - K_\Gamma)\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle}.$$

For $W \subset V$ with $|W| \leq \frac{1}{2}|V|$ and $h(\Gamma) = \frac{|\mathcal{E}(W)|}{|W|}$, defining $\varphi_0 : V \to \mathbb{C}$ as in Lemma 1.2.18, we get that

$$\begin{aligned}
\lambda_1(\Gamma) &\leq \frac{|\mathcal{E}(W)|}{\sum\limits_{x\in V} val(x)\|\varphi_0\|^2} \\
&\leq \frac{1}{\sum\limits_{x\in V} val(x)} \frac{|\mathcal{E}(W)|}{\mu_\Gamma(W)\mu_\Gamma(V\setminus W)} \\
&= \frac{1}{\sum\limits_{x\in V} val(x)} \frac{\sum\limits_{x\in V} val(x)}{\sum\limits_{x\in W} val(x)} \frac{\sum\limits_{x\in V} val(x)}{\sum\limits_{x\in V\setminus W} val(x)} |\mathcal{E}(W)| \\
&\leq \frac{v_+|V|}{v_-|V\setminus W|} \frac{1}{v_-|W|} |\mathcal{E}(W)| \\
&\leq \frac{2v_+}{v_-^2} h(\Gamma).
\end{aligned}$$

The inequality for d-regular Γ follows directly by setting $v_- = v_+ = d$. □

We are now ready to state the definition for the families of graphs used in Chapter 2.

Definition 1.3.20. *A family $(\Gamma_i)_{i\in\mathcal{I}}$ of finite non-empty connected graphs $\Gamma_i = (V_i, E_i, p_i)$ is an expander family if there exist constants $v \geq 1$ and $h > 0$, independent of i, such that the following conditions are satisfied:*

(i) For any $N \geq 1$, there are only finitely many Γ_i with at most N vertices.

(ii) For each $i \in \mathcal{I}$, we have that $\max\limits_{x\in V_i} val(x) \leq v$.

(iii) For each $i \in \mathcal{I}$, $h(\Gamma_i) \geq h > 0$.

Of particular interest to us will be the following class of expander graphs:

Definition 1.3.21. *Let $d \geq 2$ be an integer. A d-regular connected finite graph Γ is called a Ramanujan graph if $\varrho_\Gamma \leq \frac{2\sqrt{d-1}}{d}$.*

Proving the existence of expander families in general, and expander families of Ramanujan graphs in particular, is not a trivial task. Refer to [52] for the full proof of the existence of the latter for each $d > 2$.

Chapter 2

Injectivity

We begin with the following elementary characterization of injectivity that is analogous to that of Lemma 9 from [9]. We will need some additional notation. Let $C := cc^* \in \mathbb{C}^{s\times s}$, $U := diag((e^{-it_j})_{j=1}^s) \in \mathbb{C}^{s\times s}$, and $J \in \mathbb{C}^{s\times s}$ be the matrix with all entries = 1. Also, define for each $N \in \mathbb{N}$ the map $\widetilde{\Phi}_N : A \times \mathcal{H}^{s\times s} \to \mathbb{R}^{N+1}$ by $\widetilde{\Phi}_N(U, C) := tr(CU^n JU^{*n})_{n=0}^N$, where

$$A := \{U \in \mathbb{C}^{s\times s} |\, U = diag((e^{-it_j})_{j=1}^s) \text{ for some } (t_j)_{j=1}^s \in [0, 2\pi)^s\},$$

and $\mathcal{H}^{s\times s}$ is the space of $s \times s$ Hermitian matrices. Consider the restriction $\Phi_N = \widetilde{\Phi}_N|_{A\times B}$ where

$$B := \{C \in \mathcal{H}^{s\times s} | \text{rank}(C) \leq 1\}.$$

Notice that a pair $(U, C) \in A \times B$ can be naturally associated with the parameteris of a signal $\mu \in S_{[0,\Lambda],s}$. Let $J_U := UJU^*$ and $J_U^n := U^n JU^{*n}$ for each $n \in \mathbb{N} \cup \{0\}$.

Theorem 2.0.1. *Φ_N is injective if and only if there is no $(U, C) \in ker(\widetilde{\Phi}_N)$ with $rank(C) = 1$ or 2.*

Proof. Suppose there is $(U, C) \in ker(\widetilde{\Phi}_N)$ with $rank(C) = 1$. Then, $\Phi_N(U, C) = \Phi_N(U, 0_s)$, where 0_s is the $s \times s$ zero matrix. If there is $(U, C) \in ker(\widetilde{\Phi}_N)$ with $rank(C) = 2$, then $C = C_1 - C_2$ for two rank 1 Hermitian matrices C_1 and C_2 by the spectral Theorem. This means that $(tr((C_1 - C_2)J_U^n))_{n=0}^N = 0$, if and only if $(tr(C_1 J_U^n))_{n=0}^N = (tr(C_2 J_U^n))_{n=0}^N$, which implies $\Phi_N(U, C_1) = \Phi_N(U, C_2)$.

Conversely, if Φ_N is not injective, then there are $C_1, C_2 \in B \subset A$ with $\Phi_N(U, C_1) = \Phi_N(U, C_1)$, if and only if $(tr(C_1 J_U^n))_{n=0}^N = (tr(C_2 J_U^n))_{n=0}^N$, and we get that $(tr((C_1 - C_2)J_U^n))_{n=0}^N = 0$, where $C = C_1 - C_2 \in \mathcal{H}^{s \times s}$. Thus, $(U, C) \in ker(\widetilde{\Phi})$. Now, either C_1 or C_2 has at least one non-zero entry, hence $rank(C) = 1$, if $C_1 = 0_s$ or $C_2 = 0_s$, otherwise $rank(C) = 2$ since C_1 and C_2 are linearly independent. □

The measurements we have considered so far are uniformly sampled; that is, the point evaluations of the Fourier transform are assumed to be equidistant. More generally, the above theorem therefore implies that injectivity requires that we avoid C (more precisely, C^*) being orthogonal to $J_U^{v_k}$ with respect to the Hilbert-Schmidt inner product for all point evaluations of $\widehat{\mu}$ at $v_k \in \mathbb{R}$. Equivalently, we must avoid having the amplitude vector c in $ker(J_U^{v_k})$ for all v_k comprising our measurements.

It turns out that it suffices to collect $O(s)$ samples while avoiding point evaluations at roots of $\widehat{\mu}$, as our main result of this chapter shows (see Theorem 2.2.7).

2.1 Estimates for the number of roots of exponential polynomials

In order to guarantee the existence of non-zero sample values, we need an upper bound on the number of roots of $\widehat{\mu}$. Robert Tijdeman gives an optimal upper bound in [63] under the assumption that $s > \Lambda\Omega$, where the length of the sampling interval is 2Ω. However, the proof of Theorem 3 in [63] requires deriving several inequalities that we choose to omit here. Instead, we will describe one possible derivation of the desired bound via elementary methods without placing assumptions on any of the signal parameters.

In the following, let $N_R(\widehat{\mu})$ denote the number of roots $\widehat{\mu}$ has over the region $R \subseteq \mathbb{F}$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We wish to establish an upper bound on $N_{[\omega_0 - \Omega, \omega_0 + \Omega]}(\widehat{\mu})$ for each $\mu \in S_{[0,\Lambda],s}$ for any $\omega_0 \in \mathbb{R}$ and $\Omega > 0$ in terms of the signal length s. The first step is to use the fact that $\widehat{\mu}$ is analytic in conjunction with the maximum principle to get an initial estimate of an upper bound for $N_{[\omega_0-\Omega,\omega_0+\Omega]}(\widehat{\mu})$. The following lemma is a direct consequence of the claim in exercise 6.1-5 of [26]:

Lemma 2.1.1. *Let* $z^* \in \mathbb{C}$, $\Omega > 0$, *and* $\widetilde{\mu} : \mathbb{C} \to \mathbb{C}$ *be an entire function such that* $|\widetilde{\mu}(z)| \leq M$ *for some* $M > 0$ *for all* $z \in \mathbb{C}$ *with* $|z - z_0| \leq 3\Omega$, *and* $|\widetilde{\mu}(z_0)| = a > 0$. *Then,*

$$N_{|z-z_0|\leq\Omega}(\widetilde{\mu}) \leq \frac{1}{\ln(2)} \ln \frac{M}{a}. \tag{2.1}$$

Proof. By applying a suitable translation, it is sufficient to show that (2.1) holds for any non-zero entire function $\widetilde{\nu} : \mathbb{C} \to \mathbb{C}$ with $z_0 = 0$. Since $\widetilde{\nu}$ is analytic for $|z| \leq 3\Omega$, we can write

$$\widetilde{\nu}(z) = \widetilde{\varphi}(z) \prod_{k=1}^{n} (z - z_k)$$

where $\{z_k\}_{k=1}^n$ are the roots of $\widetilde{\nu}$ in the region $|z| \leq \Omega$ and $\widetilde{\varphi}$ is analytic over $|z| \leq 3\Omega$ (so we have that $n = N_{|z|\leq\Omega}(\widetilde{\nu})$). In addition, for any $z^* \in \mathbb{C}$ with $|z^*| = 3\Omega$, we have that $|z^* - z_k| \geq ||z^*| - |z_k|| \geq 2\Omega$ which implies $|z^* - z_k|^{-1} \leq \frac{1}{2\Omega}$ for each $k \in \{1, ..., n\}$, so we get that

$$\begin{aligned} |\widetilde{\varphi}(z^*)| &\leq |\widetilde{\nu}(z^*)| |\prod_{k=1}^{n} (z^* - z_k)^{-1}| \\ &\leq M |\frac{1}{2\Omega}|^n = 2^{-n}\Omega^{-n} M. \end{aligned}$$

Since $\widetilde{\varphi}$ is analytic, the maximum modulus principle implies that $\widetilde{\varphi}$ attains its maximum over $|z| \leq 3\Omega$ at a point along the boundary $|z| = 3\Omega$. In particular, the above inequality implies that

$$\frac{a}{\Omega^n} \leq \frac{a}{|z_1||z_2|...|z_n|} \leq \frac{|\widetilde{\nu}(0)|}{|z_1 z_2 ... z_n|} = |\widetilde{\varphi}(0)| \leq 2^{-n}\Omega^{-n} M$$

which gives

$$N_{|z|\leq\Omega}(\widetilde{\nu}) = n \leq \frac{1}{\ln 2} \ln \frac{M}{a}.$$

The claim follows by setting $\widetilde{\nu}(z) = \widetilde{\mu}(z - z_0)$. $\square$

It is worth noting at this point that Lemma 2.1.1 readily gives an upper bound for the number of roots of $\widehat{\mu}$ in a bounded interval in terms of $\|c\|_{\ell_1}$ instead of s. However, this bound cannot be calculated explicitly without prior knowledge of the location of the roots as the next proposition shows.

Proposition 2.1.2. *Let $\widetilde{\mu} : \mathbb{C} \to \mathbb{C}$ given by $\widetilde{\mu}(z) = \sum_{j=1}^{s} c_j e^{-2\pi i z t_j}$ be the analytic continuation of $\widehat{\mu}$ for some $\mu \in S_{[0,\Lambda]}$ over $|z| \leq \Omega$ and suppose $\widetilde{\mu}$ has the factorization*

$$\widetilde{\mu}(z) = \widetilde{\nu}(z) \prod_{k=1}^{n} (z - z_k)$$

where $\{z_k\}_{k=1}^{n}$ are the roots of $\widetilde{\mu}$ in the region $|z| \leq \Omega$. Let $z_0 \in \mathbb{C}$ such that $\widetilde{\mu}(z_0) > 0$. Let $m := \min_{k \in \{1,\ldots,N\}} |z_0 - z_k| > 0$ and $C := \widetilde{\nu}(z_0) > 0$. Then, we have that

$$N_{|z-z_0| \leq \Omega}(\widehat{\mu}) \leq \frac{\ln \|c\|_{\ell 1}}{\ln 2mC}. \tag{2.2}$$

Proof. Let $z^* \in \mathbb{C}$ satisfy $|\widetilde{\mu}(z^*)| = \max_{|z-z_0| \leq 3\Omega} |\widetilde{\mu}|$. We use the estimate from Lemma 2.1.1 to calculate

$$N_{|z-z_0| \leq \Omega}(\widehat{\mu}) \leq \frac{1}{\ln 2} \frac{\max_{|z-z_0| \leq 3\Omega} |\widetilde{\mu}|}{|\widetilde{\mu}(z_0)|} \leq \frac{1}{\ln 2} \ln \frac{\sum_{j=1}^{s} |c_j| |e^{z_j z^*}|}{|\widetilde{g}(z_0)| (\min_{k \in \{1,\ldots,N\}} |z_0 - z_k|)^N}$$

Thus, we get that

$$N(\ln 2 + \ln mC) = N \ln 2mC \leq \ln \|c\|_{\ell 1}$$

and

$$N_{|z-z_0| \leq \Omega}(\widehat{\mu}) \leq \frac{\ln \|c\|_{\ell 1}}{\ln 2mC}. \qquad \square$$

Let us return to our original derivation. The second step is to bound the RHS of the initial estimate from Lemma 2.1.1 by replacing $\widetilde{\mu}$ with a suitable polynomial and appealing to Viète's forumla [57, 36]. The following lemma is a consequence of Theorem B in [8].

Lemma 2.1.3. *Let $z^* \in \mathbb{C}$. Let $\widetilde{\mu} : \mathbb{C} \to \mathbb{C}$ be the analytic continuation of $\widehat{\mu}$ for some $\mu \in S_{[0.\Lambda],s}$ such that*

$$\widetilde{\mu}(z) = \sum_{j=1}^{s} c_j e^{-2\pi i z t_j}$$

for some $c_j \in \mathbb{C} \setminus \{0\}$ and $t_j \in [0, \Lambda]$ for each $j \in \{1, ..., s\}$. Set $z_j = -2\pi i t_j$ for each $j \in \{1, ..., s\}$ and $\Delta := \min_{j \neq k} |z_j - z_k| > 0$, then we have that $\Lambda' := 2\pi\Lambda \geq |-2\pi i t_j|$ for each $j \in \{1, ..., s\}$ and

$$|\widetilde{\mu}(z^*)| \leq e^{2\pi\Lambda|z^*| + (s-1)[\ln \frac{\Lambda}{\Delta} + 3]} \max_{j=1,...,s} |\widetilde{\mu}^{j-1}(0)|. \tag{2.3}$$

Proof. We have that

$$\widetilde{\mu}(z) = \sum_{j=1}^{s} c_j e^{z_j z}.$$

There is a unique polynomial, $P_{\widehat{\mu}} \equiv P$ of degree at most $s - 1$, where $P : \mathbb{C} \to \mathbb{C}$ and $P(z_j) = e^{z^* z_j}$ for each $j \in \{1, ..., s\}$. This polynomial may be expressed in terms of Lagrange polynomials as

$$P(z) = \sum_{j=1}^{s} e^{z^* z_j} \prod_{k \neq j} \frac{(z - z_k)}{(z_j - z_k)} := \sum_{h=0}^{s-1} p_h z^h$$

for some $p_h \in \mathbb{C}$ for each $h \in \{0, ..., s - 1\}$. This implies that

$$\sum_{j=1}^{s} c_j P(z_j) = \sum_{j=1}^{s} c_j e^{z^* z_j} = \widetilde{\mu}(z^*). \tag{2.4}$$

On the other hand, we have for each $n \in \mathbb{N}$,

$$\frac{d^n}{dz^n}(\widetilde{\mu}(z)) = \sum_{j=1}^{s} c_j z_j^n e^{z_j z},$$

where $\frac{d^n}{dz^n}(\widetilde{\mu}) := \widetilde{\mu}^{(n)}(z)$ denotes the n-th derivative of $\widetilde{\mu}$. In particular, this gives

$$|\widetilde{\mu}^{(n)}(0)| = |\sum_{j=1}^{s} c_j z_j^n| \tag{2.5}$$

for each $n \in \mathbb{N}$.

Together, (2.3) and (2.4) imply that

$$
\begin{aligned}
|\widetilde{\mu}(z^*)| = |\sum_{j=1}^{s} c_j P(z_j)| &= |\sum_{j=1}^{s} c_j (\sum_{h=0}^{s-1} p_h z_j^h)| \\
&= |\sum_{h=0}^{s-1} p_h (\sum_{j=1}^{s} c_j z_j^h)| \\
&\leq \sum_{h=0}^{s-1} |p_h| |\sum_{j=1}^{s} c_j z_j^h| = \sum_{h=0}^{s-1} |p_h| |\widetilde{\mu}^{(h)}(0)| \\
&\leq \sum_{h=0}^{s-1} |p_h| \max_{h=0,1,\ldots,s-1} |\widetilde{\mu}^{(h)}(0)|
\end{aligned}
$$

Set $\Delta' := 2\pi\Delta$. In order to find an upper bound for $\sum_{h=0}^{s-1} |p_h|$, we observe that

$$
\begin{aligned}
P(z) &= \sum_{j=1}^{s} e^{z^* z_j} \prod_{k \neq j} \frac{(z - z_k)}{(z_j - z_k)} \\
&\leq \frac{e^{|z^*|\Lambda'}}{\Delta'^{s-1}} \sum_{j=1}^{s} \prod_{k \neq j} (z - z_k) \\
&\leq \frac{e^{|z^*|\Lambda'}}{\Delta'^{s-1}} s(s\Lambda'^{s-1}) \sum_{j=0}^{s-1} z^j
\end{aligned}
$$

where the last inequality follows from Viète's forumla [57, 36]. This implies that

$$
\sum_{h=0}^{s-1} |p_h| \leq (s-1) s^2 \Lambda'^{s-1} \frac{e^{|z^*|\Lambda'}}{\Delta'^{s-1}} \leq s^3 \Lambda'^{s-1} \frac{e^{|z^*|\Lambda'}}{\Delta'^{s-1}}
$$

and we get that

$$
\begin{aligned}
|\widetilde{\mu}(z^*)| &\leq s^3 \Lambda'^{s-1} \frac{e^{|z^*|\Lambda'}}{\Delta'^{s-1}} \max_{j=1,\dots,s} |\widetilde{\mu}^{j-1}(0)| \\
&= e^{2\pi\Lambda|z^*| + 3\ln s + (s-1)[\ln \frac{\Lambda}{\Delta}]} \max_{j=1,\dots,s} |\widetilde{\mu}^{j-1}(0)| \\
&\leq e^{2\pi\Lambda|z^*| + 3(s-1) + (s-1)[\ln \frac{\Lambda}{\Delta}]} \max_{j=1,\dots,s} |\widetilde{\mu}^{j-1}(0)| \\
&= e^{2\pi\Lambda|z^*| + (s-1)[\ln \frac{\Lambda}{\Delta} + 3]} \max_{j=1,\dots,s} |\widetilde{\mu}^{j-1}(0)|. \qquad \square
\end{aligned}
$$

The third and final step is to combine the bounds from both preceding lemmas using the technique presented in Theorem 1 of [63]. This allows us to give an upper bound for the number of zeros of any $\widehat{\mu}$ with $\mu \in S_{[0,\Lambda],s}$ over a bounded interval.

Theorem 2.1.4. *Let $\omega_0 \in \mathbb{R}, \Omega \geq 1$. Let $\mu \in S_{[0,\Lambda]}$ so that $\widehat{\mu}$ be given by*

$$
\widehat{\mu}(\omega) = \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j}
$$

for some $c_j \in \mathbb{C} \setminus \{0\}$ and $t_j \in [0, \Lambda]$ for each $j \in \{1, ..., s\}$ such that $\widehat{\mu}(\omega_0) > 0$. Then,

$$
N_{[\omega_0 - \Omega, \omega_0 + \Omega]}(\widehat{\mu}) \leq 9\pi\Lambda\Omega + 2s[\ln \frac{\Lambda}{\Delta} + \ln s + 3]. \tag{2.6}
$$

Proof. Define $\widetilde{\mu}$ as in Lemma 2.1.2. Since $\widetilde{\mu}$ is an entire function, we know that it attains its maximum over the region $|z| \leq \Omega$ at a point along the boundary. Choose $z^* \in \mathbb{C}$ such that $|\widetilde{\mu}(z^*\Omega)| = \max_{|z| \leq 3\Omega} |\widetilde{\mu}(z)|$, so we have that $|z^*| = 3$. Then, Lemma 2.1.3. implies that

$$
\begin{aligned}
|\widetilde{\mu}(z^*\Omega)| &\leq e^{2\pi\Lambda|z^*\Omega| + (s-1)[\ln \frac{\Lambda}{\Delta} + 3]} \max_{j=1,\dots,s} |\widetilde{\mu}^{j-1}(0)| \\
&\leq e^{6\pi\Lambda\Omega + (s-1)[\ln \frac{\Lambda}{\Delta} + 3]} \sum_{j=1}^{s} (j-1)! \max_{j=1,\dots,s} \frac{|\widetilde{\mu}^{j-1}(0)|}{(j-1)!}
\end{aligned}
$$

We also have that

$$\max_{j=1,\ldots,s} \frac{|\widetilde{\mu}^{j-1}(0)|}{(j-1)!} \leq \sum_{j=1}^{s} \frac{|\widetilde{\mu}^{j-1}(0)|}{(j-1)!} \leq \sum_{j=1}^{s} \frac{|\widetilde{\mu}^{j-1}(0)\Omega^{j-1}|}{(j-1)!} \leq \max_{|z|\leq\Omega} |\widetilde{\mu}(z)|. \tag{2.7}$$

Applying Lemma 2.1.1 with z_0 satisfying $|\widetilde{\mu}(z_0)| = \max_{|z|\leq\Omega} |\widetilde{\mu}(z)| > 0$, we get that

$$\begin{aligned} N_{|z-z_0|\leq\Omega}(\widetilde{\mu}) &\leq \frac{1}{\ln 2} \ln \frac{|\widetilde{\mu}(z^*\Omega)|}{|\widetilde{f}(z_0)|} \\ &\leq \frac{1}{\ln 2} \ln s! e^{6\pi\Lambda\Omega+(s-1)[\ln\frac{\Lambda}{\Delta}+3]} \\ &\leq \frac{1}{\ln 2} \ln s! + \ln e^{6\pi\Lambda\Omega+(s-1)[\ln\frac{\Lambda}{\Delta}+3]} \\ &\leq \frac{3}{2}[s\ln s + 6\pi\Lambda\Omega + (s-1)[\ln\frac{\Lambda}{\Delta}+3]] \\ &\leq 9\pi\Lambda\Omega + 2s[\ln\frac{\Lambda}{\Delta} + \ln s + 3]. \end{aligned} \tag{2.8}$$

Since the RHS in (7) does not depend on z_0, choosing $z_0 = \omega_0 \in [-\Omega, \Omega] \subset \mathbb{R}$ such that $\widehat{\mu}(\omega_0) > 0$ implies that the restriction $\widetilde{\mu}|_{\mathbb{R}} = \widehat{\mu}$ satisfies

$$N_{[\omega_0-\Omega,\omega_0+\Omega]}(\widehat{\mu}) \leq 9\pi\Lambda\Omega + 2s[\ln\frac{\Lambda}{\Delta} + \ln s + 3]. \qquad \square$$

The assumption $\Omega \geq 1$ is made to simplify the proof of the theorem above, specifically to obtain the inequality (2.6). However, since the upper bound does not depend on ω_0 and applies to any finite interval of length 2Ω with $\Omega \geq 1$, we may conclude that for any $\omega_0 \in \mathbb{R}$ and $0 < \Omega < 1$,

$$N_{[\omega_0-\Omega,\omega_0+\Omega]}(\widehat{\mu}) \leq N_{[\omega_0-1,\omega_0+1]}(\widehat{\mu}) \leq 9\pi\Lambda + 2s[\ln\frac{\Lambda}{\Delta} + \ln s + 3].$$

Finally, the bound in Theorem 2.1.3 is $O(slog(s))$. Tijdeman gives a better bound under the assumption $s > \Lambda\Omega$ in his result from [63] copied below.

Theorem 2.1.5 (Theorem 3 from [63]). *Let $\Lambda, \Omega > 0$, $s \in \mathbb{N}$ with $s > \Lambda\Omega$, and $\mu \in S_{[0,\Lambda]}$ having a support of size at most s, then if $\mu \neq 0$, the number of roots of the Fourier transform $\hat{\mu}$ in $[-\Omega, \Omega]$*

cannot exceed $(1+6/\ln(s/\Lambda\Omega))s$.

2.2 Sufficient condition for injectivity

Our goal in this section is to show that for a measurement design consisting of $O(s)$ intensity measurements, the associated map is injective. Recall that our procedure is aimed at recovering the equivalence class $[\mu] \in S_{[0,\Lambda]}/\sim$ where $\sim$ is the equivalence relation defined by $\mu \sim \nu$ if and only if $\mu = c\nu$, with $c \in \mathbb{C}$, $|c| = 1$. This limitation is based on the fact that for any linear functional ϕ on $S_{[0,\Lambda]}$ and any $c \in \mathbb{C}$ with $|c| = 1$, we have that $|\phi(c\mu)|^2 = |\phi(\mu)|^2$.

The point evaluations of $\hat{\mu}$ are part of the linear functionals we include in the measurement. However, as mentioned in the previous section, if for some $\omega \in \mathbb{R}$, $\hat{\mu}(\omega) = 0$, then only limited information can be deduced from this. Thus, we rely on the bound from Theorem 2.1.5 [63] to formulate a simple consequence for signal recovery from (linear) sampling of the Fourier transform $\hat{\mu}$. Let the sampling set be $\{v_1, v_2, \ldots, v_n\} \subset [-\Omega, \Omega]$, and evaluate the Fourier transform of μ at these points, so the corresponding linear map is $\phi_v : S_{[0,\Lambda]} \to \mathbb{C}^n$, $(\phi_v\mu)_j = \hat{\mu}(v_j)$.

Proposition 2.2.1. *Let* $\Lambda, \Omega > 0$, $s \in \mathbb{N}$ *such that* $s > \Lambda\Omega$, *If* $n > (2 + 12/\ln(2s/\Lambda\Omega))s$, *then the sampling map* ϕ_v *associated with* $\{v_1, v_2, \ldots, v_n\} \subset [-\Omega, \Omega]$, *maps* $S_{[0,\Lambda],s}$ *injectively to* $\mathbb{C}^n$.

Proof. If $\mu_1, \mu_2 \in S_{[0,\Lambda],s}$ give $\phi_v\mu_1 = \phi_v\mu_2$, then $\phi_v(\mu_1 - \mu_2) = 0$. Since $\mu = \mu_1 - \mu_2 \in S_{[0,\Lambda],2s}$, $\widehat{\mu}$ has n zeros. This is only possible if $\mu = 0$, so $\mu_1 = \mu_2$. □

Next, we wish to consider squared magnitude measurements. For this purpose, we consider a finite set of points $V = \{v_1, v_2, \ldots, v_n\}$ in $[-\Omega, \Omega]$ and identify these points with the vertices in a graph Γ having an edge set E. We then consider the following data consisting of intensity values

$$\mathcal{M}_{0,\Gamma}(\mu) = (|\widehat{\mu}(v_j)|^2)_{j=1}^n,$$

and interference magnitudes

$$\mathcal{M}_{1,\Gamma}(\mu) = (|\widehat{\mu}(v_j) - \widehat{\mu}(v_k)|^2)_{\{v_j,v_k\}\in E, j<k},$$
$$\text{and} \quad \mathcal{M}_{2,\Gamma}(\mu) = (|\widehat{\mu}(v_j) - i\widehat{\mu}(v_k)|^2)_{\{v_j,v_k\}\in E, j<k}.$$

This implies that the total number of measured quantities is the sum $N = |V| + 2|E|$. It can also be easily verified that $\Phi_N(U,C) = ((\mathcal{M}_{0,\Gamma_N}(\mu))(n))_{n=0}^N = (\widehat{a}_\mu(n))_{n=0}^N$, therefore, $\mathcal{M}_{0,\Gamma_N}$ is injective if and only if the condition in Theorem 2.0.1 is satisfied. We will need the following lemma which provides the basis for the polarization identity used in the phase propagation algorithm presented in [18].

Lemma 2.2.2. *If $a, b \in \mathbb{C}^2$ and $|a_1| = |b_1|$, $|a_2| = |b_2|$ as well as $|a_1 - a_2| = |b_1 - b_2|$ and $|a_1 - ia_2| = |b_1 - ib_2|$ then the matrix identity $aa^* = bb^*$ holds. Moreover, if $a_2 \neq 0$, then $a_1/a_2 = b_1/b_2$.*

Proof. The stated identities can be viewed as $\mathrm{tr}[A_i aa^*] = \mathrm{tr}[A_i bb^*]$ with $i \in \{1,2,3,4\}$ and $A_1 = e_1e_1^*$, $A_2 = e_2e_2^*$, $A_3 = e_1e_1^* - e_1e_2^* - e_2e_1^* + e_2e_2^*$ and $A_4 = e_1e_1^* - ie_1e_2^* + ie_2e_1^* + e_2e_2^*$, where e_i is the ith canonical basis vector. Since $\{A_1, A_2, A_3, A_4\}$ spans the space of all 2×2 matrices, the rank-one matrices obtained from a and b are identical, $aa^* = bb^*$. This implies $a_1\overline{a_2} = b_1\overline{b_2}$. Assuming $a_2 \neq 0$ and using $|a_2|^2 = |b_2|^2$, we get the desired identity for the quotient. □

As a simple consequence, we have the following sufficient condition for the recovery of any $[\mu] \in S_{[0,\Lambda],1}/\sim$:

Proposition 2.2.3. *Let $\mu \in S_{[0,\Lambda],1}$ and $\{v_1, v_2\} \subset [-\Omega, \Omega]$ with $\Omega\Lambda < 1/2$, then the graph Γ with vertices $V = \{v_1, v_2\}$ and one edge $E = \{\{v_1, v_2\}\}$ permits the recovery of $[\mu]$ from $\{\mathcal{M}_{i,\Gamma}(\mu)\}_{i=0}^2$ with a total of 4 measured magnitudes.*

Proof. Equivalently to the recovery of $[\mu]$, we have to determine $\hat{\mu}(\omega) = ce^{-2\pi it\omega}$, up to a unimodular factor. We know $\mu = 0$ if and only if $\hat{\mu}(v_1) = \hat{\mu}(v_2) = 0$, so it remains to treat the case when $\hat{\mu}$ does not vanish at v_1 or v_2, but then it is non-zero at both points.

Let us fix a representative of the equivalence class of $[\mu]$ by setting $\hat{\mu}(v_2) = |\hat{\mu}(v_2)|$, then measuring the interference magnitudes together with the intensity at v_1 permits us to compute $\hat{\mu}(v_1)/\hat{\mu}(v_2) = e^{-2\pi i t(v_1-v_2)}$ and from $0 \le t|v_1 - v_2| \le 2\Lambda\Omega < 1$, we can solve for t and, in turn, deduce $c = \hat{\mu}(v_2)/e^{-2\pi i t v_2}$. □

Computing $\hat{\mu}(v_i)/\hat{\mu}(v_j)$ is possible if $\hat{\mu}(v_j) \neq 0$. We then say the "phase propagates along the edge $\{v_i, v_j\}$". If there is a path in Γ such that each vertex v_j along the path corresponds to a non-vanishing value $\hat{\mu}(v_j)$, then fixing the phase at one end determines the values of $\hat{\mu}$ at all other vertices of the path. If Γ has a spanning tree where the vanishing values of $\hat{\mu}$ are in a subset of the leaves, then fixing the phase at the root permits to propagate the phase to the entire graph. In general, this may not be possible because of vertices where $\hat{\mu}$ vanishes, but then it is still possible to propagate the phase among vertices in a subset.

Definition 2.2.4. *Let $\Gamma = (V, E)$ be a graph. The subgraph of Γ induced by a subset $W \subset V$ is given by the vertex set W and the set of edges $\{v_j, v_k\} \in E$ with $\{v_j, v_k\} \subset W$. Given a complex measure μ of finite support, we often choose $W = \{v \in V : \hat{\mu}(v) \neq 0\}$ and consider the subgraph Γ_μ induced by W.*

In order to ensure an induced subgraph with a large connected component, we exploit properties of expander graphs, as presented in [2]. As a first step, we formulate a general result.

Theorem 2.2.5. *Let $\mu \in S_{[0,\Lambda],s}$ and $\Gamma = (V, E)$ be a graph such that the induced subgraph Γ_μ has at least one connected component with $(2 + 12/\ln(2s/\Lambda\Omega))s$ vertices, then the measurement $\{\mathcal{M}_{i,\Gamma}(\mu)\}_{i=0}^{2}$ determines $[\mu]$.*

Proof. We note that phase propagation permits us to assign values to $\{\hat{\mu}(v) : v \in K\}$ where K is a connected component of W so that these values are consistent with the measurements. By phase propagation, the measurement with the subgraph of Γ induced by K provides $\hat{\mu}|_K$. If K has the size stated in the assumption, then by the injectivity result, this characterizes $[\mu]$. □

Following the measurement design in Algorithm A of [2], we model the points where the measurements $\mathcal{M}_{0,\Gamma}(f)$ are taken as vertices of a Ramanujan graph, and suppose that we are given interference measurements $\mathcal{M}_{1,\Gamma}$ and $\mathcal{M}_{2,\Gamma}$ corresponding to each edge in Γ. The size of Γ is chosen such that after removing at most $(1+6/\ln(s/\Lambda\Omega))s$ vertices and edges adjacent to them, we may invoke properties of expander graphs to show that there remains a connected component that is large enough to guarantee that phase propagation together with the Prony method returns a unique solution.

Lemma 2.2.6 (Lemma 5.2 in [44])**.** *Let $d \geq 2$. Let $\Gamma = (V, E, p)$ be a d-regular connected graph. For all $\delta \leq \frac{\lambda_1(\Gamma)}{12}$, removing at most $2\delta d|V|$ edges from the graph will leave a connected component of size at least $(1 - \frac{4\delta}{\lambda_1(\Gamma)})|V|$.*

Proof. If removing the edges divides the graph into two connected components, then there is $W \subset V$ with $|W| \leq \frac{1}{2}|V|$ and $|\mathcal{E}(W)| \leq 2\delta d|V|$. This implies by Cheeger's inequality (Theorem 1.2.19) that

$$\begin{aligned} |W|\lambda_1(\Gamma)\frac{d}{2} &\leq |W|h(\Gamma) \\ &\leq |\mathcal{E}(W)| \\ &\leq 2\delta d|V| \\ &\leq 2\frac{\lambda_1(\Gamma)}{12}d|V|, \end{aligned}$$

which gives

$$|W| \leq \frac{1}{3}|V|.$$

This shows that every connected component that gets separated from the rest of the graph and is of size $\leq \frac{1}{2}|V|$, must be of size at most $\frac{1}{3}|V|$. Moreover, if we have two connected components, W_1 and W_2, each of size at most $\frac{1}{3}|V|$ that are separated from the rest of graph, then we have that $|W_1 \cup W_2| \leq \frac{2}{3}|V|$. If $|W_1 \cup W_2| \leq \frac{1}{2}|V|$, then as shown above, $|W_1 \cup W_2| \leq \frac{1}{3}|V|$. On the other hand, $|W_1 \cup W_2| > \frac{1}{2}|V|$ implies $|(W_1 \cup W_2)^c| \leq \frac{1}{2}|V|$, and we get that $|W_1 \cup W_2| > \frac{2}{3}|V|$, a contradiction. Thus, we must have a connected component of size at least $(1 - \frac{4\delta}{\lambda_1(\Gamma)})|V| \geq \frac{2}{3}|V|$. □

Our main result in this chapter gives a condition that ensures injectivity of measurements. The proof relies on the connectivity and spectral properties of Ramanujan graphs discussed in Section

1.2.

Theorem 2.2.7. *Let* $\mu \in S_{[0,\Lambda]}$ *be a complex measure of support* $s \in \mathbb{N}$. *Let* $d \geq 3$ *such that there exists a Ramanujan graph* $\Gamma = (V, E)$ *with regularity* d *and at least* $N \geq \frac{6d(1+6/\ln(s/\Lambda\Omega))s}{d-2\sqrt{d-1}}$ *vertices in the set* $[-\Omega, \Omega]$ *with* $2\Omega\Lambda < 1$. *The* $N = (d+1)|V|$ *quadratic measurements associated with the graph then determine* $[\mu] \in S_{[0,\Lambda]}/\sim$.

Proof. Let $\Gamma = (V, E)$ be a Ramanujan graph with the claimed regularity and size, whose vertices are chosen to be points $\{v_1, v_2, \ldots, v_N\} \subset [-\Omega, \Omega]$. We first note that $\hat{\mu}$ has at most $(1 + 6/\ln(s/\Lambda\Omega))s$ zeros in $[-\Omega, \Omega]$. This means, when considering the induced subgraph Γ_μ for which only vertices are included where $\hat{\mu}$ does not vanish, then at most $d(1 + 6/\ln(s/\Lambda\Omega))s$ edges have been removed from Γ.

With the assumption on the size of the graph Γ, $d(1+6/\ln(s/\Lambda\Omega))s \leq (d-2\sqrt{d-1})|V|/6$. By the expander property of this graph, Γ_μ has a connected component whose vertex set K has a size at least

$$|K| \geq \frac{2}{3}|V| = \frac{4(1 + 6/\ln(s/\Lambda\Omega))s}{1 - 2\sqrt{d-1}/d} > 2(1 + 6/\ln(s/\Lambda\Omega))s\,.$$

Using the $N = (2d+1)|V|$ measured quantities from $\{\mathcal{M}_{i,\Gamma}\}_{i=0}^{2}$, we assign values $c\hat{\mu}(v_j)$ for each $v_j \in K$ with some residual unknown $c \in \mathbb{C}, |c| = 1$. Next, applying Theorem 2.2.5, we observe that $[\mu]$ is determined by $\hat{\mu}|_K$. □

Chapter 3

Reconstruction algorithms

Our result from Chapter 2 gives a theoretical guarantee for injectivity. However, it does not provide a method for constructing an inverse map that allows for signal recovery. In this chapter, we answer the following question: Is it possible to reconstruct every $[\mu] \in S_{[0,\Lambda],s}/\sim$ using $O(s)$ intensity measurements?

The difficulty arises from the fact that exact reconstruction methods from linear measurements require equidistant sampling at a critical rate. Unfortunately, we do not have access to this information via phase propagation due to the possibility of having zero measurements where no phase information is available. In order to avoid the zero measurements while maintaining the critical sampling frequency, we show that it is possible to resample the $\widehat{\mu}$ in a way that allows us to show that the signal recovery problem is equivalent to an optimization problem where we find the minimizer of a collection of quadratic forms defined over $S_{[0,\Lambda]/\sim}$. However, the resampling procedure will only be exact if we are given infinitely many equidistant intensity measurements of μ. After acquiring linear measurements of μ, we consider two methods to complete the reconstruction process. First is the Prony method, where we show that if $n > \frac{Cds}{d-2\sqrt{d-1}}$ and n is prime, $d \geq 3$, and Γ is a Ramanujan graph with at least $n+1$ vertices and regularity d, then there is a sampling strategy associated Γ and an iterative phase propagation algorithm resulting in $M = (d+1)(n+1)$ measured quantities provides values $\zeta\widehat{\mu}(\frac{2k\Omega}{(n+1)})$ for each $k \in \{1, 2, \ldots, \frac{(n+1)}{2}\}$ with a remaining residual unknown $\zeta \in \mathbb{C}$,

$|\zeta| = 1$. Then, the Prony algorithm can be used to determine the values t_j, $j \in \{1, 2, \ldots, s\}$, and solving a linear system then gives c, which allows us to reconstruct $[\mu]$. Due to the sensitivity of the Prony method to noise, we propose a second algorithm where we use TV-norm minimization to recover an approximation of μ. For a sufficiently large number of measurements, we show in Section 3.3 that following a similar technique, we are able to approximate μ within an error bound that is proportional to $\frac{1}{n^2}$, where n is the number of given samples by solving an optimization problem similar to [22] (see Theorem 3.3.6). In addition, we show under some conditions on the signal parameters that we may still achieve exact recovery via this method using $O(s)$ intensity measurements (see Theorems 3.2.4 and 3.2.5). In Section 3.4, we will discuss the stability of this algorithm in the presence of noise. We show that for a sufficiently large number of measurements, we are able to recover an approximation of μ with an error bound that consists of two additive parts: one decays at the same rate as $\frac{1}{n^2}$ and the other is proportional to the ratio $\frac{\|\epsilon\|_\infty}{\|\mu\|_{TV}}$, where ϵ represents the noise in the measurements (see Theorem 3.4.3).

3.1 Recovery using the Prony method

Refer to the appendix for a review of the Prony method. In particular, Appendix A gives a proof of the main result in [14]. We will use a different representation of this method (in terms of operators) in order to obtain a sufficient condition for recovery in terms of the number of measurements, n, depending on the signal length, s. See [50] for more details about this generalized version of the Prony method. Let Γ be a Ramanujan graph with vertices $\{v_1, v_2, ..., v_n\}$. We choose $v_k = \frac{k\Omega}{n}$. Let $z_j = e^{-2\pi i t_j h}$ and let M_{z_j} be a modulation operator on $\ell^2(\{1, ..., N\})$ given by

$$M_{z_j} f(k) = z_j^k f(k).$$

The function $\widehat{\mu}$ restricted to V can then be identified with a multiplication operator on $\ell^2(\{1, 2, \ldots, n\})$, a linear combination of modulation operators

$$T = \sum_{j=1}^{s} c_j M_{z_j}.$$

We will show how our intensity measurements can be translated into actions of T which allow us to determine the values of t_j for each $j \in \{1, 2, \ldots, s\}$. Once the node locations $\{t_j\}_{j=1}^{s}$ are known, recovering the coefficient vector c is equivalent to solving an overdetermined system of linear equations. In order to estimate the number of measurements that are sufficient for recovery, we will use the injectivity result to establish the linear independence of these modulation operators for sufficiently large n.

Lemma 3.1.1. *Let $\Lambda, \Omega > 0$, $\Lambda\Omega < \frac{1}{2}$, $n \in \mathbb{N}$, $\{t_1, t_2, \ldots, t_s\} \subset [0, \Lambda]$, and for each $j \in \{1, 2, \ldots, s\}$, $z_j = e^{-2\pi i t_j h}$ with $h = \frac{\Omega}{n}$. If $n > (1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s$, then the associated set of modulation operators $\{M_{z_j}\}_{j=1}^{s}$ on $\ell_2(\{1, 2, \ldots, n\})$ is a linearly independent set.*

Proof. The linear independence property is a consequence of the fact that the diagonal entries of each modulation operator is given by a complex exponential. If a linear combination of modulation operators vanishes, then all diagonal entries of this linear combination vanish. By the result on the number of roots of linear combinations of complex exponentials, if n is sufficiently large as assumed, this is only possible if the linear combination is trivial. □

Corollary 3.1.2. *Under the assumptions of Lemma 3.1.1, given the operator $T = \sum_{j=1}^{s} c_j M_{z_j}$ on $\ell^2(\{1, 2, \ldots, n\})$ and the values $\{z_1, z_2, \ldots, z_s\}$, then the identity for T has a unique solution $c \in \mathbb{C}^s$*

The Prony method uses the fact that complex exponentials are eigenvectors of the shift operator. We define the cyclic shift S by $Sf(k) = f(k-1)$ and $Sf(1) = f(n)$. We recall a version of Weyl commutation relations [64].

Lemma 3.1.3. *If $f \in \ell^2(\{1, 2, \ldots, n\})$ has support in $\{1, 2, \ldots, n-1\}$, then for any $z \in \mathbb{C}$, $|z| = 1$,*

$$S^* M_z S f = z M_z f.$$

Proof. By linearity, it is enough to prove this for each canonical basis vector e_k with $k \in \{1, 2, \ldots, n-1\}$. We note that by definition, $Se_k = e_{k+1}$ and $M_z e_k = z^k e_k$. Since S^* is the inverse of S, $S^* M_z S e_k = z^{k+1} e_k = z M_z e^k$. □

Together with linearity and iterating the application of S, the Weyl commutation relation can be used to generate an annihilating polynomial for T. To this end, we associate a map, Ψ_q with each complex polynomial q with degree m, that maps any operator X on $\ell^2(\{1, 2, \ldots, n\})$ to

$$\Psi_q(X) = \sum_{l=0}^{m} q_l (S^l)^* X S^l,$$

where $q(z) = \sum_{l=0}^{m} q_l z^l$.

Lemma 3.1.4. *If $f \in \ell^2(\{1, 2, \ldots, n\})$ has support in $\{1, 2, \ldots, m\}$, then for any $z \in \mathbb{C}$, $|z| = 1$, and any complex polynomial q of degree at most $n - m$,*

$$\Psi_q(M_z) f = \sum_{l=0}^{n-m} q_l (S^l)^* M_z S^l f = q(z) M_z f,$$

where $q(z) = \sum_{l=0}^{n-m} q_l z^l$.

Proof. For each monomial, this is true by iterating the commutation relation, because $S^l e_k = e_{k+l}$ and if $k, l \leq m$, then $k + l \leq n$. The statement for polynomials follows by linearity. □

Theorem 3.1.5. *Let $\Lambda, \Omega > 0$, $\Lambda\Omega < \frac{1}{2}$, $n \in \mathbb{N}$, $\{t_1, t_2, \ldots, t_s\} \subset [0, \Lambda]$, and for each $j \in \{1, 2, \ldots, s\}$, $z_j = e^{-2\pi i t_j h}$ with $h = \frac{\Omega}{n}$. Let $T = \sum_{j=1}^{s} c_j M_{z_j}$ on $\ell^2(\{1, 2, \ldots, n\})$ with $n \geq s + m$, $m > (1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s$ and non-vanishing complex coefficients $\{c_1, c_2, \ldots, c_s\}$, then there is a unique* *monic polynomial q of degree s such that for each $f \in \ell^2(\{1,2,\ldots,n\})$ with support in $\{1,2,\ldots,m\}$,*

$$\Psi_q(T) = \sum_{l=0}^{s} q_l (S^l)^* T S^l = 0.$$

Proof. We note that by Lemma 3.1.4

$$\sum_{l=0}^{s} q_l (S^l)^* T S^l f = \sum_{j=1}^{s} q(z_j) c_j M_{z_j} f.$$

If $q(z) = \prod_{j=1}^{s}(z - z_j)$, then indeed $q(z_j) = 0$ and the statement is true. Moreover, requiring the degree of q to be s yields only one choice of a monic polynomial that annihilates T. By the linear independence of $\{M_{z_j}\}_{j=1}^{s}$ and $c_j \neq 0$, if $\sum_{j=1}^{s} q(z_j) c_j M_{z_j} f = 0$ for each f with support size m, then $q(z_j) = 0$ for each $j \in \{1,2,\ldots,s\}$. This forces q to be the (monic) polynomial given by the factored form. □

Next, we show the uniqueness of the annihilating polynomial for this operator associated with μ.

Corollary 3.1.6. *Under the assumptions of Theorem 3.1.5, the solution of the homogeneous linear system given by the equations for each $k \in \{1,2,\ldots,m\}$*

$$\sum_{l=0}^{s} q_l \sum_{j=1}^{s} c_j z_j^{k+l} = 0$$

satisfying $q_s = 1$ gives the monic polynomial $q(z) = \prod_{j=1}^{s}(z - z_j)$.

Proof. This follows from the definition of T and from the choice $f = e_k$ for $k \in \{1,2,\ldots,m\}$. □

Theorem 3.1.7. *Let $\Lambda, \Omega > 0$, $\Lambda\Omega < \frac{1}{2}$. If $n > \frac{12d(1+\frac{6}{\ln(\frac{s}{\Lambda\Omega})})s}{d-2\sqrt{d-1}}$, $d \geq 3$ such that there exists a d-regular Ramanujan graph Γ with n vertices $V = \mathbb{Z}_n$ with $v_k = k$ for each $k \in \{0,1,\ldots,n-1\}$. Then, for almost every signal $[\mu] \in S_{[0,\Lambda]}/\sim$, phase propagation applied to the measurements $\{\mathcal{M}_{i,\Gamma}(\mu)\}_{i=0}^{2}$ provides values $\zeta\widehat{\mu}(k)$ for each $k \in \{0,1,\ldots,n\}$ with a remaining residual unknown*

$\zeta \in \mathbb{C}$, $|\zeta| = 1$. *Moreover, the unique monic polynomial q of degree s satisfying the homogeneous system of equations for $k \in \{0, 1, 2, \ldots, s+1\}$*

$$\sum_{l=0}^{s} q_l \widehat{\mu}(k+l) = 0$$

factors into $q(z) = \prod_{j=1}^{s}(z - z_j)$ with $z_j = e^{\frac{-2\pi i t_j \Omega}{n}}$, and the system $\sum_{j=1}^{s} c_j z_j^k = c\widehat{\mu}(v_k)$ with $k \in \{1, 2, \ldots, n\}$ has unique solution c.

Proof. We note that the set of all signals $\mu \in S_{[0,\Lambda]}$ that have any roots at some $k \in \{0, 1, \ldots, n-1\}$ has Lebesgue measure zero. Thus, applying phase propagation allows us to construct the values $(\zeta\widehat{\mu}(k))_{k=0}^{n-1}$ for almost all $\mu \in S_{[0,\Lambda]}$. Next,

$$\frac{(n+1)}{2} > \frac{3d(1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s}{d - 2\sqrt{d-1}} > 3(1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s$$

hence $\frac{(n+1)}{2} > m + s$ This implies that the Prony method can be applied to find the polynomial whose roots are $\{e^{-2\pi i t_j}\}_{j=1}^{s}$. Knowing the support of μ then allows us to solve for the coefficient vector c. □

In addition to this probabilistic estimate, we show in the work with Bodmann [16] that it is possible to relax the domain restriction in the above theorem by oversampling. For the remainder of this section, we introduce the inverse Fourier transform defined by

$$\widetilde{\mu}(t) = \sum_{k=1}^{m} \widehat{\mu}(\frac{k\Omega}{m}) e^{\frac{i\pi t k}{\Lambda}}.$$

We note that among all 2Λ-periodic trigonometric polynomials of degree m, $\widetilde{\mu}$ has $m+1$ vanishing coefficients, meaning we have implicitly set $\widetilde{\mu}(\frac{k\Omega}{m}) = 0$ for $k \in \{-m, -m+1, \ldots, -1, 0\}$. The idea is to use the injectivity result from Chapter 2 along with the fact that it cannot be the case that both $\widehat{\mu}$ and $\widetilde{\mu}$ have many roots in the interval $[-\Omega, \Omega]$. This notion is made exact in Terence Tao's strengthened uncertainty principle for groups of prime order.

Lemma 3.1.8 (Corollary 1.4 in [62]). *If $G = \mathbb{Z}_p$ and p is prime, and $f \mapsto \widehat{f}$ denotes the Fourier transform on G, so $\widehat{f}(\xi) = \frac{1}{p}\sum_{x\in G} f(x)e^{\frac{2\pi i x\xi}{p}}$, then if $S, \widetilde{S} \subset G$, $|S| = |\widetilde{S}|$, $F : \ell^2(S) \to \ell^2(\widetilde{S})$ is invertible, where $Ff(\xi) = \widehat{f}(\xi)$ and $\ell^2(S)$ denotes the functions on G vanishing on $G \setminus S$.*

If $\widehat{\mu}$ is known for each $k \in \{1, 2, \ldots, m + s\}$, then the Prony method can be applied to extract the values t_j and c_j. As a consequence, we note that evaluating the function $\widetilde{\mu}$ at any subset of (at least) m vertices from $\{v_k\}_{k=1}^{2m+1}$ with $v_k = \frac{2k\Lambda}{2m+1}$ determines $\{\widehat{\mu}(\frac{k\Omega}{m})\}_{k=1}^{m}$, if $2m + 1$ is prime.

Corollary 3.1.9. *Let $\Lambda, \Omega > 0$, $\Lambda\Omega < \frac{1}{2}$, and μ be a complex measure having support $\{t_1, t_2, \ldots, t_s\} \subset [0, \Lambda]$. If $n > \frac{6d(1+\frac{6}{\ln(\frac{s}{\Lambda\Omega})})s}{d-2\sqrt{d-1}}$ and n is prime, $d \geq 3$ and $V = \{v_0, v_1, \ldots, v_n\}$ with $v_k = hk$ and $h = \frac{2\Lambda}{n}$ for each v_k, then phase propagation applied to the oversampled measurement $\{\mathcal{M}_{i,\Gamma}(\widetilde{\mu})\}_{i=0}^{2}$ with a Ramanujan graph Gamma of degree d having the vertex set V provides values $\zeta\widehat{\mu}(\frac{2k\Omega}{(n+1)})$ for each $k \in \{1, 2, \ldots, \frac{(n+1)}{2}\}$ with a remaining residual unknown $\zeta \in \mathbb{C}$, $|\zeta| = 1$. Moreover, the unique monic polynomial q of degree s satisfying the homogeneous system of equations for $k \in \{1, 2, \ldots, s + 1\}$*

$$\sum_{l=0}^{s} q_l \widehat{\mu}(\frac{2(k+l)\Omega}{(n-1)})$$

factors into $q(z) = \prod_{j=1}^{s}(z - z_j)$ with $z_j = e^{\frac{-2\pi i t_j \Omega}{n}}$, and the system $\sum_{j=1}^{s} c_j z_j^k = c\widehat{\mu}(v_k)$ with $k \in \{1, 2, \ldots, n\}$ has unique solution c.

Proof. The assumptions for phase propagation and the Prony method need to be combined. We note that if the number of vertices in the Ramanujan graph is $n+1$ with $n >$, then phase propagation provides a consistent set of values $\{\zeta\widetilde{\mu}(v_k) : v_k \in K\}$ with a residual unknown factor. By the expander property, $|K| > \frac{2|V|}{3} = \frac{2(n+1)}{3}$. If n is prime, using the fact that the linear map from $(\widehat{\mu}(\frac{k\Omega}{m}))_{k=1}^{m}$ with $m = \frac{n+1}{2}$ to $(\widetilde{\mu}(v_k))_{v_k\in K}$ is invertible, we obtain sample values for $\widehat{\mu}$. Next,

$$\frac{(n+1)}{2} > \frac{3d(1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s}{d - 2\sqrt{d-1}} > 3(1 + \frac{6}{\ln(\frac{s}{\Lambda\Omega})})s$$

hence $\frac{(n+1}{2} > m + s$ This implies that the Prony method can be applied to find the polynomial whose roots are $\{e^{-2\pi i t_j}\}_{j=1}^{s}$. Knowing the support of μ then allows to solve for the coefficient vector c. □

See also [52] for constructions of Ramanujan graphs with more general sizes (number of vertices). In the next section, we will replace the second part of our recovery algorithm with a super-resolution strategy based on TV-norm minimization.

3.2 Recovery using TV-norm minimization

Even with access to linear measurements, it is known that the task of reconstructing the original signal (super-resolution) remains ill-posed under only the sparsity assumption, as noted in [22]. However, we will see that introducing a minimum separation distance between the nodes as an additional assumption, following the technique presented in [22], will allow us to obtain a more robust recovery method. As mentioned, this will come at the cost of a truncation error even in the case of exact measurements. The intensity measurements we want to consider in the noiseless case are

$$\mathcal{M}_{i,\Gamma_N}(\mu), \quad i \in \{0,1,2\} \tag{3.1}$$

associated with the linear graph $\Gamma_N = (V_N, E_N)$ where $V_N = \{\frac{n\pi}{\Lambda}\}_{n=-N}^{N}$, and $E_N = \{\frac{n\pi}{\Lambda}, \frac{(n+2)\pi}{\Lambda}\}$. This choice of sampling frequency is based on the sampling theorem we employ below. We emphasize that this approach is motivated by the sensitivity of algorithms used in extracting the roots a higher order polynomial from its coefficients to noise [42], a fact which follows from the continuous dependence of the roots on the coefficients [43]. However, this comes at the cost of introducing a truncation error, even in the case of exact measurements. To illustrate this framework for recovery, We will focus in this section on the ideal case by assuming we have access to infinitely many equidistant exact point evaluations of the $\widehat{\mu}_0(\omega) := |\widehat{\mu}(\omega)|^2$, $\widehat{\mu}_1(\omega) := |\widehat{\mu}(\omega) - \widehat{\mu}(\omega + \frac{2\pi}{\Lambda})|^2$, and $\widehat{\mu}_2(\omega) := |\widehat{\mu}(\omega) - i\widehat{\mu}(\omega + \frac{2\pi}{\Lambda})|^2$, some of which are possibly zero (so we implicitly set $N = \infty$ in (3.1)). First, we will need the following sampling theorem for time-limited distributions due to Campbell [20] in order to interpolate the given intensity measurements.

Lemma 3.2.1 (Corollary to Theorem 1 from [20]). *Let $\mu \in S_{[-\Lambda,\Lambda]}$, let $\chi_{[-\Lambda,\Lambda]} : \mathbb{R} \to \mathbb{R}$ be given by*

$$\chi_{[-\Lambda,\Lambda]}(t) = \begin{cases} 1 & \text{if } |t| \leq \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have that

$$\widehat{\mu}(\omega) = \frac{1}{2\Lambda} \sum_{n=-\infty}^{\infty} \widehat{\mu}(\frac{n\pi}{\Lambda}) \widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{2\Lambda}) \tag{3.2}$$

where the RHS in (2) converges for all $\omega \in \mathbb{R}$.

Proof. Using the Poisson summation formula (Proposition 1.3.5) [37], we get that

$$\sum_{m\in\mathbb{Z}} \mu(t - 2m\Lambda) = \frac{1}{2\Lambda} \sum_{n\in\mathbb{Z}} \widehat{\mu}(\frac{n\pi}{\Lambda}) e^{\frac{in\pi t}{\Lambda}}. \tag{3.3}$$

Integrating both sides in (10) against $\chi_{[-\Lambda,\Lambda]}(t)e^{-2\pi i\omega t}$, we get on the RHS

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\frac{in\pi t}{\Lambda}} \chi_{[-\Lambda,\Lambda]}(t) e^{-2\pi i\omega t} dt &= \int_{-\Lambda}^{\Lambda} \chi_{[-\Lambda,\Lambda]}(t) e^{-2\pi i t(\omega - \frac{n\pi}{2\Lambda})} dt \\ &= \widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{2\Lambda}) \end{aligned} \tag{3.4}$$

by definition. On the LHS, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \mu(t) \chi_{[-\Lambda,\Lambda]}(t) e^{-2\pi i\omega t} dt &= \int_{-\Lambda}^{\Lambda} \mu(t) e^{-2\pi i\omega t} dt \\ &= \widehat{\mu}(\omega) \end{aligned} \tag{3.5}$$

whereas

$$\int_{-\infty}^{\infty} \mu(t - 2m\Lambda) \chi_{[-\Lambda,\Lambda]}(t) e^{-2\pi i\omega t} dt = 0 \tag{3.6}$$

for any non-zero integer m, since the support of $\chi_{[-\Lambda,\Lambda]}(t)$ given by $[-\Lambda, \Lambda]$, is disjoint from the support of $\mu(t - 2m\Lambda)$ given by $[(-1 + 2m)\Lambda, (1 + 2m)\Lambda]$ for each m. Substituting (3.3), (3.5) and (3.6) into (3.3) and summing over all $m, n \in \mathbb{Z}$ gives (3.2) as claimed. □

Note that

$$\begin{aligned}
\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{2\Lambda}) &= \mathcal{F}(e^{2\pi \frac{n\pi}{2\Lambda} it} \chi_{[-\Lambda,\Lambda]}(t)) \\
&= \int_{-\infty}^{\infty} e^{2\pi \frac{n\pi}{2\Lambda} it} \chi_{[-\Lambda,\Lambda]}(t) e^{-2\pi i \omega t} dt \\
&= \int_{-\Lambda}^{\Lambda} e^{-2\pi i t(\omega - \frac{n\pi}{2\Lambda})} dt \\
&= \frac{-1}{2\pi i(\omega - \frac{n\pi}{2\Lambda})} e^{-2\pi i t(\omega - \frac{n\pi}{2\Lambda})} |_{-\Lambda}^{\Lambda} \\
&= \frac{-1}{2\pi i(\omega - \frac{n\pi}{2\Lambda})} (e^{-2\pi i \Lambda(\omega - \frac{n\pi}{2\Lambda})} - e^{2\pi i \Lambda(\omega - \frac{n\pi}{2\Lambda})}) \\
&= \frac{1}{\pi(\omega - \frac{n\pi}{2\Lambda})} \frac{e^{2\pi i \Lambda(\omega - \frac{n\pi}{2\Lambda})} - e^{-2\pi i \Lambda(\omega - \frac{n\pi}{2\Lambda})}}{2i} = \frac{sin(2\pi\Lambda(\omega - \frac{n\pi}{2\Lambda}))}{\pi(\omega - \frac{n\pi}{2\Lambda})}.
\end{aligned}$$

Next, we will follow the kernel method introduced in [18] in order to construct a collection of quadratic forms that are minimized precisely at $[\mu]$. We will apply Lemma 3.2.1 to choose non-zero sample values for $\widehat{\mu}_0$. Note that $\widehat{\mu}_0(\omega) = 0$ if and only if $\widehat{\mu}(\omega) = 0$ for any $\omega \in \mathbb{R}$, so we are equivalently attempting to resample in a way that avoids the roots for $\widehat{\mu}$. The quadratic forms we construct also require sample values of the function $\widehat{\nu} : \mathbb{R} \to \mathbb{C}$ given by

$$\begin{aligned}
\widehat{\nu}(\omega) :&= \widehat{\mu}(\omega)\overline{\widehat{\mu}(\omega + \frac{2\pi}{\Lambda})} \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j \overline{c_k} e^{-2\pi i t_j \omega} e^{2\pi i t_k(\omega + \frac{2\pi}{\Lambda})} \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j c_k e^{-2\pi i (t_j - t_k)\omega} e^{2\pi i t_k \frac{2\pi}{\Lambda}}.
\end{aligned}$$

We will again make use of the polarization identity from [18] to acquire these values. In order to use the sampling theorem above, it is important to note that

$$\begin{aligned}
\nu(t) = \mathcal{F}^{-1}(\widehat{\nu}(\omega))(t) &= \mathcal{F}^{-1}(\widehat{\mu}(\omega).\overline{\widehat{\mu}(\omega+\frac{2\pi}{\Lambda})})(t) \\
&= \mu(t) * e^{2\pi i t \frac{2\pi}{\Lambda}} \overline{f(-t)} \\
&= \sum_{j=1}^{s} c_j \delta_{t_j}(t) * \sum_{k=1}^{s} e^{2\pi i t_k \frac{2\pi}{\Lambda}} \overline{c_k} \delta_{t_k}(-t) \\
(&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j c_k \int_{-\infty}^{\infty} \delta_{t_j}(x) e^{2\pi i (t-x)\frac{2\pi}{\Lambda}} \delta_{t_k}(x-t)dx) \\
&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j c_k e^{-2\pi i t_k \frac{2\pi}{\Lambda}} \delta_{t_j - t_k}(t)
\end{aligned}$$

(or we can direcly calculate

$$\begin{aligned}
\mathcal{F}^{-1}(\widehat{\mu}(\omega).\overline{\widehat{\mu}(\omega+\frac{2\pi}{\Lambda})})(t) &= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j} \sum_{k=1}^{s} \overline{c_k} e^{2\pi i (\omega + \frac{2\pi}{\Lambda}) t_k} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j \overline{c_k} e^{2\pi i \frac{2\pi}{\Lambda} t_k} \int_{-\infty}^{\infty} e^{-2\pi i \omega (t_j - t_k)} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j c_k e^{2\pi i t_k \frac{2\pi}{\Lambda}} \delta_{t_j - t_k}(t))
\end{aligned}$$

which implies that $\nu \in S_{[-\Lambda,\Lambda]}$ since $t_j - t_k \in [-\Lambda, \Lambda]$ for each $j,k \in \{1,...,s\}$. Similarly, we have that

$$\begin{aligned}
\mu_0(t) = \mathcal{F}^{-1}(\widehat{\mu}_0)(t) &= \int_{-\infty}^{\infty} \widehat{\mu}(\omega)\overline{\widehat{\mu}(\omega)} e^{2\pi i \omega t} d\omega \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j} \overline{\sum_{k=1}^{s} c_k e^{-2\pi i \omega t_k}} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j \overline{c_k} \int_{-\infty}^{\infty} e^{-2\pi i \omega (t_j - t_k)} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s}\sum_{k=1}^{s} c_j \overline{c_k} \delta_{t_j - t_k}(t),
\end{aligned}$$

$$\begin{aligned}
\mu_1(t) = \mathcal{F}^{-1}(\widehat{\mu}_1)(t) &= \int_{-\infty}^{\infty} [\widehat{\mu}(\omega) - \widehat{\mu}(\omega + \frac{\pi}{\Lambda})]\overline{[\widehat{\mu}(\omega) - \widehat{\mu}(\omega + \frac{2\pi}{\Lambda})]} e^{2\pi i \omega t} d\omega \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j (e^{-2\pi i \omega t_j} - e^{-2\pi i (\omega + \frac{2\pi}{\Lambda}) t_j}) \overline{\sum_{k=1}^{s} c_k (e^{-2\pi i \omega t_k} - e^{-2\pi i (\omega + \frac{2\pi}{\Lambda}) t_k})} e^{2\pi i \omega t} d\omega \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j} (1 - e^{-2\pi i \frac{\pi}{\Lambda} t_j}) \sum_{k=1}^{s} \overline{c_k} e^{2\pi i \omega t_k} (1 - e^{2\pi i \frac{2\pi}{\Lambda} t_k}) e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} \int_{-\infty}^{\infty} c_j \overline{c_k} e^{-2\pi i \omega (t_j - t_k)} (1 - e^{-2\pi i \frac{2\pi}{\Lambda} t_j} - e^{2\pi i \frac{\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j \overline{c_k} (1 - e^{-2\pi i \frac{2\pi}{\Lambda} t_j} - e^{2\pi i \frac{2\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) \int_{-\infty}^{\infty} e^{-2\pi i \omega (t_j - t_k)} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j \overline{c_k} (1 - e^{-2\pi i \frac{\pi}{\Lambda} t_j} - e^{2\pi i \frac{2\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) \delta_{t_j - t_k}(t),
\end{aligned}$$

and

$$\begin{aligned}
\mu_2(t) = \mathcal{F}^{-1}(\widehat{\mu}_2)(t) &= \int_{-\infty}^{\infty} [\widehat{\mu}(\omega) - i\widehat{\mu}(\omega + \frac{2\pi}{\Lambda})]\overline{[\widehat{\mu}(\omega) - i\widehat{\mu}(\omega + \frac{2\pi}{\Lambda})]} e^{2\pi i \omega t} d\omega \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j (e^{-2\pi i \omega t_j} - i e^{-2\pi i (\omega + \frac{2\pi}{\Lambda}) t_j}) \overline{\sum_{k=1}^{s} c_k (e^{-2\pi i \omega t_k} - i e^{-2\pi i (\omega + \frac{2\pi}{\Lambda}) t_k})} e^{2\pi i \omega t} d\omega \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{s} c_j e^{-2\pi i \omega t_j} (1 - i e^{-2\pi i \frac{2\pi}{\Lambda} t_j}) \sum_{k=1}^{s} \overline{c_k} e^{2\pi i \omega t_k} (1 + i e^{2\pi i \frac{2\pi}{\Lambda} t_k}) e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} \int_{-\infty}^{\infty} c_j \overline{c_k} e^{-2\pi i \omega (t_j - t_k)} (1 - i e^{-2\pi i \frac{2\pi}{\Lambda} t_j} + i e^{2\pi i \frac{2\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j \overline{c_k} (1 - i e^{-2\pi i \frac{2\pi}{\Lambda} t_j} + i e^{2\pi i \frac{2\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) \int_{-\infty}^{\infty} e^{-2\pi i \omega (t_j - t_k)} e^{2\pi i \omega t} d\omega \\
&= \sum_{j=1}^{s} \sum_{k=1}^{s} c_j \overline{c_k} (1 - i e^{-2\pi i \frac{2\pi}{\Lambda} t_j} + i e^{2\pi i \frac{2\pi}{\Lambda} t_k} + e^{-2\pi i \frac{2\pi}{\Lambda} (t_j - t_k)}) \delta_{t_j - t_k}(t),
\end{aligned}$$

so we get that $\mu_i \in S_{[-\Lambda,\Lambda]}$ for each $i \in \{0, 1, 2\}$. The next lemma shows how the quadratic forms can be constructed. We will adapt this technique in Sections 3.3 and 3.4 to the cases of finite numbers of exact and noisy measurements.

Lemma 3.2.2. *Let I_N be the identity matrix in $\mathbb{C}^{N\times N}$. Let $N \in \mathbb{N}\cup\{\infty\}$ and $\mu \in S_{[0,\Lambda]}$. Define the quadratic forms $Q_{\mu,N}, Q_{\mu,-N} : S_{[0,\Lambda]} \to \mathbb{R}$ by*

$$Q_{\mu,N}(\varphi) = \|(I_{N+1} - T_{-\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}\big(\widehat{\varphi}(\omega_0 + \frac{2n\pi}{\Lambda})\big)_{n=0}^{N}\|^2_{\ell_2^{N+1}} - 1 \tag{3.7}$$

and

$$Q_{\mu,-N}(\varphi) = \|(I_{N+1} - T_{\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},-N}\big(\widehat{\varphi}(\omega_0 + \frac{2n\pi}{\Lambda})\big)_{n=-N}^{0}\|^2_{\ell_2^{N+1}} - 1$$

respectively, where $T_{-\frac{2\pi}{\Lambda}}, M_{\frac{1}{\widehat{\mu}},N} : \ell_2(\{0,1,...,N\}) \to \ell_2(\{0,1,...,N\})$ and $T_{\frac{2\pi}{\Lambda}}, M_{\frac{1}{\widehat{\mu}},-N} : \ell_2(\{-N,-(N-1),...,0\}) \to \ell_2(\{-N,-(N-1),...,0\})$ are the operators given by

$$T_{-\frac{2\pi}{\Lambda}}\big((x_n)_{n=0}^{N}\big) = (0, x_1, x_2, ..., x_{N-1}, x_{N-1}),$$

$$M_{\frac{1}{\widehat{\mu}},N}\big((x_n)_{n=0}^{N}\big) = \big(\frac{1}{\widehat{\mu}(\omega_{2n})}x_n\big)_{n=0}^{N},$$

$$T_{\frac{2\pi}{\Lambda}}\big((x_n)_{n=-N}^{0}\big) = (x_{-(N-1)}, x_{-(N-2)}, ..., x_2, x_1, 0),$$

$$M_{\frac{1}{\widehat{\mu}},-N}\big((x_n)_{n=-N}^{0}\big) = \big(\frac{1}{\widehat{\mu}(\omega_{2n})}x_n\big)_{n=-N}^{0},$$

and $\omega_n := \omega_0 + \frac{n\pi}{\Lambda}$ for a fixed $\omega_0 \in [-\frac{\pi}{\Lambda}, \frac{\pi}{\Lambda}] \subset \mathbb{R}$ satisfying $\widehat{\mu}(\omega_{2n}) \neq 0$ for all $n \in \mathbb{Z}$. Then, $Q_{\mu,N}$ and $Q_{\mu,-N}$ can be constructed using the measurements (3.1) for each $N \in \mathbb{N}\cup\{\infty\}$.

Proof. Let $N \in \mathbb{N}\cup\{\infty\}$ and fix $\mu \in S_{[0,\Lambda]}$. Let $\varphi \in S_{[0,\Lambda]}$, then for any such $\omega_0 \in \mathbb{R}$ we have that

$$\begin{aligned}
Q_{\widehat{\mu},N}(\varphi) &= \|(I_{N+1} - T_{\frac{-2\pi}{\Lambda}})M_{\frac{1}{\mu},N}\big(\widehat{\varphi}(\omega_0 + \frac{2n\pi}{\Lambda})\big)_{n=0}^{N}\|^2_{\ell_2^{N+1}} - 1 \\
&= \langle (I_{N+1} - T_{-\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}\big(\widehat{\varphi}(\omega_{2n})\big)_{n=0}^{N}, (I - T_{\frac{-2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}\big(\widehat{\varphi}(\omega_{2n})\big)_{n=0}^{N}\rangle_{\ell_2^{N+1}} - 1 \\
&= \langle [(I_{N+1} - T_{-\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}]^*[(I - T_{-\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}]\big(\widehat{\varphi}(\omega_{2n})\big)_{n=0}^{N}, \big(\widehat{\varphi}(\omega_{2n})\big)_{n=0}^{N}\rangle_{\ell_2^{N+1}} - 1.
\end{aligned}$$

In the case of $N \in \mathbb{N}$, each of the operators used in the definition of $Q_{\widehat{\mu},N}$ can be represented as a matrix of the form

$$I = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix},$$

$$T_{-\frac{2\pi}{\Lambda}} = \begin{bmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & & & & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{bmatrix},$$

$$M_{\frac{1}{\widehat{\mu}},N} = \begin{bmatrix} \frac{1}{\widehat{\mu}(\omega_0)} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\widehat{\mu}(\omega_2)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{1}{\widehat{\mu}(\omega_{2(N-1)})} & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{\widehat{\mu}(\omega_{2N})} \end{bmatrix},$$

respectively, which allows us to calculate

$$(I_{N+1} - T_{\frac{-2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N} = \begin{bmatrix} \frac{1}{\widehat{\mu}(\omega_0)} & 0 & \dots & \dots & \dots & 0 \\ -\frac{1}{\widehat{\mu}(\omega_0)} & \frac{1}{\widehat{\mu}(\omega_2)} & \ddots & & & \vdots \\ 0 & -\frac{1}{\widehat{\mu}(\omega_2)} & \frac{1}{\widehat{\mu}(\omega_4)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\frac{1}{\widehat{\mu}(\omega_{2(N-2)})} & \frac{1}{\widehat{\mu}(\omega_{2(N-1)})} & 0 \\ 0 & \dots & \dots & 0 & -\frac{1}{\widehat{\mu}(\omega_{2(N-1)})} & \frac{1}{\widehat{\mu}(\omega_{2N})} \end{bmatrix}$$

and $[(I_{N+1} - T_{\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}]^*[(I_{N+1} - T_{-\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}]$ is given by

$$\begin{bmatrix} \frac{1}{|\widehat{\mu}(\omega_0)|^2} & -\frac{1}{\widehat{\mu}(\omega_0)\overline{\widehat{\mu}(\omega_2)}} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{\widehat{\mu}(\omega_0)\overline{\widehat{\mu}(\omega_2)}} & \frac{2}{|\widehat{\mu}(\omega_2)|^2} & -\frac{1}{\overline{\widehat{\mu}(\omega_2)}\widehat{\mu}(\omega_4)} & \ddots & & \vdots \\ 0 & -\frac{1}{\widehat{\mu}(\omega_2)\overline{\widehat{\mu}(\omega_4)}} & \frac{2}{|\widehat{\mu}(\omega_4)|^2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \frac{2}{|\widehat{\mu}(\omega_{2(N-1)})|^2} & -\frac{1}{\overline{\widehat{\mu}(\omega_{2(N-1)})}\widehat{\mu}(\omega_{2N})} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{\widehat{\mu}(\omega_{2(N-1)})\overline{\widehat{\mu}(\omega_{2N})}} & \frac{1}{|\widehat{\mu}(\omega_{2N})|^2} \end{bmatrix} \quad (3.8)$$

Next, we show that each entry in (3.8) can be obtained from the measurements (3.1) with the desired properties (i.e., such that $\widehat{\mu}(\omega_{2n}) \neq 0$ for any $n \in \{0, 1, ..., N\}$). Since $\widehat{\mu_i} \in S_{[-\Lambda,\Lambda]}$ for each $i \in \{0, 1, 2\}$, Lemma 3.2.1 implies that each function can be interpolated exactly for each $\omega \in \mathbb{R}$ using the measurements (3.1) for $N = \infty$. In particular, we have that

$$\widehat{\mu}_i(\omega) = \frac{1}{2\Lambda} \sum_{n=-\infty}^{\infty} \widehat{\mu}_i(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{2\Lambda}) \quad (3.9)$$

for each $i \in \{0, 1, 2\}$ (for example, $|\widehat{\mu}(\omega)|^2 = \sum_{n=-\infty}^{\infty} |\widehat{\mu}(\frac{n\pi}{\Lambda})|^2 \frac{sin(2\pi\Lambda(\omega - \frac{n\pi}{2\Lambda}))}{2\pi\Lambda(\omega - \frac{n\pi}{2\Lambda})}$). In order to show that there is an $\omega_0 \in \mathbb{R}$ such that $Q_{\mu,N}$ is well defined, we use Theorem 2.1.4 to see that for any interval $[-n, n]$, $n \in \mathbb{N}$, the function $\widehat{\mu}$ has at most $9\pi\Lambda n + 2s[\ln \frac{\Lambda}{\Delta} + \ln s + 3]$ zeros, which implies that $\widehat{\mu}$ has at most countably infinte zeros on the real line. Let $\{z_j | j \in \mathbb{Z}\}$ be the set of roots for $\widehat{\mu}_0$. Then,

$$|\bigcup_{j\in\mathbb{Z}}([-\frac{\pi}{\Lambda}, \frac{\pi}{\Lambda}] \cap \{z_j + \frac{n\pi}{\Lambda} | n \in \mathbb{Z}\})| \leq \aleph_0 \quad (3.10)$$

which means that there exists $\omega_0 \in [\frac{-\pi}{\Lambda}, \frac{\pi}{\Lambda}]$ such that the measurements $\mathcal{N}_i : S_{[0,\Lambda]} \to \ell_\infty(\mathbb{Z})$

$$\mathcal{N}_0(\mu) = (|\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda})|^2)_{n=-\infty}^{\infty} = (\widehat{\mu}_0(\omega_0 + \frac{n\pi}{\Lambda}))_{n=-\infty}^{\infty}$$

$$\mathcal{N}_1(\mu) = (|\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda}) - \widehat{\mu}(\omega_0 + \frac{(n+2)\pi}{\Lambda})|^2)_{n=-\infty}^{\infty} = (\widehat{\mu}_1(\omega_0 + \frac{n\pi}{\Lambda}))_{n=-\infty}^{\infty} \tag{3.11}$$

$$\mathcal{N}_2(\mu) = (|\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda}) - i\widehat{\mu}(\omega_0 + \frac{(n+2)\pi}{\Lambda})|^2)_{n=-\infty}^{\infty} = (\widehat{\mu}_2(\omega_0 + \frac{n\pi}{\Lambda}))_{n=-\infty}^{\infty}$$

satisfy $(N_0(\mu))_n \neq 0$ for all $n \in \mathbb{N}$. In other words, $\{\omega_0 + \frac{n\pi}{\Lambda} | n \in \mathbb{Z}\} \cap \{z_j | j \in \mathbb{Z}\} = \emptyset$ (otherwise, $z_j - \omega_0 = \frac{n\pi}{\Lambda}$ for some $j, n \in \mathbb{Z} \iff \omega_0 \in \{z_j + \frac{n\pi}{\Lambda} | n \in \mathbb{Z}\}$). Thus, we are able to interpolate each $\widehat{\mu}_i$ for $i \in \{0, 1, 2\}$ to obtain the measurements (3.11). This allows us to compute the values

$$\begin{aligned}
&(1-i)(\mathcal{N}_0(\mu))_n + (1-i)(\mathcal{N}_0(\mu))_{n+2} - (\mathcal{N}_1(\mu))_n + i(\mathcal{N}_2(\mu))_n = \\
&\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} - i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&-[\widehat{\mu}(\omega_n) - \widehat{\mu}(\omega_{n+2})][\overline{\widehat{\mu}(\omega_n) - \widehat{\mu}(\omega_{n+2})}] + i[\widehat{\mu}(\omega_n) - i\widehat{\mu}(\omega_{n+2})][\overline{\widehat{\mu}(\omega_n) - i\widehat{\mu}(\omega_{n+2})}] \\
&= \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} - i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&-[\widehat{\mu}(\omega_n) - \widehat{\mu}(\omega_{n+2})][\overline{\widehat{\mu}(\omega_n)} - \overline{\widehat{\mu}(\omega_{n+2})}] + i[\widehat{\mu}(\omega_n) - i\widehat{\mu}(\omega_{n+2})][\overline{\widehat{\mu}(\omega_n)} + i\overline{\widehat{\mu}(\omega_{n+2})}] \\
&= \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} - i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&-[\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})} - \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})}] \\
&+i[\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})} - i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})}] \\
&= \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} - i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&-\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} + \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)} - \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&+i\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_n)} - \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)} + i\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_{n+2})} \\
&= (\widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})} + \widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)}) + (\widehat{\mu}(\omega_{n+2})\overline{\widehat{\mu}(\omega_n)} - \widehat{\mu}(\omega_n)\overline{\widehat{\mu}(\omega_{n+2})}) \\
&= 2\overline{\widehat{\mu}(\omega_n)}\widehat{\mu}(\omega_{n+2}) = 2\overline{\widehat{\nu}(\omega_n)}
\end{aligned}$$

for each $n \in \mathbb{Z}$. This shows that each entry in (3.8) can be obtained using the measurements (3.1). In particular, $\frac{1}{\widehat{\nu}(\omega_n)}, \frac{1}{\overline{\widehat{\nu}(\omega_n)}}$ and $\frac{1}{(\mathcal{N}_0(\mu))_n}$ can be computed for each $n \in 2\mathbb{N} \cup \{0\}$, which proves the claim for $Q_{\mu,N}$. The proof for $Q_{\mu,-N}$ is similar. □

The 4-term polarization identity used in the previous lemma is a consequence of the observation in Lemma 2.2.2. We note that the coefficients in the interference measurements can be chosen more generally [12]. In particular, Theorem 2.1 of [13] gives a collection of such identities corresponding to pairs of choices of modulation factors. Finally, we appeal to Lemma 3.2.1 to show that the unique minimizer of the quadratic forms constructed above over $S_{[0,\Lambda]}/\sim$ is indeed $[\mu]$.

Theorem 3.2.3. *Given $\Lambda > 0$, we have that*

$$Q_{\mu,N}(\varphi) = Q_{\mu,-N}(\varphi) = 0 \quad \forall\ N \in \mathbb{N} \cup \{0\}$$

if and only if $\varphi \in [\mu]$ for any $\mu, \varphi \in S_{[0,\Lambda]}$.

Proof. Let $\varphi \in [\mu] \in S_{[0,\Lambda]}/\sim$, so $\widehat{\varphi}(\omega) = c\widehat{\mu}$ for some $c \in \mathbb{C}$ with $|c| = 1$. Then, we can compute

$$\begin{aligned} Q_{\mu,N}(\varphi) &= \|(I_{N+1} - T_{\frac{-2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},N}\big(c\widehat{\mu}(\omega_{2n})\big)_{n=0}^{N}\|^2_{\ell_2^{N+1}} - 1 \\ &= \|(I_{N+1} - T_{\frac{-2\pi}{\Lambda}})(c)_{n=0}^{N}\|^2_{\ell_2^{N+1}} - 1 = |c|^2 - 1 = 0, \end{aligned}$$

and

$$\begin{aligned} Q_{\mu,-N}(\varphi) &= \|(I_{N+1} - T_{\frac{2\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},-N}(c\widehat{\mu}(\omega_{2n})_{n=-N}^{0})\|^2_{\ell_2^{N+1}} - 1 \\ &= \|(I_{N+1} - T_{\frac{2\pi}{\Lambda}})(c)_{n=-N}^{0}\|^2_{\ell_2^{N+1}} - 1 = |c|^2 - 1 = 0. \end{aligned}$$

Conversely, if $Q_{\mu,N}(\varphi) = Q_{\mu,-N}(\varphi) = 0$ for some $\varphi \in S_{[0,\Lambda]}$ for all $N \in \mathbb{N}$, we want to show that $\varphi \in [\mu]$. By Lemma 3.2.1, it is sufficient to show that $\widehat{\varphi}(\omega_{2n}) = c\widehat{\mu}(\omega_{2n})$ for all $n \in \mathbb{Z}$ for some $c \in \mathbb{C}$ with $|c| = 1$. We proceed by induction. For the base case, n=0, we have by $Q_{\mu,0}(\varphi) = 0$ that

$$|(I_{N+1} - T_{\frac{-2\pi}{\Lambda}})\frac{\widehat{\varphi}(\omega_0)}{\widehat{\mu}(\omega_0)}|^2 = |\frac{\widehat{\varphi}(\omega_0)}{\widehat{\mu}(\omega_0)} - 0|^2 = |\frac{\widehat{\varphi}(\omega_0)}{\widehat{\mu}(\omega_0)}|^2 = 1$$

if and only if $\widehat{\varphi}(\omega_0) = c\widehat{\mu}(\omega_0)$ for some $c \in \mathbb{C}$ with $|c| = 1$. Next, suppose $\widehat{\varphi}(\omega_{2n}) = c\widehat{\mu}(\omega_{2n})$ for all $n \in \{-(N-1), -(N-2), ..., N-2, N-1\}$. We want to show that $\widehat{\varphi}(\omega_{2N}) = c\widehat{\mu}(\omega_{2N})$ and

$\widehat{\varphi}(\omega_{-2N}) = c\widehat{\mu}(\omega_{-2N})$. By assumption, we have that

$$Q_{\mu,N}(\varphi) = Q_{\mu,-N}(\varphi) = 0,$$

hence,

$$Q_{\mu,N}(\varphi) = \sum_{n=1}^{N} |\frac{\widehat{\varphi}(\omega_{2n})}{\widehat{\mu}(\omega_{2n})} - \frac{\widehat{\varphi}(\omega_{2(n-1)})}{\widehat{\mu}(\omega_{2(n-1)})}|^2 + |\frac{\widehat{\varphi}(\omega_0)}{\widehat{\mu}(\omega_0)}|^2 - 1 = |\frac{\widehat{\varphi}(\omega_{2N})}{\widehat{\mu}(\omega_{2N})} - c|^2 = 0$$

if and only if $\frac{\widehat{\varphi}(\omega_{2N})}{\widehat{\mu}(\omega_{2N})} = c$. Similarly, we get that

$$Q_{\mu,-N}(\varphi) = \sum_{n=-N}^{-1} |\frac{\widehat{\varphi}(\omega_{2n})}{\widehat{\mu}(\omega_{2n})} - \frac{\widehat{\varphi}(\omega_{2(n+1)})}{\widehat{\mu}(\omega_{2(n+1)})}|^2 + |\frac{\widehat{\varphi}(\omega_0)}{\widehat{\mu}(\omega_0)}|^2 - 1 = |\frac{\widehat{\varphi}(\omega_{-2N})}{\widehat{\mu}(\omega_{-2N})} - c|^2 = 0$$

if and only if $\frac{\widehat{\varphi}(\omega_{-2N})}{\widehat{\mu}(\omega_{-2N})} = c$. □

Before we discuss error bounds, we demonstrate the fact that exact recovery can still be achieved if we avoid the error caused by truncation and interpolation. In [22], it is shown that $\mu \in S_{\Delta,[0,\Lambda],s}$ can be reconstructed from $4s$ equidistant point evaluations of the Fourier transform via TV-norm minimization. The first idea is to apply Theorem 1.2 of [22] to our signal class. We know that all half open intervals of the real line are homeomorphic. Thus, every signal μ with $\mu \in S_{\Delta,[0,\Lambda],s}$ can be mapped to a signal $\mu_{\mathbb{T}} : [0,1] \to \mathbb{C}$ defined by $\mu_{\mathbb{T}}(t) = \mu(\frac{t}{\Lambda})$ for each $t \in [0,\Lambda]$. Similarly each magnitude measurement can be interpreted as a quadratic measurement of the new signal $\mu_{\mathbb{T}}$ by rescaling (so we would have $\widehat{\mu}_{\mathbb{T}}(\omega) = \mathcal{F}(\mu(\frac{t}{\Lambda})) = \Lambda\widehat{\mu}(\Lambda\omega)$). Since Theorem 1.2 in [22] implies that every complex measure $\mu_{\mathbb{T}}$ over $[0,1]$ can be recovered provided that $2N+1$ linear measurements are available, with $N \geq 128$ and minimum separation $\Delta' \geq \frac{2}{N}$, which is equivalent to μ obeying a minimum separation condition with $\Delta \geq \frac{2\Lambda}{N}$, we may deduce the following result:

Theorem 3.2.4. *A total of* $2N+1$ *exact non-zero intensity measurements (as in measurements (3.13)) such that* $N \geq \max\{3(4s+3), 128, \frac{2\Delta}{\Delta}\}$ *are sufficient to reconstruct* $\mu \in S_{\Delta,[0,\Lambda],s}$ *provided that* $\Delta \geq \frac{2\Lambda}{\min\{|N_1|, N_2\}}$ *and* $\min\{|N_1|, N_2\} \geq 128$ *(or equivalently, if* $\min\{|N_1|, N_2\} \geq \max\{128, \frac{2\Lambda}{\Delta}\}$*).*

In this case, any solution to the optimization problem

$$\min_{\varphi} \|\varphi\|_{TV} \quad \textit{subject to} \quad \sum_{n=N_1}^{N_2} \widetilde{Q_{\mu,E_{\omega_0},n}}(\widehat{\varphi}) = 0$$

over the set of complex measures belongs to $[\mu] \in S_{\Delta,[0,\Lambda]}/\sim$.

Proof. As the authors point out in [22], $4s$ linear measurements of μ are sufficient for the recovery. Thus, if $N \geq 2s+1$ such that we have $3(2N+1) = 3(4s+3)$ quadratic measurements as in (3.13), then we are able to construct $4s+3$ exact linear measurements of μ, since $\|\epsilon_{\omega_0,0}\|_\infty = 0$, which allows the interpolation of the signal exactly. □

The next result considers a different approach towards a sufficient condition for recovery by oversampling on a finite interval (instead of requiring non-zero measurements) under the assumption $\|\widehat{\mu}\|_{L_\infty[-\Omega,\Omega]} > \alpha\|\mu\|_{TV}$ for some $\alpha > 0$, which can be interpreted as a restriction (in particular, a lower bound) on the minimum gap between the nodes, Δ.

Theorem 3.2.5. *Let* $\mu \in S_{\Delta,[0,\Lambda],s}$. *Suppose that* $\|\widehat{\mu}\|_{L_\infty[-\Omega,\Omega]} > \alpha\|\mu\|_{TV}$ *for some* $\alpha > 0$. *If*

$$N \geq \frac{a\alpha}{4\pi^2} \qquad \textit{and} \qquad \frac{a\alpha}{4\pi^2} \geq \frac{4s+3}{2},$$

then we can reconstruct some $\varphi \in [\mu]$.

Proof. By continuity, there is $\omega_0 \in [-\Omega,\Omega]$ such that $|\widehat{\mu}(\omega_0)| > \alpha\|\mu\|_{TV} > 0$. Since $\widehat{\mu}$ has a Lipschitz constant $2\pi\Lambda\|\mu\|_{TV}$, there is $\delta > 0$ such that $|\widehat{\mu}(\omega)| > 0$ for all $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$. In particular, if $\delta = \frac{\alpha}{4\pi\Lambda}$, then

$$||\widehat{\mu}(\omega_0)| - |\widehat{\mu}(\omega)|| \leq |\widehat{\mu}(\omega_0) - \widehat{\mu}(\omega)| \leq 2\pi(\delta)\Lambda\|\mu\|_{TV} = 2\pi\Lambda\|\mu\|_{TV}\frac{\frac{\alpha\|\mu\|_{TV}}{2}}{2\pi\Lambda\|\mu\|_{TV}}$$

so we get that

$$|\widehat{\mu}(\omega)| \geq |\widehat{\mu}(\omega_0)| - \frac{\alpha\|\mu\|_{TV}}{2} > \alpha\|\mu\|_{TV} - \frac{\alpha\|\mu\|_{TV}}{2} > 0$$

for any $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$. By choosing a sampling rate

$$\begin{aligned}
a \geq \frac{(4s+3)\pi}{2\Lambda\delta} &= \frac{(4s+3)\pi}{2\Lambda\frac{\alpha}{4\pi\Lambda}} \\
&= \frac{2\pi^2(4s+3)}{\alpha} \\
&> 2\pi^2(4s+3)\left(\frac{\|\mu\|_{TV}}{\|\widehat{\mu}\|_{L_\infty[-\Omega,\Omega]}}\right)
\end{aligned}$$

we see that there are $4s+3$ consecutive non-zero measurements in the interval $[\omega_0 - \delta, \omega_0 + \delta]$. Applying Lemma 3.2.2 to these measurements, we obtain a vector of $4s+3$ linear measurements of μ, which guarantee the uniqueness of the solution to

$$\min_{\varphi} \|\varphi\|_{TV} \quad \text{subject to} \quad \sum_{n=-N}^{N} Q_{\mu,n} = 0$$

over the set of all finite complex measures supported on $[-\Lambda, \Lambda]$ according to Theorem 1.2 of [22]. $\square$

In order to gain a better understanding of the condition stated in Theorem 3.2.5, we give a useful norm equivalence relation. We have that

$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\right)(t) &= \frac{2}{\Delta}\mathcal{F}^{-1}(\mathcal{F}(\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}).\mathcal{F}(\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}))(t) \\
&= \frac{2}{\Delta}[\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}](t) \\
&= \frac{2}{\Delta}\int_{-\infty}^{\infty} \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(x)\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(t-x)dx \\
&= \begin{cases} \frac{2}{\Delta}\int_{-\frac{\Delta}{4}}^{t+\frac{\Delta}{4}} dx & -\frac{\Delta}{2} \leq t < 0, \\ \frac{2}{\Delta}\int_{t-\frac{\Delta}{4}}^{\frac{\Delta}{4}} dx & 0 \leq t \leq \frac{\Delta}{2}, \\ 0 & else \end{cases} \\
&= \begin{cases} \frac{2}{\Delta}t + 1 & -\frac{\Delta}{2} \leq t < 0, \\ 1 - \frac{2}{\Delta}t & 0 \leq t \leq \frac{\Delta}{2}, \\ 0 & else. \end{cases}
\end{aligned}$$

Next, for $\Delta > 0$, let the triangular function $\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]} : \mathbb{R} \to \mathbb{R}$ be defined as

$$\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t) = \begin{cases} \frac{2}{\Delta}t + 1 & -\frac{\Delta}{2} \leq t < 0, \\ \frac{-2}{\Delta}t + 1 & 0 \leq t \leq \frac{\Delta}{2}, \\ 0 & else, \end{cases}$$

so we have that $\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]} = \mathcal{F}^{-1}(\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]})$ and

$$\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t - t_j) = \begin{cases} \frac{2}{\Delta}(t - t_j) + 1 & t_j - \frac{\Delta}{2} \leq t < t_j, \\ \frac{-2}{\Delta}(t - t_j) + 1 & t_j \leq t \leq t_j + \frac{\Delta}{2}, \\ 0 & else, \end{cases}$$

for any $t_j \in \mathbb{R}$. Using the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \|\mu\|_{TV} = \|c\|_{\ell_1} = \sum_{j=1}^{s} |c_j| &= \sum_{j=1}^{s} |c_j|.1 \\ &\leq (\sum_{j=1}^{s} |c_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{s} 1^2)^{\frac{1}{2}} \\ &= \|c\|_{\ell_2}\sqrt{s} \end{aligned}$$

On the other hand, we know that

$$\|c\|_{\ell_2}^2 = \sum_{j=1}^{s} |c_j|^2 \leq (\sum_{j=1}^{s} |c_j|)^2 = \|c\|_{\ell_1}^2.$$

We also have that

$$\|\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}\|_{L^2} = (\int_{\mathbb{R}} |\sum_{j=1}^{s} c_j \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(x-t_j)|^2 dx)^{\frac{1}{2}}$$
$$= (\sum_{j=1}^{s} 2\int_{t_j}^{t_j+\frac{\Delta}{2}} (\frac{2}{\Delta}(t-t_j)+1)^2 |c_j|^2 dx)^{\frac{1}{2}}$$
$$= (\sum_{j=1}^{s} (|c_j|^2 \frac{\Delta}{2})^{\frac{1}{2}} = \frac{\sqrt{\Delta}}{\sqrt{2}} \|c\|_{\ell_2}.$$

Therefore, we get that

$$\frac{\sqrt{2}}{\sqrt{\Delta}} \|\widehat{\mu}.\widehat{\lambda}_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}\|_{L^2} \leq \|\mu\|_{TV} \leq \frac{\sqrt{2s}}{\sqrt{\Delta}} \|\widehat{\mu}.\widehat{\lambda}_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}\|_{L^2}$$

or equivalently,

$$\frac{2\sqrt{2}}{\Delta^{\frac{3}{2}}} \|\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2} \leq \|\mu\|_{TV} \leq \frac{2\sqrt{2s}}{\Delta^{\frac{3}{2}}} \|\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2}.$$

This relationship is a direct consequence of the equivalence of norms on finite-dimensional spaces, under the guarantee of a minimal support separation, $\Delta > 0$. We may now modify the condition in Theorem 3.2.5 using the fact that

$$\|\mu\|_{TV} \leq \frac{2\sqrt{2s}}{\Delta^{\frac{3}{2}}} \|\widehat{\mu}.\widehat{\chi}^2\|_{L^2}$$
$$\leq \frac{2\sqrt{2s}}{\Delta^{\frac{3}{2}}} \left(\int_{-\infty}^{\infty} \|\widehat{\mu}\|_\infty^2 |\widehat{\chi}^2(\omega)|^2 d\omega \right)^{\frac{1}{2}}$$
$$\leq \frac{2\sqrt{2s}}{\Delta^{\frac{3}{2}}} (4\Lambda^2)^{\frac{1}{2}} \|\widehat{\mu}\|_\infty$$
$$= \frac{4\Lambda^2\sqrt{2s}}{\Delta^{\frac{3}{2}}} \|\widehat{\mu}\|_\infty.$$

Hence, setting $\alpha = \frac{\Delta^{\frac{3}{2}}}{4\Lambda^2\sqrt{2s}}$ means that a sufficient condition for recovery from exact measurements is

$$N \geq \frac{a\Delta^{\frac{3}{2}}}{16\pi^2\sqrt{2s}\Lambda^2} \qquad \text{and} \qquad \frac{a\Delta^{\frac{3}{2}}}{16\pi^2\sqrt{2s}\Lambda^2} \geq \frac{4s+3}{2}.$$

In particular, $N \geq \frac{\sqrt{s}(4s+3)}{2}$ intensity measurements are sufficient for reconstruction of $[\mu] \in S_{\Delta,[0,\Lambda],s}$ under the assumption that $\|\widehat{\mu}\|_{L_\infty[-\Omega,\Omega]} > \frac{\Delta^{\frac{3}{2}}}{4\Lambda^2\sqrt{2s}}\|\mu\|_{TV}$ for some $\alpha > 0$.

3.3 Error bounds for recovery by minimizing the TV norm from exact measurements

We want to consider the following restrictions on the measurements discussed in the previous section:

(i) We have access to sampling only on a finite interval $[-\frac{N+1}{\Lambda}, \frac{N+1}{\Lambda}] \subset \mathbb{R}$ for some $N \in \mathbb{N}$.

(ii) Instead of the measurements $\mathcal{M}_{i,\Gamma_N}$, we are given noisy measurements $\widetilde{\mathcal{M}_i} : S_{[0,\Lambda]} \to \ell_2(\{-N, ..., N\})$ for $i \in \{0, 1, 2\}$ where

$$\widetilde{\mathcal{M}_{0,\Gamma_N}}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda})|^2 - (\epsilon_0)_n)_{n=-N}^N$$
$$\widetilde{\mathcal{M}_{1,\Gamma_N}}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda}) - \widehat{\mu}(\frac{n\pi}{\Lambda} + \frac{\pi}{\Lambda})|^2 - (\epsilon_1)_n)_{n=-N}^N \tag{3.12}$$
$$\widetilde{\mathcal{M}_{2,\Gamma_N}}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda}) - i\widehat{\mu}(\frac{n\pi}{\Lambda} + \frac{\pi}{\Lambda})|^2 - (\epsilon_2)_n)_{n=-N}^N$$

for some $\epsilon_i \in \ell_2(\{-N, ..., N\})$ for each $i \in \{0, 1, 2\}$.

We will follow a two step procedure where we use phase propagation to construct approximate linear measurements of μ, then we resample the approximations of $\widehat{\mu}_0$, $\widehat{\mu}_1$ and $\widehat{\mu}_2$, if necessary, in a way that avoids zero measurements. The process of extrapolating the Fourier transform of μ outside of the sampling interval is known as super-resolution [27, 22]. The procedure is displayed in Figure 3.1.

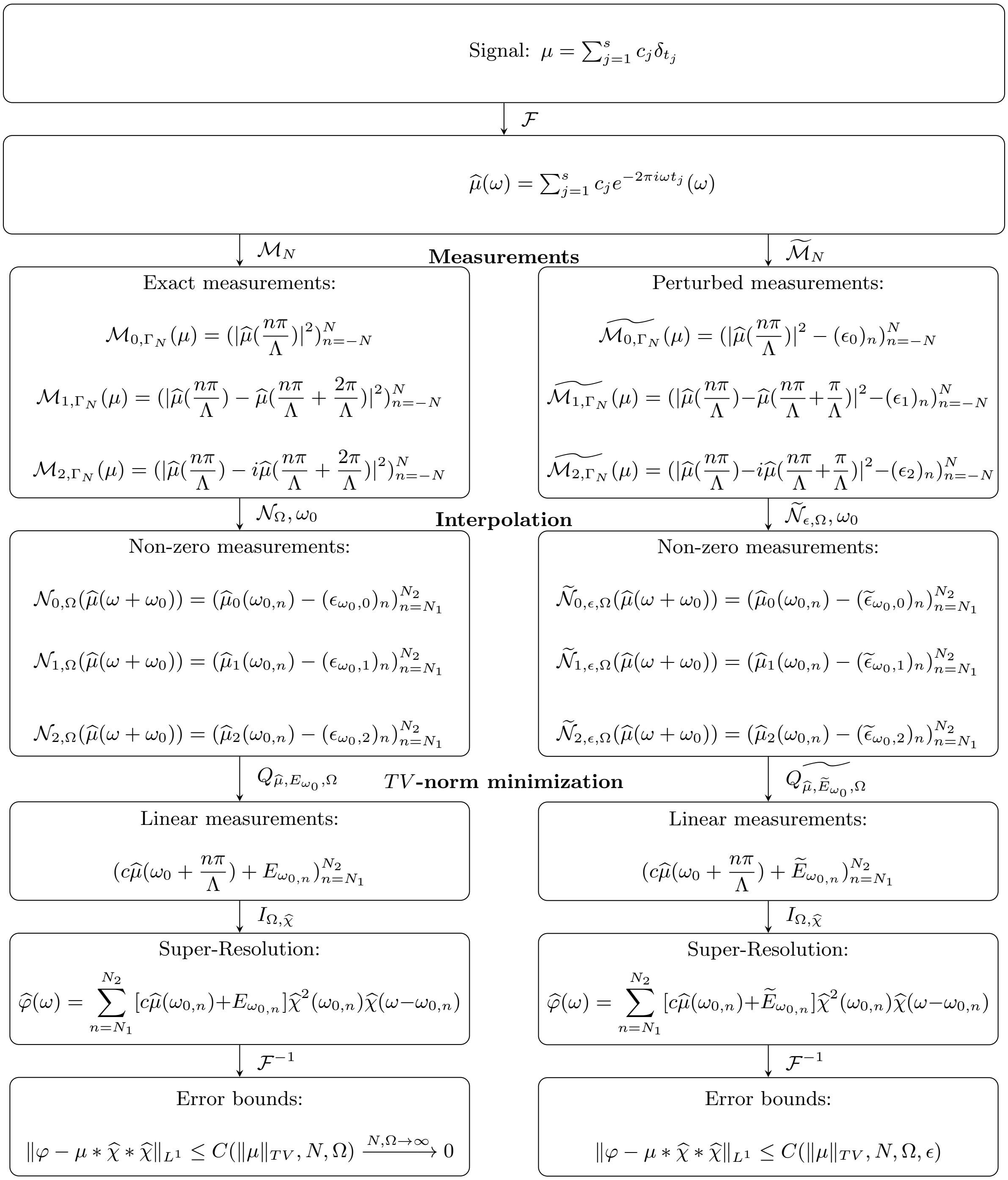

Figure 3.1: Flow chart of recovery by TV-norm minimization with or without the presence of noise.

The error bound we derive depends on the number of samples, the length of the resampling interval, as well as the signal parameters. The dependence on the number of samples is based on the approximation we construct of $\widehat{\mu}$ using the sampling theorem. In order to increase the convergence rate, we will replace $\widehat{\mu}$ with the Fourier transform of the smooth function obtained by convolving μ with the $sinc^2$-kernel. However, this can only be accomplished if we assume a minimum separation distance, $\Delta > 0$, between the nodes of μ. We also consider the effect of oversampling and show that it can be used to improve the error bound.

First, we consider the case where only the restriction (i) holds. So we assume we are given measurements as in (3.1):

$$\mathcal{M}_{0,\Gamma_N}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda})|^2)_{n=-N}^N,$$

$$\mathcal{M}_{1,\Gamma_N}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda}) - \widehat{\mu}(\frac{n\pi}{\Lambda} + \frac{\pi}{\Lambda})|^2)_{n=-N}^N, \tag{3.13}$$

and

$$\mathcal{M}_{2,\Gamma_N}(\mu) = (|\widehat{\mu}(\frac{n\pi}{\Lambda}) - i\widehat{\mu}(\frac{n\pi}{\Lambda} + \frac{\pi}{\Lambda})|^2)_{n=-N}^N$$

for a fixed $N \in \mathbb{N}$.

As mentioned, in order to reduce the truncation error, we need to choose an appropriate interpolation kernel with a fast decay rate. With the minimum separation assumption, our goal now is to recover an equivalence class $[\mu]$ in the restriction $S_{\Delta,[0,\Lambda]} \subset S_{[0,\Lambda]}$ defined as

$$S_{\Delta,[0,\Lambda]} := \{\mu \in S_{[0,\Lambda]} \,\big|\, |t_j - t_k| \geq \Delta \,\text{for all nodes}\, \{t_j\}_{j=1}^s \,\text{of}\, \mu\}$$

for some $\Delta > 0$. We will also need the following inequality to derive a uniform truncation error bound for resampling. We have that

$$\begin{aligned}\int_N^\infty \widehat{\chi}^2_{[-\Lambda,\Lambda]}(\omega-\frac{x\pi}{\Lambda})dx \leq \int_{-\infty}^{-N} \widehat{\chi}^2_{[-\Lambda,\Lambda]}(\omega-\frac{x\pi}{\Lambda})dx &= \int_{-\infty}^{-N} \frac{sin^2(2\pi\Lambda(\omega-\frac{x\pi}{\Lambda}))}{\pi^2(\omega-\frac{x\pi}{\Lambda})^2}dx \\ &\leq \int_{-\infty}^{-N} \frac{1}{(|\omega|-\frac{x\pi}{\Lambda})^2}dx \\ &\leq \int_{-\infty}^{-N} \frac{1}{(-\frac{x\pi}{\Lambda})^2}dx \\ &= \frac{\Lambda^2}{\pi^3}\int_{-\infty}^{-N}\frac{1}{x^2}dx \\ &= \frac{\Lambda^s}{\pi^3}\frac{-1}{x}\Big|_{-\infty}^{-N} \\ &= \frac{\Lambda^2}{\pi^3 N}\end{aligned}$$

for any $\omega \in \mathbb{R}$.

The first part of the next lemma is a direct consequence of Shannon's sampling theorem for band-limited functions [60]. The second part provides uniform error bounds for the truncation of the series representation of $\widehat{\mu}$.

Lemma 3.3.1. *Let $\mu \in S_{\Delta,[0,\Lambda],s}$, let $\widehat{\mu} = \sum_{j=1}^s c_j e^{-2\pi i t_j}$, and let $\|c\|_{\ell_1}$ denote the norm of the amplitude vector $(c_j)_{j=1}^s$. Then we have that*

$$\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega-\frac{n\pi}{\Lambda}) \tag{3.14}$$

where the RHS converges for all $\omega \in \mathbb{R}$, and

$$|E_{\widehat{\mu},N}(\omega)| = |\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega) - \sum_{n=-N}^{N} \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega-\frac{n\pi}{\Lambda})| \leq \frac{2\sqrt{3}\|c\|_{\ell_1}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} \tag{3.15}$$

for any all $\omega \in \mathbb{R}$.

Proof. We have that

$$\begin{aligned}\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]} = \int_{-\infty}^{\infty} \mu(x)\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t-x)dx &= \int_{\infty}^{\infty} \sum_{j=1}^{s} c_j \delta_{t_j}(x)\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t-x)dx \\ &= \sum_{j=1}^{s} c_j \int_{-\infty}^{\infty} \delta_{t_j}(x)\lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t-x)dx \\ &= \sum_{j=1}^{s} c_j \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}(t-t_j).\end{aligned}$$

We also have that

$$\begin{aligned}\mathcal{F}^{-1}(\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]})(\omega) &= \mathcal{F}(\mathcal{F}^{-1}(\widehat{\mu}) * \mathcal{F}^{-1}(\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}))(-\omega) \\ &= \widehat{\mu}.\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(-\omega).\end{aligned}$$

Thus, $\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]}$ is a piecewise smooth continuous function, and $(\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]})(t) = 0$ for all $|t| > \Lambda$. Shannon's sampling theorem [60] then implies that $\mathcal{F}^{-1}(\mu * \lambda_{[-\frac{\Delta}{2},\frac{\Delta}{2}]})(\omega) = \widehat{\mu}.\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(-\omega)$ can be reconstructed from the point evaluations $\widehat{\mu}.\frac{2}{\Delta}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})$ for $n \in \mathbb{Z}$ according the series expansion

$$\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(-\omega) = \sum_{n=-\infty}^{\infty} \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(-\omega - \frac{n\pi}{\Lambda})$$

and (3.14) follows by replacing $-\omega$ with ω.

Next, we wish to establish uniform bounds for the truncation error. For any $\omega \in \mathbb{R}$ we have

$$
\begin{aligned}
E_{\widehat{\mu},N}(\omega) &= |\widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\omega) - \sum_{-N}^{N} \widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})| \\
&= |\sum_{|n|>N} \widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})| \\
&\leq (\sum_{|n|>N} |\widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})|^2)^{\frac{1}{2}} (\sum_{|n|>N} \widehat{\chi}^2_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda}))^{\frac{1}{2}} \\
&\leq (\sum_{|n|>N} |\widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})|^2)^{\frac{1}{2}} (\int_{-\infty}^{-N} \widehat{\chi}^2_{[-\Lambda,\Lambda]}(\omega - \frac{x\pi}{\Lambda})dx + \int_{N}^{\infty} \widehat{\chi}^2_{[-\Lambda,\Lambda]}(\omega - \frac{x\pi}{\Lambda})dx)^{\frac{1}{2}} \\
&\leq (\sum_{|n|>N} |\widehat{\mu}.\widehat{\chi}^2_{[\frac{-\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})|^2)^{\frac{1}{2}} (\frac{2\Lambda^2}{\pi^3 N})^{\frac{1}{2}} \\
&\leq \|c\|_{\ell_1} (\sum_{|n|>N} |(\frac{\Lambda}{n\pi})^2|^2)^{\frac{1}{2}} \frac{\sqrt{2}\Lambda}{\pi^{\frac{3}{2}}\sqrt{N}} \\
&= \frac{\sqrt{2}\Lambda}{\pi^{\frac{3}{2}}\sqrt{N}} \|c\|_{\ell_1} (\frac{\Lambda}{\pi})^2 (\sum_{|n|>N} \frac{1}{n^4})^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}\Lambda}{\pi^{\frac{3}{2}}\sqrt{N}} \|c\|_{\ell_1} (\frac{\Lambda}{\pi})^2 (2\int_{N}^{\infty} \frac{1}{x^4}dx)^{\frac{1}{2}} \\
&= \frac{\sqrt{2}\Lambda}{\pi^{\frac{3}{2}}\sqrt{N}} \|c\|_{\ell_1} (\frac{\Lambda}{\pi})^2 \frac{\sqrt{2}}{\sqrt{3N^3}} \\
&= \frac{2\sqrt{3}\|c\|_{\ell_1}\Lambda^3}{3\pi^{\frac{7}{2}}N^2}. \qquad \square
\end{aligned}
$$

Following the procedure presented in Section 3.2, we need to show that it is possible to interpolate the data (3.13) to obtain non-zero intensity measurements that allow for phase propagation.

Proposition 3.3.2. *Let $\mu \in S_{\Delta,[-\Lambda,\Lambda],s}$ be given by $\mu = \sum_{j=1}^{s} c_j \delta_{t_j}$. Fix $N \in \mathbb{N}$ and let Ω be any real number satisfying $0 < \Omega \leq \frac{N\pi}{\Lambda}$. Then, there exists $\omega_0 \in [-\Omega, \Omega]$ such that the following inequality holds*

$$
|\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda})| \geq \widetilde{m}_n (\frac{\Omega}{N'' + 1})^{N_{\mathbb{R}}} > 0 \tag{3.16}
$$

for some $\widetilde{m}_n > 0$ and $N'', N_{\mathbb{R}} \in \mathbb{N}$ for all $n \in \mathbb{Z}$ satisfying $\omega_0 + \frac{n\pi}{\Lambda} \in [-\Omega, \Omega]$.

Proof. Let $\widetilde{\mu}$ be the analytic continuation for $\widehat{\mu}$ over $\mathbb{C}$. For any $|z| \leq \Omega$, we can write

$$\widetilde{\mu}(z) = \widetilde{\varphi}(z) \prod_{k=1}^{N'} (z - z_k)$$

where $\{z_k\}_{k=1}^{N'}$ are the roots of $\widetilde{\mu}$ in the region $|z| \leq \Omega$ and $\widetilde{\varphi}$ is an analytic function with $\widetilde{\varphi}(z) \neq 0$ for any $|z| \leq \Omega$. Let $F = \{z_k \in \mathbb{C} \setminus \mathbb{R}\}_{k=1}^{N_{\mathbb{C}}} \subseteq \{z_k\}_{k=1}^{N'}$ be the set of all non-real roots of $\widetilde{f}$. Set

$$\widetilde{\phi}(z) = \widetilde{\varphi}(z) \prod_{k=1}^{N_{\mathbb{C}}} (z - z_k)$$

where $z_k \in F$ for each $k \in \{1, \ldots, N_{\mathbb{C}}\}$. Then we have that

$$\widetilde{\mu}(z) = \widetilde{\phi}(z) \prod_{k=1}^{N_{\mathbb{R}}} (z - z_k)$$

where $z_k \in \{z_j\}_{j=1}^{N'} \setminus F$ for each $k \in \{1, \ldots, N_{\mathbb{R}}\}$ and $N_{\mathbb{R}} = N' - N_{\mathbb{C}}$. Let

$$R = \{z_{k,n} := z_k + \frac{n\pi}{\Lambda} | \widetilde{\mu}(z_k) = 0, z_k \in [-\Omega, \Omega], n \in \mathbb{Z}\}.$$

By Theorem 2.1.4, we know that $N_{\mathbb{R}} \leq N' \leq \max\{[9\pi\Lambda\Omega + 2s(\ln \frac{\Lambda}{\Delta} + \ln s + 3)], [9\pi\Lambda + 2s(\ln \frac{\Lambda}{\Delta} + \ln s + 3)]\}$, so we get that

$$N'' := |R \cap [-\Omega, \Omega]| \leq (\frac{2N\pi}{\Lambda}) \max\{[9\pi\Lambda\Omega + 2s(\ln \frac{\Lambda}{\Delta} + \ln s + 3)], [9\pi\Lambda + 2s(\ln \frac{\Lambda}{\Delta} + \ln s + 3)]\}.$$

.

This implies that there is at least one pair $z_{k,n}, z_{j,m} \in R \cap [-\Omega, \Omega]$ such that $|z_{k,n} - z_{j,m}| \geq \frac{2\Omega}{N''+1}$ for some $j, k \in \{1, ..., N_{\mathbb{R}}\}$ and $n, m \in \mathbb{Z}$, otherwise $\Omega - z_{k',n'} \geq \frac{2\Omega}{N''+1}$ or $|\Omega + z_{j',m'}| \geq \frac{2\Omega}{N''+1}$ for some $z_{k',n'} \in R \cap [0, \Omega]$ and $z_{j',m'} \in R \cap [-\Omega, 0]$ for some $j', k' \in \{1, ..., N_{\mathbb{R}}\}$ and $n', m' \in \mathbb{Z}$. In any of these cases, there is a pair of points $x, y \in \{R \cap [-\Omega, \Omega]\} \cup \{-\Omega, \Omega\}$ with $x < y$ satisfying

$y - x \geq \frac{2\Omega}{N''+1}$. Let $\omega_0 = \frac{y+x}{2}$, then we have that

$$\{\omega_{0,n} := w_0 + \frac{n\pi}{\Lambda} | n \in \mathbb{Z}\} \cap \{z_k \in \mathbb{R} | \widetilde{\mu}(z_k) = 0\} \cap [-\Omega, \Omega] = \emptyset.$$

We also get that

$$|(\omega_0 + \frac{n\pi}{\Lambda}) - z_k| = |(\omega_0 - (z_k - \frac{n\pi}{\Lambda})| \geq \frac{\Omega}{N''+1}$$

for any $n \in \mathbb{Z}$ and $k \in \{1, ..., N_{\mathbb{R}}\}$. Note that the above inequality holds for any $n \in \mathbb{Z}$ with $z_{k,-n} \in [-\Omega, \Omega]$ since $|\omega_0 - z_{k,-n}| \geq |z_0 - x|$ for any such n by choice of ω_0. On the other hand, if $z_{k,-n} \notin [-\Omega, \Omega]$, then clearly, $|\omega_0 - z_{k,-n}| > |\omega_0 + \Omega| \geq |\omega_0 - x|$ so we get the same result. It follows that

$$\begin{aligned} |\widehat{\mu}(\omega_{0,n})| = |\widetilde{\mu}(\omega_{0,n})| &= |\widetilde{\phi}(\omega_{0,n}) \prod_{k=1}^{N_{\mathbb{R}}} (\omega_{0,n} - z_k)| \\ &\geq |\widetilde{\phi}(\omega_{0,n}) \prod_{k=1}^{N_{\mathbb{R}}} (\frac{\Omega}{N''+1})| \\ &= \widetilde{m}_n (\frac{\Omega}{N''+1})^{N_{\mathbb{R}}} > 0 \end{aligned}$$

where $\widetilde{m}_n = |\widetilde{\phi}(\omega_{0,n})| > 0$ for each $\omega_{0,n} \in [-\Omega, \Omega]$ and $n \in \mathbb{Z}$. □

Lemma 3.3.3. *Let $N \in \mathbb{N}$ and set*

$$\widehat{\mu}_{(N)}(\omega) := \sum_{n=-N}^{N} \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4}, \frac{\Delta}{4}]}(\frac{n\pi}{\Lambda}) \widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda}). \tag{3.17}$$

Let $\Omega > 0$. If N is chosen sufficiently large, then there exists $\omega_0 \in [-\Omega, \Omega]$ such that the following inequality holds

$$|\widehat{\mu}_{(N)}(\omega_0 + \frac{n\pi}{\Lambda})| \geq \frac{\widetilde{M}_n}{2} (\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} > 0 \tag{3.18}$$

for some $\widetilde{M}_n > 0$ and $\widetilde{N} \in \mathbb{N}$ for all $n \in \mathbb{Z}$ satisfying $\omega_0 + \frac{n\pi}{\Lambda} \in [-\Omega, \Omega]$.

Proof. We will use the same notation as in the proof of Proposition 3.2.2 above. Observe that

$\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega) = 0$ if and only if $\widehat{\mu}(\omega) = 0$ or $\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega) = 0$. Let

$$\widetilde{R} = [\{z_{k,n} := z_k + \frac{n\pi}{\Lambda} | \widetilde{\mu}(z_k) = 0, z_k \in [-\Omega, \Omega], n \in \mathbb{Z}\} \cup \{\frac{2k}{\Delta}, k \in \mathbb{Z}\} \cup \{-\Omega, \Omega\}] \cap [-\Omega, \Omega].$$

Then, we have that

$$\widetilde{N} = |\widetilde{R}| \leq (\frac{2N\pi}{\Lambda}) \max\{[9\pi\Lambda\Omega + 2s(\ln\frac{\Lambda}{\Delta} + \ln s + 3)], [9\pi\Lambda + 2s(\ln\frac{\Lambda}{\Delta} + \ln s + 3)]\} + \Delta\Omega + 2.$$

Thus, there is at least one pair $x, y \in \widetilde{R}$ with $x < y$ and $y - x \geq \frac{2\Omega}{\widetilde{N}}$. Choosing $\omega_0 = \frac{y+x}{2}$ implies that

$$|\omega_0 - r| \geq |\omega_0 - x| \geq \frac{\Omega}{\widetilde{N} + 1}$$

for all $r \in \widetilde{R}$. In particular, we get that

$$\begin{aligned}
|\widehat{\mu}_{(N)}(\omega_{0,n})| &= |\widetilde{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_{0,n}) - E_{\widehat{\mu},N}(\omega_{0,n})| \\
&\geq ||\widetilde{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_{0,n})| - |E_{\widehat{\mu},N}(\omega_{0,n})|| \\
&\geq |\widetilde{\phi}(\omega_{0,n}) \prod_{k=1}^{N_{\mathbb{R}}} (\omega_{0,n} - z_k).\frac{sin^2(2\pi\frac{\Delta}{4}\omega_{0,n})}{\pi\omega_{0,n}^2}| - |E_{\widehat{\mu},N}| \\
&\geq \widetilde{\widetilde{M}}_n(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} - E_{\widehat{\mu},N} > 0
\end{aligned}$$

where $\widetilde{\widetilde{M}}_n = \widetilde{\phi}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_{0,n})$ for each $n \in \mathbb{Z}$, and $E_{\widehat{\mu},N} \equiv \frac{2\sqrt{3}\|c\|_{\ell_1}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} \geq 0$ is made sufficiently small by choosing a large value of N such that $E_{\widehat{\mu},N} \leq \min_{n\in\mathbb{Z}|\omega_{0,n}\in[-\Omega,\Omega]} \frac{\widetilde{\widetilde{M}}_n}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$. ☐

Lemma 3.3.3 proves the existence of non-zero sample values. The lower bound given in (3.18) cannot be explicitly computed without the knowledge of $\widehat{\mu}$ since $\widetilde{\widetilde{M}}_n$ depends on the value of the outer function in the factorization of $\widehat{\mu}$. However, once ω_0 is chosen, the interpolated values $\widehat{\mu}_{(N)}(\omega_{0,n})$ can be computed for each $n \in \mathbb{Z}$ according to (3.17).

Next, we consider the effect of truncation on phase propagation. The following lemma expresses the error of approximation in a way that is analogous to Lemma 3.5 in [18].

Lemma 3.3.4. *For any signal* $\mu \in S_{\Delta,[0,\Lambda]}$ *and* $\Omega > 0$, *a vector* $\{\widehat{\varphi}(\omega_{0,j})\}_{j=N_1}^{N_2} \in \mathbb{C}^{N_2-N_1+1}$ *can be constructed such that* $\widehat{\varphi}(\omega_{0,j}) \neq 0$,

$$(E_{\omega_0})_j := |\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| \leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{|j|}}{1-C} + \frac{C^{|j|}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}$$

and

$$\|\epsilon_{\omega_0,0}\|_\infty \leq \frac{2\sqrt{3}\|c\|_{\ell_1}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2}$$

for some $m, C > 0$ *for each* $j \in \{N_1, ..., N_2\}$ *and* $\omega_{0,j} \in [-\Omega, \Omega]$ *using the measurements (3.13) for* N *chosen sufficiently large.*

Proof. Since $\widehat{\mu}_0 \in S_{[-\Lambda,\Lambda],\Delta}$, using Lemma 3.3.3, we may choose $\omega_0 \in [-\Omega, \Omega]$ such that for all $\omega_{0,j} = \omega_0 + \frac{j\pi}{\Lambda}$ with $\omega_{0,j} \in [-\Omega, \Omega]$ we have for $\widehat{\mu}_{0,N} : \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{\mu}_{0,N}(\omega) = \sum_{n=-N}^{N} \widehat{\mu}_0.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda}) \tag{3.19}$$

that

$$|\widehat{\mu}_{0,N}(\omega_0 + \frac{j\pi}{\Lambda})| \geq \frac{\widetilde{M}_j}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} > 0 \tag{3.20}$$

for some $\widetilde{M}_j > 0$ and $\epsilon_N \geq 0$, and letting

$$m = \min_{j\in\{N_1,...,N_2\}} \frac{\frac{\widetilde{M}_j}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}}{\|c\|_{\ell_1}^2}$$

implies that

$$|\widehat{\mu}_{0,N}(\omega_{0,j})| \geq m\|c\|_{\ell_1}^2 > 0. \tag{3.21}$$

Thus, if N is chosen sufficiently large such that (27) is satisfied, then we can obtain the measurements $\mathcal{N}_{i,\Omega} : S_{\Delta,[0,\Lambda]} \to \ell_\infty(\{N_1, ..., N_2\})$ given by

$$\mathcal{N}_{0,\Omega}(e^{2\pi i\omega_0 t}\mu) = \left(\widehat{\mu}_{0,N}(\omega_0 + \frac{j\pi}{\Lambda})\right)_{j=N_1}^{N_2} = \left(\widehat{\mu}_0(\omega_{0,j}) - (\epsilon_{\omega_0,0})_j\right)_{j=N_1}^{N_2}$$

$$\mathcal{N}_{1,\Omega}(e^{2\pi i\omega_0 t}\mu) = \big(\widehat{\mu}_{1,N}(\omega_0 + \frac{j\pi}{\Lambda})\big)_{j=N_1}^{N_2} = \big(\widehat{\mu}_1(\omega_{0,j}) - (\epsilon_{\omega_0,1})_j\big)_{j=N_1}^{N_2} \tag{3.22}$$

$$\mathcal{N}_{2,\Omega}(e^{2\pi i\omega_0 t}\mu) = \big(\widehat{\mu}_{2,N}(\omega_0 + \frac{j\pi}{\Lambda})\big)_{j=N_1}^{N_2} = \big(\widehat{\mu}_2(\omega_{0,j}) - (\epsilon_{\omega_0,2})_j\big)_{j=N_1}^{N_2}$$

satisfying $\big(\mathcal{N}_{0,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_j \neq 0$ for all $j \in \{N_1, ..., N_2\}$ for some $N_1, N_2 \in \mathbb{Z}$ such that $\omega_{0,j} = \omega_0 + \frac{j\pi}{\Lambda} \in [-\Omega, \Omega]$ for each $j \in \{N_1, ..., N_2\}$, and $((\epsilon_{\omega_0,i})_j)_{j=N_1}^{N_2} \in \ell_\infty(\{N_1, ..., N_2\})$ with

$$\|\epsilon_{\omega_0,i}\|_\infty \leq \frac{2\sqrt{3}\|c\|_{\ell_1}^2 \Lambda^3}{3\pi^{\frac{7}{2}} N^2}$$

for each $i \in \{0, 1, 2\}$ by Lemma 3.3.1.

The remainder of the proof is the same as lemma 3.5 in [18] and is included for convenience. We proceed by induction. For the base case, let

$$\widehat{\varphi}(\omega_0) = \sqrt{\big(\mathcal{N}_{0,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_0} = \sqrt{\widehat{\mu}_0(\omega_0) - (\epsilon_{\omega_0,0})_0} = \sqrt{|\widehat{\mu}(\omega_0)|^2 - (\epsilon_{\omega_0,0})_0},$$

and let

$$C = \frac{\frac{3\sqrt{2}}{2}\|\epsilon_{\omega_0,0}\|_\infty + \|c\|_{\ell_1}^2}{m\|c\|_{\ell_1}^2}.$$

By the mean value theorem, there is $\xi \in (|\widehat{\mu}(\omega_0)|^2 - (\epsilon_{\omega_0,0})_0, |\widehat{\mu}(\omega_0)|^2)$ such that

$$\begin{aligned}
|\widehat{\varphi}(\omega_0) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_0)| &= |\sqrt{|\widehat{\mu}(\omega_0)|^2 - (\epsilon_{\omega_0,0})_0} - \sqrt{|\widehat{\mu}(\omega_0)|^2}| \\
&= \frac{|(\epsilon_{\omega_0,0})_0|}{2\sqrt{\xi}} \\
&\leq \frac{\|\epsilon_{\omega_0,0}\|_\infty}{2\sqrt{m}\|c\|_{\ell_1}} = \left(\frac{1 + \frac{3\sqrt{2}}{2}}{m}\frac{1 - C^0}{1 - C} + \frac{C^0}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}.
\end{aligned}$$

For the inductive step, set

$$\widehat{\nu}(\omega_{0,j}) =$$

$$\frac{1}{2}((1-i)\big(\mathcal{N}_{0,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_j + (1-i)\big(\mathcal{N}_{0,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_{j+1} - \big(\mathcal{N}_{1,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_j + i\big(\mathcal{N}_{2,\Omega}(e^{2\pi i\omega_0 t}\mu)\big)_j)$$

for each $j \in \{N_1, ..., N_2\}$, and

$$\widehat{\varphi}(\omega_{0,j}) = \frac{\widehat{\nu}(\omega_{0,j-1})}{\left(\mathcal{N}_0(e^{2\pi i \omega_0 t}\mu)\right)_{j-1}} \widehat{\varphi}(\omega_{0,j-1})$$

for all $j \in \{1, ..., N_2\}$. Suppose that the measurement $\widehat{\varphi}(\omega_{0,j})$ satisfies

$$(E_{\omega_0})_j = |\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| \leq \left(\frac{1 + \frac{3\sqrt{2}}{2}}{m} \frac{1 - C^j}{1 - C} + \frac{C^j}{2\sqrt{m}} \right) \frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}$$

for some $j \in \{0, ..., N_2 - 1\}$. Then, we can compute

$$\begin{aligned}
&|\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})| = \Big| \frac{(1-i)(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j) + (1-i)(|\widehat{\mu}(\omega_{0,j+1})|^2 - (\epsilon_{\omega_0,0})_{j+1})}{2} \\
&- \frac{(|\widehat{\mu}(\omega_{0,j}) - \widehat{\mu}(\omega_{0,j+1})|^2 - (\epsilon_{\omega_0,1})_j) - i(|\widehat{\mu}(\omega_{0,j}) - i\widehat{\mu}(\omega_{0,j+1})|^2 - (\epsilon_{\omega_0,2})_j)}{2} \\
&- \frac{(1-i)|\widehat{\mu}(\omega_{0,j})|^2 + (1-i)|\widehat{\mu}(\omega_{0,j+1})|^2}{2} \\
&+ \frac{|\widehat{\mu}(\omega_{0,j}) - \widehat{\mu}(\omega_{0,j+1})|^2 - i|\widehat{\mu}(\omega_{0,j}) - i\widehat{\mu}(\omega_{0,j+1})|^2}{2} \Big| \\
&= \frac{|(i-1)(\epsilon_{\omega_0,0})_j + (i-1)(\epsilon_{\omega_0,0})_{j+1} + (\epsilon_{\omega_0,1})_j + (\epsilon_{\omega_0,2})_j|}{2} \\
&\leq \frac{3\sqrt{2} \max\limits_{i \in \{0,1,2\}} \|\epsilon_{\omega_0,i}\|_\infty}{2},
\end{aligned}$$

which implies that

$$
\begin{aligned}
&|\widehat{\varphi}(\omega_{0,j+1}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j+1})| \\
&= \Big| \frac{\widehat{\nu}(\omega_{0,j})}{|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j}\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_{0,j})|^2|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}) \Big| \\
&= \Big| \frac{|\widehat{\mu}(\omega_{0,j})|^2\widehat{\nu}(\omega_{0,j})\widehat{\varphi}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_0)_j)}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \Big| \\
&= \Big| \frac{|\widehat{\mu}(\omega_{0,j})|^2(\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1}))\widehat{\varphi}(\omega_{0,j})}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&+ \frac{|\widehat{\mu}(\omega_{0,j})|^2\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}))}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&- \frac{\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})\widehat{\mu}(\omega_{0,j})(\epsilon_0)_j}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \Big| \\
&= \Big| \frac{(\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1}))\widehat{\varphi}(\omega_{0,j}) + \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}))}{(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&- \frac{\widehat{\mu}(\omega_{0,j+1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j}{(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \Big|.
\end{aligned}
$$

Since $|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j \geq m\|c\|_{\ell_1}^2$, we get the inequality

$$
\begin{aligned}
&|\widehat{\varphi}(\omega_{0,j+1}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j+1})| \\
&\leq \Big|\frac{(\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1}))\widehat{\varphi}(\omega_{0,j}) + \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}))}{m\|c\|_{\ell_1}^2} \\
&- \frac{\widehat{\mu}(\omega_{0,j+1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j}{m\|c\|_{\ell_1}^2}\Big| \\
&\leq \frac{|\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})||\widehat{\varphi}(\omega_{0,j})| + |\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})||\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})|}{m\|c\|_{\ell_1}^2} \\
&+ \frac{|\widehat{\mu}(\omega_{0,j+1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j|}{m\|c\|_{\ell_1}^2} \\
&\leq \frac{|\widehat{\nu}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})|(|\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| + |\widehat{\mu}(\omega_{0,j})|)}{m\|c\|_{\ell_1}^2} \\
&+ \frac{|\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j+1})||\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| + |\widehat{\mu}(\omega_{0,j+1})(\epsilon_{\omega_0,0})_j|}{m\|c\|_{\ell_1}^2}.
\end{aligned}
$$

Next, using the induction hypothesis, we get that

$$
\begin{aligned}
|\widehat{\varphi}(\omega_{0,j+1}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j+1})| &\leq \frac{\frac{3\sqrt{2}\|\epsilon_{\omega_0,0}\|_\infty}{2}((E_{\omega_0})_j + \|c\|_{\ell_1}) + \|c\|_{\ell_1}^2(E_{\omega_0})_j + \|c\|_{\ell_1}\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}^2} \\
&= \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + \frac{\frac{3\sqrt{2}}{2}\|\epsilon_{\omega_0,0}\|_\infty + \|c\|_{\ell_1}^2}{m\|c\|_{\ell_1}^2}(E_{\omega_0})_j \\
&= \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + C(E_{\omega_0})_j \\
&\leq \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + C\left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^j}{1-C} + \frac{C^j}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}} \\
&= \left(\frac{1+\frac{3\sqrt{2}}{2}}{m} + \frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{C-C^{j+1}}{1-C} + \frac{C^{j+1}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}} \\
&= \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{j+1}}{1-C} + \frac{C^{j+1}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}.
\end{aligned}
$$

This shows that we can construct $\left(\widehat{\varphi}(\omega_j)\right)_{j=0}^{N_2}$ with the required properties. For the construction of the remaining sample values, set

$$\widehat{\varphi}(\omega_{0,j}) = \frac{\overline{\widehat{\nu}(\omega_{0,j+1})}}{\left(\widetilde{\mathcal{N}}_0(e^{2\pi i\omega_0 t}\mu)\right)_{j+1}}\widehat{\varphi}(\omega_{0,j+1})$$

for each $j \in \{N_1, ..., -1\}$, and assume

$$(E_{\omega_0})_j = |\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| \leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{-j}}{1-C} + \frac{C^{-j}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}$$

for some $j \in \{N_1 + 1, ..., 0\}$.

Then, a similar argument can be used to show that

$$
\begin{aligned}
&|\widehat{\varphi}(\omega_{0,j-1}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j-1})| = \Big|\frac{\overline{\widehat{\nu}(\omega_{0,j})}}{|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j}\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_{0,j})|^2|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})\Big| \\
&= \Big|\frac{|\widehat{\mu}(\omega_{0,j})|^2\overline{\widehat{\nu}(\omega_{0,j})}\widehat{\varphi}(\omega_{0,j}) - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})(|\widehat{\mu}(\omega_{0,j})| - (\epsilon_0)_j)}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)}\Big| \\
&= \Big|\frac{|\widehat{\mu}(\omega_{0,j})|^2(\overline{\widehat{\nu}(\omega_{0,j})} - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1}))\widehat{\varphi}(\omega_{0,j})}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&+ \frac{|\widehat{\mu}(\omega_{0,j})|^2\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})))(\epsilon_{\omega_0,0})_j}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&- \frac{\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)}\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})\widehat{\mu}(\omega_{0,j})(\epsilon_{\omega_0,0})_j}{|\widehat{\mu}(\omega_{0,j})|^2(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)}\Big| \\
&= \Big|\frac{(\overline{\widehat{\nu}(\omega_{0,j})} - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1}))\widehat{\varphi}(\omega_{0,j}) + \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}))}{(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)} \\
&- \frac{\widehat{\mu}(\omega_{0,j-1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j}{(|\widehat{\mu}(\omega_{0,j})|^2 - (\epsilon_{\omega_0,0})_j)}\Big| \\
&\leq \Big|\frac{(\overline{\widehat{\nu}(\omega_{0,j})} - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1}))\widehat{\varphi}(\omega_{0,j}) + \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})(\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j}))}{m\|c\|_{\ell_1}^2} \\
&- \frac{\widehat{\mu}(\omega_{0,j-1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j}{m\|c\|_{\ell_1}^2}\Big| \\
&\leq \frac{|\overline{\widehat{\nu}(\omega_{0,j})} - \overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})||\widehat{\varphi}(\omega_{0,j})| + |\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})||\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})|}{m\|c\|_{\ell_1}^2} \\
&+ \frac{|\widehat{\mu}(\omega_{0,j-1})\frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}(\epsilon_{\omega_0,0})_j|}{m\|c\|_{\ell_1}^2} \\
&\leq \frac{|\overline{\widehat{\nu}(\omega_{0,j}) - \widehat{\mu}(\omega_{0,j})\overline{\widehat{\mu}(\omega_{0,j-1})}}|(|\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| + |\widehat{\mu}(\omega_{0,j})|)}{m\|c\|_{\ell_1}^2} \\
&+ \frac{|\overline{\widehat{\mu}(\omega_{0,j})}\widehat{\mu}(\omega_{0,j-1})||\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| + |\widehat{\mu}(\omega_{0,j-1})(\epsilon_{\omega_0,0})_j|}{m\|c\|_{\ell_1}^2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|\widehat{\varphi}(\omega_{0,j-1}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j-1})| &\leq \frac{\frac{3\sqrt{2}\|\epsilon_{\omega_0,0}\|_\infty}{2}((E_{\omega_0})_j + \|c\|_{\ell_1}) + \|c\|_{\ell_1}^2 (E_{\omega_0})_j + \|c\|_{\ell_1}\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}^2} \\
&= \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + \frac{\frac{3\sqrt{2}}{2}\|\epsilon_{\omega_0,0}\|_\infty + \|c\|_{\ell_1}^2}{m\|c\|_{\ell_1}^2}(E_{\omega_0})_j \\
&= \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + C(E_{\omega_0})_j \\
&\leq \frac{(1+\frac{3\sqrt{2}}{2})\|\epsilon_{\omega_0,0}\|_\infty}{m\|c\|_{\ell_1}} + C\left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{-j}}{1-C} + \frac{C^{-j}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}} \\
&= \left(\frac{1+\frac{3\sqrt{2}}{2}}{m} + \frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{C-C^{-j+1}}{1-C} + \frac{C^{-j+1}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}} \\
&= \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{-(j-1)}}{1-C} + \frac{C^{-(j-1)}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}.
\end{aligned}
$$

□

This allows us to construct a collection of quadratic forms that determine an approximation of μ similar to Lemma 3.2.2.

Lemma 3.3.5. *Let $\mu \in S_{\Delta,[0,\Lambda]}$ for some $\Delta, \Lambda > 0$, let $\Omega > 0$. Define the quadratic forms $\widetilde{Q_{\mu,E_{\omega_0},N_2}} : L^1(\mathbb{R}) \to \mathbb{R}$ and $\widetilde{Q_{\mu,E_{\omega_0},N_1}} : L^1(\mathbb{R}) \to \mathbb{R}$ for a pair $(N_1, N_2) \in \mathbb{Z}\times\mathbb{N}$ with $N_1 < N_2$ by*

$$
\widetilde{Q_{\mu,E_{\omega_0},N_2}}(\widehat{\varphi}(\omega)) = \|(I_{N_2+1} - T_{-\frac{\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},E_{\omega_0},N_2}\big(\widehat{\varphi}(\omega_0 + \frac{n\pi}{\Lambda})\big)_{n=0}^{N_2}\|_{\ell_2^{N_2+1}}^2 - 1 \tag{3.23}
$$

and

$$
\widetilde{Q_{\mu,E_{\omega_0},N_1}}(\widehat{\varphi}(\omega)) = \|(I_{N_1+1} - T_{\frac{\pi}{\Lambda}})M_{\frac{1}{\widehat{\mu}},E_{\omega_0},N_1}\big(\widehat{\varphi}(\omega_0 + \frac{n\pi}{\Lambda})\big)_{n=N_1}^{0}\|_{\ell_2^{|N_1|+1}}^2 - 1
$$

where $I_{N_2}, T_{-\frac{\pi}{\Lambda}}, M_{\frac{1}{\widehat{\mu}},E_{\omega_0},N_2}$ and $I_{N_1}, T_{-\frac{\pi}{\Lambda}}, M_{\frac{1}{\widehat{\mu}},E_{\omega_0},N_1}$ are the operators defined on $\ell_2(\{0,1,...,N_2\})$ and $\ell_2(\{N_1, N_1+1,...,0\})$ respectively by

$$
I_{N_2+1}\big((x_n)_{n=0}^{N_2}\big) = (x_n)_{n=0}^{N_2},
$$

$$I_{N_1+1}\big((x_n)_{n=N_1}^0\big) = (x_n)_{n=N_1}^0,$$

$$T_{-\frac{\pi}{\Lambda}}\big((x_n)_{n=0}^N\big) = \big(0, x_1, x_2, ..., x_{N-1}, x_{N-1}\big),$$

$$T_{\frac{\pi}{\Lambda}}\big((x_n)_{n=-N}^0\big) = \big(x_{-(N-1)}, x_{-(N-2)}, ..., x_2, x_1, 0\big),$$

$$M_{\frac{1}{\widehat{\mu}}, E_{\omega_0}, N_2}\big((x_n)_{n=0}^N\big) = \big(\frac{1}{\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda}) - (E_{\omega_0})_n} x_n\big)_{n=0}^{N_2},$$

and

$$M_{\frac{1}{\widehat{\mu}}, E_{\omega_0}, N_1}\big((x_n)_{n=-N}^0\big) = \big(\frac{1}{\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda}) - (E_{\omega_0})_n} x_n\big)_{n=N_1}^{0}$$

for some $\mu \in S_{\Delta,[0,\Lambda]}$, $\omega_{0,n} = \omega_0 + \frac{n\pi}{\Lambda} \in [-\Omega, \Omega]$ *satisfying* $|\widehat{\mu}(\omega_0 + \frac{n\pi}{\Lambda}) - (E_{\omega_0})_n| > 0$ *and*

$$\|E_{\omega_0}\|_\infty \leq \max_{j\in\{N_1,...,N_2\}} \left(\frac{1 + \frac{3\sqrt{2}}{2}}{m} \frac{1 - C^{|j|}}{1 - C} + \frac{C^{|j|}}{2\sqrt{m}} \right) \frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|c\|_{\ell_1}}$$

for all $n \in \{N_1, ..., N_2\}$. *Then, for* N *chosen sufficiently large, these quadratic forms can be constructed using the measurements (3.13).*

Proof. Since $\frac{\overline{\widehat{\mu}(\omega_0)}}{\widehat{\mu}(\omega_0)}\widehat{\mu} \in [\widehat{\mu}]$ for any $\omega_0 \in [-\Omega, \Omega]$, Lemma 3.3.4 implies that both $\widetilde{Q_{\mu, E_{\omega_0}, N_1}}$ and $\widetilde{Q_{\mu, E_{\omega_0}, N_2}}$ can be constructed with the above properties. □

It is important to note that Instead of the measurements (3.13) considered in this section, we can start with more general measurements and follow the same technique presented above to interpolate these measurements to construct measurements of the form (3.12), with the error in this case controlled by the Lipschitz continuity of $\widehat{\mu}_i$ for each $i \in \{0, 1, 2\}$.

Let $\Omega > 0$. We have that

$$\begin{aligned}
|\widehat{\mu}(\omega_1) - \widehat{\mu}(\omega_2)| &= |\sum_{j,k=1}^{s} c_j \bar{c_k} e^{-2\pi i \omega_1 (t_j - t_k)} - \sum_{j,k=1}^{s} c_j \bar{c_k} e^{-2\pi i \omega_2 (t_j - t_k)}| \\
&= |\sum_{j,k=1}^{s} c_j \bar{c_k} (e^{-2\pi i \omega_1 (t_j - t_k)} - e^{-2\pi i \omega_2 (t_j - t_k)}| \\
&= |\sum_{j,k=1}^{s} c_j \bar{c_k} (-2\pi i) \int_{\omega_1}^{\omega_2} e^{-2\pi i \omega (t_j - t_k)}| \\
&\leq 2\pi\Lambda \sum_{j,k=1}^{s} |c_j \bar{c_k}| \int_{\omega_1}^{\omega_2} |e^{-2\pi i \omega (t_j - t_k)}| d\omega| \\
&\leq (2\pi\Lambda \sum_{j,k=1}^{s} |c_j \bar{c_k}|) |\omega_2 - \omega_1|,
\end{aligned}$$

and if $|c_j \bar{c_k}| < 1$ for all $j, k = 1, .., s$, then

$$|\widehat{\mu}(\omega_1) - \widehat{\mu}(\omega_2)| \leq 2\pi\Lambda s(s-1) \, |\omega_2 - \omega_1|$$

and in general we obtain the Lipschitz constant $c := 2\pi\Lambda(\sum_{j,k=1}^{s} |c_j \bar{c_k}|)$ such that

$$|\widehat{\mu}(\omega_1) - \widehat{\mu}(\omega_2)| \leq c \, |\omega_2 - \omega_1|$$

for any $\omega_1, \omega_2 \in [-\Omega, \Omega]$. Thus, we get that

$$|\widehat{\mu}_i(\frac{n\pi}{\Lambda}) - \widehat{\mu}_i(v_n)| \leq 2\pi\Lambda \|c\|_{\ell_1}^2 |\frac{n\pi}{\Lambda} - v_n|$$

for each $n \in \mathbb{Z}$ and $v_n \in [-\Omega, \Omega]$, which implies that

$$\widehat{\mu}_i(v_n) = \widehat{\mu}_i(\frac{n\pi}{\Lambda}) - (\epsilon_i)_n$$

for each $i \in \{0, 1, 2\}$ with

$$\|\epsilon_i\|_\infty \leq 4\Omega\pi\Lambda \|c\|_{\ell_1}^2.$$

This shows that the measurements (3.12) can be obtained from a finite number of samples at different locations (not necessarily equidistant).

Now, we derive an error bound for reconstruction from a finite number of exact measurements.

Theorem 3.3.6. *Let $\mu \in S_{\Delta,[0,\Lambda]}$, and let N be chosen sufficiently large such that there is $\omega_0 \in [-\Omega,\Omega]$ for some $\Omega > 0$ and $(N_1,N_2) \in \mathbb{Z}\times\mathbb{N}$ satisfying*

$$\min_{j\in\{N_1,\ldots,N_2\}} \frac{\widetilde{M}_j(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} - E_{\widehat{\mu},N}}{\|\mu\|_{TV}^2} \geq \frac{\frac{\widetilde{M}_j}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}}{\|\mu\|_{TV}^2} = m > 0,$$

and let

$$C = \frac{\frac{3\sqrt{2}}{2}\|\epsilon_{\omega_0,0}\|_\infty + \|\mu\|_{TV}^2}{m\|\mu\|_{TV}^2}.$$

Let

$$T := T(\|\mu\|_{TV},\Omega) = \max_{n\in\{N_1,\ldots,N_2\}} \left(\frac{1+\frac{3\sqrt{2}}{2}}{m} \sum_{j=0}^{|n|-1} (\frac{1}{m} + \frac{\sqrt{6}\Lambda^3}{3\pi^{\frac{7}{2}}m})^j + (\frac{1}{m} + \frac{\sqrt{6}\Lambda^3}{3\pi^{\frac{7}{2}}m})^{|n|} \frac{1}{2\sqrt{m}} \right) 2\sqrt{3}\Lambda^3\|\mu\|_{TV}.$$

If

$$\widetilde{Q_{\mu,E_{\omega_0},n}}(\widehat{\varphi}) = 0 \quad \text{for all} \quad n \in \{N_1,\ldots,N_2\}$$

for some $\widehat{\varphi} \in L^1$ with $\|\widehat{\varphi}\|_{L^1[-\Omega,\Omega]^c} = 0$, then

$$\|\varphi - \mu * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1} \leq \sqrt{(N_2-N_1)\left(\frac{\Lambda T^2}{N^4}\right) + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{N_1,N_2\})^2}}.$$

Proof. If $\widetilde{Q_{\mu,E_{\omega_0},n}}(\widehat{\varphi}) = 0$ for all $n \in \{N_1,\ldots,N_2\}$, then $|\widehat{\varphi}(\omega_{0,n}) - \widehat{\mu}(\omega_{0,n})| \leq (E_{\omega_0})_n$ for each

$n \in \{N_1, ..., N_2\}$. We have that

$$
\begin{aligned}
C &= \frac{\frac{3\sqrt{2}}{2}\|\epsilon_{\omega_0,0}\|_\infty + \|\mu\|_{TV}^2}{m\|\mu\|_{TV}^2} \\
&\leq \left(\frac{\|\mu\|_{TV}^2 + \frac{3\sqrt{2}}{2}\frac{2\sqrt{3}}{3}\frac{\|\mu\|_{TV}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2}}{m\|\mu\|_{TV}^2}\right) \\
&= \frac{1}{m} + \frac{\sqrt{6}\Lambda^3}{3\pi^{\frac{7}{2}}mN^2}.
\end{aligned}
$$

Therefore, for each $n \in \{N_1, ..., N_2\}$, we get that

$$
\begin{aligned}
(E_{\omega_0})_n &= \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\frac{1-C^{|n|}}{1-C} + \frac{C^{|n|}}{2\sqrt{m}}\right)\frac{\|\epsilon_{\omega_0,0}\|_\infty}{\|\mu\|_{TV}} \\
&\leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\sum_{j=0}^{|n|-1} C^j + \frac{C^{|n|}}{2\sqrt{m}}\right)\frac{2\sqrt{3}}{3}\frac{\|\mu\|_{TV}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2}\frac{1}{\|\mu\|_{TV}} \\
&\leq \max_{n\in\{N_1,...,N_2\}}\left(\frac{1+\frac{3\sqrt{2}}{2}}{m}\sum_{j=0}^{|n|-1}(\frac{1}{m} + \frac{\sqrt{6}\Lambda^3}{3\pi^{\frac{7}{2}}m})^j + (\frac{1}{m} + \frac{\sqrt{6}\Lambda^3}{3\pi^{\frac{7}{2}}m})^{|n|}\frac{1}{2\sqrt{m}}\right)\frac{2\sqrt{3}\Lambda^3\|\mu\|_{TV}}{N^2} \\
&= \frac{T}{N^2}.
\end{aligned}
$$

Thus, any function φ that minimizes the collection of the quadratic forms in this case will satisfy

$$\begin{aligned}
\|\varphi - \mu * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1}^2 &\leq \Lambda\|\varphi - \mu * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2}^2 \\
&= \Lambda\|\widehat{\varphi} - \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2}^2 \\
&= \Lambda\|\widehat{\varphi}(\omega_0 + \frac{n\pi}{\Lambda}) - \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 + \frac{n\pi}{\Lambda})\|_{\ell_2}^2 \\
&= \Lambda\Bigg(\sum_{n=N_1}^{N_2} |\widehat{\varphi}(\omega_0 + \frac{n\pi}{\Lambda}) - \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 + \frac{n\pi}{\Lambda})|^2\Bigg) \\
&+ \Lambda\Bigg(\sum_{n\notin\{N_1,\ldots,N_2\}} |\widehat{\varphi}(\omega_0 + \frac{n\pi}{\Lambda}) - \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 + \frac{n\pi}{\Lambda})|^2\Bigg) \\
&= \Lambda\Bigg(\sum_{n=N_1}^{N_2} |\sum_{j=N_1}^{N_2} |(\widehat{\mu} - (E_{\omega_0})_j).\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 + \frac{j\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\frac{(n-j)\pi}{\Lambda}) \\
&- \widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 + \frac{n\pi}{\Lambda})|^2\Bigg) \\
&+ \Lambda\Bigg(\sum_{n\notin\{N_1,\ldots,N_2\}} |\widehat{\mu}.\widehat{\chi}^2(\omega_0 + \frac{n\pi}{\Lambda})|^2\Bigg) \\
&= \Lambda\Bigg(\sum_{n=N_1}^{N_2} |(E_{\omega_0})_n\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0 - \frac{j\pi}{\Lambda})|^2\Bigg) + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2} \\
&= \Lambda\sum_{n=N_1}^{N_2}\left(\frac{T}{N^2}\right)^2 + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2} \\
&= (N_2 - N_1)\left(\frac{\Lambda T^2}{N^4}\right) + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}.
\end{aligned}$$

□

The TV norm is equivalent to the L^1 norm for smooth functions. We want to determine how "good" the error bound derived in Theorem 3.3.6 is, in terms of providing us with direct information about the signal parameters. We observe that $S_{\Delta,[0,\Lambda]}$ is not a vector space. In particular, the difference between two signals $\mu, \varphi \in S_{\Delta,[0,\Lambda]}$ is not necessarily in $S_{\Delta,[0,\Lambda]}$. This means that the norm equivalence introduced in the previous section does not hold for $\|\mu - \varphi\|_{TV}$ in general. Fortunately, it is possible to use the L^1 approximation obtained in Theorem 3.3.6 to approximate the signal parameters as we will show at the end of the next section (see Corollary 3.4.5).

3.4 Error bounds for recovery by minimizing the TV norm from noisy measurements

Finally, we turn our attention to the case of perturbed measurements. We begin with the following lemma, which gives a truncation error bound as well as lower bound on the magnitudes of resampled values in this case. This will allow us to perform phase propagation to construct approximate linear measurements of μ.

Lemma 3.4.1. *Let $\mu \in S_{\Delta,[-\Lambda,\Lambda]}$ and define $\widehat{\mu}_{N,\epsilon} : \mathbb{R} \to \mathbb{C}$ by*

$$\begin{aligned}\widehat{\mu}_{N,\epsilon}(\omega) &:= \sum_{n=-N}^{N} [\widehat{\mu}(\frac{n\pi}{\Lambda}) - \epsilon_n].\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda}) \\ &= \sum_{n=-N}^{N} \widehat{\mu}(\frac{n\pi}{\Lambda}).\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda}) - \sum_{n=-N}^{N} \epsilon_n \widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})\end{aligned}$$

where

$$\|\epsilon\|_\infty \leq \frac{\Delta\pi}{4\Lambda} \min_{n\in\{N_1,\dots,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$$

for some $\Omega > 0$. Then we have that

$$E_{\widehat{\mu},N,\epsilon}(\omega) = |\widehat{\mu}(\omega) - \widehat{\mu}_{N,\epsilon}(\omega)| \leq \frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \min_{n\in\{N_1,\dots,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$$

for any $\omega \in \mathbb{R}$. Morever, if N is chosen sufficiently large, there exists $\omega_0 \in [-\Omega,\Omega]$ such that for each $n \in \{N_1, ..., N_2\}$ for some $N_1, N_2 \in \mathbb{Z}$ and $\omega_{0,n} = \omega_0 + \frac{n\pi}{\Lambda}$ with $\omega_{0,n} \in [-\Omega,\Omega]$, we have that

$$|\widehat{\mu}_{N,\epsilon}(\omega_{0,n})| \geq \min_{n\in\{N_1,\dots,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} > 0.$$

Proof. For any $\omega \in [-\Omega, \Omega]$, we calculate

$$
\begin{aligned}
|\widehat{\mu}(\omega) - \widehat{\mu}_{N,\epsilon}(\omega)| &= |\widehat{\mu}(\omega) - \sum_{n=-N}^{N} [\widehat{\mu}(\frac{n\pi}{\Lambda}) - \epsilon_n].\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})| \\
&= |\widehat{\mu}(\omega) - \sum_{n=-N}^{N} \widehat{\mu}(\frac{n\pi}{\Lambda}).\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda}) \\
&+ \sum_{n=-N}^{N} \epsilon_n.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})| \\
&= |E_{\widehat{\mu},N}(\omega) + \sum_{n=-N}^{N} \epsilon_n.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega - \frac{n\pi}{\Lambda})| \\
&\leq |E_{\widehat{\mu},N}(\omega)| + \|\epsilon\|_\infty \int_{-N}^{N} |\frac{sin^2(\frac{\pi\Delta}{4}\frac{x\pi}{\Lambda})}{(\frac{x\pi^2}{\Lambda})^2} \frac{sin((\pi\Lambda)(\omega - \frac{x\pi}{\Lambda}))}{\pi\omega - \frac{x\pi^2}{\Lambda}}|dx \\
&\leq |E_{\widehat{\mu},N}(\omega)| + \|\epsilon\|_\infty \int_{-N}^{N} \frac{sin^2(\frac{\pi\Delta}{4}\frac{x\pi}{\Lambda})}{(\frac{x\pi^2}{\Lambda})^2} dx \\
&\leq |E_{\widehat{\mu},N}(\omega)| + \frac{4\Lambda}{\Delta\pi}\|\epsilon\|_\infty \\
&\leq \frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \frac{4\Lambda}{\Delta\pi}\frac{\Delta\pi}{4\Lambda} \min_{n\in\{N_1,\ldots,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} \quad \text{(by Lemma 3.3.1)} \\
&\leq \frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \min_{n\in\{N_1,\ldots,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}.
\end{aligned}
$$

By Proposition 3.2.2, we know that there exists ω_0 such that

$$
\begin{aligned}
|\widehat{\mu}_{N,\epsilon}(\omega_{0,n})| &= |\sum_{n=-N}^{N}[\widehat{\mu}(\frac{n\pi}{\Lambda})-\epsilon_n].\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega_{0,n}-\frac{n\pi}{\Lambda})| \\
&= |\sum_{n=-N}^{N}\widehat{\mu}(\frac{n\pi}{\Lambda}).\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega-\frac{n\pi}{\Lambda}) \\
&- \sum_{n=-N}^{N}\epsilon_n.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega_{0,n}-\frac{n\pi}{\Lambda}))| \\
&= |\widehat{\mu}_N(\omega_{0,n}) - \sum_{n=-N}^{N}\epsilon_n.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega_{0,n}-\frac{n\pi}{\Lambda})| \\
&\geq |\widehat{\mu}_N(\omega_{0,n})| - |\sum_{n=-N}^{N}\epsilon_n.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\frac{n\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\omega_{0,n}-\frac{n\pi}{\Lambda})| \\
&\geq |\widehat{\mu}_N(\omega_{0,n})| - \frac{4\Lambda}{\Delta\pi}\|\epsilon\|_\infty \\
&\geq \min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M}_n}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} - \frac{4\Lambda}{\Delta\pi}\frac{\Delta\pi}{4\Lambda}\min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} \\
&= \min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} > 0.
\end{aligned}
$$

□

Next, we consider the error bounds for phase propagation in the case of perturbed measurements. We show that similar to the exact measurements case, the error is inversely proportional to the signal to noise ratio, provided that the truncation error is sufficiently small.

Lemma 3.4.2. *Let $\Omega > 0$. For any signal $\mu \in S_{\Delta,[0,\Lambda]}$ and $N \in \mathbb{N}$ chosen sufficiently large, a vector $\{\widehat{\varphi}(\omega_{0,j})\}_{j=N_1}^{N_2} \in \mathbb{C}^{N_2-N_1+1}$ can be constructed such that $\widehat{\varphi}(\omega_{0,j}) \neq 0$ for any $j \in \{N_1,...,N_2\}$,*

$$(\widetilde{E}_{\omega_0})_j := |\widehat{\varphi}(\omega_{0,j}) - \frac{\overline{\widehat{\mu}(\omega_0)}}{|\widehat{\mu}(\omega_0)|}\widehat{\mu}(\omega_{0,j})| \leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\frac{1-\widetilde{C}^{|j|}}{1-\widetilde{C}} + \frac{\widetilde{C}^{|j|}}{2\sqrt{\widetilde{m}}}\right)\frac{\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty}{\|\mu\|_{TV}} \tag{3.24}$$

and

$$\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty \leq \frac{2\sqrt{3}\|\mu\|_{TV}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} \tag{3.25}$$

for some $\widetilde{m}, \widetilde{C} > 0$ *for each* $j \in \{N_1, ..., N_2\}$ *and* $\omega_{0,j} \in [-\Omega, \Omega]$ *using the measurements (3.12) under the assumption that*

$$\max_{i\in\{0,1,2\}} \|\epsilon_i\| \leq \frac{\Delta\pi}{4\Lambda} \min_{n\in\{N_1,...,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}. \tag{3.26}$$

Proof. First, we construct the measurements $\widetilde{\mathcal{N}}_{i,\omega_0,\epsilon,\Omega} : S_{\Delta,[0,\Lambda]} \to \ell_\infty(\{N_1, ..., N_2\})$ given by

$$\begin{aligned} \widetilde{\mathcal{N}}_{0,\omega_0,\epsilon,\Omega}(e^{-2\pi i\omega_0 t}\mu) &= \big(\widehat{\mu}_{0,N,\epsilon}(\omega_0 + \frac{j\pi}{\Lambda})\big)_{j=N_1}^{N_2} = \big(\widehat{\mu}_0(\omega_{0,j}) - (\widetilde{\epsilon}_{\omega_0,0})_j\big)_{j=N_1}^{N_2} \\ \widetilde{\mathcal{N}}_{1,\omega_0,\epsilon,\Omega}(e^{-2\pi i\omega_0 t}\mu) &= \big(\widehat{\mu}_{1,N,\epsilon}(\omega_0 + \frac{j\pi}{\Lambda})\big)_{j=N_1}^{N_2} = \big(\widehat{\mu}_1(\omega_{0,j}) - (\widetilde{\epsilon}_{\omega_0,1})_j\big)_{j=N_1}^{N_2} \\ \widetilde{\mathcal{N}}_{2,\omega_0,\epsilon,\Omega}(e^{-2\pi i\omega_0 t}\mu) &= \big(\widehat{\mu}_{2,N,\epsilon}(\omega_0 + \frac{j\pi}{\Lambda})\big)_{j=N_1}^{N_2} = \big(\widehat{\mu}_2(\omega_{0,j}) - (\widetilde{\epsilon}_{\omega_0,2})_j\big)_{j=N_1}^{N_2} \end{aligned} \tag{3.27}$$

satisfying $\big(\widetilde{\mathcal{N}}_{0,\omega_0,\epsilon,\Omega}(e^{-2\pi i\omega_0 t}\mu)\big)_j \neq 0$ for all $j \in \{N_1, ..., N_2\}$ for some $N_1, N_2 \in \mathbb{Z}$ such that $\omega_{0,j} = \omega_0 + \frac{j\pi}{\Lambda} \in [-\Omega, \Omega]$ for each $j \in \{N_1, ..., N_2\}$, and $((\widetilde{\epsilon}_{\omega_0,i})_j)_{j=N_1}^{N_2} \in \ell_\infty(\{N_1, ..., N_2\})$ with

$$\|\widetilde{\epsilon}_{\omega_0,i}\|_{\ell_\infty} \leq \frac{2\sqrt{3}\|\mu\|_{TV}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \min_{n\in\{N_1,...,N_2\}} \frac{\widetilde{M}_n}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$$

for each $i \in \{0, 1, 2\}$ using the measurements (3.12) and Lemma 3.4.1. Then, the proof proceeds by induction as in Lemma 3.2.2 by setting $\widetilde{m} = m$ and replacing C with

$$\widetilde{C} = \frac{\frac{3\sqrt{2}}{2}\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty + \|\mu\|_{TV}^2}{\widetilde{m}\|\mu\|_{TV}^2}. \qquad \square$$

Finally, we state our main result in the case of perturbed measurements. We have the following error bounds for recovery from the measurements (3.12):

Theorem 3.4.3. *Let* $L = L(\|\mu\|_{TV}, \Omega)$ *and* $E = E(N, \Omega)$ *be given by*

$$\max_{n\in\{N_1,...,N_2\}} \Bigg(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}} \sum_{j=0}^{|n|-1} (\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^j + (\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^{|n|} \frac{1}{2\sqrt{\widetilde{m}}}\Bigg) \frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}}$$

and

$$\max_{n\in\{N_1,\dots,N_2\}}\left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\sum_{j=0}^{|n|-1}(\frac{1}{\widetilde{m}}+\frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2}+\frac{3\sqrt{2}}{2})^j+(\frac{1}{\widetilde{m}}+\frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2}+\frac{3\sqrt{2}}{2})^{|n|}\frac{1}{2\sqrt{\widetilde{m}}}\right)\frac{\widetilde{M_n}}{2\|\mu\|_{TV}}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$$

respectively. If $N\in\mathbb{N}$ *is chosen sufficiently large, and*

$$\max_{i\in\{0,1,2\}}\|\epsilon_i\|\leq\frac{\Delta\pi}{4\Lambda}\min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M_n}}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}},$$

we can construct $\widetilde{Q_{\mu,\widetilde{E}_{\omega_0},N_2}}:L^1(\mathbb{R})\to\mathbb{R}$ *and* $\widetilde{Q_{\mu,\widetilde{E}_{\omega_0},N_1}}:L^1(\mathbb{R})\to\mathbb{R}$ *using the measurements (3.12) such that if*

$$\widetilde{Q_{\mu,\widetilde{E}_{\omega_0},n}}(\widehat{\phi})=0\quad\textit{for all}\quad n\in\{N_1,...,N_2\}$$

for any $\phi\in L^1(\mathbb{R})$ *with* $\|\widehat{\phi}\|_{L^1[-\Omega,\Omega]^c}=0$*, then*

$$\|\phi-\mu*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1}\leq\sqrt{\Lambda(N_2-N_1)\left(\frac{L}{N^2}+E\right)^2+\frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}}.$$

Proof. The proof proceeds as in Theorem 3.3.6, using the bounds obtained in Lemma 3.4.2. We have that

$$\begin{aligned}\widetilde{C}&=\frac{\frac{3\sqrt{2}}{2}\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty+\|\mu\|_{TV}^2}{\widetilde{m}\|\mu\|_{TV}^2}\\&\leq\left(\frac{\|\mu\|_{TV}^2+\frac{3\sqrt{2}}{2}\frac{2\sqrt{3}\|\mu\|_{TV}^2\Lambda^3}{3\pi^{\frac{7}{2}}N^2}+\min_{\{N_1,\dots,N_2\}}\frac{3\sqrt{2}}{2}\frac{\widetilde{M_n}}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}}{\widetilde{m}\|\mu\|_{TV}^2}\right)\\&=\frac{1}{\widetilde{m}}+\frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2}+\frac{\min_{n\in\{N_1,\dots,N_2\}}\frac{3\sqrt{2}\widetilde{M_n}}{4}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}}{\frac{\min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M_n}}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}}{\|\mu\|_{TV}^2}\|\mu\|_{TV}^2}\\&=\frac{1}{\widetilde{m}}+\frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2}+\frac{3\sqrt{2}}{2}.\end{aligned}$$

Thus, we get that

$$
\begin{aligned}
\widetilde{E}_{\omega_0,n} &= \left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\frac{1-\widetilde{C}^{|j|}}{1-\widetilde{C}} + \frac{\widetilde{C}^{|j|}}{2\sqrt{\widetilde{m}}}\right)\frac{\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty}{\|\mu\|_{TV}} \\
&\leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\sum_{j=0}^{|n|-1}(\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^j\right. \\
&\left. + (\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^{|n|}\frac{1}{2\sqrt{m}}\right)\left[\frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} + \min_{n\in\{N_1,\dots,N_2\}}\frac{\widetilde{M}_n}{2\|\mu\|_{TV}}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}\right] \\
&\leq \max_{n\in\{N_1,\dots,N_2\}}\left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\sum_{j=0}^{|n|-1}(\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^j\right. \\
&\left. + (\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^{|n|}\frac{1}{2\sqrt{\widetilde{m}}}\right)\frac{2\sqrt{3}\|\mu\|_{TV}\Lambda^3}{3\pi^{\frac{7}{2}}N^2} \\
&+ \max_{n\in\{N_1,\dots,N_2\}}\left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}}\sum_{j=0}^{|n|-1}(\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^j\right. \\
&\left. + (\frac{1}{\widetilde{m}} + \frac{\sqrt{6}\Lambda^3}{\pi^{\frac{7}{2}}\widetilde{m}N^2} + \frac{3\sqrt{2}}{2})^{|n|}\frac{1}{2\sqrt{\widetilde{m}}}\right)\frac{\widetilde{M}_n}{2\|\mu\|_{TV}}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}} \\
&= \frac{L}{N^2} + E,
\end{aligned}
$$

so we can calculate similar to Theorem 3.3.6,

$$\begin{aligned}
\|\phi-\mu*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1}^2 &\leq \Lambda\|\phi-\mu*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2}^2\\
&=\Lambda\|\widehat{\phi}-\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^2}^2\\
&=\Lambda\|\widehat{\phi}(\omega_0+\frac{n\pi}{\Lambda})-\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{n\pi}{\Lambda})\|_{\ell_2}^2\\
&=\Lambda\bigg(\sum_{n=N_1}^{N_2}|\widehat{\phi}(\omega_0+\frac{n\pi}{\Lambda})-\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{n\pi}{\Lambda})|^2\bigg)\\
&+\Lambda\bigg(\sum_{n\notin\{N_1,\ldots,N_2\}}|\widehat{\phi}(\omega_0+\frac{n\pi}{\Lambda})-\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{n\pi}{\Lambda})|^2\bigg)\\
&=\Lambda\bigg(\sum_{n=N_1}^{N_2}|\sum_{j=N_1}^{N_2}|(\widehat{\mu}-\widetilde{E}_{\omega_0,j}).\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{j\pi}{\Lambda})\widehat{\chi}_{[-\Lambda,\Lambda]}(\frac{(n-j)\pi}{\Lambda})\\
&-\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{n\pi}{\Lambda})|^2\bigg)+\Lambda\bigg(\sum_{n\notin\{N_1,\ldots,N_2\}}|\widehat{\mu}.\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0+\frac{n\pi}{\Lambda})|^2\bigg)\\
&=\Lambda\bigg(\sum_{n=N_1}^{N_2}|\widetilde{E}_{\omega_0,n}\widehat{\chi}^2_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(\omega_0-\frac{j\pi}{\Lambda})|^2\bigg)+\frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}\\
&\leq\Lambda\bigg(\sum_{n=N_1}^{N_2}|\widetilde{E}_{\omega_0,n}|^2\bigg)+\frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}\\
&=\Lambda\sum_{n=N_1}^{N_2}\bigg(\frac{L}{N^2}+E\bigg)^2+\frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}\\
&=\Lambda(N_2-N_1)\bigg(\frac{L}{N^2}+E\bigg)^2+\frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}.
\end{aligned}$$

□

The phase retrieval (propagation) error is the truncation error for the auto-correlation function in disguise. Thus, the total error for recovery consists of the truncation error for the series expansion of the auto-correlation plus that of the Fourier transform of the signal. By Theorem 3.3.6., it is clear that $\|\varphi-\mu*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}*\chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1}\to 0$ as $N,\Omega\to\infty$. Similarly, in Theorem 3.4.3, we see that $\widetilde{m}$ increases as $N \to \infty$ which means that

$$E \to \max_{n \in \{N_1, \ldots, N_2\}} \frac{A_\mu \widetilde{M_n}}{2\|\mu\|_{TV}} (\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}$$

as $N \to \infty$ for some constant A_μ not depending on N. This leads to the following error bound in terms of the signal length:

Corollary 3.4.4. *For any $\mu \in S_{\Delta,[0,\Lambda],s}$, and sufficiently large $k \in \mathbb{N}$, a total of ks perturbed measurements (as in measurements (3.12)) allow for the reconstruction of a function $\phi \in L^1(\mathbb{R})$ with*

$$\|\phi - \mu * \chi_{[-\frac{\Delta}{4}, \frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4}, \frac{\Delta}{4}]}\|_{L^1} \leq \sqrt{\Lambda(ks)\left(\frac{L}{(ks)^2} + E\right)^2 + 6\Lambda^4 \frac{\|\mu\|_{TV}^2}{\Omega^2}}$$

under the assumption that and

$$\max_{i \in \{0,1,2\}} \|\epsilon_i\| \leq \frac{\Delta\pi}{4\Lambda} \min_{n \in \{N_1, \ldots, N_2\}} \frac{\widetilde{M_n}}{4} (\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}},$$

Proof. This follows from Theorem 3.4.3 by constructing ks linear measurements, as in Lemma 3.4.2, and setting $N = ks$. □

Remark. We can show that the error bound obtained above is inversely proportional to the signal-to-noise ratio, $\frac{\|\mu\|_{TV}}{\|\epsilon\|_\infty}$. By the proof of Lemma 3.4.1, we have that $\|\widetilde{\epsilon}_{\omega_0}\|_\infty \leq \frac{2\sqrt{3}\|\mu\|_{TV}^2 \Lambda^3}{3\pi^{\frac{7}{2}} N^2} + \frac{4\Lambda}{\Delta\pi}\|\epsilon\|_\infty$. This means that the linear measurements we construct in Lemma 3.4.2 have the error bound

$$\widetilde{E}_{\omega_0,n} = \left(\frac{1 + \frac{3\sqrt{2}}{2}}{\widetilde{m}} \frac{1 - \widetilde{C}^{|n|}}{1 - \widetilde{C}} + \frac{\widetilde{C}^{|n|}}{2\sqrt{\widetilde{m}}}\right) \frac{\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty}{\|\mu\|_{TV}}$$

Setting $C_n := \left(\frac{1+\frac{3\sqrt{2}}{2}}{\widetilde{m}} \frac{1-\widetilde{C}^{|n|}}{1-\widetilde{C}} + \frac{\widetilde{C}^{|n|}}{2\sqrt{\widetilde{m}}} \right)$, we get that

$$\|\phi - \mu * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]} * \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}\|_{L^1} \leq \sqrt{2\Omega\Lambda \max_{n\in\{N_1,\dots,N_2\}} \left(\widetilde{E_{\omega_0,n}}\right)^2 + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}}$$
$$= \sqrt{2\Omega\Lambda \max_{n\in\{N_1,\dots,N_2\}} (C_n)^2 \frac{\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty^2}{\|\mu\|_{TV}^2} + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}}$$
$$\leq \sqrt{2\Omega\Lambda \max_{n\in\{N_1,\dots,N_2\}} (C_n)^2 \left[\frac{4\Lambda^6\|\mu\|_{TV}^2}{N^4} + \frac{\Lambda^4\|\epsilon\|_\infty}{\Delta N^2} + \frac{16\Lambda^2}{\Delta^2}\frac{\|\epsilon\|_\infty^2}{\|\mu\|_{TV}^2}\right] + \frac{2\sqrt{3}\Lambda^4\|\mu\|_{TV}}{3\pi^{\frac{7}{2}}(\min\{|N_1|,N_2\})^2}}.$$

Finally, we show how the error bounds obtained in Theorems 3.3.6 and 3.4.3 can be used to approximate the signal parameters directly provided the signal-to-noise ratio is sufficiently large.

Corollary 3.4.5. *Let $\mu \in S_{\Delta,[0,\Lambda],s}$, let $E_{L^1} > 0$. Suppose $|c_j| > 2E_{L^1}$ for each $j \in \{1,\dots,s\}$. If*

$$\|\varphi - \mu * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]} * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]}\|_{L^1} \leq E_{L^1}$$

for a known $\varphi \in L^1$, then we are able to construct sets of values $\{\widetilde{t_j}\}_{j=1}^s$ and $\{\widetilde{c_j}\}_{j=1}^s$ such that

$$|t_j - \widetilde{t_j}| \leq \frac{\Delta}{4} \qquad \textit{and} \qquad |c_j - \widetilde{c_j}| \leq \frac{64}{\Delta^2}E_{L^1}.$$

Proof. By monotonicity, for any interval $I \subset [0,\Lambda]$,

$$\|\big(\varphi - \mu * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]} * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]}\big)\chi_I\|_{L^1} \leq E_{L^1}.$$

In particular, if $I \cup [t_j - \frac{\Delta}{8}, t_j + \frac{\Delta}{8}] = \emptyset$ for each $j \in \{1,2,\dots,s\}$, then $\|\varphi\chi_I\| \leq E_{L^1}$. Consequently, if for some $\widetilde{t} \in [0,\Lambda]$, $\|\varphi\chi_{[\widetilde{t}-\frac{\Delta}{8},\widetilde{t}+\frac{\Delta}{8}]}\| > E_{L^1}$, then there is $j \in \{1,2,\dots,s\}$ with $|\widetilde{t}-t_j| \leq \frac{\Delta}{4}$. By our minimum separation assumption, since $\Delta \leq |t_j - t_k| \leq |t_j - \widetilde{t}| + |\widetilde{t} - t_k| \leq \frac{\Delta}{4} + |\widetilde{t} - t_k|$, which implies that $\frac{3\Delta}{4} \leq |\widetilde{t} - t_k|$ for any t_k with $k \neq j$, we get that there is at most one such t_j. Thus, we can recover a set of values $\{\widetilde{t_j}\}_{j=1}^s$ with $|\widetilde{t_j} - t_j| \leq \frac{\Delta}{4}$.

The value of each c_j can then be approximated by $\widetilde{c}_j := \frac{64}{\Delta^2}\int_{\widetilde{t}_j-\frac{3\Delta}{8}}^{\widetilde{t}_j+\frac{3\Delta}{8}} \varphi(t)dt$ by observing that

$$\begin{aligned}
|\frac{\Delta^2}{64}\widetilde{c}_j - \frac{\Delta^2}{64}c_j| &= |\int_{\widetilde{t}_j-\frac{3\Delta}{8}}^{\widetilde{t}_j+\frac{3\Delta}{8}} \varphi(t)dt - \frac{\Delta^2}{64}c_j| \\
&= |\int_{\widetilde{t}_j-\frac{3\Delta}{8}}^{\widetilde{t}_j+\frac{3\Delta}{8}} \varphi(t)dt - \int_{\widetilde{t}_j-\frac{3\Delta}{8}}^{\widetilde{t}_j+\frac{3\Delta}{8}} \mu * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]} * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]}(t)dt| \\
&\leq \int_{\widetilde{t}_j-\frac{3\Delta}{8}}^{\widetilde{t}_j+\frac{3\Delta}{8}} |\varphi(t) - \mu * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]} * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]}(t)|dt \\
&= \|\big(\varphi - \mu * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]} * \chi_{[-\frac{\Delta}{16},\frac{\Delta}{16}]}\big)\chi_{[\widetilde{t}_j-\frac{3\Delta}{8},\widetilde{t}_j+\frac{3\Delta}{8}]}\|_{L^1} \leq E_{L^1}.
\end{aligned}$$

□

An alternative approximation for the amplitudes comes from solving the overdetermined perturbed linear system

$$\sum_{j=1}^{s} \widetilde{c}_j e^{-2\pi i\omega_n \widetilde{t}_j} = \widehat{\mu}(\omega_n) + (E_{\omega_0})_n$$

for each $n \in \{N_1, \ldots, N_2\}$, $N_2 - N_1 \geq s$ (in the case of noiseless measurements). Consider the Vandermonde matrices $U_\mu := (e^{-2\pi ikt_j})_{k=0,j=0}^{s-1,s-1}$ and $\widetilde{U}_\mu := (e^{-2\pi ik\widetilde{t}_j})_{k=0,j=0}^{s-1,s-1}$. We observe that U_μ is invertible if and only if $e^{t_j} \neq e^{t_k}$ for all $j,k \in \{1,\ldots,s\}$, which is granted by assumption on the node differences. Moreover, the proof of Theorem 3.3.6 (or Theorem 3.4.3) implies that $\|E_{\omega_0}\|_{\ell_1} \leq E_{L^1}$ (respectively, $\|\widetilde{E}_{\omega_0}\|_{\ell_1} \leq E_{L^1}$). Therefore, as mentioned in the introduction of [25], an elementary bound for the error in recovering the amplitudes can be expressed as

$$\frac{\|c-\widetilde{c}\|_{\ell_1}}{\|c\|_{\ell_1}} \leq \|U_\mu\|_1 \|U_\mu^{-1}\|_1 \big(\frac{\|\widetilde{U}_\mu - U_\mu\|_1}{\|U_\mu\|_1} + \frac{E_{L^1}}{\|\mu\|_{TV}}\big)$$

where $\|.\|_1$ denotes the induced matrix 1-norm. We can compute

$$\|\widetilde{U}_\mu - U_\mu\|_1 \leq \|(|(\widetilde{U}_\mu - U_\mu)_{ij}|)_{i=1,j=1}^{s,s}\|_1 \leq \max_j \sum_{k=1}^{s} |e^{-2\pi i\widetilde{t}_j k} - e^{-2\pi it_j k}| \leq s\frac{\Delta}{2},$$

and

$$\|U_\mu\|_1 = \max_j \sum_{k=1}^{s} |e^{-2\pi i t_j k}| = s.$$

Setting $C_\mu := \|U_\mu^{-1}\|_1$, we get that

$$|c_j - \widetilde{c}_j| \leq \|c - \widetilde{c}\|_{\ell_1} \leq \frac{1}{2} s C_\mu \big(\Delta \|\mu\|_{TV} + 2E_{L^1}\big).$$

Note that an easy upper bound for C_μ can be obtained using the formula for the inverse of a Vandermonde matrix. In particular, we have that $C_\mu \leq s!$. See also [3] for a discussion of the stability of solutions to linear systems of equations. We conclude with the following remark regarding the dependence of our error bounds on Λ in the main theorems of Sections 3.3 and 3.4:

Remark. One may attempt to improve the error bounds in Theorems 3.3.6 and 3.4.3 by choosing smaller values of Λ. For example, the error bound in Theorem 3.3.6 consists of two additive parts, the first part being proportional to $\sqrt{\Lambda}$ and the second proportional to $\frac{\Lambda^2}{\Omega}$. For fixed Ω, decreasing Λ implies reducing the number of measurements in the resampling procedure. In this direction, it is clear that the second term in the error bound is improved by increasing the length of the resampling interval. As shown in the proof of Theorem 3.2.4, the choice of Λ can be altered using a suitable scaling argument. However, this transformation also comes at the cost of changing the value of our minimum separation bound, Δ. Moreover, since Δ is chosen proportional to Λ, and the first term of E_{L^1} in the case of Theorem 3.3.6 is proportional to $\sqrt{\Lambda}$, we see that decreasing the value of Λ would result in worse error bounds for the approximated amplitudes $\{\widetilde{c}_j\}_{j=1}^{s}$. The error bound in Theorem 3.4.3 can be treated similarly, as the first term is also proportional to $\sqrt{\Lambda}$.

Chapter 4

Extensions

4.1 Application to spline functions

In this section, we investigate the application our current recovery framework to a wider class of functions. We show that the same techniques can be used to study linear combinations of basis functions that are compactly supported with non-trivial minimum separation between the basis points. In addition, the signal we recover is in the same class of functions as the original signal, in contrast to the approximation we construct in Theorems 3.3.6 and 3.4.3. We accomplish this using the Lipschitz continuity of $\widehat{\mu}$ to recover a signal $\varphi \in S_{\Delta,[0,\Lambda]}$ with error bounds for the recovery that are analogous to the bounds given in [22] in the discrete reconstruction from linear measurements. This method also relies on the equivalence of norms, however, it has the drawback that the bound we derive depends on the input signal φ.

In particular, it is possible to derive the following error bounds, where the length of the resampling interval is 2Ω:

Theorem 4.1.1. *We can construct* $\widetilde{Q_{\mu,\epsilon,N_2}} : S_{\Delta,[0,\Lambda],s} \to \mathbb{R}$ *and* $\widetilde{Q_{\widehat{\mu},\epsilon,N_1}} : S_{\Delta,[0,\Lambda],s} \to \mathbb{R}$ *using the measurements (3.12) such that*

$$\widetilde{Q_{\mu,\epsilon,n}}(\phi) = 0 \quad \textit{for all} \quad n \in \{N_1, ..., N_2\}$$

for any $\phi \in S_{\Delta,[0,\Lambda],s}$ *with* $\widehat{\phi}\|_{L^1[-\Omega,\Omega]^c} = 0$ *implies that*

$$\|(\widehat{\phi} - \widehat{\mu}).\widehat{\chi}^2\|_{L^2} \leq \frac{1}{(\Omega)^{\frac{3}{2}}}\sqrt{(\frac{2\pi\Lambda\Omega}{N}(\|c\|_{\ell_1} + \|c_\phi\|_{\ell_1}) + \widetilde{E}_{\max})^2 + (\|c\|_{\ell_1} + \|c_{\widehat{\varphi}}\|_{\ell_1})^2}$$

for any $\mu \in S_{\Delta,[0,\Lambda],s}$ *where*

$$\widetilde{E}_{\max} = \max_{j\in\{N_1,\ldots,N_2\}} \widetilde{E}_j$$

and $\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty$ *satisfies the inequality*

$$\|\widetilde{\epsilon}_{\omega_0,0}\|_\infty \leq \frac{\pi\Omega^2}{2\Lambda}[\min_{n\in\{N_1,\ldots,N_2\}} \frac{\widetilde{M_n}}{2}(\frac{\Omega}{\widetilde{N}+1})^{N_{\mathbb{R}}}].$$

Instead of proving Theorem 4.1.1 in general, we will show how this type of error bound can be derived in the special case of spline functions. The space of spline functions is of interest in many applications including biomedical imaging[1, chapter 2]. It is defined in terms of basis functions as follows:

$$S_d(T_\Delta) = \{p(t) = \sum_{j=1}^{s} c_j\beta^{n_j}(t-t_j) \,|\, c_j \in \mathbb{C}\setminus\{0\}, 0 \leq n_j \leq d, t_j \in T_\Delta\}$$

where

$$T_\Delta = \{t_1, ..., t_s \,|\, 0 \leq t_1 < t_2 < ... < t_s \leq \Lambda, |t_j - t_{j+1}| \geq \Delta > 0 \,\text{for all}\, j \in \{1, ..., s-1\}\}$$

and $\beta^n : \mathbb{R} \to \mathbb{R}$ is defined recursively for each $n \in \mathbb{N}$ by

$$\beta^n(t) = \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}(t) * \beta^{n-1}(t)$$

and $\beta^0 = \chi_{[-\frac{\Delta}{4},\frac{\Delta}{4}]}$. For any $p \in S_d(T_\Delta)$ such that $p(t) = \sum_{j=1}^{s} c_j\beta^{n_j}(t-t_j)$ for all $t \in \mathbb{R}$, we have that

$$\widehat{p}(\omega) = \sum_{j=1}^{s} c_j e^{-2\pi i\omega(n_j+1)t_j}\widehat{\beta^{n_j}}(\omega) = \sum_{j=1}^{s} c_j e^{-2\pi i\omega(n_j+1)t_j}\widehat{\chi}^{n_j+1}(\omega).$$

We would like to follow the analysis from Section 3.2. The first step is to consider the truncation error. Hence, $\widehat{p}$ is a piecewise smooth continuous function with $|\mathcal{F}^{-1}(\widehat{p})(t)| = |p(t)| = 0$ for all $|t| \geq \Lambda + \frac{\Delta}{2}$ and Shannon's sampling Theorem [60] implies that

$$\widehat{p}(\omega) = \sum_{n=-\infty}^{\infty} \widehat{p}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda}) \tag{4.1}$$

where the RHS in (4.1) converges for all $\omega \in \mathbb{R}$. Similar to Lemma 3.2.1, we calculate the error of truncating the infinite series in (4.1). In this case, we will consider the error bound over a bounded interval, $[-\Omega, \Omega]$. For any $\omega \in [-\Omega, \Omega]$ where $0 < \Omega \leq \frac{N\pi}{\Lambda}$, $N \in \mathbb{N}$, we get that

$$\begin{aligned}
|E_{\widehat{p},N}(\omega)| &= |\widehat{p}(\omega) - \sum_{n=-N}^{N} \widehat{p}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda})| \\
&= |\sum_{n=-\infty}^{-N} \widehat{p}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda}) + \sum_{n=N}^{\infty} \widehat{p}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda})| \\
&= |\sum_{n=-\infty}^{-N-1} \sum_{j=1}^{s} c_j e^{-2\pi i \frac{n\pi}{\Lambda}(n_j+1)t_j} \widehat{\chi}^{n_j+1}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda}) \\
&+ \sum_{n=N+1}^{\infty} \sum_{j=1}^{s} c_j e^{-2\pi i \frac{n\pi}{\Lambda}(n_j+1)t_j} \widehat{\chi}^{n_j+1}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda})| \\
&\leq 2\int_{N+1}^{\infty} |\sum_{j=1}^{s} c_j e^{-2\pi i \frac{n\pi}{\Lambda}(x_j+1)t_j} \widehat{\chi}^{n_j+1}(\frac{x\pi}{\Lambda})\widehat{\chi}(\omega - \frac{x\pi}{\Lambda}) dx| \\
&\leq 2\|c\|_{\ell_1} \sum_{j=1}^{s} \int_{N+1}^{\infty} |\widehat{\chi}^{n_j+1}(\frac{x\pi}{\Lambda})\widehat{\chi}(\omega - \frac{x\pi}{\Lambda})| dx \\
&\leq 2\|c\|_{\ell_1} \sum_{j=1}^{s} \int_{N+1}^{\infty} \frac{1}{|\omega - \frac{x\pi}{\Lambda}|^{n_j+2}} dx \\
&\leq 2\|c\|_{\ell_1} \sum_{j=1}^{s} \int_{N+1}^{\infty} \frac{1}{(\frac{x\pi}{\Lambda} - \Omega)^{n_j+2}} dx \\
&= 2\|c\|_{\ell_1} \sum_{j=1}^{s} \frac{\Lambda}{(-n_j-1)\pi} \frac{-1}{(\frac{(N+1)\pi}{\Lambda} - \Omega)^{n_j+1}} \\
&= \sum_{j=1}^{s} \frac{2\|c_p\|_{\ell_1}\Lambda}{(n_j+1)\pi(\frac{(N+1)\pi}{\Lambda} - \Omega)^{n_j+1}}.
\end{aligned}$$

Next, since the sinc function $\widehat{\chi}$ and the exponential function are entire, so is $\widehat{p}$ as

$$\begin{aligned}\widehat{p}(\omega) &= \sum_{j=1}^{s} c_j e^{-2\pi i\omega(n_j+1)t_j} \frac{sin^{n_j+1}(2\pi\frac{\Delta}{4}\omega)}{\pi^{n_j+1}\omega^{n_j+1}} \\ &= \sum_{j=1}^{s} c_j e^{-2\pi i\omega(n_j+1)t_j} \frac{e^{(i\pi\frac{\Delta}{2}\omega)} - e^{(-i\pi\frac{\Delta}{2}\omega)}}{(2\pi i\omega)^{n_j+1}} \\ &= \sum_{j=1}^{s} \frac{c_j e^{-2\pi i\omega[(n_j+1)t_j+\frac{\Delta}{4}]}}{(2\pi i\omega)^{n_j+1}} - \sum_{j=1}^{s} \frac{c_j e^{-2\pi i\omega[(n_j+1)t_j-\frac{\Delta}{4}]}}{(2\pi i\omega)^{n_j+1}}.\end{aligned}$$

Therefore, $\widehat{p}$ admits a Hadamard factorization, which means it can only have a countable number of roots over an interval $[-\Omega, \Omega]$. Following the procedure in Lemma 3.2.2, $\omega_0 \in \mathbb{R}$ can be chosen such that $|\widehat{p}(\omega_{0,n})| > 0$ for all $n \in \mathbb{Z}$ with $\omega_{0,n} = \omega_0 + \frac{n\pi}{\Lambda} \in [-\Omega, \Omega]$ where $0 < \Omega \leq \frac{N\pi}{\Lambda}$. Moreover, for the function $\widehat{p}_N : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{p}_N(\omega) = \sum_{n=-N}^{N} \widehat{p}(\frac{n\pi}{\Lambda})\widehat{\chi}(\omega - \frac{n\pi}{\Lambda})$$

we have that

$$\widehat{p}_N(\omega) = \widehat{p}(\omega) - \epsilon_N(\omega)$$

with

$$|\epsilon_N(\omega)| \leq \sum_{j=1}^{s} \frac{2\|c_p\|_{\ell_1}\Lambda}{(n_j+1)\pi(\frac{(N+1)\pi}{\Lambda} - \Omega)^{n_j+1}}$$

for all $\omega \in [-\Omega, \Omega]$. Hence, the proof of Lemma 3.2.2. implies that if N is chosen sufficiently large such that

$$|\epsilon(\omega_{0,n})| < |\widehat{p}(\omega_{0,n})|$$

for all $\omega_{0,n} \in [-\Omega, \Omega]$, then there exists $L_{\omega_0,\widehat{p}} > 0$ such that $|\widehat{p}(\omega_{0,n})| \geq L_{\omega_0,\widehat{p}} > 0$ for all $n \in \mathbb{Z}$ with $\omega_{0,n} \in [-\Omega, \Omega]$. Once the non-zero sample values are obtained, we can follow the two step procedure consisting of:

(1) (Phase Retrieval) acquiring linear measurements of p as in Lemma 3.2.2, then,

(2) (Super-Resolution) extrapolating the values of $\widehat{p}$ outside the interval $[-\Omega, \Omega]$ to recover the signal p. In particular, a vector $\{\widehat{q}(\omega_{0,n})\}_{n=N_1}^{N_2}$ can be obtained such that $N_2 - N_1 = 2N$, $\widehat{q}(\omega_{0,n}) \neq 0$, and

$$|\widehat{q}(\omega_{0,n}) - \frac{\overline{\widehat{p}(\omega_0)}}{|\widehat{p}(\omega_0)|}\widehat{p}(\omega_{0,n})| \leq \left(\frac{1+\frac{3\sqrt{2}}{2}}{m_p}\frac{1-C_p^{|n|}}{1-C_p} + \frac{C_p^{|n|}}{2\sqrt{m_p}}\right) = E_{p,n}$$

for each $n \in \{N_1, ..., N_2\}$, where

$$m_p = \frac{L_{\omega_0,\widehat{p}}}{\|c_p\|_\infty} > 0$$

and

$$C_p = \frac{\frac{3\sqrt{2}}{2} \max\limits_{n\in\{N_1,...,N_2\}} |\epsilon_N(\omega_{0,n})| + \|c\|_\infty^2}{m_p\|c_p\|_\infty^2}.$$

For any $q \in S_d(\widetilde{T}_\Delta)$ satisfying the above inequality for each $\omega_{0,n} \in [-\Omega, \Omega]$, we have for each $\omega \in [-\Omega, \Omega]$ that

$$\begin{aligned}
|\widehat{q}(\omega) - \widehat{p}(\omega)| &\leq \min_{n\in\{N_1,...,N_2\}} |\widehat{q}(\omega) - \widehat{q}(\omega_{0,n})| + |\widehat{q}(\omega_{0,n}) - \widehat{p}(\omega_{0,n})| + |\widehat{p}(\omega_{0,n}) - \widehat{p}(\omega)| \\
&\leq \min_{n\in\{N_1,...,N_2\}} L_{\widehat{q}}|\omega - \omega_{0,n}| + E_{p,n} + L_{\widehat{p}}|\omega - \omega_{0,n}| \\
&\leq \frac{2\Omega}{2N}(L_{\widehat{p}} + L_{\widehat{q}}) + E_{p,n} \\
&= \frac{\Omega}{N}(L_{\widehat{p}} + L_{\widehat{q}}) + E_{p,n}
\end{aligned}$$

for any $\omega \in [-\Omega, \Omega]$, where $L_{\widehat{p}}, L_{\widehat{q}}$ are the Lipschitz constants for each of $\widehat{p}$ and $\widehat{q}$ respectively. So we have that $L_{\widehat{p}} = 2\pi\Lambda\|c_p\|_{\ell_1}$ and $L_{\widehat{q}} = 2\pi\Lambda\|c_q\|_{\ell_1}$. This allows us to conclude the following result:

Theorem 4.1.2. *For any $p \in S_d(T_\Delta)$, we can construct a signal $q \in S_d(\widetilde{T}_\Delta)$ for some $T_\Delta \subsetneq \mathbb{R}$ from the measurements*

$$\begin{gathered}
(|\widehat{p}(\frac{n\pi}{\Lambda})|^2)_{n=-N}^N \\
(|\widehat{p}(\frac{n\pi}{\Lambda}) - \widehat{p}(\frac{n\pi}{\Lambda} + \frac{2\pi}{\Lambda})|^2)_{n=-N}^N \qquad (4.2) \\
(|\widehat{p}(\frac{n\pi}{\Lambda}) - i\widehat{p}(\frac{n\pi}{\Lambda} + \frac{2\pi}{\Lambda})|^2)_{n=-N}^N
\end{gathered}$$

such that

$$\|p-q\|_{L^2} = \|\widehat{p}-\widehat{q}\|_{L^2} \leq \sqrt{\frac{2s}{\Omega^3}[(\frac{2\pi\Lambda\Omega}{N}(\|c_p\|_{\ell_1}+\|c_{\widehat{q}}\|_{\ell_1})+E_{p,\max})^2+(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})^2]}.$$

where $E_{p,\max} = \max_{j\in\{N_1,\dots,N_2\}} E_{p,j}$

Proof. We calculate

$$\begin{aligned}
\|\widehat{p}-\widehat{q}\|_{L^2}^2 &= \int_{-\Omega}^{\Omega}|(\widehat{p}-\widehat{q})(\omega)|^2 d\omega + \int_{-\infty}^{-\Omega}|(\widehat{p}-\widehat{q})(\omega)|^2 d\omega + \int_{\Omega}^{\infty}|(\widehat{p}-\widehat{q})(\omega)|^2 d\omega \\
&\leq \int_{-\Omega}^{\Omega}\sum_{j=1}^{s}|\frac{\Omega}{N}(L_{\widehat{\mu}}+L_{\widehat{\varphi}})+E_n|^2.|\widehat{\chi}^{n_j+1}(\omega)|^2 d\omega + \int_{-\infty}^{-\Omega}\sum_{j=1}^{s}|\|c\|_{\ell_1}+\|c_{\widehat{\varphi}}\|_{\ell_1}|^2.|\widehat{\chi}^{n_j+1}(\omega)|^2 d\omega \\
&+ \int_{\Omega}^{\infty}\sum_{j=1}^{s}|\|c_p\|_{\ell_1}+\|c_{\widehat{\varphi}}\|_{\ell_1}|^2.|\widehat{\chi}^{n_j+1}(\omega)|^2 d\omega \\
&\leq s(\frac{\Omega}{N}(L_{\widehat{\mu}}+L_{\widehat{\varphi}})+E_n)^2\int_{-\Omega}^{\Omega}(\frac{1}{\omega^2})^2 d\omega + 2s(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})^2\int_{\Omega}^{\infty}(\frac{1}{\omega^2})^2 d\omega \\
&= s(\frac{\Omega}{N}(L_{\widehat{\mu}}+L_{\widehat{\varphi}})+E_n)^2(\frac{-1}{\omega^3})|_{-\Omega}^{\Omega} + 2s(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})^2(\frac{-1}{\omega^3})|_{\Omega}^{\infty} \\
&= 2s(\frac{\Omega}{N}(L_{\widehat{\mu}}+L_{\widehat{\varphi}})+E_n)^2\frac{1}{\Omega^3} + 2s(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})^2\frac{1}{\Omega^3} \\
&\leq \frac{2s}{\Omega^3}[(\frac{2\pi\Lambda\Omega}{N}(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})+E_{p,\max})^2+(\|c_p\|_{\ell_1}+\|c_q\|_{\ell_1})^2].
\end{aligned}$$

□

4.2 Conclusion and future work

In this dissertation, we have investigated the recovery of a complex distribution $\mu = \sum_{j=1}^{s} c_j\delta_{t_j}$ from the magnitude of linear measurements. We constructed an injective measurement map over $S_{[0,\Lambda],s}/\sim$ using $O(s)$ measurements. We have also shown that $O(s)$ exact intensity measurements are sufficient to recover $[\mu]$ with an error bound that is $O(\frac{1}{s^2})$ in the exact case, using TV-norm minimization, and gave explicit error bounds for recovery in the presence of noise (see Theorems 3.3.6 and 3.4.3).

Our current work is directly related to the questions posed by the authors in Section 6.2 of [22]. In particular, we have shown that under the assumption of sparsity, using a linear number (in terms of the signal length) of quadratic measurements of a function consisting of a superposition of compactly supported distributions (basis functions), we are able to perform stable recovery provided that there is a nontrivial minimum separation between the basis points.

We also highlight the fact that in our analysis the minimum separation dependence in the error estimation comes from the norm equivalence used in passing from the signed measure (signal) space to the L^2 (measurement) space. This, in turn, depends on the L^2 norm of the basis function which is convolved with the original signal. On the other hand, the dependence on the sampling interval appears in the truncation error bound, which also depends on the interpolation kernel, $\widehat{\chi}^2$, considered in the analysis. Thus, it is possible to improve the error bound obtained by using a different interpolation kernel with faster decay, for example, by convolving the original signal with $\widehat{\chi}^3$ instead.

We conclude with the following observation: The error bound obtained in Theorem 4.1.1. is comparable to the one presented in Theorem 1.5 of [22]. In the latter, the authors estimate the ℓ_1-error of reconstructing a discrete signal supported on a finite lattice in terms of the number of nodes and the number of equidistant measurements which represent the cutoff frequency. These two components are combined into what they call the Super-Resolution Factor (SRF). Analogously, the TV-error of recovering the continuous signal is expressed in terms of the minimum separation (Δ) required, which provides an upper bound for the total number of nodes the signal could possibly have, as well as the sampling interval (Ω) which acts as the cutoff frequency.

In future work, we hope to compare the performance of our proposed algorithms to existing literature. We also would like to explore the case of randomized measurements, or the probabilistic approach, rather than the deterministic one we adopted here. Exploring other signal spaces that this analysis scheme can be applied to is also interesting and could have useful applications in many areas. Finally, we would like to investigate both the stability of our methods with respect to mismatch error, and the robustness with respect to the sparsity assumption.

Appendix A

Appendix

A.1 Review of the Prony method

The Prony method relies on the symmetric properties of Hankel matrices. We will restate the main result of [14], and give an alternative proof using the following fact.

Proposition A.1.1 (Theorem 7, p.205 [35]). *The infinite Hankel matrix defined by* $H = (H_{i+j})_{i,j=0}^{\infty}$ *where* $H_i \in \mathbb{C}$ *for all* $i \in \mathbb{N} \cup \{0\}$*, is of finite rank* r *if and only if there exists* r *numbers* $\alpha_1, ..., \alpha_r$ *such that* $H_{k+1} = \sum_{j=1}^{k} \alpha_k H_{k+1-j}$ *for all* $k \geq r$*, and* r *is the least number having this property.*

Proof. Let R_i denote the ith row of H. Suppose that H has finite rank r. This means that the first $r+1$ rows of H are linearly dependent, so there is $m \leq r$ such $R_1, ..., R_m$ are linearly independent and $R_{m+1} = \sum_{j=1}^{m} \alpha_j R_{m+1-j}$ for some $\{\alpha_j\}_{j=1}^{m} \subset \mathbb{C}$. The structure of the Hankel matrix then implies that $R_{k+m+1} = \sum_{j=1}^{m} \alpha_j R_{m+k-j+1}$ holds for all $k \geq m$ since the relationship holds entry-wise for each of these rows. Thus, every row following the mth row can be expressed as a linear combination of previous rows, so we get that $m = r$.

Conversely, if $H_{k+1} = \sum_{j=1}^{k} \alpha_k H_{k+1-j}$ holds for all $k \geq r$, and r is the least number having this property, then every row following the rth row can be expressed as a linear combination of previous rows, so $rank(H) \leq r$. But the second condition implies that the same relationships do not hold for any $k < r$, so we must have $rank(H) = r$. □

Let $\mu \in S_{[0,2\pi),s}$, and suppose we are given the data

$$\widehat{a}_\mu(n) := |\widehat{\mu}(n)|^2 = \sum_{j=1}^{s}\sum_{k=1}^{s} c_j \overline{c_k} e^{-in(t_j - t_k)}$$

for $n \in \{0, ..., N\}$ through the measurement map $\mathcal{M}_{0,\Gamma_N} : \mu \mapsto (\widehat{a}_\mu(n))_{n=0}^{N}$. To resolve the ambiguities in this case, we observe as Proposition 2.1 of [14] shows, that for any $c \in \mathbb{C}$ with $|c| = 1$ and $N \in \mathbb{N}$, we have that $\mathcal{M}_{0,\Gamma_N}(c\mu) = \mathcal{M}_{0,\Gamma_N}(\mu)$. Also, $\mathcal{M}_{0,\Gamma_N}(\mu(t-b)) = \mathcal{M}_{0,\Gamma_N}(\mu)$ and $\mathcal{M}_{0,\Gamma_N}(\overline{\mu(-t)}) = \mathcal{M}_{0,\Gamma_N}(\mu)$ for any $b \in \mathbb{R}$. Hence, we aim to recover $[\mu] \in \widetilde{S}_{[0,2\pi)}$, where $\widetilde{S}_{[0,2\pi)}$ is the set of equivalence classes defined by

$$[\mu] = \{\nu \in S_{[0,2\pi)}\,|\nu(t) = \overline{\mu(t)} \text{ or } \nu(t) = e^{i\theta t}\mu(t) \text{ or } \nu(t) = e^{i\theta}\mu(t) \text{ for some } \theta \in \mathbb{R} \text{ for all } t \in \mathbb{R}\}.$$

The signal μ is completely determined by the amplitude vector $c := (c_j)_{j=1}^{s} \in \mathbb{C}^s$ and the support vector $(t_j)_{j=1}^{s} \in [0, 2\pi)^s$. Let $C := cc^* \in \mathbb{C}^{s\times s}$, $U := diag((e^{-it_j})_{j=1}^{s}) \in \mathbb{C}^{s\times s}$, and $J \in \mathbb{C}^{s\times s}$ be the matrix with all entries $= 1$. Also, define for each $N \in \mathbb{N}$ the map $\widetilde{\Phi}_N : A \times \mathcal{H}^{s\times s} \to \mathbb{R}^{N+1}$ by $\widetilde{\Phi}(U, C) := tr(CU^n JU^{*n})_{n=0}^{N}$, where

$$A := \{U \in \mathbb{C}^{s\times s}|\, U = diag((e^{-it_j})_{j=1}^{s}) \text{ for some } (t_j)_{j=1}^{s} \in [0, 2\pi)^s\},$$

and $\mathcal{H}^{s\times s}$ is the space of $s \times s$ Hermitian matrices. Consider the restriction $\Phi_N = \widetilde{\Phi}_N|_{A\times B}$ where

$$B := \{C \in \mathcal{H}^{s\times s}|\text{rank}(C) \leq 1\}.$$

Proposition A.1.2. *Let* $J_U := UJU^*$ *and* $J_U^n := U^n JU^{*n}$ *for each* $n \in \mathbb{N} \cup \{0\}$. *Then,*

(1)

$$J_U^n = \begin{bmatrix} 1 & e^{in(t_1-t_2)} & e^{in(t_1-t_3)} & \dots & e^{in(t_1-\Lambda)} \\ e^{-in(t_1-t_2)} & 1 & e^{in(t_2-t_3)} & \dots & e^{in(t_2-\Lambda)} \\ \vdots & & \ddots & & \vdots \\ e^{-in(t_1-\Lambda)} & e^{-in(t_2-\Lambda)} & e^{-in(t_3-\Lambda)} & \dots & 1 \end{bmatrix},$$

(2) J_U^n is a rank 1 Hermitian for any $U \in A$ and $n \in \mathbb{N}$,

(3) and, if we assume that $t_j - t_k = \alpha\pi$ for some irrational α, then $(J_U^n)_{(j,k)} \neq (J_U^m)_{(j,k)}$ for any $n \neq m$ for all $j, k = 1, ..., s$ with $j \neq k$.

Proof. (1) We simply compute

$$(UJU^*)_{j,k} = \sum_{m=1}^{s} u_{jm}(\sum_{p=1}^{s} J_{kp}\overline{u_{pk}}) = \sum_{m=1}^{s} u_{jm}(\sum_{p=1}^{s} \overline{u_{pk}}) = u_{jj}\overline{u_{kk}} = e^{i(t_k-t_j)}$$

for $j, k = 1, ..., s$. Since U is diagonal, we get that

$$(U^n JU^{*n})_{j,k} = e^{in(t_k-t_j)}$$

for $j, k = 1, ..., s$ and $n \in \mathbb{N} \cup \{0\}$.

(2) The fact that J_U^n is Hermitian follows from (1) by observing that

$$(U^n JU^{*n})_{j,k} = e^{in(t_k-t_j)} = \overline{e^{in(t_j-t_k)}} = \overline{(U^n JU^{*n})_{k,j}}$$

for $j, k = 1, ..., s$ and $n \in \mathbb{N}$. It is rank 1 since

$$1 \leq \text{rank}(UJU^*) \leq min\{\text{rank}(U), \text{rank}(J), \text{rank}(U^*)\} = \text{rank}(J) = 1.$$

(3) We show the contrapostive. Suppose $(J_U^n)_{(j,k)} = (J_U^m)_{(j,k)}$ for some $n, m \in \mathbb{N}$ with $n \neq m$ and $j, k \in \{1, ...s\}$. Then,

$$e^{in(t_k-t_j)} = e^{im(t_k-t_j)}$$

$$\iff n(t_k - t_j) = m(t_k - t_j) mod(2\pi)$$

$$\iff (n-m)(t_k - t_j) = 2l\pi$$

$$\iff (t_k - t_j) = \frac{2l}{n-m}\pi$$

where $\beta := \frac{2l}{n-m}$ is rational.

$\square$

Theorem A.1.3. *The restriction* $\widehat{\Phi}|_{D\times F}$ *where*

$$D := \{U \in A|\, (J_U)_{i,j} \neq (J_U)_{i',j'}\ \forall (i,j) \neq (i',j'), j > i, j' > i',$$
$$\textit{and}\ t_j - t_k = \alpha_{jk}\pi\ \textit{for some irrational}\, \alpha_{jk}\, \textit{for each}\ 1 \leq j,k \leq s\ \textit{with}\, j \neq k\}$$

and

$$F = \{C \in B|\, C_{1,1} \neq C_{s,s}\}$$

with $N \geq 2s(s-1)+1$ *is injective.*

Proof. Consider the infinite data matrix

$$H = \begin{bmatrix} \Phi_N(U,C)_0 & \Phi_N(U,C)_1 & \Phi_N(U,C)_2 & \dots \\ \Phi_N(U,C)_1 & \Phi_N(U,C)_2 & \Phi_N(U,C)_3 & \dots \\ \Phi_N(U,C)_2 & \Phi_N(U,C)_3 & \Phi_N(U,C)_4 & \dots \\ \vdots & & \ddots & \end{bmatrix}$$

Since H is a Hankel matrix, we know, by Proposition A.1.1 [35], that H is of finite rank r if and only if there exists r numbers $\alpha_1, \alpha_2, ..., \alpha_r$ such that $\Phi_N(U,C)_m = \sum_{n=1}^{r} \alpha_n \Phi_N(U,C)_{m-n}$ for all $m \in \mathbb{N}$ with $m \geq r$

and r is the least number for which this happens. This holds if and only if

$$tr(CJ_U^m) = \sum_{n=1}^{r} \alpha_n tr(CJ_U^{m-n}) \tag{A.1}$$

for $m \geq r$. We claim that if $(U, C) \in D \times F$ and $N \geq 2s(s-1)+1$, then (A.1) is satisfied for $r = s(s-1)+1$. Let $(U, C) \in D \times F$ and define

$$\Delta \coloneqq span_{\mathbb{R}}\{J_U^n | n \in \mathbb{N} \cup \{0\}\}.$$

By construction, Δ has dimension at most $s(s-1)+1$ (since every $J_U^n \in \Delta$ is a Hermitian matrix, by part 2 of Proposition A.1.2, with constant diagonal). Suppose $J_U^N = \sum_{n=0}^{N-1} \beta_n J_U^n$ for some $N < s(s-1)+1$. Then,

$$e^{iN(t_j - t_k)} = \sum_{n=0}^{N-1} \beta_n e^{in(t_j - t_k)}$$

for each $j, k \in \{1, ..., s\}$. The equation $z^N - \sum_{n=0}^{N-1} \beta_n z^n = 0$ has at most N different complex solutions. However, we have that $(J_U^n)_{(j,k)} = e^{in(t_j - t_k)}$ is a solution for all $j, k \in \{1, ..., s\}$. By the assumption $U \in D$ and part 3 of Proposition A.1.2, we know that these are $s(s-1)+1$ distinct values, which contradicts $N < s(s-1)+1$. Hence, we conclude that $dim(\Delta) = s(s-1)+1$ where the first $s(s-1)+1$ elements are linearly independent.

Thus,

$$J_U^{s(s-1)+1} = \sum_{n=1}^{s(s-1)+1} \alpha_n J_U^{[s(s-1)+1]-n}$$

for some $\{\alpha_i\}_{i=1}^{s(s-1)+1} \in \mathbb{R}^{s(s-1)+1}$ and (entry-wise)

$$e^{i[s(s-1)+1](t_j - t_k)} = \sum_{n=1}^{s(s-1)+1} \alpha_n e^{i[s(s-1)+1-n](t_j - t_k)}$$

for each $j, k \in \{1, ..., s\}$, and multiplying both sides by $e^{i(t_j - t_k)}$ gives

$$e^{im(t_j - t_k)} = \sum_{n=1}^{s(s-1)+1} \alpha_n e^{i(m-n)(t_j - t_k)} \tag{A.2}$$

for each $j, k \in \{1, ..., s\}$ and $m = s(s-1)+1, s(s-1)+2, ...$ if and only if

$$tr(CJ_U^m) = \sum_{n=1}^{s(s-1)+1} \alpha_n tr(CJ_U^{m-n})$$

for $m = s(s-1)+1, s(s-1)+2, ...,$ so we get that $r \leq s(s-1)+1$, and in the worst case when $r = s(s-1)+1$, we obtain the modified data matrix

$$H_r = \begin{bmatrix} \Phi_N(U,C)_0 & \Phi_N(U,C)_1 & \Phi_N(U,C)_2 & \dots & \Phi_N(U,C)_{s(s-1)} \\ \Phi_N(U,C)_1 & \Phi_N(U,C)_2 & \Phi_N(U,C)_3 & \dots & \Phi_N(U,C)_{s(s-1)+1} \\ \Phi_N(U,C)_2 & \Phi_N(U,C)_3 & \Phi_N(U,C)_4 & \dots & \Phi_N(U,C)_{s(s-1)+2} \\ \vdots & & \ddots & & \vdots \\ \Phi_N(U,C)_{s(s-1)} & \Phi_N(U,C)_3 & \Phi_N(U,C)_4 & \dots & \Phi_N(U,C)_{2s(s-1)} \end{bmatrix}.$$

Next, we show that the task of recovering $t_j - t_k$ is equivalent to finding the set of coefficients $\{\alpha_n\}_{n=0}^{s(s-1)}$. The equation $z^{[s(s-1)+1]} - \sum_{n=0}^{s(s-1)} \alpha_n z^n = 0$ has at most $s(s-1)+1$ distinct solutions. By (A.2), those are exactly $1, e^{in(t_j - t_k)}$ for $j, k = 1, ..., s$ with $j \neq k$.

Using the given data, we obtain the linear system of equations

$$\Phi_m = \sum_{n=1}^{r} \alpha_{n-1} \Phi_{m-n}$$

for $m = s(s-1)+1, s(s-1)+2, ...$, where $\Phi_m = \Phi(U,C)_m$, which can be represented as

$$\begin{bmatrix} (\Phi_N)_0 & (\Phi_N)_1 & (\Phi_N)_2 & \dots & (\Phi_N)_{s(s-1)} \\ (\Phi_N)_1 & (\Phi_N)_2 & (\Phi_N)_3 & \dots & (\Phi_N)_{s(s-1)+1} \\ (\Phi_N)_2 & (\Phi_N)_3 & (\Phi_N)_4 & \dots & (\Phi_N)_{s(s-1)+2} \\ \vdots & & \ddots & & \\ (\Phi_N)_{s(s-1)} & (\Phi_N)_3 & (\Phi_N)_4 & \dots & (\Phi_N)_{2s(s-1)} \end{bmatrix} \begin{bmatrix} \alpha_{s(s-1)+1} \\ \alpha_{s(s-1)} \\ \alpha_{s(s-1)-1} \\ \vdots \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} (\Phi_N)_{s(s-1)+1} \\ (\Phi_N)_{s(s-1)+2} \\ (\Phi_N)_{s(s-1)+3} \\ \vdots \\ (\Phi_N)_{2s(s-1)+1} \end{bmatrix}.$$

Since H_r has full rank by construction, we know that it is invertible which means that system admits a unique solution. Hence, the roots of the equation $z^{[s(s-1)+1]} - \sum_{n=0}^{s(s-1)} \alpha_n z^n = 0$ can be uniquely determined, and we obtain an unlabeled set of values $e^{in(t_j - t_k)}$ for $j,k \in \{1,...,s\}$. Reordering the values $t_j - t_k$ in ascending order, we obtain the sequence of distances between nodes $(\Delta_m)_{m=1}^{s(s-1)+1}$ where $\Delta_1 = t_1 - \Lambda, \Delta_2 = t_1 - t_{s-1}$ (or $t_2 - \Lambda$), $..., \Delta_{\frac{s(s-1)}{2}+1} = 0, ..., \Delta_{s(s-1)} = t_{s-1} - t_1$ (or $t_2 - \Lambda$), $\Delta_{s(s-1)+1} = \Lambda - t_1$. Finally, the unlabeled entries in C, $(C_m)_{m=1}^{m=s(s-1)+1}$ with $C_{\frac{s(s-1)}{2}+1} = \sum_{j=1}^{s} |c_j|^2$ associated with each value $e^{-i\Delta_m}$ for $m = 1, ..., s(s-1)+1$ can be recovered by solving the following overdetermined linear system

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ e^{-i\Delta_1} & e^{-i\Delta_2} & e^{-i\Delta_3} & \dots & e^{-i\Delta_{s(s-1)+1}} \\ e^{-i2\Delta_1} & e^{-i2\Delta_2} & e^{-i2\Delta_3} & \dots & e^{-i2\Delta_{s(s-1)+1}} \\ \vdots & & \ddots & & \vdots \\ e^{-i[2s(s-1)+1]\Delta_1} & e^{-i[2s(s-1)+1]\Delta_2} & e^{-i[2s(s-1)+1]\Delta_3} & \dots & e^{-i[2s(s-1)+1]\Delta_{s(s-1)+1}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_{s(s-1)+1} \end{bmatrix} = \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{2s(s-1)+1} \end{bmatrix}$$

The final stage of the recovery process is to determine the associated labels for each Δ_m. The following can be found in the proof of Theorem 3.1 of [14]. Based on the ambiguities mentioned above (and addressed in the definition of $\widetilde{S}_{[0,2\pi)}$), we may set $t_1 = 0$, which means that $\Lambda = -\Delta_1$. Next, we may have $\Delta_2 = t_1 - t_{s-1}$ or $t_2 - \Lambda$. Again, appealing to the reflection and conjugation ambiguity, we can choose $\Delta_2 = t_1 - t_{s-1}$, which gives $t_{s-1} = -\Delta_2$. This implies that there is some

Δ_k with $\Delta_k = \Lambda - t_{s-1} = -\Delta_1 + \Delta_2$. Hence, we get that

$$C_1 = c_1\overline{c_s}, \qquad C_2 = c_1\overline{c_{s-1}}, \quad \text{and} \quad C_k = c_{s-1}\overline{c_s}$$

which gives

$$\overline{c_s} = \frac{C_1}{c_1}, \qquad \overline{c_{s-1}} = \frac{C_2}{c_1}$$

and

$$|c_1|^2 = \frac{C_1\overline{C_2}}{C_k}.$$

Since we want to recover f only up to a unimodular constant, we may choose c_1 to be real and nonnegative, which then allows us to compute c_{s-1} and c_s. Finding the remaining labels is a process that requires trial and error and its details are described in [14]. This proves that given $(\Phi(U,C)_n)_{n=0}^{N}$ with $N \geq s(s-1)+1$, the matrices U and C can be uniquely recovered. $\square$

Alternatively, one may verify that the coefficients of the Prony polynomial $P : \mathbb{C} \to \mathbb{C}$ given by

$$P(z) = \prod_{j,k=1}^{s} (z - e^{-i(t_j - t_k)}) = \sum_{n=0}^{s(s-1)+1} \lambda_n z^n$$

satisfy $(\lambda_n)_{n=0}^{s(s-1)} = (\alpha_n)_{n=0}^{s(s-1)}$ where $\lambda_{s(s-1)+1} = 1$. Loosely speaking, the Hankel matrix simply asks whether old events could be used to predict new events. Can new rows be written as a linear combination of old rows? More formally, the goal is to reconstruct $\widehat{a}(\omega)$ by finding an inverse to the modified data matrix. However, this immediately places a lower bound on the number of measurements that are required, as the dimension of H_r is proportional to the number of unknowns, $s(s-1)+1$, which is quadratic in s. As mentioned, this is part of the motivation to search for other recovery techniques, as we notice that the actual number of unknowns that are needed to determine μ, as stated above, is $2s$, which is linear in s.

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