NONNEGATIVE MATRICES WITH DOUBLY STOCHASTIC POWERS

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Nancy Rosenblad Bedford

June 1966

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ABSTRACT

Let A be a nonnegative irreducible square matrix, and let m be an integer greater than one. Then it is possible to obtain the following necessary and sufficient conditions for A^m to be doubly stochastic while A is not doubly stochastic. First, A is cyclic of index h, where h and m are not relatively prime. Also, there exist positive numbers β_i and ζ_i, $i=1,\ldots,h$, such that the matrices A_i in a Frobenius normal form of A are respectively β_i/β_{i+1} stochastic, indices modulo h, and such that the matrices A_i^T are $\zeta_{h-i}/\zeta_{h-i+1}$ stochastic respectively, indices modulo h. These numbers β_i and ζ_i are such that at least one of the two sets $\{\beta_i\}$ and $\{\zeta_i\}$ contains at least two distinct elements, $\beta_i=\beta_{i+m}$ and $\zeta_i=\zeta_{i+m}$(indices mod h), and $\zeta_{h-i}/\zeta_{h-i+1}=(n_i/n_{i+1})(\beta_i/\beta_{i+1})$, where the matrices A_i are respectively $n_i \times n_{i+1}$(indices mod h).

The above results may be used to obtain the following necessary and sufficient conditions for a reducible nonnegative square matrix A to have a doubly stochastic positive integral power m, while A itself is not doubly stochastic. There exists some permutation matrix P such that PAP^T is the direct sum of irreducible nonnegative square matrices, each of which is either doubly stochastic or satisfies the conditions stated above as necessary and sufficient for A^m to be doubly stochastic while A is not; and at least one of the matrices in the direct sum is not doubly stochastic.

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CHAPTER I

INTRODUCTION

A matrix is called <u>nonnegative</u> if all the entries are nonnegative real numbers. If $B = (b_{ij})$ is a nonnegative $m \times n$ matrix such that

$$\sum_{j=i}^{n} b_{ij} = \beta, \; i = 1,\ldots,m,$$

then B is β <u>stochastic</u> or <u>generalized</u> <u>stochastic</u>. If $\beta = 1$, B is <u>stochastic</u>. The matrix obtained from B by interchanging the rows and columns is called the <u>transpose</u> of B and is denoted in this paper by B^T. An $n \times n$ nonnegative matrix is <u>doubly</u> <u>stochastic</u> if both the matrix and its transpose are stochastic; that is if both the row sums and the column sums are equal to one. This paper is concerned with square nonnegative matrices which are not doubly stochastic, but for which some positive power is doubly stochastic. The nonnegative irreducible matrices with doubly stochastic powers are considered first, and the results obtained for them are used in consideration of the reducible matrices. The results for nonnegative irreducible matrices with doubly stochastic powers are obtained by applying a theorem of D. London, [2], concerning nonnegative irreducible matrices with stochastic powers.

A permutation matrix is a matrix with one and only one 1 in each row and each column, and all other elements 0. A matrix A is <u>reducible</u> if there exists a permutation matrix

P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square. Otherwise, A is <u>irreducible</u>. For nonnegative irreducible matrices we have the very powerful Perron-Frobenius theorem, [1,p. 53], which is stated here without proof.

THEOREM 1.1. Let A be an n X n nonnegative irreducible matrix. Then:

(I) A has a real positive characteristic root r which is a simple root of the characteristic equation of A. If λ_i is any characteristic root of A, then $|\lambda_i| \le r$.

(II) There exists a positive characteristic vector corresponding to r.

(III) If A has h characteristic roots of modulus r: $\lambda_0 = r, \lambda_1, \ldots, \lambda_{h-1}$, then these are the h distinct roots of $\lambda^h - r^h = 0$.

(IV) If h = 1, A is called <u>primitive</u>.

(V) If h > 1, A is called <u>cyclic of index</u> h and there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & A_{h-1} \\ A_h & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (1.1)$$

where the null matrices on the main diagonal are square matrices of orders n_k, $k = 1,\ldots,h$.

The representation as in (1.1) is called a <u>Frobenius normal form of</u> A. With indices here and in the remainder of the paper being taken modulo h, A_i is thus an $n_i \times n_{i+1}$ matrix.

There is also a normal form for reducible matrices, [1, p. 74]. If A is a square reducible nonnegative matrix, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ A_{p1} & A_{p2} & A_{p3} & \cdots & A_{pp} \end{bmatrix},$$

where the matrices A_{ii} on the diagonal are square and irreducible; and the matrices below the diagonal are nonnegative, although some or all of them could be zero matrices.

London's theorem, [2, Th.1], on nonnegative irreducible square matrices is stated as follows:

THEOREM 1.2. Let A be a nonnegative irreducible square matrix, and let $m > 1$ be a natural number. Let H be the cyclic permutation

$$H = (12\ldots h),$$

and let

$$H^m = C_1C_2\ldots C_r \tag{1.3}$$

be the representation of H^m as the product of disjoint cycles. A is not a stochastic matrix while A^m is stochastic if and only if

(I) A is cyclic of index h, where $(h,m) > 1$.

(II) There exist positive numbers β_i, $i = 1,\ldots,h$, such that the matrices A_i in the Frobenius normal form (1.1) of A are respectively β_i/β_{i+1} stochastic. The numbers β_i fulfill the following two conditions.

(A) They are not all equal.

(B) Every two numbers with indices belonging to the same cycle in (1.3) are equal.

CHAPTER II

NONNEGATIVE IRREDUCIBLE MATRICES

Let A be a nonnegative irreducible square matrix which is not doubly stochastic. The theorems stated in the introduction and the lemmas below will be used to obtain necessary and sufficient conditions for some power of A to be doubly stochastic.

LEMMA 2.1. If $B_1, B_2, \ldots, B_n$ are respectively $\beta_1, \beta_2, \ldots, \beta_n$ stochastic $m_i \times m_{i+1}$ matrices, then the product matrix $B_1 B_2 \cdot \ldots \cdot B_n$ is $\beta_1 \beta_2 \cdot \ldots \cdot \beta_n$ stochastic.

PROOF: For $n = 1$ assertion is true, since we have B_1 is β_1 stochastic. Assume lemma is true for $n < k$. Then let $A = (a_{ij}) = B_1 B_2 \cdot \ldots \cdot B_{k-1}$; and by the inductive hypothesis, we have

$$\sum_{j=1}^{m_{k-1}} a_{ij} = \beta_1 \beta_2 \cdot \ldots \cdot \beta_{k-1} .$$

Let $B_k = (b_{ij})$ be β_k stochastic; then we have

$$\sum_{j=1}^{m_k} b_{ij} = \beta_k .$$

Then the sum of the elements in any row of AB_k is given by

$$\sum_{t=1}^{m_k} \sum_{s=1}^{m_{k-1}} a_{is} b_{st} = \sum_{s=1}^{m_{k-1}} \sum_{t=1}^{m_k} b_{st} a_{is}$$

$$= \sum_{s=1}^{m_{k-1}} \beta_k a_{is}$$

$$= \beta_k(\beta_1\beta_2\cdot\ldots\cdot\beta_{k-1})$$
$$= \beta_1\beta_2\cdot\ldots\cdot\beta_k.$$

Thus the induction is completed, and the lemma is proved.

LEMMA 2.2. Let A be a nonnegative irreducible matrix which is cyclic of index h. The following statements are equivalent.

(I) A is stochastic.

(II) The Frobenius normal form of A is stochastic.

(III) Each matrix A_i appearing in the Frobenius normal form of A is stochastic.

PROOF: (a) (I)⇒(II): Let A be a stochastic matrix. Since any permutation matrix P has exactly one 1 in each row and column, P and P^T are both stochastic. Thus by lemma 2.1, PAP^T is stochastic.

(b) (II)⇒(III): Suppose the Frobenius normal form (1.1) is stochastic. Then since A_i is the only non-zero matrix in the ith row of blocks, the row sums are all equal to 1, and each A_i is stochastic.

(c) (III)⇒(I): If each matrix A_i in the Frobenius normal form of A is stochastic, each row in (1.1) has row sum equal to 1. Then by multiplication on the left by $P^T = P^{-1}$ and multiplication on the right by P in equation (1.1) and the use of lemma 2.1, A is stochastic.

LEMMA 2.3. Denote the blocks of PAP^T in the partitioning

(1.1) by A_{ij}, $i,j = 1,\ldots,h$, and the blocks of PA^mP^T in the same partitioning by $A_{ij}^{(m)}$. Then

$$(2.1) \qquad A_{ij}^{(m)} = \begin{cases} A_iA_{i+1}\cdot\ldots\cdot A_{i+m-1}, & j = i + m \pmod h \\ 0, & j \neq i + m \pmod h \end{cases}.$$

PROOF: For $m = 1$, we have

$$A_{ij} = \begin{cases} A_i, & j = i + 1 \pmod h \\ 0, & j \neq i + 1 \pmod h \end{cases},$$

which is precisely the Frobenius normal form of A. Now assume (2.1) is true for m-1. Then

$$A_{ij}^{(m-1)} = \begin{cases} A_iA_{i+1}\cdot\ldots\cdot A_{i+m-2}, & j = i + m - 1 \pmod h \\ 0, & j \neq i + m - 1 \pmod h \end{cases}.$$

Therefore,

$$A_{ij}^{(m)} = \sum_{k=1}^{h} A_{ik}^{(m-1)} A_{kj}$$

$$= \sum_{k=1}^{h} \begin{cases} A_iA_{i+1}\cdot\ldots\cdot A_{i+m-2}, & k = i+m-1 \pmod h \\ 0, & k \neq i+m-1 \pmod h \end{cases} \cdot \begin{cases} A_k, & j = k+1 \pmod h \\ 0, & j \neq k+1 \pmod h \end{cases}.$$

The only nonzero terms are obtained when $k = i + m - 1 \pmod h$ and $j = k + 1 \pmod h$; that is when $j = i + m \pmod h$. Thus the above equation becomes

$$A_{ij}^{(m)} = \begin{cases} A_iA_{i+1}\cdot\ldots\cdot A_{i+m-2}A_{i+m-1}, & j = i + m \pmod h \\ 0, & j \neq i + m \pmod h \end{cases},$$

and the induction is completed.

THEOREM 2.1. Let A be a nonnegative irreducible square matrix, and let $m > 1$ be a positive integer. Let H be the cyclic permutation

$$H = (12\ldots h)$$

and let

(2.2) $$H^m = C_1C_2\ldots C_r$$

be the representation of H^m as the product of disjoint cycles. A^m is doubly stochastic while A is not doubly stochastic if and only if

(I) A is cyclic of index h, where $(h,m) > 1$.

(II) There exist positive numbers β_i, $i = 1,\ldots,h$, and positive numbers ζ_i, $i = 1,\ldots,h$, such that the matrices A_i appearing in the Frobenius normal form of A are respectively β_i/β_{i+1} stochastic, and the matrices A_i^T are respectively $\zeta_{h-i}/\zeta_{h-i+1}$ stochastic. The numbers β_i and ζ_i satisfy the following conditions:

(A) At least one of the following conditions (1) and (2) holds.

(1) The numbers β_i are not all equal.

(2) The numbers ζ_i are not all equal.

(B) Every two numbers β_i with indices belonging to the same cycle in (2.2) are equal, and every two numbers ζ_i with indices belonging to the same cycle are equal.

(C) If the zero matrices on the diagonal in the Frobenius normal form of A are of order n_i, then the relation between the numbers β_i and ζ_i is given by

$$\frac{\zeta_{h-i}}{\zeta_{h-i+1}} = \frac{n_i}{n_{i+1}} \cdot \frac{\beta_i}{\beta_{i+1}} \text{, indices modulo h.}$$

PROOF: Let us first prove that these conditions are necessary. Let A be a nonnegative irreducible square matrix such that A^m is doubly stochastic, but A is not doubly stochastic. Thus A^m and $(A^T)^m$ are both stochastic, but at least one of A and A^T is not stochastic. Since A is irreducible, A^T is irreducible. Thus they are both primitive or both cyclic of the same index. By theorem 1.2, a matrix which is not stochastic but has a stochastic mth power is cyclic of index h such that $(h,m) > 1$. Thus A and A^T are cyclic of index h with $(h,m) > 1$, and condition (I) is necessary.

Then since A is cyclic of index h, it has a Frobenius normal form as in (1.1). If A is not stochastic, we know from theorem 1.2 that there exist numbers β_i, $i = 1,\ldots,h$, not all equal, such that the matrices A_i in the normal form are respectively β_i/β_{i+1} stochastic. If A is stochastic, the matrices A_i are all stochastic; and for numbers β_i, all equal, A_i is β_i/β_{i+1} stochastic.

Consider the following permutation matrix:

$$K = \begin{bmatrix} 0 & 0 & \ldots & 0 & I_h \\ 0 & 0 & \ldots & I_{h-1} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & I_2 & \ldots & 0 & 0 \\ I_1 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

where I_k is an identity matrix of order n_k, $k = 1,\ldots,h$. Then since

$$(PAP^T)^T = PA^TP^T = \begin{bmatrix} 0 & 0 & \ldots & 0 & A_h^T \\ A_1^T & 0 & \ldots & 0 & 0 \\ 0 & A_2^T & \ldots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & A_{h-1}^T & 0 \end{bmatrix},$$

we have

$$KPA^TP^TK^T = \begin{bmatrix} 0 & 0 & \ldots & 0 & A_{h-1}^T & 0 \\ 0 & 0 & \ldots & A_{h-2}^T & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ A_1^T & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & A_h^T \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & 0 & I_1 \\ 0 & 0 & \ldots & I_2 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & I_{h-1} & \ldots & 0 & 0 \\ I_h & 0 & \ldots & 0 & 0 \end{bmatrix}$$

and

$$(2.3) \qquad KPA^TP^TK^T = \begin{bmatrix} 0 & A_{h-1}^T & 0 & \ldots & 0 & 0 \\ 0 & 0 & A_{h-2}^T & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ 0 & 0 & 0 & \ldots & 0 & A_1^T \\ A_h^T & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Thus, (2.3) is a Frobenius normal form for A^T. We are assuming A^m is doubly stochastic; thus if A^T is not stochastic, by theorem 1.2, there exist numbers ζ_i, $i = 1,\ldots,h$, not all equal, such that the matrices A_{h-i}^T are respectively ζ_i/ζ_{i+1} stochastic. If A^T is stochastic, then each of the matrices A_{h-1}^T is stochastic, and for numbers ζ_i, all equal, A_{h-i}^T is ζ_i/ζ_{i+1} stochastic. We have thus shown the existence of the numbers β_i and ζ_i mentioned in condition (II). Since we are assuming A is not doubly stochastic, at least one of A and A^T is not stochastic. If all the β_i were equal and all the ζ_i were equal, then A and A^T would both be stochastic. Thus condition (A) is necessary.

If A is not stochastic, we have from theorem 1.2 that the numbers β_i with indices belonging to the same cycle in (2.2) are equal. Likewise, if A^T is not stochastic, numbers ζ_i with indices belonging to the same cycle are equal. If either A or A^T is stochastic, we have respectively the numbers β_i or the numbers ζ_i all equal, making those with indices belonging to the same cycle equal. Thus condition (B) is necessary.

If the zero matrices on the diagonal in a Frobenius normal form of A are of order n_i, we have that the matrices A_i are $n_i \times n_{i+1}$ respectively. Since the $n_i \times n_{i+1}$ matrix A_i is β_i/β_{i+1} stochastic, the sum of all the elements in A_i is $n_i(\beta_i/\beta_{i+1})$. The $n_{i+1} \times n_i$ matrix A_i^T is $\zeta_{h-i}/\zeta_{h-i+1}$ stochastic, so the sum of the elements in A_i^T is $n_{i+1}(\zeta_{h-i}/\zeta_{h-i+1})$ These sums must be equal, so we have

$n_i(\beta_i/\beta_{i+1}) = n_{i+1}(\zeta_{h-i}/\zeta_{h-i+1})$, or

$$\frac{\zeta_{h-i}}{\zeta_{h-i+1}} = \frac{n_i}{n_{i+1}} \cdot \frac{\beta_i}{\beta_{i+1}} .$$

Thus condition (C) is also necessary, and we have completed the proof of necessity.

Now we will show that conditions (I) and (II) are also sufficient. Let A be a nonnegative irreducible square matrix which satisfies (I) and (II). Since $(h,m) > 1$, there is more than one cycle in (2.2). Thus it is possible to have numbers β_i or ζ_i with indices belonging to the same cycle equal but such that not all the β_i or ζ_i are equal. From condition (A) we know that at least one of the two sets of numbers $\{\beta_i\}$ and $\{\zeta_i\}$ contains at least two distinct numbers. If not all the β_i are the same, not all the A_i in the Frobenius normal form for A are stochastic. Thus by lemma 2.2, A is not stochastic. Similarly, if all the ζ_i are not equal, A^T is not stochastic. In either case, A is not doubly stochastic. If all the β_i are not equal and all the ζ_i are not equal, neither A nor A^T is stochastic; therefore, A is not doubly stochastic.

Let the blocks in the Frobenius normal form for A be denoted by A_{ij}. Letting the blocks in PA^mP^T be denoted by $A_{ij}^{(m)}$, we have from lemma 2.3

$$A_{ij}^{(m)} = \left\{ \begin{array}{ll} A_iA_{i+1}\cdot\ldots\cdot A_{i+m-1} & , j = i + m \pmod h \\ 0 & , j \neq i + m \pmod h \end{array} \right\}.$$

Since $j = i + m \pmod h$ exactly once for each i and exactly once for each j, we have exactly one nonzero matrix block in each row and in each column of blocks. Using lemmas 2.1 and 2.3 and the fact that the matrices A_i are respectively β_i/β_{i+1} stochastic, we have that the matrix $A_{i,i+m}^{(m)}$ is

$$\frac{\beta_i}{\beta_{i+1}} \cdot \frac{\beta_{i+1}}{\beta_{i+2}} \cdot\ldots\cdot \frac{\beta_{i+m-1}}{\beta_{i+1}} = \frac{\beta_i}{\beta_{i+m}}$$

stochastic. By condition (B), $\beta_i = \beta_{i+m}$, so $A_{i,i+m}^{(m)}$ is stochastic. Thus from lemma 2.2, we know that A^m is stochastic. Similarly, if the blocks in the Frobenius normal form for A^T as partitioned in (2.3) are denoted by B_{ij} and the blocks in $KP(A^T)^m P^T K^T$ are denoted by $B_{ij}^{(m)}$, we have

$$B_{ij}^{(m)} = \begin{cases} A_{h-i}^T A_{h-i-1} \cdot \ldots \cdot A_{h-i-m+1} & , j = i + m \pmod h \\ 0 & , j \neq i + m \pmod h \end{cases}.$$

From (II) we know that A_i^T is $\zeta_{h-i}/\zeta_{h-i+1}$ stochastic, so A_{h-i}^T is ζ_i/ζ_{i+1} stochastic. Thus, as above, $B_{ij}^{(m)}$ is ζ_i/ζ_{i+m} stochastic. Since $\zeta_i = \zeta_{i+m}$, by condition (B), the blocks in the mth power of the normal form for A^T are stochastic. Thus $(A^T)^m$ is also stochastic and A^m is doubly stochastic. We have thus shown that conditions (I) and (II) are sufficient for A^m to be doubly stochastic while A is not doubly stochastic.

It is helpful to consider some numerical examples illustrating the above theorem.

EXAMPLE 2.1. A is a 10×10 matrix such that neither A nor A^T is stochastic, but A^2 is doubly stochastic. Let $h = 4$; then if $H = (1234)$, $H^2 = (13)(24)$. Letting $\beta_1 = \beta_3 = 4$ and $\beta_2 = \beta_4 = 3$, we have that A_1 and A_3 are 4/3 stochastic and A_2 and A_4 are 3/4 stochastic. Letting $n_1 = n_3 = 2$ and $n_2 = n_4 = 3$, we have that A_1^T and A_3^T are $(2/3)(4/3) = 8/9$ stochastic and A_2^T and A_4^T are $(3/2)(3/4) = 9/8$ stochastic. The following matrix represents one possible choice of values of elements satisfying these conditions:

$$A = \begin{bmatrix} 0 & 0 & 2/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/9 & 5/9 & 5/9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/8 & 3/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5/9 & 5/9 & 2/9 \\ 3/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/9 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 13/24 & 11/24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11/24 & 13/24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11/36 & 11/36 & 14/36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13/36 & 13/36 & 10/36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12/36 & 12/36 & 12/36 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

which is doubly stochastic. Since $(PAP^T)^2 = PA^2P^T$ for any permutation matrix P, multiplication of the above matrix A on the left by a permutation matrix P and on the right by P^T would also result in a doubly stochastic matrix.

EXAMPLE 2.2. A is a 5 X 5 matrix which is stochastic, but not doubly stochastic. A^2 is doubly stochastic. For $H = (12)$, $H^2 = (1)\ (2)$. Letting $\beta_1 = \beta_2 = 1$, we have A_1 and A_2 stochastic. Letting $n_1 = 2$ and $n_2 = 3$, we have A_1^T 2/3 stochastic and A_2^T 3/2 stochastic. The following matrix represents one possible Frobenius normal form for a matrix satisfying these conditions:

$$A = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix},$$

which is doubly stochastic. An example in which the matrix is not stochastic, its transpose is stochastic, and its square is doubly stochastic may be obtained by using the transpose of the matrix A in this example.

CHAPTER III

NONNEGATIVE REDUCIBLE MATRICES

We shall now use the results obtained for the irreducible matrices to find necessary and sufficient conditions for a reducible nonnegative square matrix which is not doubly stochastic to have a doubly stochastic power. The following lemmas will be useful.

LEMMA 3.1. Let $B = (b_{ij})$ be a nonnegative $s \times t$ matrix with at least one nonzero element in each row. Then if $N = (n_{ij})$ is a nonnegative $r \times s$ matrix such that $NB = 0$, then $N = 0$.

PROOF: Since all the terms involved are nonnegative, and since

$$\sum_{k=1}^{s} n_{ik} b_{kj} = 0, \text{ for all } i \text{ and } j,$$

each term in each sum must be 0. Let b_{pq} be a nonzero element of A. Then for $i = 1,\ldots,r$, $n_{ip} b_{pq} = 0$. Thus $n_{ip} = 0$ for all i. Thus the pth column of N is all zeros. Since each row of B contains at least one nonzero element, each column of N is composed of zeros. Thus $N = 0$.

LEMMA 3.2. If C is an irreducible nonnegative matrix which satisfies the conditions of theorem 2.1, then each row of C^{m-1} has at least one nonzero element.

PROOF: If C satisfies the conditions of theorem 2.1, there exists a permutation matrix P such that PCP^T is in Frobenius normal form. Also there exist positive numbers β_i such that each A_i in that normal form is respectively β_i/β_{i+1} stochastic. Letting the blocks in PC^mP^T be denoted by $A_{ij}^{(m)}$, we have from equation (2.1) and Lemma 2.1 that

$$A_{i,i+m-1}^{(m-1)} \quad \text{is } \beta_i/\beta_{i+m-1} \text{ stochastic.}$$

All the numbers β_i are greater than zero, so each row sum is greater than zero, making each row have some nonzero element.

Before continuing, we need to define the direct sum of matrices. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, the <u>direct sum</u> $C = (c_{ij})$ of A and B, is the $(m+p) \times (n+q)$ matrix defined as follows:

$$c_{ij} = a_{ij}\ , \text{ if } i = 1,\ldots,m;\ j = 1,\ldots,n$$

$$c_{m+i,n+j} = b_{ij}\ , \text{ if } i = 1,\ldots,p;\ j = 1,\ldots,q$$

$$c_{ij} = 0\ , \text{ otherwise.}$$

LEMMA 3.3. If A is doubly stochastic reducible nonnegative matrix, there exists a permutation matrix P such that PAP^T is a direct sum of square irreducible matrices.

PROOF: As stated in the introduction, for any reducible nonnegative square matrix there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ A_{p1} & A_{p2} & A_{p3} & \cdots & A_{pp} \end{bmatrix},$$

where the A_{ii} on the main diagonal are square irreducible matrices. Let A_{ii} be of order n_i. Then since PAP^T is stochastic, each row sum in A_{11} is equal to one, and the sum of all the elements in A_{11} is n_i. But the column sums of PAP^T are also equal to one since PAP^T is doubly stochastic, making the sum of all the elements in the first n_i columns equal to n_i. Since this amount is already present in A_{11}, we have $A_{21} = \dots = A_{p1} = 0$; and A_{11} is doubly stochastic. Repeating this argument, it is possible to show that $A_{22},\dots,A_{pp}$ are doubly stochastic; and all the blocks below the matrices on the diagonal are zero.

THEOREM 3.1. Let A be a nonnegative reducible square matrix, and let $m > 1$ be an integer. Then A^m is doubly stochastic while A is not doubly stochastic if and only if the following conditions hold.

(I) There exists a permutation matrix P such that PAP^T is the direct sum of square irreducible matrices, each of which is either doubly stochastic or satisfies the conditions of theorem 2.1.

(II) At least one of the matrices in the direct sum satisfies the conditions of theorem 2.1.

PROOF: Let A be a nonnegative reducible square matrix which is not doubly stochastic, but such that A^m is doubly stochastic. Since A is reducible, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} C_1 & 0 \\ D_1 & E_1 \end{bmatrix},$$

where C_1 and D_1 are square matrices and C_1 is irreducible. It can be shown easily by induction on m that any positive integral power of a reducible matrix is reducible and that

$$PA^mP^T = \begin{bmatrix} C_1^m & 0 \\ \sum_{k=1}^{m} E_1^{k-1} D_1 C_1^{m-k} & E_1^m \end{bmatrix}.$$

From lemma 3.3 we know that if A^m is doubly stochastic and reducible, it is necessary that it is some permutation of a direct sum of irreducible matrices. Thus

$$\sum_{k=1}^{m} E_1^{k-1} D_1 C_1^{m-k} = 0.$$

Since all the matrices involved are nonnegative, each term in the sum must be a zero matrix. Corresponding to $k = 1$, we have $D_1 C_1^{m-1} = 0$; but from lemma 3.2 we know that C_1^{m-1} has a nonzero element in each row. Thus by lemma 3.1, $D_1 = 0$. If E_1 is irreducible, the necessity of the condition is shown. If E_1 is reducible, there is a permutation matrix Q such that

$$QE_1Q^T = \begin{bmatrix} C_2 & 0 \\ D_2 & E_2 \end{bmatrix},$$

where C_2 and E_2 are square matrices and C_2 is irreducible. By the same process used above, it follows that $D_2 = 0$. This process can be continued until some matrix E_k is irreducible; and then letting E_k be C_{k+1}, we have A as a permutation of the direct sum of the matrices $C_1,\ldots,C_{k+1}$, since the matrices $D_1,\ldots,D_k$ are all zero matrices. That is, there is some permutation matrix P such that

$$PAP^T = \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & C_{k+1} \end{bmatrix} \text{ and } PA^mP^T = \begin{bmatrix} C_1^m & 0 & 0 & \cdots & 0 \\ 0 & C_2^m & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & C_{k+1}^m \end{bmatrix}.$$

Since A^m, and thus PA^mP^T, is reducible and doubly stochastic, the matrices C_i^m , $i = 1,\ldots,k+1$, are doubly stochastic. Thus each C_i is either doubly stochastic or satisfies the conditions stated in theorem 2.1 as necessary for a matrix which is not doubly stochastic to have a doubly stochastic mth power. Thus condition (I) of the theorem is necessary. If all the matrices C_i, $i = 1,\ldots,k+1$, were doubly stochastic, A would be doubly stochastic. Thus at least one of the matrices satisfies the conditions of theorem 2.1, and condition (II) is also necessary.

It is easy to see that these conditions are also sufficient. Let A be a nonnegative reducible matrix and $m > 1$ an integer such that conditions (I) and (II) are satisfied. Since the mth power of a direct sum of matrices is the direct sum of the mth powers of the matrices involved, it sufficies to show that the mth power of each of the matrices in the direct sum referred to in (I) is doubly stochastic. Those matrices satisfying the conditions of theorem 2.1 are such that the mth power is doubly stochastic. Those matrices which are doubly stochastic have all positive integral powers doubly stochastic. This can be seen by applying lemma 2.1 to both the matrix and its transpose. Thus in either case, all the matrices in the direct sum have doubly stochastic mth powers; and A^m is therefore doubly stochastic. Condition (II) assures us that A itself is not doubly stochastic, since at least one of the matrices in the direct sum satisfies the conditions of theorem 2.1 and is therefore not doubly stochastic. We have now shown that conditions (I) and (II) are both necessary and sufficient.

Examples of reducible matrices which are not doubly stochastic but have a doubly stochastic mth power may be obtained by taking a permutation of a direct sum of matrices, each of which is either doubly stochastic or satisfies the conditions of theorem 2.1; and at least one of them must satisfy the conditions of theorem 2.1 in order to keep the matrix from being doubly stochastic.

EXAMPLE 3.1. The following matrix is a direct sum of a doubly stochastic matrix and the transpose of the matrix used in example 2.2:

$$A = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix} .$$

Then

$$A^2 = \begin{bmatrix} 5/9 & 4/9 & 0 & 0 & 0 & 0 & 0 \\ 4/9 & 5/9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} ,$$

which is doubly stochastic.

Since all nonnegative matrices are either reducible or irreducible, theorems 2.1 and 3.1 give a complete characterization of those nonnegative square matrices which are not doubly stochastic but have some doubly stochastic power.

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