LEBESGUE-STIELTJES INTEGRATION

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

George O. Golightly

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ABSTRACT

In the following, we shall develop enough of the theory of Lebesgue-Stieltjes integration to define the measure of a number set with respect to a non-decreasing function. Our development partially parallels that of F. Riesz's development of the Lebesque integral.

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CHAPTER I

THE CLASS $C_0(a)$

We shall begin by stating two preliminary assumptions and establishing some notation used in the sequel.

Assumption I. If a is a non-decreasing function from the set of all real numbers then there are at most countably many numbers x at which a is not continuous and for each such number x the limits $a(x+)$ and $a(x-)$ exist.

Assumption II. If M is a closed, bounded number set and f_1, f_2, f_3,... is a non-increasing sequence of non-negative functions, whose restrictions to M are continuous, and if the sequence (f_i) converges on M to the zero function then (f_i) converges uniformly on M.

Henceforth, all functions are assumed to be real-valued and to have domain the set of all real numbers. We shall use the letter "a" to denote some non-decreasing function.

If S is a collection, S^* denotes the union of the elements of S. If M is a number set, $\overline{M}$ denotes the closure of M.

The statement that f is a step function means that f is a function and there is a finite collection H of mutually exclusive open intervals such that $\overline{H^*}$ fills up some closed interval $[c,d]$, a is continuous at each end-number of an open interval in H, f is constant on each interval of H and zero off $[c,d]$. Such a collection H will be called a defining collection for the function f. The collection of all step functions will be denoted $C_0(a)$.

The a-length of the open interval (p,q) is the number $a(q-)-a(p+)$ and is denoted $\ell_a((p,q))$. The statement that N is an a-null set means that N is the empty set or N is a number set such that if c is a positive number there is a collection G of open intervals covering N such that the sum of the a-lengths of the elements in any finite subcollection of G is less than c.

Note 0. Suppose (A_i) is a sequence of a-null sets. Then $\{A_i | i \text{ a positive integer}\}^*$ is an a-null set.

Note 1. Suppose H and K are defining collections for the function f. Then

$$\sum (f(t_{pq})[a(q-)-a(p+)], \text{ where } t_{pq} \text{ is in } (p,q) | (p,q) \text{ in } H) =$$
$$\sum (f(t_{pq})[a(q-)-a(p+)], \text{ where } t_{pq} \text{ is in } (p,q) | (p,q) \text{ in } K).$$

The statement that j is an a-integral for the function f in $C_0(a)$ means that there is a defining collection H such that j is

$$\sum (f(t_{pq})[a(q-)-a(p+)], \text{ with } t_{pq} \text{ in } (p,q) | (p,q) \text{ in } H).$$

The a-integral for the function f in $C_0(a)$ will be denoted $L_a(f)$.

Note 2. Note 1 shows that a function f in $C_0(a)$ has only one a-integral. If b is a number and f is a function in $C_0(a)$, then bf is a function in $C_0(a)$ and $L_a(bf)$ is $bL_a(f)$. Furthermore, if f and g are functions in $C_0(a)$, then f+g is a function in $C_0(a)$ and $L_a(f+g)$ is $L_a(f)+L_a(g)$, max(f,g) and min(f,g) are functions in $C_0(a)$, and, in consequence, $|f|$ is a function in $C_0(a)$. If f is a non-negative function in $C_0(a)$, $L_a(f)$ is non-negative.

Suppose $H_1,\ldots,H_n$ are finite collections of open intervals. Suppose that if i is a positive integer in $[1,n]$ N_i denotes the set of end-numbers of open intervals in H_i. $\{N_1,\ldots,N_n\}^*$ is a finite number set $\{a_1,\ldots,a_m\}$, where $a_i < a_j$ if $i<j$ and i and j are integers in $[1,m]$. The collection $\{(a_i,a_{i+1})|$ i an integer in $[1,m-1]\}$ will be denoted $H_1 \circ H_2 \circ \ldots \circ H_n$.

Note 3. Suppose H is a collection of open intervals and c is a positive number such that if K is a finite sub-collection of H then $\sum(\ell_a(k)|k \text{ in } K) \leq c$. If P is a finite collection of mutually exclusive open intervals, each of which is a subset of H*, then $\sum(\ell_a(t)|t \text{ in } P) \leq c$.

Proof. Suppose there is a finite collection P, of mutually exclusive open intervals, each of which is a subset of H^*, such that $\sum(\ell_a(t) \mid t \text{ in } P) > c$.

There is a finite collection, T, of mutually exclusive open intervals, the closure of each member of which is a subset of P^*, such that $\sum(\ell_a(t) \mid t \text{ in } T) > c$.

There is a finite subcollection, $\{(a_1,b_1),\ldots,(a_m,b_m)\}$, of H covering $\overline{T^*}$ such that no proper subcollection of $\{(a_1,b_1),\ldots,(a_m,b_m)\}$ covers $\overline{T^*}$.

There is a collection, $\{(c_i,d_i) \mid i \text{ an integer in } [1,m]\}$, of open intervals at each end-number of which a is continuous such that if i is an integer in $[1,m]$ then (c_i,d_i) is a subset of (a_i,b_i) and such that $\overline{T^*}$ is a subset of $\{(c_i,d_i) \mid i \text{ an integer in } [1,m]\}^*$.

There is a finite collection, I, of mutually exclusive open intervals, each of which is a subset of some member of T and of only one member of $\{(c_i,d_i) \mid i \text{ an integer in } [1,m]\}$, such that $\overline{I^*}$ is $\overline{T^*}$ and $\sum(\ell_a(i) \mid i \text{ in } I) = \sum(\ell_a(t) \mid t \text{ in } T)$.

Note that if (a,b) is an open interval and S is a finite collection of mutually exclusive open intervals, each of which is a subset of (a,b), $\sum(\ell_a(s) \mid s \text{ in } S) \leq \ell_a((a,b))$.

Thus, $\sum(\ell_a(i) \mid i \text{ in } I) =$

$\sum(\sum(\ell_a(i) \mid i \text{ in } I \text{ and } i \subseteq (c_j, d_j)) \mid j \text{ an integer in } [1,m]) \leq$

$\sum(\ell_a((c_j, d_j)) \mid j \text{ an integer in } [1,m]) \leq$

$\sum(\ell_a((a_j, b_j)) \mid j \text{ an integer in } [1,m]) \leq c.$

This is a contradiction.

Theorem A. Suppose $f_1, f_2, f_3, \ldots,$ is a non-increasing sequence of non-negative functions in $C_0(a)$. Then the following two statements are equivalent:

(1) The sequence $(L_a(f_i))$ has limit zero.

(2) There exists an a-null set N such that if x is a number not in N the sequence $(f_i(x))$ has limit zero.

Proof. (1) implies (2). Suppose there is a sequence, (f_i), for which (1) holds and (2) does not hold. Suppose N denotes the set of all numbers x such that the sequence $(f_i(x))$ does not converge to zero. Then N is not an a-null set. If x is a number in N, there is a positive integer i such that if n is a positive integer $f_n(x) > 1/i$. Suppose that N_i is the set of all numbers x such that if n is a positive integer $f_n(x) > 1/i$.

Since the star of a countable collection of a-null sets is an a-null set, there is a positive integer, i, such that N_i is not an a-null set. Thus there is a positive number, c, such that if G is a collection of open intervals covering N_i there is some finite subcollection of G, the sum of the a-lengths of the elements of which is greater than c. There is a positive integer, m, such that $L_a(f_m) < c/i$.

Suppose B is a defining collection for f_m. N_i is a subset of $\overline{B^*}$. $\overline{(B^*)}-B^*$ is an a-null set. If G is a collection of open intervals covering the common part of N_i and B*, then there is a finite subcollection of G, the sum of the a-lengths of the elements of which is not less than c. Let D be the subcollection of B to which t belongs only in case t intersects N_i. D covers the common part of B* and N_i. $L_a(f_m) \geq \sum(f_m(t)\ell_a(t) | t \text{ in } D) \geq (1/i)\sum(\ell_a(t) | t \text{ in } D) \geq c/i$.

This is a contradiction. Thus, (1) implies (2).

(2) implies (1). Suppose (2) is true and (1) is not true. Then, since $(L_a(f_i))$ is a non-increasing sequence of positive numbers, there is a positive number, c, to which the sequence $(L_a(f_i))$ converges.

Suppose T is an a-null set such that if y is a number not in T the sequence $(f_i(y))$ has limit zero. Suppose G is a collection of open intervals covering T, the sum of the a-lengths of the elements of any finite subcollection of which is less than c/(4M), where M is a number such that if x is a number $|f_1(x)| < M$.

If i is a positive integer, let K_i be a defining collection for the function f_i and N_i a finite collection of open intervals covering the collection of end-numbers of open intervals in K_i such that $\Sigma(l_a(t) | t \text{ in } N_i) < c/(2^{i+3}M)$.

By assumption II, there is a positive integer, p, such that if x is a number in the complement of G* and of $\{N_i{}^* | i \text{ a positive integer}\}^*$, $f_p(x) < c/(4\Sigma(l_a(k) | k \text{ in } K_1))$.

Suppose O denotes the subcollection of K_p to which s belongs only in case s intersects the intersection of the complement of G* and of $\{N_i{}^* | i \text{ a positive integer}\}^*$. Then,
$\Sigma(f_p(x_t) l_a(t)$, with x_t in $t | t$ in O) <
$(c/(4\Sigma(l_a(k) | k \text{ in } K_1)))\Sigma(l_a(k) | k \text{ in } K_1)$. Also,
$\Sigma(f_p(x_t) l_a(t) | t \text{ in } K_p - O) < (c/2M)M$. Thus,
$\Sigma(f_p(x_t) l_a(t)$, with x_t in $t | t$ in O) +
$\Sigma(f_p(x_t) l_a(t) | t \text{ in } K_p - O) = L_a(f_p) < c$. Since $L_a(f_p) \geq c$, this is a contradiction and the proof is complete.

Theorem B. Suppose (f_i) is a non-decreasing sequence of functions in $C_0(a)$ and there is a number, K, such that if i is a positive integer $L_a(f_i) < K$. Then there is an a-null set N such that if x is a number not in N the number sequence $(f_i(x))$ converges.

Proof. Suppose that for each positive integer i C_i is a defining collection for the function f_i and M_i is the set of end-numbers of open intervals in C_i. Let N_0 be $\{M_i | i \text{ a positive integer}\}^*$. Suppose B is the least upper bound of $\{L_a(f_i) | i \text{ a positive integer}\}$. Further, suppose N is the set of all numbers x such that the sequence $(f_i(x))$ diverges.

Suppose e is a positive number and n is a positive integer such that $B-L_a(f_n) < e/2$. Let N_e be the set of all numbers x such that there is a positive integer i for which $f_i(x) > f_n(x)+1$. Note that N is a subset of $\{N_0, N_e-N_0\}^*$. Since N_0 is an a-null set, if N_e-N_0 is covered by a collection of open intervals, the sum of the a-lengths of the elements of any finite subcollection of which is less than e/2, then N is covered by a collection of open intervals, the sum of the a-lengths of the elements of any finite subcollection of which is less than e. We shall assume in the following that N_e-N_0 is not the empty set.

We assert the existence of a sequence, $K_1, K_2, \ldots$, of finite collections of open intervals such that the sum of the a-lengths of the elements of any finite subcollection of $\{K_i | i \text{ a positive integer}\}^*$ is less than e/2, and such that $\{K_i | i \text{ a positive integer}\}^*$ covers $N_e - N_0$. To support the assertion, we shall construct K_1, K_2, and K_3.

Let m_1 be the least positive integer i such that there is a number x not in $\{M_1, \ldots, M_i\}^*$ for which $f_i(x) > f_n(x)+1$. Then $m_1 > n$. Let Z be the set of all numbers x not in $\{M_1, \ldots, M_{m_1}\}^*$ such that $f_{m_1}(x) > f_n(x)+1$. There is a finite collection, X, of mutually exclusive open intervals such that X* is Z and such that X is a subset of $C_1 \circ \ldots \circ C_{m_1}$. Since $C_1 \circ \ldots \circ C_{m_1}$ is a defining collection for $f_{m_1} - f_n$, since $f_{m_1} > f_n$, and since $L_a(f_{m_1}) - L_a(f_n) < e/2$, the sum of the a-lengths of the elements of X is less than e/2. Let K_1 be X.

If K_1 covers $N_e - N_0$, let K_2 be K_1 and m_2 be m_1+1. Suppose K_1 does not cover $N_e - N_0$. There is a positive integer $i > m_1$ and a number x not in K_1* and not in $\{M_1, \ldots, M_i\}^*$ such that $f_i(x) > f_n(x)+1$. Let m_2 be the least such positive integer i and let Z be the set of all numbers x not in $\{M_1, \ldots, M_{m_2}\}^*$ and not in K_1* for which $f_{m_2}(x) > f_n(x)+1$. Note that $m_2 > m_1$.

There is a finite collection, X, of mutually exclusive open intervals whose star is Z. For if x is a number in Z then x is in some open interval s of $C_1 \circ \ldots \circ C_{m_2}$ and every number in s is in Z. Let K_2 be X.

$\{K_1,K_2\}^*$ covers the set of all numbers x not in N_0 such that there is a positive integer $i \leq m_2$ for which $f_i(x) > f_n(x)+1$. $\{K_2{}^*, K_1{}^* - M_{m_2}\}^*$ is the star of a finite collection, Y, of mutually exclusive open intervals, the sum of whose a-lengths is the sum of the a-lengths of the elements of $\{K_1,K_2\}^*$. Further, Y is a subset of a defining collection for $f_{m_2}-f_n$ and on each open interval of Y $f_{m_2}-f_n > 1$. Since $f_{m_2} \geq f_n$ and since $L_a(f_{m_2})-L_a(f_n) < e/2$ the sum of the a-lengths of the elements of $\{K_1,K_2\}^*$ is less than e/2.

Suppose $\{K_1,K_2\}^*$ does not cover N_e-N_0. There is a positive integer i such that there is a number x not in $\{M_1,\ldots,M_i\}^*$ and not in $K_1{}^*$ or in $K_2{}^*$ for which $f_i(x) > f_n(x)+1$. Suppose m_3 is the least such positive integer i. Then $m_3 > m_2$. Suppose Z is the set of all numbers x such that x is not in $\{M_1,\ldots,M_{m_3}\}^*$ and not in $K_1{}^*$ or in $K_2{}^*$ and such that $f_{m_3}(x) > f_n(x)+1$. There is a finite collection, X of mutually exclusive open intervals such that X* is Z. Let K_3 be X. If $\{K_1,K_2\}^*$ covers N_e-N_0, let m_3 be m_2+1 and K_3 be K_2.

The collection $\{K_1,K_2,K_3\}^*$ covers the set of all numbers x not in N_0 such that there is a positive integer $i \leq m_3$ for which $f_i(x) > f_n(x)+1$. The sum of the a-lengths of the elements of $\{K_1,K_2,K_3\}^* < e/2$.

Thus, if $N_e - N_0$ is not the empty set there is a sequence, $K_1,K_2,K_3,\ldots$, of finite collections of open intervals and an increasing sequence, $m_1,m_2,m_3,\ldots$, positive integers such that if i is a positive integer the sum of the a-lengths of the open intervals in $\{K_1,\ldots,K_i\}^*$ is less than e/2, and $\{K_1,\ldots,K_i\}^*$ covers the set of all numbers x such that x is not in N_0 and such that there is a positive integer j in $[1,m_i]$ for which $f_j(x) > f_n(x)+1$.

Suppose x is a number in $N_e - N_0$. Let L be the least positive integer i such that $f_i(x) > f_n(x)+1$. Note that $m_L \geq L$. Thus, x is in $\{K_1^*,\ldots,K_L^*\}$. The collection $\{K_i \mid i \text{ a positive integer}\}^*$ covers $N_e - N_0$ and the sum of the a-lengths of the elements in any finite subcollection of $\{K_i \mid i \text{ a positive integer}\}^*$ is less than e/2.

We have shown that either there is a positive number e such that $N_e - N_0$ is the empty set or that given any positive number e there is a collection of open intervals covering $N_e - N_0$, the sum of the a-lengths of the elements of any finite subcollection of which is less than e/2. If there is a positive number e such that $N_e - N_0$ is the empty set, then N is an a-null set. If the other case holds, then if e is a positive number there is a collection of open intervals covering N, the sum of the a-lengths of the elements of any finite subcollection of which is less than e. We conclude that N is an a-null set.

Note 4. Suppose N is an a-null set. Then there is a non-decreasing sequence $f_1, f_2, f_3, \ldots$ of functions in $C_0(a)$ and a number K such that if i is a positive integer $L_a(f_i) < K$ and (f_i) diverges on N.

Proof. We wish to show first that if i is a positive integer there is a countably infinite collection s_i of open intervals covering N such that a is continuous at each end-number of an open interval in s_i and such that the sum of the a-lengths of the elements in any finite subcollection of s_i is less than $1/2^i$.

Suppose i is a positive integer. There is a collection, G, of open intervals such that countably many of the elements of G have non-zero a-length, such that G covers N, and such that the sum of the a-lengths of the elements in any finite subcollection of G is less than $1/(2^{i+4})$.

Suppose the star of the set of all members of G having zero a-length is not the empty set. Then it is the star of a countably infinite collection, $\{B_k | k$ a positive integer$\}$, of open intervals, each of which has zero a-length and at the end-numbers of each of which a is continuous.

The subcollection, U, of G to which s belongs only in case $\ell_a(s)$ is positive is countable. Suppose that M is a set of positive integers and (t_p) a sequence such that if p and j are positive integers in M then t_p and t_j are in U and t_p is not t_j. If k is a positive integer in M, there is a countably infinite collection, $\{I_{kj} | j$ a positive integer$\}$, of open intervals such that $\{I_{kj} | j$ a positive integer$\}^*$ is t_k, such that a is continuous at each end-number of an open interval in the collection, and such that the sum of the a-lengths of the elements in any finite subcollection of it is less than $1/2^{k+i+4} + \ell_a(t_k)$.

If the star of the set of all members of G having zero a-length is the empty set, let s_i be $\{I_{kj} | k$ in M and j a positive integer$\}$. If it is not the empty set, let s_i be the collection of all open intervals in $\{I_{kj} | k$ in M and j a positive integer$\}$ or in $\{B_k | k$ a positive integer$\}$. s_i is a countably infinite collection of open intervals covering N, the sum of the a-lengths of the elements in any finite subcollection of which is less than $1/2^i$, and a is continuous at each end-number of an open interval in s_i.

Suppose that if i is a positive integer s_i is $\{s_{ij} | j$ a positive integer$\}$. If i and j are positive integers $\{s_{ik} | k$ an integer in $[1,j]\}^*$ is the star of a finite collection, X, of non-overlapping open intervals and the sum of the a-lengths of the open intervals in X does not exceed the sum of the a-lengths of the open intervals in $\{s_{ik} | k$ an integer in $[1,j]\}^*$. Note that a is continuous at each end-number of an open interval in X. Thus, if we define f_{ij} to be 1 on X* and 0 elsewhere, then f_{ij} is in $C_0(a)$ and X is a defining collection for f_{ij}. Further, $L_a(f_{ij}) \leq 1/2^i$. Note that the sequence whose jth term is (f_{ij}) is a non-decreasing sequence.

Define f_i to be $\sum(f_{ji} | j$ an integer in $[1,i])$. The sequence (f_i) is non-decreasing. If i is a positive integer, $L_a(f_i) < 2$. Also, if x is a number in N, there is an increasing sequence $m_1, m_2, \ldots,$ of positive integers such that if i is a positive integer x belongs to s_{i,m_i}. Suppose that M is a positive number. There is a positive integer K>M. There is a positive integer J>K such that if z is an integer in $[1,K]$ then $m_z < J$.

$f_J(x) > K > M$.

Thus, (f_i) diverges on N. This completes the verification.

CHAPTER II

THE CLASS $C_1(a)$

Let $C_1(a)$ denote the set such that f belongs to it only in case f is a function and there exists a non-decreasing sequence (f_i) of functions in $C_0(a)$, a number K such that if n is a positive integer $L_a(f_n) \leq K$, and an a-null set N such that if x is a number not in N the sequence $(f_i(x))$ converges to $f(x)$.

Theorem C. Suppose s and t are functions in $C_1(a)$, (s_i) and (t_i) are non-decreasing sequences of functions in $C_0(a)$, K is a number such that if n is a positive integer $L_a(s_n) \leq K$ and $L_a(t_n) \leq K$, and N is an a-null set such that if x is a number not in N the sequence $(s_i(x))$ converges to $s(x)$, the sequence $(t_i(x))$ converges to $t(x)$, and $s(x) \leq t(x)$. Then the number sequence $(L_a(s_i))$ converges to a number J_1, the number sequence $(L_a(t_i))$ converges to a number J_2, and $J_1 \leq J_2$.

Proof. Note that each of $(L_a(s_i))$ and $(L_a(t_i))$ is a non-decreasing sequence of numbers which is bounded above by the number, K. Thus, $(L_a(s_i))$ converges to a number, J_1, and $(L_a(t_i))$ converges to a number, J_2.

Suppose N is a positive integer. The sequence

$(s_n - \min(t_i, s_n))$ is a non-increasing sequence of non-negative functions in $C_0(a)$ such that if x is a number not in N the sequence $(s_n(x) - \min(t_i(x), s_n(x)))$ converges to 0. By Theorem A, the sequence $(L_a(s_n - \min(t_i, s_n)))$ converges to 0. The ith term of this sequence is $L_a(s_n) - L_a(\min(t_i, s_n))$. Thus, the number sequence $(L_a(\min(t_i, s_n)))$ converges to $L_a(s_n)$. Since the ith term of the sequence $(L_a(t_i))$ is not less than the ith term of the sequence $(L_a(\min(t_i, s_n)))$, the number, J_2, must be at least as great as $L_a(s_n)$. Thus, if n is a positive integer, $L_a(s_n) \leq J_2$. The limit, J_1, of the sequence $(L_a(s_n))$ does not exceed J_2. This completes the proof.

The phrase, "almost everywhere," has the same meaning as the phrase, "except on an a-null set".

Definition. The statement that j is an a-integral for the function f in $C_1(a)$ means that j is a number and there is a non-decreasing sequence of functions in $C_0(a)$ converging almost everywhere to f such that j is the limit of the corresponding sequence of a-integrals.

Note that if f is a function in $C_1(a)$ and (f_i) and (g_i) are non-decreasing sequences of functions in $C_0(a)$ converging almost everywhere to f such that both the sequence $(L_a(f_i))$ and the sequence $(L_a(g_i))$ converge then the limit of the sequence $(L_a(f_i))$ is the limit of the sequence $(L_a(g_i))$. This justifies the assertion that there is at most one a-integral for a function f in $C_1(a)$. Theorem B shows that each member of $C_1(a)$ has an a-integral.

Also note that each function f in $C_0(a)$ is in $C_1(a)$ and that the a-integral for f according to the preceding definition is the same as the a-integral for f according to the earlier definition. If f is in $C_1(a)$, $L_a(f)$ will denote the a-integral of f.

We wish to show that if f and g are functions in $C_1(a)$, then max(f,g) is in $C_1(a)$. There are non-decreasing sequences, $f_1, f_2, f_3, \ldots$ and $g_1, g_2, g_3, \ldots$, of functions in $C_0(a)$ such that (1.) there is a number K such that if i is a positive integer $L_a(f_i) < K$ and $L_a(g_i) < K$ and such that (2.) there is an a-null set N such that if x is a number not in N the number sequence $(f_i(x))$ converges to f(x) and $(g_i(x))$ converges to g(x). The sequence $(\max(f_i, g_i))$ is a non-decreasing sequence of functions in $C_0(a)$ converging almost everywhere to max(f,g). If i is a positive integer, $L_a(\max(f_i,g_i)) \leq L_a(|f_i| + |g_i|) \leq L_a(f_i + 2|f_1|) + L_a(g_i + 2|g_1|) < 2(K + L_a(|f_1| + |g_1|))$, since $|f_i| \leq 2|f_1| + f_i$ and $|g_i| \leq 2|g_1| + g_i$. We conclude that max(f,g) is in $C_1(a)$. The minimum of f and g is also in $C_1(a)$.

Note 5. If c and d are non-negative numbers and f and g are functions in $C_1(a)$, then $cf+dg$ is a function in $C_1(a)$ and $L_a(cf+dg)$ is $cL_a(f)+dL_a(g)$. If g is a function in $C_1(a)$ and g is non-negative almost everywhere, $L_a(g)$ is non-negative. If g is a function in $C_1(a)$ and f is a function which agrees with g almost everywhere, f is a function in $C_1(a)$ and $L_a(f)$ is $L_a(g)$.

Theorem D. Suppose $f_1, f_2, f_3, \ldots,$ is a non-decreasing sequence of functions in $C_1(a)$ and that there is a number, K, such that for every positive integer n $L_a(f_n) < K$. Then there is a function f in $C_1(a)$ such that the function sequence (f_i) converges almost everywhere to f and the number sequence $(L_a(f_i))$ converges to the number $L_a(f)$.

Proof. If i is a positive integer, there is a non-decreasing sequence, $(f(i,j))$, of functions in $C_0(a)$ which converges almost everywhere to f_i such that the sequence $(L_a(f(i,j)))$ converges to $L_a(f_i)$.

Suppose that for each positive integer i N_i is an a-null set such that if x is a number not in N_i the number sequence $(f(i,j)(x))$ converges to $f_i(x)$.

Suppose that if i is a positive integer s_i is defined to be $\max(f(1,i), \ldots, f(i,i))$. Then (s_i) is a non-decreasing sequence of functions in $C_0(a)$.

Suppose i is a positive integer. If x is a number not in $\{N_j | j \text{ a positive integer}\}^*$, then $s_i(x) \leq f_i(x)$. By Theorem C, $L_a(s_i) \leq L_a(f_i)$.

Thus, for i and j positive integers and $j \leq i$, we have $L_a(f(j,i)) \leq L_a(s_i) \leq L_a(f_i) \leq K$.

The sequence (s_i) converges almost everywhere to a function, g, in $C_1(a)$ and the sequence $(L_a(s_i))$ converges to $L_a(g)$.

Suppose N is an a-null set such that if x is a number not in N the sequence $(s_i(x))$ converges to g(x). Suppose x is a number not in N or in $\{N_i | i \text{ a positive integer}\}^*$. If i and j are positive integers and $i \leq j$, then $s_j(x) \geq f(i,j)(x)$. The limit of the sequence $(s_j(x))$ must be at least as great as the limit of the sequence whose jth term is f(i,j)(x). That is, $f_i(x) \leq g(x)$. Thus, the sequence $(f_i(x))$ has a limit. On the other hand, for each positive integer i, $f_i(x) \geq s_i(x)$. Thus, the limit of the sequence $(f_i(x))$ is g(x). The sequence (f_i) converges to g almost everywhere.

For each positive integer i, $f_i \leq g$ almost everywhere. Thus, $L_a(f_i) \leq L_a(g)$. The limit of the non-decreasing sequence $(L_a(f_i))$ exists and is less than or equal to $L_a(g)$.

If i is a positive integer, $L_a(s_i) \leq L_a(f_i)$. As the sequence $(L_a(s_i))$ converges to $L_a(g)$, we conclude that the limit of the sequence $(L_a(f_i))$ is $L_a(g)$. This completes the proof.

CHAPTER III

THE CLASS $C_2(a)$

Note 6. Suppose each of f_1, f_2, g_1 and g_2 is a function in $C_1(a)$. If $f_1-f_2=g_1-g_2$, then $L_a(f_1)-L_a(f_2)=L_a(g_1)-L_a(g_2)$.

$C_2(a)$ is the set of all functions f for which there is a function f_1 in $C_1(a)$ and a function f_2 in $C_1(a)$ such that f is f_1-f_2. In this case, an a-integral for f is $L_a(f_1)-L_a(f_2)$.

Note 7. Note 6 shows that there is only one a-integral for a function f in $C_2(a)$. Each function f in $C_1(a)$ is in $C_2(a)$ and the a-integral for the function f according to the preceding definition is the same as the a-integral for f according to the earlier definition. The a-integral of a function f in $C_2(a)$ will be denoted $L_a(f)$.

If c and d are numbers and f and g are functions in $C_2(a)$, then $cf+dg$ is a function in $C_2(a)$ and $L_a(cf+dg)= cL_a(f)+dL_a(g)$. If each of f and g is a function in $C_2(a)$, then max(f,g) and min(f,g) are functions in $C_2(a)$. If f is a function in $C_2(a)$ which is non-negative almost everywhere, then $L_a(f)$ is non-negative. If f is a function in $C_2(a)$ and g is a function which agrees with f almost everywhere then g is in $C_2(a)$ and $L_a(f)$ is $L_a(g)$.

Theorem E. Suppose (f_i) is a non-decreasing sequence of functions in $C_2(a)$ and there is a number, K, such that for each positive integer i $L_a(f_i) \leq K$. Then there is a function f in $C_2(a)$ such that the sequence (f_i) converges almost everywhere to f and the sequence $(L_a(f_i))$ converges to $L_a(f)$.

Proof. $(f_i - f_1)$ is a non-decreasing sequence of non-negative functions in $C_2(a)$. If there is a function g in $C_2(a)$ such that $(f_i - f_1)$ converges almost everywhere to g and $(L_a(f_i - f_1))$ converges to $L_a(g)$, then (f_i) converges almost everywhere to $g + f_1$ and $(L_a(f_i))$ converges to $L_a(g + f_1)$.

We first prove that if f is in $C_2(a)$, f is non-negative, and e is a positive number, then there are functions, ℓ_1 and ℓ_2, in $C_1(a)$, each of which is non-negative, such that $L_a(\ell_2) < e$ and f is $\ell_1 - \ell_2$.

To show this, we note that there are functions, h_1 and h_2, in $C_1(a)$ such that $f = h_1 - h_2$. Since h_2 is in $C_1(a)$, there is a non-decreasing sequence $(h(2,j))$, of functions in $C_0(a)$ converging almost everywhere to h_2 such that the sequence $(L_a(h(2,j)))$ converges to $L_a(h_2)$. Suppose that N is an a-null set and if x is a number not in N the sequence $(h(2,j)(x))$ converges to $h_2(x)$. Further, suppose n is a positive integer such that $L_a(h_2) - L_a(h(2,n)) < e$. If x is a number not in N, let $\ell_2(x)$ be $h_2(x)-h(2,n)(x)$. If x is in N, then let $\ell_2(x)$ be 0. Since $(h(2,j)-h(2,n))$ is a non-decreasing sequence of functions in $C_0(a)$ converging almost everywhere to $h_2-h(2,n)$ and since there is a bound for the corresponding sequence of a-integrals, $h_2-h(2,n)$ is in $C_1(a)$. Since ℓ_2 agrees with $h_2-h(2,n)$ almost everywhere, ℓ_2 is in $C_1(a)$. Finally, ℓ_2 is non-negative and $L_a(\ell_2) < e$.

Suppose x is a number in N. Let $\ell_1(x)$ be f(x). If x is a number not in N, let $\ell_1(x)$ be $h_1(x)-h(2,n)(x)$. Note that if x is a number in N, $\ell_1(x) \geq 0$. Suppose x is a number not in N. $h_1(x) \geq h_2(x) \geq h(2,n)(x)$. Thus, $\ell_1(x) \geq 0$. We conclude that ℓ_1 is non-negative. ℓ_1 is in $C_1(a)$. As f is $\ell_1-\ell_2$, we conclude the justification.

Suppose i is a positive integer. Let g_i be f_i-f_1. $L_a(g_i) \leq K+|L_a(f_1)|$. The sequence $g_1, g_2-g_1, g_3-g_2, \ldots$ is a non-decreasing sequence of non-negative functions in $C_2(a)$. If j is a positive integer, let h_j and k_j be non-negative functions in $C_1(a)$ such that $g_{j+i}-g=h_j-k_j$ and $L_a(k_j) < 1/2^{j+1}$. Further, suppose h_0 and k_0 are non-negative functions in $C_1(a)$ such that $g_1=h_0-k_0$ and $L_a(k_0) < 1/2$. The sequence $(\Sigma(h_j \mid j \text{ an integer in } [0,i]))$ is a non-decreasing sequence of functions in $C_1(a)$. $L_a(\Sigma(h_j \mid j \text{ an integer in } [0,i]))-1 \leq L_a(\Sigma((h_j-k_j) \mid j \text{ an integer in } [0,i]))=L_a(g_i) \leq K+|L_a(f_1)|$.

If i is a positive integer, $L_a(\Sigma(k_j | j$ an integer in $[0,i]))\leq 1$. $k_0, k_0+k_1, \ldots$ is a non-decreasing sequence of functions in $C_1(a)$. The sequence $(\Sigma(k_j | j$ an integer in $[0,i]))$ converges almost everywhere to a function, P, in $C_1(a)$ and the sequence $(L_a(\Sigma(k_j | j$ an integer in $[0,i])))$ converges to $L_a(P)$. The sequence $(\Sigma(h_j | j$ an integer in $[0,i]))$ converges almost everywhere to a function, H, in $C_1(a)$ and the sequence $(L_a(\Sigma(h_j | j$ an integer in $[0,i])))$ converges to $L_a(H)$.

If i is a positive integer, $g_i = \Sigma(g_{k+1}-g_k | k$ an integer in $[1,i-1]) + g_1 = \Sigma(h_j - k_j | j$ an integer in $[0,i])$. The sequence (g_i) converges almost everywhere to the function H-P in $C_2(a)$ and the sequence $(L_a(g_i))$ converges to $L_a(H)-L_a(P)$. This completes the proof

Corollary to Theorem E. If $f_1, f_2, f_3, \ldots$ is a monotonic sequence of functions in $C_2(a)$ which converges almost everywhere to the function, f, in $C_2(a)$, then $(L_a(f_i))$ converges to $L_a(f)$.

Proof. Suppose the sequence (f_i) is non-decreasing. Then if i is a positive integer, $L_a(f_i)\leq L_a(f)$. By Theorem E, there is a function g in $C_2(a)$ to which the sequence (f_i) converges almost everywhere and the sequence $(L_a(f_i))$ converges to $L_a(g)$. Since g is f almost everywhere, $L_a(g)$ is $L_a(f)$. Thus, the corollary is true for this case.

Now suppose (f_i) is a non-increasing sequence of functions. Then $(f-(f_i-f))$ is a non-decreasing sequence of functions converging almost everywhere to f. Thus, $(L_a(f)-L_a(f_i-f))$ converges to $L_a(f)$. So the sequence $(L_a(f_i-f))$ converges to 0 and the sequence $(L_a(f_i))$ converges to $L_a(f)$. This completes the justification of the corollary.

Theorem F. Suppose (f_i) is a sequence of functions in $C_2(a)$ which converges almost everywhere to the function, f, and there exists a function, g, in $C_2(a)$ such that for each positive integer n $|f_n| \leq g$. Then (1.) for each positive integer n the function, g_n, where g_n is $\sup\{f_n, f_{n+1}, \ldots\}$, is in $C_2(a)$ and the function h_n, where h_n is $\inf\{f_n, f_{n+1}, \ldots\}$, is in $C_2(a)$, (2.) the function sequences (g_i) and (h_i) converge almost everywhere to f, (3.) f is in $C_2(a)$, and (4.) the number sequences $(L_a(h_i))$, $(L_a(g_i))$, and $(L_a(f_i))$ all converge to $L_a(f)$.

Proof. (1.) Note that if h is a function in $C_2(a)$ and k is a function and h is k almost everywhere, then k is in $C_2(a)$ and $L_a(h)$ is $L_a(k)$. Suppose n is a positive integer. Let k_1 be f_n and for each positive integer i let k_{i+1} be $\sup\{k_i, f_{n+1}\}$. If i is a positive integer, $k_i \leq g$ and, thus,

$L_a(k_i) \leq L_a(g)$. (k_i) is a non-decreasing sequence of functions in $C_2(a)$. There is a function k in $C_2(a)$ such that (k_i) converges almost everywhere to k. k is g_n almost everywhere. Thus, g_n is in $C_2(a)$. Since $\inf\{f_n, f_{n+1}, \ldots\}$ is $-\sup\{-f_n, -f_{n+1}, \ldots\}$, h_n is in $C_2(a)$. (2.) There is an a-null set, N, such that if x is a number not in N then $(f_i(x))$ converges to f(x). Suppose x is such a number and e is a positive number. There is a positive integer m such that if n is a positive integer and $n \geq m$ then $|f(x)-f_n(x)| < e/4$. Suppose n is a positive integer and $n \geq m$. There is a positive integer $P \geq n$ such that $|f_P(x)-g_n(x)| < e/4$. Then $|f(x)-g_n(x)| \leq |f(x)-f_P(x)|+|f_P(x)-g_n(x)| < e/2$. Thus, $(g_n(x))$ converges to f(x). Also, $(h_n(x))$ converges to f(x). Thus, (2.) is established. (3.) The sequence (h_i) is a non-decreasing sequence of functions in $C_2(a)$ converging almost everywhere to a function k in $C_2(a)$ and the sequence $(L_a(h_i))$ converges to $L_a(k)$. k is f almost everywhere. Thus f is in $C_2(a)$. In establishing (4.), note that $L_a(k)$ is $L_a(f)$. Thus, $(L_a(h_i))$ converges to $L_a(f)$. By the corollary to Theorem E, $(L_a(g_i))$ converges to $L_a(f)$. If i is a positive integer, $L_a(h_i) \leq L_a(f_i) \leq L_a(g_i)$. Thus, $L_a(f_i)$ converges to $L_a(f)$.

The statement that f is an a-integrable function means that f is in $\dot{C}_2(a)$. If M is a number set, the characteristic function of M is the function f such that f(x) is 1 if x is in M and f(x) is zero if x is a number not in M. The statement that a number set is a-integrable means that its characteristic function is a-integrable. If M is an a-integrable number set, the a-measure of M, denoted $m_a(M)$, is the a-integral of the characteristic function of M. The set of all a-integrable number sets is a δ-ring of sets and the function m_a is a measure on this δ-ring in the following sense: (1.) If A and B are a-integrable number sets, $\{A,B\}^*$ is a-integrable and if, in addition, A and B are disjoint $m_a(\{A,B\}^*)$ is $m_a(A)+m_a(B)$. (2.) If A and B are a-integrable number sets and A is not a subset of B, A-B is an a-integrable number set. (3.) If (A_i) is a non-increasing sequence of a-integrable number sets, the intersection of the sets A_i is a-integrable and the a-measure of this intersection in the limit of the sequence $(m_a(A_i))$.

The statement that a function f is a-measurable means that there exists a sequence of functions in $C_0(a)$ converging to f almost everywhere. The statement that a number set is a-measurable means that its characteristic function is a-measurable.

If A is an a-measurable number set, f belongs to the collection $L_2(a,A)$ only in case f is a function, f is zero off A, and f and f^2 are a-integrable.

Suppose in the following that A is an a-measurable number set.

We shall quote a collection of definitions together with a collection of results that we have not obtained.

If f and g are functions in $L_2(a,A)$ and if c is a number, then f+g and cf are in $L_2(a,A)$.

If f and g are in $L_2(a,A)$, fg is in $C_2(a)$.

If f and g are functions in $L_2(a,A)$, then the inner product of f with g, denoted $\langle f,g\rangle$, is the number $L_a(fg)$. The norm of the function f in $L_2(a,A)$, denoted $\|f\|$ is the non-negative square root of $\langle f,f\rangle$.

If c is a number and each of f, g, and h is a function in $L_2(a,A)$, then (1.) $\langle f,g\rangle$ is a number, (2.) $\langle f,f\rangle > 0$ unless f is the zero function almost everywhere, (3.) $\langle f,g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$, (4.) $\langle cf,g\rangle = c\langle f,g\rangle$, (5.) $\langle f,g\rangle = \langle g,f\rangle$, (6.) $\|f\| > 0$ unless f is the zero function almost everywhere, (7.) $\|cf\| = |c|\,\|f\|$, and (8.) $\|f+g\| \leq \|f\| + \|g\|$.

Theorem. If (f_i) is a sequence of functions in $L_2(a,A)$, then the following two statements are equivalent: (1.) There exists a function f in $L_2(a,A)$ such that the sequence ($\| f-f_i \|$) has limit zero. (2.) For each positive number e there is a positive integer n such that if p and q are positive integers greater than n then $\| f_p - f_q \| < e$.

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