GRAPH PARAMETERS VIA OPERATOR SYSTEMS

A Dissertation Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
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Abstract

This work is an attempt to bridge the gap between the theory of operator systems and various aspects of graph theory.

We start by showing that two graphs are isomorphic if and only if their corresponding operator systems are isomorphic with respect to their order structure. This means that the study of graphs is equivalent to the study of these special operator systems up to the natural notion of isomorphism in their category. We then define a new family of graph theory parameters using this identification. It turns out that these parameters share a lot in common with the Lovász theta function, in particular we can write down explicitly how to compute them via a semidefinte program. Moreover, we explore a particular parameter in this family and establish a sandwich theorem that holds for some graphs.

Next, we move on to explore the concept of a graph homomorphism through the lens of C*-algebras and operator systems. We start by studying the various notions of a quantum graph homomorphism and examine how they are related to each other. We then define and study a C*-algebra that encodes all the information about these homomorphisms and establish a connection between computational complexity and the representation of these algebras. We use this C*-algebra to define a new quantum chromatic number and establish some basic properties of this number. We then suggest a way of studying these quantum graph homomorphisms using certain completely positive maps and describe their structure. Finally, we use these completely positive maps to define the notion of a "quantum" core of a graph.

Contents

Chapter 1

Introduction

Ever since their inception in 1969 [1], operator system have played a central role in operator algebras. Almost fifty years later, we now see an extensive theory [27] and applications [10, 34] of such mathematical objects. Still very little is known about the interaction between these objects and graph theory. This thesis is an attempt to fill this gap.

The classic work of Shannon [32] associated a *confusability graph* to a binary channel and argued that the zero error capacity of the channel was a parameter definable solely in terms of this graph and its products. Later, Lovász [22] introduced his theta function, which he showed was an upper bound for Shannon's capacity. He presented many formulas for computing his theta function, which are optimization problems over a certain vector space of matrices associated with the graph. There is now a rich literature on Lovász's theta function [20] and it plays an important role in both graph theory and binary information theory.

In analogy with the work of Shannon and Lovász, for a quantum channel, Duan, Severini, and Winter [10] have established that some notions of quantum capacity only depend on a vector space of matrices associated with the quantum channel, i.e., two quantum channels that define the same vector space have the same capacity. They argued that the study of these spaces of matrices should be treated as a kind of *non-commutative graph theory*. This is the main idea that drove our work.

The vector spaces of matrices associated with a graph by Lovász and with a quantum channel by Duan, Severini, and Winter are both examples of finite dimensional operator systems. In particular, most of the vector spaces they consider are operator systems that come from a graph. Given a graph G, we let $\mathcal{S}_G$ denote this operator system of matrices that is associated with G.

The natural notion of equivalence of operator systems is *unital, complete order isomorphism*. Our first main result shows that two graphs G and H are graph isomorphic if and only if the operator systems $\mathcal{S}_G$ and $\mathcal{S}_H$ are unitally, completely order isomorphic. Thus, there is no difference between studying graphs and studying this special family of operator systems. In particular, it should be possible to relate all graph parameters of G to properties of $\mathcal{S}_G$. In **Chapter 3**, we are more interested in the converse. Namely, we begin with parameters that are "natural" to associate with operator systems and attempt to relate them to classical graph parameters.

The Lovász theta function naturally fits this viewpoint and served as an excellent guide to look for new parameters. Quotients of operator systems [21] come equipped with two norm structures and we will show that a generalization of the theta function, introduced in [10], is an upper bound for the ratio between these two naturally occurring norms. We then define a new family of parameters of a graph using the two different quotient norms you can define on an operator system and discuss the similarities between these parameters and the Lovász theta function. More specifically, both of these norms are multiplicative with respect to the strong product of graphs and are semidefinite programs (SDP) solvable in polynomial time to some degree of precision. We end this chapter discussing a particular parameter in this family and establish a new graph theoretic condition, that if satisfied gives rise to a new Lovász "sandwich" type theorem [20].

In **Chapter 4**, we take a close look at non-local games on graphs (e.g. quantum graph homomorphisms) through the lens of C*-algebras and operator systems.

The theory of graph homomorphisms is one of the central tools of graph theory and is used in the development of the concept of the core of a graph [15]. More recently, work in quantum information theory has lead to quantum versions of many concepts in graph theory and there is an extensive literature ([7], [10], [28]). In particular, D. Roberson [31] and L. Mancinska [23] developed an extensive theory of quantum homomorphisms of graphs. D. Stahlke [34] interpreted graph homomorphisms in terms of "completely positive (CP) maps on the traceless operator space of a graph".

These papers motivated us to consider quantum and classical graph homomorphisms as special families of completely positive maps between the operator systems of the graphs.

There is not just a single quantum theory of graphs, but there are really possibly several different quantum theories depending on the validity of certain conjectures of Connes [6] and Tsirelson [19]. In earlier work on quantum chromatic numbers [29, 28], the authors studied the differences and similarities between the properties of the quantum chromatic numbers defined by the possibly different quantum theories. We wish to parallel those ideas for quantum graph homomorphisms. One technique of [28] and [11] was to show that the existence of quantum colorings was equivalent to the existence of certain types of traces on a C*-algebra affiliated with the graph and we wish to expand upon that topic here. This leads us to introduce the C*-algebra of a graph homomorphism and we will show that the existence or non-existence of various types of quantum graph homomorphisms are related to properties of this C*-algebra, e.g., whether or not it has any finite dimensional representations or has any traces. In particular, this C*-algebra helped us establish a surprising connection between the computational complexity of the quantum chromatic numbers and the representation of such algebra.

In addition, it turns out that the existence of this C*-algebra can be viewed as a new type of homomorphism between graphs. Using this new notion of a graph homomorphisms, we manage to define yet another chromatic number. We prove basic properties about this number and relate it to the quantum chromatic numbers. We also introduce an analog of the Roberson-Mancinska's projective rank [23] for this chromatic number using techniques developed in [28].

Finally, we address a question asked by Roberson in his thesis [31]: how should we define a "quantum" core of a graph? We use our correspondence between quantum graph homomorphisms and CP maps to introduce a quantum analogue of the core of a graph.

Most of the work presented in this thesis has been published and appears on [24] and [25].

Chapter 2

Preliminaries

2.1 Operator Systems

As customary, we let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on some Hilbert space $\mathcal{H}$, let $M_n := \mathcal{B}(\mathbb{C}^n)$, and let $E_{i,j}$ $1 \leq i, j \leq n$ be the canonical matrix units. We call a vector subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ ***-closed** provided $X \in \mathcal{S}$ implies that $X^* \in \mathcal{S}$, where X^* denotes the adjoint of X. We define $\mathcal{S}$ to be an **operator system** if $\mathcal{S}$ is a unital (i.e $I \in \mathcal{S}$, where I is the identity operator) $*$-closed subspace of $\mathcal{B}(\mathcal{H})$.

Operator systems are naturally endowed with a **matrix ordering** and can be axiomatically characterized in theses terms. See, for example [27]. Briefly, given any vector space $\mathcal{S}$, we let $M_n(\mathcal{S})$ denote the vector space of $n \times n$ matrices with entries from $\mathcal{S}$. We identify $M_n(\mathcal{B}(\mathcal{H})) \equiv \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ and let $M_n(\mathcal{B}(\mathcal{H}))^+$ denote the positive operators on the Hilbert space $\mathcal{H} \otimes C^n$. Given an operator system $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, we set $M_n(\mathcal{S})^+ = M_n(\mathcal{B}(\mathcal{H}))^+ \cap M_n(\mathcal{S})$. We set $\mathcal{S}^+ := \mathcal{B}(\mathcal{H}))^+ \cap \mathcal{S}$.

The natural notion of equivalence between two operator systems is **unital, complete order isomorphism**. First, we need to define the notion of a positive map.

Let $\mathcal{S}$ and $\mathcal{T}$ be two operator systems and let $\phi : \mathcal{S} \to \mathcal{T}$ be a linear map. A linear map ϕ is called **positive** if $s \in \mathcal{S}^+ \implies \phi(s) \in \mathcal{T}^+$. Also, notice that from a linear map $\phi : \mathcal{S} \to \mathcal{T}$, we can obtain maps $\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ via the formula $\phi_n((a_{i,j})) = (\phi(a_{i,j}))$. Now, a linear map $\phi : \mathcal{S} \to \mathcal{T}$ is called **completely positive** provided that for all k, ϕ_k is positive, that is, $(X_{i,j}) \in M_k(\mathcal{S})^+$ implies that $(\phi(X_{i,j})) \in M_k(\mathcal{T})^+$. The map ϕ is **unital** provided $\phi(I) = I$. The map ϕ is called a **complete order isomorphism** if and only if ϕ is one-to-one, onto and ϕ and ϕ^{-1} are both completely positive. This last condition is equivalent to requiring that for all n, $(X_{i,j}) \in M_n(\mathcal{S})^+$ if and only if $(\phi(X_{i,j})) \in M_n(\mathcal{T})^+$.

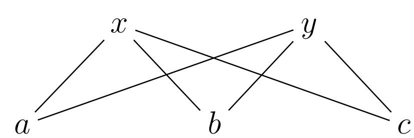
Another parameter of interest that plays an important role in the theory of operator systems is the completely bounded norm. Recall that given a linear map $\phi : \mathcal{S} \to \mathcal{T}$, the **norm** of ϕ, denoted $||\phi||$, is given by $||\phi|| = \sup\{||\phi(x)|| : ||x|| \leq 1\}$. We define the **completely bounded norm** of a linear map $\phi : \mathcal{S} \to \mathcal{T}$ to be

$$||\phi||_{cb} := \sup_{n \in \mathbb{N}} \{||\phi_n||\}.$$

For a complete discussion on the theory of operator systems, see [27].

2.2 Graphs

We define a **graph** G on n vertices to be a pair of sets, $(V(G), E(G))$, where $V(G)$ is called the *vertex set* of G, and $E(G) \subset V(G) \times V(G)$ is called the *edges* of G, and $|V(G)| = n$. All graphs we are going to be considering are *simple* (i.e. $(i, i) \notin E(G)$ for all $i \in V(G)$), *undirected* ($(i, j) \in E(G) \iff (j, i) \in E(G)$) and *finite* ($V(G)$ a finite set). We say that two vertices i and j are *connected* if $(i, j) \in E(G)$. On occasions we will write $i \sim_G j$ instead of $(i,j) \in E(G)$ and when there is no source for confusion we will simply write $i \sim j$. We let $\overline{G}$ denote the **complement of the graph** G, that is, the graph with the property that $(i,j) \in E(\overline{G}) \iff (i,j) \notin E(G)$. We let K_n denote the **complete graph** on n vertices, i.e. all vertices are connected. A **complete bipartite graph** $B(k,l)$ is a graph whose vertex set is given by two disjoint subset V_1 and V_2 of $V(B(k,l))$ where $|V_1| = l$ and $|V_2| = k$ with the property that $(i,j) \in E(B(k,l)) \iff i \in V_1$ and $j \in V_2$, i.e. for which each of the k-vertices connects to each of the l-vertices, and no other vertices are connected. For example, $B(2,3)$ looks like



We say that a graph H is a **subgraph** of G, denoted by $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Also, we say that a graph H is an **induced subgraph** of G, if $(i,j) \in E(H) \iff (i,j) \in E(G)$, for all $i,j \in V(H)$. A **path** of length k of a graph G is a subgraph $P_k \subset G$ where $V(P_k) = \{i_1, \ldots, i_{k+1}\}$, $E(P_k) = \{(i_1,i_2),(i_2,i_3),\cdots,(i_k,i_{k+1})\}$, and the i_l's have mutually different indices. A graph G is said to be **connected** if for every $x,y \in V(G)$ there exists a path P_k such that $i_1 = x$ and $i_{k+1} = y$ for some k. If a graph G is not connected, then the vertex set of G can always be partitioned into maximal disjoint sets, that is, $I_1 \dot\cup I_2 \cdots \dot\cup I_l = V(G)$ such that the induced subgraph on each I_k is connected but the induced subgraph on each $I_k \dot\cup I_h$ is disconnected, if $k \neq h$. Moreover, the induced subgraph on every I_k is called a **(connected) component** of G.

Given graphs G and H a **graph homomorphism** from G to H is a mapping

$f : V(G) \to V(H)$ such that

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H).$$

If a graph homomorphism from G to H exists we write $G \to H$. If f is a bijection and f^{-1} is a graph homorphism, then we say f is a **graph isomorphism** and the two graphs are **isomorphic**.

Given a graph G and a set $\{1, \dots, c\} \subset \mathbb{N}$, a c**-coloring** of G is a map $f\colon V \to \{1, \dots, c\}$ such that whenever $v \sim w \implies f(v) \neq f(w)$. The **chromatic** or **coloring number** of G, denoted by $\chi(G)$, is the least c for which there is a c-coloring of G. Notice that this definition is equivalent to,

$$\chi(G) = \min\{c : G \to K_c\}$$

A subgraph $C \subset G$ is called a **clique** in G if $C = K_m$ for some m. We define the **clique number**, $\omega(G)$ to be the order of the largest clique in G. Notice that,

$$\omega(G) = \max\{m : K_m \to G\}$$

Let

$$R_G := I + A_G$$

where I is the identity matrix and

$$A_G := \sum_{(i,j) \in E(G)} E_{i,j}$$

denotes the usual adjacency matrix of G.

Given a self-adjoint $n \times n$ matrix A, we let $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ denote the eigenvalues of A. It is known that $\lambda_1(A_G) \geq -\lambda_n(A_G)$, $\|A_G\| = \lambda_1(A_G)$, and $\|R_G\| = 1 + \lambda_1(A_G)$ [33].

For graphs G and H on n and m vertices, respectively. We define $G \boxtimes H$ to be the **strong product** of the graphs, that is, the graph on nm vertices, $V(G) \times V(H)$ with,

$$
\begin{aligned}
&((i,j),(k,l)) \in E(G \boxtimes H) \iff \\
&(i,k) \in E(G) \text{ and } j = l \text{ or} \\
&(j,l) \in E(H) \text{ and } i = k \text{ or} \\
&(i,k) \in E(G) \text{ and } (j,l) \in E(H)
\end{aligned}
$$

Other graph products we will briefly encounter in Chapter 4 are the tensor product and box product. We define the **tensor product** of two graphs, $G \times H$, to be the graph on nm vertices, $V(G) \times V(H)$, with,

$$
\begin{aligned}
&((i,j),(k,l)) \in E(G \times H) \iff \\
&(i,k) \in E(G) \text{ and } (j,l) \in E(H)
\end{aligned}
$$

We define the **box product** of two graphs, $G \square H$, to be the graph on nm vertices, $V(G) \times V(H)$, with,

$$
\begin{aligned}
&((i,j),(k,l)) \in E(G \square H) \iff \\
&i = k \text{ and } (j,l) \in E(H) \text{ or} \\
&(i,k) \in E(G) \text{ and } j = l
\end{aligned}
$$

2.3 The Operator System of a Graph

We define the **operator system of the graph** G to be

$$\mathcal{S}_G := \text{Span}\{E_{ij} : (i,j) \in E(G) \textit{ or } i = j\}.$$

Note that M_k is a Hilbert space with respect to the inner product $\langle a, b\rangle = \text{tr}(ab^*)$, $a, b \in M_k$. Thus, given any subspace $\mathcal{S} \subseteq M_n$, one may form the orthogonal complement $\mathcal{S}^{\perp}$ of $\mathcal{S}$. Now given a graph G on k vertices it follows that,

$$\mathcal{S}_G^{\perp} = \text{span}\{E_{i,j} : (i,j) \in E(\overline{G})\}.$$

Given that $A \subseteq B$ are unital algebras with same unit and a subspace $V \subseteq B$, we say that V is an A**-bimodule** if $AVA = \{a_1 v a_2 : a_1, a_2 \in A, v \in V\} \subseteq V$. Let $\mathcal{D}_n$ denote the set of diagonal matrices in M_n.

Proposition 2.3.1. *Given a graph G on n vertices, $\mathcal{S}_G \subseteq M_n$ is a $\mathcal{D}_n$-bimodule.*

Proof. It suffices to show that $E_{i,j} \in \mathcal{S}_G \implies D_1 E_{i,j} D_2 \in \mathcal{S}_G$ for every $D_1, D_2 \in \mathcal{D}_n$. But $D_1 E_{i,j} D_2 = \alpha E_{i,j}$ where $\alpha = [D_1]_{i,i}[D_2]_{j,j}$ and hence $D_1 E_{i,j} D_2 \in \mathcal{S}_G$. □

We are interested in giving an abstract characterization of $\mathcal{S}_G$ but first we need the following lemma:

Lemma 2.3.2. *Let $V \subseteq M_n$ be a $\mathcal{D}_n$-bimodule. Then $V = span\{E_{i,j} : E_{i,j} \in V\}$*

Proof. The backward containment is trivial. For the forward containment let $A = [a_{i,j}] \in V$. Then $A = \sum_{i,j=1}^n a_{i,j} E_{i,j}$. If we only show that $E_{i,j} \in V$ whenever $a_{i,j} \neq 0$ then we are done. To this end, note that $I = \sum_{i=1}^n E_{i,i} = \sum_{j=1}^n E_{j,j}$ so that

$A = IAI = \sum_{i,j=1}^{n} E_{i,i}AE_{j,j}$. A moment's thought will convince you that $E_{i,i}AE_{j,j}$ will be an $n \times n$ matrix with all it's entries 0 except possibly the one in $(i,j)^{th}$ slot, which is $a_{i,j}$. With this insight, we note that each $E_{i,i}AE_{j,j} \in V \implies a_{i,j}E_{i,j} \in V$. Thus, $a_{i,j} \neq 0 \implies E_{i,j} \in V$ (because then there exist two diagonal matrices $D_1 = I$ and $D_2 = \frac{1}{a_{i,j}}E_{j,j}$ so that $D_1 a_{i,j} E_{i,j} D_2 \in V$). Hence $A = \sum_{i,j=1}^{n} a_{i,j}E_{i,j} \in \text{span}\{E_{i,j} : E_{i,j} \in V\}$. Since $A \in V$ is arbitrary, $V \subseteq \text{span}\{E_{i,j} : E_{i,j} \in V\}$. This completes the proof. □

Theorem 2.3.3. *Let $\mathcal{S} \subseteq M_n$ be an operator system. Then $\mathcal{S}$ is a $\mathcal{D}_n$-bimodule if and only if there exists a graph G on n vertices such that $\mathcal{S} = \mathcal{S}_G$.*

Proof. Since $\mathcal{S} \subset M_n$ is a $\mathcal{D}_n$-bimodule, by the previous lemma $\mathcal{S} = \text{span}\{E_{i,j} : E_{i,j} \in V\}$. Define $G := (V(G), E(G))$, where $V(G) = \{1, \ldots, n\}$ and $E(G) = \{(i,j) : i \neq j, E_{i,j} \in S\}$. It follows then that $(i,j) \in E(G) \implies E_{i,j} \in \mathcal{S}$. Now $\mathcal{S}$ being an operator system and thus closed under adjoint implies $E_{i,j}^* \in \mathcal{S}$. But $E_{i,j}^* = E_{j,i} \in S$ which in turn implies that $(j,i) \in E(G)$. So, G is a graph. The Operator System $\mathcal{S}_G$ $:= \text{span}(\{E_{i,j} : i \neq j, E_{i,j} \in \mathcal{S}\} \cup \{E_{i,i} : 1 \leq i \leq n\})$. Note that since $I_n \in \mathcal{S}$, we have for each $1 \leq i \leq n$, $E_{i,i} = E_{i,i}I_nE_{i,i} \in \mathcal{S}$ and hence we have $\mathcal{S}_G := \text{span}(\{E_{i,j} : i \neq j, E_{i,j} \in \mathcal{S}\} \cup \{E_{i,i} : 1 \leq i \leq n\}) = \text{span}(\{E_{i,j} : i \neq j, E_{i,j} \in \mathcal{S}\} \cup \{E_{i,i} : E_{i,i} \in \mathcal{S}\}) = \text{span}(\{E_{i,j} :, E_{i,j} \in \mathcal{S}\}) = \mathcal{S}$. This completes the proof.

□

Remark 2.3.4. There are finite dimensional operator systems that can not be embedded in M_n. For example, $\mathcal{S} = \text{Span}\{1, e^{i\theta}, e^{-i\theta}\} \subset C(\mathbb{T})$ is a finite dimensional operator system that cannot be embedded in the matrices [27].

Finally, given two graphs G and H, it turns out that the operator system of the graph $G \boxtimes H$ correspond to tensoring the operator systems of G and H, namely,

$$\mathcal{S}_{G\boxtimes H} = \mathcal{S}_G \otimes \mathcal{S}_H$$

This is an important property that we will be using in the Chapter 3.

2.4 Lovász Theta Function

The **Lovász theta function** of a graph G is defined to be

$$\vartheta(G) = \min_{||c||=1,\{u_i\}} \left(\max_{1\leq i\leq n} \frac{1}{|\langle u_i, c\rangle|^2} \right), \tag{2.1}$$

where $\{u_i\}_{i=1}^n$ is an orthonormal representation of G on a real Hilbert space. An **orthogonal representation** of G in a Hilbert space $\mathcal{H}$ is a subset $\{\psi(i)\}_{i\in V(G)} \subset \mathcal{H}$, where $\psi : V(G) \to \mathcal{H}$ is an injective map with the property that if $(i, j) \notin E(G)$ then $\langle \psi(i), \psi(j)\rangle = 0$.

In his paper Lovász [22] only considered real Hilbert spaces and real matrices. We will set $\vartheta_{\mathbb{C}}(G)$ equal to the same quantity as above but where we allow the Hilbert spaces to be complex. It turns out that both quantities are equal, which we show below.

We present a couple of equivalent formulations of $\vartheta(G)$ that we will be using throughout our discussion. The proof that all these equvalent formulations hold in the complex case is identical to the original proof Lovász provided in his paper. See [22] for the details.

Theorem 2.4.1. *[22, Theorem 3]*

$$\vartheta(G) = \min\{\lambda_1(A)\colon A = A^t \in M_n(\mathbb{R}), a_{ij} = 1, \text{ for } i = j \text{ or } (i,j) \notin E(G)\}.$$

Theorem 2.4.2. *[22, Theorem 5]*

$$\vartheta(G) = \max\left\{ \sum_{i=1}^{n} |\,\langle v_i, d\rangle\,|^2 \colon ||d|| = 1, \right.$$
$$\left. \{v_i\} \text{ is a real orthonormal representation of } \overline{G} \right\}.$$

Corollary 2.4.3.

$$\vartheta(G) = \max\{||I + K||\colon K \in \mathcal{S}_G^{\perp} \cap M_n(\mathbb{R}), I + K \geq 0\}$$

where $\|A\|$ is the operator norm of the matrix A.

Proof. For $V = (v_1 \dots v_n)$, where $\{v_i\}$ is an real orthonormal representation of G, observe that

$$V^*d = \begin{pmatrix} \langle d, v_1\rangle \\ \vdots \\ \langle d, v_n\rangle \end{pmatrix}$$

and $||V^*d||^2 = \sum_{i=1}^{n} |\,\langle v_i, d\rangle\,|^2$. By the last theorem and letting d vary, $\vartheta(G) = \max\{||V^*||^2\}$, over all orthonormal representations $\{v_i\}$ of $\overline{G}$. The desired formula now follows from the C*-identity and the fact that $V^*V = (\langle v_j, v_i\rangle) = I + K$, for some $K \in \mathcal{S}_G^{\perp} \cap M_n(\mathbb{R})$. □

Corollary 2.4.4.

$$\vartheta_{\mathbb{C}}(G) = \max\{||I+K||\colon K \in \mathcal{S}_G^{\perp}, I+K \geq 0\},$$

where $||A||$ *is the operator norm of the matrix* A*.*

Proof. Use the same proof as above. □

Theorem 2.4.5. $\vartheta(G) = \vartheta_{\mathbb{C}}(G)$.

Proof. The original definition expresses $\vartheta(G)$ and $\vartheta_{\mathbb{C}}(G)$ as the minimum of a quantity over all real, respectively, complex, orthonormal representations of the graph. Since there are more complex representations, we have that $\vartheta_{\mathbb{C}}(G) \leq \vartheta(G)$.

But the last result expresses these quantities as the maximum norm of a family of matrices. Since there are more such complex matrices than real matrices, we have that, $\vartheta(G) \leq \vartheta_{\mathbb{C}}(G)$, and equality follows. □

2.5 Non-local Games on Graphs

In this section we present the background necessary for Chapter 4.

A finite family of operators $\{M_m\}_{m=1}^n$ on a Hilbert Space $\mathcal{H}$ is called a **measurement system** provided that $\sum_{m=1}^n M_m^* M_m = I$.

Remark 2.5.1. The motivation behind this definition stems from the interpretation of the number $\|M_m\phi\|^2 = \langle \phi, M_m^* M_m \phi\rangle$ as the probability of observing outcome m starting from a state $\phi \in \mathcal{H}$. As the sum of the probabilities of all possible outcomes must equal 1, we have $\langle \phi, \sum_m M_m^* M_m \phi\rangle = 1$ for each $\phi \in \mathcal{H}$ of norm one, and this forces $\sum_m M_m^* M_m = I$.

A set $\{P_i\}_{i=1}^n$ of operators is called a **projection valued measure (PVM)** provided they are a measurement system and $P_i = P_i^* = P_i^2$, for each i.

Remark 2.5.2. For the purposes of our discussion we will only consider PVMs and not Positive Operator Value Measures (POVMs) since given POVM we can always apply Stinespring's dilation theorem to get a PVM on a bigger Hilbert space that when restricted to the original space yields the POVM. Moreover, if the original space is finite dimensional, the bigger space is finite dimensional. See [27] and [4].

Consider the following scenario. Suppose two non-communicating players, Alice and Bob, each receives an input from some finite set I and each must produce an output belonging to some finite set O.

The "rules" of the game are given by a function

$$\lambda : I \times I \times O \times O \to \{0, 1\}$$

where $\lambda(v, w, x, y) = 0$ means that if Alice and Bob receive inputs v, w, respectively, then producing respective outputs x, y is "disallowed". We define a **game** $\mathcal{G}$ to be the tuple $\mathcal{G} = (I, O, \lambda)$.

A **strategy** for such a game is a conditional probability density p where $p(x, y|v, w)$ represents the probability that if Alice receives input v and Bob receives input w, then they produce outputs x and y, respectively.

Such a strategy is called **winning** or **perfect** provided:

$$\lambda(v, w, x, y) = 0 \implies p(x, y|v, w) = 0.$$

We call p **synchronous** if $p(x, y|v, v) = 0,\ \forall x \neq y$ [28].

Let us look at a couple of examples of games:

Example 2.5.3. Let $G = (V, E)$ be a graph with vertex set V and edges $E \subset V \times V$, the inputs are $I = V$ and the outputs O are a set of colors. The rules are that,

- $\lambda(v, v, x, y) = 0$, $\forall v \in V$, $\forall x \neq y$
- $\lambda(v, w, x, x) = 0$, $\forall (v, w) \in E$, $\forall x \in O$

We call this game the **graph coloring game**.

The reason behind the name is that any winning strategy p for this game with values in $\{0, 1\}$ gives you a $|O|$-coloring.

Example 2.5.4. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the game where inputs are $V(G)$ and the outputs are $V(H)$ and the rules are that,

- $\lambda(v, v, x, y) = 0$, $\forall v \in V(G)$, $\forall x \neq y$
- $\lambda(v, w, x, y) = 0$, $\forall (v, w) \in E(G)$, $\forall (x, y) \notin E(H)$.

We call this game the **graph homomorphism game**.

Again, the reason behind defining this game is because it extends the notion of a graph homomorphism. Any winning strategy p for this game with values in $\{0, 1\}$ gives you a graph homomorphism from G to H. Also, notice that the coloring game is a special case of the homomorphism game, where $H = K_c$, $c = |O|$ is the number of colors. For an overview of non-local games, see [3].

A density p is called a **local** or **classical correlation** if there is a probability space (Ω, μ) and random variables,

$$f_v, g_w : \Omega \to O \text{ for each } v, w \in I$$

such that,

$$p(x,y|v,w) = \mu(\{\omega \mid f_v(\omega) = x, g_w(\omega) = y\})$$

A density p is called a **quantum correlation** if it arises as follows:

Suppose Alice and Bob have finite dimensional Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ and for each input $v \in I$ Alice has PVMs $\{F_{v,x}\}_{x\in O}$ on $\mathcal{H}_A$ and for each input $w \in I$ Bob has PVMs $\{G_{w,y}\}_{y\in O}$ on $\mathcal{H}_B$ and they share a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$, then

$$p(x,y|v,w) = \langle F_{v,x} \otimes G_{w,y}\psi, \psi\rangle$$

This is the probability of getting outcomes x, y given that they conducted experiments v, w.

A density p is called a **quantum commuting correlation** if there is a single Hilbert space $\mathcal{H}$, such that for each $v \in I$ Alice has PVMs $\{F_{v,x}\}_{x\in O}$ on $\mathcal{H}$ and for each $w \in I$ Bob has PVMs $\{G_{w,y}\}_{y\in O}$ on $\mathcal{H}$ satisfying,

$$F_{v,x}G_{w,y} = G_{w,y}F_{v,x},\ \forall v, w, x, y$$

and

$$p(x,y|v,w) = \langle F_{v,x}G_{w,y}\psi, \psi\rangle$$

where $\psi \in \mathcal{H}$ is a shared state.

Let $n := |I|$ and $m := |O|$, we let:

- $C_{loc}(n,m)$ denote the set of all densities that are local correlations
- $C_q(n,m)$ denote the set of all densities that are quantum correlations

- Set $C_{qa}(n,m) := \overline{C_q(n,m)}$, the closure of $C_q(n,m)$.
- $C_{qc}(n,m)$ denote the set of all densities that are quantum commuting correlations
- For $x \in \{loc, q, qa, qc\}$, we let $C_x^s(n,m)$ denote the set of synchronous correlations in $C_x(n,m)$.

Remark 2.5.5. Note that we can view each density p as a n^2m^2-tuple where each value is given by $p(x,y|v,w)$.

Here is what is known and why these objects are interesting.

- $C_{loc}(n,m) \subseteq C_q(n,m) \subseteq C_{qa}(n,m) \subseteq C_{qc}(n,m)$.
- $C_{loc}(n,m)$ and $C_{qc}(n,m)$ are closed.
- "Bounded Entanglement Problem": Is $C_q(n,m) = C_{qa}(n,m)\ \forall n,m$?
- Tsirelson conjecture [19]: Is $C_q(n,m) = C_{qc}(n,m)\ \forall n,m$?
- Ozawa [26] proved that Connes' embedding conjecture [6] is true if and only if $C_{qa}(n,m) = C_{qc}(n,m)$, $\forall n,m$.
- Paulsen and Dykema [11] proved that Connes' embedding conjecture is true if and only if $\overline{C_q^s(n,m)} = C_{qc}^s(n,m)$, $\forall n,m$.
- "The synchronous approximation problem": Is $\overline{C_q^s(n,m)} = C_{qa}^s(n,m)\ \forall n,m$?
- Tsirelson's conjecture $\implies$ Connes' embedding conjecture.

Remark 2.5.6. In [28] the authors asked the following questions: Can we distinguish these sets of correlations by studying existence of winning strategies for highly combinatorial games? Or conversely provide some evidence for the truth of these conjectures by showing no difference in existence? In Chapter 4, we look for clues to answer these question by studying the graph homomorphism game.

Chapter 3

Lovász Theta-Type Norms via Operator Systems

3.1 The Isomorphism Theorem

In this section we prove that two graphs are isomorphic if and only if their operator systems are unitally, completely order isomorphic. This shows that the morphism $G \to \mathcal{S}_G$ in a certain sense loses no information. It suggests that there should be a dictionary for translating graph theoretical parameters into parameters of these special operator systems, which one could then hope to generalize to all operator systems. In particular, the "isomorphism" problem for operator subsystems of M_n is at least as hard as the isomorphism problem for graphs.

First, we do the "easy" equivalence. Suppose that we are given two graphs G_1, G_2 on n vertices that are isomorphic via a permutation $\pi : V(G_1) \to V(G_2)$, so that $E(G_2) = \{(\pi(i), \pi(j)) : (i,j) \in E(G_1)\}$. If we define a linear map $U_\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ via $U_\pi(e_j) = e_{\pi(j)}$, where $\{e_j : 1 \leq j \leq n\}$ denotes the canonical orthonormal basis for

$\mathbb{C}^n$, then it is not hard to see that U_π is a unitary matrix and that $U_\pi^* \mathcal{S}_{G_2} U_\pi = \mathcal{S}_{G_1}$. Moreover, the map $\phi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ defined by $\phi(X) = U_\pi^* X U_\pi$ is a unital, complete order isomorphism. Hence, the restriction $\phi : \mathcal{S}_{G_2} \to \mathcal{S}_{G_1}$ is a unital, complete order isomorphism between the operator systems of the graphs.

Conversely, if there exists a permutation such that $U_\pi^* \mathcal{S}_{G_2} U_\pi = \mathcal{S}_{G_1}$, then G_1 and G_2 are isomorphic via π. To see this, note that we have

$$\begin{aligned}(U_\pi E_{i,j} U_\pi^{-1})(e_k) &= U_\pi E_{i,j} e_{\pi^{-1}(k)} \\ &= U_\pi e_i \quad (\text{whenever} \quad j = \pi^{-1}(k) \quad \text{and} \quad 0 \quad \text{otherwise}). \\ &= e_{\pi(i)} \quad (\text{since} \quad j = \pi^{-1}(k) \implies \pi(j) = k)\end{aligned}$$

Thus $U_\pi E_{i,j} U_\pi^{-1} = E_{\pi(i),\pi(j)}$.

The next result arrives at the same conclusion even when the unitary is not induced by a permutation.

Proposition 3.1.1. *Let G_1 and G_2 be graphs on n vertices. If there exists a unitary U such that $U^* \mathcal{S}_{G_1} U = \mathcal{S}_{G_2}$, then G_1 and G_2 are isomorphic.*

Proof. Let $P_k = U^* E_{k,k} U$, $k = 1, \ldots, n$ and $\mathcal{C} = \text{span}\{P_k : k = 1, \ldots, n\}$. Since $\mathcal{S}_{G_1}$ is a bimodule over the algebra $\mathcal{D}_n$ of all diagonal matrices by 2.3.3, $\mathcal{S}_{G_2}$ is a bimodule over $\mathcal{C}$. Note that each P_k is a rank one operator.

Write $P_1 = (\lambda_i \overline{\lambda_j})_{i,j=1}^n$. Set $\Lambda_1 = \{i : \lambda_i \neq 0\}$, and renumber the vertices of G_2 so that $\Lambda_1 = \{1, 2, \ldots, k\}$, for some $k \leq n$. Suppose that $E_{i,j} \in \mathcal{S}_{G_2}$ for some $i \in \{1, \ldots, k\}$ and some $j > k$. We have that the matrix $P_1 E_{i,j}$ has as its (l, j)-entry, where $l \in \{1, \ldots, k\}$, the scalar $\lambda_l \overline{\lambda_i} \neq 0$. Since S_{G_2} is a $\mathcal{D}_n$-bimodule, it follows that if $(i, j) \in E(G_2)$, where $i \in \{1, \ldots, k\}$ and $j > k$, then $(l, j) \in E(G_2)$ for all

$l = 1, \dots, k$.

It now follows that if $W_1 \in M_n$ is a unitary matrix of the form $W_1 = V \oplus I_{n-k}$, where $V \in M_k$ is unitary and I_{n-k} is the identity of rank $n-k$, then $W_1^* \mathcal{S}_{G_2} W_1 = \mathcal{S}_{G_2}$. Choose such a W_1 with the property that $V^* P_1 V = E_{1,1}$. Then $W_1^* U^* \mathcal{S}_{G_1} U W_1 = \mathcal{S}_{G_2}$ and $W_1^* U^* E_{1,1} U W_1 = E_{1,1}$.

Now let $Q_2 = W^* U^* E_{2,2} U W$; then Q_2 is a rank one operator in $\mathcal{S}_{G_2}$; write $Q_2 = (\mu_i \overline{\mu_j})_{i,j=1}^n$ and set $\Lambda_2 = \{i : \mu_i \neq 0\}$. Since $E_{1,1} E_{2,2} = E_{2,2} E_{1,1} = 0$, we have that $E_{1,1} Q_2 = Q_2 E_{1,1} = 0$. This implies that $1 \notin \Lambda_2$. Now proceed as in the previous paragraph to define a unitary $W_2 \in M_n$ such that $W_2^* W_1^* U^* \mathcal{S}_{G_1} U W_1 W_2 = \mathcal{S}_{G_2}$ and, after a relabeling of the vertices of G_2, we have that $W_2^* W_1^* U^* E_{1,1} U W_1 W_2 = E_{1,1}$ and $W_2^* W_1^* U^* E_{2,2} U W_1 W_2 = E_{2,2}$.

A repeated use of the above argument shows that, up to a relabeling of the vertices of G_2, we may assume that there exists a unitary $W \in M_n$ such that $W^* \mathcal{S}_{G_1} W = \mathcal{S}_{G_2}$ and $W^* E_{i,i} W = E_{i,i}$ for each i. But this means that $We_i = \lambda_i e_i$ with $|\lambda_i| = 1$ for each i (here $\{e_i\}$ is the standard basis of $\mathbb{C}^n$). Hence W is a diagonal unitary, and so $W^* \mathcal{S}_{G_1} W = \mathcal{S}_{G_1}$ and so up to re-ordering, $\mathcal{S}_{G_1} = \mathcal{S}_{G_2}$, which implies that G_1 is isomorphic to G_2.

□

Given any operator system $\mathcal{S}$, each time we choose a unital complete order embedding $\gamma : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ we can consider the C*-algebra generated by the image, $C^*(\gamma(S)) \subseteq \mathcal{B}(\mathcal{H})$. The theory of the C*-envelope guarantees that among all such generated C*-algebras, there is a universal quotient, denoted $C_e^*(\mathcal{S})$ and called the *C*-envelope* of $\mathcal{S}$. See [27, Chapter 15].

Before we move on to our next theorem we need the following remark:

Remark 3.1.2. Given $(i_1, i_2), (i_2, i_3) \in E(G)$, notice that $E_{i_1i_2}E_{i_2i_3} \in \mathcal{S}_G$ and $E_{i_1i_3} = E_{i_1i_2} \cdot E_{i_2i_3}$.

More generally, for any path P_k in G, with $E(P_k) = \{(i_1, i_2), \cdots, (i_k, i_{k+1})\}$, we have $E_{i_1i_2} \cdots E_{i_ki_{k+1}} \in \mathcal{S}_G$ and $E_{i_1i_{k+1}} = E_{i_1i_2} \cdot \cdots \cdot E_{i_ki_{k+1}}$. In particular, we see that $E_{i_1i_{k+1}}$ can be written as a product of k elements in $\mathcal{S}_G$.

Conversely, suppose $E_{ij} \in M_n$ $(i \neq j)$ and we can write E_{ij} as a product of k elements of form E_{lm} $(l \neq m)$ in $\mathcal{S}_G$, i.e. $E_{ij} = E_{ii_2} \cdot \cdots \cdot E_{i_kj}$, where $i_l \neq i, j$ and $i_l \neq i_h$ for $l \neq h$, then it can be easily seen that $\{(i, i_2), (i_2, i_3), \cdots, (i_k, j)\}$ defines a path of length k connecting i with j.

Theorem 3.1.3. *Let G be a graph on n vertices. Then the C*-subalgebra of M_n generated by $\mathcal{S}_G$ is the C*-envelope of $\mathcal{S}_G$.*

Proof. Let $C^*(\mathcal{S}_G) \subseteq M_n$ be the C*-subalgebra generated by $\mathcal{S}_G$. By the general theory of the C*-envelope, there is a *-homomorphism $\pi : C^*(\mathcal{S}_G) \to C^*_e(\mathcal{S}_G)$ that is a complete order isomorphism when restricted to $\mathcal{S}_G$.

First assume that G is connected. Then for any $i, j \in V(G)$ if one uses a path from i to j in G then this path gives a way to express $E_{i,j}$ as a product of matrix units that belong to $\mathcal{S}_G$ by the above remark. Thus, the C*-subalgebra of M_n generated by $\mathcal{S}_G$ is all of M_n. But since M_n is irreducible, π must be an isomorphism.

For the general case, assume that G has connected components of sizes $n_1, ..., n_k$ with $n_1 + \cdots + n_k = n$. By the argument above one can see that $C^*(\mathcal{S}_G) \equiv M_{n_1} \oplus \cdots \oplus M_{n_k}$. If for each component C_j one lets $P_j = \sum_{i \in C_j} E_{i,i}$, then these projections belong to the center of $C^*(\mathcal{S}_G)$ and $P_jC^*(\mathcal{S}_G)P_j$ is *-isomorphic to M_{n_j}. Also, their images $\pi(P_j)$ belong to the center of $C^*_e(\mathcal{S}_G)$.

Thus, $\pi(P_j)C^*_e(\mathcal{S}_G)\pi(P_j)$ is either 0 or *-isomorphic to M_{n_j}.

Look at the diagonal matrices $\mathcal{D}_n \subseteq \mathcal{S}_G$. Since π is a *-homomorphism on the subalgebra and a complete order isomorphism on this subalgebra, it is a *-isomorphism when restricted to $\mathcal{D}_n$. Thus, $\pi(P_j) \neq 0$ and so, these central projections allow us to decompose $C_e^*(\mathcal{S}_G) = A_1 \oplus \cdots A_k$, with $A_j \equiv M_{n_j}$. □

Theorem 3.1.4. *Let G_1 and G_2 be graphs on n vertices. The following are equivalent:*

1. *G_1 is isomorphic to G_2,*
2. *there exists a unitary U such that $U^*\mathcal{S}_{G_1}U = \mathcal{S}_{G_2}$,*
3. *$\mathcal{S}_{G_1}$ is unitally, completely order isomorphic to $\mathcal{S}_{G_2}$.*

Proof. We have shown above that (1) implies (3) and that (2) implies (1). It remains to prove that (3) implies (2).

So assume that (3) holds and let $\phi : \mathcal{S}_{G_1} \to \mathcal{S}_{G_2}$ be a unital, complete order isomorphism. In this case, by [27, Theorem 15.6] ϕ extends uniquely to a *-isomorphism, which we will denote by ρ, between their C*-envelopes. Since, by the previous theorem, the C*-envelopes are just the C*-subalgebras that they generate, we have $\rho : C^*(\mathcal{S}_{G_1}) \to C^*(\mathcal{S}_{G_2})$ is a unital *-isomorphism.

Suppose first that G_1 is connected. Then $M_n = C^*(\mathcal{S}_{G_1})$ is all of M_n. Thus, $dim(C^*(\mathcal{S}_{G_2})) = dim(C^*(\mathcal{S}_{G_1})) = n^2$, which forces $C^*(\mathcal{S}_{G_2}) = M_n$.

Hence, $\rho : M_n \to M_n$ is a *-isomorphism. But every *-isomorphism of M_n is induced by conjugation by a unitary, and so (2) holds.

Now assume that G_1 has connected components of sizes $n_1, ..., n_k$, with $n_1 + \cdots + n_k = n$. In this case, applying the last theorem, we see that $C^*(\mathcal{S}_{G_1}) \equiv M_{n_1} \oplus \cdots \oplus M_{n_k} \equiv C_e^*(\mathcal{S}_{G_1}) \equiv C_e^*(\mathcal{S}_{G_2}) \equiv C^*(\mathcal{S}_{G_2})$. Since $C^*(\mathcal{S}_{G_2}) \equiv M_{n_1} \oplus \cdots \oplus M_{n_k}$ one sees that G_2 has components of sizes $n_1, ..., n_k$ as well.

The central projections onto these components decomposes $\mathbb{C}^n$ into a direct sum of subspaces of dimensions $n_1, ..., n_k$ in two different ways and on each subspace the complete order isomorphism is implemented by conjugation by a unitary. Thus, the complete order isomorphism is implemented by conjugation of the direct sum of these unitaries. □

3.2 Quotients and the Lovász Theta Function

In this section we introduce some natural operator system parameters, which when specialized to graphs we will see are related to Lovász's theta function.

Given an operator system $\mathcal{S}$, a subspace $\mathcal{J} \subseteq \mathcal{S}$ is called a *kernel* if there is an operator system $\mathcal{T}$ and a unital, completely positive (UCP) map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{J} = ker(\phi)$. Since every operator system $\mathcal{T}$ has a unital complete order embedding into $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. There is no loss in generality in assuming that $\mathcal{T} = \mathcal{B}(\mathcal{H})$ in the definition of a kernel.

In [21], it was shown that the vector space quotient $\mathcal{S}/\mathcal{J}$ can be turned into an operator system, called the *quotient operator system* as follows. Let $\mathcal{D}_n(\mathcal{S}/\mathcal{J})$ be the set of all $(x_{i,j} + \mathcal{J}) \in M_n(\mathcal{S}/\mathcal{J})$ for which there exists $(y_{i,j}) \in M_n(\mathcal{J})$ such that $(x_{i,j} + y_{i,j}) \in M_n(\mathcal{S})^+$. Let $M_n(\mathcal{S}/\mathcal{J})^+$ be the *Archimedeanisation* of $\mathcal{D}_n(\mathcal{S}/\mathcal{J})$; that is $(x_{i,j} + \mathcal{J}) \in M_n(\mathcal{S}/\mathcal{J})^+$ if and and only if for every $\epsilon > 0$, $(x_{i,j} + \epsilon 1_n + \mathcal{J}) \in \mathcal{D}_n(\mathcal{S}/\mathcal{J})$. Here, 1_n is the element of $M_n(\mathcal{S})$ whose diagonal entries are all equal to 1 and all other entries are zero. Also, if $\mathcal{J}$ is finite dimensional, then we know that $\mathcal{D}_n(\mathcal{S}/\mathcal{J}) = M_n(\mathcal{S}/\mathcal{J})^+$ and this Archimedeanisation process is unnecessary by [18].

Every operator system is also an operator space. For this reason, the quotient $\mathcal{S}/\mathcal{J}$ carries two, in general distinct, operator space structures. One is the canonical quotient operator space structure on $\mathcal{S}/\mathcal{J}$ arising from the fact that $\mathcal{S}$ and $\mathcal{J}$ are operator spaces. On the other hand, the operator system quotient $\mathcal{S}/\mathcal{J}$ is an operator system and so carries a norm. Examples have been given to show that these two norms can be quite different. See [12] for some important examples of this phenomenon.

To simplify notation, given $x \in \mathcal{S}$ we shall set $\dot{x} := x + \mathcal{J} \in \mathcal{S}/\mathcal{J}$, and for $X = (x_{i,j}) \in M_n(\mathcal{S})$ we set $\dot{X} := (x_{i,j} + \mathcal{J}) \in M_n(\mathcal{S}/\mathcal{J})$.

Following [21], given $X \in M_n(\mathcal{S})$ so that $\dot{X} \in M_n(\mathcal{S}/\mathcal{J})$ we let $\|\dot{X}\|_{\text{osp}}$ (resp. $\|\dot{X}\|_{\text{osy}}$) denote the operator space (resp. the operator system) quotient norm. It is known that $\|\dot{X}\|_{\text{osy}} \leq \|\dot{X}\|_{\text{osp}}$ for every $X \in M_n(\mathcal{S})$ and every n [21].

We identify a kernel $\mathcal{J}$ in the operator system $\mathcal{S}$ with a kernel $\mathcal{K}$ in the operator system $\mathcal{T}$ provided the operator systems $\mathbb{C}1 + \mathcal{J}$ and $\mathbb{C}1 + \mathcal{K}$ are unitally completely order isomorphic.

Definition 3.2.1. *Let $\mathcal{S}$ be an operator system and let $\mathcal{J} \subseteq \mathcal{S}$ be a kernel. Then the* **relative n-distortion** *is*

$$\delta_n(\mathcal{S}, \mathcal{J}) = \sup\{\frac{\|\dot{X}\|_{\text{osp}}}{\|\dot{X}\|_{\text{osy}}} : X \in M_n(\mathcal{S})\}$$

and we call $\delta_{cb}(\mathcal{S}, \mathcal{J}) = \sup\{\delta_n(\mathcal{S}, \mathcal{J}) : n \in \mathbb{N}\}$ the **relative complete distortion**. *We call*

$$\delta_n(\mathcal{J}) = \sup\{\delta_n(\mathcal{S}, \mathcal{J})\}$$

the **absolute n-distortion** *and $\delta_{cb}(\mathcal{J}) = \sup\{\delta_n(\mathcal{J}) : n \in \mathbb{N}\}$ the* **complete distortion**, *where the supremum is taken over all operator systems $\mathcal{S}$ that contain $\mathcal{J}$ as a kernel.*

When $n = 1$ we simplify the notation by setting $\delta(\mathcal{S}, \mathcal{J}) = \delta_1(\mathcal{S}, \mathcal{J})$ and $\delta(\mathcal{J}) =$

$\delta_1(\mathcal{J})$. We now wish to relate this to a Lovász theta type parameter, which was first introduced in [10].

Definition 3.2.2. *Let $\mathcal{S}$ be an operator system and let $\mathcal{J} \subseteq \mathcal{S}$ be a kernel. Then we set*

$$\vartheta_n(\mathcal{J}) = \sup\{\|1_n + J\|_{M_n(\mathcal{S})} : J \in M_n(\mathcal{J}) \text{ and } 1_n + J \geq 0\}$$
$$\text{and } \vartheta_{cb}(\mathcal{J}) = \sup\{\vartheta_n(\mathcal{J}) : n \in \mathbb{N}\}.$$

Again when $n = 1$ we set $\vartheta(\mathcal{J}) := \vartheta_1(\mathcal{J})$.

Remark 3.2.3. If we let $\mathcal{S} = M_n$ and let $\mathcal{J}$ denote the set of diagonal matrices of trace 0, then $\mathcal{J}$ is a kernel and it follows from the characterization of the quotient $M_n/\mathcal{J}$ in [12] that $n \leq \delta(M_n, \mathcal{J})$. For any $J \in \mathcal{J}$ we see that $tr(I_n + J) = n$ and so when $I_n + J \geq 0$ we see that $\|I_n + J\| \leq n$. Letting J be the diagonal matrix with diagonal entries, $(n-1, -1, \ldots, -1)$ we see have $\|I_n + J\| = n$, and so $\vartheta(\mathcal{J}) = n$.

Theorem 3.2.4. *We have that $\delta(\mathcal{J}) \leq \vartheta(\mathcal{J})$ and $\delta_{cb}(\mathcal{J}) \leq \vartheta_{cb}(\mathcal{J})$.*

Proof. Let $x \in \mathcal{S}$ be such that $\|\dot{x}\|_{\text{osy}} = 1$. Then

$$\begin{pmatrix} \dot{1}_{\mathcal{S}} & \dot{x} \\ \dot{x}^* & \dot{1}_{\mathcal{S}} \end{pmatrix} \in M_2(\mathcal{S}/\mathcal{J})^+.$$

Thus, for every $\epsilon > 0$,

$$\begin{pmatrix} (1+\epsilon)\dot{1}_{\mathcal{S}} & \dot{x} \\ \dot{x}^* & (1+\epsilon)\dot{1}_{\mathcal{S}} \end{pmatrix} \in D_2(\mathcal{S}/\mathcal{J})$$

and so there exists $\left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right) \in M_2(\mathcal{J})$ such that

$$\begin{pmatrix} (1+\epsilon)1_{\mathcal{S}} + a & x+b \\ x^*+b^* & (1+\epsilon)1_{\mathcal{S}} + c \end{pmatrix} \in M_2(\mathcal{S})^+.$$

But then

$$\|x+b\| \leq \max\{\|(1+\epsilon)1_{\mathcal{S}} + a\|, \|(1+\epsilon)1_{\mathcal{S}} + c\|\}$$

with $(1+\epsilon)1_{\mathcal{S}}+a, (1+\epsilon)1_{\mathcal{S}}+c \in \mathcal{S}^+$. Since ϵ was arbitrary, we have that $\|x+b\| \leq \vartheta(\mathcal{J})$ On the other hand,

$$\|x+b\| \geq \inf\{\|x+y\| : y \in \mathcal{J}\} = \|\dot{x}\|_{\text{osp}}$$

and it follows that $\vartheta(\mathcal{J}) \geq \|\dot{x}\|_{\text{osp}}$. Thus, $\delta(\mathcal{S}, \mathcal{J}) \leq \vartheta(\mathcal{J})$ for every $\mathcal{S}$ and so $\delta(\mathcal{J}) \leq \vartheta(\mathcal{J})$.

Note that $M_n(\mathcal{J})$ is a kernel in $M_n(\mathcal{S})$ and $\delta_n(\mathcal{S}, \mathcal{J}) = \delta_1(M_n(\mathcal{S}), M_n(\mathcal{J}))$. Also, $\vartheta(M_n(\mathcal{J})) = \vartheta_n(\mathcal{J})$. Hence,

$$\delta_{cb}(\mathcal{J}) = \sup_n\{\delta(M_n(\mathcal{J}))\} \leq \sup_n\{\vartheta(M_n(\mathcal{J}))\} = \vartheta_{cb}(\mathcal{J}).$$

□

Corollary 3.2.5. *For any $X \in M_n(\mathcal{S})$,*

$$||\dot{X}||_{osp} \leq \vartheta_n(\mathcal{J}) \cdot ||\dot{X}||_{osy}$$

We now compute these parameters in one case.

Corollary 3.2.6. *If $\mathcal{J} \subseteq M_n$ denotes the diagonal matrices of trace 0, then*

$$n = \delta(M_n, \mathcal{J}) = \delta(\mathcal{J}) = \vartheta(\mathcal{J}) = \vartheta_{cb}(\mathcal{J}).$$

Proof. By Remark 3.2.3 and the above result, we have that

$$n \leq \delta(M_n, \mathcal{J}) \leq \delta(\mathcal{J}) \leq \vartheta(\mathcal{J}) = n.$$

So all that remains is to show that $\vartheta_{cb}(\mathcal{J}) = n$.

If we let $\mathcal{D}_n \subseteq M_n$ denote the diagonal matrices, then for each p, $M_p(\mathcal{D}_n)$ can be thought of as the C*-algebra of functions from the set $\{1, ..., n\}$ into M_p. From this it can be seen that every $(J_{k,l}) \in M_p(\mathcal{D}_n)$ is unitarily equivalent via an element in this algebra to a diagonal element $diag(J_1, ..., J_p)$ of this algebra. Moreover, since each J_i is a linear combination of the matrices $J_{k,l}$ it follows that if $tr(J_{k,l}) = 0$ for all k, l, then $tr(J_i) = 0$ for all i. Since unitaries preserve norms, we see that if $J_{k,l} \in \mathcal{J}$ and $diag(I_n, ..., I_n) + (J + k, l) \geq 0$, then $I_n + J_i \geq 0$. Also, $\|diag(I_n, ..., I_n) + (J_{k,l})\| = \max\{\|I_n + J_1\|, ..., \|I_n + J_p\|\} \leq \vartheta(\mathcal{J})$.

This shows that $\vartheta_{cb}(\mathcal{J}) = \vartheta(\mathcal{J})$ and the result follows. □

Results in [10] imply that $\mathcal{S}_G^{\perp}$ is a kernel in our sense. Below is a direct proof in the language of operator systems, that also characterizes the quotient as the operator system dual of $\mathcal{S}_G$.

We recall that given a finite dimensional operator system, $\mathcal{S}$, the dual space $\mathcal{S}^d$ is also an operator system [30]. The matrix ordering on the dual space is defined by $(f_{i,j}) \in M_n(\mathcal{S}^d)^+$ if and only if the map $F : \mathcal{S} \to M_n$ given by $F(x) = (f_{i,j}(x))$ is completely positive.

Proposition 3.2.7. *Let G be a graph on k vertices. Then $\mathcal{S}_G^\perp$ is a kernel in M_k and the quotient $M_k/\mathcal{S}_G^\perp$ is completely order isomorphic to the operator system dual $\mathcal{S}_G^d$.*

Proof. It is proven in [30, Thm. 6.2] that M_k is self-dual as an operator system via the map $\rho : M_k \to M_k^d$ that sends the matrix unit $E_{i,j} \in M_k$ to the dual functional $\delta_{i,j} \in M_k^d$. Let $\iota : \mathcal{S}_G \to M_k$ be the inclusion map; it is clearly a complete order embedding. Thus its dual $\iota^d : M_k^d \to \mathcal{S}_G^d$ is a complete quotient map by [12, Prop. 1.8]. Let $\mathcal{J}$ be its kernel. A functional $f = \sum_{i,j} \lambda_{i,j}\delta_{i,j}$ is in the kernel of ι^d if and only if $f(E_{i,j}) = 0$ whenever $(i,j) \in E(G)$ or $i = j$. Thus, f is in the kernel of ι^d if and only if $\lambda_{i,j} = 0$ whenever $(i,j) \in E(G)$ or $i = j$. Thus,

$$\ker \iota^d = \operatorname{span}\{\delta_{i,j} : (i,j) \in E(\overline{G})\}.$$

Thus,

$$\rho^{-1}(\ker \iota^d) = \operatorname{span}\{E_{i,j} : (i,j) \in E(\overline{G})\} = \mathcal{S}_G^\perp.$$

It follows that $\mathcal{S}_G^d \equiv M_k^d/\ker \iota^d \equiv M_k/\mathcal{S}_G^\perp$. □

Corollary 3.2.8. *Let G be a graph on k vertices, let $x = \sum_{i,j=1}^k x_{i,j}E_{i,j} \in M_k$ and let $f = \sum_{i,j=1}^k x_{i,j}\delta_{i,j} : \mathcal{S}_G \to \mathbb{C}$ denote the corresponding functional. Then*

$$\|f\| = \|\dot{x}\|_{\text{osy}} \geq \delta(M_k, \mathcal{S}_G^\perp)^{-1}\|\dot{x}\|_{\text{osp}} \geq \vartheta(\mathcal{S}_G^\perp)^{-1}\|\dot{x}\|_{\text{osp}}.$$

We now see that

$$\vartheta(\mathcal{S}_G^\perp) = \sup\{\|I + K\| : I + K \geq 0, K \in \mathcal{S}_G^\perp\} = \vartheta(G)$$

by [10]. Similarly, $\vartheta_{cb}(\mathcal{S}_G^\perp) = \tilde{\vartheta}(\mathcal{S}_G)$ is the "complete" Lovász number of G introduced in [10].

In [10] it is shown that for graphs,

$$\vartheta_{cb}(\mathcal{S}_G^{\perp}) = \vartheta(\mathcal{S}_G^{\perp}).$$

It is useful to recall their argument.

First, note that $M_p(\mathcal{S}_G) = \mathcal{S}_{G \boxtimes K_p}$, where K_p denotes the complete graph on p vertices. Also notice that

$$\mathcal{S}_{G \boxtimes K_p}^{\perp} = M_p(\mathcal{S}_G^{\perp})$$

Hence,

$$\vartheta_{cb}(\mathcal{S}_G^{\perp}) = \sup_p \vartheta(\mathcal{S}_{G \boxtimes K_p}^{\perp}) = \sup_p \vartheta(G \boxtimes K_p) = \sup_p \vartheta(G)\vartheta(K_p) = \vartheta(G),$$

using Lovász famous result that ϑ is multiplicative for strong products of graphs and the fact that $\vartheta(K_p) = 1$.

We now get a lower bound on the distortion in terms of a graph theoretic parameter.

Theorem 3.2.9. *Let G be a graph on k vertices and let $K_{p,q}$ be an induced complete bipartite of G. Then*

$$\sqrt{pq} \leq \delta(M_k, \mathcal{S}_G^{\perp}).$$

Proof. Let the vertices for the subgraph be numbered $1, ..., p$ for the first set and $p+1, ..., p+q$ for the remainder. Let $X = (x_{i,j})$ be the matrix with $x_{i,j} = 1$ for $1 \leq i \leq p$ and $p+1 \leq j \leq p+q$ and 0 otherwise. Let $K = (k_{i,j})$ with $k_{i,j} = 1$ for $1 \leq i, j \leq p$ and $i \neq j$ and 0 otherwise. Let $R = (r_{i,j})$ be the matrix such that $r_{i,j} = 1$

for $p+1 \leq i, j \leq p+q$, $i \neq j$ and 0 otherwise. Then $K, R \in \mathcal{S}_G^{\perp}$ and

$$\begin{pmatrix} I+K & X \\ X^* & I+R \end{pmatrix}$$

is positive. Hence, $\|X\|_{osy} \leq 1$. However,

$$\|\dot{X}\|_{osp} = dist(X, \mathcal{S}_G^{\perp}) = \|X\| = \sqrt{pq}.$$

Hence, $\frac{\|X+S_G^{\perp}\|_{osp}}{\|X+\mathcal{S}_G^{\perp}\|_{osy}} \geq \sqrt{pq}$.

□

Remark 3.2.10. Haemers [14] introduces the parameter $\Phi(G) = \max\{\sqrt{pq} : K_{p,q} \subseteq G\}$, i.e., the maximum over all complete bipartite subgraphs of G, that are not necessarily induced subgraphs. He proves that $\Phi(G) \leq \vartheta'(G)$, which is another variant of the Lovasz theta function. We have been unable to find any relationship between his parameters and ours.

If we let $\mathcal{S} = M_n$ and let $\mathcal{T} = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in M_n\}$ then these operator systems are unitally, completely order isomorphic, but $\vartheta(\mathcal{S}^{\perp}) = 1$, while $\vartheta(\mathcal{T}^{\perp}) = 2$. However, $\delta(M_n, \mathcal{S}^{\perp}) = \delta(M_{2n}, \mathcal{T}^{\perp}) = 1$. This motivates the following problems:

Problem 3.2.11. *If $\mathcal{S} \subseteq M_n$ and $\mathcal{T} \subseteq M_n$ are unitally completely order isomorphic operator systems, then is $\vartheta(\mathcal{S}^{\perp}) = \vartheta(\mathcal{T}^{\perp})$?*

Problem 3.2.12. *If $\mathcal{S} \subseteq M_n$ and $\mathcal{T} \subseteq M_m$ are unitally completely order isomorphic, then is $\delta(M_n, \mathcal{S}^{\perp}) = \delta(M_m, \mathcal{T}^{\perp})$?*

Problem 3.2.13. *Is $\delta_{cb}(\mathcal{J}) = \vartheta_{cb}(\mathcal{J})$?*

3.3 Multiplicativity of Graph Parameters

One of the great strengths of the Lovász theta function is the fact that it is multiplicative for strong graph product. Recall that,

$$\vartheta(G) = \vartheta(\mathcal{S}_G^{\perp}) = \sup\{\|I + K\| : K \in \mathcal{S}_G^{\perp}, I + K \in M_n^+\}.$$

In this section we wish to examine multiplicativity of the quotient norms when interpreted as graph parameters. We have been unable to determine if our general theta function is multiplicative for tensor products of kernels or if any of the various distortions are multiplicative.

Instead we focus more closely on the graph theory case where we get some multiplicativity results using general facts about tensor products of operator spaces and operator systems. Let us examine more closely the case when $\mathcal{S} = M_n$ and $\mathcal{J} = \mathcal{S}_G^{\perp}$. Throughout this section let $X \in M_n$ and $Y \in M_m$. This means we can define the following two families of parameters,

$$\sigma(G, X) := ||X + \mathcal{S}_G^{\perp}||_{osy}$$

$$d_\infty(G, X) := ||X + \mathcal{S}_G^{\perp}||_{osp}.$$

We will prove that given two graphs G and H:

$$\sigma(G \boxtimes H, X \otimes Y) = \sigma(G, X)\sigma(H, Y),$$

and

$$d_\infty(G \boxtimes H, X \otimes Y) = d_\infty(G, X)d_\infty(H, Y),$$

for any matrices X and Y.

In parallel with Lovász's work, of special interest are the cases when these matrices are I, A_G, and R_G, which are all real symmetric matrices. Finally, for real matrices we give formulas for these quotient norms in terms of SDP's which are then easy to implement and find numerically.

Remark 3.3.1. Our results can be extended to $||(X_{i,j} + \mathcal{S}_G^{\perp})||$, in either the operator space or operator system case, by using the graph $G \boxtimes K_m$.

Before tackling our next result we need the following elementary lemma.

Lemma 3.3.2. *Let G be a graph on n vertices and let H be a graph on m vertices. Then*

$$\mathcal{S}_G^{\perp} \otimes M_m + M_n \otimes \mathcal{S}_H^{\perp} = \mathcal{S}_{G \boxtimes H}^{\perp}$$

Proof. Let $X \otimes Y \in M_n \otimes \mathcal{S}_H^{\perp}$ and $N \otimes M \in \mathcal{S}_G \otimes \mathcal{S}_H$. Notice that,

$$\langle X \otimes Y, N \otimes M \rangle = \langle X, N \rangle \langle Y, M \rangle = 0$$

This implies that,

$$M_n \otimes \mathcal{S}_H^{\perp} \perp \mathcal{S}_G \otimes \mathcal{S}_H$$

Similarly, $\mathcal{S}_G \otimes M_m \perp \mathcal{S}_G \otimes \mathcal{S}_H$. Hence

$$\mathcal{S}_G \otimes M_m + M_n \otimes \mathcal{S}_H^{\perp} \subseteq (\mathcal{S}_G \otimes \mathcal{S}_H)^{\perp}.$$

Equality holds since they have the same dimensions.

□

Theorem 3.3.3. *Let G be a graph on n vertices with $X \in M_n$ and let H be a graph on m vertices with $Y \in M_m$. Then*

$$||X \otimes Y + \mathcal{S}_{G\boxtimes H}^{\perp}||_{osp} = ||X + \mathcal{S}_G^{\perp}||_{osp} \cdot ||Y + \mathcal{S}_H^{\perp}||_{osp},$$

that is, $d_\infty(G \boxtimes H, X \otimes Y) = d_\infty(G, X) \cdot d_\infty(H, Y)$.

Proof. Let $K \in \mathcal{S}_G^{\perp}$ and $L \in \mathcal{S}_H^{\perp}$ and notice the following,

$$\begin{aligned} ||X + K|| \cdot ||Y + L|| &= ||(X + K) \otimes (Y + K)|| \\ &= ||X \otimes Y + X \otimes L + K \otimes Y + K \otimes L|| \end{aligned}$$

Note that $X \otimes L + K \otimes Y + K \otimes L \in \mathcal{S}_G^{\perp} \otimes M_m + M_n \otimes \mathcal{S}_H^{\perp}$ and by 3.3.2 we have that $\mathcal{S}_G^{\perp} \otimes M_m + M_n \otimes \mathcal{S}_H^{\perp} = \mathcal{S}_{G\boxtimes H}^{\perp}$. Now if we take the infimum on both sides of the above equation, over all K and L, we get,

$$||X + \mathcal{S}_G^{\perp}||_{osp} \cdot ||Y + \mathcal{S}_H^{\perp}||_{osp} \geq \inf\{||X \otimes Y + R|| : R \in \mathcal{S}_{G\boxtimes H}\} = ||X \otimes Y + \mathcal{S}_{G\boxtimes H}^{\perp}||_{osp}.$$

The other inequality requires some results from the theory of operator spaces. Let $Q_1 : M_n \to M_n/\mathcal{S}_G^{\perp}$ and $Q_2 : M_m \to M_m/\mathcal{S}_H^{\perp}$ denote the quotient maps. Since both of these maps are completely contractive by [27, Thm. 12.3] the map $Q_1 \otimes Q_2 : M_n \otimes_{min} M_m \to (M_n/\mathcal{S}_G^{\perp}) \otimes_{min} (M_m/\mathcal{S}_H^{\perp})$ is completely contractive. But $M_n \otimes_{min} M_m = M_{nm}$ and the kernel of $Q_1 \otimes Q_2$ is $\mathcal{S}_{G\boxtimes H}^{\perp}$. Hence,

$$\|X \otimes Y + \mathcal{S}_{G\boxtimes H}^{\perp}\| \geq \|Q_1(X) \otimes Q_2(Y)\| = \|Q_1(X)\| \cdot \|Q_2(Y)\|,$$

where the last equality follows from the fact [2] that the min tensor norm is a crossnorm. We have that $\|Q_1(X)\| = \|X + \mathcal{S}_G^\perp\|_{\text{osp}}$ and $\|Q_2(Y)\| = \|Y + \mathcal{S}_H^\perp\|_{\text{osp}}$ and so the proof is complete. $\square$

We now turn our attention to the operator space quotient norm in $M_n/\mathcal{S}_G^\perp$. Recall that [21]

$$||X + \mathcal{S}_G^\perp||_{osy} = \inf\left\{\lambda : \begin{pmatrix} \lambda I + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I + K_3 \end{pmatrix} \in M_2(M_n)^+, \text{ for } K_i \in \mathcal{S}_G^\perp\right\}.$$

Theorem 3.3.4. *Let G and H be graphs on n and m vertices, respectively, and let $X \in M_n$ and $Y \in M_m$. Then*

$$||X \otimes Y + \mathcal{S}_{G\boxtimes H}^\perp||_{osy} = ||X + \mathcal{S}_G^\perp||_{osy}||Y + \mathcal{S}_H^\perp||_{osy},$$

that is, $\sigma(G \boxtimes H, X \otimes Y) = \sigma(G, X) \cdot \sigma(H, Y)$.

Proof. We use the fact that [21, Prop. 4.1],

$$||X + \mathcal{S}_G^\perp||_{osy} = \sup\{||\phi_G(X)|| : \phi_G : M_n \to \mathcal{B}(\mathcal{H}),\ \phi_G(\mathcal{S}_G^\perp) = 0, \phi_G\ UCP\ \}\ (*)$$

where the supremum is over all Hilbert spaces H and UCP stands for "unital, completely positive". Note that,

$$\begin{aligned} ||X + \mathcal{S}_G^\perp||_{osy}||Y + \mathcal{S}_H^\perp||_{osy} &= \sup_{\phi_G,\phi_H} \{||\phi_G(X)|| \cdot ||\phi_H(Y)||\} \\ &= \sup_{\phi_G,\phi_H} \{||\phi_G(X) \otimes \phi_H(Y)||\} \\ &= \sup_{\phi_G,\phi_H} \{||\phi_G \otimes \phi_H(X \otimes Y)||\} \end{aligned}$$

where this supremum is over all maps that satisfy property $(*)$ and $\phi_G \otimes \phi_H(X \otimes Y)$ is the map that takes elementary tensors to the tensor of the corresponding images of the maps. Notice $\phi_G \otimes \phi_H$ is a UCP map that vanishes on $\mathcal{S}_G^\perp \otimes M_m + M_n \otimes \mathcal{S}_H^\perp = \mathcal{S}_{G\boxtimes H}^\perp$ (3.3.2). Finally note,

$$\sup_{\phi_G,\phi_H} \{\|\phi_G \otimes \phi_H(X \otimes Y)\|\} \leq ||X \otimes Y + \mathcal{S}_{G\boxtimes H}^\perp||_{osy}.$$

Thus,

$$\|X \otimes Y + \mathcal{S}_{G\boxtimes H}^\perp\|_{osy} \geq \|X + \mathcal{S}_G^\perp\|_{osy}\|Y + \mathcal{S}_H^\perp\|_{osy}.$$

We now prove the other inequality: $||X+\mathcal{S}_G^\perp||_{osy}||Y+\mathcal{S}_H^\perp||_{osy} \geq ||X\otimes Y+\mathcal{S}_{G\boxtimes H}^\perp||_{osy}$. Let $\lambda > \|X + \mathcal{S}_G^\perp\|_{\text{osy}}$ and pick $K_i \in \mathcal{S}_G^\perp$ such that the $2n \times 2n$ block matrix

$$\begin{pmatrix} \lambda I_n + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I_n + K_3 \end{pmatrix} \geq 0.$$

Similarly, let $\mu > \|Y + \mathcal{S}_H^\perp\|_{\text{osy}}$ and pick $L_i \in \mathcal{S}_H^\perp$ such that the $2m \times 2m$ matrix

$$\begin{pmatrix} \mu I_m + L_1 & Y + L_2 \\ Y^* + L_2^* & \mu I_m + L_3 \end{pmatrix} \geq 0.$$

Tensoring these matrices we have that the $4mn \times 4mn$ block matrix,

$$\begin{pmatrix} \lambda I_n + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I_n + K_3 \end{pmatrix} \otimes \begin{pmatrix} \mu I_m + L_1 & Y + L_2 \\ Y^* + L_2^* & \mu I_m + L_3 \end{pmatrix} \geq 0.$$

Restricting to the 4 blocks that occur in the corners we see that

$$\begin{pmatrix} (\lambda I_n + K_1) \otimes (\mu I_m + L_1) & (X + K_2) \otimes (Y + L_2) \\ (X^* + K_2^*) \otimes (Y^* + L_2^*) & (\lambda I_n + K_3) \otimes (\mu I_m + L_3) \end{pmatrix} \geq 0$$

But this matrix is of the form

$$\begin{pmatrix} \lambda \cdot \mu(I_n \otimes I_m) + Q_1 & X \otimes Y + Q_2 \\ (X \otimes Y + Q_2)^* & \lambda \cdot \mu(I_n \otimes I_m) + Q_3 \end{pmatrix}$$

for some $Q_i \in \mathcal{S}_G^\perp \otimes M_m + M_n \otimes \mathcal{S}_H^\perp = \mathcal{S}_{G \boxtimes H}^\perp$. From this it follows that

$$\|X \otimes Y + \mathcal{S}_{G \boxtimes H}^\perp\|_{\text{osy}} \leq \lambda\mu.$$

Since λ and μ were arbitrary,

$$||X + \mathcal{S}_G^\perp||_{osy}||Y + \mathcal{S}_H^\perp||_{osy} \geq ||X \otimes Y + \mathcal{S}_{G \boxtimes H}^\perp||_{osy}$$

and the proof is complete. □

For the purposes of numerical calculation it is often convenient to have dual formulations for computing $||X + \mathcal{S}_G^\perp||_{osp}$ and $\|X + \mathcal{S}_G^\perp\|_{\text{osy}}$, especially in the case that X is a real matrix. We write $M_n(\mathbb{R})$ for the set of real matrices and X^T for the transpose of the matrix X.

Proposition 3.3.5. *Let G be a graph on n vertices and let $X \in M_n(\mathbb{R})$. Then $||X + \mathcal{S}_G^\perp||_{osp} = \|X + H\|$ for some $H \in \mathcal{S}_G^\perp \cap M_n(\mathbb{R})$.*

Proof. Given a matrix $Y = (y_{i,j})$ we set $\overline{Y} = (\overline{y}_{i,j})$. Since $\mathcal{S}_G^\perp$ is a subspace of M_n

we know that there is a $K \in \mathcal{S}_G^\perp$ such that $||X + \mathcal{S}_G^\perp||_{osp} = ||X + K||$. Now since $||X + K|| = ||\overline{X + K}|| = \|X + \overline{K}\|$ and $\overline{K} \in \mathcal{S}_G^\perp$ we get that,

$$||X + \mathcal{S}_G^\perp||_{\text{osp}} \geq ||X + \frac{K + \overline{K}}{2}||$$

so we have that $||X + \mathcal{S}_G^\perp||_{osp} = ||X + H||$ where $H = \frac{K+\overline{K}}{2} \in \mathcal{S}_G^\perp \cap M_n(\mathbb{R})$. □

Proposition 3.3.6. *Let G be a graph on n vertices and let $X \in M_n(\mathbb{R})$ be a real matrix. Then*

$$||X + \mathcal{S}_G^\perp||_{osp} = \max\{Tr(X^T Q) : Q \in \mathcal{S}_G^\perp \cap M_n(\mathbb{R}), Tr(|Q|) \leq 1\}$$

Proof. This follows from general facts about the "dual of a quotient" in Banach space theory together with the fact that the trace norm is the dual of the operator norm. Alternatively, this is a consequence of Example (34) in [35], which states that for the following minimization problem,

$$||X + \mathcal{S}_G^\perp||_{osp} = \min\{||X + \sum_{i,j} k_{i,j} E_{ij}|| : E_{ij} \in \mathcal{S}_G^\perp, k_{ij} \in \mathbb{R}\}$$

its dual is given by,

$$\begin{aligned} \text{maximize} \quad & Tr(X^T Q) \\ \text{subject to} \quad & Tr((E_{ij})^T Q) = 0, \; E_{ij} \in \mathcal{S}_G^\perp \\ & Tr(|Q|) \leq 1, \end{aligned}$$

where $E_{i,j}$ denote the usual matrix units. Now since $Tr((E_{ij})^T Q) = q_{ij} = 0$, where q_{ij} is the ij-entry of Q, we get our result. □

We now turn our attention to a dual formulation of $||X + \mathcal{S}_G^{\perp}||_{osy}$ as an SDP, but just like in the case of the operator space norm we first need the following lemma,

Lemma 3.3.7. *Let G be a graph on n vertices and let $X \in M_n(\mathbb{R})$. Then the value of $||X + \mathcal{S}_G^{\perp}||_{osy}$ is achieved for some choice of $K_i \in \mathcal{S}_G^{\perp} \cap M_n(\mathbb{R})$, $i = 1, 2, 3$ with $K_1 = K_1^T$, $K_3 = K_3^T$.*

Proof. Suppose $||X + \mathcal{S}_G^{\perp}||_{osy} = \lambda$. By definition,

$$\begin{pmatrix} \lambda I + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I + K_3 \end{pmatrix} \geq 0$$

for some choice of $K_i \in \mathcal{S}_G^{\perp}$ where $K_i = K_i^*$ for $i = 1, 2, 3$. Now note that,

$$0 \leq \overline{\begin{pmatrix} \lambda I + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I + K_3 \end{pmatrix}} = \begin{pmatrix} \lambda I + \overline{K_1} & X + \overline{K_2} \\ X^* + \overline{K_2^*} & \lambda I + \overline{K_3} \end{pmatrix}$$

Finally if we average over this two positive matrices,

$$\frac{1}{2}\Big[\begin{pmatrix} \lambda I + K_1 & X + K_2 \\ X^* + K_2^* & \lambda I + K_3 \end{pmatrix} + \begin{pmatrix} \lambda I + \overline{K_1} & X + \overline{K_2} \\ X^* + \overline{K_2^*} & \lambda I + \overline{K_3} \end{pmatrix}\Big] = \begin{pmatrix} \lambda I + \frac{K_1 + \overline{K_1}}{2} & X + \frac{K_2 + \overline{K_2}}{2} \\ X^* + \frac{K_2^* + \overline{K_2^*}}{2} & \lambda I + \frac{K_3 + \overline{K_3}}{2} \end{pmatrix}$$

we get our desired result. □

Proposition 3.3.8. *Let G be a graph on n vertices and let $X \in M_n(\mathbb{R})$, then*

$$||X + \mathcal{S}_G^{\perp}||_{osy} = \max\{2 \cdot Tr(X^T B) : \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in M_2(\mathcal{S}_G)^+, Tr(A + C) = 1\},$$

with $A, B, C \in M_n(\mathbb{R})$.

Proof. Notice that we can write $||X + \mathcal{S}_G^{\perp}||_{osy}$ as the following SDP:

minimize $\langle x, c\rangle$

subject to

$$\begin{pmatrix} \lambda I & X \\ X^* & \lambda I \end{pmatrix} + \sum_{(i,j)\in E(\overline{G})} \begin{pmatrix} k_{i,j}(E_{ij}+E_{ji}) & z_{i,j}E_{i,j} \\ z_{i,j}E_{j,i} & y_{i,j}(E_{i,j}+E_{j,i}) \end{pmatrix} \geq 0$$

for $c = \begin{cases} 1, & \text{if } l = 1 \\ 0, & \text{if } l \neq 1 \end{cases}$ and $x = \begin{cases} \lambda, & \text{if } l = 1 \\ k_{ij}, & \text{if } 2 \leq l \leq \lfloor \frac{dim(\mathcal{S}_G^{\perp})}{2} \rfloor \\ y_{ij}, & \text{if } \lfloor \frac{dim(\mathcal{S}_G^{\perp})}{2} \rfloor < l \leq dim(\mathcal{S}_G^{\perp}) \\ z_{ij}, & \text{if } dim(\mathcal{S}_G^{\perp}) < l \leq \lfloor \frac{3dim(\mathcal{S}_G^{\perp})}{2} \rfloor \end{cases}$.

Now by [35] the dual of the above program is given by,

$$\begin{aligned} \text{maximize} \quad & 2 \cdot Tr(X^T B) \\ \text{subject to} \quad & \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \in M_2(\mathcal{S}_G)^+ \\ & Tr(A+C) = 1. \end{aligned}$$

Finally, we see that strong duality also holds for this SDP since we can always pick,

$$x = \begin{cases} \lambda = \max_j \big\{ \sum_{i=1}^{n} |X_{ij}| \big\} + 1, & \text{if } l = 1 \\ 0, & \text{if } l \neq 1 \end{cases}$$

(X_{ij} is the ij-entry of X) such that our constraint satisfies,

$$\begin{pmatrix} \lambda \cdot I & X \\ X^* & \lambda \cdot I \end{pmatrix} > 0.$$

□

Remark 3.3.9. The two multiplicativity theorems, Theorem 3.3.3 and Theorem 3.3.4, can be proven for real matrices X and Y using these two dual formulations.

3.4 Quotient Norms as Graph Parameters

Lovász's famous sandwich theorem [20] says that

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G),$$

where $\omega(G)$ is the size of the largest clique in G and $\chi(G)$ is the chromatic number of G. One of the many formulas for Lovász's theta function is that

$$\vartheta(\overline{G}) = \min\{\lambda_1(R_G + K) : K = K^* \in \mathcal{S}_G^{\perp}\},$$

where λ_1 denotes the largest eigenvalue. Note that by Proposition 3.3.5,

$$d_\infty(G, R_G) = \inf\{||R_G + K|| : K = K^* = K^t \in \mathcal{S}_G^{\perp}\}.$$

Since for self-adjoint matrices their norm is the maximum of the absolute values of their eigenvalues,

$$\vartheta(\overline{G}) \leq d_\infty(G, R_G).$$

The only potential difference between these two quantities is that for any matrix $K = K^* \in \mathcal{S}_G^\perp$ with $\lambda_1(R_G + K) = \vartheta(\overline{G})$ we have that $-\lambda_n(R_G + K) > \lambda_1(R_G + K)$.

This suggests we should examine the question of equality of these two parameters and study the role that the potentially larger $d_\infty(G, R_G)$ could play in sandwich type theorems.

We begin with an example where $\vartheta(\overline{G}) < d_\infty(G, R_G)$. For $G = C_6$, the 6-cycle, we know that $\vartheta(\overline{G}) = 2$, but $d_\infty(G, R_G) = 2.25$. To see that this is the case notice that for any $K = K^* = K^t \in \mathcal{S}_G^\perp$

$$A = \frac{\sum_{k=0}^{5} (S^*)^k (R_G + K) S^k}{6} = \begin{pmatrix} 1 & 1 & a & b & a & 1 \\ 1 & 1 & 1 & a & b & a \\ a & 1 & 1 & 1 & a & b \\ b & a & 1 & 1 & 1 & a \\ a & b & a & 1 & 1 & 1 \\ 1 & a & b & a & 1 & 1 \end{pmatrix}$$

where S is the cyclic forward shift mod 6. Since K is real and symmetric, $a, b \in \mathbb{R}$ by 3.3.5. Now since $||A|| \leq ||R_G + K||$ for any $K \in \mathcal{S}_G^\perp$, we have that $d_\infty(G, R_G)$ achieves its minimum value at such a matrix A for some choice of a and b. A similar argument shows that $\lambda_1(R_G + K)$ achieves its minimum at such a matrix A. Now notice that for this matrix we can explicitly compute its spectrum

$$Spec(A) = \{-a-b+2, -a-b+2, 2a-b-1, b-a, b-a, 2a+b+3\}$$

and hence if we perform the following minimization we get that,

$$d_\infty(G, R_G) = \min_{a,b\in\mathbb{R}} \max\{|-a-b+2|, |2a-b-1|, |b-a|, |2a+b+3|\} = 2.25$$

achieved when $a = -0.25$ and $b = 0.5$.

Similarly, minimizing $\lambda_1(A)$ over all a and b yields the well-known fact that $\vartheta(\overline{G}) = 2$.

This fact gives rise to a new condition on the graph, namely, what happens when $\vartheta(\overline{G}) = d_\infty(G, R_G)$?

Note that the orthogonal projection, $P_G : M_n \to \mathcal{S}_G$ is given by Schur product with R_G. Although P_G has norm one when we regard M_n as a Hilbert space, in general, when we endow M_n with the usual operator norm then $\|P_G\|$ can be much larger than 1 [36]. It is this latter norm that we are interested in. For operator theorists, this is known as the *Schur multiplier norm of* R_G [8], sometimes denoted $\|R_G\|_m$. For graph theorists, this is sometimes denoted $\gamma(G)$.

Proposition 3.4.1. *If* $\vartheta(\overline{G}) = d_\infty(G, R_G)$, *then*

$$\frac{1 + \lambda_1(A_G)}{\|P_G\|} \leq \vartheta(\overline{G}).$$

Proof. In [22] it was show that there exists a self-adjoint matrix of the form $R_G + K$ with $K \in \mathcal{S}_G^\perp$ such that $\vartheta(\overline{G}) = \lambda_1(R_G + K)$. Now, if $\vartheta(\overline{G}) = d_\infty(G, R_G)$, then

$\|R_G + K\| = \lambda_1(R_G + K)$, and we get that

$$\|R_G\| = \|P_G(R_G + K)\| \leq \|P_G\| \cdot \|R_G + K\| = \|P_G\|\vartheta(\overline{G}),$$

so that

$$\frac{\|R_G\|}{\|P_G\|} \leq \vartheta(\overline{G}).$$

Also, it is the case that $\|A_G\| = \lambda_1(A_G)$ [33], from which it follows that $\|R_G\| = \|I + A_G\| = 1 + \lambda_1(A_G)$. □

Corollary 3.4.2. *If $\vartheta(\overline{G}) = d_\infty(G, R_G)$, then*

$$\frac{\chi(G)}{\|P_G\|} \leq \vartheta(\overline{G}) \leq \chi(G).$$

Proof. By Wilf's theorem [33], $1 + \lambda_1(A_G) \geq \chi(G)$. □

We now give at least one condition for when these parameters are equal, although it is very restrictive.

Theorem 3.4.3. *If $\vartheta(\overline{G}) \leq 2$ then there exists a matrix A satisfying $a_{ij} = 1$ when i=j or $i \nsim j$ with $\lambda_1(A) = \vartheta(G) = ||A||$.*

Proof. By [5, Theorem 3] there exist a matrix A satisfying (1) and $\vartheta(G) = \lambda_1(A)$ such that

$$\vartheta(G)I - A = (c - \sqrt{\vartheta(G)} \cdot u_i)^T(c - \sqrt{\vartheta(G)} \cdot u_j) \qquad (*)$$

with optimal orthonormal representation $(u_1, u_2, ..., u_n)$ of G with handle c such that $\vartheta(G) = \frac{1}{(c^T u_1)^2} = \cdots = \frac{1}{(c^T u_n)^2}$. We must show that $-\lambda_n(A) \leq \vartheta(G)$. By $(*)$ we get that,

$$\begin{aligned}
-A &= (c - \sqrt{\vartheta(G)} \cdot u_i)^T (c - \sqrt{\vartheta(G)} \cdot u_j) - \vartheta(G)I \\
&= c^T c - \sqrt{\vartheta(G)} \cdot u_i^T c - \sqrt{\vartheta(G)} \cdot c^T u_j + \vartheta(G) \cdot u_i^T u_j - \vartheta(G)I \\
&= 1 - 1 - 1 + \vartheta(G) \cdot u_i^T u_j - \vartheta(G)I \\
&= \vartheta(G) \cdot u_i^T u_j - \vartheta(G)I - J
\end{aligned}$$

Now pick a unit vector h such that $-\lambda_n(A) = \langle -Ah, h \rangle$ and notice that,

$$\begin{aligned}
-\lambda_n(A) = \langle -Ah, h \rangle &\leq \vartheta(G) \left\langle u_i^T u_j h, h \right\rangle - \vartheta(G) \langle h, h \rangle \\
&\leq \vartheta(G) ||u_i^T u_j|| - \vartheta(G) = \vartheta(G) ||I + H|| - \vartheta(G).
\end{aligned}$$

for some $H \in \mathcal{S}_G$. Now since $\vartheta(\overline{G}) = \max\{||I + H|| : H \in \mathcal{S}_G\}$ we get,

$$\vartheta(G)\vartheta(\overline{G}) - \vartheta(G) \leq \vartheta(G).$$

□

Corollary 3.4.4. *If $\vartheta(G) \leq 2$ then $\vartheta(\overline{G}) = d_\infty(G, R_G)$.*

The condition $\vartheta(G) \leq 2$ is quite restrictive. It is met by $K_n, C_4, K_{2,\ldots,2}$ and some graphs that are "nearly" complete.

3.5 Conclusion and Open Problems

In this section we managed to establish a connection between graph theory and operator algebras via the operator system of a graph. We managed to define two new families of parameters of G via the two quotient norms you can define on the operator system $M_n/\mathcal{S}_G$. We also saw how these norms are similar to the Lovász theta function $\vartheta(G)$ e.g. SDP, Multiplicative, etc. Finally, we discussed one particular parameter, namely $d_\infty(G, I + A_G)$, and discussed in detail how this parameter gives rise to a new graph theoretic condition.

In Section 3, we posed a couple of interesting problems while discussing the distortion of an operator system. We still do not know necessary and sufficient conditions such that $d_\infty(G, I + A_G) = \vartheta(\overline{G})$. Also, we do not know if for a different $X \in M_n$, $||X + S_G^\perp||_{osp/osy}$ tells us anything new about G.

Chapter 4

Quantum Graph Homomorphisms via Operator Systems

4.1 The Homomorphism Game

Recall that given graphs G and H a **graph homomorphism** from G to H is a mapping $f : V(G) \to V(H)$ such that

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H).$$

When a graph homomorphism from G to H exists we write $G \to H$.

Paralleling the work on quantum chromatic numbers [29], we study a graph homomorphism game, played by Alice, Bob, and a Referee. Given graphs G and H, the Referee gives Alice and Bob a vertex of G, say v and w, respectively, and they each respond with a vertex from H, say x and y, respectively. Alice and Bob win provided that:

$$v = w \implies x = y,$$

$$v \sim_G w \implies x \sim_H y.$$

If they have some random strategy and we let $p(x, y|v, w)$ denote the probability that we get outcomes x and y given inputs v and w, then these equations translate as:

1. $p(x \neq y|v = w) = 0$

2. $p(x \nsim_H y|v \sim_G w) = 0$

Now say G has n vertices and H has m vertices. We consider the sets of correlations studied in [28] and [29]:

$$C_l(n, m) \subseteq C_q(n, m) \subseteq C_{qa}(n, m) \subseteq C_{qc}(n, m) \subseteq C_{vect}(n, m).$$

For $t \in \{l, q, qa, qc, vect\}$ we define:

$$G \xrightarrow{t} H,$$

provided that there exists

$$p(x, y|v, w) \in C_t(n, m)$$

satisfying (1) and (2). Notice that when we write $p(x, y|v, w) \in C_t(n, m)$ we really mean $\big(p(x, y|v, w)\big)_{v,w,x,y} \in C_t(n, m)$. Any $p(x, y|v, w) \in C_t(n, m)$ satisfying these conditions we call a **winning t-strategy** and say that there exists a **quantum t-homomorphism** from G to H.

The condition (1) is easily seen to be the **synchronous condition** defined in [28] and the subset of correlations satisfying this condition was denoted $C_t^s(n,m)$. Thus, $p(x,y|v,w)$ is a winning t-strategy if and only if $p(x,y|v,w) \in C_t^s(n,m)$ and satisfies (2).

The following result is known, but we provide a proof since we are using a slightly different(but equivalent) characterization of $C_l(n,m)$.

Theorem 4.1.1. *Let G and H be graphs. Then $G \to H$ if and only if $G \xrightarrow{l} H$.*

Proof. First assume that $G \to H$. Let $f : V(G) \to V(H)$ be a graph homomorphism. Let $\Omega = \{t\}$ be the singleton probability space. For each $v \in V(G)$ let Alice have the "random variable", $f_v(t) = f(v)$ and for each $w \in V(G)$ let Bob have the random variable $g_w(t) = f(w)$. Then

$$p(x,y|v,w) := Prob(x = f_v(t), y = g_w(t)) = \begin{cases} 1, & \text{when } x = f(v), y = f(w) \\ 0, & \text{else} \end{cases}.$$

From this it easily follows that $p(x,y|v,w)$ satisfies (1) and (2).

Conversely, assume that we have a probability space (Ω, P) and random variables $f_v, g_w : \Omega \to V(H) = \{1, ..., m\}$ so that $p(x,y|v,w) = P(x = f_v(\omega), y = g_w(\omega))$ satisfies (1) and (2). By (1), for each v the set $B_v = \{\omega : f_v(\omega) = g_v(\omega))$ has probability 1. Similarly, for each $(v,w) \in E(G)$ the set $C_{v,w} = \{\omega : (f_v(\omega), g_w(\omega)) \in E(H)\}$ has probability 1. Thus,

$$D = \left(\cap_{v \in V(G)} B_v\right) \cap \left(\cap_{(v,w) \in E(G)} C_{v,w}\right)$$

has measure 1, and so in particular is non-empty. Fix any $\omega \in D$ and define f :

$V(G) \to V(H)$ by $f(v) := f_v(\omega) = g_v(\omega)$. Then whenever $(v,w) \in E(G)$ we have that $(f(v), f(w)) = (f_v(\omega), g_w(\omega)) \in E(H)$. Thus, f is a graph homomorphism. □

Thus, quantum l-homomorphisms correspond to classical graph homomorphisms.

Remark 4.1.2. In [7] several notions of graph homomorphisms were also introduced, including $G \xrightarrow{B} H$, $G \xrightarrow{V} H$ and $G \xrightarrow{+} H$. A look at their definition shows that

$$G \xrightarrow{vect} H \text{ if and only if } G \xrightarrow{V} H$$

Corollary 4.1.3. *Let G and H be graphs. Then*

$$G \longrightarrow H \implies G \xrightarrow{q} H \implies G \xrightarrow{qa} H \implies G \xrightarrow{qc} H \implies G \xrightarrow{vect} H$$

Proof. This is a direct consequence of the above definitions, Theorem 4.1.1, and the corresponding set containments. □

4.2 Quantum Homomorphisms and CP Maps

Recall that the operator system of a graph G on n vertices is the subspace of the $n \times n$ complex matrices M_n given by

$$\mathcal{S}_G = span\{E_{v,w} : v = w \text{ or } (v,w) \in E(G)\},$$

where $E_{v,w}$ denotes the $n \times n$ matrix that is 1 in the (v,w)-entry and 0 elsewhere.

We now wish to use a winning x-strategy for the homomorphism game to define a CP map from $\mathcal{S}_G$ to $\mathcal{S}_H$. It will suffice to do this in the case of winning vect-strategies since every other strategy is a subset.

Proposition 4.2.1. *Let* $p(x,y|v,w) \in C^s_{vect}(n,m)$*, let* $E_{v,w} \in M_n$ *and* $E_{x,y} \in M_m$ *denote the canonical matrix unit bases. Then the linear map* $\phi_p : M_n \to M_m$ *defined on the basis by*

$$\phi_p(E_{v,w}) = \sum_{x,y} p(x,y|v,w)E_{x,y},$$

is completely positive.

Proof. By Choi's theorem [27], to prove that ϕ_p is CP it is enough to prove that the Choi matrix,

$$P := \sum_{v,w} E_{v,w} \otimes \phi_p(E_{v,w}) = \sum_{v,w,x,y} p(x,y|v,w)E_{v,w} \otimes E_{x,y} \in M_n \otimes M_m = M_{nm}$$

is positive semidefinite.

By the definition and characterization of vector correlations satisfying the synchronous condition in [29] there exists a Hilbert space and vectors $\{h_{v,x}\}$ satisfying:

- $h_{v,x} \perp h_{v,y}$ for all $x \neq y$,
- $\sum_x h_{v,x} = \sum_x h_{w,x}$ for all v, w,

such that $p(x,y|v,w) = \langle h_{v,x}, h_{w,y}\rangle$.

Now let $\{e_v\}$ and $\{f_x\}$ denote the canonical orthonormal bases for $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, let $a_{v,x} \in \mathbb{C}$ be arbitrary complex numbers, so that $k = \sum_{v,x} a_{v,x} e_v \otimes f_x$ is an arbitrary vector in $\mathbb{C}^n \otimes \mathbb{C}^m$. We have that

$$\langle Pk, k\rangle = \sum_{v,w,x,y} \overline{a_{v,x}} a_{w,y} p(x,y|v,w) = \sum_{v,w,x,y} \overline{a_{v,x}} a_{w,y} \langle h_{v,x}, h_{w,y}\rangle = \langle h, h\rangle,$$

where $h = \sum_{v,x} \overline{a_{v,x}} h_{v,x}$.

Thus, P is positive semidefinite and ϕ_p is CP. □

Theorem 4.2.2. *Let G and H be graphs, let $p(x,y|v,w) \in C^s_{vect}(n,m)$ be a winning vect-strategy for a quantum vect-homomorphism from G to H and let $\phi_p : M_n \to M_m$ be the CP map defined in Proposition 4.2.1. Then $\phi_p(\mathcal{S}_G) \subseteq \mathcal{S}_H$ and ϕ_p is trace-preserving on $\mathcal{S}_G$.*

Proof. To see that ϕ_p is trace preserving on $\mathcal{S}_G$ it will be enough to show that $tr\big(\phi_p(E_{v,v})\big) = tr(E_{v,v}) = 1$, and for $v \sim_G w$, $tr\big(\phi_p(E_{v,w})\big) = tr(E_{v,w}) = 0$.

When $v = w$ we have that

$$tr\big(\phi_p(E_{v,v})\big) = tr\Big(\sum_{x,y} p(x,y|v,v)E_{x,y}\Big) = \sum_x p(x,x|v,v) = 1 = tr(E_{v,v}),$$

by definition of p.

Finally, if $v \neq w$ and $E_{v,w} \in \mathcal{S}_G$, then

$$tr\big(\phi_p(E_{v,w})\big) = \sum_x p(x,x|v,w) = 0 = tr(E_{v,w}),$$

by (2) and the fact that $x \nsim_H x$.

Hence, ϕ_p is trace-preserving on $\mathcal{S}_G$.

Now we prove that $\phi(\mathcal{S}_G) \subseteq \mathcal{S}_H$. First, $\phi_p(E_{v,v}) = \sum_{x,y} p(x,y|v,v)E_{x,y}$, but since p is synchronoous, $p(x,y|v,v) = 0$ for $x \neq y$. Hence, $\phi_p(E_{v,v})$ is a diagonal matrix and so in $\mathcal{S}_H$. To finish the proof it will be enough to show that when $v \sim_G w$, we have $\phi_p(E_{v,w}) \in \mathcal{S}_H$. But by property (2), $p(x,y|v,w) = 0$ when $x \nsim_H y$. Thus, $\phi_p(E_{v,w}) \in \mathcal{S}_H$. In fact, it is a matrix with 0-diagonal in $\mathcal{S}_H$. □

Corollary 4.2.3. *Let* $x \in \{l, q, qa, qc, vect\}$*. If* $p(x, y|v, w) \in C_x^s(n, m)$ *is a winning x-strategy, then the map* $\phi_p : M_n \to M_m$ *is CP,* $\phi_p(\mathcal{S}_G) \subseteq \mathcal{S}_H$ *and* ϕ_p *is trace-preserving on* $\mathcal{S}_G$*. We say that the correlation* $p(x, y|v, w)$ **implements** *the quantum t-homomorphism.*

Example 4.2.4. Suppose we have a graph homomorphism $G \to H$ given by $f : V(G) \to V(H)$. If we let $\Omega = \{t\}$ be a one point probability space and define Alice and Bob's random variables $f_v, g_w : \Omega \to V(H)$ by $f_v(t) = f(v), g_w(t) = f(w)$ as in the proof of Theorem 4.1.1, then we obtain $p(x, y|v, w) \in Q_l^s(n, m)$ with

$$p(x, y|v, w) = Prob(f_v = x, g_w = y) = \begin{cases} 1 & x = f(v), y = f(w) \\ 0 & \text{else} \end{cases}.$$

The corresponding CP map satisfies

$$\phi_p(E_{v,w}) = E_{f(v),f(w)}.$$

We now wish to turn our attention to the composition of quantum graph homomorphisms. First we need a preliminary result.

Proposition 4.2.5. *Let* $x \in \{l, q, qa, qc, vect\}$*, let* $p(x, y|v, w) \in C_x(n, m)$ *and let* $q(a, b|x, y) \in C_x(m, l)$*. Then*

$$r(a, b|v, w) := \sum_{x,y} q(a, b|x, y)p(x, y|v, w) \in C_x(n, l).$$

Moreover, if p *and* q *are synchronous, then* r *is synchronous.*

Proof. First we show the synchronous condition is met by r. Suppose that $v = w$

and $a \neq b$. Since p is synchronous, all the terms $p(x, y|v, v)$ vanish unless $x = y$. Thus, $r(a, b|v, v) = \sum_x q(a, b|x, x)p(x, x|v, v)$. But because q is synchronous, each $q(a, b|x, x) = 0$. Hence, if $a \neq b$, then $r(a, b|v, v) = 0$.

When $x = l$ we are dealing with classical distributions.

Let (Ω_1, λ_1) be a probability space such that there exist $\{f_v\}_{1\leq v\leq n}$ and $\{g_w\}_{1\leq w\leq n}$, $f_v, g_w : \Omega_1 \to \{1, \dots, m\}$, with

$$p(x, y|v, w) = \lambda_1(f_v^{-1}(x) \cap g_w^{-1}(y))$$

Similarly, let (Ω_2, λ_2) be a probability space such that there exist $\{h_x\}_{1\leq x\leq m}$ and $\{k_y\}_{1\leq y\leq m}$, $h_x, k_y : \Omega_2 \to \{1, \dots, l\}$, with

$$q(a, b|x, y) = \lambda_2(h_x^{-1}(a) \cap k_y^{-1}(b))$$

Define $F_v, G_w : \Omega_1 \times \Omega_2 \to \{1, \dots, l\}$ on the probability space $(\Omega_1 \times \Omega_2, \lambda_1 \times \lambda_2)$ by

$$F_v(\omega_1, \omega_2) = h_{f_v(\omega_1)}(w_2) \text{ and } G_w(\omega_1, \omega_2) = k_{g_w(\omega_1)}(w_2)$$

If we define,

$$r(a, b|v, w) := \lambda_1 \times \lambda_2(F_v^{-1}(a) \cap G_w^{-1}(b))$$

then by definition $r(a, b|v, w) \in C_l(n, l)$. Finally notice that,

$$r(a, b|v, w) := \lambda_1 \times \lambda_2(F_v^{-1}(a) \cap G_w^{-1}(b))$$

$$= \lambda_1 \times \lambda_2(\{(\omega_1, \omega_2) : F_v(\omega_1, \omega_2) = a \text{ and } G_w(\omega_1, \omega_2) = b\})$$

$$= \lambda_1 \times \lambda_2(\{(\omega_1, \omega_2) : h_{f_v(\omega_1)}(w_2) = a \text{ and } k_{g_w(\omega_1)} = b\})$$

$$= \lambda_1 \times \lambda_2([\cup_x f_v^{-1}(x) \times h_x^{-1}(a)] \cap [\cup_y g_w^{-1}(y) \times k_y^{-1}(b)])$$

[Suppose $x \neq y$ and let $(\omega_1, \omega_2) \in [f_v^{-1}(x) \times h_x^{-1}(a)] \cap [f_v^{-1}(y) \times h_y^{-1}(a)]$. This means $f_v(\omega_1) = x$, $h_x(\omega_2) = a$, $f_v(\omega_1) = y$, and $h_y(\omega_2) = a$, a contradiction. Hence, these sets are disjoint. Similarly, $[g_w^{-1}(x) \times k_x^{-1}(b)] \cap [g_w^{-1}(y) \times k_y^{-1}(b)] = \varnothing$.]

$$= \sum_{x,y} \lambda_1 \times \lambda_2([f_v^{-1}(x) \times h_x^{-1}(a)] \cap [g_w^{-1}(y) \times k_y^{-1}(b)])$$

$$= \sum_{x,y} \lambda_1 \times \lambda_2([f_v^{-1}(x) \cap g_w^{-1}(y)] \times [h_x^{-1}(a) \cap k_y^{-1}(b)])$$

$$= \sum_{x.y} \lambda_2(h_x^{-1}(a) \cap k_y^{-1}(b)) \cdot \lambda_1(f_v^{-1}(x) \cap g_w^{-1}(y))$$

$$= \sum_{x,y} q(a, b|x, y)p(x, y|v, w)$$

Let us move on to the case where $x = qc$. For $p(x, y|v, w)$, we have a Hilbert space $\mathcal{H}_1$ a unit vector $\eta_1 \in \mathcal{H}_1$ and for each $1 \leq v, w \leq n$ we have PVM's on $\mathcal{H}_1$, $\big(A_{v,x}\big)_{1\leq x\leq m}$ and $\big(B_{w,y}\big)_{1\leq y\leq m}$ such that $A_{v,x}B_{w,y} = B_{w,y}A_{v,x}$ with $p(x, y|v, w) = \langle A_{v,x}B_{w,y}\eta_1, \eta_1\rangle$. Similarly, for $q(a, b|x, y)$ we have a Hilbert space $\mathcal{H}_2$ and a unit vector $\eta_2 \in \mathcal{H}_2$, and for each $1 \leq x, y \leq m$, PVM's $\big(C_{x,a}\big)_{1\leq a\leq l}$ and $\big(D_{y,b}\big)_{1\leq b\leq l}$ so that the C's and D's commute and $q(a, b|x, y) = \langle C_{x,a}D_{y,b}\eta_2, \eta_2\rangle$.

Now consider the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, unit vector $\eta_1 \otimes \eta_2$ and operators $P_{v,a} =$

$\sum_x A_{v,x} \otimes C_{x,a}$, $Q_{w,b} = \sum_x B_{w,x} \otimes D_{x,b}$. We have

$$P_{v,a}^2 = \sum_{x,y} A_{v,x}A_{v,y} \otimes C_{x,a}C_{y,a} = P_{v,a},$$

since $A_{v,x}A_{v,y} = 0$ unless $x = y$ and $C_{x,a}^2 = C_{x,a}$. Also,

$$\sum_a P_{v,a} = \sum_x \sum_a A_{v,x} \otimes C_{x,a} = \sum_x A_{v,x} \otimes I_{\mathcal{H}_2} = I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2},$$

so that $\left(P_{v,a}\right)_{1\le a\le l}$ is a PVM. Similarly, $\left(Q_{w,b}\right)_{1\le b\le l}$ is a PVM, and it is not hard to see that the P's commute with the Q's.

Finally,

$$\begin{aligned}\langle P_{v,a}Q_{w,b}\eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2\rangle &= \langle \sum_{x,y}(A_{v,x}B_{w,y} \otimes C_{x,a}D_{y,b})\eta_1 \otimes \eta_2, \eta_1 \otimes \eta_2\rangle \\ &= \sum_{x,y}\langle A_{v,x}B_{w,y}\eta_1, \eta_1\rangle\langle C_{x,a}D_{y,b}\eta_2\eta_2\rangle = r(a,b|x,y).\end{aligned}$$

This proves the case for $x = qc$.

The case that $x = q$ is similar. Suppose now that $p \in C_{qa}(n,m), q \in C_{qa}(m,l)$. Then we may pick sequences $p_k \in C_q(n,m)$ and $q_k \in C_q(m,l)$ such that $\lim_k p_k(x,y|v,w) = p(x,y|v,w)$ and $\lim_k q_k(a,b|x,y) = q(a,b|x,y)$ for all v,w,x,y,a,b. Then $r_k(a,b|v,w) = \sum_{x,y} q_k(a,b|x,y)p_k(x,y|v,w)$ belongs to $C_q(n,l)$ and converges to r. Hence, $r \in C_{qa}(n,l)$.

Finally, to see the case that $x = vect$. In this case, we are given a Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, unit vectors $\eta_1 \in \mathcal{H}_1, \eta_2 \in \mathcal{H}_2$, and vectors $h_{v,x}, k_{w,y} \in \mathcal{H}_1$, $f_{x,a}, g_{y,b} \in \mathcal{H}_2$ such that:

$$h_{v,x} \perp h_{v,y}, k_{v,x} \perp k_{v,y}, \forall x \neq y, \;\; f_{x,a} \perp f_{x,b}, g_{x,a} \perp g_{x,b}, \forall a \neq b,$$

$$\sum_x h_{v,x} = \sum_x k_{v,x} = \eta_1, \forall v, \quad \sum_a f_{x,a} = \sum_a g_{x,a} = \eta_2, \forall x$$

such that $p(x,y|v,w) = \langle h_{v,x}, k_{w,y}\rangle$ and $q(a,b|x,y) = \langle f_{x,a}, g_{y,b}\rangle$.

We set $\alpha_{v,a} = \sum_x h_{v,x} \otimes f_{x,a}$ and $\beta_{w,b} = \sum_y k_{w,y} \otimes g_{y,b}$. Now one checks that these vectors satisfy all the necessary conditions, e.g., $\alpha_{v,a} \perp \alpha_{v,b}, \forall a \neq b$ and $\sum_a \alpha_{v,a} = \eta_1 \otimes \eta_2, \forall v$, and that

$$\langle \alpha_{v,a}, \beta_{w,b}\rangle = \sum_{x,y} \langle h_{v,x}, k_{w,y}\rangle \langle f_{x,a}, g_{y,b}\rangle = r(a,b|x,y).$$

□

Corollary 4.2.6. *Let $x \in \{l, q, qa, qc, vect\}$, let $p(x,y|v,w) \in C_x(n,m)$, $q(a,b|x,y) \in C_x(m,l)$ and let $r(a,b|v,w) = \sum_{x,y} q(a,b|x,y)p(x,y|v,w) \in C_x(n,l)$. If $\phi_p : M_n \to M_m$, $\phi_q : M_m \to M_l$ and $\phi_r : M_n \to M_l$ are the corresponding linear maps, then $\phi_r = \phi_q \circ \phi_p$.*

The following is now immediate:

Theorem 4.2.7. *Let $x \in \{l, q, qa, qc, vect\}$, let G, H and K be graphs on n, m and l vertices, respectively, and assume that $G \stackrel{x}{\to} H$, $H \stackrel{x}{\to} K$. If $p(x,y|v,w) \in C_x(n,m)$ and $q(a,b|x,y) \in Q_x(m,l)$ are winning quantum x-strategies for homomorphisms from G to H and H to K, respectively, then $r(a,b|v,w) = \sum_{x,y} q(a,b|x,y)p(x,y|v,w) \in C_x(n,l)$ is a winning x-strategy for a homomorphism from G and K, so that $G \stackrel{x}{\to} K$. In summary,*

$$\text{if } G \stackrel{x}{\to} H \text{ and } H \stackrel{x}{\to} K, \text{ then } G \stackrel{x}{\to} K.$$

4.3 C*-algebras and Graph Homomorphisms

We wish to define a C*-algebra $\mathcal{A}(G,H)$ generated by the relations arising from a winning strategy for the graph homomorphism game.

Definition 4.3.1. *Let G and H be graphs. A set of projections $\{E_{v,x} : v \in V(G),\ x \in V(H)\}$ on a Hilbert space $\mathcal{H}$ satisfying the following relations:*

1. *for each $v \in V(G)$, $\sum_x E_{v,x} = I_{\mathcal{H}}$,*

2. *if $(v,w) \in E(G)$ and $(x,y) \notin E(H)$ then $E_{v,x}E_{w,y} = 0$,*

is called a **representation of the graph homomorphism game from G to H**. *If no set of projections on any Hilbert space exists satisfying these relations, then we say that the* **graph homomorphism game from G to H is not representable**.

Definition 4.3.2. *Let G and H be graphs. If a representation of the graph homomorphism game exists, then we let $\mathcal{A}(G,H)$ denote the "universal" C*-algebra generated by such sets of projections. If the graph homomorphism game from G to H is not representable, then we say that $\mathcal{A}(G,H)$* **does not exist**. *We write $G \xrightarrow{C^*} H$ if and only if $\mathcal{A}(G,H)$ exists.*

By "universal" we mean that $\mathcal{A}(G,H)$ is a unital C*-algebra generated by projections $\{e_{v,x} : v \in V(G), x \in V(H)\}$ satisfying

1. for each $v \in V(G)$, $\sum_x e_{v,x} = 1$,

2. if $(v,w) \in E(G)$ and $(x,y) \notin E(H)$, then $e_{v,x}e_{w,y} = 0$,

with the property that for any representation of the graph homomorphism game on a Hilbert space $\mathcal{H}$ by projections $\{E_{v,x}\}$ satisfying the above relations, there exists a *-homomorphism $\pi : \mathcal{A}(G,H) \to B(\mathcal{H})$ with $\pi(e_{v,x}) = E_{v,x}$.

Here is one result that relates to existence. Let E_m be the "empty" graph on m vertices, i.e., the graph with no edges.

Proposition 4.3.3. *Let G be a graph with at least one edge, $(v, w) \in E(G)$. Then $\mathcal{A}(G, E_m)$ does not exist.*

Proof. By definition we have that $e_{v,x}e_{w,y} = 0$ for all x, y. Thus,

$$0 = \sum_{x,y} e_{v,x}e_{w,y} = \big(\sum_x e_{v,x}\big)\big(\sum_y e_{w,y}\big) = 1,$$

contradiction. □

Proposition 4.3.4. *If $G \xrightarrow{C^*} H$, then $G \xrightarrow{B} H$, as defined in [7].*

Proof. Let $\{E_{v,x} : v \in V(G), x \in V(H)\}$ be a set of projections that yields a representation of the graph homomorphism game on a Hilbert space $\mathcal{H}$ and let $h \in \mathcal{H}$ be any unit vector.

If we set $h^v_x = E_{v,x}h$, then set of vectors $\{h^v_x\}$ satisfies all the properties of the definition of $G \xrightarrow{B} H$ in [7, Definition 2]. □

Remark 4.3.5. We do not know necessary and sufficient conditions for $\mathcal{A}(G, H)$ to exist. In particular, we do not know if $G \overset{B}{\to} H$ implies $G \xrightarrow{C^*} H$.

Proposition 4.3.6. *If $G \overset{C^*}{\to} H$ and $H \overset{C^*}{\to} K$, then $G \overset{C^*}{\to} K$.*

Proof. Since $G \overset{C^*}{\to} H$ and $H \overset{C^*}{\to} K$, then we know that there exist projections $\{E_{v,x}\}$ and $\{F_{y,a}\}$ with $v \in V(G)$, $x, y \in V(H)$ and $a \in V(K)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, satisfying (1) and (2). Consider the set of self-adjoint operators on $\mathcal{H} \otimes \mathcal{K}$ defined by $G_{v,a} = \sum_{x \in V(H)} E_{v,x} \otimes F_{x,a}$ for $x \in V(G)$ and $a \in V(K)$. Notice that,

$$G_{v,a}G_{v,a} = (\sum_x E_{v,x} \otimes F_{x,a})(\sum_y E_{v,y} \otimes F_{y,a}) =$$

$$\sum_{x,y} E_{v,x}E_{v,y} \otimes F_{x,a}F_{y,a} = \sum_x E_{v,x} \otimes F_{x,a} = G_{v,a}$$

by (1) and the fact that $E_{v,x}$ and $F_{x,a}$ are projections. Thus, each $G_{v,a}$ is a projection. Furthermore, for each $v \in V(G)$,

$$\sum_a G_{v,a} = \sum_a \sum_x E_{v,x} \otimes F_{x,a} = \sum_x E_{v,x} \otimes (\sum_a F_{x,a}) = (\sum_x E_{v,x}) \otimes I_{\mathcal{K}} = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$$

by (1). Moreover, for each $(v, w) \in E(G)$ and $(a, b) \notin E(K)$,

$$G_{v,a}G_{w,b} = (\sum_x E_{v,x} \otimes F_{x,a})(\sum_y E_{w,y} \otimes F_{y,b}) = \sum_x \sum_y (E_{v,x} \otimes F_{x,a})(E_{w,y} \otimes F_{y,b})$$

$$= \sum_x \sum_y E_{v,x}E_{w,y} \otimes F_{x,a}F_{y,b} = \sum_{x \sim y} E_{v,x}E_{w,y} \otimes F_{x,a}F_{y,b} = 0$$

by (2). Hence, $\{G_{v,a} : v \in V(G), a \in V(K)\}$ is a representation of a graph homomorphism game from G to K. □

Recall that a **trace** on a unital C*-algebra $\mathcal{B}$ is any state τ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{B}$.

Theorem 4.3.7. *Let G be a graph and let $x \in \{l, q, qa, qc, vect\}$.*

1. *$G \stackrel{qc}{\to} H$ if and only if there exists a tracial state on $\mathcal{A}(G, H)$,*

2. *if $G \stackrel{qc}{\to} H$, then $G \stackrel{C^*}{\to} H$,*

3. $G \stackrel{q}{\to} H$ *if and only if* $\mathcal{A}(G,H)$ *has a finite dimensional representation,*

4. $G \to H$ *if and only if* $\mathcal{A}(G,H)$ *has an abelian representation.*

Proof. We have that $G \stackrel{qc}{\to} H$ if and only if there exists a winning qc-strategy $p(x,y|v,w) \in C^s_{qc}(n,m)$. By [28] this strategy must be given by a trace on the algebra generated by Alice's operators with $p(x,y|v,w) = \tau(A_{v,x}A_{w,y})$. Moreover, in the GNS representation, this trace will be faithful.

We now wish to show that these operators satisfy the necessary relations to induce a representation of $\mathcal{A}(G,H)$.

By the original hypotheses, we will have that $A_{v,x}A_{v,y} = 0$ for $x \neq y$. When $(v,w) \in E(G)$ and $(x,y) \notin E(H)$, we will have that $\tau(A_{v,x}A_{w,y}) = p(x,y|v,w) = 0$ and hence, $A_{v,x}A_{w,y} = 0$.

Thus, Alice's operators give rise to a representation of $\mathcal{A}(G,H)$ and composing this *-homomorphism with the tracial state on the algebra generated by Alice's operators gives the trace on $\mathcal{A}(G,H)$.

Clearly, (2) follows from (1).

The proof of (3) is similar to the proof of (1). In this case since $p(x,y|v,w) \in C^s_q(n,m)$ the operators all live on a finite dimensional space and hence generate a finite dimensional representation.

The proof of (4) first uses the fact that $G \to H$ if and only if $G \stackrel{l}{\to} H$ (4.1.1). If we let (Ω,λ) be the corresponding probability space and let $f_v, g_w : \Omega \to V(H)$ be the random variables for Alice and Bob, respectively, then the conditions imply that $f_v = g_v$ a.e. If we let $E_{v,x}$ denote the characteristic function of the set $f^{-1}(\{x\})$, then it is easily checked that these projections in $L^\infty(\Omega,\lambda)$ satisfy all the conditions needed to give an abelian representation of $\mathcal{A}(G,H)$. □

Note that saying that $\mathcal{A}(G,H)$ has an abelian representation is equivalent to requiring that it has a one-dimensional representation.

We now apply these results to coloring numbers. Let K_c denote the complete graph on c vertices.

Proposition 4.3.8. *Let $x \in \{l, q, qa, qc, vect\}$, then $\chi_x(G)$ is the least integer c for which $G \stackrel{x}{\to} K_c$.*

Proof. Any winning x-strategy for a homomorphism from G to H is a winning strategy for a x-coloring. □

The above result motivates the following definition.

Definition 4.3.9. *Define $\chi_{C^*}(G)$ to be the least integer c for which $G \stackrel{C^*}{\to} K_c$. Similarly, define $\omega_{C^*}(G)$ to be the biggest integer c for which $K_c \stackrel{C^*}{\to} G$.*

Proposition 4.3.10. *Let G be a graph, then*

$$\omega_{C^*}(G) \leq \vartheta(\overline{G}) \leq \chi_{C^*}(G).$$

Proof. Let $c := \chi_{C^*}(G)$. If we combine 4.3.4 with [7, Theorem 6] we know that,

$$G \stackrel{C^*}{\to} K_c \implies G \stackrel{B}{\to} K_c \iff \vartheta(\overline{G}) \leq \vartheta(\overline{K_c}) = c.$$

Similarly, if you apply the above proof to $K_d \stackrel{C^*}{\to} G$, where $d := \omega_{C^*}(G)$, you get the remaining inequality. □

Remark 4.3.11. Since $G \stackrel{qc}{\to} K_c \implies G \stackrel{C^*}{\to} K_c$, we have that $\chi_{qc}(G) \geq \chi_{C^*}(G)$, but we don't know the relation between $\chi_{C^*}(G)$ and $\chi_{vect}(G)$.

This leads to the following results:

Theorem 4.3.12. *Let G be a graph.*

1. $\chi(G)$ *is the least integer* c *for which there is an abelian representation of* $\mathcal{A}(G, K_c)$,

2. $\chi_q(G)$ *is the least integer* c *for which* $\mathcal{A}(G, K_c)$ *has a finite dimensional representation.*

3. $\chi_{qc}(G)$ *is the least integer* c *for which* $\mathcal{A}(G, K_c)$ *has a tracial state.*

4. $\chi_{C^*}(G)$ *is the least integer* c *for which* $\mathcal{A}(G, K_c)$ *exists.*

Theorem 4.3.13. *Let G be a graph.*

1. *The problem of determining if* $\mathcal{A}(G, K_3)$ *has an abelian representation is NP-complete.*

2. *The problem of determining if* $\mathcal{A}(G, K_3)$ *has a finite dimensional representation is NP-hard.*

3. *The problem of determining if* $\mathcal{A}(G, K_c)$ *has a trace is solvable by a semidefinite programming problem.*

Proof. We have shown that $\mathcal{A}(G, K_3)$ has an abelian representation if and only if G has a 3-coloring and this latter problem is NP-complete [9].

In [17, Theorem 1], it is proven that an NP-complete problem is polynomially reducible to determining if $\chi_q(G) = 3$. Hence, this latter problem is NP-hard.

In [28], it is proven that for each n and c there is a spectrahedron $S_{n,c} \subseteq \mathbb{R}^{n^2c^2}$ such that for each graph G on n vertices there is a linear functional $L_G : \mathbb{R}^{n^2c^2} \to \mathbb{R}$ with the property that $\chi_{qc}(G) \leq c$ if and only if there is a point $p \in S_{n,c}$ with $L_G(p) = 0$.

Thus, determining if $\chi_{qc}(G) \leq c$ is solvable by a semidefinite programming problem. But we have seen that $\chi_{qc}(G) \leq c$ if and only if $\mathcal{A}(G, K_c)$ has a trace. $\square$

Remark 4.3.14. Currently, there are no known algorithms for determining if $\chi_q(G) \leq 3$, i.e., for determining if $\mathcal{A}(G, K_3)$ has a finite dimensional representation.

Remark 4.3.15. We do not know the complexity level of determining if $\mathcal{A}(G, H)$ exists. In particular, we do not know the complexity level of determining if $G \stackrel{C^*}{\to} K_3$, or any algorithm.

Remark 4.3.16. In [7] it is proven that $\chi_{vect}(G) = \lceil \vartheta^+(\overline{G}) \rceil$, which is solvable by an SDP.

Remark 4.3.17. There is a family of finite input, finite output games that are called **synchronous games**[11], of which the graph homomorphism game is a special case. For any synchronous game $\mathcal{G}$ we can construct the C*-algebra of the game $\mathcal{A}(\mathcal{G})$ and there are analogues of many of the above theorems. For instance, the game will have a winning qc-strategy, q-strategy or l-strategy if and only if $\mathcal{A}(\mathcal{G})$ has a trace, finite dimensional, or abelian representation, respectively.

4.4 Tensor Product

In [13] the authors discuss the problem of whether or not, $\chi_q(G \times H) = \min\{\chi_q(G), \chi_q(H)\}$ (Hedetniemi conjeture [16] for χ_q). They proved the conjecture holds for some special graphs. Here, we explain why the trivial inequality also holds for $\chi_{C^*}(G)$.

Since $G \times H \to G$ and $G \times H \to H$ always exist, we know that $G \times H \stackrel{C^*}{\to} G$ and

$G \times H \overset{C^*}{\to} H$ exist by 4.1.3. Suppose $G \overset{C^*}{\to} K_a$ and $H \overset{C^*}{\to} K_b$, for some $a, b \in \mathbb{N}$, then

$$G \times H \overset{C^*}{\to} G \overset{C^*}{\to} K_a \implies G \times H \overset{C^*}{\to} K_a$$

by our composition theorem. Similarly, $G \times H \overset{C^*}{\to} K_b$. Now, since a, b are arbitrary we get,

$$\chi_{C^*}(G \times H) \geq \min\{\chi_{C^*}(G), \chi_{C^*}(H)\}$$

4.5 Cartesian Product

In [13] it was shown that $\chi_q(G \square H) = \max\{\chi_q(G), \chi_q(H)\}$. Here, we adapt their proof to get the same result for χ_{C^*}.

Lemma 4.5.1. *Suppose $G \overset{C^*}{\to} F$ and $H \overset{C^*}{\to} K$, then $G \square H \overset{C^*}{\to} F \square K$.*

Proof. Since $G \overset{C^*}{\to} F$ and $H \overset{C^*}{\to} K$, then we know that there exist projections $\{E_{v,a}\}$ and $\{F_{x,r}\}$ with $v \in V(G)$, $x \in V(H)$, $a \in V(F)$, and $r \in V(K)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, satisfying (1) and (2) of definition 4.3.1. Consider the set of self-adjoint operators on $\mathcal{H} \otimes \mathcal{K}$ defined by $G_{(v,x),(a,r)} = E_{v,a} \otimes F_{x,r}$ for $(v, x) \in V(G \square H)$ and $(a, r) \in V(F \square K)$. Notice that,

$$\begin{aligned} G_{(v,x),(a,r)} G_{(v,x),(a,r)} &= (E_{v,a} \otimes F_{x,r})(E_{v,a} \otimes F_{x,r}) \\ &= E_{v,a} E_{v,a} \otimes F_{x,r} F_{x,r} = E_{v,a} \otimes F_{x,r} = G_{(v,x),(a,r)} \end{aligned}$$

since $E_{v,x}$ and $F_{a,r}$ are projections. Thus, each $G_{(v,x),(a,r)}$ is a projection. Furthermore, for each $(a,r) \in V(F\square K)$,

$$\sum_{(v,x)} G_{(v,x),(a,r)} = \sum_{(v,x)} E_{v,a} \otimes F_{x,r} = (\sum_{v} E_{v,a}) \otimes (\sum_{x} F_{x,r}) = (\sum_{x} E_{v,x}) \otimes I_{\mathcal{K}} = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$$

by (1).

Now, suppose $\big((v,x),(w,y)\big) \in E(G\square H)$ and $\big((a,r),(b,s)\big) \notin E(K\square F)$. By definition,

$$\big((v,x),(w,y)\big) \in E(G\square H) \iff v = w \text{ and } (x,y) \in E(H) \text{ or } (v,w) \in E(G) \text{ and } x = y$$

and

$$\big((a,r),(b,s)\big) \notin E(K\square F) \iff a \neq b \text{ or } (r,s) \notin E(K) \text{ and } (a,b) \notin E(F) \text{ or } r \neq s$$

$$G_{(v,x),(a,r)}G_{(w,y),(b,s)} = (E_{v,a} \otimes F_{x,r})(E_{w,b} \otimes F_{x,s}) = E_{v,a}E_{w,b} \otimes F_{x,r}F_{y,s}$$

We need to show $E_{v,a}E_{w,b} = 0$ or $F_{x,r}F_{y,s} = 0$. Without lost of generality, suppose $(v,w) \in E(G)$ and $x = y$. We know $F_{x,r}F_{x,s} = 0$, unless $r = s$ (by (1), we can pick $F_{x,r}$ and $F_{x,s}$ orthogonal from each other). Now, if $r = s$ and $\big((a,r),(b,s)\big) \notin E(K\square F)$, then $(a,b) \notin E(F)$, forcing $E_{v,a}E_{w,b} = 0$ by (2). Hence, $\{G_{v,a} : v \in V(G), a \in V(K)\}$ is a representation of the graph homomorphism game from $G\square H$ to $F\square K$.

□

Theorem 4.5.2.

$$\chi_{C^*}(G\square H) = \max\{\chi_{C^*}(G), \chi_{C^*}(H)\}$$

Proof. Since $G \to G\square H$ and $H \to G\square H$ always exist, $G \overset{C^*}{\to} G\square H$ and $H \overset{C^*}{\to} G\square H$ exist. Suppose $G \overset{C^*}{\to} K_n$ and $H \overset{C^*}{\to} K_m$ then,

$$G \overset{C^*}{\to} G\square H \overset{C^*}{\to} K_n \implies G \overset{C^*}{\to} K_n$$

$$H \overset{C^*}{\to} G\square H \overset{C^*}{\to} K_m \implies H \overset{C^*}{\to} K_m$$

and hence,

$$\chi_{C^*}(G\square H) \leq \max\{\chi_{C^*}(G), \chi_{C^*}(H)\}$$

If we let $c := \max\{\chi_{C^*}(G), \chi_{C^*}(H)\}$, then $G \overset{C^*}{\to} K_c$ and $H \overset{C^*}{\to} K_c$. If we apply the above lemma we get,

$$G\square H \overset{C^*}{\to} K_c\square K_c \overset{C^*}{\to} K_c$$

since $K_c\square K_c \to K_c$ and we get the desired result. □

4.6 C*-rank

We will use [28, Theorem 6.11] as a guiding principle to define a projective rank that corresponds to $\chi_{C^*}(G)$.

Definition 4.6.1. *Define the number $\xi_{C^*}(G)$ to be the minimum positive real number t such that there exist a Hilbert space $\mathcal{H}$, a unit vector $\eta \in \mathcal{H}$, a unital C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$, and projections $\{E_v\}_{v\in V(G)} \subset \mathcal{A}$ satisfying:*

1. $E_vE_w = 0$ *if* $(v, w) \in E(G)$
2. $\langle E_v\eta, \eta\rangle \geq \frac{1}{t}$ *for all* $v \in V(G)$.

Remark 4.6.2. Notice that there is an analog to condition (17) in [28] in this case too, namely that the map $s : A \to \mathbb{C}, s(X) = \langle X\eta, \eta\rangle$ is a state. But this is equivalent to the above condition since for any C^*-algebra we may apply the GNS construction to obtain the Hilbert space representation from the state.

Theorem 4.6.3. *For any graph G,*

$$\xi_{C^*}(G) \leq \chi_{C^*}(G)$$

Proof. Let $c := \chi_{C^*}(G)$. We know that there exists a Hilbert space $\mathcal{H}$, a unit vector η and projections $E_{v,x}, 1 \leq x \leq c$ satisfying

1. $\sum_x E_{v,x} = 1$ for all $v \in V(G)$

2. If $(v, w) \in E(G)$ and $x = y$, then

$$E_{v,x}E_{w,y} = 0.$$

Let $\tilde{\mathcal{H}}$ be the direct sum of c copies of $\mathcal{H}$, and set $\tilde{E}_v := E_{v,1} \oplus E_{v,2} \oplus \cdots \oplus E_{v,c}$. Now if we let $\tilde{\eta} := \eta \oplus \cdots \oplus \eta$ (c-times) notice that,

$$\langle \tilde{E}_v\tilde{\eta}, \tilde{\eta}\rangle = \frac{1}{c}\sum_{k=1}^{c}\langle E_{v,k}\eta, \eta\rangle = \frac{1}{c}$$

Finally we see that the set of projections $\{\tilde{E}_v : v \in V(G)\}$ along with $\tilde{\mathcal{H}}$ and $\tilde{\eta}$ satisfies all the conditions in the definition 4.6.1 and hence we get,

$$\xi_{C^*}(G) \leq \chi_{C^*}(G)$$

□

4.7 Factorization of Graph Homomorphisms

In this section, we show that the CP maps that arise from graph homomorphisms have a canonical factorization involving $\mathcal{A}(G,H)$.

Proposition 4.7.1. *Let G and H be graphs on n and m vertices, respectively. The map $\Gamma : M_n \to M_m(\mathcal{A}(G,H))$ defined on matrix units by $\Gamma(E_{v,w}) = \sum_{x,y} E_{x,y} \otimes e_{v,x}e_{w,y}$ is CP.*

Proof. Let $E_{v,x}, v \in V(G), x \in V(H)$ denote the $n \times m$ matrix units. Let $Z = \sum_{w,y} E_{w,y} \otimes e_{w,y} \in M_{n,m}(\mathcal{A}(G,H))$. Then

$$\Gamma(\sum_{v,w} c_{v,w}E_{v,w}) = Z^*\big(c_{v,w}E_{v,w} \otimes I\big)Z,$$

where I denotes the identity of $\mathcal{A}(G,H)$ and $\big(c_{v,w}E_{v,w} \otimes I\big) \in M_n(\mathcal{A}(G,H))$. □

Let $p(x,y|v,w) \in C^s_{qc}(n,m)$ be a winning qc-strategy for a graph homomorphism from G to H. Then there is a tracial state $\tau : \mathcal{A}(G,H) \to \mathbb{C}$ such that ϕ_p factors as $\phi_p = (id_m \otimes \tau) \circ \Gamma$, where $id_m \otimes \tau : M_m(\mathcal{A}(G,H)) \to M_m$. Conversely, if $\tau : \mathcal{A}(G,H) \to \mathbb{C}$ is any tracial state, then $(id_m \otimes \tau) \circ \Gamma = \phi_p$ for some winning qc-strategy $p(x,y|v,w) \in C^s_{qc}(n,m)$.

Similarly, this map ϕ_p arises from a winning q-strategy if and only if it arises from a τ that has a finite dimensional GNS representation and from a winning l-strategy if and only if it arises from a τ with an abelian GNS representation.

This factorization leads to the following result. Recall that $\vartheta(G)$ denotes the Lovasz theta function of a graph and $\|\phi\|_{cb}$ denotes the completely bounded norm of a map.

Lemma 4.7.2. *Let G be a graph on n vertices, let $\mathcal{H}$ be a Hilbert space, let $P_{v,w} \in B(\mathcal{H})$, $\forall v, w \in V(G)$ and regard $P = (P_{v,w})$ as an operator on $\mathcal{H} \otimes \mathbb{C}^n$. If*

1. $P = (P_{v,w}) \geq 0$,
2. $P_{v,v} = I_{\mathcal{H}}$,
3. $(v, w) \in E(G) \implies P_{v,w} = 0$,

then $\|P\| \leq \vartheta(G)$.

Proof. Given any vector $k = \sum_v e_v \otimes k_v$, let $h_v = k_v/\|k_v\|$,(set $h_v = 0$ when $k_v = 0$) and $\lambda_v = \|k_v\|$. Let $y = \sum_v \lambda_v e_v \in \mathbb{C}^n$ so that $\|y\|_{\mathbb{C}^n} = \|k\|$. Set $B_k = \big(\langle P_{v,w} h_w, h_v\rangle\big) \in M_n = B(\mathbb{C}^n)$, so that $\langle Pk, k\rangle_{\mathcal{H}\otimes\mathbb{C}^n} = \langle B_k y, y\rangle_{\mathbb{C}^n}$.

This observation shows that if for any $h_v \in \mathcal{H}$, $\forall v \in V(G)$ with $\|h_v\| = 1$ we let $\big(\langle P_{v,w} h_w, h_v\rangle\big) \in M_n = B(\mathbb{C}^n)$. Then

$$\|P\| = \sup\{\|(\langle P_{v,w} h_w, h_v\rangle)\|_{M_n} : \|h_v\| = 1\}.$$

Now by the above hypotheses each matrix $(\langle P_{v,w} h_w, h_v\rangle) \geq 0$, has all diagonal entries equal to 1 and $(v, w) \in E(G) \implies \langle P_{v,w} h_w, h_v\rangle = 0$. Thus, by [22], $\|(\langle P_{v,w} h_w, h_v\rangle)\| \leq \vartheta(G)$. □

Proposition 4.7.3. *Let $p(x, y|v, w) \in C^s_{qc}(n, m)$ be a winning qc-strategy for a graph homomorphism from G to H. Then $\|\phi_p\|_{cb} \leq \vartheta(G)$.*

Proof. Since $id_m \otimes \tau$ is a completely contractive map, we have that $\|\phi_p\|_{cb} \leq \|\Gamma\|_{cb}$. Since this map is CP, by [27] we have that

$$\|\Gamma\|_{cb} = \|\Gamma(I)\| = \|Z^*Z\| = \|ZZ^*\|.$$

Since $e^*_{w,y} = e_{w,y}$, we have

$$ZZ^* = \sum_{v,w,x,y} (E_{v,x} \otimes e_{v,x})(E_{w,y} \otimes e_{w,y})^* = \sum_{v,w} E_{v,w} \otimes \big(\sum_x e_{v,x}e_{w,x}\big).$$

Now if we let $p_{v,w}$ denote the (v, w)-entry of the above matrix in $M_n(\mathcal{A}(G, H))$, then $p_{v,v} = \sum_x e_{v,x} = I$. When $(v, w) \in E(G)$, then by Definition 4.3.1(3), we have that $p_{v,w} = 0$.

Hence, by the above lemma, $\|ZZ^*\| \leq \vartheta(G)$. □

4.8 Quantum Cores of Graphs

A **retract** of a graph G is a subgraph H of G such that there exists a graph homomorphism $f : G \to H$, called a **retraction** with $f(x) = x$ for any $x \in V(H)$. A **core** is a graph which does not retract to a proper subgraph [15].

Note that if $f : G \to G$ is an idempotent graph homomorphism and we define a graph H by setting $V(H) = f(V(G))$ and defining $(x, y) \in E(H)$ if and only if there exists $(v, w) \in E(G)$ with $f(v) = x, f(w) = y$, then H is a subgraph of G and f is a retraction onto H. We denote H by $f(G)$.

The following result is central to proofs of the existence of cores of graphs.

Theorem 4.8.1 ([15])**.** *Let f be an endomorphism of a graph G. Then there is an n*

such that f^n *is idempotent and a retraction onto* $R = f^n(G)$.

Our goal in this section is to attempt to define a quantum analogue of the core using completely positive maps, in particular we will use the above theorem as a guiding principle.

For $A = (a_{ij}) \in M_n$, denote $||A||_1 = \sum_{i,j} |a_{ij}|$ and $\sigma(A) = \sum_{i,j} a_{ij}$. Let $\phi_p : M_n \to M_m$, $\phi_p(E_{vw}) = \sum_{x,y} p(x,y|v,w)E_{xy}$, for some $p(x,y|v,w) \in C^s_{vect}(n,m)$. Before we continue our discussions on cores we will need the following facts,

Lemma 4.8.2.

$$\sigma(\phi_p(A)) = \sigma(A)$$

Proof. By linearity it is enough to show the claim for matrix units,

$$\sigma(\phi_p(E_{vw})) = \sum_{x,y} p(x,y|v,w) = \sum_{x,y} \langle h_{v,x}, h_{w,y} \rangle =$$

$$\langle \sum_x h_{v,x}, \sum_y h_{w,y} \rangle = \langle \eta, \eta \rangle = 1 = \sigma(E_{vw})$$

□

Lemma 4.8.3. *Let* $A = (a_{vw})$ *be a matrix, then*

$$||\phi_p(A)||_1 \leq ||A||_1$$

If the entries of A *are non-negative, then* $\|\phi_p(A)\|_1 = \|A\|_1$.

Proof. We have,

$$||\phi_p(A)||_1 = \sum_{x,y} |\sum_{v,w} p(x,y|v,w)a_{v,w}| \leq \sum_{v,w} |a_{v,w}|(\sum_{x,y} p(x,y|v,w))$$

$$= \sum_{v,w} |a_{vw}| = ||A||_1$$

When the entries of A are all non-negative, the first inequality is an equality. □

For the next step in our construction, we need to recall the concept of a *Banach generalized limit*. A Banach generalized limit is a positive linear functional f on $\ell^\infty(\mathbb{N})$, such that:

- if $(a_k) \in \ell^\infty(\mathbb{N})$ and $\lim_k a_k$ exists, then $f((a_k)) = \lim_k a_k$,
- if $b_k = a_{k+1}$, then $f((b_k)) = f((a_k))$.

The existence and construction of these are presented in [5], along with many of their other properties. Often a Banach generalized limit functional is written as *glim*.

Now fix a Banach generalized limit *glim*, assume that $n = m$, and that $\phi_p : M_n \to M_n$, $\phi_p(E_{vw}) = \sum_{x,y} p(x,y|v,w)E_{xy}$, for some $p(x,y|v,w) \in C^s_{qc}(n,n)$. Fix a matrix $A \in M_n$ and set

$$a_{x,y}(k) = \langle \phi_p^k(A)e_y, e_x \rangle$$

so that $\phi_p^k(A) = \sum_{x,y} a_{x,y}(k)E_{x,y}$. By Lemma 4.8.3, for every pair, (x,y) the sequence $(a_{x,y}(k)) \in \ell^\infty(\mathbb{N})$.

We define a map, $\psi_p : M_n \to M_n$ by setting

$$\psi_p(A) = \sum_{x,y} glim((a_{x,y}(k)))E_{x,y}.$$

Alternatively, we can write this as,

$$\psi_p(A) = (id_n \otimes glim)\phi_p^k(A).$$

Proposition 4.8.4. *Let* $\big(p(x,y|v,w)\big) \in C^s_{vect}(n,n)$ *and let* $\psi_p : M_n \to M_n$ *be the map obtained as above via some Banach generalized limit, glim. Then:*

1. ψ_p *is CP,*
2. $\sigma(\psi_p(A)) = \sigma(A)$ *for all* $A \in M_n$,
3. $\|\psi_p(A)\|_1 \le \|A\|_1$,
4. $\psi_p \circ \phi_p = \phi_p \circ \psi_p = \psi_p$,
5. $\psi_p \circ \psi_p = \psi_p$.

Proof. The first two properties follow from the linearity of the glim functional. For example, if $A = (a_{x,y})$ and $h = (h_1, ..., h_n) \in \mathbb{C}^n$, then

$$\langle \psi_p(A)h, h\rangle = \sum_{x,y} glim((a_{x,y}(k)))h_y\overline{h_x} = glim\big(\sum_{x,y} a_{x,y}(k)h_y\overline{h_x}\big)$$

$$= glim\big(\langle \phi_p^k(A)h, h\rangle\big)$$

If $A \ge 0$, then $\phi^k(A) \ge 0$ for all k, and so is the above function of k. Since $glim$ is a positive linear functional, we find $A \ge 0$ implies $\langle \psi_p(A)h, h\rangle \ge 0$, for all h. This shows that ψ_p is a positive map. The proof that it is CP is similar, as is the proof that it preserves σ.

The proof of the third property is similar to the proof of Lemma 4.8.3.

For the next claim, we have that

$$\psi_p(\phi_p(A)) = (id \otimes glim)(\phi_p^{k+1}(A)) = (id \otimes glim)(\phi_p^k(A)) = \psi_p(A).$$

If we set $\psi_p(A) = \sum_{v,w} b_{v,w}E_{v,w}$, with $b_{v,w} = glim(a_{v,w}(k))$, then

$$\phi_p(\psi_p(A)) = \sum_{x,y,v,w} p(x,y|v,w) b_{v,w} E_{x,y}$$

$$= \sum_{x,y} glim\big(\sum_{v,w} p(x,y|v,w) a_{v,w}(k)\big) E_{x,y} = \sum_{x,y} glim\big(a_{x,y}(k+1)\big) E_{x,y} = \psi_p(A)$$

Finally, to see the last claim, we have that

$$\psi_p(\psi_p(A)) = (id \otimes glim)(\phi_p^k(\psi_p(A))) = (id \otimes glim)(\psi_p(A))) = \psi_p(A),$$

since the *glim* of a constant sequence is equal to the constant. □

Theorem 4.8.5. *Let G be a graph on n vertices, let $x \in \{l, qa, qc, vect\}$ and let $p(x,y|v,w) \in Q_x^s(n,n)$ be a winning x-strategy implementing a quantum graph x-homomorphism from G to G. Set $p_1(x,y|v,w) = p(x,y|v,w)$ and recursively define,*

$$p_{k+1}(x,y|v,w) = \sum_{a,b} p(x,y|a,b) p_k(a,b|v,w).$$

If we set $r(x,y|v,w) = glim\big(p_k(x,y|v,w)\big)$, then $r(x,y|v,w) \in C_x^s(n,n)$ is a winning x-strategy implementing a graph x-homomorphism from G to G such that:

1. $\psi_p = \phi_r$,
2. $r(x,y|v,w) = \sum_{a,b} r(x,y|a,b) r(a,b|v,w)$.

Proof. By Theorem 4.2.7, $\phi_p^k = \phi_{p_k}$, and p_k is a winning x-strategy for a graph x-homomorphism from G to G. Thus,

$$\psi_p(E_{v,w}) = (id \otimes glim)(\phi_p^k(E_{v,w})) = (id \otimes glim)(\phi_{p_k}(E_{v,w}))$$

$$= \sum_{x,y} glim\big(p_k(x,y|v,w)\big)E_{x,y} = \phi_r(E_{v,w}).$$

Thus, (1) follows.

Since $\phi_r \circ \phi_r = \psi_p \circ \psi_p = \psi_p = \phi_r$, the second claim follows from Proposition 4.2.6.

Finally, if a bounded sequence of matrices $A_k = \big(a_{v,w}(k)\big) \in M_n$ all belong to a closed set, then it is not hard to see that $A = \big(glim(a_{v,w}(k))\big)$ also belongs to the same closed set. Thus, since $\big(p_k(x,y|v,w)\big)$ is in the closed set $C_x^s(n,n)$ for all k, we have that $\big(r(x,y|v,w)\big) \in C_x^s(n,n)$. Also, since p_k is a winning x-strategy for a graph x-homomorphism of G, for all k, we have that for all k, $\big(p_k(x,y|v,w)\big)$ is zero in certain entries. Since the *glim* of the 0 sequence is again 0, we will have that $\big(r(x,y|v,w)\big)$ is also 0 in these entries. Hence, r is a winning x-strategy for a graph x-homomorphism. □

Remark 4.8.6. In the case that p is a winning q-strategy implementing a graph q-homomorphism, all we can say about r is that it is a winning qa-strategy implementing a graph qa-homomorphism, since we do not know if the set $C_q^s(n,n)$ is closed.

There is a natural partial order on idempotent CP maps on M_n. Given two idempotent maps $\phi, \psi : M_n \to M_n$ we set $\psi \leq \phi$ if and only if $\psi \circ \phi = \phi \circ \psi = \psi$.

Theorem 4.8.7. *Let $x \in \{l, qa, qc, vect\}$, then there exists $r(x,y|v,w) \in C_x^s(n,n)$ implementing a quantum x-homomorphism, such that $\phi_r : M_n \to M_n$ is idempotent and is minimal in the partial order on idempotent maps of the form ϕ_p implemented by a quantum x-homomorphism of G.*

Remark 4.8.8. It is important to note that we are not claiming that ϕ_r can be chosen minimal among all idempotent CP maps, just minimal among all such maps that implement a quantum x-homomorphism of G.

Proof. Quantum x-homomorphisms always exist, since the identity map on G belongs to the l-homomorphisms, which is the smallest set. By the last theorem, we see that beginning with any correlation p implementing a quantum x-homomorphism, there exists a correlation r implementing a quantum x-homomorphism with ϕ_r idempotent.

It remains to show the minimality claim. We will invoke Zorn's lemma and show that every totally ordered set of such correlations has a lower bound. Let $\{p_t(x,y|v,w) : t \in T\} \subset C_x^s(n,n)$ and T a totally ordered set, where all $p_t(x,y|v,w)$ implement a quantum x-homomorphisms, with ϕ_{p_t} idempotent, and $\phi_{p_t} \leq \phi_{p_s}$, whenever $s \leq t$.

These define a net in the compact set $C_x^s(n,n)$ and so we may choose a convergent subnet. Now it is easily checked that if we define $p(x,y|v,w)$ to be the limit point of this subnet, then it implements a quantum x-homomorphism, ϕ_p is idempotent, and $\phi_p \leq \phi_{p_t}$ for all $t \in T$. □

Definition 4.8.9. *Let $x \in \{l, qa, qc, vect\}$, then a* **quantum x-core for** *G is any $r(x,y|v,w) \in C_x^s(n,n)$ that implements a quantum x-homomorphism such that ϕ_r is idempotent and minimal among all ϕ_p implemented by a quantum x-homomorphism of G.*

4.9 Conclusion and Open Problems

Our main goal in this Chapter was to provide a unified framework in which one can study graph homomorphisms through the lens of Operator Algebra theory. We saw how you can use CP maps to study individual correlations and how to use such maps to defined a generalization of the core of a graph. We also saw how to abstract the notion of a quantum graph homomorphism by defining a C*-algebra that encodes the important information about the homomorphisms. A surprising observation in this Chapter is the fact that by studying the representation of $\mathcal{A}(G, K_n)$ we managed to link the representation of these algebras with the computational complexity of the quantum chromatic numbers. In addition, we introduced a new chromatic number via this generalized notion of a homomorphism.

There is still work to be done. The set $C_{qa}(n, m)$ continues to be mystery. We where unable to characterize $G \stackrel{qa}{\to} H$, nor were we able to say anything about the type of representations of the C*-algebra generated by such homomorphism. Moreover, we still know very little about the C*-algebra $A(G, H)$ (e.g. Nuclear?, AF?). Also, we where unable to determine the complexity level of determining if $\mathcal{A}(G, K_m)$ exists, i.e. $\chi_{c^*}(G) \leq m$.

Also, if we could show that whenever families of projections on an infinite dimensional Hilbert space exist that satisfy the relations for $\mathcal{A}(G, K_m)$ to exist, then these relations could be met by projections on a finite dimensional space, then it follows that

$$\chi_q(G) = \chi_{qa}(G) = \chi_{qc}(G) = \chi_{c^*}(G).$$

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