EQUIVALENCE OF PARACOMPACTNESS, STRONG SCREENABILITY, FULL NORMALITY AND COLLECTIONWISE NORMALITY, IN A MOORE SPACE

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Dionysios M. Phocas

August, 1966

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ABSTRACT

The purpose of this thesis is to prove that in a regular, developable, topological space (MOORE SPACE) the properties of paracompactness, strong screenability, full normality, and collectionwise normality are equivalent, and each one implies normality, complete normality, and screenability.

An example is given of a screenable Moore space which is not paracompact.

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INTRODUCTION

Regular, developable topological spaces (MOORE SPACES) have recieved considerable attention since this class of spaces includes all metric spaces.

Several properties, which are known to be strong enough to establish a metric in a Moore space, are proved equivalent in this thesis.

In particular, the properties of paracompactness, strong screenability, full normality, and collectionwise normality are proved equivalent and, in addition, it is proved that each of these properties implies normality, complete normality, and screenability in a Moore space.

It remains unknown as to whether normality implies screenability, and it remains unknown as to whether normality implies collectionwise normality in a Moore space.

An example is given of a screenable Moore space which is not strongly screenable.

Although each of the results is known, each of the proofs and the construction of the examples were obtained independently by the writer of this thesis.

CHAPTER I

NOTATION, DEFINITIONS, AND STATEMENT OF AXIOMS

Point, region, and set will be taken as undefined terms.[1]

DEFINITION 1. The statement that p is a limit point of the point set M means that, if R is a region containing p, then R contains a point of M distinct from p.

DEFINITION 2. The point set 0 is said to be an open set, or domain, if for each point p of 0 there exists a region R containing p, such that R is a subset of 0.

DEFINITION 3. The point p is said to be a boundary point of the point set M if and only if every region that contains p contains a point of M and a point that does not belong to M.

DEFINITION 4. The boundary of a point set M is the set of all boundary points of M.

DEFINITION 5. Two point sets are said to be mutually separated if they are mutually exclusive and neither of them contains a limit point of the other.

DEFINITION 6. The collection G is said to be an open covering of the point set S if every element of G is an open set; and it is true that if p is a point of S, then p is contained in some element of the collection G.

DEFINITION 7. The point set H is said to be the intersection of the point sets M and N, if and only if, H is the set of all points p such that p is a point of both point sets M and N.

NOTATION: The intersection of two point sets M and N will be denoted by $M \cdot N$.

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[1] Moore, Foundations of Point Set Theory, p. 1.

If M is a point set, then $\overline{M}$ is meant the point set such that p belongs, if and only if p is a point of M, or p is a limit point of M. If G is a collection of point sets, then by G* is meant the point set such that p is a point of G*, if and only if p is a point of some point set of the collection G.

If H and K are two point sets, then by H - K is meant the set, such that p is a point of H - K, if and only if, p is a point of H and p is not a point of K.

Let Λ be an index set, then small Greek letters will denote elements of Λ.

DEFINITION 8. The collection G of sets is said to be discrete if the elements of G are mutually separated; and if G'' is any subcollection of G, then $\overline{G''*} = G''*$.

DEFINITION 9. The collection H' of point sets is said to be a refinement of the collection H, if and only if every element of H' is a subset of some element of H.

DEFINITION 10. The star of a point set A with respect to an open covering H is the sum of all elements of H that intersect A.

AXIOM 0. Every region is a point set.

AXIOM 1. There exists a sequence G_1, G_2, G_3..., such that (1) for each n, G_n is a collection of regions covering the space, (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is a region, A is a point of R, and B is a point of R, then there exists a positive integer m such that if g is any region belonging to G_n for $n \geq m$ and g contains A, then $\overline{g}$ is a subset of R, and $\overline{g}$ does not contain B.

DEFINITION 11. The statement that the topological space S is a Moore space means that there exists a sequence of collections of regions satisfying Axiom 0 and Axiom 1.

DEFINITION 12. The statement that the space S is screenable[1] means that if H is an open covering of space S, there exists a sequence H_1, H_2, H_3, ... such that H_i is a collection of mutually exclusive domains, each element of which is a subset of some element of H, and $\sum_{i=1}^{\infty} H_i$ covers S.

DEFINITION 13. The statement that the space S is strongly screenable[1] means that if H is an open covering of the space, there exists a sequence H_1, H_2, H_3, ... such that H_i is a discrete collection of mutually exclusive domains each element of which is a subset of some element of H, and $\sum_{i=1}^{\infty} H_i$ covers the space.

DEFINITION 14. The statement that the space S is collectionwise normal[1] means that if H is a discrete collection of point sets there then exists a collection G of mutually exclusive domains, such that G^* covers H^* and no element of G intersects two of the sets of H.

DEFINITION 15. The statement that the space S is paracompact[1] means that if H is an open covering of S, there then exists an open covering H' and H'' of S such that H' refines H and no element of H'' intersects infinitely many elements of H'.

REMARK: The open covering of definition 15 is also called locally finite refinement of H.

DEFINITION 16. The statement that the space S is fully normal[1] means that if H is an open covering of S there then exists an open covering H' of S such that the star of each point of S with respect to H' is a subset of some element of H.

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[1] Bing, Metrization of Topological Spaces, pp. 176-177.

DEFINITION 17. The statement that the space S is normal means that if H and K are two mutually exclusive closed point sets, there then exist two domains, D_H and D_K , covering H and K respectively, such that D_H and D_K are mutually exclusive.

DEFINITION 18. The statement that S is completely normal means that if H and K are two mutually exclusive closed point sets there then exist two mutually exclusive domains, D_H and D_K , covering H and K respectively, such that $\overline{D_H} \cdot \overline{D_K}$ does not exist.

CHAPTER II

EQUIVALENCE OF PARACOMPACTNESS

THEOREM 1. If S is a strongly screenable Moore space, then S is paracompact.

PROOF: Let H be an open covering of S. We must show that there exists an open covering H' of S, such that H' is a locally finite refinement of H.

Let H_1, H_2, H_3, ... be a sequence of discrete collections of open sets satisfying the conditions of strong screenability with respect to H.

Denote by W_1 the collection of open sets such that w belongs to W_1 if and only if w is an element of H_1. Denote by β_1 the boundary of $W_1{*}$. For each point belonging to β_1, let g_p be a region of G_1 containing p, and denote by G_{β_1} the collection of all such regions g_p. Let $D_1 = G_{\beta_1}^*$; clearly D_1 is a domain.

Let $W_1' = W_1^* - D_1$ and let W_2 be a collection of open sets such that $w \epsilon W_2$ if and only if there is an element h of H_2 such that $w = h - h \cdot W_1'$.

Denote by β_2 the boundary of $(W_1 + W_2)^*$. For each point p that belongs to β_2 let g_p denote a region of G_2 that contains p, such that if p belongs to β_1, then $\overline{g}_p$ is a subset of some region of G_{β_1}; otherwise, g_p is simply a region of G_2. Let G_{β_2} denote the collection of all such regions g_p, and denote by D_2 the domain such that $D_2 = G_{\beta_2}^*$.

Let $W_2' = (W_1 + W_2)^* - D_2$ and denote by W_3 the collection of open sets, such that w is an element of W_3 if and only if there is an element h of H_3 such that $w = h - h \cdot W_2'$.

In general, let $S_{n-1} = (W_1 + W_2 + \ldots W_{n-1})$, and denote by β_{n-1} the boundary of S_{n-1}. For each point p that belongs to β_{n-1} let g_p denote a region of G_{n-1} that contains p, and if p is a point of β_{n-2}, then $\overline{g_p}$ is a subset of some region of $G_{\beta_{n-2}}$. Let D_{n-1} denote the domain such that $D_{n-1} = G_{\beta^*_{n-1}}$.

Let $W'_{n-1} = S^*_{n-1} - D_{n-1}$, and denote by W_n the collection such that w is an element of W_n if and only if there is an element h of H_n such that $w = h - h \cdot W'_{n-1}$.

Let $H' = W_1, W_2, W_3, \ldots$.

Suppose p is a point of S. There exists a least integer n such that some element of H_n contains p. Therefore, some element of W_n contains p; moreover, every element of W_n is a subset of some element of H_n and hence, is a subset of some element of H.

It follows then that $\sum_{i=1}^{\infty} W_i$ covers the space and H' is a refinement of H.

It remains to be proved that H' is locally finite.

Assuming the contrary, let p be a point of S such that if R is a region containing p, then R intersects infinitely many elements of H'. There exists a least integer m such that p belongs to some element of W_m. The point p is not a limit point of any sequence of elements of W_i since W_i is discrete (discreteness of W_i follows from the discreteness of H_i) and neither is p a limit point of the sum of the elements of $W = W_1 + W_2 + \ldots W_m$, since W is the sum of finitely many discrete collections.

Since p does not belong to β_m, the boundary of W^*_m, there then exists an integer n such that if g is a region of G_n and contains p, then $\overline{g}$ is a subset of W^*_m, and there exist an integer N such that p does not belong to D_N.

There exists an integer J, such that if g is a region of G_i, $i > J$ and g contains p, then $\overline{g}$ is a subset of W_m^* and does not intersect $\overline{D_N}$, and hence, g does not intersect W_N^* for $N > m$.

So p is not a limit point of any sequence of elements belonging to $\{W_i\}_{i=m+1}^{\infty}$.

It then follows that for each point p of S there exists a region R_p containing p, such that R_p intersects at most finitely many elements of H'.

It was shown that given an open covering H of S, there exists an open covering H' of S such that H' is a locally finite refinement of H.

Hence, S is paracompact.

<u>THEOREM 2</u>: If S is a paracompact Moore space, then S is fully normal.

PROOF: Let H be an open covering of the space, and let H' be an open refinement of H, such that H' covers the space, and the closure of each element of H' is a subset of some element of H.

There exists an open covering H'' of S, such that H'' is locally finite refinement of H' and H; moreover, the closure of each element of H'' is a subset of some element of H. Since H'' is locally finite, every point p of S belongs to at most finitely many elements of H'', say g_p^1, g_p^2, ... g_p^N. Let $g_p = g_p^1 \cdot g_p^2 \cdot g_p^3 \cdot \ldots g_p^N$, since g_p is the intersection of finitely many open sets, it then follows that g_p is an open set containing p.

Let G_p denote the collection such that g is an element of G_p if and only if, g is an element of H'', and $\overline{g}$ does not contain p. Since H'' is locally finite it follows that $\overline{G_p^*}$ does not contain p, and $\overline{G_p^*}$ is a closed set. Let $C_p = S - \overline{G_p^*}$, then C_p is an open set, and contains the point P.

For each point p of S, let $h_p = C_p \cdot g_p$; h_p is the intersection of two open sets, it then follows that h_p is an open set and contains p.

Let H_1 denote the collection such that g is an element of H_1 if and only if g is some h_p for some point p of S. It is clear that H_1 is an open covering of the space, and H_1 refines H.

It remains to be shown that the star of each point x of the space with respect to H_1 is a subset of some element of H.

Assuming the contrary, let x be a point of S, and let h_x be some element of H'' that contains x and assume that some element h_p of H_1 contains x, such that h_p is not a subset of h_x.

Since h_x does not contain p, it follows from the construction that h_x is a subset of G^*_p , and therefore h_x does not intersect $S - \overline{G^*_p}$.

Since h_p is a subset of $S - \overline{G^*_p}$ then h_p does not contain x, and this is a contradiction. Therefore, the star of x with respect to H_1 is a subset of h_x , where h_x is a subset of some element of H. Therefore, the star of x with respect to H_1 is a subset of some element of H. Since x was an arbitrary point of S, it then follows that H_1 is a star refinement of H.

Therefore, S is fully normal.

<u>THEOREM 3</u>: If S is a fully normal Moore space, then S is collectionwise normal.

PROOF: Let G be a discrete collection of mutually exclusive closed point sets. Denote by H an open covering of S, such that the elements of H are regions, and if h is an element of H and h intersects g, some element of G then $\overline{h}$ does not intersect $G^* - g$.

Since S is fully normal, there exists a refinement H' of H such that if p is a point of S then the star of p with respect to the open covering H' is a subset of some element of H.

For each element g of the collection G let O_g denote the sub-collection of H', such that O_g consists of all elements of H' that intersect g.

Let $O = \{O_g\}_{g \in G}$. Consider any two elements of O, say O_{g_α} and O_{g_w} .

Assume that O_{g_α} and O_{g_w} have a point p in common. Some element, say h_p of H' contains the star of p with respect to H'. If h_p intersects g_α , then O_{g_w} cannot contain p; for if it did, then the star of p with respect to H would not be a subset of h_p. So h_p does not intersect g_α , and for the same reason h_p does not intersect g_w. It follows then that the star of p with respect to H' is not a subset of any element of H. Hence, O_{g_α} and O_{g_w} are mutually exclusive.

Then the collection O is a collection of mutually exclusive domains covering G, such that no element of O intersects two elements of G. Therefore, S is collectionwise normal.

<u>THEOREM 4</u>: If S is a collectionwise normal Moore space, then S is strongly screenable.

PROOF: Let H be an open covering of S. Let W be a well-ordering of the elements of H. Denote by $P_{h,i}$ the point set, such that p is a point of $P_{h,i}$, if and only if (a) h is the first element of H in W that contains p, and (b) every region g of G_i that contains p is such that $\overline{g}$ is a subset of h. Then each $P_{h,i}$ is a closed set. Now assuming the contrary, suppose that p is a limit point of $P_{h,i}$ such that p does not belong to $P_{h,i}$. Then there exist a region g of G_i such that g contains p and $\overline{g}$ is not a subset of h. Since p is a limit point of $P_{h,i}$, then g must contain a point of $P_{h,i}$, and this is contradictory.

For each i let X_i denote the collection such that X is an element, if and only if there exists an element h of W such that X is $P_{h,i}$. The fact that the elements of X_i are mutually exclusive follows from the construction.

To see that X_i is a discrete collection of mutually exclusive closed point sets, assume that there is a subcollection X_i' of X_i and a point p of S - $X_i'^*$ such that p is a limit point of $X_i'^*$.

Let h_α be the first element of H in W that contains p. If g is a region of G_i such that g contains p, then g cannot intersect $P_{h_\beta,i}$ for $\beta < \alpha$. For if it did, then g would contain a point of $P_{h_\beta,i}$ and a point not in h_β , since h_α follows h_β in W and h_α was the first element of H in W containing the point p. Therefore, $\overline{g}$ would not be a subset of h_β , contrary to the construction.

Assume that X_i' is a subcollection of X_i such that the elements of X_i' are subsets of elements of H that follow h_α in W. There exist a g of G_i, such that g contains p, and $\overline{g}$ is a subset of h_α . Since p is a limit point of X' , then g intersects some element $P_{h_\beta, i}$ of X_i' , for $\beta > \alpha$.

Let $Y = g \cdot P_{h_\beta, i}$; then h_β is the first element of H in W containing Y; but h_α precedes h_β in W, and Y is a subset of h_α . Therefore, the assumption that p was a limit point of $X_i'^*$ leads to a contradiction. Hence, X_i is a discrete collection of mutually exclusively closed point sets.

Let p be a point of S, and let h_p be the first element of H in W containing p. Then there exists an integer m such that, if g is a region of G_m and g contains p then $\overline{g}$ is a subset of h_p. Therefore, p is a point of $P_{h_p, m}$.

$$\text{Hence,} \quad \sum_{i=1}^{\infty} X_i = S \quad .$$

Since S is a collectionwise normal, and for each i, X_i is a discrete collection of mutually exclusive closed point sets, there exists a collection O_i of mutually exclusive domains covering X_i^* such that no element of O_i intersects two elements of X_i . Now, if $P_{h_\alpha, i}$ is an element of X_i let $O_{\alpha, i}$ be the element of O_i that covers $P_{h_\alpha, i}$, and let $O'_{\alpha, i}$ be such that $O'_{\alpha, i} = O_{\alpha, i} \cdot h_\alpha$.

Let O_i' be the collection of all $O'_{\alpha, i}$. It is clear that O_i' covers X_i ; moreover, the elements of O_i' are mutually exclusive and if $O'_{\alpha, i}$ is an element of O_i' , then $O'_{\alpha, i}$ is a subset of h_α , where h_α is an element of H.

Let L_i denote the point set such that p is a point of L_i, if and only if (a) p is a point of the boundary of some element of O_i' , or (b) p is a limit point of the set whose elements are the elements of O_i' .

L_i is a closed set; moreover, L_i and X_i^* have no point in common. By theorems 5 and 6 of Chapter III, S is normal, so there exists two domains D_{L_i} and $D^*_{X_i}$, covering L_i and X_i^* respectively such that $\overline{D_{L_i}}$ and $D^*_{X_i}$ have no point in common.

Let H_i denote the collection such that x belongs to H_i, if and only if there exists an element O of O_i' such that, $x = O \cdot D^*_{X_i}$. Then H_i is a discrete collection of mutually exclusive domains covering X_i, and each element of H_i is a subset of some element of H.

The sequence H_1, H_2, H_3, ... satisfies all the conditions of strong screenability, and the theorem follows.

CHAPTER III

NORMALITY, COMPLETE NORMALITY, SCREENABILITY

THEOREM 5. If S is a normal Moore space, then S is completely normal.

PROOF: Let H and K be two mutually exclusive closed point sets. Since S is normal, there exist two mutually exclusive domains, D_H and D_K, covering H and K respectively. Let β_H and β_K denote the boundary of D_H and D_K respectively.

H and β_H are two mutually exclusive closed point sets, and since S is normal, there exist two mutually exclusive domains, D'_H and D_{β_H}, covering H and β_H respectively. Let D denote the domain such that $D = D_H - \overline{D_{\beta_H}}$. It is clear that D and D_K are mutually exclusive domains where D covers H and D_K covers K. Moreover, $\overline{D}$ and $\overline{D}_K$ have no point in common.

Hence, S is completely normal.

THEOREM 6. If S is a collectionwise normal Moore space, then S is normal.

PROOF: Let H and K be two mutually exclusive closed point sets. Now the collection consisting of only the sets H and K is a discrete collection of closed point sets. Since S is collectionwise normal, there exist two domains, D_H and D_K, covering H and K respectively, such that D_H and D_K are mutually exclusive.

Hence, S is normal.

COROLLARY 1. If S is a strongly screenable Moore space, then S is screenable.

PROOF: This is an immediate consequence of the definition of strong screenability.

CHAPTER IV

AN EXAMPLE OF A SCREENABLE MOORE SPACE THAT IS NOT PARACOMPACT

Denote by S_1 the points of the cartesian plane of y-coordinate greater than zero. Denote by S_2 and S_3 the rationals and the irrationals of the x-axis respectively. Let S denote the space such that p is a point of S if and only if p is a point of either S_1, S_2, or S_3.

A region of G_n that contains a point x of S_1 is the degenerate region, X. If x is a point of S_2, then a region of G_n that contains x is a circle with center at (x,y), $0 < y \leq \frac{1}{n}$ and tangent to the x-axis at the point $(x,0)$.

If x is a point of S_3 then a region of G_n that contains x is a vertical line at x of height less than $\frac{1}{n}$.

It is clear that the axioms are satisfied at all the points of S_1. It is also clear that axiom 0 and the first two parts of axiom 1 are satisfied at all points of S_2 and S_3.

Let p be any point of S_2 and R be a region containing p, and let L be the radius of R, then there exists an integer N greater than zero such that $\frac{1}{N} < L$, and if R_1 is a circle of radius less than $\frac{1}{N}$ tangent to the x-axis at the point p, then $\overline{R}_1$ is a subset of R. If x is a point of S_3, and L is a vertical line at the point x, then there is a line L_1 vertical at x of length less than $\frac{L}{2}$ so L_1 is a subset of L. This shows that the third part of axiom 1 is also satisfied at every point of S_2 and S_3.

It then follows that S is a Moore space.

1. S is screenable.

Let H be an open covering of S. Let H_1 denote the points of S_1. Let H_2 be a collection of regions of G_1 such that g is an element of H_2 if and only if (a) g contains a point of S_2, and (b) $\overline{g}$ is a subset of some element of H. Let H_3 denote a collection of regions of G_1 such that g is an element of H_3 if and only if (a) g contains a point of S_3 and, (b) $\overline{g}$ is a subset of some element of H. Since S_2 is countable then H_2 is countable.

Let $H_2 = h_1, h_2, h_3, \ldots$. It is clear that H_1 and H_3 are collections of mutually exclusive open sets.

2. S is not normal.

Assume the contrary. S_2 and S_3 are two mutually exclusive closed point sets. Since S is assumed to be normal then there exists two mutually exclusive domains, H_2 and H_3, covering S_2 and S_3 respectively. S_2 is a countable discrete collection of points, and since S is normal there exist a collection O such that o_i is an element of O if and only if (a) o_i is an open set and contains a point x_i of S_2 and (b) $\overline{o_i}$ is a subset of H_2. Moreover O* covers S_2 and no element of O contains two points of S_2.

Let $R = r_1, r_2, r_3, \ldots$ be a simply infinite sequence such that the elements of R are the points of S_2. Let O_1 be the element of O that covers r_1, and let ρ_1 denote the radius of O_1.

Let T_1 denote the set bounded by the interval $[x_1, x_1 + \rho_1]$, and the arc of O_1 over the interval $[x_1, x_1 + \rho_1]$.

Let r_{n_1} be the first element of R that follows r_1 such that r_{n_1} belongs to the interval $[x_1, x_1 + \rho_1]$ and O_{n_1} the element of O that contains r_{n_1} is a subset of T_1. Let ρ_{n_1} denote the radius of O_{n_1}.

Denote by I_1 the interval $[r_{n_1} - \rho_{n_1}, r_1]$, and by ℓ_1 the vertical line $x = r_{n_1} - \rho_1$. Let T_2 denote the set bounded by I_1, ℓ_1, and the arc of O_{n_1} over the interval I_1. Let r_{n_2} be the first element of R that follows r_{n_1}

such that r_{n_2} is a point of the interval I_1 and O_{n_2} the element of O that contains r_{n_2} is a subset of T_2. Denote by ρ_3 the radius of O_{n_2}, and by I_2 the interval $[r_{n_2}, r_{n_2} + \rho_3]$. Let T_3 be the set bounded by the interval I_2 the vertical line ℓ_2, $x = r_{n_2} + \rho_3$ and the arc of O_{n_2} over the interval I_2.

Let r_{n_3} be the first element of R that follows r_{n_2} such that r_{n_3} is a point of the interval I_2 and O_{n_3} the element of O that contains r_{n_3} is a subset of T_3. Continue this process.

We obtain a sequence $O' = o_1, o_{n_2}, o_{n_3}, \ldots$ and $R' = r_1, r_{n_2}, \ldots$. Both sequences R', and o' converge to some point p.

Assume that p is a point of R. It is clear that p is not a point of R', for if it was then for some integer n, p would not be a point of I_n. From the assumption then follows that either p follows every point of R' in R or that the element O_p of O that contains p is not a subset of T_n for any n, and therefore O_p intersects every element of O'.

Since R is a simple infinite sequence, then p cannot follow every element of R' in R_j from the nature of the collection O it follows that O_p cannot intersect any of the elements of O'. So p is not a point of S_2. It then follows that p is a point of S_3.

Let L be a vertical line that contains p such that L is a subset of H_3. Let d be the length of L. There is a $k > o$ such that the point r_{n_j} of R' is such that $|p - r_{n_j}| < \frac{d}{4}$, $j = k, (k+1), (k+2), \ldots$ and p is a point of the interval $I_{n_j}, j = k, k+1, \ldots$. Consider now the circle O_{n_j} of radius p_{n_j} tangent to the x-axis at the point r_{n_j}; then L intersects O_{n_j}. And since L is a subset of H_3, it then follows that H_3 intersects O_{n_j} and therefore H_2. This is a contradiction. So H_2 and H_3 are not mutually exclusive. Therefore, S is not normal.

Since paracompactness is equivalent to collectionwise normality, then by theorem 6 it follows that S is not collectionwise normal, and hence, not paracompact.

CHAPTER V

AN EXAMPLE OF A SEMI-METRIZABLE SPACE THAT IS NOT PARACOMPACT

In this chapter it will be established that every metric space is paracompact, and also that there exists a semi-metrizable space that is not paracompact.

DEFINITION 1. A topological space S is said to be semi-metrizable if there exists a distance function ρ, such that the domain of ρ is $S \times S$ and its range is the positive real axis, satisfying the following conditions:

1. $\rho(x,y) = 0$ if and only if $x = y$ for all points x of S.
2. $\rho(x,y) = \rho(y,x) \geq 0$ for all points x and y of S.

DEFINITION 2. If x and y are two distinct points of S, then denote by N(x,y) the least integer such that if g is a region of $G_m, m \geq N(x,y)$ and g contains x then $\overline{g}$ does not contain y.

NOTATION: The integer N(x,y) is assured by axiom I_3. It is also obvious that $N(x,y) = N(y,x)$.

DEFINITION 3. A point p is said to be a limit point of the point set M in a Moore space if and only if every region that contains p contains a point x of M distinct from p.

DEFINITION 4. The point p is said to be a limit point of the point set M in a semi-metrizable space if and only if, given $\epsilon > 0$ there exists a point x of M such that $\rho(p,x) < \epsilon$.

THEOREM 6. Every Moore space is semi-metrizable.

PROOF: Let S be a Moore space, and define a distance function ρ on $S \times S$ as follows:

1. $\rho(x,x) = 0$ for all points x of S.

2. If x and y are two distinct points of S then $\rho(x,y) = \frac{1}{N(x,y)}$. Since $N(x,y) = N(y,x)$ it follows that $\rho(x,y) = \rho(y,x)$. Therefore, part 2 of Definition 1 is satisfied.

Let $\rho(x,y) = 0$. Then show that the point x is identical with the point y. Assuming the contrary, let $\rho(x,y) = 0$ and assume that x is different from y. Then $\rho(x,y) = \frac{1}{N(x,y)} > 0$ contrary to the hypothesis. Therefore, the function ρ satisfies both parts of Definition 1.

It remains to be shown that the function ρ does not alter the notion of the limit point.

Let M be a subset of S and assume that p is a limit point of M in the Moore space. We wish then to show that given $\epsilon > 0$ there exists a point x of M such that $\rho(p,x) < \epsilon$. Let $\epsilon > 0$ be given, and let N be an integer such that $\frac{1}{N} < \epsilon$. From Axiom I_2 some region of G_N contains p and a point x of M distinct from p. Let $N_1 = N_1(p,x)$, then $N_1 > N$ and $\rho(p,x) = \frac{1}{N_1} < \frac{1}{N} < \epsilon$. This then shows that p is a limit point of M in the semi-metrizable space.

Assume now that M is a subset of S and p is not a limit point of M in the Moore space. We then wish to show that p is not a limit point of M in the semi-metrizable space. Since p is not a limit point of M in the Moore space, then there exists a least integer N_p such that no region of G_m $m > N_p$ contains both p and a point of M. It then follows that, if x is a point of M, then $\rho(p,x) \geq \frac{1}{N_p}$. Therefore, p is not a limit point of M in the semi-metrizable space, and the theorem follows.

Lemma 1: There exists a semi-metrizable space S, such that S is not paracompact.

PROOF: Let S denote the example on page . It was shown that S is a non-paracompact Moore space. By Theorem 6, S is semi-metrizable, and the lemma follows.

<u>THEOREM 7</u>: Every metric space is paracompact.

PROOF: Let H be an open covering of S. Let α be a well-ordered sequence, such that the terms of α are the elements of H. For each n denote by H_n a collection of subsets of S defined as follows:

Denote by h_{w_1} the first element of H in α, such that h_{w_1} contains a point p with the property that the distance $d(p, S - h_{w_1}) > \frac{1}{2^n}$, and let $O(w_1, n)$ denote the set of all such points of h_{w_1}.

Let h_{w_2} be the first element of H that follows h_{w_1} in α such that h_{w_2} contains a point p with the property that the distance $d(p, [S - [h_{w_2} + O(w_1, n)]]) > \frac{1}{2^n}$. Let h_{w_3} be the first element of H that follows h_{w_2} in α, such that h_{w_3} contains a point p with the property that the distance $d(p, [S - [h_{w_3} + O(w_1, n) + O(w_2, n)]] > \frac{1}{2^n}$. Continue the process. Let $H_n = O(w_1, n), O(w_2, n), \ldots O(w_\beta, n), \ldots$.

Let O be an element of H_n, and let p be a point of O. O is a subset of some element of H, say h_w; denote by d the distance of p from $S - h_w$. By definition of O, $d > \frac{1}{2^n}$.

Let $d_1 = d - \frac{1}{2^n}$, and let g be a sphere of radius $\frac{d_1}{2}$ and center at p. By the triangular inequality it then follows that if x is a point of g other than p and d' is the distance of x from $S - h_w$, then $d' > d - \frac{d_1}{2}$ or $d' > \frac{1}{2^n}$; hence x is a point of O. Therefore, O is an open set.

By construction, the elements of H_n are mutually exclusive and no subsequence of H_n has a limit point. So, H_n is a discrete collection of mutually exclusive open sets each element of which is a subset of some element of H.

Let p be a point of S, and let h_p be the first element of H in α such that h_p contains p. There exists a least integer n_p such that the distance $d(p, S - h_p) > \frac{1}{2^{n_p}}$, it then follows that p belongs to $O(h_p, n_p)$, where $O(h_p, n_p)$ is an element of H_{n_p}.

Therefore, $\sum_{i=1}^{\infty} H_i^* = S$.

Hence, S is strongly screenable, and the theorem follows.

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