Subharmonics and ultraharmonics in the forced oscillations of weakly nonlinear systems

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The characteristics of subharmonic and ultraharmonic modes appearing in the forced, steady-state oscillations of weakly nonlinear systems are considered from the physical, rather than mathematical, viewpoint. A simple explanation of the differences between the two modes, and in particular of the threshold effect usually exhibited by subharmonic oscillations, is presented. The fundamental resonance in the case of weak excitation is also briefly considered.

I. INTRODUCTION

Some degree of nonlinearity is present virtually in every practical problem involving oscillations. Sometimes this complication can be ignored and a linearized treatment is adequate, but in very many cases the interest lies in the very nonlinear aspects of the oscillations. In view of the large number of applications to electronic circuits, control theory, mechanical vibrations, and other branches of physics, it appears desirable to develop a simple and direct understanding of the essential characteristics of such processes. The advantage would be twofold, making possible an intuitive appraisal of the effect of nonlinearities in specific circumstances, and allowing the introduction of such topics at an earlier stage in the educational curriculum. The present paper is intended as a step in this direction.

We discuss the forced, steady state oscillations of weakly nonlinear systems, with particular emphasis on ultraharmonic and subharmonic modes and their differences. Duffing's equation is used as a model for the discussion, which however is conducted in such a way as to allow immediate generalization to other nonlinear equations.

The central part of this paper is constituted by Secs. III and V. In the first one a simple method for identifying the resonance frequencies of weakly nonlinear systems is given; in the second one the physical reasons for the differences exhibited by ultraharmonic and subharmonic oscillations are explained. Section II deals with some aspects of linear oscillations relevant for the following discussion, Sec. IV contains a succinct presentation of some of the salient features of nonlinear oscillations, and Sec. VI deals with the fundamental resonance in the case of weak excitation. Finally, Sec. VII presents a method for the higher order perturbation analysis of weakly nonlinear systems in steady state regime.

The amount of literature on nonlinear oscillations is so extensive that no attempt has been made to give exhaustive bibliographical indications. Furthermore, in all the books of an introductory level known to this author, the problem of nonlinear forced oscillations is approached by "guessing" rather than deriving the form of the approximate solution.[1] In spite of this difference in outlook, it has been deemed advisable to list some standard references which the interested reader may find useful for a deeper discussion.[2-10]

For the sake of brevity, physical examples of nonlinear systems have only been considered in a footnote. In this connection a series of papers by Ludeke[11] is recommended.

II. LINEAR OSCILLATIONS

Consider the equation describing the motion of a damped harmonic oscillator under the action of a sinusoidal forcing function:

$$\frac{d^2X}{dt_d^2}+f\frac{dX}{dt_d}+\Omega_0^2X=F\cos\Omega t_d$$

where t_d is the (dimensional) time, Ω_0 the natural frequency of the system, and f the friction coefficient. If L and T are suitable length and time scales for the problem, one can define the following dimensionless quantities:

$$x=X/L,\quad t=t_d/T,\quad 2b=fT,$$
$$\omega_0=\Omega_0T,\quad \omega=\Omega T,\quad P=FT^2/L,$$

in terms of which the equation can be rewritten as

$$\ddot{x}+2b\dot{x}+\omega_0^2x=P\cos\omega t \tag{1}$$

where the dots denote differentiation with respect to dimensionless time t. The general solution of this equation consists of damped oscillations at frequency $(\omega_0^2-b^2)^{1/2}$, plus oscillations at the impressed frequency ω:

$$x(t)=a_0\exp(-bt)\cos[(\omega_0^2-b^2)^{1/2}t+\psi_0]+Q(\omega,b)\cos(\omega t+\varphi), \tag{2}$$

where a_0, ψ_0 are two constants determined by the initial conditions and $Q(\omega,b)$, φ are given by

$$Q(\omega,b)=P[(\omega_0^2-\omega^2)^2+4b^2\omega^2]^{-1/2}, \tag{3a}$$

$$\varphi=\tan^{-1}[2b\omega/(\omega^2-\omega_0^2)]. \tag{3b}$$

The quantity $Q(\omega,b)$ is the response function of the linear oscillator, and it presents the familiar resonance structure with a sharp maximum at $\omega^2=\omega_0^2-2b^2$ (Fig. 1).

It is evident from (2) that, as time gets large in com-

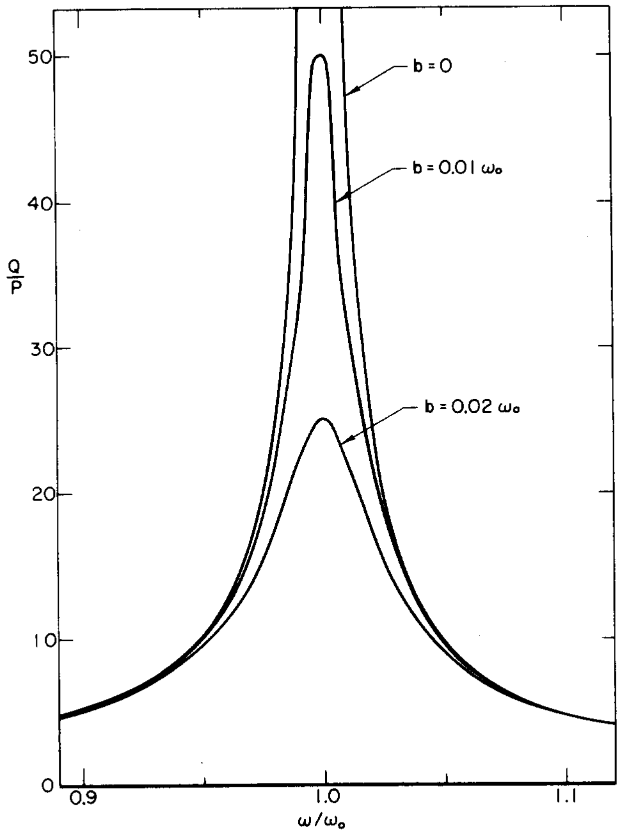

Fig. 1. The response function $Q(\omega,b)$ of the linear oscillator for various values of the damping parameter b [Eq. (3a)].

parison with b^{-1}, the first term (and with it the influence of the initial conditions) disappears so that in the limit $t \to \infty$ we obtain the *steady state* solution of (1) as

$$x_0 = Q(\omega, b)\cos(\omega t + \varphi).$$

The disappearence of the oscillations at frequency $(\omega_0^2 - b^2)^{1/2}$ is a consequence of the fact that in a linear system there is no mechanism of energy transfer between different modes, so that only the mode corresponding to the driving frequency can sustain itself in the presence of dissipative forces.

In view of later considerations we should like to note here that the resonance structure of the response allows an amplification of the excitation amplitude, so that even a (dimensionless) weak driving force (of order ϵ, say, with $|\epsilon| << 1$) can produce a response of order one[12] if the dimensionless damping is sufficiently small (of order ϵ). Indeed, if we let $P = \epsilon P'$, $b = \epsilon\beta$, with P', β quantities of order one, we get from (3a)

$$Q(\omega, \epsilon\beta) \sim P'/2\beta\omega_0 \quad \text{for } \omega \sim \omega_0$$

which is of order one.

III. NONLINEAR OSCILLATIONS

To illustrate some general features of nonlinear systems, let us now consider a particular nonlinear oscillator described by the well-known Duffing equation[13]:

$$\frac{d^2X}{dt_a^2} + f\frac{dX}{dt_a} + \Omega_0^2 X = F\cos\Omega t_a + \Gamma X^3,$$

where Γ is a real constant with dimensions (length $\times$ time)$^{-2}$. An equation of this type occurs very frequently in problems involving oscillations, as soon as one tries to take into account departures from linearity of the restoring force. For instance, in the case of a pendulum for which the restoring force is proportional to $\sin X$, the approximation $\sin X \sim X - X^3/6$ is very accurate for amplitudes of oscillation smaller than about 30°.[14]

We shall confine ourselves to the case of weak damping and weak nonlinearity, rewriting the equation in dimensionless form as

$$\ddot{x} + 2\epsilon\beta\dot{x} + \omega_0^2 x = P\cos\omega t + \epsilon x^3, \qquad (4)$$

where $\epsilon = L^2T^2\Gamma$, $|\epsilon| << 1$, and β is a dimensionless quantity of order one. Notice that the appearance of the small parameter ϵ in the damping term is not to be interpreted as suggesting a relationship between the physical mechanisms giving rise to the nonlinear effects and the energy dissipation, but only as indicating the order of magnitude of the dimensionless damping, which can be adjusted by varying β.

It may be expected that the steady state solution of this equation will be related to the solution of the linear equation obtained as $\epsilon \to 0$, i.e., Eq. (1), so that it is natural to introduce a new unknown y through

$$x = x_0 + y = Q\cos(\omega t + \varphi) + y, \qquad (5)$$

where Q and φ are given by Eqs. (3). Upon substitution into (4), we are led to the following equation for y:

$$\ddot{y} + 2\epsilon\beta\dot{y} + \omega_0^2 y = \epsilon[\tfrac{1}{4}Q^3(\cos 3\omega t + 3\cos\omega t) + \tfrac{3}{2}Q^2 y(1 + \cos 2\omega t) + 3Qy^2\cos\omega t + y^3]. \quad (6)$$

For simplicity of writing, the time origin has been shifted by φ/ω, so that $\cos(\omega t + \varphi) \to \cos\omega t$, and elementary trigonometric relations have been used to express $\cos^2\omega t$, $\cos^3\omega t$ in terms of the multiple angles $2\omega t$, $3\omega t$. It is easy to see that, in spite of the small quantity ϵ multiplying the "forcing function" in the right-hand side (RHS) of this equation, the response y is not necessarily small, because of the amplification effects mentioned in Sec. II. For instance, the first term in the RHS of Eq. (6) would make a contribution that would be of order one whenever $3\omega \sim \omega_0$. Notice that the appearance of the new frequency 3ω is an effect of the nonlinearity x^3, which causes a coupling of the first term of (5), x_0, with itself. We may now suspect that other resonating frequencies are present in the RHS of (6), arising from the coupling of y with x_0, and of y with itself. They can be determined without actually solving the equation by the following simple reasoning.

Suppose that we had the correct expression for the y appearing in the RHS of (6), y_R, say, and that we are then left to solve the resulting linear equation for the y appearing in the left-hand side (LHS), y_L. Of course this should be done in such a way that, in the end, $y_R = y_L$. Suppose also that we are interested only in terms of order one, neglecting all terms of order ϵ and smaller. The discussion of Sec. II shows that, in order to produce a component of order one in y_L, a term in the bracket in the RHS of (6) should satisfy two requirements:

Table I. The new frequencies ω_k introduced to first order by the nonlinearity in the RHS of Eq. (6).

Term	ω_k	Condition for resonance
$(1/4)Q^3 \cos^3\omega t$	3ω	$\omega \sim \omega_0/3$
	ω	$\omega \sim \omega_0$
$(3/2)Q^2(1 + \cos 2\omega t)y$	ω_0	none
	$2\omega + \omega_0$	impossible
	$2\omega - \omega_0$	$\omega \sim \omega_0$
$3Qy^2 \cos\omega t$	ω	$\omega \sim \omega_0$
	$2\omega_0 + \omega$	impossible
	$2\omega_0 - \omega$	$\omega \sim 3\omega_0$
y^3	ω_0	none
	$3\omega_0$	impossible

(a) it should have a frequency close to ω_0;

(b) it should have an amplitude of order one.

It follows from (a) that the only term of order one in y_L will have a frequency close to ω_0. However, since y_L must equal y_R, it also follows from (b) that this same term is the only one present in y_R that we should consider to determine the resonant frequencies to lowest order in ϵ. We therefore let $y \propto \cos\omega_0 t$ in the RHS of (6) and compute the new frequencies introduced by the nonlinearity by means of the trigonometric relation $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$. Whenever one of these frequencies is close to ω_0, the corresponding term will produce a response of order one. In this way we get the results shown in Table I.

We shall not be concerned here with the more complex case of strong excitation of the fundamental resonance, $\omega \sim \omega_0$, but only with the *ultraharmonic* and *subharmonic* resonances occurring when $3\omega \sim \omega_0$ and $\omega \sim 3\omega_0$, respectively. The reason for the naming is that in the first case the strong response at $\sim \omega_0$ is at three times the exciting frequency ω, while in the second case it is at one-third of ω.

IV. ULTRAHARMONICS AND SUBHARMONICS

In the ultraharmonic region, $\omega \sim \omega_0/3$, neglecting terms that cannot produce resonance, we get from (6)

$$\ddot{y} + 2\epsilon\beta\dot{y} + (\omega_0^2 - \tfrac{3}{2}\epsilon Q^2)y = \tfrac{1}{4}\epsilon Q^3 \cos 3\omega t + \epsilon y^3. \quad (7)$$

This equation is very similar to the original one, except that the order of magnitude of the driving amplitude has been lowered from one to ϵ. In agreement with the discussion in the previous section, to determine the solution of (7) to order one we now let

$$y = C_3 \cos(3\omega t + \varphi_3) + O(\epsilon)$$

and retain only terms oscillating with frequency 3ω. The result is

$$[(\omega_0^2 - 9\omega^2 - \tfrac{3}{2}\epsilon Q^2)C_3 - \tfrac{3}{4}\epsilon C_3^3 - \tfrac{1}{4}\epsilon Q^3 \cos\varphi_3]\cos(3\omega t + \varphi_3) - [6\epsilon\omega\beta C_3 + \tfrac{1}{4}\epsilon Q^3 \sin\varphi_3]\sin(3\omega t + \varphi_3) = O(\epsilon) \cong 0,$$

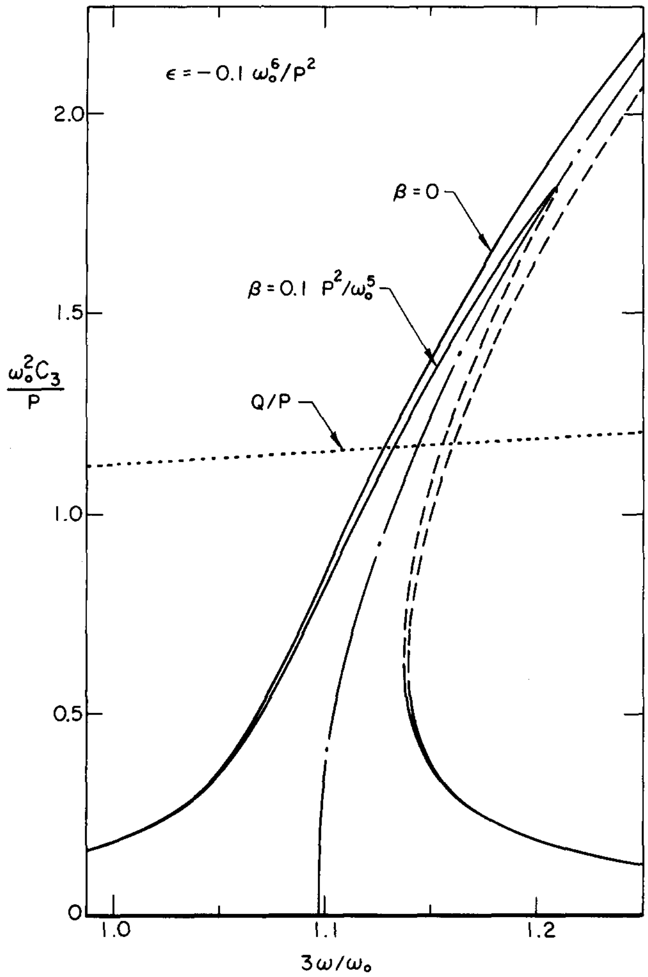

Fig. 2. Amplitude of the ultraharmonic, C_3, as a function of $3\omega/\omega_0$, as determined by Eqs. (8). The dashed portions of the curves correspond to unstable oscillations. The dash-and-dot line is the "backbone curve," Eq. (9), and the dotted line $Q(\omega,0)$, Eq. (3a).

from which

$$(\omega_0^2 - 9\omega^2 - \tfrac{3}{2}\epsilon Q^2)C_3 - \tfrac{3}{4}\epsilon C_3^3 = \tfrac{1}{4}\epsilon Q^3 \cos\varphi_3, \quad (8a)$$

$$-6\epsilon\beta\omega C_3 = \tfrac{1}{4}\epsilon Q^3 \sin\varphi_3. \quad (8b)$$

The amplitude C_3 determined by these equations is plotted as a function of $3\omega/\omega_0$ in Fig. 2 for two values of the damping parameter β. In the undamped case, $\beta = 0$, the phase φ_3 can be either 0 or π, so that the first equation becomes

$$(3\omega/\omega_0)^2 = 1 - \tfrac{3}{2}\epsilon\omega_0^{-2}Q^2 - \tfrac{3}{4}\epsilon\omega_0^{-2}C_3^2 \pm \tfrac{1}{4}\epsilon\omega_0^{-2}C_3^{-1}Q^3.$$

It is clear from this equation that, as C_3 increases, the two branches tend asymptotically to the curve

$$(3\omega/\omega_0)^2 = 1 - \tfrac{3}{2}\epsilon\omega_0^{-2}Q^2 - \tfrac{3}{4}\epsilon\omega_0^{-2}C_3^2 \quad (9)$$

also shown in Fig. 2. If $\beta > 0$, the two branches join together across this curve and the amplitude has a maximum. On the other hand, as $|3\omega - \omega_0|$ increases, the corresponding value of C_3 decreases. The situation is therefore very similar to the resonance phenomenon in the linear case (Fig. 1), the only substantial difference being the fact that the asymptotic curve is not a vertical straight line, but is bent to the right or to the left according as $\epsilon < 0$ or $\epsilon > 0$.[15] The bending of the resonance peak has the important consequence (typical of nonlinear oscillations) that the function $C_3(\omega)$ is multivalued in a certain frequency range. A stability analysis shows that the intermediate value of C_3 corresponds to an unstable state[16] which therefore cannot be observed because any infinitesimal disturbance will grow leading the amplitude towards one of the other two (stable) values. The appearance of one or the other of these values in the steady state oscillations is determined by the initial conditions of the motion. In contrast with the linear case, therefore, the steady state nonlinear oscillations retain some memory of the initial values of displacement and velocity.

To discuss the subharmonic oscillations,[17] when $\omega \sim 3\omega_0$, we start again from (6) retaining only the terms capable of producing resonance according to Table I:

$$\ddot{y} + 2\epsilon\beta\dot{y} + (\omega_0^2 - \tfrac{3}{2}\epsilon Q^2)y = \tfrac{3}{2}\epsilon Q y^2 \cos\omega t + \epsilon y^3. \quad (10)$$

The structure of this equation is very different from (7) because all terms of the RHS contain y: this circumstance causes very profound differences between ultraharmonic and subharmonic oscillations. If we let

$$y = C_{1/3}\cos(\tfrac{1}{3}\omega t + \varphi_{1/3})$$

in (10) and retain only terms oscillating with frequency $\omega/3$ we get, in place of (8),

$$(\omega_0^2 - \tfrac{1}{9}\omega^2 - \tfrac{3}{2}\epsilon Q^2)C_{1/3} - \tfrac{3}{4}\epsilon C_{1/3}^3 = \tfrac{3}{4}\epsilon Q C_{1/3}^2 \cos 3\varphi_{1/3}, \quad (11a)$$

$$-\tfrac{2}{3}\epsilon\omega\beta C_{1/3} = \tfrac{3}{4}\epsilon Q C_{1/3}^2 \sin 3\varphi_{1/3}. \quad (11b)$$

Two examples of the response curves determined by these equations are plotted for the case $\epsilon < 0$ in Fig. 3.

The first important remark to be made about Eqs. (11) is that $C_{1/3} = 0$ is a solution for any value of $\varphi_{1/3}$. Therefore, for any value of ω and P, the steady state oscillations ordinarily will not contain a subharmonic component, which however will be present if the initial values of displacement and velocity lie in suitable ranges. For this to happen, however, it is usually necessary that the equilibrium state (or a pre-existing, steady, purely harmonic oscillation) be perturbed quite substantially *(shock excitation* of the subharmonic). Another possibility for the appearance of the subharmonic at a particular frequency is when the purely harmonic oscillations, $C_{1/3} = 0$, are unstable. Although this does not occur in the case of Duffing's equation, it is nevertheless commonly found to happen in other nonlinear systems. An example is shown in Fig. 4, where the subharmonic response of order ½ (i.e., occurring for $\omega \sim 2\omega_0$) exhibited by the radial oscillations of a spherical gas bubble in an incompressible

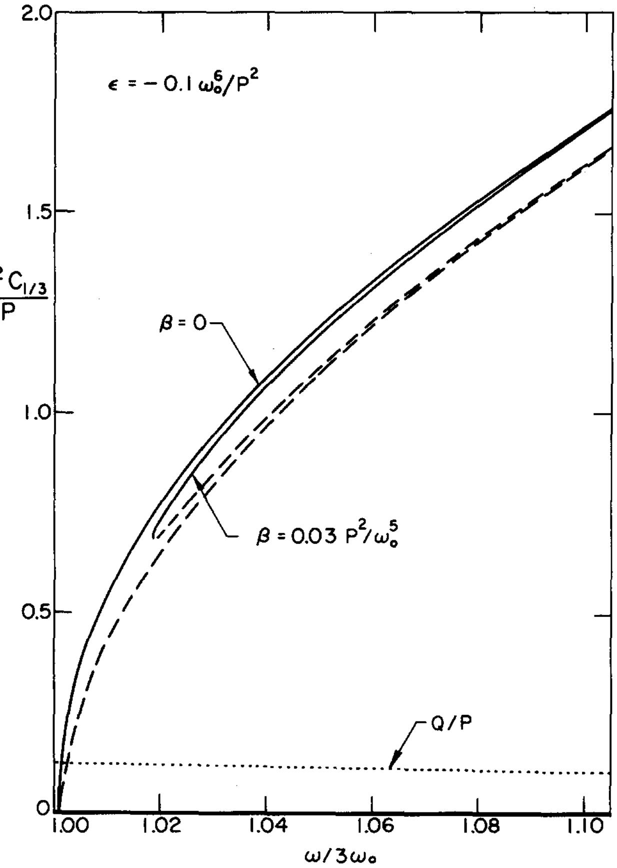

Fig. 3. Amplitude of the subharmonic, $C_{1/3}$, as a function of $\omega/3\omega_0$, as determined by Eqs. (11). The dashed portions of the curves correspond to unstable oscillations. The dotted line represents $Q(\omega,0)$, Eq. (3a), which, to zero order in ϵ, is the total amplitude of the response in the absence of the subharmonic.

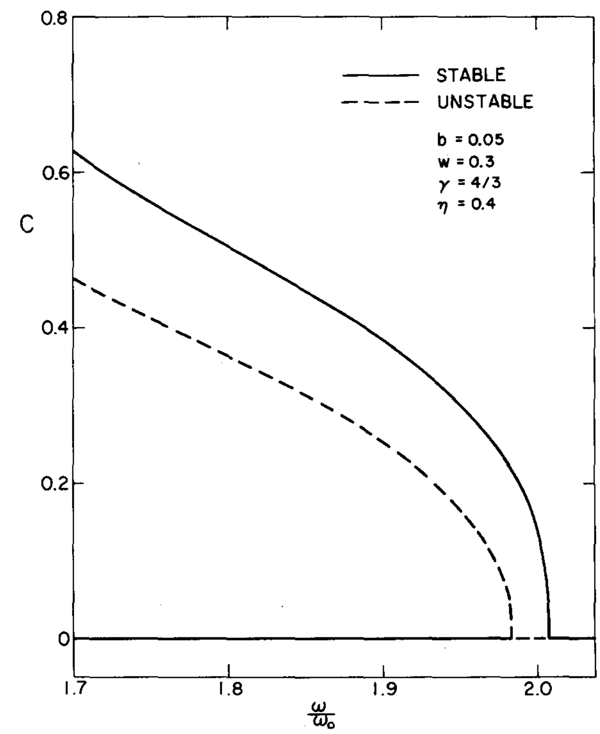

Fig. 4. Amplitude of the subharmonic response of order ½ (i.e., for $\omega \sim 2\omega_0$) for the oscillations of a spherical gas bubble in an incompressible, viscous liquid (from Ref. 16, Fig. 1). For a given liquid, the natural frequency of the bubble, ω_0, depends only on its radius. The bubbles whose radii are such that the corresponding ω/ω_0 lies in the dashed interval in the vicinity of $\omega/\omega_0 = 2$ develop a strong subharmonic component because the purely harmonic, $C = 0$, mode is unstable there. For bubbles of other radii both modes of oscillation, with and without the subharmonic, are stable.

liquid[18] is shown. In the dashed frequency interval on the abscissa axis, only the subharmonic oscillations correspond to a stable mode.

Another striking characteristic of the subharmonic response is obtained by considering the reality conditions for $C_{1/3}$. It is a simple matter to show from Eqs. (11) that no subharmonic component can be present unless the following *threshold condition* for the driving amplitude is fulfilled:

$$Q^2 > \frac{8}{21}\epsilon^{-1}\{\omega_0{}^2 - \frac{1}{9}\omega^2 - [(\omega_0{}^2 - \frac{1}{9}\omega^2)^2 - \frac{28}{9}\epsilon^2\beta^2\omega^2]^{1/2}\}.$$

It will be noticed that the RHS of this equation reduces to zero if no damping is present.

In Fig. 3 the dotted line shows the function $Q(\omega,\epsilon\beta)$ in the subharmonic region. This quantity would be the total amplitude of the response if no subharmonic were present. Its comparison with the subharmonic response gives an idea of the violence of subharmonic oscillations, which is an important reason for their practical importance.[19]

V. DISCUSSION

It has been shown that in a particular nonlinear system large amplitude oscillations can occur at a frequency different from the driving frequency provided that there exists a mechanism capable of transferring efficiently (i.e., via resonance) the energy introduced at frequency ω into a mode close to the natural frequency of the system. Obviously this result holds true also for more general nonlinearities of the form $\epsilon x^m \dot{x}^n$, with m, n integers such that $m + n > 1$. In this case, too, we let

$$x = x_0 + y = Q\cos(\omega t + \varphi) + y \qquad (12)$$

where the first term is the steady solution of the linearized equation. Since the ultraharmonic oscillations occur at a frequency higher than the driving frequency ω, it is obvious that the term x_0 in (12) is by itself sufficient to feed energy into such modes. Indeed, upon substitution into $\epsilon x^m \dot{x}^n$, it will give rise to a term of the form $\epsilon Q^{m+n} \cos^m(\omega t + \varphi)\sin^n(\omega t + \varphi)$ which, when expressed in terms of multiple angle functions, will contain frequencies $k\omega$ capable of inducing large responses whenever $k\omega \sim \omega_0$. Under suitable conditions, also the coupling between x_0 and y caused by the nonlinearity can introduce additional "channels" through which energy can be transmitted to a particular ultraharmonic mode, but even if these couplings were absent, the mode in question would still be capable of sustaining itself. In this respect the ultraharmonic oscillations behave basically like ordinary linear forced oscillations, and they will occur whenever the frequency is in an appropriate range, just as resonance occurs in the linear case when $|\omega - \omega_0|$ is not too large. Their amplitude can always adjust itself in such a way that the energy dissipated by the viscous forces equals the energy input by the driving force because the dissipation, $-2\epsilon\beta\dot{y}^2$, is proportional to $C_j{}^2$ (where C_j is the amplitude of the jth ultraharmonic), while the energy intake, $x_0{}^k\dot{y}$, is proportional to $Q^k C_j$ (plus possibly terms containing the second and higher powers of C_j).

For the subharmonic oscillations the situation is fundamentally different. Since they occur at a frequency lower than the driving frequency, a mechanism for frequency demultiplication of the energy input is required. Mathematically, this mechanism is furnished by the couplings $x_0{}^i y^j$ through the second term in the trigonometric identities $2\cos\alpha\cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$, etc. It follows that the energy input term is now of the form $(x_0{}^i y^j)\dot{y}$, and is no longer proportional to the first power of the amplitude, but to the second or higher, while the characteristics of the energy dissipation are unchanged. It may happen therefore that, for a given driving amplitude, the resulting subharmonic amplitude is too low for the system to absorb energy at a rate sufficient to balance the dissipation. These considerations explain why subharmonic oscillations usually exhibit a threshold effect, which however disappears as the damping is reduced to zero.

VI. THE FUNDAMENTAL RESONANCE

An exhaustive treatment of the fundamental resonance for $\omega \sim \omega_0$ in the case of strong excitation is beyond the scope of the present considerations. Nevertheless the case in which the driving force is weak (of order ϵ) can easily be discussed. Consider therefore the equation

$$\ddot{x} + 2\epsilon\beta\dot{x} + \omega_0{}^2 x = \epsilon P\cos\omega t + \epsilon x^3, \qquad (13)$$

for the case when $\omega \sim \omega_0$. Following a line of reasoning similar to the one adopted in Sec. III we may let

$$x = C_1\cos(\omega t + \varphi_1),$$

disregarding other harmonics which give no contribution to zero order in ϵ. Upon substitution into (13) the following two equations are obtained:

$$(\omega_0{}^2 - \omega^2 - \tfrac{3}{4}\epsilon C_1{}^2)C_1 = \epsilon P\cos\varphi_1, \qquad (14a)$$

$$-2\epsilon\beta\omega C_1 = \epsilon P\sin\varphi_1. \qquad (14b)$$

It is interesting to note the similarity of these equations with Eqs. (8) for the ultraharmonic case. This analogy illustrates from another point of view the affinity between ultraharmonic and ordinary resonance. The shape of the peak is similar to the one shown in Fig. 2, and can readily be computed from (14).

VII. HIGHER ORDER APPROXIMATIONS

It is not difficult to extend the considerations of Sec. III to obtain steady state solutions of a higher accuracy. To this end we begin by letting

$$y = y_0 + \epsilon x_1 = C_i\cos(i\omega + \varphi_i) + \epsilon x_1$$

in (6), where $i = ⅓$ in the subharmonic case and $i = 3$ in the ultraharmonic one; the amplitudes C_i and phases φ_i are given by Eqs. (11) and Eqs. (8) respectively. As an example of the procedure, let us consider the subharmonic case here. The following equation for x_1 is obtained:

$$\ddot{x}_1 + 2\epsilon\beta\dot{x}_1 + \omega_0{}^2 x_1 = \tfrac{1}{4} Q^3(\cos 3\omega t + 3\cos\omega t) + \tfrac{3}{4} Q^2 C[\cos(\tfrac{7}{3}\omega t + \varphi) + \cos(\tfrac{5}{3}\omega t + \varphi)]$$
$$+ \tfrac{3}{2} QC^2[\cos\omega t + \tfrac{1}{2}\cos(\tfrac{5}{3}\omega t + \varphi)] + \tfrac{1}{4} C^3 \cos(\omega t + 3\varphi)$$
$$+ \epsilon(\{\tfrac{3}{2} Q^2(1 + \cos 2\omega t) + 3QC[\cos(\tfrac{4}{3}\omega t + \varphi) + \cos(\tfrac{2}{3}\omega t - \varphi)] + \tfrac{3}{2} C^2[1 + \cos(\tfrac{2}{3}\omega t + 2\varphi)]\}x_1$$
$$+ 3\epsilon[Q\cos\omega t + C\cos(\tfrac{1}{3}\omega t + \varphi)]x_1{}^2 + \epsilon^2 x_1{}^3),$$

where for convenience of writing the subscript ⅓ has been omitted. This is now a nonlinear equation for x_1, which can be dealt with in the same way indicated in Sec. III. We therefore let

$$x_1 = x_1{}^0 + y_1,$$

where $x_1{}^0$ is the solution of the equation obtained by neglecting the terms in the curly brackets, and y_1 the correction of order one introduced by the terms in the curly brackets oscillating with frequency close to ω_0. This procedure can clearly be continued, and an asymptotic expansion of the solution in the form

$$x = x_0 + y_0 + \epsilon(x_1{}^0 + y_1{}^0) + \epsilon^2(x_2{}^0 + y_2{}^0) + \cdots$$

can be obtained. It should be noted that all the terms $y_k{}^0$ in this expansion oscillate with frequency close to the natural frequency ω_0.

To obtain a solution in a frequency region different from the principal harmonic, ultraharmonic or subharmonic domains already considered, it is sufficient to omit the term y_0 and to start by letting

$$x = Q\cos(\omega t + \varphi) + \epsilon x_1$$

in Eq. (4). The following equation for x_1 is then obtained:

$$\ddot{x}_1 + 2\epsilon\beta\dot{x}_1 + \omega_0{}^2 x_1 = \tfrac{1}{4} Q^3(\cos 3\omega t + 3\cos\omega t)$$
$$+ \epsilon[\tfrac{3}{2} Q^2(1 + \cos 2\omega t)x_1 + 3\epsilon Q\cos\omega t x_1{}^2 + \epsilon^2 x_1{}^3].$$

(In writing this equation the time origin has again been shifted by φ/ω.) By the same method used for the construction of Table I it is easily found that, aside from the resonance regions already considered, no other resonating frequency is introduced by the terms in brackets to this order.[20]

The procedure outlined in this section essentially amounts to an algorithm for the iterative solution of the system of equations that would be obtained by expanding x in a truncated Fourier series. Since the equations that are obtained in this way are usually highly nonlinear, the above method may have some computational advantages.

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[1]Perhaps the fact should be mentioned here that, although our results are well known, it has not been possible to find in the literature any discussion of the methods applied in this paper (especially in Secs. III, V, and VII) to derive them.

[2]J. J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems* (Interscience, New York, 1950).

[3]N. N. Bogoliubov and N. Kryloff, *Introduction to Nonlinear Mechanics* (Princeton U. P. , Princeton, 1943).

[4]N. W. McLachlan, *Ordinary Nonlinear Differential Equations in Engineering and Physical Sciences* (Claredon, Oxford, 1956), Chaps. 4 and 5.

[5]N. Minorski, *Nonlinear Oscillations* (Van Nostrand, Princeton and New York, 1962), Chaps. 15 and 19.

[6]J. K. Hale, *Ordinary Differential Equations* (Wiley–Interscience, New York, 1969), Chaps. 4–6.

[7]C. Hayashi, *Nonlinear Oscillations in Physical Systems* (McGraw-Hill, New York, 1964).

[8]A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators* (Pergamon, Oxford, 1966), Chap. 9.

[9]L. A. Pipes and L. R. Harvill, *Applied Mathematics for Engineers and Physicists* (McGraw-Hill, New York, 1970), 3rd ed., Chap. 15.

[10]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968), Chap. 14.

[11]C. A. Ludeke, J. Appl. Phys. **13,** 418 (1942); **17,** 603 (1946); **20,** 600 (1949); **22,** 1321 (1951); Trans. Am. Soc. Mech. Eng. **79,** 439 (1957); Am. J. Phys. **16,** 430 (1948); with W. Pong, J. Appl. Phys. **24,** 96 (1953); with J. D. Blades, J. Appl. Phys. **28,** 1326 (1957).

[12]By saying that a certain (dimensionless) quantity is *of order one* we mean that its absolute value is large compared with $|\epsilon|$. Similarly, *of first order* will mean that its absolute value is of the order of magnitude of $|\epsilon|$, but much larger than ϵ^2, etc.

[13]This equation or the following one, Eq. (4), is considered in all the Refs. 2–10. In particular, one might mention Ref. 2, pp. 83ff, and Ref. 10, pp. 829ff. The original reference is: G. Duffing, *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz* (Vieweg, Braunschweig, 1918).

[14]Another example is furnished by an oscillatory circuit with a nonlinear inductor described by $i = L^{-1}\Phi + g\Phi^3$, where i is current, Φ the magnetic flux, L the inductance and g a constant characterizing the amount of nonlinearity. A second example is the forced oscillations of a body on top of an elastic bar, the constitutive law of which contains a cubic term. More in general, if $f(X)$ is a symmetric restoring force, i.e. such that $f(X) = f(-X)$, then one can write by a Taylor series expansion $f(X) \sim X + f'(0)X^3/6$. This approximation will be valid for oscillations of moderate amplitude about the equilibrium position $X = 0$.

[15]In the first case the restoring force, $-\omega_0{}^2 x + \epsilon x^3$, increases with displacement ("hardening spring"), so that on the average it is larger than its linear part, $-\omega_0{}^2 x$. This circumstance causes an increase in the "average" natural frequency $\omega_0{}^*$ which, clearly, becomes amplitude dependent: $\omega_0{}^* = 1 - 3\epsilon\omega_0{}^2 Q^2/2 - 3\epsilon\omega_0{}^{-2} C_3{}^2/4$. It is evident that this fact is responsible for the bending of the peak, which would be vertical if the scale of the abscissa axis were in terms of $3\omega/\omega_0{}^*$ instead of $3\omega/\omega_0$. Similarly, for $\epsilon > 0$ ("softening spring"), the average value of the natural frequency decreases and the peak leans to the left.

[16]The stability analysis can be carried out in several ways. See, e.g., Ref. 5, pp. 375–380.

[17]The subharmonic oscillations are discussed to some extent in all the Ref. 2–10, and practically in all books on nonlinear oscillations. The

treatments given in Ref. 2, pp. 103ff, and in Ref. 10, pp. 851ff, may be found particularly helpful.

[18]A. Prosperetti, J. Acoust. Soc. Am. **56,** 878 (1974).

[19]Since resonance is responsible for the transfer of energy from the mode at ω to the mode at $\sim\omega_0$, ω_0 should not be too different from the appropriate multiple or submultiple of ω for the present method to apply. Indeed, the assumption that $|3\omega - \omega_0|$ or $|\omega - 3\omega_0|$ is of order $\epsilon\omega_0$ is implicit in all our results. If these conditions are not satisfied, the response y is of order ϵ, as can be seen from Figs. 2 and 3.

[20]It can be shown, however, that Duffing's equation has an infinity of resonant frequencies. See, e.g., M. E. Levenson, J. Appl. Phys. **20,** 1045 (1949). These frequencies could be found by carrying the present method to higher orders in ϵ.