MATRICES WHICH COMMUTE WITH MENON OPERATORS

A Thesis

Presented to

the Faculty of the College of Arts and Sciences

The University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematics

by

Gerald Edward Suchan

August 1971

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ABSTRACT

If A is a nonnegative square matrix and X is a vector, then the Menon operator associated with A, denoted by T_A, is defined by $(T_A X)_i = (\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-1}$. A close relation is known to exist between doubly stochastic matrices and Menon operators. The following problem is investigated: If each of E and F is a matrix, when is ET_AF a Menon operator? It is conjectured, but not proven, that if A is a nonnegative square matrix satisfying certain criterion, and each of E and F is a nonnegative matrix such that ET_AF is a Menon operator, then each of E and F is the product of a diagonal matrix with positive diagonal and a permutation matrix. This conjecture is supported by examples, and also by theorems which show that if A is doubly stochastic and $ET_A = T_AE$ then either there is a number r such that rE is doubly stochastic or there is a permutation matrix P such that P^tEP can be partitioned into a certain block form. A condition is defined on a doubly stochastic matrix which implies that $ET_A = T_AE$ if and only if there is a number r such that rE is a permutation matrix.

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CHAPTER I

NUMBERS. Let N be the set of all nonnegative real numbers. It will be convenient to extend N to include ∞ and to order and topologize this set N_∞ in the usual way. Multiplication and addition in N will be extended to N_∞ by the following conventions [1, pg. 34] : $0^{-1} = \infty$, $\infty^{-1} = 0$, $\infty + \infty = \infty$, $0 \cdot \infty = 0$, and if $r > 0$ then $r \cdot \infty = \infty$. It is recognized that multiplication on N_∞ is not continuous at 0. It is understood that each of 0^{-1} and ∞ is an alternate symbol for $\frac{1}{0}$ and it is also understood that 0^{-1} is not the multiplicative inverse of 0 since $0 \cdot 0^{-1} = 0$.

MATRICES. Let each of m and n be a positive integer and let A be a m × n matrix. If i is in {1,...,m} and j is in {1,...,n} then $(A)_{ij}$ is the element in the i th row and j th column of A. A is 0 provided $(A)_{ij} = 0$ for i in {1,...,m} and j in {1,...,n}, in which case one may write A = 0. A is positive provided $0 < (A)_{ij} < \infty$ for i in {1,...,m} and j in {1,...,n}, in which case one may write A >> 0. A is nonnegative provided $0 \leq (A)_{ij} < \infty$ for i in {1,...,m} and j in {1,...,n}, in which case one may write $A \geq 0$. $A > 0$ provided $A \geq 0$ and $A \neq 0$. The transpose of A, denoted by A^t, is defined by $(A^t)_{ij} = (A)_{ji}$. If A is a nonsingular matrix then A^{-1} denotes the multiplicative inverse of A. A is a permutation matrix provided A is n × n and there is a permutation σ on {1,...,n} such that $(A)_{ij} = 1$ if $\sigma(j) = i$ and $(A)_{ij} = 0$ if $\sigma(j) \neq i$. If σ is the identity permutation on {1,...,n} then the corresponding permutation matrix, denoted by I, is the n × n identity matrix.

If A is a $m \times n$ matrix, p is in $\{1,\ldots,m\}$, q is in $\{1,\ldots,n\}$, each of $\{r_i\}_{i=1}^{p}$ and $\{c_j\}_{j=1}^{q}$ is a positive integer sequence, $\sum_{i=1}^{p} r_i = m$, and $\sum_{j=i}^{q} c_j = n$, then A can be represented in block form as

$$\begin{pmatrix} A_{11} & \cdots & A_{1q} \\ \vdots & & \vdots \\ A_{p1} & \cdots & A_{pq} \end{pmatrix} .$$

If A is represented in block form then A is said to be partitioned into block form. If $p = 1$ then A is represented in block form as $[A_{11} \ldots A_{1q}]$ and if $q = 1$ then A is represented in block form as

$$\begin{pmatrix} A_{11} \\ \vdots \\ A_{p1} \end{pmatrix} .$$

A is reducible provided A is a $n \times n$ nonnegative matrix and there is a permutation matrix P such that

$$P^t AP = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

and each of A_1 and A_2 is a square nonempty matrix. A is *irreducible* provided A is a $n \times n$ nonnegative matrix and A is not reducible.

A proof of the following Theorem of Perron and Frobenius is provided by Gantmacher [2,pg 65].

THEOREM 1.1. *An irreducible* $n \times n$ *nonnegative matrix* A *always has a positive characteristic number* r, *which is a simple root of the characteristic equation. The moduli of all the other characteristic numbers are at most* r. *A characteristic vector* Z, *unique to within a scalar factor, with positive coordinates, coresponds to the dominant*

<u>characteristic number</u> r. <u>If in addition</u> A <u>has precisely</u> h <u>characteristic numbers</u> $\lambda_0 = r, \lambda_1, \ldots, \lambda_{h-1}$, <u>of modulus equal to</u> r, <u>then these characteristic numbers are different from each other and are roots of the equation</u> $\lambda^h - r^h = 0$, <u>and, in general, the entire spectrum</u> $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ <u>of</u> A, <u>when plotted as a system of points in the complex plane, is carried into itself when the plane is rotated by the angle</u> $\frac{2\pi}{h}$. <u>When</u> h > 1, <u>there is a permutation matrix</u> P <u>such that</u>

$$P^tAP = \begin{pmatrix} 0 & A_1 & 0 & \ldots & 0 \\ 0 & 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & A_{h-1} \\ A_h & 0 & 0 & \ldots & 0 \end{pmatrix}$$

<u>where the</u> 0 <u>blocks on the main diagonal are square</u>.

A is a primitive matrix provided A is an irreducible matrix with only one characteristic number having modulus the modulus of the dominant characteristic number of A. The following Theorem provides a useful property of primitive matrices [2,pg 97].

THEOREM 1.2. <u>A nonnegative</u> $n \times n$ <u>matrix</u> A <u>is primitive if and only if there is a positive integer</u> p <u>so that</u> A^p <u>is positive</u>.

A $m \times n$ matrix A is <u>row stochastic</u> provided $A \geq 0$ and $\sum_{j=1}^{n} (A)_{ij} = 1$ for i in $\{1,\ldots,m\}$. The following Theorem provides a useful property of $n \times n$ row stochastic matrices [2, pg 100].

THEOREM 1.3. <u>A nonnegative</u> $n \times n$ <u>matrix</u> A <u>is row stochastic if</u>

and only if the vector

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

is a characteristic vector of A, _with corresponding characteristic number_ 1. _For a row stochastic matrix,_ 1 _is the dominant characteristic root._

A m × n matrix A is column stochastic provided $A \geq 0$ and $\sum_{i=1}^{n}(A)_{ij} = 1$ for j in {1,...,n}. A is doubly stochastic provided A is n × n, A is row stochastic, and A is column stochastic. The set of all n × n doubly stochastic matrices is denoted by Ω_n. A proof of the following famous Theorem of G. Birkhoff may be found in [3,pg 98].

THEOREM 1.4. _The set of all_ n × n _doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices._

The n × n flat matrix, denoted by J_n, is defined by $(J_n)_{ij} = \frac{1}{n}$ for i and j in {1,...,n}. A matrix A is _idempotent_ provided $A^2 = A$. The following useful Theorem was proven by R. Sinkhorn in [4].

THEOREM 1.5. A ε Ω_n _is idempotent if and only if there exist positive integers_ $n_1,\ldots,n_s$ _with sum_ n _and a permutation matrix_ P _such that_

$$A = P\begin{bmatrix} J_{n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{n_s} \end{bmatrix}P^t.$$

A matrix A is partly decomposable provided $A > 0$, A is $n \times n$, and there is a permutation matrix P and a permutation matrix Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

and each of A_1 and A_2 is a square nonempty matrix. A matrix A is fully indecomposable provided $A > 0$, A is $n \times n$, and A is not partly decomposable. By convention every 1×1 matrix is irreducible but a 1×1 matrix is fully indecomposable only if it is positive.

If A is an $n \times n$ matrix and σ is a permutation on $\{1,\ldots,n\}$ then the sequence $\{(A)_{i\sigma(i)}\}_{i=1}^{n}$ is the dianonal of A corresponding to σ. If σ is the identity permutation then the corresponding diagonal is the main diagonal. A $n \times n$ matrix A is a diagonal matrix provided $(A)_{ij} = 0$ if $i \neq j$. A is said to have total support if $A > 0$ and every positive element of A lies on a positive diagonal. In [5] R. Sinkhorn and P. Knopp prove the following Theorem.

THEOREM 1.6. *A necessary and sufficient condition that there exist a doubly stochastic matrix B of the form $D_1 A D_2$ where D_1 and D_2 are diagonal matrices with positive main diagonals is that A has total support. If B exists then it is unique. Also, D_1 and D_2 are unique up to a scalar multiple if and only if A is fully indecomposable.*

VECTORS. Let V_∞ be the set of all $n \times 1$ matrices with elements taken from N_∞. X is a vector provided X is in V_∞. If X is a vector then X is a 0 vector provided X is a 0 matrix, X is a positive vector provided X is a positive matrix, and X is a nonnegative vector provided

X is a nonnegative matrix. If X is a vector and i is in $\{1,\ldots,n\}$ then $(X)_i = (X)_{i1}$. If i is in $\{1,\ldots,n\}$ then δ_i is defined by $(\delta_i)_j = 1$ for $i = j$ and $(\delta_i)_j = 0$ for $i \neq j$. e is the vector $\sum_{i=1}^{n} \delta_i$.

OPERATORS. T is an <u>operator</u> provided T is a function with domain and range a subset of V_∞. Let X be a vector. The <u>inverse operator</u>, denoted by U, is defined by $(UX)_i = (X)_i^{-1}$ for i in $\{1,\ldots,n\}$. Let A be a n × n matrix such that A > 0. Note that if r is in N_∞ then $UrX = r^{-1}UX$, $UU = I$, $(AUX)_i = \sum_{j=1}^{n}(A)_{ij}(X)_j^{-1}$, $(UAX)_i = (\sum_{j=1}^{n} (A)_{ij}(X)_j)^{-1}$, if A is a diagonal matrix with positive main diagonal then $AU = UA^{-1}$, and if A is a permutation matrix then $AU = UA$. The <u>Menon operator associated with A</u> [1, pg 34], denoted by T_A, is UA^tUA. Note that $(T_AX)_i = (\sum_{j=1}^{n} (A)_{ji}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-1}$, if X is a nonnegative vector and r is a nonnegative number then $T_A rX = rT_AX$, if $A \in \Omega_n$ then $T_Ae = e$, and if A is the product of a permutation matrix and a diagonal matrix with positive main diagonal then $T_A = I$. The following Theorem provides a motivation for the study of Menon operators.

THEOREM 1.7. <u>Let</u> A <u>be a</u> n × n <u>matrix such that</u> A > 0. <u>There is a positive vector</u> X <u>such that</u> $T_AX = X$ <u>if and only if</u> A <u>has total support</u>. <u>If</u> A <u>has</u> m <u>nonzero rows and if there is a positive number</u> λ <u>and a positive vector</u> X <u>so that</u> $T_AX = \lambda X$, <u>then</u> $m\lambda = n$ <u>and hence</u> $\lambda = 1$ <u>if and only if</u> A <u>has total support</u>.

PROOF. Let m be in $\{1,\ldots,n\}$ and let A be an n × n matrix such that A > 0 and such that A has only m nonzero rows.

Suppose there is a positive number λ and a positive vector X so that $T_A X = \lambda X$. Since $T_A X$ is positive then each column of A contains a positive number.

$$\text{Let } D_1 = \begin{pmatrix} (UAX)_1 & 0 & \cdots & 0 \\ 0 & (UAX)_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (UAX)_n \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} (X)_1 & 0 & \cdots & 0 \\ 0 & (X)_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (X)_n \end{pmatrix}.$$

Let i be in $\{1,\ldots,n\}$. Then $\sum_{j=1}^{n}(D_1AD_2)_{ji} = \sum_{j=1}^{n}(UAX)_j(A)_{ji}(X)_i = (\sum_{j=1}^{n}(A)_{ji}(\sum_{j=1}^{n}(A)_{jk}(X)_k)^{-1})^{-1}(X)_i = (T_AX)_i^{-1}(X)_i = \lambda^{-1}$, and if the $j^{\underline{th}}$ row of A is not 0 then $\sum_{j=1}^{n}(D_1AD_2)_{ij} = \sum_{j=1}^{n}(UAX)_i(A)_{ij}(X)_j = (AX)_i = 1$. Thus $\sum_{i=1}^{n}\sum_{j=1}^{n}(D_1AD_2)_{ij} = m$ and $\lambda\sum_{j=1}^{n}\sum_{i=1}^{n}(D_1AD_2)_{ij} = m$. Hence $m\lambda = n$ so that $\lambda = 1$ only if each row of A contains a positive number. If $\lambda = 1$ then $T_AX = X$ so that $D_1AD_2 \in \Omega_n$ and hence, by Theorem 1.6, A has total support. Now suppose A has total support. By Theorem 1.6 there is an $n \times n$ diagonal matrix D_1' and a $n \times n$ diagonal matrix D_2', each with a positive diagonal, such that $D_1'AD_2' \in \Omega_n$. Let the vector X' be defined by $(X)_i = (D_2')_{ii}$ and let

$$D_3 = \begin{pmatrix} (UAX')_1 & 0 & \cdots & 0 \\ 0 & (UAX')_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (UAX')_n \end{pmatrix}$$

Then D_3AD_2' is row stochastic. Hence $1 = \sum_{j=1}^{n}(D_1'AD_2')_{ij} = \sum_{j=1}^{n}\sum_{s=1}^{n}(D_1')_{is}\sum_{k=1}^{n}(A)_{sk}(D_2')_{kj} = \sum_{s=1}^{n}(D_1')_{is}\sum_{k=1}^{n}(A)_{sk}\sum_{j=1}^{n}(D_2')_{kj} = (D_1')_{ii}\sum_{k=1}^{n}(A)_{ik}(D_2')_{kk}$. Similarily $1 = (D_3)_{ii}\sum_{k=1}^{n}(A)_{ik}(D_2')_{kk}$. Thus

$(D_3)_{ii} = (D_1')_{ii}$ and $D_3 = D_1'$. Thus $D_3AD_2' \in \Omega_n$ and hence $1 =$ $\sum_{j=1}^{n}(D_3AD_2')_{ji} = \sum_{j=1}^{n}(UAX')_j\ (A)_{ji}(X')_i = (\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}(X')_k)^{-1})^{-1}(X')_i =$ $(T_AX')_i^{-1}(X')_i$. Therefore $T_AX = X$.

The following examples substantiste Theorem 1.7.

EXAMPLE 1. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\lambda = 2$. Then $T_AX = \lambda X$, $m = 1$, $n = 2$ and hence $\lambda m = n$. Note that $\lambda \neq 1$ and A does not have total support.

EXAMPLE 2. Let $A \in \Omega_n$. Since $Ae = A^te = e$, then $T_Ae = e$. Furthermore, since $m = n = \lambda = 1$ then $\lambda m = n$. Note that $\lambda = 1$ and, by Theorem 1.6, A must have total support.

Brualdi, Parter and Schneider [1, pg 42] proved the following useful Theorem.

THEOREM 1.8. If A is fully indecomposable then 1 is an eigenvalue of T_A with unique eigenvector X, and furthermore, X is the unique positive eigenvector of T_A.

Theorems 1.6 and 1.7 clearly imply that A is a fully indecomposable matrix if and only if there is a positive vector X which is an eigenvalue of T_A and X is unique to within a scalar multiple.

OPERATORS OF THE FORM ET_AF.

EXAMPLE 3. Suppose A is a n × n matrix, A > 0, and there is a nonsingular nonnegative matrix E and a n × n nonnegative matrix B such that $E^{-1}T_AE = T_B$. Then $T_AE = ET_B$ and hence if B has total support then there is a positive eigenvector X of T_B so that EX is a positive eigenvector of T_A, and thus A also has total support.

The above observation, along with a prevaling interest in doubly stochastic matrices, encouraged an interest in the following problem.

PROBLEM 1. Let n be a positive integer and let S be the set to which T_A belongs only if A is a n × n nonnegative matrix and T_A is the Menon operator associated with A. For T_A in S, under what conditions is it possible to find a matrix E and a matrix F so that ET_AF is in S?

While the solution to Problem 1 has proven to be quite elusive, certain related questions have yielded answers.

EXAMPLE 4. Suppose A is a n × n matrix such that A > 0. Let P be a n × n permutation matrix and let D be a diagonal matrix with positive main diagonal. Since PU = UP and $DU = UD^{-1}$ then

(i) $T_A = UA^tUA$

$= UA^t(DP^t)(PD^{-1})UA = UA^tDP^tUPDA = T_{PDA}$

$= UA^t(P^tD)(D^{-1}P)UA = UA^tP^tDUDPA = T_{DPA}$.

(ii) $PDT_A = PDUA^tUA = PDUA^tUA(D^{-1}P^t)(PD) =$

$(UPD^{-1}A^tUAD^{-1}P^t)PD = T_{AD^{-1}P^t}PD = T_{A(PD)^{-1}}PD$

(iii) $DPT_A = DPUA^tUA = DPUA^tUA(P^tD^{-1})DP = (UD^{-1}PA^tUAP^tD^{-1})DP =$

$$T_{AP^tD^{-1}}DP = T_{A(DP)^{-1}}DP.$$

Hence (iv) $T_A = T_{PDA} = T_{PA} = T_{DA}$

(v) $(PD)T_A(PD)^{-1} = T_{A(PD)^{-1}} = T_{(PD)A(PD)^{-1}}$

(vi) $(DP)T_A(DP)^{-1} = T_{A(DP)^{-1}} = T_{(DP)A(DP)^{-1}}$.

Consideration of (iv) above demonstrates that $T_A = T_B$ may not imply that A = B. In fact, R. Sinkhorn (unpublished papers) has proven that if $A \in \Omega_n$ and $B \in \Omega_n$ then $T_A = T_B$ if and only if there is a permutation matrix P such that A = PB.

EXAMPLE 5.

If $A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ then $ET_A = T_A$.

EXAMPLE 6.

If $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $T_A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and hence if each of α and β is a number and $E = \begin{pmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ then $ET_A = T_AE$.

EXAMPLE 7. If A is the product of a diagonal matrix with positive main diagonal and a permutation matrix then $T_A = I$ and hence if E is a matrix then $ET_A = T_AE$. In particular, if E is nonsingular then $ET_AE^{-1} = T_A$.

EXAMPLE 8. If s is a positive integer, $\{m_i\}_{i=1}^{s}$ is a positive integer sequence, $\{J_{m_i}\}_{i=1}^{s}$ is a sequence of flat matrices, and

$$A = \begin{pmatrix} J_{m_1} & 0 & \dots & 0 \\ 0 & J_{m_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & J_{m_s} \end{pmatrix}$$

then $T_A = A$. Hence if $\{E_i\}_{i=1}^{s}$ is a sequence of matrices such that if i is in $\{1,\dots,s\}$ then $E_i \in \Omega_{m_i}$, and

$$E = \begin{pmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & E_s \end{pmatrix}$$

then $ET_A = T_AE$. In particular, if E is nonsingular then $ET_AE^{-1} = T_A$.

The above examples have suggested the following unproven conjecture.

CONJECTURE. Let A be a nonnegative $n \times n$ matrix with a positive number in each column, and suppose A is such that if each of D_1 and D_2 is a diagonal matrix with positive main diagonal and each of P_1 and P_2 is a permutation matrix then $P_1D_1AD_2P_2$ is not idempotent. If each of E and F is a nonnegative matrix such that ET_AF is a Menon operator, then each of E and F is the product of a diagonal matrix with a positive main diagonal and a permutation matrix.

The theorems in the following chapter provide limited support for the conjecture.

CHAPTER II

MATRICES WHICH COMMUTE WITH MENON OPERATORS. If A is a n × n matrix with total support then by Theorem 1.6 there is a diagonal matrix D_1 and a diagonal matrix D_2, each with a positive main diagonal, so that $D_1AD_2 \in \Omega_n$. Hence $D_2^{-1}T_AD_2 = T_{D_1AD_2}$ and therefore if there is a matrix E which commutes with $T_{D_1AD_2}$ then $D_2ED_2^{-1}$ commutes with T_A. This observation, and the search for the solution to Problem 1, encouraged an interest in the following Problem.

PROBLEM 2. If $A \in \Omega_n$ and E is a matrix such that $E > 0$, under what conditions does E commute with T_A?

The following theorems investigate Problem 2.

LEMMA 1 TO THEOREM 2.1. *If A is a fully indecomposable matrix then A^tA is irreducible.*

PROOF. Let A be a fully indecomposable matrix and suppose that A^tA is reducible. Then there is a permutation matrix P, a positive integer n_1, and a positive integer n_2 so that

$$P^tA^tAP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

A_{11} is $n_1 \times n_1$, and A_{22} is $n_2 \times n_2$. Partition AP into

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

so that F_{11} is $n_1 \times n_1$. Then

$$(AP)^tAP = \begin{pmatrix} F_{11}^t & F_{21}^t \\ F_{12}^t & F_{22}^t \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} (F_{11}^tF_{11} + F_{21}^tF_{21}) & (F_{11}^tF_{12} + F_{21}^tF_{22}) \\ (F_{12}^tF_{11} + F_{22}^tF_{21}) & (F_{12}^tF_{12} + F_{22}^tF_{22}) \end{pmatrix}.$$

Since $F_{11}^tF_{12} + F_{21}^tF_{22} = 0$ then $F_{11}^tF_{12} = 0$ and $F_{21}^tF_{22} = 0$. Since A is fully indecomposable then AP is fully indecomposable and therefore there is an integer i_1 and an integer j_1 so that $(F_{12})_{i_1j_1} \neq 0$. Since $F_{11}^tF_{12} = 0$ then $\sum_{k=1}^{n_1} (F_{11}^t)_{ik}(F_{12})_{kj_1} = 0$ for i in $\{1,\ldots,n_1\}$. Thus $(F_{11}^t)_{ii_1} = 0$ for i in $\{1,\ldots,n_1\}$ and therefore the i_1th row of F_{11} is 0. Similarily, there is an integer i_2 and an integer j_2 so that $(F_{21})_{i_2j_2} \neq 0$. Since $F_{21}^t F_{22} = 0$ then $\sum_{k=1}^{n_2} (F_{21}^t)_{i_2k}(F_{22})_{kj} = 0$ for j in $\{1,\ldots,n_2\}$. Thus $(F_{22})_{i_2j} = 0$ for j in $\{1,\ldots,n_2\}$ and therefore the i_2th row of F_{22} is 0. Since A is fully indecomposable then there is an integer m_1 in $\{1,\ldots,n_1\}$ and an integer m_2 in $\{1,\ldots,n_2\}$ so that there are only m_1 rows of F_{12} which contain a positive element and only m_2 rows of F_{21} which contain a positive element. Since there are m_1 rows of F_{12} which contain a positive element then there are m_1 rows of F_{11} which are 0 and hence there are $n_1 - m_1$ rows of F_{11} which contain a positive element. Thus, since there are m_2 rows of F_{21} which contain a positive element then there are $n_1 - m_1 + m_2$ rows of

$$\begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix}$$

which contain a positive element. Hence there is a permutation matrix Q so that QAP can be partitioned into

$$\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

and so that B_{11} is $(n_1 - m_1 + m_2) \times n_1$. Since A is fully indecomposable then $n_1 - m_1 + m_2 \neq n_1$. If $n_1 - m_1 + m_2 < n_1$ then there is a positive integer k so that $n_1 - m_1 + m_2 + k = n_1$. Since QAP can be partitioned into

$$\begin{pmatrix} B'_{11} & B'_{12} \\ 0 & B'_{22} \end{pmatrix}$$

so that B'_{11} is $(n_1 - m_1 + m_2 + k) \times n_1$, then A is not fully indecomposable. But this contradicts the hypothesis that A is fully indecomposable and therefore $n_1 - m_1 + m_2 \not< n_1$. If $n_1 - m_1 + m_2 > n_1$ then $n_2 + m_1 - m_2 < n_2$ and hence there is a positive integer k so that $n_2 + m_1 - m_2 + k = n_2$. Since QAP can be partitioned into

$$\begin{pmatrix} B'_{11} & 0 \\ B'_{21} & B'_{22} \end{pmatrix}$$

so that B'_{22} is $(n_2 + m_1 - m_2 + k) \times n_2$, then A is not fully indecomposable. But this contradicts the hypothesis that A is fully indecomposable and hence $n_1 - m_1 + m_2 \not> n_1$. Thus m_1 and m_2 do not exist and therefore A^tA is irreducible.

Note that if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then A is irreducible but A^tA is reducible. Hence it is not true in general that " A^tA is irreducible whenever A is an irreducible matrix".

LEMMA 2 TO THEOREM 2.1. *If A is an irreducible matrix in Ω_n and E is a matrix such that $E > 0$ and $EA = AE$ then there is a positive number r such that $rE \in \Omega_n$.*

PROOF. Let A be an irreducible matrix in Ω_n and suppose that there is a matrix $E > 0$ and $EA = AE$. Since $E > 0$ then $E^te > 0$, and since $EA = AE$

then $A^tE^te = E^tA^te = E^te$. Thus E^te is a characteristic vector of A^t corresponding to the characteristic number 1. Hence by Theorems 1.1 and 1.3 there is a positive number r such that $rE^te = e$. Therefore rE is column stochastic and so $\sum_{i=1}^{n} \sum_{j=1}^{n} r(E)_{ij} = n$. Similarily $AEe = EAe = Ee$ and thus there is a positive number r' such that $r'Ee = e$. Therefore $r'E$ is row stochastic and so $\sum_{j=1}^{n} \sum_{i=1}^{n} r'(E)_{ij} = n$. Therefore $r' = r$ and $rE \in \Omega_n$.

THEOREM 2.1. Let E be a nonnegative matrix with a positive number in each row. If A is a fully indecomposable matrix in Ω_n and $ET_A = T_AE$ then there is a real number r so that $rE \in \Omega_n$, $EA^tA = A^tAE$, and $E^tE[(A^tA)^2 - A^tA] = (A^tA)^2 - A^tA$.

PROOF. Let E be a nonnegative matrix with a positive number in each row. Suppose A is a fully indecomposable matrix in Ω_n and suppose $ET_A = T_AE$. Then $ET_Ae = T_AEe$ and since $A \in \Omega_n$ then $T_Ae = e$ so that $T_AEe = Ee$. By Theorem 1.8 there is a real number r so that rE is row stochastic. For the remainder of the proof it may be assumed, without loss of generality, that E is row stochastic. Let i be in $\{1,\ldots,n\}$ and let X be a vector. Since $(ET_AX)_i = (T_AEX)_i$ then

$$\sum_{m=1}^{n}(E)_{im}\Big(\sum_{j=1}^{n}(A)_{jm}\big(\sum_{k=1}^{n}(A)_{jk}(X)_k\big)^{-1}\Big)^{-1} = \Big(\sum_{j=1}^{n}(A)_{ji}\big(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m\big)^{-1}\Big)^{-1}.$$

Let s be in $\{1,\ldots,n\}$. Since

$$\frac{\partial(ET_AX)_i}{\partial(X)_s} = \frac{\partial(T_AEX)_i}{\partial(X)_s}$$

then

$$\sum_{m=1}^{n}(E)_{im}\Big(\sum_{j=1}^{n}(A)_{jm}\big(\sum_{k=1}^{n}(A)_{jk}(X)_k\big)^{-1}\Big)^{-2}\Big(\sum_{j=1}^{n}(A)_{jm}\big(\sum_{k=1}^{n}(A)_{jk}(X)_k\big)^{-2}(A)_{js}\Big) =$$

$$\Big(\sum_{j=1}^{n}(A)_{ji}\big(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m\big)^{-1}\Big)^{-1}\Big(\sum_{j=1}^{n}(A)_{ji}\big(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m\big)^{-1}.$$

$(\sum_{k=1}^{n}(A)_{jk}(E)_{ks}))$. Hence evaluating $\frac{\partial(ET_A X)_i}{\partial(X)_s}$ at X = e gives

$$\left.\frac{\partial(ET_A X)_i}{\partial(X)_s}\right|_{X=e} = \left.\frac{\partial(T_A EX)_i}{\partial(X)_s}\right|_{X=e}$$

so that

$\sum_{m=1}^{n}(E)_{im}\sum_{j=1}^{n}(A)_{jm}(A)_{js} = (\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km})^{-1})^{-2}\cdot$

$(\sum_{j=1}^{n}(A)_{ji}\sum_{k=1}^{n}(A)_{jk}(E)_{ks})$. Hence $(EA^tA)_{is} = (T_A Ee)_i^2(A^tAE)_{is}$ so that $(EA^tA)_{is} = 1\cdot(A^tAE)_{is}$ and thus $EA^tA = A^tAE$. Therefore by Lemmas 1 and 2 to Theorem 2.1, $E \in \Omega_n$. Let u be in $\{1,\ldots,n\}$. Then

$\frac{\partial^2(T_A EX)_i}{\partial(X)_u\partial(X)_s} = [(-2)(\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m)^{-1})^{-3}(\sum_{j=1}^{n}(A)_{ji}(-1)\cdot$

$(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m)^{-2}(\sum_{k=1}^{n}(A)_{jk}(E)_{ku}))(\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m)^{-2}\cdot$

$(\sum_{k=1}^{n}(A)_{jk}(E)_{ks}))] + [(\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m)^{-1})^{-2}\cdot$

$(\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}(E)_{ks})(-2)(\sum_{k=1}^{n}(A)_{jk}\sum_{m=1}^{n}(E)_{km}(X)_m)^{-3}(\sum_{k=1}^{n}(A)_{jk}(E)_{ku}))]$.

And $\frac{\partial^2(ET_A X)_i}{\partial(X)_u\partial(X)_s} = \sum_{m=1}^{n}(E)_{im}[(-2)(\sum_{j=1}^{n}(A)_{jm}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-3}\cdot$

$(\sum_{j=1}^{n}(A)_{jm}(-1)(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-2}(A)_{ju})(\sum_{j=1}^{n}(A)_{jm}(A)_{js}(\sum_{k=1}^{n}(A)_{jk}(X)_j)^{-2}) +$

$(\sum_{j=1}^{n}(A)_{jm}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-2}(\sum_{j=1}^{n}(A)_{jm}(A)_{js}(-2)(\sum_{k=1}^{n}(A)_{jk}(X)_j)^{-3}(A)_{ju})]$.

Hence evaluating $\frac{\partial^2(ET_A X)_i}{\partial(X)_u\partial(X)_s}$ at X = e gives $\left.\frac{\partial^2(ET_A X)_i}{\partial(X)_u\partial(X)_s}\right|_{X=e} = \left.\frac{\partial^2(T_A EX)_i}{\partial(X)_u\partial(X)_s}\right|_{X=e}$

so that $\sum_{m=1}^{n}(E)_{im}[(\sum_{j=1}^{n}(A)_{jm}(A)_{ju})(\sum_{j=1}^{n}(A)_{jm}(A)_{js})-(\sum_{j=1}^{n}(A)_{jm}(A)_{js}(A)_{ju})] =$

$= (\sum_{j=1}^{n}(A)_{ji}\sum_{k=1}^{n}(A)_{jk}(E)_{ku})(\sum_{j=1}^{n}(A)_{ji}\sum_{k=1}^{n}(A)_{jk}(E)_{ks}) - (\sum_{j=1}^{n}(A)_{ji}(\sum_{k=1}^{n}(A)_{jk}(E)_{ku})\cdot$ $(\sum_{k=1}^{n}(A)_{jk}(E)_{ks}))$. Hence $(A^tAE)_{iu}(A^tAE)_{is} - \sum_{j=1}^{n}(A)_{ji}(AE)_{ju}(AE)_{js} =$ $\sum_{m=1}^{n}(E)_{im}[(A^tA)_{mu}(A^tA)_{ms} - \sum_{j=1}^{n}(A)_{jm}(A)_{js}(A)_{ju}]$. Since

$$\sum_{i=1}^{n}\left(\frac{\partial^2(ET_AX)_i}{\partial(X)_u\partial(X)_s}\right)_{X=e} = \sum_{i=1}^{n}\left(\frac{\partial^2(T_AEX)_i}{\partial(X)_u\partial(X)_s}\right)_{X=e}$$

then $((A^tAE)^t(A^tAE))_{us} - ((AE)^t(AE))_{us} = ((A^tA)^t(A^tA))_{us} - (A^tA)_{us}$ so that $E^t[(A^tA)^2 - (A^tA)]E = (A^tA)^2 - (A^tA)$. Therefore $E^tE[(A^tA)^2 - (A^tA)] = (A^tA)^2 - (A^tA)$.

THEOREM 2.2. Let E be a nonnegative matrix with a positive number in each row. If A is a partly decomposable matrix in Ω_n which is not a permutation matrix and $ET_A = T_AE$ then there is an integer s in $\{2,\ldots,n-1\}$, a positive integer sequence $\{m_i\}_{i=1}^{s}$ such that $\sum_{i=1}^{s}m_i = n$, a permutation matrix P such that

$$P^tEP = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix}$$

and E_{ij} is $m_i \times m_j$, a $s \times s$ matrix R and a $s \times s$ matrix C such that each of $(R)_{ij}$ and $(C)_{ij}$ is a positive number such that $(R)_{ij}m_i = (C)_{ij}m_j$ and such that if $E_{ij} \neq 0$ then $(R)_{ij}E_{ij}$ is row stochastic and $(C)_{ij}E_{ij}$ is column stochastic.

PROOF. Let A be a partly decomposable matrix in Ω_n which is not a permutation matrix. Then there is a permutation matrix Q, a permutation matrix P, a positive integer sequence $\{m_i\}_{i=1}^{s}$ such that

$$QAP = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix},$$

$A_i \in \Omega_{m_i}$, and A_i is fully indecomposable.Since A is partly decomposable then $s > 1$. If $s = n$ and $A \in \Omega_n$ then A is a permutation matrix and thus, since A is not a permutation matrix, then $s < n$. Therefore s is in $\{2,\ldots,n-1\}$ and $n \geq 3$. Let E be a matrix such that $E > 0$, each row of E contains a positive number, and such that $ET_A = T_AE$. Then $P^tEPUP^tA^tQ^tUQAP = UP^tA^tQ^tUQAPP^tEP$ and so $P^tEPT_{QAP} = T_{QAP}P^tEP$. Partition P^tEP into

$$\begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix}$$

so that E_{ij} is $m_i \times m_j$. Let the vector X be partitioned into

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{pmatrix}$$

so that X_i is $m_i \times 1$. Then $P^tEPT_{QAP}X =$

$$\begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix} U \begin{pmatrix} A_1^t & 0 & \cdots & 0 \\ 0 & A_2^t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s^t \end{pmatrix} U \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{pmatrix} =$$

$$\begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix} \begin{pmatrix} UA_1^tUA_1X_1 \\ UA_2^tUA_2X_2 \\ \vdots \\ UA_s^tUA_sX_s \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix} \begin{pmatrix} T_{A_1}X_1 \\ T_{A_2}X_2 \\ \vdots \\ T_{A_s}X_s \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{s} E_{1k}T_{A_k}X_k \\ \sum_{k=1}^{s} E_{2k}T_{A_k}X_k \\ \vdots \\ \sum_{k=1}^{s} E_{sk}T_{A_k}X_k \end{pmatrix}$$

and $T_{QAP}P^tAPX = U\begin{pmatrix} A_1^t & 0 & \cdots & 0 \\ 0 & A_2^t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s^t \end{pmatrix} U \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}\begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ E_{21} & E_{22} & \cdots & E_{2s} \\ \vdots & \vdots & & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{pmatrix} =$

$$U\begin{pmatrix} A_1^t & 0 & \cdots & 0 \\ 0 & A_2^t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s^t \end{pmatrix} U \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}\begin{pmatrix} \sum_{k=1}^{s} E_{1k}X_k \\ \sum_{k=1}^{s} E_{2k}X_k \\ \vdots \\ \sum_{k=1}^{s} E_{sk}X_k \end{pmatrix} = \begin{pmatrix} T_{A_1}\sum_{k=1}^{s} E_{1k}X_k \\ T_{A_2}\sum_{k=1}^{s} E_{2k}X_k \\ \vdots \\ T_{A_s}\sum_{k=1}^{s} E_{sk}X_k \end{pmatrix}.$$

Let i be in $\{1,\ldots,s\}$. Then $\sum_{k=1}^{s} E_{ik}T_{A_k}X_k = T_{A_i}\sum_{k=1}^{s} E_{ik}X_k$. Let e_{m_k} be the e – vector of length m_k. Since $A_k \in \Omega_{m_k}$ then $T_{A_k}e_{m_k} = e_{m_k}$ and thus $\sum_{k=1}^{s} E_{ik}e_{m_k} = T_{A_i}\sum_{k=1}^{s} E_{ik}e_{m_k}$. Since each row of E contains a positive number then $\sum_{k=1}^{s} E_{ik}e_{m_k}$ is positive. Since A_i is fully indecomposable then by Theorem 1.8 there is a positive number r_i so that $r_i \sum_{k=1}^{s} E_{ik}e_{m_k} = e_{m_k}$. Therefore there is a positive number sequence $\{r_i\}_{i=1}^{s}$ so that $r_i[E_{i1} \ldots E_{is}]$ is row stochastic. Let g be in $\{1,\ldots,m_i\}$. Then $(\sum_{k=1}^{s} E_{ik}T_{A_k}X_k)_g = \sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{jk})_{gu}(\sum_{v=1}^{m_k}(A_k)_{vu}(\sum_{q=1}^{m_k}(A_k)_{vq}(X_k)_q)^{-1})^{-1}$. Let j be in $\{1,\ldots,s\}$ and let w be in $\{1,\ldots,m_j\}$. Then

$$\frac{\partial(\sum_{k=1}^{s} E_{ik}T_{A_k}X_k)_g}{\partial(X_j)_w} = \sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{jk})_{gu}(-1)(\sum_{v=1}^{m_k}(A_k)_{vu}(\sum_{q=1}^{m_k}(A_k)_{vq}(X_k)_q)^{-1})^{-2}\cdot$$

$$(\sum_{v=1}^{m_k}(A_k)_{vu}(-1)(\sum_{q=1}^{m_k}(A_k)_{vq}(X_k)_q)^{-2}(\sum_{q=1}^{m_k}(A_k)_{vq}\frac{\partial(X_k)_q}{\partial(X_j)_w})).$$

Since $\frac{\partial (X_k)_q}{\partial (X_j)_w} = 0$ unless $j = k$ and $q = w$ then $\frac{\partial(\sum_{k=1}^{s} E_{ik} T_{A_k} X_k)_g}{\partial (X_j)_w} =$

$$\sum_{u=1}^{m_j} (E_{ij})_{gu} \Big(\sum_{v=1}^{m_j}(A_j)_{vu}\Big(\sum_{q=1}^{m_j}(A_j)_{vq}(X_j)_q\Big)^{-1}\Big)^{-2}\Big(\sum_{v=1}^{m_j}(A_j)_{vu}\Big(\sum_{q=1}^{m_j}(A_j)_{vq}(X_j)_q\Big)^{-2}(A_j)_{vw}\Big).$$

Thus $\left.\frac{\partial(\sum_{k=1}^{m} E_{ik} T_{A_k} X_k)_g}{\partial (X_j)_w}\right|_{X_j=e_{m_j}} = \sum_{u=1}^{m_j}(E_{ij})_{gu} \sum_{v=1}^{m_j}(A_j)_{vu}(A_j)_{vw} = (E_{ij}A_j^tA_j)_{gw}.$

Similarily $(T_{A_i}\sum_{k=1}^{s} E_{ik}X_k)_g = \Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}(X_k)_u\Big)^{-1}\Big)^{-1}.$

Thus $\frac{\partial(T_{A_i}\sum_{k=1}^{s} E_{ik}X_k)_g}{\partial (X_j)_w} = (-1)\Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}(X_k)_u\Big)^{-1}\Big)^{-2}\cdot$

$$\Big(\sum_{v=1}^{m_i}(A_i)_{vg}(-1)\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}(X_k)_u\Big)^{-2}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}\cdot$$

$$\frac{\partial (X_k)_u}{\partial (X_j)_w}\Big)\Big) = \Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}(X_k)_u\Big)^{-1}\Big)^{-2}\cdot$$

$$\Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}\sum_{k=1}^{s}\sum_{u=1}^{m_k}(E_{ik})_{qu}(X_k)_u\Big)^{-2}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}(E_{ij})_{qw}\Big)\Big).$$

Thus $\left.\frac{\partial(T_{A_i}\sum_{k=1}^{s} E_{ik}X_k)_g}{\partial (X_j)_w}\right|_{X_j=e_{m_j}} = \Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(r_i)^{-1}(A_i)_{vq}\Big)^{-1}\Big)^{-2}\cdot$

$$\Big(\sum_{v=1}^{m_i}(A_i)_{vg}\Big(\sum_{q=1}^{m_i}(r_i)^{-1}(A_i)_{vq}\Big)^{-2}\Big(\sum_{q=1}^{m_i}(A_i)_{vq}(E_{ij})_{qw}\Big)\Big) = \sum_{v=1}^{m_i}(A_i)_{vg}\sum_{q=1}^{m_i}(A_i)_{vq}(E_{ij})_{qw} =$$

$(A_i^tA_iE_{ij})_{gw}$. Therefore $E_{ij}A_j^tA_j = A_i^tA_iE_{ij}$. Now suppose $E_{ij} \neq 0$. By Lemma 1 to Theorem 2.1 $A_iA_i^t$ is irreducible, and since $E_{ij}e_{m_j} = A_i^tA_iE_{ij}e_{m_j}$ then by Theorems 1.1 and 1.3 there is a positive number $(R)_{ij}$ so that $(R)_{ij}E_{ij}$ is row stochastic. Furthermore, since $A_j^tA_jE_{ij}^t = E_{ij}^tA_i^tA_i$ then $A_j^tA_jE_{ij}^te_{m_i} = E_{ij}^te_{m_i}$ and so there is a positive number $(C)_{ij}$ so that

$(C)_{ij}E_{ij}$ is column stochastic. Since $\sum_{q=1}^{m_j}\sum_{v=1}^{m_i} (R)_{ij}(E_{ij})_{vq} = m_j$ and $\sum_{v=1}^{m_i}\sum_{q=1}^{m_j} (C)_{ij}(E_{ij})_{vq} = m_i$ then $(R)_{ij}m_i = (C)_{ij}m_j$. Hence there is a $s \times s$ matrix R and a $s \times s$ matrix C such that if each of i and j is in $\{1,\ldots,s\}$ then each of $(R)_{ij}$ and $(C)_{ij}$ is a positive number, $(R)_{ij}m_i = (C)_{ij}m_j$, and if $E_{ij} \neq 0$ then $(R)_{ij}E_{ij}$ is row stochastic and $(C)_{ij}E_{ij}$ is column stochastic.

LEMMA 1 TO THEOREM 2.3. If A is a matrix in Ω_n such that A^tA is idempotent then there is a permutation matrix Q so that QA is idempotent.

PROOF. Let A be a matrix in Ω_n and be such that A^tA is idempotent. By Theorem 1.5 there is a positive integer s, a positive integer sequence $\{m_i\}_{i=1}^{s}$ such that $\sum_{i=1}^{s} m_i = n$, and a permutation matrix P such that

$$P^tA^tAP = \begin{pmatrix} J_{m_1} & 0 & \cdots & 0 \\ 0 & J_{m_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{m_s} \end{pmatrix}.$$

Partition AP into

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ B_{21} & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{ss} \end{pmatrix}$$

so that B_{ij} is $m_i \times m_j$. Let each of i and j be in $\{1,\ldots,s\}$.

$$\text{Then } \sum_{k=1}^{s} B_{ki}^{t} B_{kj} = \begin{cases} 0 \text{ matrix of size } m_i \times m_j & \text{if } i \neq j \\ J_{m_i} & \text{if } i = j \end{cases}$$

Let k be in {1,...,s} and suppose that $i \neq j$. Then $B_{ki}^{t}B_{kj} = 0$. Let v be in $\{1,\ldots,m_k\}$, let w be in $\{1,\ldots,m_j\}$, and suppose that $(B_{kj})_{vw} \neq 0$. Let g be in $\{1,\ldots,m_i\}$. Since $\sum_{u=1}^{m_k}(B_{ki}^{t})_{gu}(B_{kj})_{uw} = 0$ then $(B_{ki}^{t})_{gv} = 0$ and thus the v th row of B_{ki} is 0. Since $\sum_{k=1}^{s} B_{ki}^{t}B_{ki} = J_{m_i}$ then

$$\begin{pmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{si} \end{pmatrix} > 0$$

and thus there is a permutation matrix R such that RAP can be partitioned into

$$\begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix}$$

so that $C_i > 0$ and C_i contains only m_i columns. Suppose C_i has f_i rows. Since $RAP \in \Omega_n$ then $\sum_{w=1}^{f_i}\sum_{v=1}^{m_i}(C_i)_{vw} = f_i$ and $\sum_{v=1}^{m_i}\sum_{w=1}^{f_i}(C_i)_{vw} = m_i$ so that $f_i = m_i$. Since the rank of P^tA^tAP is the rank of A then the rank of A is s. Since $C_i > 0$ then the rank of C_i is greater than or equal to 1. Therefore since the rank of RAP is s then the rank of C_i is 1 and thus $C_i = J_{m_i}$. Hence RAP is idempotent and therefore $P(RAP)P^t = PRA$ so that $(PR)A$ is idempotent.

THEOREM 2.3. *If E is a primitive matrix in Ω_n, $A \varepsilon \Omega_n$, and $ET_A = T_AE$ then A^tA is idempotent and so there is a permutation matrix Q so that QA is idempotent.*

PROOF. Let $A \varepsilon \Omega_n$, let E be a primitive matrix in Ω_n, and suppose that $ET_A = T_AE$. Let m be a positive integer and suppose that $E^{m-1}T_A = T_AE^{m-1}$. Then $EE^{m-1}T_A = ET_AE^{m-1} = T_AEE^{m-1}$ and thus $E^mT_A = T_AE^m$. Since E is a primitive matrix in Ω_n then 1 is a simple characteristic root of E and is the dominant root of E. Thus if λ is a characteristic root of E, not 1, then $|\lambda| < 1$, and therefore $\lim_{m \to \infty} E^m$ exists and has rank 1. Let α be a positive number and let i be in $\{1,\ldots,n\}$. Then there is a positive integer q such that $\left|\sum_{j=1}^{n}(E^q)_{ij} - \sum_{j=1}^{n}(\lim_{m \to \infty} E^m)_{ij}\right| < \alpha$ and $\left|\sum_{j=1}^{n}((E^q)^t)_{ij} - \sum_{j=1}^{n}((\lim_{m \to \infty} E^m)^t)_{ij}\right| < \alpha$. Hence $\lim_{m \to \infty} E^m \varepsilon \Omega_n$ and therefore $\lim_{m \to \infty} E^m = J_n$. Since $\lim_{m \to \infty} E^mT_A = \lim_{m \to \infty} T_AE^m$ then $J_nT_A = T_AJ_n$. Let X be a vector and let each of u and v be in $\{1,\ldots,n\}$. Then

$T_AJ_nX = T_A(\frac{1}{n}\sum_{k=1}^{n}(X)_k)e = (\frac{1}{n}\sum_{k=1}^{n}(X)_kT_Ae = (\frac{1}{n}\sum_{k=1}^{n}(X)_k)e$. Since

$$\left.\frac{\partial^2(J_nT_AX)_i}{\partial(X)_v\partial(X)_u}\right|_{X=e} = \left.\frac{\partial^2(T_AJ_nX)_i}{\partial(X)_v\partial(X)_u}\right|_{X=e} \text{ then } \left.\frac{\partial^2(\sum_{w=1}^{n}(\sum_{j=1}^{n}(A)_{jw}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-1})}{\partial(X)_v\partial(X)_u}\right|_{X=e} =$$

$$\left.\frac{\partial^2(\sum_{w=1}^{n}(X)_w)}{\partial(X)_v\partial(X)_u}\right|_{X=e}.$$ Thus

$$\frac{\partial(\sum_{w=1}^{n}(-1)(\sum_{j=1}^{n}(A)_{jw}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-2}(\sum_{j=1}^{n}(A)_{jw}(-1)(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-2}(A)_{ju}))}{\partial(X)_v}\Bigg|_{X=e} =$$

$\frac{\partial(1)}{\partial(X)_v}\Big|_{X=e}$. Thus $\Big[\sum_{w=1}^{n}(-2)(\sum_{j=1}^{n}(A)_{jw}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-3}\cdot$

$\cdot(\sum_{j=1}^{n}(A)_{jw}(-1)(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-2}(A)_{jv})(\sum_{j=1}^{n}(A)_{jw}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-2}(A)_{ju})$ +

$\sum_{w=1}^{n}(\sum_{j=1}^{n}(A)_{jw}(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-1})^{-2}(\sum_{j=1}^{n}(A)_{jw}(A)_{ju}(-2)(\sum_{k=1}^{n}(A)_{jk}(X)_k)^{-3}(A)_{jv}\Big]_{X=e} = 0.$

Thus $\sum_{w=1}^{n}(\sum_{j=1}^{n}(A)_{jw}(A)_{jv})(\sum_{j=1}^{n}(A)_{jw}(A)_{ju}) = \sum_{w=1}^{n}\sum_{j=1}^{n}(A)_{jw}(A)_{ju}(A)_{jv}$. Thus $\sum_{w=1}^{n}(A^tA)_{wv}(A^tA)_{wu} = (A^tA)_{vu}$ so that $((A^tA)A^tA)_{vu} = (A^tA)_{vu}$. Hence $(A^tA)^2 = A^tA$ and therefore by Lemma 1 to Theorem 2.3 there is a permutation matrix Q so that QA is idempotent.

Theorems 2.1, 2.2, and 2.3 investigate Problem 2 under somewhat general conditions. The following theorem considers Problem 2 for a case in which A is more specifically defined than in previous theorems. Since it is well known, i.e. Birkhoff's Theorem, that if $A \in \Omega_n$ then A is a convex combination of permutation matrices, then the theorem to follow may point the way to the total solution of Problem 2.

THEOREM 2.4. _Let_ Q _be a_ $n \times n$ _permutation matrix, let each of_ α _and_ β _be a nonnegative number, and let_ $A = \beta J_n + (\alpha - \frac{1}{n}\beta)Q$ _be a matrix in_ Ω_n _which is not a permutation matrix or_ J_n. _If_ E _is a nonnegative matrix such that each row of_ E _contains a positive number, then_ $ET_A = T_AE$ _if and only if there is a number_ r _so that_ rE _is a permutation matrix._

PROOF. Let Q be a $n \times n$ permutation matrix, let each of α and β be a nonnegative number, and let $A = \beta J_n + (\alpha - \frac{1}{n}\beta)Q$ be a matrix in Ω_n which is not a permutation matrix or J_n. Since $A \in \Omega_n$ and A is not a permutation matrix then $n > 1$, $\beta > 0$, and if $n = 2$ then $\alpha \neq 0$. Since $T_{(\beta J_n+(\alpha-\frac{1}{n}\beta)I)} = U(\beta J_n+(\alpha-\frac{1}{n}\beta)I)Q^t QU(\beta J_n+(\alpha-\frac{1}{n}\beta)I) =$

$U(\beta J_n+(\alpha-\frac{1}{n}\beta)Q)^t U(\beta J_n+(\alpha-\frac{1}{n}\beta)Q) = T_A$ then it is sufficent to prove the theorem for $Q = I$.If there is a number r and a $n \times n$ matrix E so that rE is a permutation matrix then clearly $ET_A = T_AE$.

Now suppose there is a nonnegative matrix E such that each row of E contains a positive number and such that $ET_A = T_AE$. Since $\beta \neq 0$ then A is fully indecomposable and therefore by Theorem 2.1 there is a number r so that $rE \in \Omega_n$. For the remainder of the proof it may be assumed, without loss of generality, that $r = 1$ and thus $E \in \Omega_n$. By Theorem 1.4 (Birkhoff's Theorem) there is a positive integer s, a positive number sequence $\{r_m\}_{m=1}^{s}$, and a reversible sequence of permutation matrices $\{P_m\}_{m=1}^{s}$ so that $\sum_{m=1}^{s} r_m = 1$ and

$\sum_{m=1}^{s} r_m P_m = E$. Since permutation matrices commute with T_A then

$\sum_{m=1}^{s} r_m P_m T_A = \sum_{m=1}^{s} T_A r_m P_m$ and therefore since $ET_A = T_AE$ then

$\sum_{m=1}^{s} T_A r_m P_m = T_A \sum_{m=1}^{s} r_m P_m$. For m an integer in $\{1,\ldots,s\}$ let σ_m be the permutation on $\{1,\ldots,n\}$ which defines P_m.

CASE I. Suppose that E is a primitive matrix. Then by Theorem 2.3 there is a permutation matrix R such that RA is idempotent and hence, by Theorem 1.5, $A = J_n$. However, $A = J_n$ contradicts the hypothesis that $A \neq J_n$ and hence there is no primitive matrix which commutes with T_A.

CASE II. Suppose E is not a primitive matrix. Then by Theorem 1.2

there is an integer $i_\circ$ and an integer $j_\circ$ so that $(E)_{i_\circ j_\circ} = 0$. By Theorem 1.4, if m is in $\{1,\ldots,s\}$ then $\sigma_m(j_\circ) \neq i_\circ$. Let Φ be the set to which m belongs only if m is the least number in $\{1,\ldots,s\}$ such that if q is in $\{1,\ldots,s\}$ then $\sigma_q(j_\circ) = \sigma_m(j_\circ)$. Let $|\Phi|$ be the cardinality of Φ. For m in Φ let Θ_m be the set to which q belongs only if q is in $\{1,\ldots,s\}$ and $\sigma_q(j_\circ) = \sigma_m(j_\circ)$. For m in Φ let $\sum_{q\varepsilon\Theta_m} r_q = R_m$. Clearly $\sum_{m\varepsilon\Phi} R_m = 1$. $T_A E\delta_{j_\circ} = T_A \sum_{m=1}^{s} r_m P_m \delta_{j_\circ} = T_A \sum_{m=1}^{s} r_m \delta_{\sigma_m(j_\circ)} = T_A \sum_{m\varepsilon\Phi} \sum_{q\varepsilon\Theta_m} r_q \delta_{\sigma_q(j_\circ)} = T_A \sum_{m\varepsilon\Phi} R_m \delta_{\sigma_m(j_\circ)}$. Let Λ be the set to which j belongs only if j is in $\{1,\ldots,n\}$ and there is a number m in Φ such that $\sigma_m(j_\circ) = j$. Let $|\Lambda|$ be the cardinality of Λ. If j is in Λ then there is only one number m in Φ so that $\sigma_m(j_\circ) = j$, and if m is in Φ then there is only one number j in Λ such that $\sigma_m(j_\circ) = j$. Therefore $|\Lambda| = |\Phi|$. $(T_A E\delta_{j_\circ})_{i_\circ} = \left(\sum_{j=1}^{n} (A)_{ji_\circ}\left(\sum_{k=1}^{n} (A)_{jk} \sum_{m\varepsilon\Phi} R_m (\delta_{\sigma_m(j_\circ)})_k\right)^{-1}\right)^{-1} =$

$$\left(\sum_{j=1}^{n} (A)_{ji_\circ}\left(\sum_{m\varepsilon\Phi} R_m \sum_{k=1}^{n} (A)_{jk} (\delta_{\sigma_m(j_\circ)})_k\right)^{-1}\right)^{-1} = \left(\sum_{j=1}^{n} (A)_{ji_\circ}\left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)}\right)^{-1}\right)^{-1} =$$

$$\left(\sum_{\substack{j=1\\ j\neq i_\circ\\ j\notin\Lambda}}^{n} (A)_{ji_\circ}\left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)}\right)^{-1} + (A)_{i_\circ i_\circ}\left(\sum_{m\varepsilon\Phi} R_m (A)_{i_\circ\sigma_m(j_\circ)}\right)^{-1} +\right.$$

$$\left.\sum_{j\varepsilon\Lambda} (A)_{ji_\circ}\left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)}\right)^{-1}\right)^{-1} = \left(n-1-|\Lambda|+n\frac{\alpha}{\beta}+\frac{\beta}{n} \sum_{j\varepsilon\Lambda}\left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)}\right)^{-1}\right)^{-1},$$

and $ET_A\delta_{j_\circ} = \sum_{m=1}^{s} T_A r_m P_m \delta_{j_\circ} = \sum_{m=1}^{s} T_A r_m \delta_{\sigma_m(j_\circ)} = \sum_{m\varepsilon\Phi} \sum_{q\varepsilon\Theta_m} T_A r_q \delta_{\sigma_q(j_\circ)} =$ $\sum_{m\varepsilon\Phi} T_A R_m \delta_{\sigma_m(j_\circ)}$ so that $(ET_A\delta_{j_\circ})_{i_\circ} = \sum_{m\varepsilon\Phi}\left(\sum_{j=1}^{n} (A)_{ji_\circ}\left(\sum_{k=1}^{n} (A)_{jk} R_m \delta_{\sigma_m(j_\circ)}\right)^{-1}\right)^{-1} =$

$$= \sum_{m\varepsilon\Phi} R_m \left(\sum_{j=1}^{n} (A)_{ji_\circ} (A)^{-1}_{j\sigma_m(j_\circ)} \right)^{-1} = \sum_{m\varepsilon\Phi} \Big(\sum_{\substack{j=1 \\ j\neq i_\circ \\ j\neq\sigma_m(j_\circ)}}^{n} (A)_{ji_\circ} (A)^{-1}_{j\sigma_m(j_\circ)} +$$

$$(A)_{i_\circ i_\circ} (A)^{-1}_{i_\circ\sigma_m(j_\circ)} + (A)_{\sigma_m(j_\circ) i_\circ} (A)^{-1}_{\sigma_m(j_\circ)\sigma_m(j_\circ)} \Big)^{-1} =$$

$$\sum_{m\varepsilon\Phi} R_m \Big(\sum_{\substack{j=1 \\ j\neq i_\circ \\ j\neq\sigma_m(j_\circ)}}^{n} \frac{\beta}{n}\frac{n}{\beta} + n\frac{\alpha}{\beta} + \frac{1}{n}\frac{\beta}{\alpha} \Big)^{-1} = \left(n - 2 + n\frac{\alpha}{\beta} + \frac{1}{n}\frac{\beta}{\alpha} \right)^{-1}.$$

If $\alpha = 0$ then $\sum_{j\varepsilon\Lambda} \left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)} \right) = \infty$ and hence there is an integer j' in Λ such that $\sum_{m\varepsilon\Phi} R_m (A)_{j'\sigma_m(j_\circ)} = 0$. If m is in Φ then $(A)_{j'\sigma_m(j_\circ)} = 0$ and therefore $\sigma_m(j_\circ) = j'$. Hence $|\Phi| = 1$ and $(E)_{j'j_\circ} > 0$. If there is an integer j'' in Λ so that $\sigma_m(j_\circ) = j''$ then $j' = j''$ and hence the $j_\circ$ th column of E is $\delta_{j'}$. Since $E \varepsilon \Omega_n$ then the j' th row of E is $\delta_{j_\circ}$ and hence every column of E contains a 0 entry. Hence every column of E is a δ-vector and therefore E is a permutation matrix.

If $\alpha \neq 0$ then $n - 1 - |\Lambda| + n\frac{\alpha}{\beta} + \frac{1}{n}\beta \sum_{j\varepsilon\Lambda} \left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)} \right)^{-1} =$ $n - 2 + n\frac{\alpha}{\beta} + \frac{1}{n}\frac{\beta}{\alpha}$. Hence $0 = |\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \frac{\beta}{n} \sum_{j\varepsilon\Lambda} \left(\sum_{m\varepsilon\Phi} R_m (A)_{j\sigma_m(j_\circ)} \right)^{-1} =$

$$|\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \frac{\beta}{n} \sum_{j\varepsilon\Lambda} \Big(\sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)\neq j}} R_m (A)_{j\sigma_m(j_\circ)} + \sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)=j}} R_m (A)_{j\sigma_m(j_\circ)} \Big)^{-1} =$$

$$|\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \frac{\beta}{n} \sum_{j\varepsilon\Lambda} \Big(\frac{\beta}{n} \sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)\neq j}} R_m + \sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)=j}} R_m \Big)^{-1} =$$

$$|\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \sum_{j\varepsilon\Lambda} \Big(\sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)\neq j}} R_m + n\frac{\alpha}{\beta} \sum_{\substack{m\varepsilon\Phi \\ \sigma_m(j_\circ)=j}} R_m \Big)^{-1} =$$

$$= |\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \sum_{j\in\Lambda}\Big(1 - \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)\neq j}} R_m + n\frac{\alpha}{\beta}\sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m \Big)^{-1} =$$

$$|\Lambda| - 1 + \frac{1}{n}\frac{\beta}{\alpha} - \sum_{j\in\Lambda}\Big(1 + \big(n\frac{\alpha}{\beta} - 1 \big) \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m \Big)^{-1}.$$ Let $x = n\frac{\alpha}{\beta} - 1$.

Then $0 = |\Lambda| - 1 + (x + 1) - \sum_{j\in\Lambda}\Big(1 + x \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ) = j}} R_m\Big)^{-1} =$

$$(|\Lambda| - 1)(x + 1) + 1 - (x + 1) \sum_{j\in\Lambda}\Big(1 + x \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m \Big)^{-1}.$$

For $z > -1$ let f be the function defined by

$f(z) = (|\Lambda| - 1)(z + 1) + 1 - (z + 1) \sum_{j\in\Lambda}\Big(1 + z \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m\Big)^{-1}$. Then

$$f'(z) = (|\Lambda| - 1) - \sum_{j\in\Lambda}\Big(1 - \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m\Big)\Big(1 + z \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m\Big)^{-2}$$ and

$$f''(z) = 2 \sum_{j\in\Lambda}\Big(1 - \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m\Big)\Big(1 + z \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m\Big)^{-3} \sum_{\substack{m\in\Phi \\ \sigma_m(j_\circ)=j}} R_m.$$

Now, since $\alpha \neq 0$ and $f(x) = f(n\frac{\alpha}{\beta} - 1)$ then $f(x) = 0$ only if $(T_A E\delta_{j_\circ})_{i_\circ} = (ET_A\delta_{j_\circ})_{i_\circ}$. Furthermore, $s = 1$ only if E is a permutation matrix. If E is a permutation matrix then $s = |\Lambda| = |\Phi| = 1$ and so $f = 0$. If E is not a permutation matrix then $s > 1$ and hence $f(0) = 0$, $f'(0) = 0$, and $f'' > 0$. Therefore if $s > 1$ then $(T_A E\delta_{j_\circ})_{i_\circ} = (ET_A\delta_{j_\circ})_{i_\circ}$ only if $x = 0$ and hence only if $A = J_n$. However,

$A = J_n$ sontradicts the hypothesis that $A \neq J_n$ and therefore if E is not a primitive matrix then $s = 1$ and E is a permutation matrix.

CONCLUSION

For A a nonnegative n × n matrix, nontrivial examples are given to demonstrate the existence of a nonnegative matrix E and a nonnegative matrix F so that if T_A is the Menon operator associated with A, then ET_AF is also a Menon operator. It is conjectured, but not proven, that if A is a n × n nonnegative matrix with a positive number in each column, there are not permutation matrices P_1 and P_2 and diagonal matrices D_1 and D_2 with positive diagonals such that $P_1D_1AP_2D_2$ is idempotent, and if E and F are nonnegative matrices such that ET_AF is a Menon operator, then each of E and F is the product of a diagonal matrix with positive diagonal and a permutation matrix. Theorems supporting this conjecture are proven which show that if A is a doubly stochastic matrix and E is a nonnegative matrix which commutes with T_A then there is a permutation matrix P such that P^tEP can be partitioned into a certain block form, and if A is fully indecomposable then there is a positive number r such that rE is a doubly stochastic matrix. It is further shown that if E is a primitive doubly stochastic matrix, A is a doubly stochastic matrix, and E commutes with T_A, then there is a permutation martix Q such that QA is idempotent. Finally it is proven that if A assumes a certain doubly stochastic form, then the only nonnegative matrix E which commutes with T_A is a constant multiple of a permutation matrix. It is also suggested that the technique used in the proof of this last result might be applied profitably to a more general case in which A is suitably defined.

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