ON M-SPACES AND Δ-SPACES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

C. Bandy

August 1972

ACKNOWLEDGEMENT

The author wishes to express his appreciation to his advisor, Professor J. N. Younglove, for his patience and assistance in the preparation of this thesis.

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ABSTRACT

This thesis is concerned with properties of M-spaces. In particular, products, paracompactness, metrization, and certain subspaces are discussed. Also Δ-spaces, a generalization of M-spaces, are considered.

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INTRODUCTION

A problem of long standing in topology is, under what conditions on two topological spaces X and Y is the product X×Y normal. Recently M.E. Rudin [8] has proved that there is a normal space whose product with the unit interval is not normal. In [5] K. Morita defines an M-space and proves that for X×Y to be normal for any paracompact M-space Y it is necessary and sufficient that X be a paracompact P-space. Since 1964 M-spaces have become of increasing interest in their own right. In [1] C.J.R. Borges introduces two generalizations of M-spaces, Δ-spaces and $\omega\Delta$-spaces. These have proved useful in the various theorems on metrization of Moore spaces. In this paper necessary and sufficient conditions are given for the existence of two M-spaces X and Y so that the product X×Y is not an M-space. An example is given of two M-spaces whose product is not an M-space. Lemmas 1 through 8 lead to Theorem 9 where a meta-Lindelof M-space is proved to be paracompact. Corollary 10 improves a result of C.J.R. Borges. Theorem 13 partially answers a question asked by V.J. Mancuso and also improves a theorem of K. Morita.

DEFINITIONS

All spaces are Hausdorff. The set of positive integers will be denoted by N. A collection of sets U, <u>star-refines</u> a collection of sets V, provided for each point x in the union of the members of U there is a member of U containing the union of all members of V containing x. The sequence $\{U_n : n \in N\}$ is a <u>normal sequence</u> of open covers of the topological space X provided for each n in N, U_n is an open cover of X and U_{n+1} star-refines U_n. The union of all members of a collection V intersecting x will be denoted by <u>$st(x,V)$</u>. A point x is a <u>cluster point</u> of a point-sequence $\{x_n : n \in N\}$ provided each open set containing x contains x_n for infinitely many members of N. The sequence $\{U_n : n \in N\}$ is an M-sequence provided that if x is in X and x_n is in $st(x,U_n)$ for each n in N, then $\{x_n : n \in N\}$ has a cluster point. A space X is called an M-space provided X has a normal sequence of open covers that is also an M-sequence. The space X is a <u>Δ-space</u> provided X has an M-sequence of open covers $\{U_n : n \in N\}$ such that for each x in X, $\overline{st(x,U_{n+1})}$ is contained in $st(x,U_n)$ for each n in N. A collection U of sets is <u>point-countable</u> provided each point is in at most countably many members of U. A collection U <u>refines</u> a collection V provided each member of U is contained in some member of V. A topological space X is <u>meta-Lindelof</u> provided each open cover has a point-countable open refinement covering X. A subset S of a topological space X is an <u>inner-limiting</u> ($=G_\delta$) set provided S is the intersection of countably many open subsets of X. A topological space X is <u>countably compact</u> provided each sequence of points in X has a cluster point in X. A collection U of subsets of a topological space X is <u>locally-finite</u> provided for each point of X there is an open set containing the point and intersecting at most finitely many members of U. A topological space X is <u>paracompact</u> provided each open cover of X has a locally-finite open refinement covering X. A topological space X is <u>totally paracompact</u> provided each open basis of X has a locally-finite subcollection covering X. The symbol $st^2(x,U)$ will denote $st(st(x,U),U)$.

In the results to follow I show how an M-space and a Δ-space can be decomposed. This decomposition is used to prove a theorem about products of M-spaces. Next, properties are developed to aid in proving that a meta-Lindelof M-space is paracompact. These properties are also used to: prove if each open subset of an M-space is an M-space then each inner-limiting set is an M-space, give a metrization theorem for M-spaces, and show that a meta-Lindelof Δ-space with a G_δ-diagonal is a first countable regular space.

RESULTS

Lemma 1. An M-space can be decomposed into closed, pairwise disjoint, countably compact inner-limiting sets.

Let $\{U_n : n \varepsilon N\}$ be a normal M-sequence of open covers of an M-space X. For each x in X, $\bigcap_n st(x, U_n)$ is closed since $st^2(x, U_{n+1})$ is contained in $st(x, U_n)$ so $\overline{st(x, U_{n+1})}$ is contained in $st(x, U_n)$ hence $\bigcap_n st(x, U_n)$ equals $\bigcap_n \overline{st(x, U_n)}$. The set $\bigcap_n st(x, U_n)$ is countably compact because if $\{x_n : n \varepsilon N\}$ is a sequence of points in $\bigcap_n st(x, U_n)$ then x_n is in $st(x, U_n)$ and since $\{U_n : n \varepsilon N\}$ is an M-sequence, $\{x_n : n \varepsilon N\}$ has a cluster point. To show $\{\bigcap_n st(x, U_n) : x \varepsilon X\}$ is pairwise disjoint suppose there is a point x common to $\bigcap_n st(p, U_n)$ and $\bigcap_n st(q, U_n)$ and a point y in $\bigcap_n st(q, U_n)$ but not in $\bigcap_n st(p, U_n)$. This implies that there is a positive integer m so that y is not in $st(p,U_m)$. Since q is in $st(x,U_{n+1})$ and y is in $st(q,U_{n+1})$, y is in $st^2(x,U_{n+1})$ which is a subset of $st(x,U_n)$ for each positive integer n, this puts y in $st(x,U_{m+1})$ and since x is in $st(p,U_{m+1})$ y is in $st^2(p,U_{m+1})$ giving the contradiction y is in $st^2(p,U_{m+1})$ which is a subset of $st(p,U_m)$.

Theorem 2. There exists two M-spaces whose product is not an M-space if and only if there exists two countably compact spaces whose product is not countably compact.

Let X and Y be M-spaces such that $X\times Y$ is not an M-space and $\{U_n: n\varepsilon N\}$ and $\{V_n: n\varepsilon N\}$ be normal M-sequences for X and Y respectively. The set $\{W_n: W_n= \{U\times V : U\varepsilon U_n$ and $V\varepsilon V_n\} : n\varepsilon N\}$ is a normal sequence for $X\times Y$, because if (x,y) is a point of $X\times Y$ then $\{U : U\varepsilon U_n$ for $n>1$ and $x\varepsilon U\}$ is contained in some member u, of U_{n-1} and $\{V : V\varepsilon V_n$ for $n>1$ and $y\varepsilon V\}$ is contained in some member v, of V_{n-1}, and therefore $\{U\times V : x\varepsilon U\varepsilon U_n$ and $y\varepsilon V\varepsilon V_n\}$ is contained in $u\times v$.

Since $X\times Y$ is not an M-space there is a point (x,y) in $X\times Y$ and an infinite discrete set

$\{(x_n, y_n) : n \varepsilon N\}$ contained in $X \times Y$ such that (x_n, y_n) is in $st((x,y),W_n)$. The set $\{x_n : n \varepsilon N\} \cup [\bigcap_n st(x,U_n)]$ is countably compact since $\bigcap_n st(x,U_n)$ is countably compact by Lemma 1 and $\{x_n : n \varepsilon N\}$ has a cluster point in $\bigcap_n st(x,U_n)$. Similarly $\{y_n : n \varepsilon N\} \cup [\bigcap_n st(y,V_n)]$ is countably compact but the product of $\{x_n : n \varepsilon N\} \cup [\bigcap_n st(x,U_n)]$ and $\{y_n : n \varepsilon N\} \cup [\bigcap_n st(y,V_n)]$ is not countably compact since it contains the infinite discrete set $\{(x_n, y_n) : n \varepsilon N\}$.

Although it has already been proved that the product of M-spaces need not be an M-space [2] the following example is simpler and is necessary in the corollary to follow. In [7] J. Novak proved the existence of two countably compact subsets A and B of $\beta(N)$, the Stone-Cech compactification of the positive integers, whose product is not countably compact. Let $[1,\Omega]$ be the space of countable ordinals together with the first uncountable ordinal Ω and having the order topology. Define a topological space S by replacing each non-limit ordinal in $[1,\Omega]$ with a homeomorphic copy of A. An open set containing a point of S will be an open subset of the homeomorphic copy of A containing the point, if the point lies in a homeomorphic copy of A. If the point λ of S is a limit ordinal of $[1,\Omega]$ then an open set containing λ is λ together with the union of all homeomorphic copies of A replacing the ordinals α through λ in $[1,\Omega]$ for some α preceding λ together with the union of all limit ordinals of $[1,\Omega]$ between α and λ. Define a topological space T similarly but using sets homeomorphic to B. Then each of S and T is countably compact since each sequence of points in S has the property that either infinitely many members of the sequence are in some homeomorphic copy of A, in which case the sequence has a cluster point, or each homeomorphic copy of A contains at most finitely many members of the sequence, in which case the least limit ordinal greater than infinitely many members of the sequence is a cluster point of the sequence; where λ is greater than α in S provided λ and α are limit ordinals of $[1,\Omega]$ and λ is greater than α in $[1,\Omega]$, and a limit ordinal λ of $[1,\Omega]$ is greater than each point of a homeomorphic copy of A replacing an ordinal of $[1,\Omega]$ less than λ, and each point of a homeomorphic copy of A replacing an ordinal α of $[1,\Omega]$ is less than each point of a homeomorphic copy of A replacing an ordinal β greater than α. Now S and T are M-spaces but $S\times T$ is not an M-space since $\bigcap_n st((\Omega,\Omega),W_n)$, for any sequence of open covers $\{W_n: n\in N\}$ of $S\times T$, contains a homeomorphic copy of $A\times B$. This follows since the intersection of countably many open subsets of S, each containing Ω, contains a homeomorphic copy of A and the intersection of countably many open subsets of T, each containing Ω, contains a homeomorphic copy of B, hence the set $\bigcap_n st((\Omega,\Omega),W_n)$ contains a homeomorphic copy of $A\times B$.

Corollary 3. There exists two normal M-spaces whose product is not an M-space if and only if there exists two normal countably compact spaces whose product is not countably compact.

Corollary 3 follows from Theorem 2 because if A and B in the proof of Theorem 2 were normal, then S and T would be normal.

Lemma 4. A meta-Lindelof countably compact space is compact.

Suppose R is a point-countable open cover of a countably compact space X such that no countable subcollection of R covers X. Let x_1 be a point of X and let x_2 be a point of X not in $st(x_1,R)$. Since $st(x_1,R)$ is the union of only countably many members of R and therefore cannot cover X, the point x_2 exists. Similarly, for each positive integer n, let x_n be a point of X not in $st(x_k,R)$ for each k less than n. The sequence $\{x_n : n \varepsilon N\}$ has the property that each member of R can contain at most one point of $\{x_n : n \varepsilon N\}$. Since X is countably compact the sequence $\{x_n : n \varepsilon N\}$ has a limit point and any member of R containing the limit point will contain infinitely many members of $\{x_n : n \varepsilon N\}$, giving a contradiction.

<u>Lemma</u> 5. A meta-Lindelof Δ-space is regular.

Suppose C is a closed subset of a Δ-space X and p is a point of X not in C.

Case 1. The set $\bigcap_n st(p,U_n)$ does not intersect C. Then there is an integer m so that $st(p,U_m)$ does not intersect C because $\overline{st(p,U_{k+1})}$ is contained in $st(p,U_k)$ for each positive integer k. Therefore $st(p,U_{m+1})$ is an open set containing p whose closure does not intersect C.

Case 2. The set $\bigcap_n st(p,U_n)$ intersects C. The set $\bigcap_n st(p,U_n)$ is compact by Lemmas 1 and 5, therefore there exists two disjoint open sets O_1 and O_2 containing $\bigcap_n st(p,U_n) \cap C$ and p respectively. The points of C not in O_1 is a closed set that does not intersect $\bigcap_n st(p,U_n)$ so by case 1 there are two disjoint open sets Q_1 and Q_2 containing the points of C not in O_1 and p respectively. Hence $O_1 \cup Q_1$ and $O_2 \cap Q_2$ have the desired property.

Corollary 6. A meta-Lindelof M-space is regular.

An M-space is a Δ-space by Lemma 2.1 of [1].

Lemma 7. If O is an open subset of an M-space X containing $\bigcap_n st(x,U_n)$ for x in X, then there is an integer m so that for each point y in $\bigcap_n st(x,U_n)$, $st(y,U_m)$ is contained in O.

Suppose the contrary. That is, for each positive integer m there is a point x_m of $\bigcap_n st(x,U_n)$ such that $st(x_m,U_m)$ does not lie in O. The sequence $\{x_n : n \varepsilon N\}$ has a cluster point p in $\bigcap_n st(x,U_n)$. There is a positive integer k so that $st^2(p,U_k)$ is contained in O. Then for some integer j greater than or equal k, x_j is in $st(x_j,U_j)$ which is contained in $st^2(p,U_j)$. This means $st(x_j,U_j)$ is contained in O, a contradiction.

Theorem 8. A meta-Lindelof M-space is paracompact.

Let X be a meta-Lindelof M-space with normal M-sequence $\{U_n : n \in N\}$. Let G be an open cover of X. Let A be an index set for $\{\bigcap_n st(x, U_n) : x \in X\}$ i.e., α is in A if and only if α is $\bigcap_n st(x, U_n)$ for some x in X. Using lemma 4 select a finite set of members of G, $G(\alpha,1)$, $G(\alpha,2), \ldots, G(\alpha,n_\alpha)$ that covers α for each α in A. Using lemma 7 let V_m be the set to which v belongs only in case v belongs to U_m and there is an α in A so that v intersects α and v is contained in $\bigcup\{G(\alpha,i) : 1 \leq i \leq n_\alpha\}$ and each member of U_m that intersects α is contained in $\bigcup\{G(\alpha,i) : 1 \leq i \leq n_\alpha\}$. Let W_m be a locally finite open refinement of U_m. This is possible using A.H. Stone's construction in [9], since all that is needed in his construction is for an open cover O of X there exist a sequence $\{O_n : n \in N\}$, of open covers of X, such that O_1 star-refines O and O_{n+1} star-refines O_n for each n in N. Each U_n in $\{U_n : n \in N\}$ has this property. Let H_m be the set to which w belongs only in case w is a member of W_n and w is a subset of some member of V_m. The set H_m is a locally finite open refinement of V_m and $\{H_m : m \varepsilon N\}$ covers X. The set $\{H_m : m \varepsilon N\}$ covers X because if p is a point of X then p belongs to α for some α in A. Therefore there is a positive integer k so that each member of U_k that intersects α is a subset of $\{G(\alpha, i) : 1 \leq i \leq n_\alpha\}$ hence each member of U_k that intersects α is a member of V_k and p belongs to some member w of W_k, which is contained in some member of U_k but this member of U_k is a member of V_k so w is a member of H_k. According to Theorem 1 of [4], $\{H_m : m \varepsilon N\}$ has a locally finite open refinement R covering X. For each r in R select only one v from $\{V_m : m \varepsilon N\}$ containing p, next select an α in A corresponding to v and intersect r with each member of the finite set $G(\alpha,1), G(\alpha,2), \ldots, G(\alpha,n_\alpha)$ previously chosen. The set of intersections yield a locally finite open refinement of G covering X. Let q be a point of X. Then there is an open set in X containing q and intersecting at most finitely many members of R and since each member of R is intersected with only finitely many members of G the open set can intersect at most finitely many of these intersections.

The following corollary improves Theorem 2.13 of [1].

Corollary 9. A meta-Lindelof M-space is metrizable if and only if it has a G_δ-diagonal.

Theorem 2.13 of [1] states that a regular pointwise paracompact M-space is metrizable if and only if it has a G_δ-diagonal. So corollary 9 follows from theorem 8 together with theorem 2.13 of [1].

Corollary 10. If each of X and Y is an M-space and one is meta-Lindelof, then $X \times Y$ is an M-space.

Theorem 6.4 of [5] states that the product of countably many paracompact M-spaces is a paracompact M-space. So Corollary 10 follows from Theorem 8 together with Theorem 6.4 of [5].

Theorem 11. A locally compact meta-Lindelof M-space is totally paracompact.

Let β be a basis for X. Since X is locally compact we may select an open cover of X each member of which has the compact closure. Using theorem 8, let $\{B_\alpha : \alpha \varepsilon A\}$ be an open cover of X so that $\{B_\alpha : \alpha \varepsilon A\}$ is locally finite and refines the cover previously selected. Let $\{C_\alpha : \alpha \varepsilon A\}$ be a collection of open sets covering X so that $\overline{C_\alpha}$ is a subset of B_α for each α in A. For each α in A select a finite subcollection of β covering C_α so that each member selected is also a subset of the corresponding B_α. The subcollection of β selected this way is locally finite and covers X, since for each point of X there is an open set containing the point and intersecting at most finitely many members of $\{C_\alpha : \alpha \varepsilon A\}$ and hence can intersect at most finitely many members of the selected subcollection of β.

In [3] V.J. Mancuso asks, if each open subset of an M-space is an M-space, then is each subset an M-space? Theorem 12 gives a partial answer. Also Theorem 12 improves Theorem 5.1(b) of [6].

Theorem 12. If each open subset of an M-space X is an M-space, then each inner-limiting set in X is an M-space.

Suppose H is an inner-limiting subset of X and $H = \bigcap_n O_n$. Let $\{U_{mn} : n \varepsilon N\}$ be a normal M-sequence for O_m for each m in N. Define $G_n = \{\bigcap_{ij} u_{ij} : i+j = n \text{ and } u_{ij} \varepsilon U_{ij}\}$. The set $\{G_n : n \varepsilon N\}$ is a normal M-sequence for H. To show G_{n+1} star-refines G_n for each n in N, let p be a point of X. Then $st(p,G_{n+1})$ is contained in

$\bigcap\{u_{ij}: u_{ij} \varepsilon U_{ij}$ with $i+j = n$ where $st(p,U_{i+1,j}) \subset u_{ij}\}$ which is a member of G_n. And if x_n is in $st(x,G_n)$ then $\{x_n: n \varepsilon N\}$ has a cluster point in H since x_n is in $st(x,U_{mn})$ for each n in N.

<u>Theorem 13</u>. If S is a countably compact subset of an M-space X, then the closure of S in X is countably compact.

Suppose $\{x_n: n \varepsilon N\}$ is a countably infinite subset of $\bar{S}$ not in S. Let u_n be a member of U_n containing x_n and choose distinct points y_n common to U_n and S. Let p be a limit point of $\{y_n: n \varepsilon N\}$. Since x_k is in $st^2(p,U_n)$ which is contained in $st(p,U_{n-1})$ for some $k \geq n$, $\{x_n: n \varepsilon N\}$ has a cluster point because $\{U_n: n \varepsilon N\}$ is a normal sequence.

<u>Theorem 14</u>. An M-space X is metrizable if and only if there is a normal sequence of open covers $\{U_n: n \varepsilon N\}$ such that x_n is in $st(x,U_n)$ implies that $\{x_n: n \varepsilon N\}$ has sequential limit.

To show X is metric, suppose $\{x_n: x_n \varepsilon st(p,U_n)\}$

has y not p as sequential limit. Then $\{x_{2n} : n \varepsilon N\}$ also has y as sequential limit. Define a sequence $\{y_n : n \varepsilon N\}$ by y_n equals p if n is odd and y_n equals x_n if n is even. Then y_n is in $st(p, U_n)$ for each n but $\{y_n : n \varepsilon N\}$ has no sequential limit. Therefore $\{x_n : n \varepsilon N\}$ has p as sequential limit. Define $G_k = \bigcup_{n=k}^{\infty} U_n$. Since $st^2(p, U_n)$ is contained in $st(p, U_{n-1})$, $\{G_n : n \varepsilon N\}$ is a basis for X satisfying Moore's metrization theorem. Moore's metrization theorem states that a topological space X is metrizable if and only if X has a sequence of open covers $\{G_n : n \varepsilon N\}$ having the property that $\{st^2(x, G_n) : x \varepsilon X \text{ and } n \varepsilon N\}$ is a basis for X.

To show that a metric space has this property let U_{n+1} be a star-refinement of U_n made from spheres with diameters less than $\frac{1}{n+1}$. Then $\{U_n : n \varepsilon N\}$ has the desired property.

In [1] C.J.R. Borges conjectures that there is a meta-Lindelof Δ-space with a G_δ-diagonal that is not a Moore space. Theorem 15 gives a result in this direction but does not settle the conjecture.

Theorem 15. A meta-Lindelof Δ-space with a G_δ-diagonal is a first countable regular space.

A meta-Lindelof Δ-space is regular by Lemma 5.

Let $\{U_n : n \varepsilon N\}$ be an M-sequence of open covers of X. Since $\bigcap_n st(x,U_n)$ is compact, by Lemma 4, and has a G_δ-diagonal, $\bigcap_n st(x,U_n)$ is metrizable. Let ρ_x be a topology preserving metric for $\bigcap_n st(x,U_n)$. Let $S(x,n)$ be an open subset of X such that if p is a point of $S(x,n)$ and $\bigcap_n st(x,U_n)$, then $\rho_x(p,x)$ is less than $\frac{1}{n}$ and $\overline{S(x,n+1)}$ is contained in $S(x,n)$. Define $O(x,n)$ to equal $st(x,U_n) \cap S(x,n)$. The set

$$\{O(x,n) : x \varepsilon X \text{ and } n \varepsilon N\}$$

is a first countable base for X. Let p be a point of X and G an open subset of X containing p, and suppose that for no positive integer n is $O(p,n)$ a subset of G. Let x_n be a point of $O(p,n)$ not in G for each n in N. The set $\{x_n : n \varepsilon N\}$ has a cluster point y in $\bigcap_n st(p,U_n)$ since $\{U_n : n \varepsilon N\}$ is an M-sequence. There is a positive integer k so that $S(p,k)$ does not contain y hence y is not in $\overline{O(p,k+1)}$. The set of points in X not in $\overline{O(p,k+1)}$ is an open set containing p but at most finitely many members of $\{x_n : n \varepsilon N\}$. This is a contradiction.

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