

WEAK CONFLUENCE AND  
W-SETS

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A Dissertation  
Presented to  
the Faculty of the Department of Mathematics  
University of Houston

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In Partial Fulfillment  
of the Requirements for the Degree  
Doctor of Philosophy

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By  
Van Clyde Nall, III  
August, 1983

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## ABSTRACT

If  $K$  is a subcontinuum of a continuum  $M$  such that  $K$  is contained in every continuum in  $M$  which intersects  $K$  and its complement, then  $K$  is called a  $C$ -set. It has been observed that every map from a continuum onto  $M$  is confluent with respect to a  $C$ -set. That is, for every map of a continuum onto  $M$  every component of the preimage of a  $C$ -set  $K$  maps onto  $K$ . A map of a continuum onto  $M$  is said to be weakly confluent with respect to a subcontinuum  $K$  if some component of the preimage of  $K$  maps onto  $K$ . If  $K$  is a subcontinuum of  $M$  such that every map of a continuum onto  $M$  is weakly confluent with respect to  $K$ , then  $K$  will be called a  $W$ -set. In this paper some fundamental properties of  $W$ -sets are discussed, and characterizations of  $W$ -sets in atriodic continua are given.

## TABLE OF CONTENTS

CHAPTER I.	INTRODUCTION.....	1
CHAPTER II.	GENERAL OBSERVATIONS ABOUT W-SETS.....	6
CHAPTER III.	CHARACTERIZATIONS OF W-SETS.....	17
CHAPTER IV.	A CHARACTERIZATION OF THE SIMPLE CLOSED CURVE.....	34
BIBLIOGRAPHY.....		42

## CHAPTER I

### INTRODUCTION

A mapping between continua is weakly confluent if for each subcontinuum  $K$  of the range some component of the preimage of  $K$  maps onto  $K$ .  $\text{Class}[W]$  is the class of all continua which are the images of weakly confluent maps only. The notion of  $\text{Class}[W]$  was introduced by Andrej Lelek in 1972. Since then it has been widely explored and some characterizations of these continua have been given. J. Grispolakis and E.D. Tymchatyn have given a characterization in terms of hyperspaces [4]. J. Davis has shown that acyclic atriodic continua are in  $\text{Class}[W]$ ; therefore, atriodic tree-like continua are in  $\text{Class}[W]$  [2]. G. Feuerbacher has shown that non chainable circle-like continua are in  $\text{Class}[W]$  if and only if they are not weakly chainable [3,Thm.7,p.21]. Here a new approach is taken, and some further results about atriodic continua and  $\text{Class}[W]$  are obtained.

The following conventions and definitions will be used. A mapping is a continuous function and a continuum is a compact connected metric space. A continuum  $M$  is a triod if  $M$  contains a subcontinuum  $K$  such that  $M-K$  has at least

three components, and the subcontinuum  $K$  will be called a core of the triod. An atriodic continuum is one which does not contain a triod. A continuum is unicoherent if it is not the union of two continua whose intersection is not connected. A continuum  $M$  is irreducible if there are two points  $a$  and  $b$  in  $M$  such that no proper subcontinuum of  $M$  contains  $a$  and  $b$ . Finally, a continuum is indecomposable if it is not the union of two of its proper subcontinua.

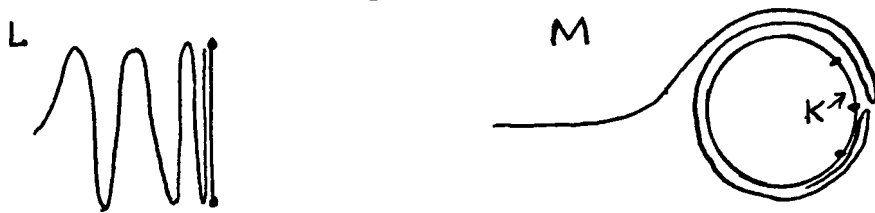
A map between continua is confluent if for each subcontinuum  $K$  of the range each component of the preimage of  $K$  maps onto  $K$ . In a recent paper W.T. Ingram made an interesting observation concerning confluent mappings. In Ingram's terms a map is confluent with respect to a subcontinuum  $K$  of its range if every component of the preimage of  $K$  maps onto  $K$ . An older notion in continuum theory is that of a  $C$ -set. A subcontinuum  $K$  of a continuum  $M$  is a  $C$ -set in  $M$  if every subcontinuum of  $M$  that intersects  $K$  and  $M-K$  contains  $K$ . Ingram has observed that if  $K$  is a subcontinuum of a continuum  $M$  such that every map of a continuum onto  $M$  is confluent with respect to  $K$  then  $K$  is a  $C$ -set in  $M$  [5,Thm.4,p.89]. In addition to giving a meaning for the  $C$  in  $C$ -set, this suggests the importance of looking at confluence and weak confluence on a one subcontinuum at a time basis.

Definition. A map between continua is weakly confluent with respect to a subcontinuum  $K$  of the range if some component of the preimage of  $K$  maps onto  $K$ .

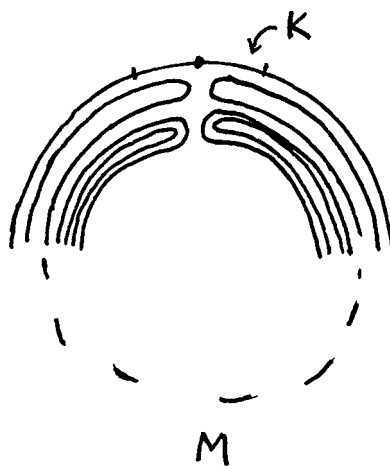
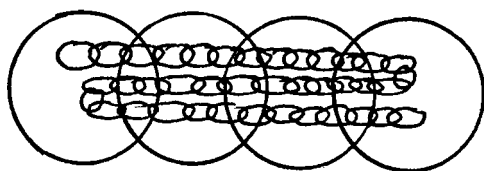
Definition. A subcontinuum  $K$  of a continuum  $M$  is a  $W$ -set in  $M$  if every map from a continuum onto  $M$  is weakly confluent with respect to  $K$ .

Example 1. If a continuum is in  $\text{Class}[W]$ , every proper subcontinuum of it is a  $W$ -set in it. For example every subcontinuum of an arc is a  $W$ -set in that arc.

Example 2. The continuum  $L$  below is a  $\sin(1/x)$  curve together with its limit line. The continuum  $M$  is obtained by identifying the endpoints of the limit line. Thus  $M$  is a ray limiting to a circle. The circle is a  $C$ -set in  $M$ . If  $K$  is an arc in the circle and the point of identification is in  $K$ , but it is not an endpoint of  $K$ , then  $K$  is not a  $W$ -set in  $M$  since the identification map is not weakly confluent with respect to  $K$ .



Example 3. A simple example of an indecomposable continuum can be obtained by taking the intersection of chain coverings where each chain loops through the previous chain from the first link to the last and back to the first link. This is the well known Knaster continuum. The intersection of the first links is an end point. If each chain were to continue after returning to the first link and end in the last link of the previous chain, the intersection of such a sequence of chain covers would still be indecomposable. Now there are two endpoints. Obtain  $M$  by identifying these two endpoints. The continuum  $M$  is still indecomposable, thus it is unicoherent. If  $K$  is one of the arcs in  $M$  which contains the point of identification, but not as an endpoint, then  $K$  is not a  $W$ -set in  $M$  since the identification map is not weakly confluent with respect to  $K$ .





In general, if an atriodic continuum contains a subcontinuum  $K$  which is not a  $W$ -set in it, then it is possible to split the continuum open as in examples 1 and 2 leaving  $K$  in two pieces with these pieces lying in a set which is still connected. Exactly what is meant by this characterization of  $W$ -sets is explained in the statement and proof of Theorem 8. Clearly, any characterization of  $W$ -sets gives a characterization of  $\text{Class}[W]$ .

## CHAPTER II

### GENERAL OBSERVATIONS ABOUT W-SETS

Though most of the results in this paper depend on the absence of triods, the first three lemmas do not.

Lemma 1. Suppose  $K$  is a subcontinuum of the continuum  $M$ , and  $p$  and  $q$  are points in  $K$  such that every continuum in  $M$  from  $p$  to  $M-K$  contains  $q$ . Then, if  $f$  is a map of a continuum  $L$  onto  $M$  and  $E$  is a component of  $f^{-1}(K)$  such that  $p$  is in  $f(E)$ ,  $q$  is also in  $f(E)$ .

Proof. Let  $E(1), E(2), E(3), \dots$  be a sequence of continua in  $L$  which properly contain  $E$ , whose intersection is  $E$ , and such that  $E(i+1)$  is contained in  $E(i)$  for each positive integer  $i$ . Then  $f(E(i))$  contains  $p$ , and  $f(E(i))$  intersects  $M-K$  for each  $i$ . So  $q$  is in  $f(E(i))$  for each  $i$ . But  $f(E)$  is the intersection of the  $f(E(i))$ 's. So  $q$  is in  $f(E)$ .

Lemma 2. Suppose  $K$  is a subcontinuum of  $M$ , and  $p$  is a point in  $M$  such that every continuum from  $p$  to  $M-K$  contains  $K$ . Then  $K$  is a W-set in  $M$ .

Proof. Let  $f$  be a map of a continuum onto  $M$ . Let  $E$  be a component of  $f^{-1}(K)$  such that  $p$  is in  $f(E)$ . Then, by Lemma 1,  $K$  is contained in  $f(E)$ .

Lemma 3. Suppose  $K$  and  $J$  are subcontinua of the continuum  $M$  such that  $K$  is contained in  $J$ , and there is a point  $p$  in  $J$  and a point  $q$  in  $K$  such that every continuum in  $J$  from  $q$  to  $J-K$  contains  $K$ . Then  $K$  is a  $W$ -set in  $M$ .

Proof. Let  $f$  be a map of a continuum onto  $M$ . Let  $E$  be a component of  $f^{-1}(J)$  such that  $p$  is in  $f(E)$ . By Lemma 1,  $K$  is contained in  $f(E)$ , and  $K$  is a  $W$ -set in  $f(E)$  by Lemma 2. So  $K$  is a  $W$ -set in  $M$ .

The following is due to Sorgenfrey [7,Thm.1.8,p.443] and will be used throughout the rest of this paper.

Theorem. If a continuum  $M$  is the union of three of its subcontinua which contain a point in common, and no one of these subcontinua is contained in the other two, then  $M$  contains a triod.

Definition. A continuum  $M$  is atriodic at a subset  $S$  of  $M$  provided no triod in  $M$  has a core in  $S$ .

Lemma 4 and Lemma 5 state facts about triods and are presented without proof.

Lemma 4. If  $K$  is a subcontinuum of the continuum  $M$  such that  $M$  is atriodic at  $K$ , and  $G$  and  $H$  are continua in  $M$  which intersect  $K$  and  $M-K$  such that  $G-K$  does not intersect  $H-K$ , then the intersection of  $K$  with  $G$  and the intersection of  $K$  with  $H$  are connected.

Lemma 5. If  $K$  is a subcontinuum of the continuum  $M$  such that  $M$  is atriodic at  $K$ , and  $J$  is a continuum in  $M$  which intersects  $K$  and  $M-K$ , then the intersection of  $J$  with  $K$  has at most two components.

Lemma 6. Suppose  $K$  is a subcontinuum of the continuum  $M$  such that  $M$  is atriodic at  $K$ . Suppose also that there is a continuum  $J$  which intersects  $K$  and  $M-K$  such that  $K$  is not contained in  $J$ , and every continuum which intersects  $K-J$  and  $M-K$  contains the intersection of  $K$  with  $J$ . Then  $K$  is a  $W$ -set in  $M$ .

Proof. Let  $f$  be a map of a continuum onto  $M$ . Let  $B$  and  $C$  be components of  $f^{-1}(K)$  such that  $f(B)$  and  $f(C)$

intersect  $K-J$ . Let  $p$  be a point in both  $f(B)$  and  $K-J$ . If  $q$  is a point in  $(K \cap J)$ , then every continuum from  $p$  to  $M-K$  contains  $q$ . Therefore, by Lemma 1,  $q$  is contained in  $f(B)$ . So  $f(B)$  contains  $(K \cap J)$ , and similarly  $f(C)$  contains  $(K \cap J)$ . The sets  $f(B)-J$  and  $f(C)-J$  must be nested, or the union of  $f(B)$ ,  $f(C)$ , and  $J$  would contain a triod with core in  $K$ . Let  $D(1), D(2), D(3) \dots$  be a countable dense subset of  $K-J$ , and for each  $i$  let  $C(i)$  be a component of  $f^{-1}(K)$  such that  $D(i)$  is in  $f(C(i))$ . Then some subsequence of  $C(1), C(2), C(3) \dots$  converges to a continuum  $D$  such that  $K=f(D)$ .

Lemma 7. Suppose  $K$  is a subcontinuum of the continuum  $M$ , and  $K$  is contained in an open set  $A$  at which  $M$  is atriodic. Then, if  $K$  is not a  $W$ -set in  $M$ , there are continua  $L$  and  $R$  in  $M$  which intersect  $K$  and  $M-K$  such that  $L-K$  and  $R-K$  do not intersect and  $K$  is not contained in  $L$  or  $R$ .

Proof. Since  $K$  is not a  $C$ -set in  $M$ , there is a continuum  $L$  in  $M$  which intersects  $K$  and  $M-K$  and which does not contain  $K$ . Assume that  $L$  is chosen so that the intersection of  $L$  with  $K$  is connected and  $L$  is contained in  $A$ . By Lemma 6, there is a continuum  $D$  which intersects  $K-L$

and  $M-K$  such that  $(K \cap L)$  is not contained in  $D$ . Again assume  $D$  is chosen so that the intersection of  $D$  with  $K$  is connected. By Lemma 5,  $(D \cap L)$  has at most two components. If  $(D \cap K)$  does not intersect  $(L \cap K)$  then  $D$  contains a continuum  $R$  which intersects both  $(D \cap K)$  and  $M-K$  and does not intersect  $L$ .  $L$  and  $R$  satisfy the conditions of the theorem. So assume  $(D \cap K)$  intersects  $(L \cap K)$ . Let  $E$  be a component of  $(D \cap L)$  such that  $E$  intersects  $K$ . If  $E$  also intersects  $M-K$ , then the set

$$(E \cup (D \cap K) \cup (L \cap K))$$

contains a triod with core in  $A$ . Therefore,  $E$  is contained in  $K$ . If the intersection of  $D$  with  $L$  is contained in  $K$ , let  $R=D$ . If it is not, then  $D$  contains a continuum  $R$  such that  $R$  intersects  $E$  and  $M-K$ , and the intersection of  $L$  and  $R$  is contained in  $E$  which is contained in  $M$ .

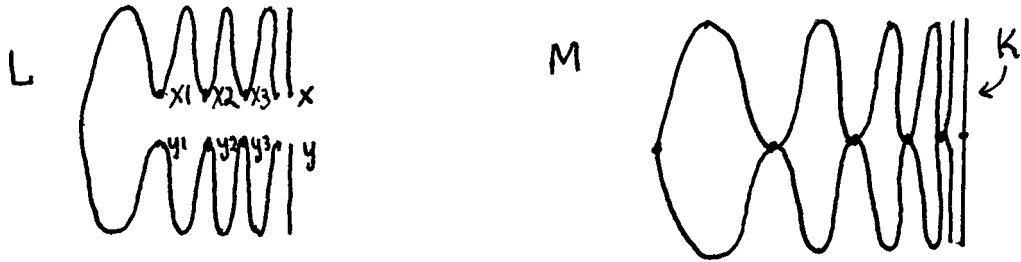
The next theorem is almost a restatement of Lemma 7.

Theorem 1. If the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic, and no subcontinuum of  $M$  is separated by  $K$ , then  $K$  is a  $W$ -set in  $M$ .

In Lemma 6 it is assumed that the continuum  $M$  is

atriodic at its subcontinuum  $K$ . In Lemma 7 this assumption is changed to say that  $M$  is atriodic at an open set containing  $K$ . The following example shows that this strengthening is necessary.

Example 4.



In the continuum labeled  $M$  the subcontinuum  $K$  is the limit line. The map  $f$  from  $L$  to  $M$  is an identification for each  $i$  of the point  $x_i$  with  $y_i$  and  $x$  with  $y$ . No subcontinuum of  $M$  is separated by  $K$ , and  $K$  does not contain the core of a triod in  $M$ . However, every open set containing  $K$  contains the core of a triod in  $M$ , and the map  $f$  is not weakly confluent with respect to  $K$ .

Theorem 2. If the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic and  $M$  is separated by  $K$ , then  $K$  is a  $W$ -set in  $M$ .

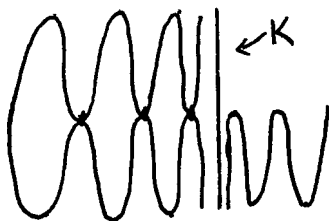
Proof. Let  $C$  and  $D$  be the components of  $M-K$ . If  $H$  is

a subcontinuum of  $M$  which intersects  $C-K$  and  $D-K$ , then  $H$  is separated by  $K$ . If  $K$  is not contained in  $H$ , then the union of  $K$  with  $H$  contains a triod with core in  $K$ . So every subcontinuum of  $M$  which intersects  $C-K$  and  $M-(C \cup K)$  contains  $K$ . It follows from Lemma 6 that  $(C \cup K)$  is a  $W$ -set in  $M$ . If  $K$  separates a subcontinuum  $H$  of  $(C \cup K)$ , then  $(H \cup K \cup D)$  contains a triod with core in  $K$ . So no subcontinuum of  $(C \cup K)$  is separated by  $K$ , and  $K$  is a  $W$ -set in  $(C \cup K)$  by Theorem 1. It follows that  $K$  is a  $W$ -set in  $M$ .

Remark. It was observed in the proof of the previous theorem that if  $K$  does not contain the core of a triod, and  $K$  separates the continuum  $M$  into two components  $C$  and  $D$ , then every continuum in  $M$  which intersects  $C$  and  $D$  contains  $K$ . This fact will be used in several of the proofs that follow.

Theorem 2 also requires the assumption that  $M$  is atriodic at an open set which contains  $K$ . In Example 5 the continuum labeled  $K$  separates the whole continuum and it does not contain the core of a triod, but it is not a  $W$ -set.



Example 5.

Actually Theorem 2 can be strengthened quite a bit.

Theorem 3. Suppose  $K$  and  $J$  are subcontinua of the continuum  $M$  such that  $K$  intersects  $J$ ,  $J$  is not contained in  $K$ , and there is an open set  $A$  containing  $(K \cup J)$  such that  $M$  is atriodic at  $A$ . If  $K$  separates  $M$ , then  $J$  is a  $W$ -set in  $M$ .

Proof. Let  $D$  and  $E$  be the components of  $M-K$ .

Case 1.  $K$  is contained in  $J$ .

In the proof of Theorem 2 it was shown that if  $J=(D \cup K)$  or  $J=(E \cup K)$ , then  $J$  is a  $W$ -set in  $M$ . So assume  $J$  is not equal to  $(D \cup K)$  or  $(E \cup K)$ . If  $J$  does not contain  $D$  or  $E$ , then  $J$  separates  $M$ . Also if  $K=J$ , then  $J$  separates  $M$ . In either case  $J$  is a  $W$ -set by Theorem 2. So assume  $(D \cup K)$  is contained in  $J$  and  $J$  intersects  $E$ . By Lemma 6,  $J$  is a  $W$ -set in  $M$  if every continuum which intersects  $J-(K \cup E)$  and  $M-J$  contains the intersection of  $J$

with  $(K \cup E)$ . So assume  $H$  is a subcontinuum of  $M$  which intersects  $J - (K \cup E)$  and  $M - J$ . Since  $H$  intersects  $C$  and  $D$  it must contain  $K$ . If  $(J \cap E)$  is not contained in  $H$  let  $P$  be a component of  $H - (K \cup E)$ , let  $Q$  be a component of  $H - (K \cup D)$  which intersects  $M - J$ , and let  $R$  be a component of  $J - (K \cup D)$  which intersects  $M - H$ . Then the set

$$(P \cup K) \cup (Q \cup K) \cup (R \cup K)$$

contains a triod with core in  $(J \cup K)$ .

Case 2.  $K$  is not contained in  $J$ .

As in Case 1,  $J$  does not intersect both  $D$  and  $E$ , but  $J$  does intersect either  $D$  or  $E$ . So assume  $J$  intersects  $D$  but not  $E$ . By Lemma 3, both  $(J \cap K)$  and  $(E \cap K)$  are continua.

Either  $(K \cup J)$  separates  $M$  or  $(K \cup J) = (D \cup K)$ . In either case  $(K \cup J)$  is a  $W$ -set in  $M$ . But  $(K \cap J)$  is a continuum which separates  $(K \cup J)$ . So by Case 1,  $J$  is a  $W$ -set in  $(K \cup J)$ , which is a  $W$ -set in  $M$ . Therefore,  $J$  is a  $W$ -set in  $M$ .

The following union theorem for  $\text{Class}[W]$  is due to J. Davis [1]. It is included here since it follows easily from Theorem 3.

Theorem 4. If  $H$  and  $K$  are continua in  $\text{Class}[W]$ ,  $(H \cap K)$  is connected, and  $(H \cup K)$  is an atriodic continuum, then

$(H \cup K)$  is in  $\text{Class}[W]$ .

Proof. If either  $H$  is contained in  $K$  or  $K$  is contained in  $H$ , then  $(H \cup K)$  is clearly in  $\text{Class}[W]$ . So assume  $H-K$  and  $K-H$  are not empty. Then  $(H \cap K)$  separates  $(H \cup K)$  so it is a  $W$ -set in  $(H \cup K)$  by Theorem 2. By Theorem 3, both  $H$  and  $K$  are  $W$ -sets in  $(H \cup K)$ . So if  $J$  is a subcontinuum of  $(H \cup K)$  either  $J$  is contained in one of  $H$  and  $K$ , in which case  $J$  is a  $W$ -set in a  $W$ -set in  $(H \cup K)$ , or  $J$  is not contained in  $H$  or  $K$  and  $J$  is a  $W$ -set by Theorem 3. Since every subcontinuum of  $(H \cup K)$  is a  $W$ -set,  $(H \cup K)$  is in  $\text{Class}[W]$ .

Theorem 5. If  $K$  is an indecomposable subcontinuum of the continuum  $M$  and  $M$  is atriodic at  $K$ , then  $K$  is a  $W$ -set in  $M$ .

Proof. Suppose  $K$  is not a  $W$ -set in  $M$ , and suppose  $p$ ,  $q$ , and  $r$  are points in  $K$  which lie in different composants of  $K$ . By Lemma 2, there are continua  $H$ ,  $I$ , and  $J$  which contain the points  $p$ ,  $q$ , and  $r$  respectively, which intersect  $M-K$ , and which do not contain  $K$ . The continua  $H$ ,  $I$ , and  $J$  can be chosen so that each has a connected intersection with  $K$ . Therefore  $H$ ,  $I$ , and  $J$  each intersect

only one component of  $K$ . Thus  $(H \cap K)$ ,  $(I \cap K)$ , and  $(J \cap K)$  are pairwise disjoint, and the union of  $K$  with  $H$ ,  $I$ , and  $J$  contains a triod with core in  $K$ .

## CHAPTER III

### CHARACTERIZATIONS OF W-SETS

The ultimate goal of this discussion is to give a characterization of W-sets in atriodic continua which is based on "internal" properties of the continua, and which is not stated in terms of mappings. The next theorem gives such a characterization for those W-sets in atriodic continua which have interior.

Theorem 6. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic, and  $K$  has interior. A necessary and sufficient condition that  $K$  be a W-set in  $M$  is that  $K$  have at least one of the following properties:

- 1)  $K$  separates  $M$
- 2)  $K$  is indecomposable
- 3)  $\text{cl}(M-K) \cap K$  is connected

Proof. That properties 1 and 2 are sufficient is established in Theorem 2 and Theorem 5. The complement of  $\text{cl}(M-K) \cap K$  in  $M$  has two components,  $\text{int}(K)$  and  $M-K$ . If  $\text{cl}(M-K) \cap K$  is connected, then  $K$  is a W-set in  $M$  by

Theorem 3.

In order to see that it is necessary to have at least one of the conditions assume  $K$  is decomposable,  $M-K$  is connected, and  $\text{cl}(M-K) \cap K$  is not connected. By Lemma 4,  $\text{cl}(M-K) \cap K$  has only two components  $C$  and  $D$ . Let  $H$  and  $J$  be proper subcontinua of  $K$  such that  $(H \cup J) = K$ .

Case 1. The interior of  $K$  is contained in  $H$ .

Then  $H$  intersects  $C$  and  $D$ , and  $J-H$  is contained in  $(C \cup D)$ . So  $J-H$  intersects either  $C$  or  $D$ . Assume  $J-H$  intersects  $C$ . Let  $I$  be a continuum obtained by identifying in the set

$$(\text{cl}(M-K) \times \{0\}) \cup (H \times \{1\}),$$

for every  $p$  in  $(H \cup D)$ , the point  $(p,0)$  with the point  $(p,1)$ . If  $f$  is the projection of  $I$  onto  $M$ , then  $f$  is not weakly confluent with respect to  $K$ .

Case 2. The interior of  $K$  is not contained in  $H$ .

If  $H$  intersects  $C$  and  $D$ , let  $b(C)$  and  $b(D)$  be continua which intersect  $K$  and  $M-K$  such that  $b(C)$  does not intersect  $b(D)$ ,  $b(C)$  intersects  $C$ , and  $b(D)$  intersects  $D$ . Then the set

$$(b(C) \cup C \cup H) \cup (b(D) \cup D \cup H) \cup (J \cup H)$$

contains a triod with core in  $K$ .

So either  $H$  does not intersect  $C$  or  $H$  does not intersect  $D$ , and, similarly, either  $J$  does not intersect  $C$  or  $J$  does

not intersect  $D$ . But, if  $H$  does not intersect either  $C$  or  $D$ , then  $J$  must intersect both  $C$  and  $D$ . Therefore,  $H$  intersects either  $C$  or  $D$ , and, similarly,  $J$  intersects either  $C$  or  $D$ . Assume  $H$  intersects  $C$  and not  $D$  and  $J$  intersects  $D$  and not  $C$ . Let  $I$  be the continuum obtained by identifying in the set

$$(\text{cl}(M-K) \times \{0\}) \cup (H \times \{1\}) \cup (J \times \{2\}),$$

for every  $p$  in  $(H \cap C)$ , the point  $(p,0)$  with  $(p,1)$ , and for every  $q$  in  $(J \cap D)$ , the point  $(q,0)$  with  $(q,2)$ . If  $f$  is the projection of  $I$  onto  $M$ , then  $f$  is not weakly confluent with respect to  $K$ .

Corollary. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. Suppose also that  $M$  is unicoherent. Then, if  $K$  has interior,  $K$  is a  $W$ -set in  $M$ .

Theorem 7. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. If  $K$  has interior and  $K$  is a  $W$ -set in  $M$ , then there are points  $p$  and  $q$  in  $M$  such that every continuum in  $M$  which contains  $p$  and  $q$  also contains  $K$ .

Proof. By Theorem 6, either  $K$  separates  $M$ ,  $K$  is

indecomposable, or  $\text{cl}(M-K) \cap K$  is connected.

Case 1.  $K$  separates  $M$ .

Let  $C$  and  $D$  be the components of  $M-K$ . If  $\text{cl}(C)$  intersects  $\text{cl}(D)$ , then  $M$  contains a triod with core in  $K$ . According to the remark which follows Theorem 2,  $K$  is irreducible from  $C$  to  $D$ . Let  $p$  be a point in  $C$  and  $q$  be a point in  $D$ . Then every continuum in  $M$  which contains  $p$  and  $q$  contains  $K$ .

Case 2.  $K$  is indecomposable.

Since  $M$  is atriodic at  $K$ , the set  $\text{cl}(M-K) \cap K$  has at most two components. If  $\text{cl}(M-K)$  intersects more than two components of  $K$ , then one component of  $\text{cl}(M-K)$  intersects two components of  $K$ , and  $K$  is contained in  $\text{cl}(M-K)$ . This contradicts the assumption that  $K$  has interior. Thus, all but two of the components of  $K$  are contained in the interior of  $K$ . Let  $p$  and  $q$  be two points which are contained in different components of the interior of  $K$ . Then every continuum in  $M$  which contains  $p$  and  $q$  also contains  $K$ .

Case 3.  $\text{cl}(M-K) \cap K$  is connected.

Every atriodic unicoherent continuum is irreducible [7,Thm.32,p.456]. If  $K$  is not irreducible, then  $K$  is not unicoherent, and by Theorem 2 of [5],  $K$  is a  $C$ -set in  $M$ . But, since  $C$ -sets do not have interior,  $K$  is irreducible



from some point  $p$  to some point  $q$ .

Suppose  $D$  is a continuum in  $M$  which contains  $p$  and  $q$ . If the intersection of  $D$  with  $K$  is not connected, then there are two continua  $G$  and  $H$  in  $(D \cap \text{cl}(M-K))$  which intersect  $K$  and  $M-K$ . But, then  $(K \cup G \cup H)$  contains a triod with core in  $K$ . Thus,  $(D \cap K)$  is connected, and since  $p$  and  $q$  are in  $(D \cap K)$ ,  $K$  is contained in  $D$ .

Lemma 8. Suppose  $K$  is a subcontinuum of the continuum  $M$ , and  $L$  and  $R$  are subcontinua of  $M$  which intersect  $K$  and  $M-K$  such that  $L-K$  does not intersect  $R-K$ , and such that  $(L \cup K \cup R)$  is contained in a open set  $A$  at which  $M$  is atriodic. Then there are continua  $P(L)$  in  $(L \cap K)$  and  $P(R)$  in  $(R \cap K)$  such that every continuum in  $(L \cup K \cup R)$  which intersects  $K$  and  $L-K$  contains  $P(L)$  and every continuum in  $(L \cup K \cup R)$  which intersects  $K$  and  $R-K$  contains  $P(R)$ . Moreover,  $P(L)$  and  $P(R)$  are  $W$ -sets in  $M$ .

Proof. Suppose  $H$  and  $J$  are continua in  $(L \cup K \cup R)$  which intersect  $K$  and  $L-K$ . If  $H-(K \cup R)$  and  $J-(K \cup R)$  are not nested, the set

$$(H \cup K) \cup (J \cup K) \cup R$$

contains a triod. So, assume  $H-(K \cup R)$  is contained in  $J-(K \cup R)$ . Note that  $(H \cap (K \cup R))$  and  $(J \cap (K \cup R))$  are

connected, and, if they are not nested, the set

$$(H - (K \cup R)) \cup (H \cap (K \cup R)) \cup (J \cap (K \cup R))$$

contains a triod. Let  $P(L)$  be the intersection of the collection of all continua of the form  $(H \cap (K \cup R))$  where  $H$  is a continuum in  $(L \cup K \cup R)$  which intersects  $K$  and  $L-K$ . Since this collection is nested,  $P(L)$  is a continuum, and every continuum in  $(L \cup K \cup R)$  which intersects  $K$  and  $L-K$  contains  $P(L)$ .

Suppose  $D$  is a subcontinuum of  $(L \cup K)$  which intersects  $L-K$  and  $K$ . Then  $(D \cap K)$  is connected and  $P(L)$  is contained in  $(D \cap K)$ . If  $P(L)$  separates a subcontinuum  $H$  of  $(D \cap K)$ , then, since every continuum from  $L-K$  to  $K$  contains  $P(L)$ , there is a continuum  $F$  in  $(L \cup K)$  which contains  $P(L)$ , intersects  $L-K$ , and such that  $(F \cap H)$  separates  $H$ . So,  $(F \cup H)$  contains a triod. Therefore, no subcontinuum of  $(D \cap K)$  is separated by  $P(L)$ , and by Theorem 1,  $P(L)$  is a  $W$ -set in  $(D \cap K)$ . If  $(D \cap K)$  separates a subcontinuum  $H$  of  $D$ , then  $H$  is contained in  $L$  and  $(H \cup K \cup R)$  contains a triod. Thus,  $(D \cap K)$  is a  $W$ -set in  $D$ . We have shown that  $P(L)$  is a  $W$ -set in every subcontinuum of  $(L \cup K)$  which intersects  $L-K$  and  $K$ .

Let  $f$  be a map of a continuum  $I$  onto  $M$ . If  $C$  is a component of  $f^{-1}(L \cup K)$  such that  $f(C)$  intersects  $L-K$  and  $K$ , then  $P(L)$  is a  $W$ -set in  $f(C)$ , and some subcontinuum of  $C$

maps onto  $P(L)$ . So assume that for every component  $C$  of  $f^{-1}(L \cup K)$  if  $f(C)$  intersects  $L-K$ , then  $f(C)$  is contained in  $L-K$ . This assumption will lead to a contradiction.

Suppose  $B$  and  $C$  are two components of  $f^{-1}(L \cup K)$  whose images intersect  $L-K$ , then  $f(B)$  and  $f(C)$  are nested. For if they were not nested, then there are two continua  $B'$  and  $C'$  in  $I$  such that  $B$  is a proper subcontinuum of  $B'$ ,  $C$  is a proper subcontinuum of  $C'$ , neither  $f(B')$  nor  $f(C')$  intersects  $(K \cup R)$ ,  $f(B')$  and  $f(C')$  are contained in  $A$ , and  $f(B')$  and  $f(C')$  are not nested. But, if  $f(B')-L$  and  $f(C')-L$  are not nested, then the set

$$(f(B') \cup L) \cup (f(B') \cap L) \cup (K \cup R)$$

contains a triod with core in  $A$ . So assume  $f(B')-L$  is contained in  $f(C')-L$ . Then the set

$$\text{cl}(f(B')-L) \cup (f(B') \cap L) \cup (f(C') \cap L)$$

contains a triod with core in  $A$ . Therefore,  $f(B)$  and  $f(C)$  must be nested.

Let  $x(1), x(2), x(3) \dots$  be a sequence of points in  $L-K$  which converges to a point  $x$  in  $K$ . For each  $i$  let  $C(i)$  be a component of  $f^{-1}(L \cup K)$  such that  $x(i)$  is contained in  $f(C(i))$ . Then some subsequence of  $C(1), C(2), C(3) \dots$  converges to a continuum  $C$  such that  $f(C)$  intersects  $K$  and  $L-K$  contrary to an earlier assumption. Thus,  $P(L)$  is a  $W$ -set in  $M$ . Similarly,  $P(R)$  is a  $W$ -set in  $M$ .

Theorem 8 gives a characterization of W-sets in atriodic continua. Though it is stated in terms of a map onto a continuum  $M$  the theorem really shows that if  $K$  is not a W-set in  $M$ , then  $M$  can be taken apart and put back together so that the resulting continuum can be projected onto  $M$  with a projection map that is not weakly confluent with respect to  $K$ . If  $M$  can be taken apart in this way, then  $M$  is not in class[W].

Theorem 8. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. If  $K$  is not a W-set in  $M$ , then there is an open set  $W$  such that  $K$  is contained in  $W$  and  $W$  is contained in  $A$ , and there is map  $f$  of a continuum  $I$  onto  $M$  which has the following properties:

- 1) The map  $f$  is not weakly confluent with respect to  $K$ .
- 2)  $f^{-1}(K)$  has only two components.
- 3) Only one point is contained in  $f^{-1}(p)$  for each point  $p$  in  $M-W$ .
- 4) At most two points are contained in  $f^{-1}(p)$  for each point  $p$  in  $M$ .
- 5) For each point  $q$  in  $I$  there is an open set containing  $q$  on which  $f$  is one-to-one.

Proof. Since  $K$  is not a  $W$ -set in  $M$ , there is a map  $g$  from a continuum  $N$  onto  $M$  such that no component of  $g^{-1}(K)$  maps onto  $K$ . By Lemma 7, there are continua  $L$  and  $R$  which intersect  $K$  and  $M-K$  such that  $L-K$  does not intersect  $R-K$  and  $K$  is not contained in  $L$  or  $R$ . Choose  $L$  and  $R$  such that  $(L \cup K \cup R)$  is contained in  $A$ . Let  $V(L)$  and  $V(R)$  be open sets in  $A$  such that  $V(L)$  intersects  $L$  but not  $(K \cup R)$  and  $V(R)$  intersects  $R$  but not  $(K \cup L)$ .

Claim 1. There is an open set  $Q$  such that  $K$  is contained in  $Q$  and  $Q$  is contained in  $A$ , and such that if  $D$  is a component of  $M-(V(L) \cup V(R))$  which intersects  $Q$ , then  $D$  is contained in  $A$ .

If the claim is false, then there is a sequence of components of  $M-(V(L) \cup V(R))$  which converges to a continuum  $E$  which intersects  $K$  and  $M-A$ . But, since  $E$  is contained in  $M-(V(L) \cup V(R))$ , the set

$$(L \cup K) \cup (R \cup K) \cup (E \cup K)$$

contains a triod.

Claim 2. There is an open set  $S$  which contains  $K$  and is contained in  $Q$  such that if  $D$  is a component of  $M-(V(L) \cup V(R))$  which intersects  $S$  and  $C$  is a component of  $g^{-1}(D)$ , then  $g(C)$  does not intersect both  $cl(V(L))$  and  $cl(V(R))$ .

Assume the claim is false. Then there is a sequence  $D(1), D(2), D(3), \dots$  of components of  $M - (V(L) \cup V(R))$  which converges to a continuum  $D$  which is contained in  $(L \cup K \cup R)$  such that for each  $i$  there is a component  $C(i)$  of  $g(D(i))$  for which  $g(C(i))$  intersects  $cl(V(L))$  and  $cl(V(R))$ . Some subsequence of  $C(1), C(2), C(3), \dots$  converges to a continuum  $C$  in  $N$ . But  $g(C)$  is contained in  $(L \cup K \cup R)$  and  $g(C)$  intersects  $cl(V(L))$  and  $cl(V(R))$ . So  $K$  separates  $g(C)$ , and by Theorem 2, some component of  $f^{-1}(K)$  maps onto  $K$ . This is a contradiction, so the claim is proven.

Suppose  $D$  is a component of  $M - (V(L) \cup V(R))$  and  $D$  intersects  $S$ ,  $cl(V(L))$ , and  $cl(V(R))$ . Then, by Claim 2,  $D$  is not a  $W$ -set in  $M$ . Therefore, by Lemma 7, there are continua  $H$  and  $J$  in  $M$  which intersect  $D$  and  $M - D$  such that  $H - D$  does not intersect  $J - D$ . Choose  $H$  and  $J$  so that  $(H \cup D \cup J)$  is contained in  $A$ . By Lemma 8, there is a continuum  $P(H)$  in  $(H \cap D)$  such that every continuum in  $(H \cup D \cup J)$  which intersects  $D$  and  $H - D$  contains  $P(H)$ , and there is the corresponding continuum  $P(J)$  in  $(D \cap J)$ . Since  $D$  is a component of  $M - (V(L) \cup V(R))$ , every continuum which intersects  $D$  and  $M - D$  also intersects either  $cl(V(L))$  or  $cl(V(R))$ . Therefore,  $P(H)$  must intersect either  $cl(V(L))$  or  $cl(V(R))$ , and  $P(J)$  must intersect either  $cl(V(L))$  or

$\text{cl}(V(R))$ . Since  $P(H)$  and  $P(J)$  are  $W$ -sets in  $M$ , neither  $P(H)$  nor  $P(J)$  intersects both  $\text{cl}(V(L))$  and  $\text{cl}(V(R))$  (recall claim 2). So assume that the naming of  $H$  and  $J$  is such that  $P(H)$  intersects  $\text{cl}(V(L))$  and does not intersect  $\text{cl}(V(R))$ , and  $P(J)$  intersects  $\text{cl}(V(R))$  and does not intersect  $\text{cl}(V(L))$ .

Next, it will be shown that there are components of  $g^{-1}(D)$  which will be called  $CL[D]$  and  $CR[D]$  such that if  $C$  is a component of  $g^{-1}(D)$  and  $P(L)$  is contained in  $g(C)$ , then  $g(C)$  is contained in  $g(CL[D])$ , and if  $C$  is a component of  $g^{-1}(D)$  and  $P(R)$  is contained in  $g(C)$ , then  $g(C)$  is contained in  $g(CR[D])$ . Suppose  $B$  and  $C$  are components of  $g^{-1}(D)$  such that both  $g(B)$  and  $g(C)$  contain  $P(H)$ . Then  $g(B)$  and  $g(C)$  are nested. For, if they were not nested, there is a continuum  $F$  in  $(H \cup D)$  such that  $F$  contains  $P(H)$ ,  $F$  intersects  $H-D$ , and  $g(B)-F$  and  $g(C)-F$  are not nested. But then the set  $(F \cup g(B) \cup g(C))$  contains a triod. Therefore, the components of  $g^{-1}(D)$  which contain  $P(H)$  are totally ordered by the containment relation on their images. Let  $CL[D]$  be a largest component according to this ordering. The component  $CR[D]$  is obtained in a similar manner. Note that  $D=(g(CL[D]) \cup g(CR[D]))$ ,  $g(CL[D])$  intersects  $\text{cl}(V(L))$ , and  $g(CR[D])$  intersects  $\text{cl}(V(R))$ .

Claim 3. Let  $D(K)$  be the component of  $M-(V(L) \cup V(R))$

which contains  $K$ . Suppose  $D(1), D(2), D(3) \dots$  is a sequence of components of  $M - (V(L) \cup V(R))$  which are not components of  $M - V(L)$  or  $M - V(R)$ , and which are contained in  $A$ . Suppose the sequence converges to a continuum in  $D(K)$ . If  $F$  is a limit of a subsequence of  $g(CL[D(1)]), g(CL[D(2)]), g(CL[D(3)]) \dots$ , then  $F$  is contained in  $g(CL[D(K)])$ , or if  $F$  is a limit of a subsequence of  $g(CR[D(1)]), g(CR[D(2)]), g(CR[D(3)]) \dots$ , then  $F$  is contained in  $g(CR[D(K)])$ .

Suppose  $F$  is a limit of a subsequence of  $g(CL[D(1)]), g(CL[D(2)]), g(CL[D(3)]) \dots$  and  $F$  is not contained in  $g(CL[D(K)])$ . There is a continuum  $E$  in  $N$  such that  $g(E) = F$ . Since  $g(E)$  does intersect  $L$ , if  $g(E)$  also intersects  $R$ , some continuum in  $E$  maps onto  $K$ . This is a contradiction. But then  $g(E)$  is contained in  $g(CL[D(K)])$  which is a contradiction. So the claim is proven.

Claim 4. Suppose  $D(1), D(2), D(3) \dots$  is a sequence of components of  $M - (V(L) \cup V(R))$  which are also components of  $M - V(L)$  (alternately  $M - V(R)$ ) such that the sequence converges to a continuum  $D$  in  $(L \cup K \cup R)$ . Then  $D$  is contained in  $g(CL[D(K)])$  (alternately  $D$  is contained in  $g(CR[D(K)])$ ).

Suppose  $D$  is not contained in  $g(CL[D(K)])$ . Let  $x(1), x(2), x(3) \dots$  be a sequence of points in  $M$  which



converges to a point  $x$  in  $D-g(CL[D(K)])$  such that  $x(i)$  is in  $D(i)$  for each  $i$ . For each  $i$  there is a component  $E(i)$  of  $g(D(i))$  such that  $g(E(i))$  intersects  $cl(V(L))$  and  $x(i)$  is in  $g(E(i))$ . Some subsequence of  $E(1), E(2), E(3), \dots$  converges to a continuum  $E$  such that  $g(E)$  is contained in  $(L \cup K \cup R)$ ,  $g(E)$  intersects  $cl(V(L))$ , and  $g(E)$  is not contained in  $g(CL[D(K)])$ . This is a contradiction as in Claim 3. So the claim is proven.

For each component  $D$  of  $M-(V(L) \cup V(R))$  which intersects  $S$  define the sets  $EL[D]$  and  $ER[D]$  as follows. If  $D$  is not a component of  $M-V(L)$  or  $M-V(R)$ , let  $EL[D]=g(CL[D])$  and  $ER[D]=g(CR[D])$ . If  $D$  is a component of  $M-V(L)$ , let  $EL[D]=D$  and let  $ER[D]$  be empty, and if  $D$  is a component of  $M-V(R)$ , let  $ER[D]=D$  and let  $EL[D]$  be empty.

By Claim 3 and Claim 4, there are open sets  $B(L)$  and  $B(R)$  such that  $B(L)$  intersects  $K-ER[D(K)]$  and  $B(L)$  does not intersect  $ER[D(K)]$ ,  $B(R)$  intersects  $K-EL[D(K)]$  and  $B(R)$  does not intersect  $EL[D(K)]$ , and there is an open set  $T$  contained in  $S$  which contains  $K$  such that if  $D$  is a component of  $M-(V(L) \cup V(R))$  which intersects  $cl(T)$ , then  $EL[D]$  does not intersect  $(V(R) \cup B(R))$  and  $ER[D]$  does not intersect  $(V(L) \cup B(L))$ .

Define the set  $FL$  to be the union of  $EL[D]$ 's for all components  $D$  of  $M-(V(L) \cup V(R))$  which intersect  $cl(T)$ , and

define  $FR$  to be the union of  $ER[D]$ 's for all components  $D$  of  $M-(V(L) \cup V(R))$  which intersect  $cl(T)$ . Let  $G=(FL \cup FR)$ . Then  $K$  is contained in the interior of  $G$ , and neither  $FL$  or  $FR$  contains  $K$ . The set  $I$  is obtained by identifying in the set

$$[cl(M-G) \times \{0\}] \cup [FL \times \{1\}] \cup [FR \times \{2\}],$$

for each point  $p$  in  $(FL \cap cl(M-G))$ , the point  $(p,0)$  with  $(p,1)$ , and for each point  $q$  in  $(FR \cap cl(M-G))$ , the point  $(q,0)$  with  $(q,2)$ . It will be shown that  $I$  is a continuum, and that if  $f$  is the projection of  $I$  onto  $M$  and  $W=int(G)$ , then  $f$  has all of the properties in the statement of the theorem.

The function  $g'$  from  $N$  onto  $I$  is defined as follows:

1) For each  $x$  in  $g^{-1}(M-W)$  let  $g'(x)=(g(x),0)$

2) For each  $x$  in  $g^{-1}(W)$  let  $D$  be the component of  $M-(V(L) \cup V(R))$  which contains  $g(x)$ . Let  $C$  be the component of  $g^{-1}(D)$  which contains  $x$ . If  $g(C)$  intersects  $cl(V(L))$ , then  $g(C)$  is contained in  $EL[D]$ . Now,  $ER[D]$  does not intersect  $cl(V(L))$  so  $g(C)$  is not contained in  $ER[D]$ . In this case, let  $g'(x)=(g(x),1)$ . Similarly, if  $g(C)$  intersects  $cl(V(R))$ , let  $g'(x)=(g(x),2)$ .

Claim 5.  $g'$  is continuous.

Let  $x$  be a point in  $N$  and let  $x(1),x(2),x(3)\dots$  be a sequence of points in  $N$  which converges to  $x$ . If  $x$  is in

$g^{-1}(M-W)$ , clearly  $g'(x(1)), g'(x(2)), g'(x(3))..$  converges to  $g'(x)=(g(x),0)$ . So assume  $x$  is in  $g^{-1}(W)$ . Since  $g^{-1}(W)$  is open, assume  $x(i)$  is in  $g^{-1}(W)$  for each  $i$ .

If  $g'(x(1)), g'(x(2)), g'(x(3))...$  does not converge to  $g'(x)$ , then either  $g'(x)=(g(x),1)$  and  $g'(x(i))=(g(x(i)),2)$  for infinitely many  $i$ 's or  $g'(x)=(g(x),2)$  and  $g'(x(i))=(g(x(i)),1)$  for infinitely many  $i$ 's. Assume the former, only assume it is true for all  $i$  to avoid reindexing. Let  $D$  be the component of  $G$  which contains  $g(x)$ , and let  $C$  be the component of  $g^{-1}(D)$  which contains  $x$ . Then  $g(C)$  is contained in  $EL[D]$ . For each  $i$  let  $D(i)$  be the component of  $G$  which contains  $f(x(i))$  and let  $C(i)$  be the component of  $g^{-1}(D(i))$  which contains  $x(i)$ . Some subsequence of  $C(1), C(2), C(3)...$  converges to a continuum  $E$  in  $N$ . Now,  $g(E)$  is contained in  $M-(V(L) \cup V(R))$  and  $g(x)$  is in  $g(E)$  so  $g(E)$  is contained in  $D$ . But  $x$  is in  $E$ , and  $E$  is contained in  $C$ . Since  $g(C(i))$  intersects  $cl(V(R))$  for each  $i$ ,  $g(E)$  intersects  $cl(V(R))$ . Therefore,  $g(C)$  intersects  $cl(V(R))$ . But  $g(C)$  is contained in  $EL[D]$ , and  $EL[D]$  does not intersect  $cl(V(L))$ . So  $g'$  is continuous.

That  $I$  is a continuum follows from the fact that  $g'$  is continuous. Let  $f$  be the projection of  $I$  onto  $M$ . The two components of  $f^{-1}(K)$  are

- 1) the set of all points  $(p,1)$  in  $I$  such that  $p$  is in

the intersection of  $EL[D(K)]$  with  $K$ ,

- 2) The set of all points  $(q,2)$  in  $I$  such that  $q$  is in the intersection of  $ER[D(K)]$  with  $K$ .

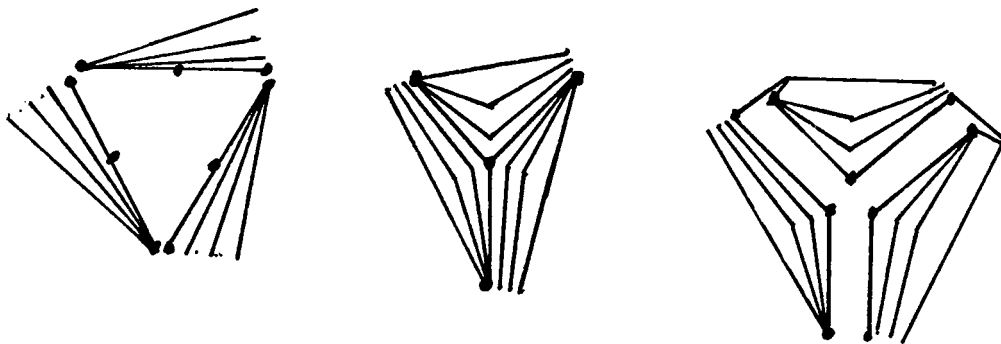
The set  $K$  is not contained in the projection of either of these sets onto  $M$  since  $K$  is not contained in  $EL[D(K)]$  or  $ER[D(K)]$ . Since all of the identification in obtaining  $I$  was for points with first coordinate in  $cl(M-G)$ , and since  $K$  is contained in the interior of  $G$ , the two sets above do not intersect. If  $p$  is a point in  $M-W$ , then  $f^{-1}(p)$  is the point  $(p,0)$ . If  $p$  is a point in  $W$ , then  $f^{-1}(p)$  contains at most two points, one of the form  $(p,1)$  and another of the form  $(p,2)$ .

Suppose  $(p,i)$  is in  $I$ . If  $i=0$ , then  $p$  is in  $M-G$ , and  $f$  is one-to-one on the open set  $f^{-1}(M-G)$ . If  $i=1$ , then the set of all points  $(q,1)$  in  $I$  where  $q$  is in  $W$  is an open set containing  $(p,i)$  on which  $f$  is one-to-one, and if  $i=2$ , then the set all of points  $(q,2)$  in  $I$  where  $q$  is in  $W$  is an open set containing  $(p,i)$  on which  $f$  is one-to-one. This concludes the proof.

In the following corollary a map  $f$  is called two-to-one if for each point  $p$  in the range of  $f$ , at most two points are contained in  $f^{-1}(p)$ .

Corollary. An atriodic continuum  $M$  is in  $\text{class}[W]$  if and only if every two-to-one map from a continuum onto  $M$  is weakly confluent.

Example 6. The continuum  $M$  is obtained by identifying three "fan" continua along their limit lines as indicated. Even after the identifications the three sets that came from the limit lines of the fans are  $W$ -sets since each is a limit of a sequence of arcs which are  $W$ -sets in  $M$ . The continuum  $L$  is obtained by splitting  $M$  apart two times. The obvious map from  $L$  to  $M$  is not weakly confluent with respect to the simple triod at the center of  $M$ . However, for any map of a continuum onto  $M$  the preimage of that simple triod has at least three components or the map is weakly confluent with respect to that simple triod.



## CHAPTER IV

### A CHARACTERIZATION OF THE SIMPLE CLOSED CURVE

Class[W], viewed as the class of all continua for which every subcontinuum is a W-set, is one extreme of a spectrum. The other extreme of that spectrum is inhabited by the class of all continua for which every nondegenerate proper subcontinuum is not a W-set. This latter class contains many continua which are not atriodic, but it will be shown that an atriodic continuum for which every nondegenerate proper subcontinuum is not a W-set is a simple closed curve.

Lemma 9. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. If  $f$  is a two-to-one map of a continuum  $I$  onto  $M$ , then  $f^{-1}(K)$  has at most four components, and there are components  $B$  and  $C$  of  $f^{-1}(K)$  such that  $(f(B) \cup f(C))=K$ .

Proof. If  $K$  is a W-set in  $M$ , then there is a component  $B$  of  $f^{-1}(K)$  such that  $K=f(B)$ . If  $C$ ,  $D$ , and  $E$  are three components of  $f^{-1}(K)$ , each different from  $B$ , then no two of the images of  $C$ ,  $D$ , or  $E$  intersect since  $f$  is two-to-one.

So let  $C'$ ,  $D'$ , and  $E'$  be continua in  $I$  whose images are in  $A$ , which properly contain  $C$ ,  $D$ , and  $E$  respectively, and such that no two of  $f(C')$ ,  $f(D')$ , and  $f(E')$  intersect. Then the set

$$(f(C') \cup f(D') \cup f(E') \cup K)$$

contains a triod in  $A$ . So, in fact, if  $K$  is a  $W$ -set in  $M$ , then  $f^{-1}(K)$  has at most three components.

If  $K$  is not a  $W$ -set in  $M$ , then let  $L$  and  $R$  be the continua from Lemma 7, and let  $P(L)$  be the continuum in  $(L \cap K)$  and  $P(R)$  be the continuum in  $(R \cap K)$  from Lemma 8. Since  $f$  is two-to-one at most two components of  $f^{-1}(K)$  can have images which contain  $P(L)$  and at most two can have images which contain  $P(R)$ . But, for each component  $C$  of  $f^{-1}(K)$  either  $P(L)$  or  $P(R)$  is contained in  $f(C)$ . Therefore,  $f^{-1}(K)$  has at most four components.

There are components  $B$  and  $C$  of  $f^{-1}(K)$  such that  $P(L)$  is contained in  $f(B)$ ,  $P(R)$  is contained in  $f(C)$ , and  $f(B)$  intersects  $f(C)$ .  $K$  is irreducible from  $P(L)$  to  $P(R)$ ; therefore,  $K=(f(B) \cup f(C))$ .

Lemma 10. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. If  $J$  is a nondegenerate subcontinuum of  $K$  such that for each two-to-one map  $f$  of a continuum onto  $M$  there is a component

$C$  of  $f^{-1}(K)$  such that  $J$  is contained in  $f(C)$ , then there is a subcontinuum  $H$  of  $K$  which contains  $J$  such that  $H$  is a  $W$ -set in  $M$ .

Proof. If  $K$  is a  $W$ -set in  $M$ , let  $H=K$ . Otherwise, from Lemma 7, we have the continua  $L$  and  $R$ , and from Lemma 8 we have the continua  $P(L)$  in  $(L \cap K)$  and  $P(R)$  in  $(R \cap K)$  which have the properties given in those lemmas. According to Theorem 8, in order to show that a subcontinuum of  $K$  is a  $W$ -set in  $M$ , it suffices to show that every two-to-one map of a continuum onto  $M$  is weakly confluent with respect to that continuum. Also, if  $K$  is not a  $W$ -set in  $M$ , then there is a two-to-one map which is not weakly confluent with respect to  $K$ . Let  $f$  be a two-to-one map of the continuum  $I$  onto  $M$  which is not weakly confluent with respect to  $K$ .

Case 1.  $J$  intersects  $P(L)$ .

Let  $C$  be a component of  $f^{-1}(K)$  such that  $J$  is contained in  $f(C)$ . Either  $P(L)$  is contained in  $f(C)$  or  $P(R)$  is contained in  $f(C)$ . But, since  $K$  is irreducible from  $P(L)$  to  $P(R)$ , if  $P(R)$  is in  $f(C)$ , then  $K=f(C)$ . This contradicts the assumption that  $f$  is not weakly confluent with respect to  $K$ . So  $P(L)$  is contained in  $f(C)$ . If  $P(L)$  separates a subcontinuum  $D$  of  $f(C)$ , then there is a continuum  $G$  in  $(L \cup K)$  which intersects  $L-K$  and  $K$  such that



$(D \cup (P(L) \cup J) \cup G)$  contains a triod. Therefore,  $(P(L) \cup J)$  is a  $W$ -set in  $f(C)$ . Let  $H=(P(L) \cup J)$ .

Case 2.  $J$  does not intersect  $P(L)$  or  $P(R)$ .

Let  $I(L)$  be a subcontinuum of  $K$  which is irreducible with respect to containing both  $P(L)$  and  $J$ , and let  $I(R)$  be a subcontinuum of  $K$  which is irreducible with respect to containing both  $P(R)$  and  $J$ . Since  $K$  is irreducible from  $P(L)$  to  $P(R)$ ,  $K=(I(L) \cup I(R))$ .  $K$  is not a  $W$ -set in  $M$ , so  $K$  is unicoherent [5,Thm 2]. Thus,  $(I(L) \cap I(R))$  is connected.

Subcase 1.  $I(L)$  is contained in  $I(R)$ .

By the hypothesis of this lemma, there is a component  $C$  of  $f(K)$  such that  $J$  is contained in  $f(C)$ . Either  $P(L)$  is contained in  $f(C)$  or  $P(R)$  is contained in  $f(C)$ , and, in either case,  $I(L)$  is contained in  $f(C)$ . Therefore, since  $I(L)$  intersects  $P(L)$ , by Case 1,  $I(L)$  is a  $W$ -set in  $M$ . Let  $H=I(L)$ .

Subcase 2.  $I(L)$  and  $I(R)$  are not nested.

Let  $C$  be a component of  $f(K)$  such that  $J$  is contained in  $f(C)$ , Then either  $P(L)$  or  $P(R)$  is contained in  $f(C)$ . So either  $I(L)$  or  $I(R)$  is contained in  $f(C)$ , and, in either case,  $(I(L) \cap I(R))$  is contained in  $f(C)$ . If  $(I(L) \cap I(R))$  separates  $f(C)$ , then  $(I(L) \cap I(R))$  is a  $W$ -set in  $f(C)$ . If  $(I(L) \cap I(R))$  does not separate  $f(C)$ , then  $f(C)$  is not  $K$  since  $(I(L) \cap I(R))$  does separate  $K$ . Thus, if  $I(L)$  is

contained in  $f(C)$ , then  $I(R)$  is not contained in  $f(C)$ . And if  $(I(L) \cap I(R))$  separates a subcontinuum  $D$  of  $f(C)$ , then the set  $(I(R) \cup (I(L) \cap I(R)) \cup D)$  contains a triod. In any case,  $(I(L) \cap I(R))$  is a  $W$ -set in  $f(C)$ . Let  $H=(I(L) \cap I(R))$ .

Lemma 11. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. Suppose also that  $J$  is a subcontinuum of  $K$  such that if  $D$  and  $E$  are two proper subcontinua of  $K$  whose union is  $K$ , then either  $J$  is contained in  $D$  or  $J$  is contained in  $E$ . Then there is a subcontinuum  $H$  of  $K$  which contains  $J$  such that  $H$  is a  $W$ -set in  $M$ .

Proof. Let  $f$  be a two-to-one map of a continuum onto  $M$ . Then, by Lemma 9, there are two components  $B$  and  $C$  of  $f^{-1}(K)$  such that  $(f(B) \cup f(C))=K$ . Thus, either  $J$  is contained in  $f(B)$  or  $J$  is contained in  $f(C)$ . The lemma follows from Lemma 10.

Lemma 11 will be used later to prove Theorem 10. The following strengthening of Lemma 11 is not used in Theorem 10, but it is included for completeness.

Theorem 9. Suppose the subcontinuum  $K$  of the continuum

$M$  is contained in an open set  $A$  at which  $M$  is atriodic. Suppose also that  $J$  is a subcontinuum of  $K$  such that if  $D$  and  $E$  are two proper subcontinua of  $K$  whose union is  $K$ , then either  $J$  is contained in  $D$  or  $J$  is contained in  $E$ . If, in addition, there are two proper subcontinua  $D$  and  $E$  of  $K$  whose union is  $K$  and whose intersection is  $J$ , then  $J$  is a  $W$ -set in  $M$ .

*Proof.* By Lemma 11, there is a  $W$ -set  $H$  in  $M$  such that  $J$  is contained in  $H$  and  $H$  is contained in  $K$ . If  $H$  intersects  $D-J$  and  $H$  intersects  $E-J$ , then  $J$  separates  $H$  so  $J$  is a  $W$ -set in  $M$  by Theorem 2. Suppose  $H$  does not intersect  $D-J$ . Then, if there is a subcontinuum  $F$  of  $H$  which is separated by  $J$ , the set  $(D \cup J \cup F)$  contains a triod. Therefore,  $J$  is a  $W$ -set in  $H$  by Theorem 1. Thus,  $J$  must be a  $W$ -set in  $M$ .

Lemma 12. If  $K$  is a unicoherent atriodic continuum such that for each nondegenerate subcontinuum  $J$  of  $K$  there is a pair of proper subcontinua  $D$  and  $E$  of  $K$  whose union is  $K$ , and such that  $J$  is not contained in either  $D$  or  $E$ , then  $K$  is an arc.

*Proof.* Let  $a$  and  $b$  be two points of  $K$  such that  $K$  is

irreducible from  $a$  to  $b$  [7,Thm.32,p.456]. Let  $x$  be a point of  $K$  which is not  $a$  or  $b$ . Let  $I(a)$  be a subcontinuum of  $K$  which is irreducible from  $a$  to  $x$ , and let  $I(b)$  be a subcontinuum of  $K$  which is irreducible from  $b$  to  $x$ . Since  $K$  is unicoherent and decomposable,  $(I(a) \cap I(b))$  is a proper subcontinuum of  $K$ . If  $(I(a) \cap I(b))$  is nondegenerate, then there are two proper subcontinua  $D$  and  $E$  of  $K$  whose union is  $K$ , and such that neither  $D$  nor  $E$  contains  $(I(a) \cap I(b))$ . Now,  $x$  is in either  $D$  or  $E$ . If  $x$  is in  $D$ , then  $D$  also contains  $I(a)$  or  $I(b)$ , since  $D$  contains either  $a$  or  $b$  ( $E$  cannot contain both  $a$  and  $b$ ). This is a contradiction, so  $(I(a) \cap I(b))$  is  $x$ , and  $x$  separates  $K$ . Since  $K$  is separated by all but two of its points,  $K$  is an arc.

Theorem 10. Suppose the subcontinuum  $K$  of the continuum  $M$  is contained in an open set  $A$  at which  $M$  is atriodic. If  $K$  does not contain a nondegenerate  $W$ -set in  $M$ , then  $K$  is an arc.

Proof. Since  $K$  does not contain a nondegenerate  $W$ -set,  $K$  does not contain a continuum with the properties of the continuum  $J$  in Lemma 11. Also,  $K$  is unicoherent since it is not a  $C$ -set [5,Thm.2]. Therefore,  $K$  is an arc by Lemma 12.

Corollary. An atriodic continuum  $M$  is a simple closed curve if and only if no proper subcontinuum of  $M$  is a  $W$ -set in  $M$ .

Proof. Suppose no nondegenerate proper subcontinuum of  $M$  is a  $W$ -set in  $M$ . Let  $K$  be a nondegenerate proper subcontinuum of  $M$ .  $K$  is an arc and  $K$  does not separate  $M$  by Theorem 2. Thus,  $\text{cl}(M-K)$  is also an arc. Since  $M$  is atriodic,  $M$  is either an arc or a simple closed curve. But arcs are in  $\text{Class}[W]$ , so  $M$  is a simple closed curve.

It is obvious that if  $M$  is a simple closed curve,  $M$  does not contain a nondegenerate proper subcontinuum which is a  $W$ -set in  $M$ .

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