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Linear oscillations of constrained drops, bubbles, and plane liquid surfaces

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The small-amplitude oscillations of constrained drops, bubbles, and plane liquid surfaces are studied theoretically. The constraints have the form of closed lines of zero thickness which prevent the motion of the liquid in the direction normal to the undisturbed free surface. It is shown that, by accounting explicitly for the singular nature of the curvature of the interface and the force exerted by the constraint, methods of analysis very close to the standard ones applicable to the unconstrained case can be followed. Weak viscous effects are accounted for by means of the dissipation function. Graphical and numerical results for the oscillations of constrained drops and bubbles are presented. Examples of two- and three-dimensional gravity-capillary waves are treated by the same method. A brief consideration of the Rayleigh-Taylor unstable configuration shows that the nature of the instability is not affected, although its growth rate is decreased. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3697796]

I. INTRODUCTION

In recent work,[1,2] Bostwick and Steen considered the shape oscillations of a drop on the surface of which an undeformable ring constrains the radial motion of the liquid (see Figure 1). Problems involving the oscillations of drops constrained in other ways have also formed the object of recent investigations. Lyubimov and co-workers[3,4] and Fayzrakhmanova and Straube[5] studied the oscillations of hemispherical drops on a vibrating plate. Here the constraint consists in the conditions imposed on the contact line, at which a partial slip was allowed. Experimental results on a system of this type have been reported by Vukasinovic *et al.*[6]

These authors made use of a spherical harmonic expansion, but their analysis had to deal with the difficulty of a possible singularity in the curvature of the free surface at the location of the constraint. This difficulty, which prevents the direct use of the well-known standard approaches (see, e.g., Refs. 7 and 8), has been dealt with in different and not totally straightforward ways. The purpose of the present paper is to show that, by accounting explicitly for the singularity, methods of analysis very similar to the standard ones are applicable with a considerable simplification of the solution procedure. Although we do not specifically consider the case of a moving contact line, the same approach may be expected to work in this case as well.

In addition to a constrained drop, in order to illustrate the efficacy of the method we consider in the same way a constrained bubble, two problems of gravity-capillary waves on a plane liquid surface and a simple instance of the Rayleigh-Taylor instability. Numerical results for the first four normal modes of constrained drops and bubbles are shown in graphical and tabular form.

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II. MATHEMATICAL FORMULATION

We consider the linear shape oscillations of a spherical drop the surface of which is prevented from moving radially along a circle seen under an angle θ_0 from the unperturbed drop center (Figure 1). The extension to more than one such constraint is immediate and is addressed later in Sec. VI.

We adopt spherical coordinates (r, θ, ϕ) centered at the center of the undisturbed drop and express the drop shape as a sum of normalized spherical harmonics $Y_n^m(\theta, \phi)$:

$$S_d(\mathbf{x}, t) \equiv r - a\left[1 + \epsilon \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_{nm}(t) Y_n^m(\theta, \phi)\right] = 0\,. \tag{1}$$

Here ϵ is a small parameter and a the equilibrium radius of the drop; the term $n = 0$ is omitted from the summation to enforce volume conservation.

Following the mathematical model of Ref. 7, which has also been adopted in the recent papers mentioned in the Introduction, we neglect viscosity and assume the flow induced by the surface oscillations to be inviscid, irrotational, and incompressible. There is therefore a velocity potential φ which satisfies Laplace's equation, $\nabla^2\varphi = 0$. The general solution of this equation regular at the origin $r = 0$ has the well-known form (see e.g., Ref. 9)

$$\varphi = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_{nm}(t) r^n Y_n^m(\theta, \phi)\,. \tag{2}$$

In the following an explicit expression for the spherical harmonics is required. We write

$$Y_n^m(\theta, \phi) = N_{nm} P_n^m(\mu)\, e^{im\phi}\,, \tag{3}$$

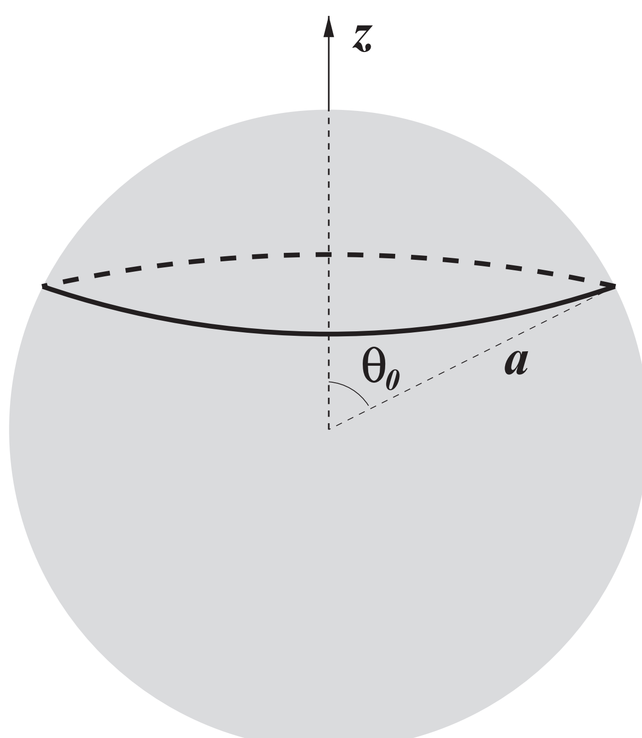

FIG. 1. A drop, or bubble, of radius a the radial motion of which is prevented by the rigid circular constraint located at $\theta = \theta_0$.

in which $\mu = \cos\theta$, P_n^m is an associated Legendre function and the normalization constant

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \tag{4}$$

ensures that the Y_n^m's have norm 1 (see, e.g., Ref. 9):

$$\left(Y_\ell^k, Y_n^m\right) \equiv \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \left[Y_\ell^m(\theta,\phi)\right]^* Y_n^m(\theta,\phi) = \delta_{\ell n}\delta_{km}\,; \tag{5}$$

here the asterisk denotes the complex conjugate.

The requirement that the liquid velocity $\partial\varphi/\partial r$ normal to the undisturbed free surface vanish on the line $\theta = \theta_0$ is expressed by

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n} n b_{nm}(t) a^{n-1} Y_n^m(\mu_0,\phi) = 0 \tag{6}$$

with $\mu_0 = \cos\theta_0$. Note that, while this condition restrains the flow normal to the undisturbed interface, it does not restrain it in the tangential direction. It is not possible to achieve this objective within the framework of irrotational flow.

From the linearized kinematic boundary condition $\partial S_d/\partial t = -\partial\varphi/\partial r|_{r=a}$ we obtain

$$b_{nm}(t) = \frac{1}{n}\frac{\dot{c}_{nm}}{a^{n-2}}\,, \tag{7}$$

in which the dot denotes the time derivative. Upon recognizing that the interface is subjected to a singular pressure along the line $\theta = \theta_0$, we write the dynamic boundary condition on the perturbation pressure p' as

$$p' = \sigma\mathcal{C} - \Delta_p(\phi,t)\,\delta(\mu-\mu_0)\,, \tag{8}$$

where σ is the surface tension coefficient, $\mathcal{C}$ is the local surface curvature, and Δ_p represents the action of the constraint. Upon making use of the linearized Bernoulli integral in the usual way (see, e.g., Refs. 1, 7, 8, 10, and 11), we re-write this boundary condition in the form

$$\rho\frac{\partial\varphi}{\partial t} = -\sigma\mathcal{C} + \Delta_p\delta(\mu-\mu_0)\,, \tag{9}$$

with ρ the liquid density.

The standard way to calculate the curvature is to take the divergence of the unit normal to the interface, $\mathbf{n} = \nabla S_d/|\nabla S_d|$. For a free drop this calculation produces the well-known result (see, e.g., Refs. 1, 7, 8, and 10)

$$\mathcal{C} = \nabla\cdot\mathbf{n} = \frac{1}{a}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)c_{nm}Y_n^m\,. \tag{10}$$

This relation is, however, incomplete in the present case as, while the drop surface is continuous, its θ-derivative is not, so that the second differentiation necessary to calculate the curvature produces a δ-singularity. We thus modify (10) to

$$a\mathcal{C} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)c_{nm}Y_n^m - \Delta_\sigma(\phi,t)\delta(\mu-\mu_0)\,, \tag{11}$$

where $\Delta_\sigma(\phi, t)$ is the magnitude of the discontinuity of the first derivative of S_d.

Upon substitution into the dynamic condition (9) we then find

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\dot{b}_{nm}(t)a^n Y_n^m = -\frac{\sigma}{\rho a}\left(\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)c_{nm}Y_n^m - \Delta(\phi,t)\delta(\mu-\mu_0)\right) \tag{12}$$

in which we have set

$$\Delta(\phi,t) = \Delta_\sigma(\phi,t) + \frac{a}{\sigma}\Delta_p(\phi,t)\,. \tag{13}$$

Upon using Eq. (7), Eq. (12) becomes

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{\ddot{c}_{nm}}{n}Y_n^m = -\frac{\sigma}{\rho a^3}\left(\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)c_{nm}Y_n^m - \Delta(\phi,t)\delta(\mu-\mu_0)\right). \tag{14}$$

We now take the scalar product of the two sides by the generic spherical harmonic Y_ℓ^k and use the orthonormality relation (5) satisfied by these functions to find

$$\frac{\ddot{c}_{\ell k}}{\ell} = -\frac{\sigma}{\rho a^3}\left[(\ell-1)(\ell+2)c_{\ell k} - \Delta_{\ell k}(t)\right], \tag{15}$$

or

$$\ddot{c}_{\ell k} + \Omega_\ell^2 c_{\ell k} = \frac{\sigma}{\rho a^3}\ell\Delta_{\ell k}(t), \tag{16}$$

where

$$\Omega_\ell^2 = \ell(\ell-1)(\ell+2)\frac{\sigma}{\rho a^3} \tag{17}$$

is the well-known oscillation frequency of a free drop (see, e.g., Ref. 7) and we have set

$$\Delta_{\ell k}(t) = N_{\ell k}P_\ell^k(\mu_0)\int_0^{2\pi}\Delta(\eta,t)e^{-ik\eta}\,d\eta. \tag{18}$$

For $\ell = 1$ (17) gives $\Omega_1 = 0$. This well-known result reflects the fact that, to the first order in ϵ inherent in the present linearized theory, $\ell = 1$ corresponds to a translation of the drop which, being free and therefore unimpeded, cannot have a restoring force nor, therefore, a finite frequency.

III. NORMAL MODES

To determine the normal modes of the constrained drop we look for solutions depending on time proportionally to $e^{i\omega t}$, with ω the angular frequency of the normal mode in question. With this ansatz (16) gives

$$\left(\Omega_\ell^2 - \omega^2\right)c_{\ell k} = \frac{\sigma}{\rho a^3}\ell\Delta_{\ell k}(t). \tag{19}$$

If $\Delta = 0$ this relation implies that the shape of the drop surface while oscillating according to the jth normal mode is

$$S_d^{(j)} = r - a\left(1 + \epsilon\sum_{m=-j}^{j}c_{jm}Y_j^m\right) = 0, \tag{20}$$

with arbitrary values of the amplitudes c_{jm} and eigenfrequency $\omega = \Omega_j$. Each mode has therefore a degeneracy of order $2j + 1$ as is well known.

If, for some value of $\ell = \ell_0$, $P_{\ell_0}^k(\mu_0) = 0$, according to (18) $\Delta_{\ell k} = 0$ as well, the constraint is immaterial, and one of the modes is given again by Eq. (20) with $j = \ell_0$ and is therefore also degenerate. Since the zeros of the associated Legendre functions are dense on the interval $-1 < \mu < 1$ (see, e.g., Eq. (26) below), it is likely that, whatever the value of μ_0, at least some of the modes will have this nature although modes with large values of ℓ_0 will be strongly damped by viscous effects (see, e.g., Ref. 8 and Sec. VII).

Other than in these two cases, we deduce from Eq. (19) that

$$c_{\ell k} = \frac{\sigma}{\rho a^3}\frac{\ell\Delta_{\ell k}}{\Omega_\ell^2 - \omega^2}. \tag{21}$$

Upon use of Eq. (7) and substitution into the constraint Eq. (6) we have

$$\frac{i\omega\sigma}{\rho a^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{n\Delta_{nm}}{\Omega_n^2-\omega^2}Y_n^m(\mu_0,\phi) = 0 \tag{22}$$

or, more explicitly and omitting inconsequential multiplicative constants,

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{nN_{nm}^2[P_n^m(\mu_0)]^2}{\Omega_n^2-\omega^2}e^{im\phi}\int_0^{2\pi}\Delta(\eta,t)e^{-im\eta}\,d\eta\;=\;0\,. \tag{23}$$

This equation can only be satisfied for all ϕ if the summation over m reduces to the single term $m = 0$. For this to happen it is necessary that Δ be actually independent of the angular variable. From now on we drop the second index and write, e.g., c_ℓ in place of $c_{\ell,\,0}$. The final form of the characteristic equation, again omitting inconsequential multiplicative constants, is then

$$\sum_{n=1}^{\infty}\frac{n(2n+1)[P_n(\mu_0)]^2}{\Omega_n^2-\omega^2}\;=\;0\,. \tag{24}$$

The degeneracy encountered in the free-drop case has therefore been removed as already noted for a related case in Ref. 3. Due to the appearance of P_n^2 the spectrum is unchanged if μ_0 is replaced by $-\mu_0$, i.e., θ_0 by $\pi - \theta_0$, as expected.

To examine the convergence of the series we note that, for a fixed ω and a sufficiently large n, it will ultimately differ by a finite constant from the series

$$\frac{2\rho a^3}{\sigma}\sum_{n=1}^{\infty}\frac{[P_n(\mu_0)]^2}{n}\,. \tag{25}$$

For large n and $\cos\theta \neq \pm 1$ one has (see, e.g., Ref. 12)

$$P_n(\cos\theta)\;=\;\left(\frac{2}{\pi n\sin\theta}\right)^{1/2}\cos\left[\left(n+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]+O(n^{-3/2})\,, \tag{26}$$

so that the terms of the series (25) are dominated by const./n^2 and the series is therefore uniformly convergent (see, e.g., Ref. 9). This argument does not apply to the cases $\theta_0 = 0$ or $\theta_0 = \pi$ in which the series fails to converge.

The shape of the free surface corresponding to the normal mode with frequency ω is found by inserting c_{nm} as given by Eq. (21) into the expression (1) of the free surface shape and is therefore

$$r(\theta)\;=\;a\left[1+\frac{\sqrt{\pi}\sigma}{\rho a^3}\Delta\epsilon\sum_{n=1}^{\infty}\frac{n\sqrt{2n+1}P_n(\mu_0)}{\Omega_n^2-\omega^2}Y_n^0(\theta,\phi)\right], \tag{27}$$

where, it will be noticed, $Y_n^0(\theta,\phi)$ is actually independent of ϕ. The multiplicative constant in front of the summation remains undetermined in keeping with the arbitrariness of the oscillation amplitude in the present linear theory.

It may also be of some interest to check explicitly the divergence of the series appearing in the expression (11) of the curvature after substitution of the result (21) for c_n. For large n and ω fixed this series is asymptotic to

$$\sum_{n=1}^{\infty}\frac{n^4}{n^3}P_n \tag{28}$$

and is, therefore, divergent as expected.

IV. THE SPECTRUM

In order to gain some insight into the nature of the solutions of the characteristic equation (24) let us rewrite it by separating out the $n = 1$ and $n = 2$ terms noting that, according to (17), $\Omega_1 = 0$:

$$-\frac{3[P_1(\mu_0)]^2}{\omega^2}+\frac{10[P_2(\mu_0)]^2}{\Omega_2^2-\omega^2}\;=\;-\sum_{n=3}^{\infty}\frac{n(2n+1)[P_n(\mu_0)]^2}{\Omega_n^2-\omega^2}\,. \tag{29}$$

As ω varies in the interval $0 < \omega < \Omega_2$, the left-hand side of the equation varies between $-\infty$ and ∞, while the right-hand side remains a finite negative-definite function of ω. It is therefore obvious that the lowest normal mode will be in this interval and will have a frequency smaller than Ω_2. This consideration proves the existence of a non-zero frequency and, therefore, a non-zero restoring force for the first mode, unlike the case of a free drop. This result is expected as the presence of the constraint removes the translational degree of freedom possessed by a free drop.

By proceeding in a similar way for $\Omega_N < \omega < \Omega_{N+1}$ we may write

$$-\frac{N(2N+1)[P_N(\mu_0)]^2}{\omega^2-\Omega_N^2}+\frac{(N+1)(2N+3)[P_{N+1}(\mu_0)]^2}{\Omega_{N+1}^2-\omega^2}$$
$$=\sum_{n=1}^{N-1}\frac{n(2n+1)[P_n(\mu_0)]^2}{\omega^2-\Omega_n^2}-\sum_{n=N+2}^{\infty}\frac{n(2n+1)[P_n(\mu_0)]^2}{\Omega_n^2-\omega^2}. \tag{30}$$

Again, the left-hand side varies between $-\infty$ and ∞ as ω ranges between Ω_N and Ω_{N+1} while the right-hand side is a finite function of ω. Hence, there will be a root of the equation in this interval.

It is convenient to introduce the dimensionless angular frequency

$$\omega_* = \sqrt{\frac{\rho a^3}{\sigma}}\,\omega \tag{31}$$

in terms of which the characteristic equation can be written as

$$\sum_{n=1}^{\infty}\frac{n(2n+1)[P_n(\mu_0)]^2}{n(n-1)(n+2)-\omega_*^2} = 0. \tag{32}$$

Figure 2 shows the dependence of the dimensionless frequency

$$f_* = \frac{1}{2\pi}\sqrt{\frac{\rho a^3}{\sigma}}\,\omega \tag{33}$$

of the first 4 normal modes of oscillation on the parameter $\mu_0 = \cos\theta_0$ (see Figure 1 for the definition). The dots are the values for a free drop, namely, $\sqrt{n(n-1)(n+2)}/2\pi$ and they occur for μ_0 equal to a zero of the Legendre polynomials of the corresponding order, as explained at the beginning of Sec. III. The presence of a constraint away from the natural nodal lines of the unconstrained case stiffens the system and, therefore, it increases the natural frequency as expected.

Table I shows a comparison of some numerical values generated from Eq. (24) with the results of Ref. 2. The differences are small and confirm the accuracy of both sets of results.

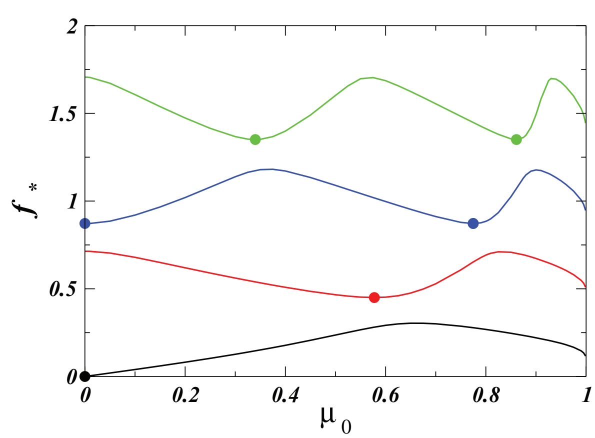

FIG. 2. The dimensionless frequencies f_*, Eq. (33), of the first four normal modes of a constrained drop vs. $\mu_0 = \cos\theta_0$, with θ_0 the angle defined in Figure 1. The lines are, in ascending order, for modes of order $n = 1, 2, 3$, and 4. The dots mark the position of the normal mode frequencies of a free drop.

TABLE I. Comparison between the present numerical results for the dimensionless eigenfrequency f_* of a constrained drop, defined in Eq. (33), and those given by Bostwick (Ref. 2). Here $\mu_0 = \cos\theta_0$ with the angle θ_0 defined in Figure 1.

	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
μ_0	Ref. 2	Present	Ref. 2	Present	Ref. 2	Present	Ref. 2	Present
0.	0	0	0.7041	0.7138	0.8717	0.8717	1.697	1.706
0.2	0.0807	0.0814	0.6143	0.6197	1.013	1.020	1.467	1.474
0.5	0.2322	0.2363	0.4651	0.4659	1.121	1.089	1.590	1.602
0.7	0.2954	0.3002	0.5522	0.5287	0.9097	0.9116	1.550	1.555
0.9	0.2193	0.2207	0.6912	0.6728	1.170	1.177	1.492	1.492

V. HEMISPHERICAL DROP ON A PLANE

The preceding analysis can be readily adapted to the case of a hemispherical drop on a plane considered in Ref. 4. We limit ourselves to the case of a fixed contact line, although the more general partial slip case considered in the cited work can be readily handled by the same method.

By reflecting the problem in the plane we can replace the hemispherical by a spherical drop provided the fluid velocity normal to the plane, vanishes. Thus, we require that

$$\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\bigg|_{\theta=\pi/2} = \sum_{n=1}^{\infty} N_{n0} b_n r^{n-1} P_n'(0) = 0 . \tag{34}$$

In writing this equation we have used the fact that only the harmonics with $m = 0$ contribute. For this relation to be satisfied for all $r \leq a$ it is necessary that $P_n'(0) = 0$, namely, that only even polynomials be included in the summation. By setting $n = 2\ell$ in Eq. (24) and recalling that

$$P_{2\ell}(0) = (-1)^{\ell}\frac{(2\ell-1)!!}{2^{\ell}\ell!} , \tag{35}$$

we find

$$2\sum_{\ell=1}^{\infty}\frac{\ell(4\ell+1)}{\Omega_\ell^2-\omega^2}\left[\frac{(2\ell+1)!!}{2^{\ell}\ell!}\right]^2 = 0 , \tag{36}$$

which is the same result given in Ref. 4.

VI. TWO CONSTRAINTS

The previous analysis is readily adapted to a situation in which the drop surface is constrained at two or more locations. As an example we consider two constraints of the form (6) acting at $\cos\theta_1 = \mu_1$ and $\cos\theta_2 = \mu_2$. On the basis of the previous results we assume that only axisymmetric modes are allowed.

The curvature equation (11) is now modified to

$$a\mathcal{C} = \sum_{n=1}^{\infty}(n-1)(n+2)c_n Y_n^0 - \Delta_{\sigma 1}\delta(\mu-\mu_1) - \Delta_{\sigma 2}\delta(\mu-\mu_2) , \tag{37}$$

with a corresponding modification for the dynamic condition (8), so that the equation for the amplitudes c_n is, in place of Eq. (16),

$$\ddot{c}_\ell + \Omega_\ell^2 c_\ell = \frac{\sigma}{\rho a^3}\ell N_{\ell 0}\left[\Delta_1 P_\ell(\mu_1) + \Delta_2 P_\ell(\mu_2)\right] . \tag{38}$$

All the amplitudes contributing to the same normal mode have the same time dependence proportional to $e^{i\omega t}$ and, therefore,

$$c_\ell = \frac{\sigma}{\rho a^3(\Omega_\ell^2-\omega^2)}\ell N_{\ell 0}\left[\Delta_1 P_\ell(\mu_1) + \Delta_2 P_\ell(\mu_2)\right] . \tag{39}$$

Upon substituting into the expression of the two constraints, after eliminating an inconsequential multiplicative constant, we find

$$\Delta_1 \sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{\Omega_\ell^2 - \omega^2} [P_\ell(\mu_1)]^2 + \Delta_2 \sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{\Omega_\ell^2 - \omega^2} P_\ell(\mu_1) P_\ell(\mu_2) = 0, \tag{40}$$

and

$$\Delta_1 \sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{\Omega_\ell^2 - \omega^2} P_\ell(\mu_1) P_\ell(\mu_2) + \Delta_2 \sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{\Omega_\ell^2 - \omega^2} [P_\ell(\mu_2)]^2 = 0\,. \tag{41}$$

This is a homogeneous linear algebraic system in Δ_1 and Δ_2 which will have non-zero solutions only if its determinant vanishes. In this case the characteristic equation is therefore

$$\left[\sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{(\Omega_\ell^2 - \omega^2)} [P_\ell(\mu_1)]^2\right] \left[\sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{(\Omega_\ell^2 - \omega^2)} [P_\ell(\mu_2)]^2\right] - \left[\sum_{\ell=1}^{\infty} \frac{\ell(2\ell+1)}{(\Omega_\ell^2 - \omega^2)} P_\ell(\mu_1) P_\ell(\mu_2)\right]^2 = 0\,. \tag{42}$$

VII. VISCOUS EFFECTS

Viscous effects can be approximately incorporated in the previous theory by having recourse to Rayleigh's dissipation function method (see, e.g., Ref. 7). The rate of energy dissipation $\mathcal{D}$ in the course of irrotational motion is given by (see, e.g., Ref. 13)

$$\mathcal{D} = -\rho\nu \oint (\nabla u^2) \cdot \mathbf{n}\, dS\,, \tag{43}$$

in which the integral is over the surface of the drop and ν denotes the kinematic viscosity of the liquid. For a free drop this relation accounts only for the dissipation in the boundary layer adjacent to the drop surface, which is the dominant dissipating region only when the viscous boundary layer, with a thickness of the order of $\delta \sim \sqrt{\nu/\omega}$, is much thinner than the drop radius. In the present application, it also neglects the effect of the no-slip condition on the constraint. However, for a constraint with an immersed volume v_c much smaller than the vortical region at the drop surface, of order $4\pi a^2 \delta$, the error may be expected to be small.

Allowing now for the complex nature of the amplitudes c_n, it is readily found that

$$\mathcal{D} = -8\pi\rho\nu a^3 \sum_{n=1}^{\infty} \frac{n-1}{n} |\dot{c}_n(t)|^2\,. \tag{44}$$

The contribution of the dissipation function to the equation of motion (14) of the amplitudes is proportional to

$$\frac{\partial \mathcal{D}}{\partial \dot{c}_\ell^*} = -16\pi \frac{\ell - 1}{\ell} \rho\nu a^3 \dot{c}_\ell\,, \tag{45}$$

in which the asterisk denotes the complex conjugate. This contribution must be inserted in the right-hand side of Eq. (14). To identify the proper proportionality constant it is sufficient to calculate the kinetic energy $\mathcal{K}$ of the flow, given by

$$\mathcal{K} = \frac{1}{2} \oint (\phi \nabla \phi) \cdot \mathbf{n}\, dS\,. \tag{46}$$

One readily finds

$$\mathcal{K} = 2\rho\pi a^5 \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} |\dot{c}_n|^2\,, \tag{47}$$

which will contribute to the equation of motion of each c_ℓ a term

$$\frac{d}{dt}\left(\frac{\partial \mathcal{K}}{\partial \dot{c}_\ell^*}\right) - \frac{\partial \mathcal{K}}{\partial c_\ell^*} = \frac{4\rho\pi a^5}{\ell(2\ell+1)}\ddot{c}_\ell\,. \tag{48}$$

This result shows that, casting the final equation in a form similar to Eq. (16), requires division by $4\rho\pi a^5/[\ell(2\ell+1)]$ and the final result is therefore

$$\ddot{c}_\ell + 2(\ell-1)(2\ell+1)\frac{\nu}{a^2}\dot{c}_\ell + \Omega_\ell^2 c_n = \frac{\sigma}{2a^3\rho}\ell(2\ell+1)\Delta P_\ell(\mu_0)\,. \tag{49}$$

The action of viscosity on each amplitude is therefore the same as in the case of a free drop.[7,8]

VIII. A CONSTRAINED BUBBLE

The analysis proceeds in a very similar way for the case of a bubble. The main difference is that, now, the expression (2) of the velocity potential is replaced by (see, e.g., Ref. 9)

$$\varphi = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_{nm}(t) r^{-n-1} Y_n^m(\theta,\phi)\,, \tag{50}$$

so that the expression of the constraint becomes

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n+1)b_{nm}(t)a^{-n-2}Y_n^m(\mu_0,\phi) = 0 \tag{51}$$

and the kinematic condition (7) is replaced by

$$b_{nm}(t) = -\frac{a^{n+2}}{n+1}\dot{c}_{nm}\,. \tag{52}$$

Since now the normal is directed inside the sphere the curvature has the opposite sign and the dynamic equation (14) becomes

$$-\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{\ddot{c}_{nm}}{n+1}Y_n^m = \frac{\sigma}{\rho a^3}\left(\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)c_{nm}Y_n^m - \Delta(\phi,t)\delta(\mu-\mu_0)\right)\,, \tag{53}$$

while, after recognizing that only $m = 0$ gives rise to non-zero coefficients, Eq. (15) is replaced by

$$\frac{\ddot{c}_\ell}{\ell+1} + \frac{\sigma}{\rho a^3}(\ell-1)(\ell+2)c_\ell = \frac{\sigma}{\rho a^3}\Delta_\ell\,, \tag{54}$$

or

$$\ddot{c}_\ell + \Omega_\ell^2 c_\ell = \frac{\sigma}{\rho a^3}(\ell+1)\Delta_\ell\,, \tag{55}$$

in which, now,

$$\Omega_\ell^2 = (\ell-1)(\ell+1)(\ell+2)\frac{\sigma}{\rho a^3} \tag{56}$$

is the well-known expression for the inviscid oscillation frequency of a free bubble.[7,10] Proceeding as before we obtain the following characteristic equation for the eigenmode frequency:

$$\sum_{n=1}^{\infty}\frac{(n+1)(2n+1)[P_n(\mu_0)]^2}{\Omega_n^2-\omega^2} = 0\,. \tag{57}$$

Numerical results for the eigenfrequency f_* of the first four normal modes non-dimensionalized according to Eq. (33) as before are shown in Figure 3 and in Table II. The dots are the values for a free bubble, namely, $\sqrt{(n-1)(n+1)(n+2)}/2\pi$, and they occur at the zeros of the Legendre polynomials of the corresponding order. The constraint increases the frequency as in the case of a drop.

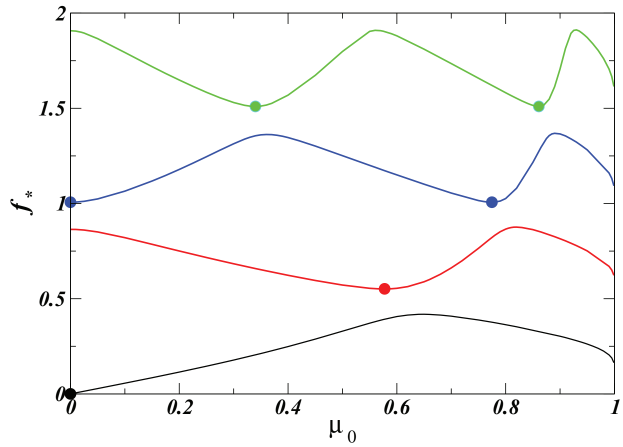

FIG. 3. The dimensionless frequencies f_*, Eq. (33), of the first four normal modes of a constrained bubble vs. $\mu_0 = \cos\theta_0$, with θ_0 the angle defined in Figure 1. The lines are, in ascending order, for modes of order $n = 1$, 2, 3, and 4. The dots mark the position of the normal mode frequencies of a free bubble.

Similarly to the derivation of Sec. VII, the effects of a weak viscosity can simply be lifted from the known theory for a free bubble. On this basis we are led to modify Eq. (55) to the form

$$\ddot{c}_\ell + 2(\ell+2)(2\ell+1)\frac{\nu}{a^2}\dot{c}_\ell + \Omega_\ell^2 c_\ell \; = \; \frac{\sigma}{\rho a^3}(\ell+1)\Delta_{\ell k}(t)\,. \tag{58}$$

IX. CAPILLARY-GRAVITY WAVES – TWO DIMENSIONS

As a further illustration of the present method we turn now to capillary-gravity waves in a finite domain, first in two and, in Sec. X, in three dimensions. We consider the situation depicted in Figure 4. The domain of interest extends between $-L$ and L in the horizontal direction and the depth of the liquid is assumed infinite for simplicity. The z-axis is directed upward with $z = 0$ on the undisturbed free surface.

We allow for surface disturbances by writing

$$S_d \; \equiv \; z - \sum_{n=1}^{\infty}\left[p_n(t)\, c_n(x) + q_n(t) s_n(x)\right] \; = \; 0\,, \tag{59}$$

in which

$$c_n \; = \; \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}\,, \qquad s_n \; = \; \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\,, \tag{60}$$

TABLE II. Some numerical results for the dimensionless eigenfrequency f_* of a constrained bubble, defined in Eq. (33). Here $\mu_0 = \cos\theta_0$ with the angle θ_0 defined in Figure 1.

μ_0	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0.	0.	0.8641	1.007	1.907
0.2	0.1146	0.7506	1.179	1.648
0.5	0.3293	0.5718	1.251	1.798
0.7	0.4084	0.6617	1.055	1.735
0.9	0.3028	0.8150	1.365	1.707

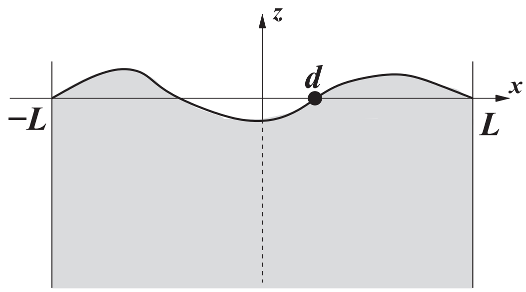

FIG. 4. Surface waves on the constrained surface of a deep liquid pool. The motion of the liquid normal to the undisturbed surface is prevented by the constraint located at $x = d$.

are normalized orthogonal eigenfunctions. The term with $n = 0$ is absent from Eq. (59) in order to eliminate flow at $z \to -\infty$. It proves convenient here to use the real form of the Fourier series so as to describe at the same time the situation in which the surface is free to slip along the planes $x = \pm L$, in which case we take $q_n = 0$ for all n, and the situation in which the surface is pinned at $x = \pm L$, which is included in the analysis by taking $p_n = 0$. It may be noted that, in this latter case, there is actually liquid flow across the planes $x = \pm L$.

We write a similar expansion for the velocity potential

$$\varphi = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\dot{p}_n(t)\, c_n(x) + \dot{q}_n(t)\, s_n(x)\right] e^{\pi n z/L}, \tag{61}$$

in which the form of the coefficients already accounts for the kinematic condition $\partial\varphi/\partial z = -\partial S_d/\partial t$ on $z = 0$. The constraint is located at $x = d$ and it prevents motion in the vertical direction so that

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = \sum_{n=1}^{\infty}\left[\dot{p}_n c_n(d) + \dot{q}_n s_n(d)\right] = 0. \tag{62}$$

It is evident from Eq. (59) that, as a consequence of this condition, $dS_d/dt = 0$ at $x = d$, as expected given that Eq. (61) satisfies the kinematic condition. Accounting for the discontinuity in the surface slope caused by the constraint, we write the curvature associated with Eq. (59) as

$$\mathcal{C} = \frac{\pi^2}{L^2}\sum_{n=1}^{\infty} n^2\left[p_n(t) s_n(x) + q_n(t) s_n(x)\right] - \Delta_\sigma \delta(x-d). \tag{63}$$

We now substitute Eqs. (61) and (63) into the dynamic condition (9) augmented by the effect of gravity:

$$\frac{\partial\varphi}{\partial t} + gz = -\frac{\sigma}{\rho}\mathcal{C} + \frac{\Delta_p}{\rho}, \tag{64}$$

in which g is the acceleration of gravity. The result is

$$\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\ddot{p}_n(t)\, c_n(x) + \ddot{q}_n(t)\, s_n(x)\right] + g\sum_{n=1}^{\infty}\left[p_n(t)\, c_n(x) + q_n(t)\, s_n(x)\right]$$
$$= -\frac{\pi^2\sigma}{\rho L^2}\sum_{n=1}^{\infty} n^2\left[p_n(t) s_n(x) + q_n(t) s_n(x)\right] + \frac{\sigma}{\rho}\Delta\delta(x-d), \tag{65}$$

in which Δ is defined as in Eq. (13) with a replaced by L. We may note that, if the singular pressure contribution were not added to the right-hand side of Eq. (64), in the absence of surface tension the constraint would not affect the motion, which would be clearly incorrect.

The scalar product of Eq. (65) with c_ℓ gives

$$L\frac{\ddot{p}_\ell}{\pi\ell} + g p_\ell = -\frac{\pi^2\ell^2\sigma}{\rho L^2} p_\ell + \frac{\sigma}{\rho}\Delta\, c_\ell(d), \tag{66}$$

while the scalar product with s_ℓ results in

$$L\frac{\ddot{q}_\ell}{\pi\ell} + gq_\ell = -\frac{\pi^2\ell^2\sigma}{\rho L^2}q_\ell + \frac{\sigma}{\rho}\Delta\, s_\ell(d)\,. \tag{67}$$

Upon introducing the frequencies of oscillation in the absence of the constraint

$$\Omega_\ell^2 = \frac{\pi\ell}{L}\left(g + \frac{\pi^2\ell^2\sigma}{\rho L^2}\right) \tag{68}$$

these equations become, respectively,

$$\ddot{p}_\ell + \Omega_\ell^2 p_\ell = \ell\frac{\pi\sigma}{\rho L}\Delta c_\ell(d)\,, \tag{69}$$

and

$$\ddot{q}_\ell + \Omega_\ell^2 q_\ell = \ell\frac{\pi\sigma}{\rho L}\Delta s_\ell(d)\,. \tag{70}$$

The same argument leading to the characteristic equation (24) leads us now to

$$\sum_{n=1}^{\infty}\frac{n\cos^2(n\pi d/L)}{\omega^2 - \Omega_n^2} = 0 \tag{71}$$

for a contact angle of $\pi/2$, and to

$$\sum_{n=1}^{\infty}\frac{n\sin^2(n\pi d/L)}{\omega^2 - \Omega_n^2} = 0 \tag{72}$$

for an interface pinned at the end points $x = \pm L$. If only periodicity is assumed, without further conditions at $x = \pm L$, it is found that the result is the sum of these two expressions:

$$\sum_{n=1}^{\infty}\frac{n}{\omega^2 - \Omega_n^2} = 0 \tag{73}$$

and is independent of d as could be anticipated.

If we recall the asymptotic form (26) of the Legendre polynomials, we see that the contribution of the high-order terms to the characteristic equations (24) and (57) for drops and bubbles is, approximately,

$$\frac{n\ \cos^2(n\theta_0)}{n^3 - \omega_*^2}. \tag{74}$$

For large n the curvature of the undisturbed interface is immaterial and this expression becomes therefore the same as the generic term of Eq. (71) aside, of course, from the contribution of gravity.

X. CAPILLARY-GRAVITY WAVES – THREE DIMENSIONS

We consider gravity-capillary waves on the surface of a liquid contained in an infinitely deep cylindrical container the radius of which is taken as 1 for convenience. We use cylindrical coordinates with the upward-directed z-axis coincident with that of the container and $z = 0$ on the undisturbed free surface (Figure 5). On the basis of the analysis presented for the drop case we anticipate that only axisymmetric modes will be compatible with the presence of a constraint in the form of a ring of radius $r = d < 1$.

We assume that the free surface slides on the cylinder wall always maintaining a contact angle of $\frac{1}{2}\pi$. Thus, we write the disturbed free surface in the form

$$S_d \equiv z - \sum_{n=1}^{\infty} c_n(t)J_0(j_n r) = 0\,, \tag{75}$$

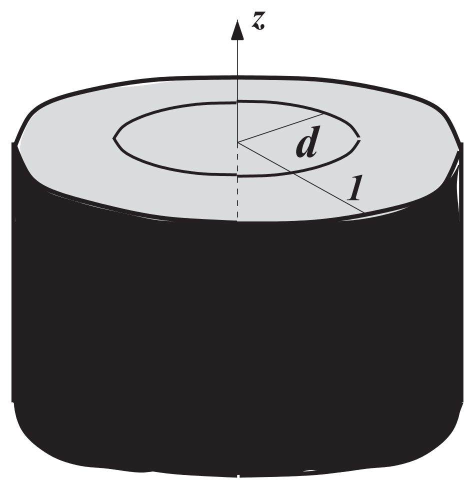

FIG. 5. Surface of a deep cylindrical liquid pool in a container with unit radius. The motion of the liquid normal to the undisturbed surface is prevented by the circular constraint located at $r = d$.

where J_0 is the Bessel function of order 0 and the j_n are the positive zeros of $J_0' = -J_1$. The velocity potential kinematically compatible with this representation has the form

$$\varphi = \sum_{n=1}^{\infty} \frac{\dot{c}_n}{j_n} J_0(j_n r)\, e^{j_n z} . \tag{76}$$

The constraint prevents the vertical motion of the surface along a circle of radius d, so that

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = \sum_{n=1}^{\infty} \dot{c}_n J_0(j_n d) = 0 . \tag{77}$$

The curvature is

$$\mathcal{C} = \sum_{n=1}^{\infty} j_n^2 c_n J_0(j_n r) - \Delta_\sigma \delta(r-d) , \tag{78}$$

and the dynamic condition (64) becomes

$$\sum_{n=1}^{\infty} \frac{\ddot{c}_n}{j_n} J_0(j_n r) + g \sum_{n=1}^{\infty} c_n(t) J_0(j_n r) = -\frac{\sigma}{\rho}\left[\sum_{n=1}^{\infty} j_n^2 c_n J_0(j_n r) - \Delta \delta(r-d)\right] , \tag{79}$$

with Δ as in Eq. (13). The scalar product with $J_0(j_\ell r)$ gives

$$\frac{\ddot{c}_\ell}{j_\ell} + \left(g + \frac{\sigma}{\rho} j_\ell^2\right) c_\ell = \frac{\sigma \Delta}{N_\ell^2 \rho} J_0(j_\ell d) , \tag{80}$$

in which

$$N_\ell^2 = \int_0^1 r J_0^2(j_\ell r)\, dr = \frac{1}{2} J_0^2(j_\ell) . \tag{81}$$

The characteristic equation follows as in the previous cases and is found to be

$$\sum_{n=1}^{\infty} \frac{j_n}{N_n^2} \frac{J_0^2(j_n d)}{\Omega_n^2 - \omega^2} = 0 , \tag{82}$$

where

$$\Omega_n^2 = j_n \left(g + \frac{\sigma}{\rho} j_n^2\right) \tag{83}$$

are the natural frequencies of the axially symmetric normal modes of a circular basin.

Since, for large n, $j_n \simeq \pi n$, with the aid of the asymptotic expression (see, e.g., Ref. 12)

$$J_0(j_n d) \simeq \sqrt{\frac{2}{\pi j_n d}} \cos\left(j_n d - \frac{\pi}{4}\right) \simeq \sqrt{\frac{2}{\pi^2 n d}} \cos(n\pi d)\,, \tag{84}$$

the generic term of the series (82) is seen to be asymptotic to

$$4\pi \frac{n\,\cos^2(n\pi d)}{\Omega_n^2 - \omega^2} \tag{85}$$

as before.

For a cylindrical basin of radius R rather than 1, these relations are modified to

$$\sum_{n=1}^{\infty} \frac{j_n}{N_n^2} \frac{J_0^2(j_n d/R)}{\Omega_n^2 - \omega^2} = 0\,, \tag{86}$$

and

$$\Omega_n^2 = \frac{j_n}{R}\left(g + \frac{\sigma}{\rho R^2} j_n^2\right), \qquad N_n^2 = \frac{R^2}{2} J_0^2(j_n)\,. \tag{87}$$

XI. THE RAYLEIGH-TAYLOR INSTABILITY

By the simple change of the sign of the gravity acceleration in Eqs. (68) and (83) one can study the unstable configuration in which the liquid is above the low-density medium.

Let N be largest integer such that

$$\Omega_N^2 = \frac{\pi N}{L}\left(-g + \frac{\pi^2 N^2 \sigma}{\rho L^2}\right) \tag{88}$$

is negative and, for $1 \le n \le N$ let us write $\tilde{\Omega}_n^2 = -\Omega_n^2$; we also set $\tilde{\omega} = i\omega$. Then the characteristic equation (71), for example, may be written

$$\sum_{n=1}^{N} \frac{n \cos^2(n\pi d/L)}{\tilde{\Omega}_n^2 - \tilde{\omega}^2} = \sum_{n=N+1}^{\infty} \frac{n \cos^2(n\pi d/L)}{\tilde{\omega}^2 + \Omega_n^2}\,. \tag{89}$$

The right-hand side is finite and positive definite while the left-hand side ranges from ∞ to $-\infty$ N times as $\tilde{\omega}^2$ increases past each one of the $\tilde{\Omega}_n^2$. Thus, this equation always has N real roots, each one corresponding to an unstable mode, and the qualitative features of the instability are not affected by the constraint. However, since for the two sides of the equation to be equal the left-hand side must be positive, on the basis of a simple graphical argument it is concluded that equality would always occur for $\tilde{\omega}^2 < \tilde{\Omega}_n^2$ implying that the growth rate of the nth mode is decreased by the constraint, as could be anticipated.

XII. SUMMARY

We have considered several situations, depicted in Figures 1, 4, and 5, in which the free surface of a liquid is prevented from moving along a closed line lying on the undisturbed free surface. By recognizing that, in this situation, the curvature of the surface at the location of the constraint is singular, and by accounting explicitly for this singularity and for the additional pressure exerted by the constraint, we have shown how problems of this type can be solved in a very straightforward way essentially following the usual steps applicable to an unconstrained surface.

The characteristic equation of the normal modes of the system is changed by the presence of the constraint. The numerical results obtained for the case of drops (Figure 2) and bubbles (Figure 3) show that the constraint increases the frequency of the modes, which may be expressed by saying that the constrained system is "stiffer" than the unconstrained one. Of course, when the constraint lies along a nodal line of the unconstrained system there is no effect on the normal modes.

Among other results, it has been shown that the degeneracy affecting the normal modes of free drops and bubbles is removed by the presence of a constraint in the form envisaged in this study. Unlike the unconstrained situation, the lowest mode for drops and bubbles acquires a finite non-zero frequency. A brief consideration of the unstable case shows that the growth rate of the Rayleigh-Taylor instability is reduced by the constraint.

ACKNOWLEDGMENTS

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