ON THE GEL'FAND REPRESENTATION OF A COMMUTATIVE BANACH ALGEBRA

A Thesis

Presented to

The Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

William Charles Haase

August, 1966

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ABSTRACT

In this paper the Gel´fand representation of a commutative Banach algebra is developed. The fundamental results are as follows. 1) Any complex commutative Banach algebra A is homomorphic to an algebra of continuous complex valued functions on a locally compact Hausdorff space. If A has an identity then the space is compact and in any case the functions vanish at infinity. The representation is norm decreasing. 2) If A is semi-simple the representation is an isomorphism. 3) If A is such that $||x^2|| = ||x||^2$ then the Gel´fand representation of A is isometric to A. Finally the Gel´fand representation is used to prove the Banach-Stone Theorem and the essential uniqueness of the Stone-Cech compactification, and the Gel´fand representation of an element of $L_1(-\infty,\infty)$ is seen to be the Fourier transform of that element.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In 1941, I. M. Gel´fand published his result that every commutative Banach algebra with identity can be represented as an algebra of complex valued functions on a compact Hausdorff space. These results were later extended to the case where an identity is not present. Some of Gel´fand's results can be extended to the noncommutative case, but the representation cannot be obtained because the quotient of a ring modulo a maximal ideal is not necessarily a field. This paper develops the representation of a commutative Banach algebra.

A Banach algebra is a complex Banach space in which a multiplication is defined such that the Banach space is also an algebra in which scalars are complex numbers. The additional requirement is that the norm of a product must be less than or equal to the product of the norms. In any Banach algebra multiplication is continuous.

The key to the Gel´fand representation is the fact that, in any commutative Banach algebra, every homomorphism of the algebra to the complex numbers is continuous and the norm of the homomorphism is less than or equal to one. Furthermore, there is a one to one correspondence between the regular maximal ideals of the algebra and the homomorphisms onto the complex numbers. These results are established in Section 7 of Chapter II.

Sections 1 through 6 of Chapter II are devoted to establishing the preliminary results needed for Section 7. Sections 2 through 5 establish the algebraic results needed to show that the quotient of a commutative Banach algebra modulo a regular maximal ideal is a field. Section 6 establishes analytic results which lead to the conclusion that a complex commutative Banach algebra which is also a field is isometrically isomorphic to the complex numbers.

The results of Section 7 lead immediately to the Gel´fand representation, namely, that any commutative Banach algebra is homomorphic to an algebra of continuous complex valued functions on a locally compact Hausdorff space. This homomorphism is in general norm decreasing. The representation has the further property that if the algebra has an identity, the space is compact, and that in any case the functions of the representation vanish at infinity. The points of the space are the regular maximal ideals of the algebra.

In Section 9 the spectrum of an element is defined. The spectrum of an element is shown to be the range of the function which is the Gel´fand representation of the element. The spectral norm of an element becomes the supremum of the absolute values of the range of the function which is the representation of the element.

The radical of a commutative Banach algebra is the intersection of the regular maximal ideals of the algebra. An algebra is semi-simple if and only if the radical contains only the zero element. In Section 10 it is shown that the Gel´fand representation is an isomorphism if and only if the algebra is semi-simple.

Questions concerning the isometry of a Banach algebra and its Gel´fand representation are considered in Section 11. A necessary and sufficient condition that the Gel´fand representation of a commutative Banach algebra be isometric to the algebra is that the norm of the square of each element of the algebra be equal to the square of the element's norm.

An analogue for analytic functions of an element of a commutative Banach algebra is developed in Section 12. It is shown that, with proper restrictions, a Banach algebra is closed under the application of analytic functions to elements of the algebra.

In Chapter III examples of the Gel´fand representation are given. In Section 1 the fact that the space of maximal ideals depends only on the structure of the algebra is used to draw some topological conclusions. In Section 2 the Fourier transform of an absolutely integrable function from the real line is seen to be the Gel´fand representation of the function.

CHAPTER II

DEFINITIONS & THEOREMS

SECTION 1 DEFINITIONS

Definition: A Banach algebra is set A such that:

a) A is a Banach space in which scalars are complex numbers,

b) A is an algebra,

c) For all x, y ε A, $||xy|| \leq ||x||\cdot||y||$.

Definition: A Banach algebra with identity is a Banach algebra A which contains an element e such that $ex = xe = x$ for all $x \varepsilon A$.

Definition: A commutative Banach algebra is a Banach algebra A such that $xy = yx$ for all $x,y \varepsilon A$.

Notation: Throughout Chapter II, A will be used for a Banach algebra (with or without identity), A_e for a Banach algebra with identity, e for the identity of a Banach algebra with identity, and C for the complex numbers.

SECTION 2 ELEMENTARY PROPERTIES

In this section some elementary properties of Banach algebras are discussed, namely that multiplication is continuous, that any Banach algebra may be imbedded in a Banach algebra with identity, that a Banach algebra with identity may be renormed such that $||e|| = 1$, and that any Banach algebra may be viewed as a ring of operators on a Banach space.

Proposition 1: In any Banach algebra multiplication is continuous.

Proof: Let $\{x_n\}$ be a sequence such that $x_n \to x$. Let $\{y_n\}$ be a sequence such that $y_n \to y$. Then

$$||x_ny_n - xy|| = ||x_ny_n - xy_n + xy_n - xy||$$
$$\leqq ||x_ny_n - xy_n|| + ||xy_n - xy||$$
$$\leqq ||y_n||\cdot||x_n - x|| + ||x||\cdot||y_n - y||.$$

Therefore

$$\lim_{n\to\infty} ||x_ny_n - xy|| \leqq \lim_{n\to\infty} (||y_n||\cdot||x_n - x||) + \lim_{n\to\infty} (||x||\cdot||y_n - y||)$$

$$= ||y|| \lim_{n\to\infty} ||x_n - x|| + ||x|| \lim_{n\to\infty} ||y_n - y|| = 0.$$

Thus, $x_ny_n \to xy$ as $n\to\infty$, or multiplication is continuous.

Proposition 2: If A is a Banach algebra without identity, then A can be imbedded isometrically and isomorphically in a Banach algebra A_e with identity.

Proof: Let $A_e = \{(\lambda,x) \mid \lambda \in C, x \in A\}$. No confusion will arise if $\lambda e + x$ is written instead of (λ,x) for $(\lambda,x) \in A_e$. Define

$$(\lambda_1 e + x_1) + (\lambda_2 e + x_2) = (\lambda_1 + \lambda_2)\, e + (x_1 + x_2),$$

$$\lambda(\lambda_1 e + x_1) = (\lambda\lambda_1)\, e + \lambda x_1,$$

and

$$(\lambda_1 e + x_1)\cdot(\lambda_2 e + x_2) = (\lambda_1\lambda_2)e + (\lambda_1 x_2 + \lambda_2 x_1 + x_1 x_2).$$

It is easily verified that A_e is an algebra and that $e = e + 0$ is an identity for A_e.

Let $||\lambda e + x|| = |\lambda| + ||x||$. It is easily verified that $||\lambda e + x||$ is a norm. The algebra A_e is complete in this norm, since $||(\lambda_m e + x_m) - (\lambda_n e + x_n)|| = |\lambda_m - \lambda_n| + ||x_m - x_n|| \to 0$, if and only if $|\lambda_m - \lambda_n| \to 0$ and $||x_m - x_n|| \to 0$. If $\{\lambda_n e + x_n\}$ is a Cauchy sequence, then $\lambda_n \to \lambda_o$ and $x_n \to x_o$ and $(\lambda_n e + x_n) \to \lambda_o + x_o$, so A_e is complete. It follows that

$$||(\lambda_1 e + x_1)(\lambda_2 e + x_2)|| = |\lambda_1 \lambda_2| + ||\lambda_1 x_2 + \lambda_2 x_1 + xx_2||$$

$$\leqq |\lambda_1| \cdot |\lambda_2| + |\lambda_1| \cdot ||x_2|| + |\lambda_2| \cdot ||x_1|| + ||x_1|| \cdot ||x_2||$$

$$= ||\lambda_1 e + x_1|| \cdot ||\lambda_2 e + x_2||.$$

Henceforth, whenever a Banach algebra without identity is said to be extended to a Banach algebra with identity, the extension will be understood to be performed in the manner described in Proposition 2.

Proposition 3: Any set satisfying a) and b) in the definition of a Banach algebra which also has an identity and continuous multiplication is isomorphic and homeomorphic to a Banach algebra with identity such that $||e|| = 1$. Moreover, the isomorphism is such that any Banach algebra may be viewed as an algebra of linear transformationson a Banach space where composition is multiplication.

Proof: Let A_e be a set satisfying a) and b) with identity and continuous multiplication.

Let $\mathrm{Hom}(A_e, A_e)$ be the set of all bounded linear transformations from A_e to A_e with the usual operator norm. Let $A_e' = \{x' \in \mathrm{Hom}(A_e, A_e) \mid$ there is an $x \in A_e$ such that $x'(y) = xy$ for all $y \in A_e\}$. Now for all $x \in A_e$ there is an $x' \in A_e'$ satisfying the above and the mapping is one to one, since if $x_1 \neq x_2$ in A_e then $x_1 e \neq x_2 e$, which implies that $x_1'(e) \neq x_2'(e)$, or $x_1' \neq x_2'$.

Now it is sufficient to show that A_e' is closed in $\mathrm{Hom}(A_e, A_e)$. If $a_o = \lim_{n\to\infty} a_n$, a_n A_e', then for each a_n, there is a $w_n \in A_e$ such that $a_n(x) = w_n x$ for all $x \in A_e$. Now for all $x, y \in A_e$,

$$a_o(xy) = \lim_{n\to\infty} a_n(xy) = \lim_{n\to\infty} w_n xy = (\lim_{n\to\infty} w_n x)y$$

$$= (\lim_{n\to\infty} a_n(x))\cdot y = (a_o(x))\cdot y.$$

Letting $x_o = a_o(e)$, then for all $y \in A_e$

$$a_o(y) = a_o(ey) = (a_o(e))\cdot y = x_o y,$$

therefore $a_o \in A_e'$, or A_e' is closed, hence A_e' is a Banach algebra.

Let $\Phi : A_e' \to A_e$ by $\Phi(x') = x$. Then

$$||x'|| = \sup_{||y|| \leq 1} ||xy|| \geq \left|\left| x\cdot \frac{e}{||e||} \right|\right| = \left|\left| \frac{x}{||e||} \right|\right| = \frac{||x||}{||e||}.$$

Thus $||x|| \leq ||e||\cdot||x'||$ or Φ is bounded. But since Φ is a one to one bounded linear transformation from a Banach space A_e' onto a Banach space A_e, Φ^{-1} is continuous.

Therefore A_e' is isomorphic and homeomorphic to A_e. Moreover, A_e' is a ring of operators as promised, and for all $x', y' \varepsilon A_e'$ and $w \varepsilon A_e$

$$||(x' \circ y')\ (w)|| = ||x'(y'(w))|| \leqq ||x'||\cdot||y'(w)||$$

$$\leqq ||x'||\cdot||y'||\cdot||w||, \text{ or } ||x' \circ y|| \leqq ||x'||\cdot||y'||.$$

It also follows that $||e'(y)|| = ||ey|| = ||y||$, or $||e'|| = 1$.

In any Banach algebra with identity, e, $||e|| \geqq 1$, since otherwise $||e|| \leqq ||e||^n \to 0$ as $n \to \infty$. Since any Banach algebra with identity satisfies the hypothesis of Proposition 3, it may be renormed such that $||e|| = 1$. Henceforth, in any Banach algebra with identity, the norm of the identity will be assumed to be equal to 1.

SECTION 3 INVERSE AND ADVERSE

Definition: The inverse of an element x of a Banach algebra with identity is an element x^{-1} such that $xx^{-1} = x^{-1}x = e$.

It will be shown that in a Banach algebra with identity, the set of elements having inverses is open.

Lemma 1; If x is an element of a Banach algebra such that $||x-e|| < 1$, then x has an inverse.

Proof: Consider the sequence $\{x_n\}$ where $x_n = e + \sum_{i=1}^{n} (e - x)^i$; $\{x_n\}$ is a Cauchy sequence, since, given $\varepsilon > 0$, there is an N such that for $m > n > N$,

$$||x_m - x_n|| = ||\sum_{i=n}^{m}(e - x)^i|| = ||(e - x)^N \sum_{i=n-N}^{m-N}(e - x)^i||$$

$$\leq ||e - x||^N ||\sum_{i=n-N}^{m-N}(e - x)^i|| \leq ||e - x||^N \sum_{i=n-N}^{m-N} ||e - x||^i$$

$$\leq ||e - x||^N \left(\frac{1}{1 - ||e - x||}\right) < \varepsilon$$

for sufficiently large N. Let $x^{-1} = \lim_{n\to\infty} x_n = e + \sum_{i=1}^{\infty}(e - x)^i$.

Then $x = e - (e - x)$ and

$$xx^{-1} = (e - (e - x))x^{-1}$$

$$= e + \sum_{i=1}^{\infty}(e - x)^i - (e - x)\sum_{i=1}^{\infty}(e - x)^i$$

$$= e + \sum_{i=1}^{\infty}(e - x)^i - \sum_{i=1}^{\infty}(e - x)^i = e, \text{ or } xx^{-1} = e.$$

Proposition 1: If A_e is a Banach algebra with identity, then the set of elements of A_e having inverses is open.

Proof: Let $V = \{x \mid x \in A_e \text{ and } x^{-1} \text{ exists}\}$.

Let $U_e = \{x \mid x \in A_e \text{ and } ||x - e|| < 1\}$. The set U_e is open and $U_e \subset V$ by Lemma 1. If $x \in V$ then x has an inverse x^{-1} such that $xx^{-1} = e$. Since multiplication is continuous, there is a neighborhood U of x such that $Ux^{-1} = \{y \mid y = x'x^{-1} \text{ for some } x' \in U\} \subset U_e$. Therefore for all $z \in U$, $zx^{-1} \in U_e$ or zx^{-1} has an inverse w. Since $e = (zx^{-1})w = z(x^{-1}w)$, then $x^{-1}w$ is an inverse for z; therefore $U \subset V$.

But x was any element of V, so V is open. Of course, V is not empty, since $e = e^{-1} \varepsilon V$.

Since a Banach algebra may not have an identity, the properties of inverses are not meaningful in that general setting. The notion of inverses does admit generalization in the following sense.

Definition: If A is a Banach algebra, and $x \varepsilon A$, then an element $y \varepsilon A$ is a right adverse of x if and only if $x + y - xy = 0$. The element x is said to be a left adverse of y.

Now, if y is a right adverse of x in a Banach algebra A, and A is imbedded in an algebra with identity, then $e - y$ is a right inverse of $e - x$, since

$$(e - x)(e - y) = e - ex - ey + xy = e - (x + y - xy) = e.$$

In any commutative Banach algebra a left adverse is also a right adverse. This is not true in general in the non-commutative case, i.e. an element might have a left adverse but not a right adverse. Nevertheless, the following may be said of any Banach algebra.

Proposition 2: In any Banach algebra, if an element x has both a left and a right adverse, then the adverses are equal and consequently unique.

Proof: If u is a left adverse of x and y is a right adverse of x, then (after imbedding the algebra in an algebra with identity if necessary) $e - u$ is a left inverse of $(e - x)$; and $e - y$ is a right inverse of $(e - x)$. Now

$$e - u = (e - u)\cdot(e - x)\cdot(e - y) = e - y.$$

Hence $e - u = e - y$ or $u = y$.

If x has a right and left adverse, then this unique element will be called the adverse of x and will be denoted by x'. Theorems concerning adverses often follow the same line as theorems concerning inverses.

Lemma 2: If x is an element of a Banach algebra such that $||x|| < 1$, then x has an adverse.

Proof: Let $y_n = -\sum_{i=1}^{n} x^i$. For $m > n$

$$||y_m - y_n|| = ||-\sum_{n+1}^{m} x^i|| \leq \sum_{n+1}^{m} ||x||^i = ||x||^n \sum_{i=1}^{m} ||x||^i$$

$$\leq ||x||^n\left(\frac{||x||}{1 - ||x||}\right) \to 0 \text{ as } n \to \infty.$$

Therefore $\{y_n\}$ is a Cauchy sequence.

If $y_0 = \lim_{n\to\infty} y_n = -\sum_{i=1}^{\infty} x^i$, then

$$x + y_0 - xy_0 = x - \sum_{i=1}^{\infty} x^i + x\sum_{i=1}^{\infty} x^i$$

$$= x - \sum_{i=1}^{\infty} x^i + \sum_{i=2}^{\infty} x^i = -\sum_{i=1}^{\infty} x^i + \sum_{i=1}^{\infty} x^i = 0.$$

So y_0 is a right adverse of x. Similarly $x = y_0 - y_0 x = 0$, so y_0 is an adverse of x and

$$||y_0|| = ||-\sum_{i=1}^{\infty} x^i|| = ||\lim_{n\to\infty} -\sum_{i=1}^{n} x^i|| = \lim_{n\to\infty}||-\sum_{i=1}^{n} x^i||$$
$$\leq ||x||\lim_{n\to\infty}\sum_{i=0}^{n} ||x||^i = ||x||/(1 - ||x||)$$

Proposition 3: If A is a Banach algebra, then the set of elements having adverses is open.

Proof: Define $x \circ y = x + y - xy$ for all $x,y \in A$. Then y is a right adverse of x if and only if $x \circ y = 0$. It follows immediately that $\circ$ is associative.

Let V' be the set of all elements of A having adverses. Since 0 has an adverse, $V' \neq \phi$.

If $y \in V'$ let $x \in A$ be such that $||x|| < (1 + ||y'||)^{-1}$. Now $||x-xy'|| \leq ||x||\cdot(1 + ||y'||) < 1$. By Lemma 2, $u = x - xy'$ has an adverse, u'. Now

$$(y + x) \circ y' = y + x + y' - yy' - xy' = x - xy';$$

consequently

$$(y + x) \circ (y' \circ u') = ((y + x) \circ y') \circ u' = 0,$$

or $y' \circ u'$ is a right adverse of $y + x$.

Similarly, if $||x|| < (1 + ||y'||)^{-1}$ then $||x-y'x|| < 1$ and $v = x - y'x$ has a left adverse v' and $y' \circ (y + x) = x - y'x = v$. Now $v' \circ y'$ is a left adverse of $y + x$.

Hence, $y + x$ has an adverse. Thus, if $z \in A$ and $||z - y|| < (1 + ||y'||)^{-1}$ then z has an adverse, and thus V' is open.

Proposition 4: In any Banach algebra, the mapping $x \to x'$ is continuous.

Proof: Using the notation of Proposition 3,

$$(y + x)' - y' = (y' \circ u') - y' = u' - y'u'.$$

Thus,

$$||(y + x)' - y'|| \leqq ||u'||\cdot(1 + ||y'||)$$
$$\leqq ||u||(1 - ||u||)^{-1}(1 + ||y'||)$$
$$\leqq \frac{||x||(1 + ||y'||)^2}{1 - ||x||(1 + ||y'||)}.$$

Letting $a = (1 + ||y'||)^{-1}$; then

$$||(y + x)' - y'|| \leqq \frac{||x||}{1 - a^{-1}\ ||x||}\ a^{-2} = \frac{||x||}{(a - ||x||)_a} \to 0$$

as $||x|| \to 0$, or the mapping $x \to x'$ is continuous.

Proposition 5: In any Banach algebra with identity, the mapping $x \to x^{-1}$ is continuous.

Proof: Let V be the set of elements having inverses. Let V' be the set of elements having adverses. Let $\Phi : V' \to V$ be defined by $\Phi(x) = e - x$. Let $\psi : V \to V$ be defined by $\psi(x) = x'$. Now if $x \in V$ then $e - x \in V'$ and $(e - x)' = e - x^{-1}$, since

$$(e - x) + (e - x^{-1}) - (e - x)(e - x^{-1})$$
$$= e - x + e - x^{-1} - e + x + x^{-1} - xx^{-1} = 0.$$

The mapping $x \to x^{-1}$ is the mapping $\Phi\circ\psi\circ\Phi$, since

$$\Phi(\psi(\Phi(x))) = \phi(\psi(e - x)) = \phi(e - x^{-1}) = x^{-1}.$$

But Φ is continuous, and ψ is continuous, therefore the mapping $x \to x^{-1}$ is continuous.

Since the mappings $x \to x'$ and $x \to x^{-1}$ are one to one, and since these mappings are their own inverses, the mappings are in fact homeomorphisms.

SECTION 4 MAXIMAL IDEALS

In the first part of this section, certain properties of Banach algebras are discussed which are consequences of the fact that a Banach algebra is a ring. In the second part, the topological properties of maximal ideals are discussed.

Definition: If R is a ring, a subset I of R is a right ideal of R, if and only if the following are satisfied:

1) if $x \varepsilon I$ and $y \varepsilon I$ then $x - y \varepsilon I$,

2) if $x \varepsilon I$ and $z \varepsilon R$ then $xz \varepsilon I$,

3) I is a proper subset of R.

A left ideal is a subset I of R satisfying 1) and 3) and the requirement 2') if $x \varepsilon I$ and $z \varepsilon R$ then $zx \varepsilon I$. An ideal of R is maximal if and only if it is properly contained in no other ideal of R. (Maximal right and left ideals are defined analogously).

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Proposition 1: If R is a ring with identity, and I is an ideal of R, then there exists a maximal ideal M_I containing I.

Proof: Let S be the set of ideals of R containing I, partially ordered by inclusion. Let $\mathcal{C}$ be any chain in S. Then $\bigcup_{I_c \varepsilon \mathcal{C}} I_c$ is a proper ideal of R for $e \notin \bigcup_{I_c \varepsilon \mathcal{C}} I_c$. (Every I_c is proper and hence cannot contain e.) Therefore $\mathcal{C}$ is bounded above, and by Zorn's Lemma, there is a maximal element of S.

Definition: Let R be a ring, and let I be an ideal of R. An element u of R is a left identity mod I if and only if for all $x \varepsilon R$, $(ux - x) \varepsilon I$. An ideal I of a ring R is regular if and only if R has a left identity mod I.

The following proposition is proved in the same manner as Proposition 1.

Proposition 2: If R is a ring and I a regular (right) ideal, then there exists a maximal regular (right) ideal containing I.

Proof: Let u be a left identity mod I. Now $u \notin I$ for if $u \varepsilon I$, then, for all $x \varepsilon R$, $ux \varepsilon I$. But $ux - x \varepsilon I$; therefore every element of R would be an element of I. Similarly, if J is any ideal containing I then J is regular, and $u \notin J$ since u is a left identity mod J. If $\mathcal{C}$ is any chain of ideals containing I, u is not in $\bigcup_{c \varepsilon \mathcal{C}} J_c$ and $\bigcup_{c \varepsilon \mathcal{C}} J_c$ is an ideal containing I. Hence, by Zorn's Lemma, there is a maximal ideal containing I.

In an algebra with identity, maximal ideals will play a key role in the representation of a Banach algebra. The analogous role in an algebra without identity is played by the regular maximal ideals. If an algebra has an identity, then every ideal is, of course, regular, since e is a left identity modulo any ideal. There is furthermore a correspondence between the regular maximal ideals of an algebra without identity and certain of the maximal ideals of the extension of this algebra, which is discussed now.

Proposition 3: If A, an algebra without identity, is extended to A_e, an algebra with identity, then there is a one to one correspondence between the regular ideals of A and the ideals of A_e which are not subsets of A.

Proof: Let I_e be any right ideal of A_e not included in A. Then there is an element of the form $-e+x$ in I_e, where $x \in A$. Let $v = -e + x$; then x is a left identity mod I_e in A_e, since, for all $y \in A_e$, $xy - y = (x - e)y = vy \in I_e$. But $x \in A$ and, for all $y \in A$, $xy - y \in A$; hence, $I_e \cap A$ is a regular ideal in A.

If I is a regular (right) ideal in A, and u is a left identity mod I in A, define $I_e = \{y \mid uy \in I\}$. I_e is a subring of A_e, and if $x \in A_e$ then $u(yx) = (uy)x$. But $uy \in I$, and I is an ideal in A; hence, $uyx \in I$. Since $u \notin I$, $e \notin I_e$. Thus I_e is an ideal of A_e. Moreover, $u - e \in I_e$ since $(u - e)u = uu - u \in I$.

But $u - e \notin A$, thus I_e is not a subset of A.

Proposition 4: In a ring R with identity, an element x has a right inverse if and only if x lies in no maximal right ideal of R.

Proof: If x has an inverse and I is a maximal ideal containing x, then $xx^{-1} = e \in I$ and $z = ez \in I$ or $I = R$; so if x has an inverse, then x is an element of no maximal ideal.

Now if x lies in no maximal ideal then x has an inverse, for if x does not have an inverse, then $I = \{xy \mid y \in R\}$ is a right ideal, and $(e \notin I)$, and I is contained in a maximal right ideal.

Proposition 5: In any ring R an element x has a right adverse if and only if there is no regular maximal right ideal modulo which x is a left identity.

Proof: If x has a right adverse, x', and if x is a left identity modulo a maximal ideal I, then $x = xx' - x' \in I$, and for all $y \in R$, $y = xy - (xy - y) \in I$. So if x has a right adverse, then there is no regular maximal right ideal modulo which x is a left identity. If there is no maximal right ideal modulo which x is a left identity, then x has an adverse, for if x does not have an adverse, then $I = \{xy - y \mid y \in R\}$ is a regular right ideal and x is a left identity mod I. This follows from $xz - z \in I$ for any $z \in R$, and $x \notin I$, since if $x \in I$, $x = xx' - x'$ for some $x' \in R$, and x' is a right adverse of x. Now I is contained in a regular maximal right ideal modulo which x is a left identity, which contradicts the hypothesis, so x must have a right adverse.

Similar theorems concerning left inverses and adverses could be formulated, and these could be used in conjunction to lead to the statement that the set of elements having inverses is the set of elements which lie in no maximal ideal and that the set of elements having adverses is the set of elements x such that there is no maximal ideal modulo which x is a relative identity.

The following propositions assert some topological properties of maximal ideals.

Proposition 6: If I is an ideal in a Banach algebra A, then its topological closure $\overline{I}$ is an ideal of A or is A.

Proof: If $x,y \in \overline{I}$, then there is a sequence $\{x_n\} \subset I$, and a sequence $\{y_n\} \subset I$, such that $x_n \to x$ and $y_n \to y$. Then

$$x + y = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n + y_n) \in \overline{I}$$

since $x_n + y_n \in I$ for all n. Also, $xz = (\lim_{n\to\infty} x_n)z = \lim_{n\to\infty}(x_n \cdot z) \in \overline{I}$ since $x_n \in I$ for all n.

If $\overline{I} \neq A$, then $\overline{I}$ is an ideal, otherwise $\overline{I} = A$.

Now maximal ideals in rings with identity and regular maximal ideals in rings without identity are closed.

Proposition 7: If A_e is a Banach algebra with identity, and M is a maximal ideal of A_e, then M is closed.

Proof: By Proposition 6, $\overline{M}$ is an ideal and by Proposition 4, $M \subset A_e - V$ where V is the set of elements having inverses. Since V is open, $\overline{M} \subset \overline{A_e - V} = A_e - V$ which is a proper subset of A_e. Thus $\overline{M}$ is an ideal containing M; but M is maximal. Hence $\overline{M} = M$, or M is closed.

Proposition 8: If A is a Banach algebra and M is a regular maximal ideal, then M is closed.

Proof: If x is a relative identity for M, then define $\rho(M, x) = \inf \{||y - x|| \mid y \in M\}$. Now $\rho(M, x) \geqq 1$, since if there is an element $y \in M$ such that $||x - y|| < 1$, then $x - y$ has an adverse a.. Then $(x - y)a - a - (x - y) = 0$, but $(x - y)a - a - (x - y) = (xa - a) - ya + y - x \in I$, and $xa - a, ya, y \in I$, so $x \in I$. But if $x \in I$ then, since $xz - z \in I$ for all $z \in A$, it follows that $z \in I$ for all $z \in A$. So $\rho(M, x) \geqq 1$, then $\rho(\overline{M}, x) \geqq 1$. Thus, $\overline{M}$ is a regular ideal containing M, but M is maximal, so $\overline{M} = M$, or M is closed.

SECTION 5 QUOTIENTS AND FIELDS

If R is a ring and I is an ideal of R, then R/I will denote the quotient ring of R by the ideal I, and (x) will denote the coset of I containing the element x.

Proposition 1: If R is a ring and I is an ideal of R, then a subset of R/I is an ideal of R/I if and only if it is of the form J/I where J is an ideal of R containing I. Moreover, J/I is regular and/or maximal if and only if J is a regular and/or maximal ideal of R and J contains I.

Proof: J is an ideal of R containing I if and only if

1) $x + y \in J$ for all $x, y \in J$ which occurs if and only if $(x) + (y) \in J/I$ for all $(x), (y) \in J/I$,

2) $xz \in J$ for all $x \in J$, $z \in R$, which occurs if and only if $(x)\cdot(z) \in J/I$ for all $(x) \in J/I$, $(z) \in R/I$.

3) $J \neq R$ which occurs if and only if $J/I \neq R/I$.

Thus, J is an ideal of R containing I if and only if J/I is an ideal of R/I. Maximality and regularity follow in a similar manner.

Proposition 2: A commutative ring R with identity is a field if and only if there are no nonzero ideals of R.

Proof: Since R is a commutative ring, it suffices to show that every nonzero element of R has an inverse. There are no nonzero ideals of R if and only if $\{0\}$ is a maximal ideal of R. By Proposition 4 of Section 4, $\{0\}$ is a maximal ideal if and only if for all $x \in R - \{0\}$, x^{-1} exists.

Proposition 3: If R is a commutative ring and M is a regular ideal of R, then R/M is a commutative ring with identity.

Proof: Obviously R/M is a commutative ring. If u is a relative identity of R mod M, then $ux - x \in M$ for all x in R, then $(u)\cdot(x) - (x) = (0)$ for all $(x) \in R/M$, or $(u)\cdot(x) = (x)$ for all $(x) \in R/M$.

Proposition 4: If R is a commutative ring and M is a regular maximal ideal, then R/M is a field.

Proof: By Proposition 3, R/M is a commutative ring with identity. Since M is a maximal ideal, $\{(0)\}$ is the only ideal of R/M, by Proposition 1. Therefore R/M is a field.

If A is a Banach algebra and I is an ideal of A, then for any $(x) \in A/I$ let $||(x)|| = \inf \{||y|| \mid y \in (x)\}$ and let $\Phi: A \to A/I$ be defined by $x \to (x)$.

Proposition 5: If A is a Banach algebra, and I is a closed ideal of A, then A/I is a Banach algebra and $\Phi: A \to A/I$ is continuous, and $||\Phi|| \leq 1$.

Proof: For $(x) \in A/I$, (x) is a closed subset of A, since $(x) = x + I$, and the mapping, $f_x(y) = x + y$ of A to A, is a homeomorphism. Now,

1) That $||x|| \geq 0$ is clear. Now $||x|| = 0$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty} \subset I$, such that $||x_n|| \to 0$. Since (x) is closed, $||x_n|| \to 0$ if and only if $0 \in (x)$ or $(x) = I$.

2) $||(x) + (y)|| = \inf \{||x + y|| \mid x \in (x), y \in (y)\}$

$\leq \inf \{||x|| + ||y|| \mid x \in (x), y \in (y)\}$

$= \inf \{||x|| \mid x \in (x)\} + \inf \{||y|| \mid y \in (y)\}$

$= ||(x)|| + ||(y)||$

3) $||\lambda(x)|| = \inf \{||\lambda x|| \mid x \in \lambda(x)\}$

$= \inf \{|\lambda| \cdot ||x|| \mid x \in (x)\}$

$= |\lambda| \inf \{||x|| \mid x \in (x)\} = |\lambda| \cdot ||x||$

Consequently, $||x||$ is a norm.

If $\{(x)_n\}$ is a Cauchy sequence in A/I, there is a subsequence $(y)_n$ of $(x)_n$ such that $||(y)_{n+1} - (y)_n|| < 2^{-n}$. Form a sequence in A as follows: Let $y_1 \in (y)_1$. Choose y_{n+1} by noticing that since $(y)_{n+1} - y_n = (y)_{n+1} - (y)_n$, there is a $y_{n+1} \in (y_{n+1})$ such that $||y_{n+1} - y_n|| = ||(y)_{n+1} - (y)_n|| < 2^{-n}$. Then y_n is a Cauchy sequence, and if $y_0 = \lim_{n\to\infty} y_n$, then (y_0) $= \lim_{n\to\infty} (y)_n = \lim_{n\to\infty} (x)_n$ since (x_n) is a Cauchy sequence and (y_n) is a subsequence of $(x)_n$. Consequently, A/I is complete.

The norm behaves properly under multiplication since

$||(x)\cdot(y)|| = \inf \{||xy|| \mid x \in (x), y \in (y)\}$

$\leq \inf \{||x|| \cdot ||y|| \mid x \in (x), y \in (y)\}$

$= \inf \{||x|| \mid x \in (x)\} \cdot \inf \{||y|| \mid y \in (y)\}$

$= ||(x)|| \cdot ||(y)||.$

Thus, A/I is a Banach algebra.

Now, for all $x \in A$, $||(x)|| = \inf \{||y|| \mid y \in (x)\}$ $\leq ||x||$ or $||\Phi|| \leq 1$ and Φ is continuous.

Now if M is a regular maximal ideal of a commutative Banach algebra, then A/M is a field which is also a Banach algebra, (Propositions 4 and 5). In Section 7 this normed field will be shown to be isometrically isomorphic to the field of complex numbers. In order to prove this isomorphism, some results concerning analytic functions will be required.

SECTION 6 ABSTRACT ANALYTIC FUNCTIONS

Definition: A subset D of the complex plane C is a region of C if and only if it is an open connected subset of C.

Definition: A function $\Gamma:[0,1] \to C$ is a simple closed curve if and only if both of the following are true:

1) Γ is continuous
2) $\Gamma(x_1) = \Gamma(x_2)$ if and only if $x_1 = 0$ and $x_2 = 1$.

If Γ is a simple closed curve, then the image of Γ will be denoted by Γ^*.

Definition: If D is a region of C and Γ is a simple closed curve such that $\Gamma^* \subset D$ then Γ is said to be a simple closed curve on D.

Definition: A simple closed curve Γ is a closed path if and only if there are finitely many points $0 = s_0 < s_1 < \ldots < s_n = 1$

such that for each interval $[s_{j-1}, s_j]$, $j = 1, 2, \ldots n$,

Γ has a continuous derivative on $[s_{j-1}, s_j]$.

Definition: If D is a region and Γ is a closed path on D and

$\lambda \in D - \Gamma^*$, then the number $\frac{1}{2\pi i} \int_\Gamma \frac{d\,\xi}{\xi - \lambda}$ is called the

index of λ and is written Ind (λ).

The set of all such λ that Ind $(\lambda) = 1$ is called the region enclosed by Γ.

Definition: A function Φ from a region D to a Banach algebra A $(\Phi: D \to A)$ is analytic on D if and only if for all $\lambda \in D$,

$\lim_{|h| \to 0} \frac{\Phi(\lambda+h) - \Phi(h)}{h}$ exists, where the limit is taken in the sense of convergence in the norm of A.

If Φ is analytic on a region D then the above limit is called the derivative of Φ at the point λ. (The derivative is an element of A).

Higher order derivatives are defined in a manner analogous to the usual definition. The symbol $\Phi^{(n)}(\lambda)$ will be used for the nth derivative of Φ at the point λ.

Definition: If Γ is a closed path on a region D, then a subset $\{\lambda_0 \ldots \lambda_n\}$ of Γ^* is a partition of Γ if and only if there exist

$0 = t_0 \leqq t_0 \leqq t \quad \ldots\ldots\ldots \leqq t_n = 1$ such that $\Gamma(t_i) = \lambda_i$ for $i = 0, 1, \ldots n$.

Definition: If $\{\lambda_o, \lambda_1, \ldots . \lambda_n\}$ is a partition of a closed path Γ on a region D then a point λ_k' is said to be between λ_k and λ_{k+1} if and only if there is a t_k' such that

$t_k \leqq t_k' \leqq t_{k+1}$ and $\Gamma(t_k') = \lambda_k'$.

Definition: If A is a Banach algebra, D is a region, $\Phi: D \to A$, Γ is a closed path on D and $I_\Gamma(\Phi)$ exists where

$$I_\Gamma(\Phi) = \lim_{\max|\lambda_{k+1} - \lambda_k| \to 0} \sum_{k=0}^{n-1} \Phi(\lambda_k')\ (\lambda_{k+1} - \lambda_k)$$

$\{\lambda_o, \lambda_1, \lambda_n\}$ is any partition of Γ and λ_k' is between λ_k and λ_{k+1} and where the limit is taken in the sense of convergence in the norm of A, then the above limit is called the integral of Φ over Γ and is written $\int_\Gamma \Phi(\lambda) d\lambda$. (Define $\int_\Gamma \Phi(\lambda)|d\lambda|$ analogously.)

The above generalized definitions of the derivative and the integral of a function from C to a Banach algebra A permit the development of a theory of abstract analytic functions. This theory parallels very closely the theory of analytic functions of a complex variable. In particular, a useful technique in proving a given theorem for an abstract analytic function will be to shift the range of the function to the complex numbers and use the analogous regular theorem.

If D is a region of the complex plane, A is a Banach algebra, Γ is a closed path on D, and $\Phi: D \to A$ is continuous, then since Γ^* is compact, Φ is uniformly continuous on Γ^*. The existence and uniqueness of the integral over Γ of such a continuous function follows in the same manner as the existence and uniqueness of the ordinary Riemann integral. The linearity of the integral is also straightforward, as is the inequality $\left\|\int_\Gamma \Phi(\lambda)d\lambda\right\| \leq \int_\Gamma \|\Phi(\lambda)\| \, |d\lambda|$.

The next two propositions establish the validity of the technique mentioned above since C, the complex numbers, is itself a Banach algebra.

Proposition 1: If A and A′ are Banach algebras, D is a region of C, $\Phi:D \to A$ is analytic on D, and $f:A \to A'$ is a bounded linear function, then $f\circ\Phi$ is analytic on D, and $(f\circ\Phi)' = f\circ\Phi'$.

Proof: $$\frac{f(\Phi(\lambda + h) - f(\Phi(\lambda))}{h} = f\left(\frac{\Phi(\lambda + h) - \Phi(\lambda)}{h}\right).$$

Let $\Phi'(\lambda) = \lim\limits_{|h|\to 0} \dfrac{\Phi(\lambda + h) - \Phi(\lambda)}{h}$. Since f is continuous

$$\lim_{|h|\to 0} \frac{f(\Phi(\lambda + h)) - f(\Phi(\lambda))}{h}$$

$$= \lim_{|h|\to 0} f\left(\frac{\Phi(\lambda + h) - \Phi(\lambda)}{h}\right)$$

$$= f\left(\lim_{|h|\to 0}\left(\frac{\Phi(\lambda + h) - \Phi(\lambda)}{h}\right)\right) = f(\Phi'(\lambda))$$

for any $\lambda \in D$.

Thus $f \circ \Phi$ is analytic on D and

$$(f \circ \Phi)'(\lambda) = f(\Phi'(\lambda)) = (f \circ \Phi')(\lambda).$$

Proposition 2: If A and A′ are Banach algebras, D is a region of C, Γ is a closed path on D, $\Phi : D \to A$ is continuous, and $f : A \to A'$ is a bounded linear function, then

$\int_\Gamma f(\Phi(\lambda))d\lambda$ exists and $\int_\Gamma f(\Phi(\lambda))d\lambda = f\left(\int_\Gamma \Phi(\lambda)d\lambda\right).$

Proof: The composition $f \circ \Phi : D \to A'$ is continuous; therefore, $\int_\Gamma f(\Phi(\lambda))d\lambda$ exists. For any partition of Γ,

$$\sum_{k=0}^{n-1} f(\Phi(\lambda'_k))(\lambda_{k+1} - \lambda_k)$$

$$= f\left(\sum_{k=0}^{n-1} \Phi(\lambda_k')(\lambda_{k+1} - \lambda_k)\right)$$

since f is linear. Since f is continuous

$$\int_\Gamma f(\Phi(\lambda))d\lambda = f\left(\int_\Gamma \Phi(\lambda)d\lambda\right).$$

Theorem (Cauchy): If A is a Banach algebra, D is a region of C, Γ is a closed path on D, and $\Phi : D \to A$ is analytic on D, then $\int_\Gamma \Phi(\lambda)d\lambda = 0$.

Proof: $\int_\Gamma \Phi(\lambda)d\lambda$ exists. Suppose $\int_\Gamma \Phi(\lambda)d\lambda = y \neq 0$. By the Hahn-Banach theorem, there is a bounded linear transformation $f : A \to C$ such that $f(y) \neq 0$. By Proposition 1, $f \circ \Phi : D \to C$ is analytic on D.

By the ordinary Cauchy's Theorem, $\int_\Gamma (f\circ\Phi)(\lambda)d\lambda = 0$. But by Proposition 2, $\int_\Gamma (f\circ\Phi)(\lambda)d\lambda = f(\int_\Gamma \Phi(\lambda)d\lambda) = f(y) \neq 0$. This is a contradiction, so $\int_\Gamma \Phi(\lambda)d\lambda = 0$.

Cauchy's Integral Formula: If A is a Banach algebra, D is a region of C, Γ is a closed path on D, $\Phi:D \to A$ is analytic on Γ and λ is inside of Γ, then

$$\Phi(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(\xi)d\xi}{\xi-\lambda}.$$

Proof: Let $y = \Phi(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(\xi)d\xi}{\xi-\lambda}$. Suppose $y \neq 0$. By the Hahn-Banach theorem there is a bounded linear functional $f:A \to C$ such that $f(y) \neq 0$. By Proposition 1, $f\circ\Phi$ is an ordinary analytic function on D. By the ordinary Cauchy's Integral Formula

$$0 = (f\circ\Phi)(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{(f\circ\Phi)(\lambda)\ d\xi}{\xi - \lambda}$$

$$= f(\Phi(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(\lambda)\ d\xi}{\xi - \lambda}) = f(y) \neq 0.$$

But this is a contradiction. Thus the formula is valid.

Proposition 3: If A is a Banach algebra, D is a region of C, Γ is a closed path on D, $\Phi:D \to A$ is continuous on Γ^*, then the function $F_n(\lambda) = \int_\Gamma \frac{\Phi(\xi)d\xi}{(\xi-\lambda)^n}$ is analytic on the region enclosed by Γ and its derivative $F_n'(\lambda) = nF_{n-1}(\lambda)$.

Proof: The analycity of F_n follows in the same manner as in the regular case replacing absolute values by norms, cf. Ahlfors [1, p. 97]. The validity of the formula can be verified as follows: Let $y = F_n'(\lambda) - nF_{n+1}(\lambda)$. If $y \neq 0$ by the Hahn-Banach theorem there is a bounded linear functional $f: A \to C$ such that $f(y) \neq 0$. Then

$$0 \neq f(y) = f(F_n'(\lambda) - nF_{n+1}(\lambda))$$
$$= (f \circ F)'(\lambda) - n(f \circ F)(\lambda) = 0.$$

But this is a contradiction.

The above formula establishes the validity of the formula for derivatives of analytic functions.

$$\phi^{(n)}(\lambda) = \frac{n!}{2\pi i} \int_\Gamma \frac{\Phi(\xi)\,d\xi}{(\xi - \lambda)^n}$$

where λ is a point enclosed by the closed path Γ on the region D.

Theorem (Liouville): If $\Phi: C \to A$ is analytic and bounded on C, the whole plane, ($||\Phi(\lambda)|| \leq M$ for all $\lambda \in C$), then there is an $x \in A$ such that $\Phi(\lambda) = x$ for all $\lambda \in C$.

Proof: Suppose Φ is not a constant, then there are $\lambda_1, \lambda_2 \in C$ such that $\Phi(\lambda_1) \neq \Phi(\lambda_2)$. By the Hahn-Banach theorem there is a bounded linear functional $f: A \to C$ such that $f(\Phi(\lambda_1)) \neq f(\Phi(\lambda_2))$. Yet again this leads to a contradiction of the regular Liouville Theorem, since for all $\lambda \in C$

$|(f \circ \Phi)(\lambda)| \leqq ||f|| \cdot ||\Phi|| \leqq ||f|| \cdot M$ and

$0 \neq f(\Phi(\lambda_1) - f(\Phi(\lambda_2) = (f \circ \Phi)(\lambda_1) - (f \circ \Phi)(\lambda_2) = 0.$

√

Proposition 4: If A_e is a Banach algebra with identity, and $x \in A$ then the mapping $\Phi_x(\lambda) = (x - \lambda e)^{-1}$ is analytic on the points of C where it is defined.

Proof: When $|\lambda| > ||x||$, x/λ has an adverse and $x/\lambda - e$ has an inverse. Thus when $|\lambda| > ||x||$, $x - \lambda e$ has an inverse. Thus there is a λ for which $\Phi_x(\lambda)$ is defined. The mapping $\lambda \to x - \lambda e$ is continuous as is the mapping $(x - \lambda e) \to (x - \lambda e)^{-1}$. Thus the set of points where the mapping is defined is open. Moreover

$$\frac{\Phi_x(\lambda + h) - \Phi_x(\lambda)}{h} = \frac{(x - (\lambda + h)e)^{-1} - (x - \lambda e)^{-1}}{h}$$

$$= (x - (\lambda + h)e)^{-1} (x - \lambda e)^{-1}$$

But the mapping $x \to x^{-1}$ is continuous, so

$$\lim_{|h| \to 0} \frac{\Phi(\lambda + h) - \Phi(\lambda)}{h} = \lim_{|h| \to 0} (x - (\lambda + h)e)^{-1} (x - \lambda e)^{-1}$$

$$= (x - \lambda e)^2.$$

SECTION 7 HOMOMORPHISMS AND ISOMORPHISMS

Definition: An algebra homomorphism is a ring homomorphism which preserves scalars, i.e.: $\Phi(\lambda x) = \lambda\Phi(x)$.

The following proposition establishes the key to Gel'fand's theorem.

Proposition 1: If A is a commutative Banach algebra, and M is a regular maximal ideal of A, then A/M is isometrically isomorphic to C, the field of complex numbers.

Proof: Recall that by Proposition 4 of Section 5 that A/M is a field. It is sufficient to show that for any $x \in A/M$ there is a $\lambda \in C$ such that $x = \lambda e$. Suppose, on the contrary, that there is an $x \in A/M$ such that for all $\lambda \in C$, $x \neq \lambda e$. Then $x - \lambda e$ is never zero and therefore $(x - \lambda e)^{-1}$ exists for all $\lambda \in C$. By Proposition 4 of Section 6, the mapping $\Phi_x(\lambda) = (x - \lambda e)^{-1}$ is analytic on C. Notice that $\lambda^{-1} \to 0$ as $\lambda \to \infty$ and $(x\lambda^{-1} - e) \to -e$ as $\lambda \to \infty$. Thus, since

$$||\Phi_x(\lambda)|| = ||(x - \lambda e)^{-1}|| = |\lambda^{-1}|\cdot||(x\lambda^{-1} - e)^{-1}|| \to 0$$

as $\lambda \to \infty$, $\Phi_x(\lambda)$ is bounded. By Liouville's theorem $\Phi_x(\lambda) = x_o$, a constant. Since $||x_o|| \to 0$ as $\lambda \to \infty$, $x_o = 0$ or $(x - \lambda e)^{-1} = 0$, a contradiction. Consequently, A/M is isomorphic to C.

Proposition 2: If $\Phi:A \to C$ is a homomorphism of a commutative Banach algebra A onto C, then $\Phi^{-1}(0)$ is a regular maximal ideal of A.

Proof: $\Phi^{-1}(0)$ is a regular ideal since an element of A whose image is 1 is a relative identity mod $\Phi^{-1}(0)$. But $A/[\Phi^{-1}(0)]$ is a field. Therefore, there are no nontrivial ideals of $A/[\Phi^{-1}(0)]$ or $[\Phi^{-1}(0)]$ is a maximal ideal of A.

Since the kernel of any homomorphism of a commutative Banach algebra onto the complex numbers is a regular maximal ideal, the fundamental homomorphism theorem for rings and Proposition 1 show that any such homomorphism is continuous and has norm less than or equal to 1. Further, by Propositions 1 and 2, there is a one to one correspondence between the regular maximal ideals of a commutative Banach algebra and the homomorphisms of the algebra to the complex numbers.

SECTION 8 THE GEL´FAND REPRESENTATION

From this section on, only commutative Banach algebras will be discussed. In any statement the term Banach algebra will be understood to mean commutative Banach algebra.

Let $\mathfrak{m}$ be the set of all regular maximal ideals of a Banach algebra A; let Δ be the set of all (continuous) homomorphisms of A onto the complex numbers C. Since there is a one to one correspondence between the set of regular maximal ideals and the continuous homomorphisms of A onto C, let h_M be the homomorphism corresponding to the maximal ideal M, and let M_h be the maximal ideal corresponding to the homomorphism h.

In this section, a normed function algebra which is homomorphic to A will be constructed. The domain of the functions will be $\mathfrak{M}$, (or equivalently Δ). If $x \in A$, define $\hat{x}: \mathfrak{M} \to C$, $(\hat{x}: \Delta \to C)$, by $\hat{x}(M_h) = \hat{x}(h_M) = h_M(x)$.

Let $\hat{A}$ be the set of all such functions. The function $\hat{x}$ will be called the Gel'fand representation of the element $x \in A$. The mapping $A \to \hat{A}$, $(x \to \hat{x})$ will be called the Gel'fand representation of A.

$\hat{A}$ is an algebra of complex valued functions when addition, multiplication, and scalar multiplication are defined pointwise.

Proposition 1: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A, then the Gel'fand representation is a homomorphism.

Proof: For any $h \in \Delta$ and any $x,y \in A$ and any $\lambda \in C$

$$\widehat{x + y}\,(h) = h(x + y) = h(x) + h(y) = \hat{x}(h) + \hat{y}(h)$$

$$\widehat{\lambda x}\,(h) = h(\lambda x) = \lambda h(x) = \lambda \hat{x}(h)$$

$$\widehat{xy}\,(h) = h(xy) = h(x)h(y) = \hat{x}(h)\hat{y}(h)$$

But h was any point in Δ, so $\widehat{x + y} = \hat{x} + \hat{y}$, $\widehat{\lambda x} = \lambda\hat{x}$, and $\widehat{xy} = \hat{x}\hat{y}$.

Thus the mapping $A \to \hat{A}$, the Gel'fand representation of A, is a homomorphism.

Proposition 2: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A, then the functions of $\hat{A}$ are bounded.

Proof: Let $\hat{x} \in A$. There is an $x \in A$ such that $x \to \hat{x}$. Now for all $h \in \Delta$, $||h|| \leq 1$ by Section 7. Now $\hat{x}$ is bounded since

$|\hat{x}(h)| = |h(x)| \leq ||x||$.

Since the functions of $\hat{A}$ are bounded, let

$$||\hat{x}|| = \sup \{|\hat{x}(h)| \;|h \in \Delta\}.$$

A is a normed linear space and the mapping $x \to \hat{x}$ is norm decreasing (hence continuous). In Section 10 conditions under which the Gel′fand representation is an isomorphism are discussed. In Section 11 conditions under which the algebra is complete are discussed.

The functions of A can be used to generate a topology on the set $\mathfrak{M}$. Let $\mathcal{T}_w$ be the weak topology generated by the functions of A. The sets of the form

$$U_{\hat{x},\varepsilon,M_o} = \{M \in \mathfrak{M} \mid |\hat{x}(M) - \hat{x}(M_o)| < \varepsilon\}$$

for $x \in A$, $\varepsilon > 0$, $M_o \in \mathfrak{M}$ are a subbase for $\mathcal{T}_w$. Since A is also an algebra of functions on Δ, the weak topology may be induced on Δ. A subbasic open set for the weak topology is of the form

$$U_{\hat{x},\varepsilon,h_o} = \{h \in \Delta \mid |\hat{x}(h) - \hat{x}(h_o)| < \varepsilon\}.$$

Since the subbasic open sets of $\mathfrak{M}$ and Δ are exactly the same under the one to one correspondence, $h_M \leftrightarrow M_h$, this correspondence is a homeomorphism. Since for all $\hat{x} \in \hat{A}$, $\hat{x}(h_M) = \hat{x}(M_h)$, the algebra A behaves exactly the same under the correspondence. The spaces $\mathfrak{M}$ and Δ will be used interchangeably.

The space $\mathfrak{M}$ with the above topology will be called the maximal ideal space of a Banach algebra A; the space Δ will be called the space of continuous homomorphisms of a Banach algebra A.

The following propositions demonstrate that $\mathfrak{M}$ is either a compact or a locally compact Hausdorff space.

Proposition 3: $\mathfrak{M}$ is a Hausdorff space under $\mathcal{T}_w$.

Proof: Suppose $M_1, M_2, \in \mathfrak{M}$, $M_1 \neq M_2$. Then there is an $x \in A$ such that $x \in M_1$ and $x \notin M_2$, (or vice versa). Then $\hat{x}(M_1) = 0 \neq \hat{x}(M_2)$

Let $U_{\hat{x},\varepsilon,M_1} = \{M \in \mathfrak{M} \mid |\hat{x}(M) - \hat{x}(M_1)| < \frac{|\hat{x}(M_2)|}{3}\}$

$U_{\hat{x},\varepsilon,M_2} = \{M \in \mathfrak{M} \mid |\hat{x}(M) - \hat{x}(M_2)| < \frac{|\hat{x}(M_2)|}{3}\}$

Now $U_{\hat{x},\varepsilon,M_1}$ is a neighborhood of M_1, and $U_{\hat{x},\varepsilon,M_2}$ is a neighborhood of M_2. Furthermore, $U_{\hat{x},\varepsilon,M_1} \cap U_{\hat{x},\varepsilon,M_2} = \phi$, for suppose M_o is in the intersection.

This leads to a contradiction since

$$|\hat{x}(M_2)| = |\hat{x}(M_2) - \hat{x}(M_1)| \leq |\hat{x}(M_2) - x(M_o)| + |\hat{x}(M_o) - \hat{x}(M_1)|$$

$$\leq \frac{|\hat{x}(M_2)|}{3} + \frac{|\hat{x}(M_2)|}{3}.$$

Thus, $\mathfrak{m}$ is a Hausdorff topological space.

A homomorphism of a Banach algebra to the complex numbers is automatically a bounded linear functional, hence a member of A^*, the set of bounded linear functions on A viewed as a Banach space. Since any homomorphism has norm less than or equal to 1, any set of homomorphisms is a subset of the unit sphere $S \subset A^*$, $S = \{x^* \varepsilon A^* \mid ||x^*|| \leq 1\}$.

The weak* topology on A^* is the weak topology induced on A^* by the elements of A when they are imbedded in A^{**}. It is the topology of pointwise convergence of elements of A^*, i.e. if $\{f_n\}$ is a sequence in A^* and f_o is the weak* limit of $\{f_n\}$, then for each $\varepsilon > o$ and for each $x \varepsilon A$ there is N such that for $n > N$, $|f_n(x) - f_o(x)| < \varepsilon$. The norm topology on the other hand is the topology of uniform convergence. The fact that the unit sphere is compact in the weak* topology will be used to show that $\mathfrak{m}$ is either compact or locally compact.

Proposition 4: If H is a set of homomorphisms of a Banach algebra A to the complex numbers C, and h is an element in the closure of H, as a subset of A^* with the weak* topology, then h is a homomorphism of A to C.

Proof: Since $h \varepsilon A^*$, h is linear. It is sufficient to prove that $h(xy) = h(x)h(y)$ for all $x,y \varepsilon A$. If x and y are elements of A and $\varepsilon > 0$, there is an $h_o \in H$ such that

$$|h(x) - h_o(x)| < \varepsilon$$

$$|h(y) - h_o(y)| < \varepsilon$$

$$|h(xy) - h_o(xy)| < \varepsilon.$$

Then $|h(x)h(y) - h(xy)|$

$$\leqq |h(x)h(y) - h_o(x)h(y)| + |h_o(x)h(y) - h_o(x)h_o(y)| + |h_o(xy) - h(xy)|$$

$$\leqq |h(y)|\cdot|h(x) - h_o(x)| + |h_o(x)|\cdot|h(y) - h_o(y)| + |h_o(xy) - h(xy)|$$

$$\leqq |h(y)|\cdot \varepsilon + |h(x)|\cdot \varepsilon + \varepsilon \leqq \varepsilon\cdot(|h(y)| + ||x|| + 1).$$

Thus for all $\varepsilon > 0$, $|h(x)h(y) - h(xy)| \leqq \varepsilon\ (|h(y)| + ||x|| + 1)$.

Consequently, $h(x)h(y) - h(xy) = 0$, or $h(x)h(y) = h(xy)$, so h is a homomorphism.

If the set H in Proposition 4 is Δ, the collection of all homomorphisms of A onto the complex numbers, then the only other homomorphism of A to the complex numbers is the zero homomorphism, which sends every element to zero. The set Δ is either closed, or the closure of Δ consists of Δ and only one additional point, 0.

If the elements of A are imbedded in the second conjugate space of A, A^{**}, then the weak topology of Δ induced by $\hat{A}$ is a subset of the weak* topology of A^*induced on Δ, $\mathcal{T}_{w^*}$. If Δ is compact (locally compact) under the weak* topology, then, since $\mathcal{T}_w \subset \mathcal{T}_{w^*}$, Δ is compact (locally compact) under the weak topology, $\mathcal{T}_w$.

Proposition 5: If A is a Banach algebra, $\mathcal{M}$, (Δ), the maximal ideal space of A endowed with the weak topology induced on $\mathfrak{M}$ by the functions of the Gel'fand representation $\hat{A}$, then $\mathfrak{M}$ is locally compact. If A has an identity, $\mathfrak{M}$ is compact. In any case, the functions of $\hat{A}$ vanish at infinity.

Proof: Since $\bar{\Delta}$ is a closed subset of the unit sphere, $\bar{\Delta}$ is compact under $\mathcal{T}_{w^*}$. If $\Delta = \bar{\Delta}$ then Δ is compact under $\mathcal{T}_{w^*}$; otherwise $\bar{\Delta} = \Delta \cup \{0\}$ is a compact Hausdorff space under $\mathcal{T}_{w^*}$; if $\bar{\Delta} = \Delta - \{0\}$, then Δ is locally compact under $\mathcal{T}_{w^*}$. Since Δ is compact or locally compact under $\mathcal{T}_{w^*}$, and $\mathcal{T}_w \subset \mathcal{T}_{w^*}$, Δ is compact or locally compact under $\mathcal{T}_w$. If A has an identity, then $h(e) = 1$ for all $h \in \Delta$. Then if h_o is the zero homomorphism, $U_{e,1/2,0} = \{x^* \in A^* \mid |x^*(e)| < 1/2\}$ is a weak* neighborhood of 0, which does not intersect Δ, or Δ is weak* closed. Then Δ is weak* compact and hence weakly compact.

If Δ is locally compact but not compact, the zero homorphism corresponds to the point at infinity in the Alexandroff one-point compactification, and if $x \in A$ and $\varepsilon > 0$, then $U_{x,\varepsilon,0} = \{M \in \mathfrak{M} \mid |\hat{x}(M)| < \varepsilon\}$ is weakly open in $\bar{\Delta}$, hence

$\bar{\Delta} - U_{\hat{x},\varepsilon,0}$ is weak* closed and weak* compact. But $\bar{\Delta} - U_{\hat{x},\varepsilon,0} = \{M \in \mathfrak{M} \mid |\hat{x}(M)| \geq \varepsilon\}$ so $\hat{x}$ vanishes at infinity.

The results of this section can be restated as in the following.

Theorem (Gel´fand): Any commutative Banach algebra is homomorphic to an algebra of continuous functions on a locally compact Hausdorff space. If the algebra has an identity, the space is compact; if the algebra does not have an identity, the functions vanish at infinity.

Henceforth, $\mathfrak{M}_0$ or Δ_0 will denote the one point compactification of $\mathfrak{M}$ or Δ and the functions of $\hat{A}$ will be continuously extended by $\hat{x}(\infty) = 0$.

SECTION 9 THE SPECTRUM AND THE SPECTRAL NORM

Definition: If A_e is a Banach algebra with identity, and λ is a complex number, then λ is in the spectrum of x if and only if $(x - \lambda e)$ does not have an inverse.

If A is a Banach algebra without identity, then a complex λ is in the spectrum of x if and only if $(x - \lambda e)^{-1}$ does not exist when A is extended to a Banach algebra with identity. The spectrum of an element, x, of a Banach algebra is the set of all complex numbers λ such that λ is in the spectrum of x. The spectrum of an element x will be denoted $\sigma(x)$.

If A_e is an algebra with identity, then by Proposition 4 of Section 4, $\lambda \notin \sigma(x)$ if and only if $(x - \lambda e)$ lies in no maximal ideal of A_e. In particular, an element, x, has an inverse if and only if $0 \notin \sigma(x)$. In a Banach algebra without identity a correspondence between the spectrum of an element and adverses is established below.

Proposition 1: If A is a commutative Banach algebra without identity, and $x \in A$, then $0 \in \sigma(x)$ and $0 \neq \lambda \in \sigma(x)$ if and only if x/λ does not have an adverse in A.

Proof: First, $0 \in \sigma(x)$, for suppose $y + \lambda e$ is an inverse of $x + 0e = x$ in A_e, the extension of A, then $e = x(y + \lambda e) = xy + \lambda x \in A$, a contradiction. If $\lambda \neq 0$, then $(x - \lambda e) = \lambda(x/\lambda - e)$ and $\lambda \notin \sigma(x)$ if and only if $(x - \lambda e)^{-1}$ exists. But $(x - \lambda e)^{-1} = \lambda^{-1}(x/\lambda - e)^{-1}$exists if and only if x/λ has an adverse.

Another extremely useful characterization of the spectrum of an element is the following.

Proposition 2: If A is a commutative Banach algebra and $x \in A$, then the spectrum of an element x corresponds to the range of $\hat{x}$, the Gel'fand representation of x. $(\sigma(x) = \hat{x}(\mathfrak{M}_0))$.

Proof: A complex number λ is in $\sigma(x)$ if and only if $x - \lambda e$ lies in some maximal ideal M. This occurs if and only if $h_M(x - \lambda e) = 0$, or $h_M(x) = \lambda$. Thus $\lambda \in \sigma(x)$ if and only if there is an $M \in \mathfrak{M}_0$ such that $\hat{x}(M) = \lambda$.

Using the results of Proposition 2, the spectrum of an element is compact, since it is the continuous image of a compact set. Moreover, the spectrum is closed and bounded.

Definition: If A is a BAnach algebra, $x \varepsilon A$, then the spectral norm $||x||_{sp}$ is defined by $||x||_{sp} = \sup \{|\lambda| \mid \lambda \varepsilon \sigma(x)\}$

By Proposition 2, $||x||_{sp} = \sup\{|\lambda| \mid \lambda \varepsilon \sigma(x)\}$

$= \sup \{|\lambda| \mid \lambda = x(M) \text{ for some } M \varepsilon \mathfrak{M}_o\} = ||\hat{x}||_\infty$.

Since $||x||_{sp} = ||\hat{x}||$, and $||\hat{x}|| \leqq ||x||$, $||x||_{sp} \leqq ||x||$.

Proposition 2 also leads immediately to the following conclusion.

Proposition 3: If A is a Banach algebra, and $x \varepsilon A$, then $\sigma(x^n) = (\sigma(x))^n$.

Proof: $\sigma(x^n) = \widehat{x^n}(\mathfrak{M}_o)$; since $x \to \hat{x}$ is a homomorphism,

$$\widehat{x^n}(\mathfrak{M}_o) = \hat{x}^n(\mathfrak{M}_o) = (\hat{x}(\mathfrak{M}_o))^n = (\sigma(x))^n.$$

The spectral norm of an element x of a Banach algebra is related to the norm of the element by the following.

Proposition 4: If A is a commutative Banach algebra and $x \varepsilon A$ then

$$||x||_{sp} = \lim_{n\to\infty} ||x^n||^{1/n}$$

Proof: For all n, Proposition 3 shows that

$||x||^n_{sp} = ||x^n||sp$, but $||x^n||sp \leqq ||x^n||$. Thus $||x||^n_{sp} \leqq ||x^n||$, or $||x||_{sp} \leqq ||x^n||^{1/n}$ for all n. Therefore $||x||_{sp} \leqq \underline{\lim}_{n\to\infty} ||x^n||^{1/n}$

It will be shown that for all but a finite number of n's, if a is such that $||x||_{sp} \leqq a$, then $||x^n||^{1/n} \leqq a$. Define $\Phi_x(\lambda) = (x - \lambda e)^{-1}$ (Embedding A in an algebra with identity if necessary). The function Φ_x is defined for all λ such that $|\lambda| > ||x||_{sp}$. Let $D = \{\lambda | \lambda \in C,\ |\lambda| \geqq ||x||_{sp}\}$.

Then $\Phi_x : D \to A$ is an abstract analytic function on D.

$$\Phi_x(\lambda) = -\lambda^{-1}(e - x/\lambda)^{-1} = -\lambda^{-1}(e + \sum_{n=1}^{\infty} (x/\lambda)^n)$$

Now if f is any bounded linear functional on A, $f(\Phi_x(\lambda))$ is analytic on D and

$$f(\Phi_x(\lambda)) = -\lambda^{-1}(f(1) + \sum_{n=1}^{\infty} f(\frac{x^n}{\lambda^n})$$

$$= -\lambda^{-1}(f(1) + \sum_{n=1}^{\infty} \lambda^{-n} f(x^n)) \text{ for all } |\lambda| \geqq ||x||_{sp}.$$

Now for any α such that $||x||_{sp} < \alpha < a$, $\sum_{n-1}^{\infty} f(x^n/\alpha^n)$ converges for any $f \in A^*$. So by the uniform boundedness principle $||x^n/\alpha^n||$ is bounded for all n by say K. Thus $||x^n|| \leq K\alpha^n$ or $||x^n||^{1/n} \leq K^{1/n}\alpha$. But as $n \to \infty$, $K^{1/n} \to 1$ and for all but a finite number of the n's, $K^{1/n} < a$, or $||x^n||^{1/n} < a$.

But a was any number such that $||x||_{sp} < a$, so

$\overline{\lim_{n\to\infty}} \; ||x^n||^{1/n} \leq ||x||_{sp}$.

Thus $||x||_{sp} = \lim_{n\to\infty} ||x^n||^{1/n}$.

SECTION 10 THE RADICAL AND SEMI-SIMPLICITY

Definition: An element x of a Banach algebra is a generalized nilpotent if and only if $\lim_{n\to\infty} ||x^n||^{1/n} = 0$.

Definition: The set of all generalized nilpotents of a Banach algebra A is the radical of A.

An equivalent definition of the radical is the following.

Proposition 1: The intersection of all regular maximal ideals of a commutative Banach algebra A is the radical of A.

Proof: An element x of A is in the radical of A if and only if

$\lim_{n\to\infty}||x^n||^{1/n} = 0$. But

$$\lim_{n\to\infty}||x^n||^{1/n} = ||x||_{sp} = ||\hat{x}|| = 0$$

if and only if $\hat{x}(M) = 0$ for all $M \in \mathfrak{M}$. But $\hat{x}(M) = 0$ for all $M \in \mathfrak{M}$ if and only if $h_M(x) = 0$ for all $M \in \mathfrak{M}$, or, equivalently, $x \in M$ for all $M \in \mathfrak{M}$.

Definition: A Banach algebra A is semi-simple if and only if the radical of A is zero (contains only the zero element).

Proposition 2: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A then $\hat{A}$ is isomorphic to A/radical(A).

Proof: If $\Phi: A \to \hat{A}$ is the Gel'fand representation of A, then

$$\ker(\Phi) = \{x \in A \mid \hat{x} = 0\} = \{x \in A \mid x(M) = 0 \text{ for all } M \in \mathfrak{M}\}$$
$$= \{x \in A \mid x \in M \text{ for all } M \in \mathfrak{M}\} = \text{radical}(A).$$

Thus, $\hat{A}$ is isomorphic to $A/\ker(\Phi)$ = A/radical(A).

Proposition 3: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A, then $\hat{A}$ is isomorphic to A if and only if A is semi-simple.

Proof: Now $\hat{A}$ is isomorphic to A if and only if $\ker(\Phi) = 0$, but $\ker(\Phi)$ = radical(A). So $\hat{A}$ is isomorphic to A if and only if radical(A) = 0, or A is semi-simple.

The results of this section can be summarized as:

In a commutative Banach algebra the following conditions are equivalent:

1) A is semi-simple

2) The radical of A is zero

3) A contains no generalized nilpotents

4) A is isomorphic to $\hat{A}$.

Proposition 4: If A is a commutative Banach algebra and $\hat{A}$ is the Gel´fand representation of A, then $\hat{A}$ is semi-simple.

Proof: If $\hat{x}_o$ is in $\hat{A}$ and $\hat{x}_o \neq 0$, then there is an $M_o \in \mathfrak{M}$ such that $\hat{x}(M_o) \neq 0$. The mapping $h_{M_o}: \hat{A} \to C$, determined by $h_{M_o}(\hat{x}) = \hat{x}(M_o)$ is a homomorphism of $\hat{A}$ onto the complex numbers. The kernel of h_{M_o} is a regular maximal ideal and $\hat{x}_o \notin \ker(h_{M_o})$. But $\hat{x}_o$ was any nonzero element of $\hat{A}$; thus $\hat{A}$ is semi-simple.

11) ISOMETRIES OF A AND $\hat{A}$

In this section, necessary and sufficient conditions that the Gel´fand representation be a homeomorphism are developed.

Definition: If A_1 and A_2 are Banach spaces and T is a linear function from A_1 to A_2 then T* the adjoint of T is the mapping from A_2^* to A_1^* defined by $T^*(x_2^*)$ for $x_2^* \in A_2^*$ is the element $x_1^* \in A_1^*$ determined by

$$x_1^*(y) = (T^*(x_2^*)(y) = x_2^*(T(y))$$

for each $y \in A_1$.

Proposition 1: If A_1 and A_2 are commutative Banach algebras and T is an algebra homomorphism of A_1 onto a dense subset of A_2, then the adjoint T* defines a homeomorphism of $\mathfrak{M}_2$, the maximal ideal space of A_2, onto a closed subset of $\mathfrak{M}_1$, the maximal ideal space of A_1.

Proof: The mapping T* of this proposition is the adjoint mapping restricted to $\Delta_2 \subset A_2^*$. This mapping is well defined on Δ_2 and has a subset of Δ_1 as its range, for if $h_2 \varepsilon \Delta_2$ then for all $y \varepsilon A_1$,

$$(T^*(h_2))(y) = h_2(T(y)) = (h_2 \circ T)(y).$$

But $h_2 \circ T$ is a homomorphism of A_1 to the complex numbers. Since h_2 maps A_2 onto the complex numbers and $T(A_1)$ is dense in A_2, there is an element $y \varepsilon A$ such that $h_2(T(y)) \neq 0$. Thus the image of h_2 is a homomorphism of A_1 different from the zero homomorphism. By the results of Section 7, this homomorphism is a continuous homomorphism of A_1 onto the complex numbers. Since the homomorphisms in Δ_2 are continuous and $T(A_1)$ is dense in A_2, then if $h_1 \neq h_2 \varepsilon \Delta_2$, there is a $y \varepsilon A_1$ such that $h_1(T(y)) \neq h_2(T(y))$. Thus the mapping of Δ_2 to Δ_1 is one to one.

Notice that if $y \varepsilon A_1$ and $h \varepsilon \Delta_2$ the function $(T(y))^\wedge$ is given by

$(T(y))^{\wedge}(h) = h(T(y)) = (T^*h)(y) = \hat{y}(T^*h)$. Since $T(A_1)$ is dense in A_2 and since the Gel′fand representation is norm decreasing, the functions in $\hat{A}_2$ determined by elements of $T(A_1)$ are dense in $\hat{A}_2$. This dense subset of $\hat{A}_2$ is sufficient to determine the weak topology of Δ_2, i.e., all sets of the form

$$U_{\hat{x},\varepsilon,h_o} = \{h \in \Delta_2 \mid |\hat{x}(h) - \hat{x}(h_o)| < \varepsilon \text{ and } \hat{x} \text{ is the Gel′fand representation of an element of } T(A_1)\}.$$

are a subbase for the weak topology of Δ_2. But each set $T^*(U_{\hat{x},\varepsilon,h_o}) = (U_{y,\varepsilon,T^*(h_o)})$ where $T(y) = x$ and $x \to \hat{x}$. Thus, T^* gives a one to one correspondence between elements of subbases for the topologies for Δ_2 and $T^*(\Delta_2)$. Consequently, T^* is a homeomorphism of Δ_2 onto $T^*(\Delta_2)$.

Now $T^*(\Delta_2)$ is closed in Δ_1 since if $g_o \in T^*(\Delta_2)$ in the weak topology of Δ_1, then g_o is such that given $\varepsilon > 0$, $x_1, x_2, \ldots x_n \in A_1$, there is an $\alpha \in \Delta_2$ such that $|g_o(x_i) - \alpha(Tx_i)| < \varepsilon$ for $i = 1,2,\ldots n$.

Define $h_o : T(A_1) \to C$ by $h_o(T(x)) = g_o(x)$. The mapping is well defined since if $T(x_1) = T(x_2)$ then $g_o(x_1) = g_o(x_2)$, and $|h_o(y)| \leqq ||y||$ for $y \in T(A_1)$. Thus h_o is a continuous homomorphism of a dense subset of A_2 onto the complex numbers, and can be uniquely extended to all of A_2. Thus $g_o = T^*(h_o)$ or $T^*(\Delta_2)$ is closed.

Proposition 2: When both A_1 and A_2 are commutative Banach algebras, A_2 is semi-simple, and T is a homomorphism of A_1 onto a dense subset of A_2 then T is continuous.

Proof: The proof of this proposition uses the Closed Graph Theorem. If Γ is the graph of T in $A_1 \times A_2$, and $(x_o, y_o) \in \overline{\Gamma}$ then there is a sequence $\{(x_n, y_n)\} \in \Gamma$ such that $(x_n, y_n) \to (x_o, y_o)$. In particular $x_n \to x_o$ and $T(x_n) \to y_o$. It must be shown that $T(x_o) = y_o$. But if $x_n \to x_o$ then $\hat{x}_n \to \hat{x}_o$ uniformly, and if $y_n \to y_o$ then $\hat{y}_n \to \hat{y}_o$ uniformly. But for all $z \in A_1$ and for all $h \in \Delta_2$, $\hat{z}(T^*(h)) = (Tz)\hat{\ }(h)$. Thus

$$(Tx_o)\hat{\ }(h) = \hat{x}_o(T^*(h)) = \lim_{n\to\infty} \hat{x}_n(T^*(h) = \lim_{n\to\infty}(Tx_n)\hat{\ }(h) = y_o(h).$$

Thus $T(x_o)\hat{\ } = \hat{y}_o$. But since A_2 is semi-simple, $T(x_o)\hat{\ } = \hat{y}_o$ implies that $T(x_o) = y_o$ or $(x_o, y_o) \in \Gamma$. By the closed Graph Theorem T is continuous.

Proposition 3: If A is a semi-simple commutative complex algebra, then there is at most one norm (to within topological equivalence) with respect to which A is a Banach algebra.

Proof: Suppose there are two norms such that A is a Banach algebra with respect to each. Now A is semi-simple with respect to each and the identity mapping is an algebra homomorphism. Thus the identity map is continuous from A with either norm to the other. Consequently, the identity is a homeomorphism or the norms are equivalent.

An immediate consequence of Proposition 2 is that every automorphism of a semi-simple Banach algebra is a homeomorphism.

Proposition 4: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A then a necessary and sufficient condition that A be semi-simple and $\hat{A}$ be uniformly closed is that there exist a constant $K > 0$ such that $||x||^2 \leqq K||x^2||$ for every $x \varepsilon A$.

Proof: If a is semi-simple and $\hat{A}$ is uniformly closed, then the Gel'fand representation is a one to one, continuous linear transformation of a Banach space onto a Banach space. Consequently, the inverse mapping is continuous. Thus there is a $K > 0$ such that $||x|| \leqq K||\hat{x}||$. Then

$$||x||^2 \leqq K^2||\hat{x}||^2 = K^2||(x^2)^\wedge|| \leqq K^2||x^2||.$$

Conversely, if $||x||^2 \leqq K||x^2||$ then

$$||x|| \leqq K^{1/2}||x^2||^{1/2} \leqq K^{1/2\,+\,1/4}||x^4||^{1/4} \leqq \ldots$$
$$\leqq K^{1/2\,+\,\ldots\,2^{-n}}||x^{2n}||^{2-n}.$$

Thus $||x|| \leqq K \lim\limits_{n\to\infty} ||x^n||^{1/n} = K||\hat{x}||$ and the Gel'fand representation is uniformly closed. By Proposition 4 of Section 10, $\hat{A}$ is semi-simple. But $\hat{A}$ is isomorphic to A. Consequently, A is semi-simple.

Proposition 5: If A is a commutative Banach algebra and $\hat{A}$ is the Gel'fand representation of A, then a necessary and sufficient condition that A be isometric to $\hat{A}$ is that $||x||^2 = ||x^2||$ for every $x \varepsilon A$.

Proof: If A is isometric to $\hat{A}$ then $||x||^2 = ||\hat{x}||^2 = ||(x^2)^\wedge|| = ||x^2||$. If $||x||^2 = ||x^2||$ then just as in Proposition 4 with $K = 1$, $||x||^2 \leqq ||\hat{x}||^2 \leqq ||x^2|| = ||x||^2$ or $||x|| = ||\hat{x}||$.

SECTION 12 ABSTRACT ANALYTIC FUNCTIONS IN $\hat{A}$

In this section it is shown that the Gel'fand representation of a commutative Banach algebra A is closed under the application of analytic functions.

Proposition 1: If A_e is a commutative Banach algebra with identity, $\hat{A}_e$ is the Gel'fand representation of A_e, $\hat{x} \varepsilon \hat{A}_e$, D is a region of C containing the spectrum of x $(\hat{x}(\mathfrak{M}))$, $f:D \to C$ is analytic on D, then there is a $y \varepsilon A_e$ which is such that $f(\hat{x}(M)) = \hat{y}(M)$ for all $M \varepsilon \mathfrak{M}$. Moreover, the element y of A_e is given by

$$y = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda)}{(\lambda e - x)} d\lambda$$

where the integral is the generalized integral defined in Section 6 and Γ is a closed path on D which encloses the spectrum of x.

Proof: Since no point of the spectrum of x is a member of Γ^*, the element $(\lambda e - x)^{-1}$ exists and is continuous on Γ^*. Moreover, since f is analytic on D, the function $g:\Gamma^* \to A_e$ defined by $g(\lambda) = f(\lambda)\cdot(\lambda e - x)^{-1}$ is uniformly continuous on Γ^*. Thus, $\int_{\Gamma} g(\lambda)d\lambda$ exists.

Since the Gel'fand representation $\Phi:A_e \to \hat{A}_e$ is in particular a bounded linear function from the commutative Banach algebra A_e to the commutative Banach algebra $\hat{A}_e$, Proposition 2 of Section 6 can be applied to yield

$$\Phi\left[\frac{1}{2\pi i}\int_{\Gamma} g(\lambda)\,d\lambda\right] = \frac{1}{2\pi i}\int_{\Gamma} (\Phi\circ g)(\lambda)\,d\lambda$$

or

$$\hat{y} = \frac{1}{2\pi i}\int_{\Gamma} \hat{g}(\lambda)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma} \widehat{f(\lambda)(\lambda e - x)^{-1}}\,d\lambda.$$

But $\Phi(x) = \hat{x}$ is a homomorphism, and thus

$$\hat{y} = \frac{1}{2\pi i}\int f(\lambda)(\lambda e - x)^{-1}\,d\lambda = \frac{1}{2\pi i}\int f(\lambda)(\lambda - \hat{x})^{-1}\,d\lambda.$$

Now if M is any element of $\mathfrak{M}$, the mapping $h_M : A_e \to C$ is a homomorphism of A_e to C. In particular h_M is a bounded linear functional from A_e to C. Using the same reasoning as above

$$\hat{y}(M) = h_M(\hat{y}) = \frac{1}{2\pi i}\int f(\lambda)(\lambda - \hat{x}(M))^{-1} = f(\hat{x}(M)).$$

In the setting of a commutative Banach algebra without identity, the above proposition may be reformulated using adverses. Additional restrictions must be placed on either f or Γ, however.

Proposition 2: If A is a commutative Banach algebra without identity, $\hat{A}$

is the Gel'fand representation of A, $\hat{x} \varepsilon \hat{A}$, D is a region containing the spectrum of x, $f: D \to C$ is analytic on D, Γ is a closed path which encloses the spectrum of x, and either

a) $f(0) = 0$ or

b) Γ does not enclose 0,

then there is a $y \varepsilon A$ which is such that $\hat{y}(M) = f(\hat{x}(M))$ for all $M \varepsilon \mathfrak{M}$.

Moreoever, the element $y \varepsilon A$ is given by

$$y = -\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda} \left(\frac{x}{\lambda}\right)' d\lambda$$

where $(x/\lambda)'$ is the adverse of x/λ.

Proof: Let A_e be the extension of A to an algebra with identity. Now $(\lambda e - x)^{-1} = \lambda^{-1}(e - x/\lambda)^{-1} = \lambda^{-1}e - \lambda^{-1}(x/\lambda)'$. By Proposition 1,

$$y = \left[\frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda} d\lambda\right] e - \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{\lambda} \left(\frac{x}{\lambda}\right)' d\lambda .$$

The first term is an element of A_e, the second is an element of A. If the term in brackets is zero, the result is in A, and by Proposition 1, $\hat{y}(M) = f(\hat{x}(M))$ for all $M \varepsilon \mathfrak{M}$. But the term in brackets is zero whenever either a) or b) is satisfied.

CHAPTER III

EXAMPLES AND APPLICATIONS

SECTION 1 ALGEBRAS OF FUNCTIONS

If X is a nonvoid set, then a set of bounded complex valued functions on X will be a commutative Banach algebra if addition, multiplication, and scalar multiplication are defined pointwise, the norm is the supremum norm, the set is closed under addition, multiplication, and scalar multiplication, and the set is closed with respect to uniform convergence. An algebra satisfying the above will be called a function algebra. Given a function algebra on a set X, the weak topology induced on X by the elements of the algebra will make the algebra one of continuous bounded complex valued functions on a topological space. Thus any function algebra may be viewed as an algebra of continuous bounded complex valued functions on a topological space. If the set X already has a topology then the weak topology might have no relation to the original topology. If the functions are continuous in the original topology, then the weak topology will be a subset of the original topology.

In this section, certain function algebras will be discussed and relations between the algebraic properties of the function algebra and topological properties of the space X will be drawn.

The Algebra C(X)

If $(X,\mathcal{T})$ is a compact Hausdorff space, then the set of all continuous complex valued functions on X is a Banach algebra. Let C(X) denote the set of all continuous complex valued functions on X. This algebra is commutative and has an identity, the function which is 1 everywhere.

In this section it will be shown that C(X) is isomorphic to $\widehat{C(X)}$, the Gel'fand representation of C(X), and that X is homeomorphic to $\mathfrak{M}$. Notice that points of X determine maximal ideals of C(X) since if p is a point of X the mapping $h_p : C(X) \to C$ determined by $h_p(f) = f(p)$ for $f \in C(X)$ is a homomorphism of C(X) onto the complex numbers. The kernel of h_p is the set of functions which are zero at the point p. Let M_p denote the kernel of h_p.

Proposition 1: If C(X) is the algebra of continuous complex valued functions on a compact Hausdorff space and I is an ideal of C(X), then there is a point p of X such that I is a subset of M_p.

Proof: Suppose I is an ideal of C(X) such that I is a subset of no M_p. Then for each $p \in X$, there is an $f \in I$ such that $f(p) \neq 0$. But $\overline{f}(p) = \overline{f(p)} \in C(X)$; thus $|f|^2 = f \cdot \overline{f} \in I$ and $|f|^2(p) > 0$. Thus for each $p \in X$ there is a $g_p \in I$ such that $g_p(p) > 0$. For each g_p there is an open set containing p such that g_p is positive on that open set containing p. The collection of all sets of this form are an open cover of X. Since X is compact there is a finite subcover of X. The sum of the function associated with the open sets of this subcover is an element of I. But this sum is a function which is bounded away from zero. Call this function f_o. Then

$1/f_o \in C(X)$ and $f_o \cdot (1/f_o) = e \in I$. (e is the identity for $C(X)$.) But then $I = C(X)$, a contradiciton. Thus there is a $p \in X$ such that $I \subset M_p$.

Now in particular maximal ideals of $C(X)$ are determined by points of X; that is, if M is a maximal ideal of $C(X)$, then there is a point p of X such that $M = M_p$. Moreover, since the functions of $C(X)$ separate points, there is a one to one correspondence between the points of X and the maximal ideals of $C(X)$, $p \leftrightarrow M_p$.

Notice that the algebra $C(X)$ is semi-simple since $\bigcap_{p \in X} M_p = 0$. Thus by Section 10 of Chapter II the Gel´fand representation is an isomorphism. Since the representation is an isomorphism and since there is a one to one correspondence between the points of X and the points of $\mathfrak{M}$, the space X endowed with the weak topology of the functions of $C(X)$ is homeomorphic to $\mathfrak{M}$. The weak topology is a subset of the original topology. But the weak topology is a Hausdorff topology and the original topology is a compact topology, thus the two topologies must be the same. The one to one correspondence $p \leftrightarrow M_p$ is thus a homeomorphism.

It has been demonstrated that the algebra $C(X)$ is isomorphic to the Gel´fand representation and that the space X is homeomorphic to the space $\mathfrak{M}$ of maximal ideals; thus the Gel´fand representation of $C(X)$ is faithful in the sense that all algebraic and topological properties are preserved. Moreover in any algebra of functions, $||f^2|| = ||f||^2$ for any f in the algebra. By Proposition 5 of Section 11, Chapter II, C(X) is isometric to its Gel´fand representation.

One immediate consequence of the fact that the Gel´fand representation is faithful is the Banach-Stone Theorem.

Theorem (Banach-Stone): Two compact Hausdorff spaces are homeomorphic if and only if their corresponding function algebras C(X) and C(Y) are isomorphic.

Proof: If X is homeomorphic to Y then C(X) is clearly isomorphic to C(Y). Moreover, if C(X) is isomorphic to C(Y), C(X) and C(Y) have the same Gel´fand representation. But X and Y are each homeomorphic to the space $\mathfrak{M}$ of this Gel´fand representation. Consequently, X and Y are homeomorphic.

The Algebra $C_o(X)$

If $(X, \mathcal{T})$ is a locally compact but not compact Hausdorff space, then the set of all continuous complex valued functions which vanish at infinity is a commutative Banach algebra. Let $C_o(X)$ be the function algebra of continuous complex valued functions which vanish at infinity on X. The algebra $C_o(X)$ does not have an identity. The extension of $C_o(X)$ to an algebra with identity is the algebra of all elements of the form $\lambda + f$, where λ is a complex number and f is an element of $C_o(X)$. If X´ is the Alexandroff one point compactification of X, then every element g of $C(X')$ is an element of the form $g(\infty) + f$, where f is in $C_0(X)$. The isomorphism of the extension of $C_0(X)$ and $C(X')$ shows that the Alexandroff one point compactification corresponds in the above sense to the extension of $C_0(X)$ to an algebra with identity. Thus, two locally compact Hausdorff spaces X and Y are homeomorphic if and only if $C_0(X)$ is isomorphic to $C_0(Y)$.

<u>Completely Regular Spaces</u>

A topological space $(X,\mathcal{T})$ is completely regular if and only if for any point $x \in X$ and any closed subset F of X which does not contain x there is a continuous real valued function f such that $f:X \to [0,1]$ and $f(x) = 0$ and $f(F) = \{1\}$. In particular, every compact or locally compact Hausdorff space is completely regular.

If X is a completely regular space, let C(X) denote the continuous complex valued functions on X. Now C(X) has the following properties:

1) C(X) is a commutative Banach algebra with identity.

2) If p is a point of X, then M_p is a maximal ideal of p.

3) If p_1 and p_2 are points of X, then there is an $f \in C(X)$ such that $f(p_1) \neq f(p_2)$

4) If $f \in C(X)$ then $\overline{f} \in C(X)$

5) If $f \in C(X)$ and $\mathrm{glb}\{|f(p)| \; |p \in X\} > 0$, then $1/f \in C(X)$

It follows from statement 1 that the Gel'fand representation of C(X) is an algebra of continuous complex valued functions on a compact

Hausdorff space. Since any function algebra is semi-simple, the Gel'fand representation is an isomorphism.

It follows from statements 2) - 5) that distinct points of X determine distinct maximal ideals. The points of X can thus be imbedded in $\mathfrak{M}$. Since the closure in $\mathfrak{M}$ of the image of X is a compact Hausdorff space, the proof of Proposition 1 shows that the maximal ideals of C(X) are given by points of this closure. Consequently, the image of X is dense in $\mathfrak{M}$. Since this image is dense in $\mathfrak{M}$, $C(\mathfrak{M})$, the algebra of continuous complex valued functions on $\mathfrak{M}$ must be just $\widehat{C(X)}$.

Recall that the Stone-Cech compactification βX of a completely regular space X is such that the functions of X can be uniquely extended to continuous function on the compactification. This unique extension property is equivalent to the statement that C(X) is isomorphic to $C(\beta X)$. Thus the Gel'fand representation is isomorphic to $C(\beta X)$. But since C(X) is isomorphic to $C(\mathfrak{M})$, $C(\mathfrak{M})$ is isomorphic to $C(\beta X)$. By the Banach-Stone Theorem, βX is homeomorphic to $\mathfrak{M}$, the maximal ideal space of the Gel'fand representation.

SECTION 2 THE FOURIER TRANSFORM

An important class of Banach algebras can be characterized as convolution algebras. An example of convolution algebra is the space of functions from the real line to the complex numbers whose absolute values are Lebesque integrable. Let $L_1(-\infty,\infty)$ denote this set. This space is a Banach space when addition and scalar multiplication are defined pointwise and the norm of an element of $L_1(-\infty,\infty)$ is the integral of the absolute value of that element,

$$||f|| = \int_{-\infty}^{\infty} |f(x)|dx.$$

Henceforth, whenever the limits of integration are $-\infty$ and ∞, the symbol $\int$ will be used in place of $\int_{-\infty}^{\infty}$.

If the product of two elements is defined as convolution,

$$(f*g)(y) = \int f(x)g(y - x)dx$$

for $y \in R$, the real numbers, and $f,g \in L_1(-\infty,\infty)$; the Banach space $L_1(-\infty,\infty)$ becomes a Banach algebra. Since

$$||f*g|| = \int|\int f(x)g(y - x)dx|dy \leq \int\int|f(x)|\cdot|g(y - x)|dxdy,$$

but by the Tonelli Theorem this is

$$= \int\int|f(x)|\cdot|g(y - x)|dydx.$$

Letting $w = y - x$ the above is

$$= \int\int|f(x)|\cdot|g(w)|dwdx$$

$$= (\int|f(x)|dx)\cdot(\int|g(w)|dw)$$

$$= ||f||\cdot||g||.$$

Consequently, $L_1(-\infty,\infty)$ is closed under multiplication, and the norm inequality on products is satisfied.

The linearity of the integral leads immediately to the fact that multiplication is distributive and that the mixed associative law holds. The Tonelli Theorem leads to the fact that multiplication is associative.

The above discussion indicates that $L_1(-\infty,\infty)$ is a Banach algebra with convolution for multiplication. Moreover, multiplication is commutative since for any $f,g, \varepsilon L_1(-\infty,\infty)$ and any $y \varepsilon R$,

$$(f*g)(y) = \int f(x)g(y - x)dx.$$

Letting $w = y - x$, the above is

$$= \int_{\infty}^{-\infty} - f(y - w)g(w)dw$$

$$= \int f(y - w)g(w)dw$$

$$= (g*f)(y),$$

or $f*g = g*f$ for any $f,g \varepsilon L_1(-\infty,\infty)$.

The algebra $L_1(-\infty,\infty)$ does not have an identity. In order to prove this result the following Lemma is needed.

Lemma 1: If $g,f \varepsilon L_1(-\infty,\infty)$ and g is bounded ($|g(t)| \leqq C$ for all t) then $g*f$ is a continuous function.

Proof: Let $z(t) = (g*f)(t)$, then

$$|z(t + h) - z(t)|$$

$$= |\int g(x)f(t + h - x)dx - \int g(x)f(t - x)dx|$$

$$\leqq \int |g(x)(f(t + h - x) - f(t - x))|dx$$

$$\leqq C\int |f(t + h - x) - f(t - x)|dx$$

Letting $w = x - t$ the preceding is

$$C\int |f(h - w) - f(-w)|dw.$$

Since any function can be approximated by a simple function and when f in the above integral is a simple function, the integral approaches 0 as $h \to 0$. Consequently z is a uniformly continuous function.

Suppose that $L_1(-\infty,\infty)$ contained an identity element. Then multiplication by the identity maps every bounded function to a continuous function. Moreover, this continuous function must equal the bounded function almost everywhere. This is a contradiction, since in particular, if χ_I is the characteristic function of $[0,1]$ then χ_I is in $L_1(-\infty,\infty)$ and there is no continuous function which is equal to χ_I almost everywhere. This contradicts the assumption that $L_1(-\infty,\infty)$ has an identity. Consequently $L_1(-\infty,\infty)$ does not have an identity.

Henceforth L_1 will denote $L_1(-\infty,\infty)$, $\hat{L}_1$ will denote the Gel´fand representation of L_1, $\mathfrak{M}$ will denote the maximal ideal space of L_1.

At least some of the points of $\mathfrak{M}$ are given by points of $(-\infty,\infty)$ in the following sense. Consider the mapping h_y from $L_1(-\infty,\infty)$ to the complex numbers $(h_y : L_1 \to C)$ defined by

$$h_y(f) = \int f(x)e^{ixy}dx \text{ for } f \in L(-\infty,\infty).$$

The mapping is linear and preserves scalars since the integral is linear.

Moreover, the mapping is a homomorphism of $L_1(-\infty,\infty)$ onto the complex numbers since

$$h_y(f*g) = \int e^{ixy}(\int f(t)g(x - t)dt)dx$$

$$= \int\int f(t)g(x - t)e^{ixy}dxdt.$$

Letting $w = x - t$; then $h_y(f*g)$

$$= \int\int f(t)g(w)e^{i(w + t)y}dwdt$$

$$= (\int f(t)e^{ity})\cdot(\int g(w)e^{iwy}dw)$$

$$= h_y(f)\cdot h_y(g).$$

Thus homomorphisms of $L_1(-\infty,\infty)$ to C are given by points in the above sense. The points of $(-\infty,\infty)$ can thus be identified with a subset of the maximal ideal space $\mathfrak{m}$ of $L_1(-\infty,\infty)$. Moreover, the value of a function f at the image of the point y in the maximal ideal space is given by

$$\hat{f}(h_y) = \hat{f}(y) = \int f(x)e^{ixy}dx.$$

This is the Fourier transform of the function f at the point y.

It will be shown that the continuous homomorphisms of $L_1(-\infty,\infty)$ onto the complex numbers are precisely those given by points of $(-\infty,\infty)$. Suppose h is any homomorphism of L_1 onto the complex numbers. Then in particular h is a bounded linear functional on L_1. Consequently, h can be represented as an integral of the form $h(f) = \int f(x)\Phi(x)dx$ for all $f \in L_1$ where Φ is an essentially bounded function on $(-\infty,\infty)$, $\Phi \in L_\infty(-\infty,\infty)$. Moreover,

$$h(f)h(g) = h(f)\int g\ (y)\Phi(y)\ dy = \int h(f)g(y)\Phi(y)dy$$

and

$$h(f)h(g) = h(f*g) = \int (f*g)(x)\cdot\Phi(x)dx$$

$$= \int g(y)(\int f(x - y)\Phi(x)dx)dy$$

$$= \int g(y)h(f_y)dy$$

where f is the translate of f defined by $f_y(x) = f(x - y)$ for all $x \in (-\infty,\infty)$. Consequently, $h(f)\Phi(y) = h(f_y)$ almost everywhere in $(-\infty,\infty)$.

But the mapping $y \to h(f_y)$ is a continuous function on $(-\infty,\infty)$ for each $f \in L_1$. Since f was any function in L_1, choose f such that $h(f) \neq 0$. Then, letting $\Phi(y) = h(f_y)/h(f)$, Φ is a continuous function of y almost everywhere in $(-\infty,\infty)$.

Letting $y = x + z$,

$$h(f)\Phi(x + z) = h(f_X) = h((f_X)) = h(f_X)\Phi = h(f)\Phi(x)\Phi.$$

Thus $\Phi(x + z) = \Phi(x)\Phi$; or Φ is a continuous homomorphism of the real numbers as an additive group to the complex numbers as a multiplicative group. But every such homomorphism is of the form $\Phi(x) = e^{ixy}$. Thus every homomorphism of L_1 is given by a point of $L_1(-\infty,\infty)$.

Now distinct points of $(-\infty,\infty)$ determine distinct homomorphisms; that is, if $y_1, y_2 \in (-\infty,\infty)$ then there is an $f \in L_1$ such that $h_{y_1}(f) \neq h_{y_2}(f)$. Thus the identification of points of y_1 with maximal ideals is one to one and onto.

Moreover, the functions of $\hat{L}_1$ are continuous functions of y

in the usual topology of the real line since if $h_n \to 0$

$$|\hat{f}(y + h_n) - \hat{f}(y)| \leq |\int f(x)(e^{ix(y + h_n)} - e^{ixy})dx|$$

$$\leq \int |e^{ixh_n} - 1| \cdot |f(x)e^{ixy}|dx$$

$$\leq \int |e^{ixh_n} - 1| \cdot |f(x)|dx.$$

But the integrand is dominated by $2|f(x)|$. Thus

$$\lim_{n\to\infty} |\hat{f}(y + h_n) - \hat{f}(y)| \leq \lim_{n\to\infty} \int |e^{ixh_n} - 1| \cdot |f(x)|dx,$$

and by the Lebesgue Dominated Convergence Theorem the last limit is

$$\int \lim_{n\to\infty} |e^{ixh_n} - 1| \cdot |f(x) \, dx = 0;$$

thus $\hat{f}$ is continuous.

Since the functions of $\hat{L}_1$ are continuous under the topology of and under the topology of $(-\infty,\infty)$ when identified with $\mathfrak{m}$ and since by the Riemann-Lebesgue Lemma the functions of $\hat{L}_1$ vanish at infinity, the the functions of $\hat{L}_1$ are continuous on both the real line with the Alexandroff one point compactification and the maximal ideal space $\mathfrak{m}_o$. But the topology of $\mathfrak{m}_o$ is a subset of the topology of the compactification of the real line. Thus the mapping of points of the compactification of the real line to points of the maximal ideal space is a one to one continuous and onto mapping of a compact space onto a Hausdorff space. Hence the inverse is continuous and the spaces are homeomorphic.

Thus the Gel'fand representation of L_1 is just the algebra of Fourier transforms of the elements of L_1.

The proof that L_1 is semi-simple is too long to be included here. The interested reader is referred to Loomis. Nevertheless, since L_1 is semi-simple the Fourier transform is an isomorphism. Thus different elements of L_1 have different Fourier transforms.

The results of Section 12 can be applied to yield the following proposition. If $\hat{f}$ is an element of $\hat{L}_1$ and $\Phi:D \to A$ is analytic on a region containing the range of $\hat{f}$ and $\Phi(0) = 0$, then there is an absolutely integrable function g such that the Fourier transform of g is such that $\hat{g}(y) = \Phi(\hat{f}(y))$ for all $y \in (-\infty,\infty)$.

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