# AN IMPROVED DEFECT RELATION FOR HOLOMORPHIC CURVES IN PROJECTIVE VARIETIES

A Dissertation Presented to
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In Partial Fulfillment
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Doctor of Philosophy

By
Charles David Mills

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# AN IMPROVED DEFECT RELATION FOR HOLOMORPHIC CURVES IN PROJECTIVE VARIETIES

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# Abstract

In this dissertation we improve Min Ru's defect relation (as well as the Second Main Theorem) for holomorphic curves $f : \mathbb{C} \to X$ intersecting $D := D_1 + \cdots + D_q$, where $D$ is a divisor of equi-degree, and $D_1, \ldots, D_q$ are big, nef, and have no components in common. Our results will decrease the number of divisors $D_i$ that $f$ is needed to omit in order to conclude that $f$ is degenerate. The corresponding arithmetic results are also obtained.

# Contents

# Chapter 1

# Introduction

When one first begins to study complex analysis, one quickly learns that holomorphic functions have a high degree of regularity. For example, if $f$ is holomorphic at $x \in \mathbb{C}$, then $f$ has a power series in some neighborhood of $x$, i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z-x)^n$ in some neighborhood of $x$. Another form of regularity is that a holomorphic function of the form $f = u + iv$ must also have that the $u$ and $v$ be harmonic. Perhaps one of the more surprising results is the Little Picard Theorem. The theorem is the following.

**Theorem 1.1** (Little Picard Theorem [GK06], page 322). *Let $f$ be an entire function, and suppose that the image of $f$ omits two distinct complex values. Then $f$ must be identically constant.*

There are many proofs of this theorem. One proof uses the universal cover, $\mathbb{D}$, of $\mathbb{C}$ with two points removed. A geometric proof, which can be seen as the root of Nevanlinna theory, can be found in [Kra04], page 78. This proof constructs a metric of negative curvature on $\mathbb{C}$ with two points removed, and then applies the Ahlfors-Schwarz lemma to conclude the function must be constant. This theorem tells us that holomorphic functions have a high degree of rigidity, and that a holomorphic function is very different from a real-valued smooth function as there are many real-valued smooth functions that omit two distinct values which are not constants, even degree polynomials, $\sin(x)$, and $\cos(x)$ to name a few.

The results of this dissertation will give us a generalization of the Little Picard Theorem. Note that the function in the theorem was entire. This result was extended to the case when $f$ is meromorphic, i.e. $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. The conclusion then is that $f$ can omit at most two points in $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. Over the past near century the question of how much this theorem can be generalized has been a very interesting one. One might ask the question "What if $f$ is a holomorphic map from $\mathbb{C}$ into $\mathbb{P}^2$, then what must $f$ omit in order to deduce that it is a constant function?" What if the target space were a general projective variety? That is the case this dissertation is concerned with. In this dissertation we will prove an improved defect relation for holomorphic curves $f : \mathbb{C} \to X$ where $X$ is a complex projective variety. In our setting, divisors will take the place of the points from the Little Picard Theorem. Thus we need to study the properties of the divisors required for $f$ to omit. Also, since our results will apply to a complex projective variety of arbitrary dimension, we will not be concluding directly the function $f$ is constant, but that $f$ is degenerate.

In order to obtain our result we employ the use of Nevanlinna theory. Nevanlinna theory deals with the asymptotic behavior of meromorphic funtions by defining a growth function and relating it to the proximity function, both entirely constructs of Nevanlinna theory. After one accepts the first results of Nevanlinna theory, the Little Picard Theorem above is obtained as an immediate corollary. Moreover, the definitions of the growth and proximity functions are easily modified to more general settings, allowing us to work with more complicated manifolds, rather than just $\mathbb{P}^1$.

Using Nevanlinna theory, generalized to the target space of a complex projective variety, Min Ru obtained a defect relation for holomorphic curves intersecting general divisors (see [Ru15]). In this dissertation we use his methods, as well as some additions, to obtain an improved defect relation when the divisors are big, nef, and have no common components.

We will begin this dissertation with an introduction to some basic concepts in algebraic geometry, namely sheaves, line bundles, divisors, and cohomology. We will then recap some important notions of positivity of line bundles and divisors paramount to understanding our main theorem. These topics will include ampleness, bigness, and nefness.

In chapter 3 we will give a recap of the development of Nevanlinna theory that has occurred over the past century. We will begin by discussing the original notions of Nevanlinna theory as it applies to a holomorphic map $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. We will then cover the generalization of this theory developed by Cartan when $f : \mathbb{C} \to \mathbb{P}^n$. After that, we will recap the recent results of Min Ru for the case when $f : \mathbb{C} \to X$, where $X$ is a complex projective variety, that will allow us to obtain our result. In this setting Ru made the following definition.

**Definition 1.0.1** ([Ru15])**.** Let $X$ be a normal complex projective variety, and $D$ be an effective Cartier divisor on $X$. The *Nevanlinna constant* of $D$, denoted by $Nev(D)$, is given by

$$Nev(D) := \inf_{N} \left( \inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right), \tag{1.1}$$

where the infimum "$\inf_N$" is taken over all positive integers $N$, and the infimum "$\inf_{\{\mu_N, V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$, where $\mu_N$ is a positive real number, and $V_N \subset H^0(X, \mathcal{O}(ND))$ is a linear subspace with $\dim V_N \geq 2$ such that, for all

$P \in \text{supp} D$, there exists a basis $B$ of $V_N$ with

$$\sum_{s \in B} \text{ord}_E(s) \geq \mu_N \, \text{ord}_E(ND)$$

for all irreducible components $E$ of $D$ passing through $P$. If $\dim H^0(X, \mathcal{O}(ND)) \leq 1$ for all positive integers $N$, then we define $Nev(D) = +\infty$.

In general, $Nev(D)$ proves difficult to compute. However, a theorem from [Ru15] tells us that if $Nev(D) < 1$, then any holomorphic map $f : \mathbb{C} \to X \backslash D$ must be degenerate.

Chapter 4 will contain the main results of this dissertation. Our main theorem will be the following.

**Main Theorem.** *Let $X$ be a complex normal projective variety of dimension $n \geq 2$. Let $D_1, \ldots, D_q$ be effective, big and nef Cartier divisors on $X$, and that the linear system $|ND_i|$ ($i = 1, \ldots, q$) is base-point free for $N \geq N_0$. We further assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in $l$-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ is equidegree. Let $f : \mathbb{C} \to X$ be holomorphic and Zariski dense. Then*

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \left(\frac{2n[(l+1)/2]}{q(1+\alpha)}\right) \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r)\right) \qquad \|_E$$

*with*

$$\alpha = \frac{2^{-3n-2} \min_{1 \leq i,j \leq q}(r_i^{n-2} r_j^2 (D_i^{n-2}.D_j^2)) \min_{1 \leq i,j \leq q}(r_i^{n-1} r_j (D_i^{n-1}.D_j))}{(nD^n)^2} > 0,$$

*where $[x]$ denotes the smallest integer greater than $x$.*

The coefficient on the right hand side of the inequality, $2n[(l+1)/2]/q(1+\alpha)$, will give us our generalization of Theorem 1.1, and tell us that if $q > 2n[(l+1)/2]$, then any holomorphic map $f : \mathbb{C} \to X \backslash D$ must be degenerate.

There is a conjecture that gives criteria in terms of the geometric properties of a projective variety $X$, and a divisor $D$, for when a holomorphic map $f : \mathbb{C} \to X \backslash D$ is degenerate. The conjecture is the following.

**Conjecture 1.2** (Griffiths Conjecture)**.** *Let $X$ be a projective variety. If $K_X$ is the canonical divisor of $X$, $D$ is a normal crossing effective divisor, and $K_X + D$ is ample, then any holomorphic map $f : \mathbb{C} \to X \backslash D$ must be degenerate.*

For example, if $X = \mathbb{P}^1$, and $a$ is a point in $\mathbb{P}^1$, then $K_X = -2\{a\}$, and we immediately obtain the result that if $f$ omits 3 points in $\mathbb{P}^1$, then it must be constant. The key difference between this conjecture and our result, is that our result mostly focuses on the geometry of the divisor $D$, and largely ignores the geometry of $X$. Note that the geometry of $X$ only shows up in the quantity $2n[(l+1)/2]/q(1+\alpha)$ with the dimension $n$. Whereas the geometry of $X$ is critical in the Griffiths Conjecture with the inclusion of the canonical divisor.

As mentioned above, the main results of this dissertation will apply to a complex projective variety of arbitrary dimension $n$. The dimension $n = 2$ results are contained in [Liao], and the results of this dissertation are contained in [MR17].

# Chapter 2

# Divisors, Line Bundles, and Positivity

We will use this chapter as a brief recap of sheaves, divisors, line bundles, and the necessary theorems we will need to prove our result. A more in depth exposition on these subjects can be found in [GH78] and [Laz04].

## 2.1 Sheaves and Cohomology

**Definition 2.1.1** ([GH78], page 35)**.** Let $X$ be a topological space. A *sheaf* $\mathscr{F}$ on $X$ associates to each open set $U$, an abelian group $\mathscr{F}(U)$, called the sections of $\mathscr{F}$ over $U \subset X$, and a map (called the restriction map) $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ for any open sets $U \subset V$, satisfying

i. For any open sets $U \subset V \subset W$,

$$r_{V,U} \circ r_{W,V} = r_{W,U}.$$

We will write $\sigma|_U$ for $r_{V,U}(\sigma)$.

ii. For any pair of open sets $U, V \subset X$ and sections $\sigma \in \mathscr{F}(U)$, $\tau \in \mathscr{F}(V)$ such that

$$\sigma|_{U\cap V} = \tau|_{U\cap V},$$

there exists a section $\rho \in \mathscr{F}(U \cup V)$ with

$$\rho|_U = \sigma, \qquad \rho|_V = \tau.$$

iii. If $\sigma \in \mathscr{F}(U \cup V)$ and

$$\sigma|_U = \sigma|_V = 0,$$

then $\sigma = 0$.

If the topological space $X$ is a complex manifold (or a complex projective variety), and $U \subset X$ is an open set, then we have a few key examples of sheaves which we will use throughout this dissertation.

i. The sheaf $\mathcal{O}$ of holomorphic functions where

$$\mathcal{O}(U) = \{\text{holomorphic functions on } U\};$$

ii. The multiplicative sheaf $\mathcal{O}^*$ of nowhere zero holomorphic functions where

$$\mathcal{O}^*(U) = \{\text{holomorphic functions } f \text{ on } U \text{ where } f(p) \neq 0 \text{ for any } p \in U\};$$

iii. The multiplicative sheaf $\mathcal{M}^*$ where

$$\mathcal{M}^*(U) = \{\text{meromorphic functions } f \text{ on } U \text{ such that } f \not\equiv 0\}.$$

We also have maps between sheaves defined via group homomorphisms as follows.

**Definition 2.1.2** ([GH78], page 36)**.** Let $\mathscr{E}$ and $\mathscr{F}$ be two sheaves on $X$. A *sheaf map*, or *sheaf morphism*, $f : \mathscr{E} \to \mathscr{F}$ is a collection of group homomorphisms

$$\{f_U : \mathscr{E}(U) \to \mathscr{F}(U)\},$$

such that for open sets $U \subset V$ and $\sigma \in \mathscr{E}(V)$, we have

$$f_V(\sigma)|_U = f_U(\sigma|_U).$$

Then the *kernel sheaf* and *image sheaf* of a map between sheaves are also well defined as

$$\ker(f)(U) = \{\ker(f_U : \mathscr{E}(U) \to \mathscr{F}(U))\},$$

and

$$\mathrm{Im}(f)(U) = \{s \in \mathscr{F}(U) \mid \forall p \in U, \exists V \subset U \text{ with } p \in V \text{ s.t. } s|_V \in \mathrm{Im}(f_V)\}.$$

Before we continue towards defining the cohomology of sheaves, observe the following diagram

$$0 \to \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G} \to 0,$$

where $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ are sheaves, and $\alpha$, $\beta$ are sheaf maps. We say that this is a *short exact sequence* if $\ker(\alpha) = \{0\}$, $\mathrm{Im}(\beta) = \mathscr{G}$, and $\mathrm{Im}(\alpha) = \ker(\beta)$. These special maps and sheaves will be a useful tool in analyzing the cohomology soon to be defined.

**Example 2.1.** Let $X$ be a complex manifold. Then the sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0,$$

where $i$ denotes inclusion, and $\exp(f) = e^{2\pi\sqrt{-1}f}$ for $f \in \mathcal{O}(U)$, is a short exact sequence.

Given a sheaf $\mathscr{F}$ on $X$, let us define a cochain group as follows,

**Definition 2.1.3.** Let $\underline{U} = \{U_\alpha\}$ be an open covering of $X$. Define the *k-th cochain group*, $C^k(\underline{U}, \mathscr{F})$, by

$$C^k(\underline{U}, \mathscr{F}) := \prod_{\alpha_0,\ldots,\alpha_k} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}).$$

An element $\sigma \in C^k(\underline{U}, \mathscr{F})$ consists then of a section $\sigma_{\alpha_0,\ldots,\alpha_k} \in \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$. Furthermore, we can define a map $\delta : C^k(\underline{U}, \mathscr{F}) \to C^{k+1}(\underline{U}, \mathscr{F})$, called the *coboundary map*, as in [GH78], page 38, by

$$(\delta\sigma)_{\alpha_0,\ldots,\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \sigma_{\alpha_0,\ldots,\hat{\alpha_j},\ldots,\alpha_{k+1}}|_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_{k+1}}}.$$

**Example 2.2.** Let $\underline{U} = \{U_1, U_2, U_3\}$ be an open covering of a topological space $X$. Then we have that for a cochain element $\sigma \in C^0(\underline{U}, \mathscr{F})$,

$$(\delta\sigma)_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathscr{F}(U_i \cap U_j),$$

and if $\sigma \in C^1(\underline{U}, \mathscr{F})$,

$$(\delta\sigma)_{ijk} = \sigma_{ij} + \sigma_{jk} - \sigma_{ik} \in \mathscr{F}(U_i \cap U_j \cap U_k).$$

A cochain $\sigma$ is called a *cocycle* if $\delta\sigma = 0$, and a *coboundary* if there exists a $\tau$ such that $\delta\tau = \sigma$. The coboundary map can be seen as analogous to the differential map on sheaves by noting the following lemma.

**Lemma 2.3.** *A coboundary is a cocycle. That is,* $\delta \circ \delta = 0$.

*Proof.* For the sake of the reader, we will only proof this for the case of example 2.2. The essence of the proof is the same for the general case, but the notation becomes a burden. In the setting of example 2.2, we have

$$\begin{aligned} ((\delta \circ \delta)\sigma)_{123} &= (\delta\sigma)_{23} - (\delta\sigma)_{13} + (\delta\sigma)_{12} \\ &= (\sigma_3 - \sigma_2) - (\sigma_3 - \sigma_1) + (\sigma_2 - \sigma_1) \\ &= 0 \in \mathscr{F}(U_1 \cap U_2 \cap U_3). \end{aligned}$$

We omit the restriction notation here as we will in the future. $\square$

We can now define the cohomology of a sheaf with respect to a cover $\underline{U}$.

**Definition 2.1.4** ([GH78], page 39)**.** Define the *k-th cohomology group* $H^k(\underline{U}, \mathscr{F})$ by

$$H^k(\underline{U}, \mathscr{F}) := \frac{\ker(\delta_k)}{\mathrm{Im}(\delta_{k-1})}.$$

Note that this definition depends on the open covering $\underline{U}$. We can however rectify this by passing to the direct limit and defining the *k-th Čech cohomology group* as

$$H^k(X, \mathscr{F}) = \lim_{\rightarrow} H^k(\underline{U}, \mathscr{F}).$$

We can even further simplify this by imposing a condition on $\underline{U}$.

**Theorem 2.4** (Leray's Theorem [GH78], page 40)**.** *Let $\mathscr{F}$ be a sheaf on $X$, and suppose $\underline{U}$ is an open cover of $X$ such that $H^p(U_{i_1} \cap \cdots \cap U_{i_p}, \mathscr{F}) = 0$ for all integers $p > 0$, and all finite intersections $U_{i_1} \cap \cdots \cap U_{i_p}$, then for all integers $k > 0$,*

$$H^k(\underline{U}, \mathscr{F}) \cong H^k(X, \mathscr{F}).$$

What this means is that in practice, we can choose a fine enough cover $\underline{U}$, and work with $H^k(\underline{U}, \mathscr{F})$ instead of having to worry about the direct limit.

Let $A_i$ be groups. We say that a sequence of homomorphisms

$$\cdots \to A_{n-1} \stackrel{\alpha_{n-1}}{\to} A_n \stackrel{\alpha_n}{\to} A_{n+1} \to \ldots$$

is a *long exact sequence* if $\mathrm{Im}(\alpha_{n-1}) = \ker(\alpha_n)$ for each $n$. As with other cohomology theories, we can associate a short exact sequence of sheaves to a long exact sequence of cohomology in the following way: Suppose we have a short exact sequence of sheaves

$$0 \to \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G} \to 0.$$

Then $\alpha$ and $\beta$ induce maps

$$\alpha : C^k(\underline{U}, \mathscr{E}) \to C^k(\underline{U}, \mathscr{F}),$$

and

$$\beta : C^k(\underline{U}, \mathscr{F}) \to C^k(\underline{U}, \mathscr{G}).$$

Furthermore, $\alpha$ and $\beta$ commute with $\delta$, thus they send a cocycle to a cocycle, and a coboundary to a coboundary. Thus they also induce maps for cohomology

$$\alpha_* : H^k(X, \mathscr{E}) \to H^k(X, \mathscr{F}),$$

and

$$\beta_* : H^k(X, \mathscr{F}) \to H^k(X, \mathscr{G}).$$

The only thing left to define is the coboundary map

$$\delta_* : H^k(X, \mathscr{G}) \to H^{k+1}(X, \mathscr{E}).$$

For $\sigma \in C^k(\underline{U}, \mathscr{G})$ sastisfying $\delta\sigma = 0$, we can refine $\underline{U}$ such that there exists $\tau \in C^k(\underline{U}, \mathscr{F})$ satisfying $\beta(\tau) = \sigma$, since $\beta$ is surjective. Then $\beta(\delta\tau) = \delta(\beta(\tau)) = \delta\sigma = 0$, thus after refining further, there exists $\mu \in C^k(\underline{U}, \mathscr{E})$ satisfying $\alpha(\mu) = \delta\tau$. Now since $\alpha(\delta\mu) = \delta(\alpha(\mu)) = \delta\delta(\tau) = 0$ and $\alpha$ is injective, $\mu$ is a cocycle and $\mu \in \ker(\delta)$. Then we can define $\delta_*\sigma := [\mu] \in H^{k+1}(X, \mathscr{E})$. We then have the following theorem.

**Theorem 2.5** ([GH78], page 40)**.** *Given a short exact sequence of sheaves $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$*

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0,$$

*the associated long sequence of cohomology*

$$\begin{aligned} 0 \;\to\;& H^0(X,\mathscr{E}) \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{G}) \\ \to\;& H^1(X,\mathscr{E}) \to H^1(X,\mathscr{F}) \to H^1(X,\mathscr{G}) \to \dots \\ &\vdots \\ \to\;& H^p(X,\mathscr{E}) \to H^p(X,\mathscr{F}) \to H^p(X,\mathscr{G}) \to \dots \end{aligned}$$

*is exact.*

## 2.2 Effective Cartier Divisors

A *complex projective (algebraic) variety* $X \subset \mathbb{P}^N$ is the locus in $\mathbb{P}^N$ of a finite collection of homogeneous polynomials $\{F_\alpha(X_0, \dots, X_N)\}$ ([GH78], page 166). In this setting, we have the following definition of Cartier divisors.

**Definition 2.2.1** ([Laz04], page 8)**.** Let $X$ be a projective variety. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. We denote by $\mathrm{Div}(X)$ the set of all such sections, so that

$$\mathrm{Div}(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*).$$

However, this definition is fairly abstract and not very illustrative. Specifically, given a divisor $D \in \mathrm{Div}(X)$, it is represented by a collection of pairs $\{(U_i, f_i)\}$, where $\{U_i\}$ is an open covering of $X$, and $f_i \in \mathcal{M}^*(U_i)$ with $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. We will call the function $f_i$ the "local defining function" for $D$ on $U_i$. We define the *support* of a divisor $D$, denoted $\mathrm{supp}(D) \subset X$, by

$$\mathrm{supp}(D) \cap U_i = \{x \in U_i \mid f_i(x) = 0\}.$$

**Definition 2.2.2.** Let $D = \{(U_i, f_i)\}$. We say that $D$ is *effective*, denoted $D \geq 0$, if each of its local defining functions $f_i$ is holomorphic on $U_i$.

$\mathrm{Div}(X)$ in fact forms a group with respect to the following addition operation: given two divisors $D_1, D_2 \in \mathrm{Div}(X)$, which are represented by $\{(U_{1i}, f_{1i})\}$ and $\{(U_{2i}, f_{2i})\}$ respectively, the new divisor $D_1 + D_2$ is given by the collection $\{(U_i, f_{1i}f_{2i})\}$.

There will also be mention of Weil divisors.

**Definition 2.2.3.** A *Weil divisor* is a formal sum of codimension one irreducible subvarieties of $X$. That is, a Weil divisor is of the form

$$\sum_{V \subset X} n_V [V],$$

where $V$ is a codimension one, irreducible subvariety of $X$, and $n_V$ are integers with all but finitely many equal to zero. We say that a Weil divisor is *effective* if all of the

$n_V$ are non-negative.

Note that we can associate a Weil divisor to any Cartier divisor in the following way: Let $D$ be a Cartier divisor, then define the associated Weil divisor as

$$\sum_{V \subset X} \mathrm{ord}_V(D)[V].$$

It is then clear that our notions of effectiveness for Cartier divisors and Weil divisors coincide in this case. In the case that $X$ is smooth, one can also construct a Cartier divisor from a Weil divisor. We can also define a Weil divisor to a meromorphic function as follows,

$$(f) = \sum_{V \subset X} \mathrm{ord}_V(f)[V].$$

**Definition 2.2.4.** We say that two divisors $D_1$ and $D_2$ are *linearly equivalent*, denoted by $D_1 \sim D_2$, if $D_1 - D_2 = (f)$ for some (global) meromorphic function $f$ on $X$.

## 2.3 Line Bundles

Let $M$ be a compact complex manifold. It is known, from the maximum principle, that there are no non-constant holomorphic functions on $M$. So, instead, we study (holomorphic) sections of holomorphic line bundles. We have the following definition of holomorphic line bundles.

**Definition 2.3.1** ([GH78], page 132-133)**.** Let $M$ be a compact complex manifold. A *holomorphic line bundle* on $M$ is a complex manifold $L$ together with a surjective holomorphic map $\pi : L \to M$ such that there exists an open covering $\{U_\alpha\}$ of $M$ and fiber-preserving biholomorphic maps (i.e. $\pi(\phi_\alpha^{-1}(x,a)) = x$ for all $x \in U_\alpha$ and $a \in \mathbb{C}$)

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C},$$

such that

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$

is a non-zero linear map on each $\{x\} \times \mathbb{C}$. The map $\phi_\alpha$ is called a *trivialization* of $L$ over $U_\alpha$.

We define the *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ by

$$x \mapsto (\phi_\alpha \circ \phi_\beta^{-1})|_{\{x\}\times\mathbb{C}} \in GL(1,\mathbb{C}) = \mathbb{C}^*,$$

where the $GL(n,\mathbb{C})$ is the complex general linear group of degree $n$ (i.e. the set of $n \times n$ invertible complex matrices). The maps $g_{\alpha\beta}$ are then holomorphic and nowhere vanishing, i.e. $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, and will necessarily satisfy the following identities

$$g_{\alpha\beta}(x)g_{\beta\alpha}(x) = 1 \quad \text{for all } x \in U_\alpha \cap U_\beta$$
$$g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = 1 \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

On the other hand, given an open cover $\underline{U} = \{U_\alpha\}$ of $M$, and holomorphic functions $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying these identities, we can construct a line bundle $L$ with transition functions $g_{\alpha\beta}$ by taking the union of $U_\alpha \times \mathbb{C}$ over all $\alpha$ and identifying $\{x\} \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $g_{\alpha\beta}(x)$. Thus we may also refer to a line bundle as a collection $\{U_\alpha, g_{\alpha\beta}\}$.

**Definition 2.3.2.** Let $\pi : L \to M$ be a holomorphic line bundle over $M$. A *holomor-*

*phic section* (resp. *meromorphic section*) $s$ of $L$ is a holomorphic (meromorphic) map $s : M \to L$ such that $\pi \circ s = s \circ \pi = \mathrm{id}$. Let $H^0(M, L)$ be the set of all holomorphic sections of $L$.

Alternatively, let $L$ be a holomorphic line bundle with transition functions $\{g_{\alpha\beta}\}$, and let $e_\alpha(x) = \phi_\alpha^{-1}(x, 1)$ for $x \in U_\alpha$, where $\phi_\alpha$ is the local trivialization of $L$ over $U_\alpha$. Then we can write, for each $s \in H^0(M, L)$ (resp. meromorphic section of $L$), $s = s_\alpha e_\alpha$ where $s_\alpha$ is a holomorphic (resp. meromorphic) function on $U_\alpha$. It is easy to check that $s_\alpha = g_{\alpha\beta} s_\beta$. Hence we can give this alternative definition: A *holomorphic section* (resp. meromorphic section) $s$ of $L$ is a collection of holomorphic (resp. meromorphic) functions $s_\alpha : U_\alpha \to \mathbb{C}$ such that $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$. These definitions also extend to a complex projective variety $X$.

There is an important interplay between line bundles and divisors. First of all, for any meromorphic section $s$ of $L$, the zero locus $[s = 0] \subseteq X$ gives a divisor on $X$. Conversely, let $D = \{(U_\alpha, f_\alpha)\}$ be Cartier divisor $D$ on $X$ where $X$ is a complex projective space, we can construct a line bundle associated to $D$, denoted by $[D]$, over $X$ as follows: we define our transition functions as

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}.$$

Then we have $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Furthermore, the collection $\{g_{\alpha\beta}\}$ does in fact satisfy the conditions of transition functions. Notice that $f_\alpha = g_{\alpha\beta} f_\beta$, and we see that $\{f_\alpha\}$ is a meromorphic section of $[D]$. This (special) section is called the *canonical section of* $[D]$ and is denoted by $s_D$. Furthermore, when $D$ is effective, $s_D$ is a holomorphic section of $[D]$.

We can also associate a Cartier divisor with a sheaf in the following way: let $D$

be a Cartier divisor on $X$, we define the sheaf $\mathcal{O}_X(D)$ as

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{M}(U) : (f) + D|_U \geq 0\}.$$

$\mathcal{O}_X(D)$ also has vector space structure since if $(f)+D|_U \geq 0$ and $(g)+D|_U \geq 0$, then $(af+bg)+D|_U \geq 0$ for scalars $a$ and $b$.

**Theorem 2.6** ([GH78], page 133-137)**.** *Let $D$ be a Cartier divisor on $X$. Then there is an isomorphism of vector spaces $H^0(X,[D]) \to H^0(X,\mathcal{O}_X(D))$ given by $s \mapsto s/s_D$ where $s_D$ is the canonical section of $[D]$.*

*Proof.* From the definitions above, a global section $f \in H^0(X, O_X(D))$ is a meromorphic function $f$ on $X$ satisfying

$$(f) + D \geq 0.$$

Let $D = \{(U_\alpha, f_\alpha)\}$. Recall that the associated line bundle $[D]$ has transition functions

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}.$$

Given $s \in H^0(X,[D])$, i.e. a collection $s = \{s_\alpha \in \mathcal{O}(U_\alpha)\}$ with

$$\frac{s_\beta}{s_\alpha} = g_{\alpha\beta} = \frac{f_\beta}{f_\alpha},$$

then $\{s_\alpha/f_\alpha\}$ defines a global meromorphic function $g$ on $X$. Since $(s_\alpha) \geq 0$ in every $U_\alpha$, we have

$$(g|_{U_\alpha}) + (f_\alpha) = (s_\alpha) \geq 0.$$

Thus, $(g)+D \geq 0$ globally on $X$, and thus $g \in H^0(X, O_X(D))$. Note that $s_D = \{f_\alpha\}$, hence $s/s_D = g$, so $s/s_D \in H^0(X, \mathcal{O}_X(D))$. It is easy to see that the map $s \mapsto s/s_D$ is injective. To show the map is surjective, let $f \in H^0(X, O_X(D))$, then the collection $\{ff_\alpha\}$ defines a section $s$ of $[D]$. Since

$$(ff_\alpha) = (f) + (f_\alpha) \geq 0$$

in every $U_\alpha$, $s$ is a holomorphic section of $[D]$. Obviously, $s/s_D = f$ since $s_D = \{f_\alpha\}$. This proves that the map $s \mapsto s/s_D$ is surjective.

□

Since we now have this correspondence, we will use the sheaf $\mathcal{O}_X(D)$ and the line bundle $[D]$ interchangeably in the future, and whether we are referring to the line bundle or the sheaf will be clear from context. That is, we will make no distinction between the notations $H^0(X, \mathcal{O}_X(D))$ and $H^0(X, [D])$, or even $H^0(X, D)$.

We can also develop a notion of a "norm" on a line bundle.

**Definition 2.3.3.** Let $L = \{U_\alpha, g_{\alpha\beta}\}$ be a line bundle over $X$ where $U_\alpha$ is an open covering, and $g_{\alpha\beta}$ are transition functions. A *metric* on $L$ is a collection of positive smooth functions

$$h_\alpha : U_\alpha \to \mathbb{R}^+,$$

such that on $U_\alpha \cap U_\beta$ we have

$$h_\beta = |g_{\alpha\beta}|^2 h_\alpha.$$

We will use $h$ to denote the collection $\{h_\alpha\}$. A holomorphic line bundle together with a Hermitian metric $h$ is called a *Hermitian line bundle*.

**Definition 2.3.4.** If $h$ is a metric on a line bundle $L$, then the global form $c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha$ is called the *first Chern form* of $L$ with respect to the metric $h$. We say that a holomorphic line bundle $L$ is *positive* if $L$ admits a metric $h$ such that its first Chern form is positive definite everywhere on $M$.

We have the following landmark theorem by Kodaira.

**Theorem 2.7** (Kodaira Embedding Theorem [GH78], page 176-181)**.** *Let $M$ be a compact complex manifold, and let $L$ be a positive line bundle over $M$. Then there exists $k_0$ such that for $k > k_0$, the map*

$$i_{L^k} : M \to \mathbb{P}^{N-1}$$

*is an embedding. Here $i_{L^k}$ is defined in the following way: choose a basis $\{s_1, \dots, s_N\}$ of $H^0(X, L^k)$, then define the induced map $i_{L^k} : M \to \mathbb{P}^{N-1}$ by*

$$x \mapsto [s_1(x) : \cdots : s_N(x)],$$

*where the choice of homogeneous coordinates on $\mathbb{P}^{N-1}$ corresponds to the basis $\{s_1, \dots, s_N\}$ of $H^0(X, L^k)$.*

Note that when $\{t_1, \dots, t_N\}$ is a different basis for $H^0(X, L^k)$, then the induced map is different, but it only differs by composition with an element of $\mathbb{P}GL(N, \mathbb{C})$.

## 2.4 Big and Nef

Let $D_i, 1 \leq i \leq k$, be Cartier divisors on an $n$-dimensional complex projective variety $X$, and let $V$ be a $k$-cycle (a linear combination of subvarieties of dimension $k$), then the *intersection number* $D_1.D_2.\ldots.D_k.[V] \in \mathbb{Z}$ can be defined. The definition is rather technical, we refer the reader to [Laz04], page 15, for the precise definition. There are a few key properties to note from the definition. The number $D_1.D_2.\ldots.D_k.[V]$ is symmetric, multilinear, and only depends on the linear equivalence class of the $D_i$. When $X = V$ we will use the abbreviation $D_1.D_2.\ldots.D_n.[X] = D_1.D_2.\ldots.D_n \in \mathbb{Z}$. We can now define the term "numerically effective" (nef).

**Definition 2.4.1.** We say that a Cartier divisor D on a complex projective variety $X$ is *nef* if

$$D.C \geq 0 \tag{2.1}$$

for any algebraic curve $C$ in $X$.

**Theorem 2.8.** *The nef divisors on a complex projective variety $X$ form a closed convex cone.*

*Proof.* Let $D_1, \ldots, D_l$ be a finite collection of nef divisors on $X$, and $a_1, \ldots, a_l \geq 0$ be real numbers. Then for any algebraic curve $C$ in $X$, we have the following:

$$(a_1D_1 + \cdots + a_lD_l).C = a_1D_1.C + \cdots + a_lD_l.C \geq 0.$$

□

We can now define what it means for a divisor to be big. Let $X$ be a complex projective variety. Let $L$ be a holomorphic line bundle on $X$. We will use the notation that $h^0(L) = h^0(X, L) = \dim H^0(X, L)$. Then we have the following definition.

**Definition 2.4.2.** Let $L$ be a line bundle on a complex projective variety $X$ of dimension $n$. Then $L$ is *big* if and only if there exists $C > 0$ such that

$$h^0(L^{\otimes m}) \geq Cm^n$$

for all sufficiently large positive integers $m$.

Since we have already established the link between $D$ and $[D]$, going forward, if we mention that a divisor is big, what we clearly mean is that the associated line bundle is big. We can now prove a well known lemma from Kodaira.

**Theorem 2.9** (Kodaira [BS95], page 61)**.** *Let $D$ be a big Cartier divisor, and $E$ be an arbitrary effective Cartier divisor on a complex projective variety $X$. Then*

$$H^0(X, \mathcal{O}_X(mD - E)) \neq 0$$

*for all sufficiently large $m$.*

*Proof.* Consider the short exact sequence

$$0 \to \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_E \to 0.$$

From the Theorem 2.5, we have an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(mD - E)) \to H^0(X, \mathcal{O}_X(mD)) \to H^0(E, \mathcal{O}_X(mD)|_E).$$

Since $D$ is big, $h^0(\mathcal{O}_X(mD))$ grows as $m^{dim(X)}$. On the other hand, $dim(E) <$

$dim(X)$, so $\dim H^0(E, \mathcal{O}_X(mD)|_E)$ grows at most as $m^{dim(X)-1}$. Additionally, from the long exact sequence of cohomology above, we know that

$$h^0(\mathcal{O}_X(mD-E)) \geq h^0(\mathcal{O}_X(mD)) - \dim H^0(E, \mathcal{O}_X(mD)|_E),$$

which is positive. Thus $H^0(X, \mathcal{O}_X(mD-E)) \neq 0$. □

## 2.5 Ampleness

**Definition 2.5.1.** Let $D$ be a Cartier divisor on a projective variety $X$. The *complete linear system of* $D$, denoted $|D|$, is given by

$$|D| = \{D' \mid D' \text{ is an effective divisor, and } D' \sim D\}.$$

The *base locus* of $|D|$ is the intersection of the support of all elements of $|D|$, and we say $|D|$ is *base-point free* if the base locus is empty.

Let $D$ be a Cartier divisor on a projective variety $X$ with $h^0(X, \mathcal{O}_X(D)) > 0$. We can define a map, as in theorem 2.7, in the following way: choose a basis $\{\sigma_1, \ldots, \sigma_N\}$ of $H^0(X, \mathcal{O}_X(D))$, then $D$ defines a rational map (it is defined outside the base locus of $|D|$)

$$\phi : X \to \mathbb{P}^{N-1},$$

where $N = \dim H^0(X, \mathcal{O}_X(D))$, by the rule

$$x \to [\sigma_1(x) : \sigma_2(x) : \cdots : \sigma_N(x)].$$

Again we note that the induced map will differ if we choose a different basis, but it only differs by composition with an element of $\mathbb{P}GL(N, \mathbb{C})$.

**Definition 2.5.2.** We say that a Cartier divisor $D$ on $X$ is *semiample* if $|mD|$ is base-point free for some $m \in \mathbb{N}$, *very ample* if the $\phi$ map defined above is an embedding of $X$, and we say that $D$ is *ample* if $mD$ is very ample for some $m \in \mathbb{N}$.

The above definition also extends to holomorphic line bundles over $X$. With this definition, from Theorem 2.7, we have the following corollary.

**Corollary 2.10.** *If a holomorphic line bundle $L$ is positive, then it is also ample.*

This is perhaps our simplest notion of positivity to understand immediately from the definition. However, from this simple definition we obtain the following theorem.

**Theorem 2.11** (Cartan-Serre-Grothendieck [Laz04]). *Let $X$ be a complex projective variety, and let $D$ be a Cartier divisor on $X$. Then the following are equivalent:*

*i. $D$ is ample;*

*ii. For every coherent sheaf $\mathcal{F}$ on $X$, there is a positive integer $m$ such that*

$$H^i(X, \mathcal{F}(mD)) = 0$$

*for all $m > m_0$ and $i > 0$ (and these cohomology groups are finite dimensional vector spaces);*

iii. *For every coherent sheaf $\mathcal{F}$ on $X$, there is a positive integer $m_0$ such that the natural map*

$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \to \mathcal{F}(mD)$$

*is surjective for all $m$ divisible by $m_0$.*

Finally, we note that if $D$ is ample, then $D$ is big and nef. But the converse statement is not true. See [Laz04] for details.

# Chapter 3

# Nevanlinna Theory

## 3.1 Jensen Formula and First Main Theorem

In the traditional theory of rational functions of one complex variable, one first learns the importance of the degree. Many properties are controlled by this value. For example, if $f$ is a rational function on $\mathbb{C}$, then the number of solutions to the equation $f(z) = a$, for any $a \in \mathbb{C}$, is equal to $d$, counting multiplicity, where $d$ is the degree of $f$. While this result is extremely simple and elegant, it is limited to a very specific set of functions. What if one wanted to make a similar conclusion for a broader set of functions, for example, all transcendental meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$? It was to this end that Nevanlinna theory was developed. Obviously such functions do not have a "degree" in the classical sense. In this case, we will define what is called the characteristic function, $T_f(r)$, which will take on the role of the classical degree. We will denote the number of solutions to the equation $f(z) = a$ in the disc $\{z : |z| \leq r\}$, counting multiplicity, by $n(r, a) = n_f(r, a)$ for any $a \in \hat{\mathbb{C}}$. Let

$$N_f(r, a) = \int_0^r \frac{n_f(t, a)}{t} dt. \tag{3.1}$$

Then we have the equation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta}) - a| d\theta = \log |f(0) - a| + N_f(r, a) - N_f(r, \infty). \tag{3.2}$$

This is called the Jensen formula, which will be used to obtain our main Nevanlinna theories. First let us introduce some notation. Let $x^+ = \max\{x, 0\}$. We define

$$m_f(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta,$$

and

$$m_f(r, a) = m_{\frac{1}{f-a}}(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta.$$

Using the fact that

$$\log^+(x \pm y) \leq \log^+ x + \log^+ y,$$

we obtain

$$m_f(r, \infty) = m_{f-a}(r, 0) + O(1).$$

Thus we can rewrite the Jensen formula (3.2) as

$$m_f(r,\infty) + N_f(r,\infty) = m_f(r,a) + N_f(r,a) + O(1), \qquad r \to \infty.$$

This is the First Main Theorem of Nevanlinnna theory (FMT), and it gives us our motivation for defining the *Nevanlinna characteristic* $T_f(r)$ by

$$T_f(r) := m_f(r,\infty) + N_f(r,\infty).$$

Since the two functions in the definition of our Nevanlinna characteristic count $a$-points and the average proximity of $f$ to $a$ on the circle $|z| = r$, we will refer to $N_f(r,a)$ and $m_f(r,a)$ as the *counting function* and *proximity function*, respectively.

## 3.2 Second Main Theorem

The main result of this dissertation will be a generalization of the Second Main Theorem (SMT), which we are now ready to state and prove. First, let $n_1(r) = n_{1,f}(r)$ denote the number of critical points of a meromorphic function $f$ in the disc $|z| \leq r$, counting multiplicity. Then we have that

$$n_{1,f}(r) = n_{f'}(r,0) + 2n_f(r,\infty) - n_{f'}(r,\infty). \tag{3.3}$$

Now just as in (3.1), we can define $N_1(r)$ as

$$N_1(r) = N_{1,f}(r) := \int_0^r \frac{n_1(t)}{t} dt. \tag{3.4}$$

Then we have the following statement of the SMT.

**Theorem 3.1** (Second Main Theorem [Nev29]). *For every finite set* $\{a_1, \ldots, a_q\} \subset \hat{\mathbb{C}}$ *we have*

$$\sum_{j=1}^{q} m_f(r, a_j) + N_1(r) \leq 2T_f(r) + S(r), \tag{3.5}$$

*where* $S(r)$ *is a small error term,* $S(r) = O(log(rT_f(r)))$ *when* $r \to \infty, r \notin E$*, where* $E$ *is a set of finite measure.*

In order to prove the SMT we will need the following lemma.

**Lemma 3.2.** *If* $g$ *is an increasing function on* $[0, \infty]$ *tending to* $+\infty$*, and* $\epsilon > 0$*, then* $g' \leq g^{1+\epsilon}(x)$ *for all* $x \notin E$*, where* $E$ *is a set of finite measure.*

*Proof.* Let $E$ be the set where $g' \geq g^{1+\epsilon}(x)$, then

$$\int_E dz \leq \int_E \frac{g'(x)}{g^{1+\epsilon}}(x)dx = \int \frac{dy}{y^{1+\epsilon}} < \infty.$$

$\square$

*Proof of the Second Main Theorem.* [Ahl39] Consider the area element

$$d\rho = \rho^2(w)\frac{dxdy}{\pi(1+|w|^2)^2},$$

where

$$\log \rho(w) := \sum_{j=1}^{q} \log \frac{1}{[w, a_j]} - 2\log\left(\sum_{j=1}^{q} \frac{1}{[w, a_j]}\right) + C, \tag{3.6}$$

and $[x, y]$ is the chordal distance between $x$ and $y$, and $C > 0$ is chosen so that

$$\int_{\mathbb{C}} d\rho = 1.$$

Then we can use $f$ to pull back our area element $d\rho$, and change variables to obtain

$$\int_{\mathbb{C}} n_f(r, a)d\rho(a) = \int_0^r \int_{-\pi}^{\pi} \rho^2(w)\frac{|w'|^2}{(1+|w|^2)^2} td\theta dt, \tag{3.7}$$

where $w = f(te^{i\theta})$. Next apply the derivative with respect to $r$ to the double integral on the right and divide by $2\pi r$ to obtain

$$\lambda(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho^2(w)\frac{|w'|^2}{(1+|w|^2)^2} d\theta. \tag{3.8}$$

Now note the following inequality,

$$\frac{1}{b-a}\int_a^b \log(g(x))dx \leq \log\Big(\frac{1}{b-a}\int_a^b g(x)dx\Big). \tag{3.9}$$

Combining (3.9) with (3.8) yields

$$\log\lambda(r) \geq \frac{1}{\pi}\int_{-\pi}^{\pi} \log\rho(w)d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} \log(1+|w|^2)d\theta + \frac{1}{\pi}\int_{-\pi}^{\pi} \log|w'|d\theta.$$

The first integral can be approximated using (3.6). The second term in (3.6) becomes irrelevant as it contains a double log, and we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log\rho(f(re^{i\theta}))d\theta = 2\sum_{j=1}^{q} m_f(r, a_j) + O(\log T_f(r)).$$

The second integral equals $4m_f(r,\infty)$, and the third can be evaluated with Jensen's formula. This gives the following relation

$$2\sum_{j=1}^{q} m_f(r,a_j) + 2\{N_{f'}(r,0) - N_{f'}(r,\infty) - 2m_f(r,\infty)\} \leq \log\lambda(r) + O\log T_f(r).$$

The expression in brackets is equal to $N_1(r) - 2T_f(r)$ by the FMT. Thus

$$\sum_{j=1}^{q} m_f(r,a_j) + N_1(r) - 2T_f(r) \leq \frac{1}{2}\log\lambda(r).$$

This is almost our desired result. All that is left to do is to estimate $\lambda$. To this end, let $d\rho$ be a probability measure in $\mathbb{C}$. Now integrate the FMT with respect to $a \in \mathbb{C}$ against $d\rho$ to obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(f(re^{i\theta}))d\theta = U(f(0)) + \int_{\mathbb{C}} N_f(r,a)d\rho(a) - N_f(r,\infty),$$

where

$$U(w) = \int_{\mathbb{C}} \log|w-a|d\rho(a).$$

Then we can estimate the expression in (3.7) by

$$\int_0^r \frac{dt}{t}\int_0^t \lambda(s)sds = \int_{\mathbb{C}} N_f(r,a)d\rho(a) \leq N_f(r,\infty) + \frac{1}{2\pi}\int_{-\pi}^{\pi} U(f(re^{i\theta}))d\theta.$$

But $U(w)$ is a potential of a probaility measure $d\rho$, so $U(w) \leq \log^+|w| + O(1)$, and we obtain that

$$\int_0^r \frac{dt}{t} \int_0^t \lambda(s) s ds \leq T_f(r) + O(1).$$

Now applying Lemma 3.2 twice we obtain that $\lambda(r) < rT_f(r)^{1+\epsilon}, r \notin E$. Thus we can conclude that $\log \lambda(r) = S(r)$ which proves the theorem. □

We can rewrite the SMT using the FMT in the following way (as in [YY03], page 23): let $\bar{N}_f(r,a)$ be the averaged counting function of distinct solutions of $f(z) = a$. Then $\sum N_f(r, a_j) \leq \sum \bar{N}_f(r,a) + N_1(r)$, and we have

$$(q-2)T_f(r) \leq \sum_{j=1}^{q} \bar{N}_f(r,a) + S(r). \tag{3.10}$$

Now that we have the machinery of the FMT and SMT, the Little Picard Theorem is an immediate corollary. If a meromorphic function $f$ were to omit three values, then the left hand side of (3.10) would be equal to $T_f(r)$. Thus $T_f(r) \leq S(r)$, which implies $f$ is a constant. We also obtain another, similar lemma.

**Lemma 3.3.** *Let $a_1, \ldots, a_5$ be five points on the Riemann sphere, then at least one of the equations $f(z) = a_j$ has simple solutions.*

*Proof.* If all five equations have multiple solutions, then $N_1(r,f) \geq (1/2)\sum_{j=1}^{5} N_f(r,a_j)$. Combining this with the SMT implies that $(5/2)T_f(r) \leq 2T_f(r) + S(r)$, thus $f$ must be constant. □

## 3.3 Generalizing the First Main Theorem

We now have the FMT and SMT for the case when $f$ is a meromorphic function on $\mathbb{C}$. Our next step is to generalize the FMT to the case when $X$ is a complex projective variety, $(L, h)$ is a Hermitian line bundle on $X$, and $f : \mathbb{C} \to X$ is a holomorphic map. Before we do this however, we will need to recall the following theorem.

**Theorem 3.4.** *Let $g$ be a function of class $C^2$ on $\bar{\mathbf{D}}(r)$, or a sub-harmonic function on $\bar{\mathbf{D}}(r)$. Then*

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<r} dd^c[g] = \frac{1}{2} g(re^{i\theta}) \frac{d\theta}{2\pi} - g(0).$$

*Proof.* Let $Z$ denote the set of singularities of $g$, and $S(Z, \epsilon)$ be the union of small circles of radius $\epsilon$ around singularites with $\epsilon$ small enough to remain in $\mathbf{D}(t)$. Then Stokes' theorem implies that

$$\begin{aligned}
\int_{|\zeta|<r} dd^c[g] &= \int_{|\zeta|=r} d^c[g] - \lim_{\epsilon \to 0} \int_{S(X,\epsilon)(t)} d^c g \\
&= \frac{1}{2} \int_{|\zeta|=r} r \frac{\partial g}{\partial r} \frac{d\theta}{2\pi} - \lim_{\epsilon \to 0} \int_{S(X,\epsilon)(t)} d^c g,
\end{aligned}$$

which we can integrate with respect to $1/t$ to obtain

$$\begin{aligned}
\int_0^r \frac{dt}{t} \int_{|\zeta|<r} dd^c[g] &= \int_0^r \frac{dt}{t} \int_{|\zeta|=t} \frac{1}{2} t \frac{\partial g}{\partial t} \frac{d\theta}{2\pi} - \int_0^r \frac{dt}{t} \lim_{\epsilon \to 0} \int_{S(X,\epsilon)(t)} d^c g \\
&= \frac{1}{2} \int_0^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} - g(0) - \int_0^r \frac{dt}{t} \mathrm{Sing}_g(t).
\end{aligned}$$

Thus,

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g] = \frac{1}{2} \int_0^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} - g(0),$$

and by definition we have that

$$\begin{aligned} \int_{|\zeta|<t} dd^c[g] &= \int_{|\zeta|<t} dd^c g + \mathrm{Sing}_g(t) \\ &= \int_{|\zeta|<t} dd^c g + \lim_{\epsilon \to 0} \int_{S(X,\epsilon)(t)} dd^c g. \end{aligned}$$

Thus the theorem is proved. □

**Lemma 3.5** (Poincare-Lelong Formula [GH78], page 388)**.** *Let $f$ be a holomorphic function on $\mathbf{D}(r)$, then*

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<r} dd^c[\log|f|^2] = N_f(r, 0).$$

*Which we can write as*

$$dd^c[\log|f|^2] = [f = 0],$$

*where $[f = 0] = \sum_p (ord_p f) \cdot p$ is the divisor associated with $f$.*

In order to define the Nevanlinna functions in this new setting, we will need to be familiar with the concept of a Weil function. Let $X$ be a complex projective variety and $D$ be an effective Cartier divisor on $X$. A *Weil function* for $D$ is a function $\lambda_D : (X \backslash \mathrm{supp} D) \to \mathbb{R}$ such that for every $x \in X$ there is an open neighborhood $U$ of $x$ in $X$, a nonzero rational function $f$ on $X$ with $D|_U = (f)$, and a continuous function $\alpha : U \to \mathbb{R}$ such that

$$\lambda_D(x) = -\log|f(x)| + \alpha(x)$$

for all $x \in (U \backslash \mathrm{supp} D)$. A continuous (fiber) metric $\|\cdot\|$ on the line sheaf $\mathcal{O}_X(D)$ determines a Weil function for $D$ given by

$$\lambda_D(x) = -\log\|s(x)\|,$$

where $s$ is the rational section of $\mathcal{O}_X(D)$ such that $D = (s)$. As an example, the Weil function for the hyperplane $H = \{a_0x_0 + \cdots + a_nx_n = 0\}$ on $\mathbb{P}^n(\mathbb{C})$ is given by

$$\lambda_H(x) = \log \frac{\max_{0\leq i\leq n}|x_i| \max_{0\leq i\leq n}|a_i|}{|a_0x_0 + \cdots + a_nx_n|}$$

for $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C})\backslash H$.

**Definition 3.3.1** (Nevanlinna Functions)**.** Let $D$ be an effective Cartier divisor on a projective variety $X$. We can now define the Nevalinna functions in our new setting.

i. The characteristic function $T_{f,L}$ of $f$ with respect to $(L, h)$ is defined by

$$T_{f,L}(r) = \int_0^r \frac{dt}{t} \int_{|\zeta|\leq r} f^*c_1(L, h)$$

**Remark 3.6.** *This definition of the characteristic function behaves precisely as desired, namely, if $L$ is ample and $T_{f,L}(r)$ is bounded, then $f$ must be a constant;*

ii. The proximity function of $f$ with respect to $D$ is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi},$$

where $\lambda_D$ is the Weil function associated to $D$;

iii. The counting function of $f$ with respect to $D$ is defined by

$$N_f(r, D) = \int_0^r n_f(t, D)\frac{dt}{t},$$

where $n_f(t, D)$ denotes the number of points of $f^{-1}(D)$ in the disc $|z| < t$ counting multiplicity.

These definitions lead us to the first main theorem which appears just as it did in the previous section.

**Theorem 3.7** (First Main Theorem)**.** *Let $X$ be a complex projective variety. Let $(L, h)$ be a Hermitian line bundle over $X$. Let $s$ be a holomorphic section of $L$, and let $D = [s = 0]$. Then for any holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C})$ not in $D$,*

$$T_{f,L}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

*Proof.* By definition, on $U_\alpha$, $||s||^2 = |s_\alpha|^2 h_\alpha$, so by the Poincare-Lelong formula,

$$dd^c[\log ||s||^2] = -c_1(L, h) + [D].$$

The FMT is then obtained by applying Theorem 3.4. $\square$

## 3.4 Generalizing the Second Main Theorem

Now that we have generalized the FMT to the case when $f : \mathbb{C} \to X$ where $X$ is a complex projective variety, we will now state and prove the SMT for the case when $M = \mathbb{P}^n(\mathbb{C})$ with the divisors as hyperplanes. In order to do this, we will need the help of the following lemma.

**Lemma 3.8** (Logarithmic Derivative Lemma [Ru01], page 8)**.** *Let $f(z)$ be a meromorphic function. Then for $\delta > 0$*

$$\int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq \left(1 + \frac{(1+\delta)^2}{2}\right) \log^+ T_f(r) + \frac{\delta}{2}\log(r) + O(1)||_{E(\delta)}.$$

*Proof.* For $w \in \mathbb{C}$, we define the surface element as follows:

$$\Phi = \frac{1}{(1+\log^2|w|)|w|^2} \frac{\sqrt{-1}}{2\pi^2} dw \wedge d\bar{w}.$$

This is a (1,1) form on $\mathbb{C}$ with singularities at $w = 0, \infty$. By computation,

$$\int_{\mathbb{C}} \Phi = \int_{\mathbb{C}} \frac{1}{(1+\log^2|r|)|r|^2} \frac{1}{2\pi^2} r dr d\theta = 1.$$

By the change of variable formula, we have

$$\int_{\Delta(t)} f^*\Phi = \int_{w \in \mathbb{C}} n_f(t,w)\Phi(w).$$

Thus, if we let $\mu(r) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\Phi$, we obtain that

$$\mu(r) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \frac{|f'|^2}{(1+\log^2|f|)|f|^2} \frac{\sqrt{-1}}{4\pi^2} dz \wedge d\bar{z}$$

$$= \int_{w\in\mathbb{C}} \int_1^r \frac{dt}{t} n_f(t,w)\Phi(w) = \int_{w\in\mathbb{C}} N_f(r,w)\Phi(w) \leq T_f(r) + O(1).$$

Where the last inequality holds due to the FMT. By Lemma 3.2, we have that

$$\frac{1}{\pi}\int_{|z|=r} \frac{|f'|^2}{(1+\log^2|f|)|f|^2}\frac{d\theta}{2\pi} \leq (\mu(r))^{(1+\delta)^2} r^\delta b^\delta ||_{E_\delta},$$

where $b$ is a constant. As a result of this, Lemma 3.2, and the concavity of log, we can compute the following:

$$\begin{aligned}
\int_0^{2\pi} \log^+\left|\frac{f'}{f}(re^{i\theta})\right|\frac{d\theta}{2\pi} &= \frac{1}{4\pi}\int_{|z|=r} \log^+\left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2}(1+\log^2|f|)\right)d\theta \\
&\leq \frac{1}{4\pi}\int_{|z|=r} \log^+\left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\right)d\theta \\
&+ \frac{1}{4\pi}\int_{|z|=r} \log^+(1+(\log^+|f|+\log^+(1/|f|))^2)d\theta \\
&\leq \frac{1}{4\pi}\int_{|z|=r} \log\left(1+\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\right)d\theta \\
&+ \frac{1}{2\pi}\int_{|z|=r} \log^+(\log^+|f|+\log^+(1/|f|))d\theta + \frac{1}{2}\log 2 \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi}\int_{|z|=r}\frac{|f'|^2}{(1+\log^2|f|)|f|^2}d\theta\right) \\
&+ \frac{1}{2\pi}\int_{|z|=r} \log(1+\log^+|f|+\log^+(1/|f|))d\theta + \frac{1}{2}\log 2 \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2}\mu^{(1+\delta)^2}(r)r^\delta b^\delta\right) \\
&+ \log(1+m_f(r)+m_{1/f}(r)) + \frac{1}{2}\log 2 ||_{E_\delta} \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2}(\mu(r))^{(1+\delta)^2}r^\delta b^\delta\right) + \log^+ T_f(r) + O(1)||_{E_\delta}
\end{aligned}$$

$$\leq \quad \left(1 + \frac{(1+\delta)^2}{2}\right) \log^+ T_f(r) + \frac{\delta}{2} \log r + O(1)||_{E_\delta}.$$

Which proves the theorem. □

We will now ensure that the reader is familiar with how our generalizations of the Nevanlinna functions appear in this special case when $M = \mathbb{P}^n(\mathbb{C})$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$. Let $L = \mathcal{O}_{\mathbb{P}^n}(1)$ be the hyperplane line bundle with transition functions $g_{\alpha\beta} = w_\alpha / w_\beta$, where $U_\alpha = \{w_\alpha = 0\}$. The sections of $L$ are $s_H = \{\langle \mathbf{a}, \mathbf{w} \rangle / w_\alpha\}$ with $[s_H = 0] = H = \{a_0 w_0 + \cdots + a_n w_n = 0\}$. The metric on $L$ is given by $h_\alpha = |w_\alpha|^2 / ||\mathbf{w}||^2$. The first Chern form of this metric is given by

$$c_1(L, h) = -dd^c \log h_\alpha = dd^c \log ||\mathbf{w}||^2.$$

This is the so called Fubini-Study metic on $\mathbb{P}^n$. By Theorem 3.4, the characteristic function takes the form

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{|\zeta|<r} dd^c \log ||\mathbf{f}||^2 = \int_0^{2\pi} \log ||\mathbf{f}(re^{i\theta})|| \frac{d\theta}{2\pi} + O(1),$$

where $\mathbf{f} = (f_o, \ldots, f_n)$ is a reduced representation of $f$, that is, $f_o, \ldots, f_n$ have no common zeros. The proximity function will take the form

$$m_f(r, H) = \int_0^{2\pi} \log \frac{1}{||s_H \circ f(re^{i\theta})||} \frac{d\theta}{2\pi} = \int_0^{2\pi} \log \frac{||f(re^{i\theta})|| \cdot ||H||}{|\langle \mathbf{a}, \mathbf{f}(re^{i\theta}) \rangle|} \frac{d\theta}{2\pi},$$

as $\lambda_H(f(z)) = \log \frac{||\mathbf{f}(z)|| \cdot ||\mathbf{a}||}{|\langle \mathbf{f}(z), \mathbf{a} \rangle|}$ is the Weil-function. Lastly, the counting function of $f$

with respect to $H$ is given by

$$N_f(r, H) = \int_0^r (n_f(t, H) - n_f(0, H)) \frac{dt}{t} + n_f(0, H) \log r,$$

where $n_f(t, H)$ is the number of points where $\langle \mathbf{a}, \mathbf{f} \rangle = 0$ in the disc $|z| < t$, counting multiplicity. And by Jensen's formula,

$$N_f(r, H) = \int_0^{2\pi} \log |\langle \mathbf{f}(re^{i\theta}), \mathbf{a} \rangle| \frac{d\theta}{2\pi} + O(1).$$

We are now ready to state the SMT in this setting.

**Theorem 3.9** (Cartan's Second Main Theorem [Car])**.** *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve. Then for any $\delta > 0$, we have*

$$\begin{aligned} &\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \\ &\leq (n+1)T_f(r) + O(\log^+ T_f(r)) + \delta \log r + O(1) ||_{E_\delta}, \end{aligned}$$

*where the Wronskian of $f_0, \ldots, f_n$ is denoted by $W(f_0, \ldots, f_n)$.*

This version of the SMT can be derived from a more general version which we will now state, prove, and then prove the derivation.

**Theorem 3.10** (General Second Main Theorem)**.** *Let $f = [f_0 : \cdots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Let $H_1, \ldots, H_q$ (or $\mathbf{a}_1, \ldots, \mathbf{a}_q$) be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then for any $\delta > 0$, we* *have the inequality*

$$\begin{aligned}&\int_0^{2\pi} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\frac{d\theta}{2\pi} + N_W(r,0)\\ &\le (n+1)T_f(r) + O(\log T_f(r)) + \delta \log r + O(1)||_{E_\delta},\end{aligned}$$

*where the maximum is taken over all subsets $K$ of $\{1,\ldots,q\}$ such that $\mathbf{a}_j, j\in K$, are linearly independent.*

*Proof.* Let $H_1,\ldots,H_q$ be the given hyperplanes with coefficient vectors $\mathbf{a}_1,\ldots,\mathbf{a}_q \in \mathbb{C}^{n+1}$. Denote $K \subset \{1,\ldots,q\}$ such that $\mathbf{a}_j, j \in K$ are linearly independent. Without loss of generality, we may assume that $q \ge n+1$ and that $\#K = n+1$. Let $T$ denote all injective maps $\mu : \{0,\ldots,n\} \to \{1,\ldots,q\}$ such that $\mathbf{a}_{\mu(0)},\ldots,\mathbf{a}_{\mu(n)}$ are linearly independent. Then

$$\begin{aligned}&\int_0^{2\pi} \max_K \sum_{k\in K} \lambda_{H_k}(f(re^{i\theta}))\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi} \max_{\mu\in T} \sum_{j=0}^{n} \log\left(\frac{||\mathbf{f}(re^{i\theta})||\cdot||\mathbf{a}_{\mu(j)}||}{|\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{\mu(j)}\rangle}\right)\frac{d\theta}{2\pi}\\ &= \int_0^{2\pi} \log\left\{\max_{\mu\in T}\left(\frac{||\mathbf{f}(re^{i\theta})||^{n+1}}{\prod_{j=0}^n |\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{\mu(j)}\rangle}\right)\right\}\frac{d\theta}{2\pi} + O(1)\\ &\le \int_0^{2\pi} \log\left\{\sum_{\mu\in T}\left(\frac{||\mathbf{f}(re^{i\theta})||^{n+1}}{\prod_{j=0}^n |\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{\mu(j)}\rangle|}\right)\right\}\frac{d\theta}{2\pi} + O(1)\\ &= \int_0^{2\pi} \log\left\{\sum_{\mu\in T}\frac{|W(\langle \mathbf{f}, \mathbf{a}_{\mu(0)}\rangle,\ldots,\langle \mathbf{f}, \mathbf{a}_{\mu(n)}\rangle)(re^{i\theta})}{\prod_{j=0}^n |\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{\mu(j)}\rangle|}\right\}\frac{d\theta}{2\pi}\\ &+ \int_0^{2\pi} \log\left\{||\mathbf{f}(re^{i\theta})||^{n+1}/|W(f_0,\ldots,f_n)|(re^{i\theta})\right\}\frac{d\theta}{2\pi} + O(1),\end{aligned} \tag{3.11}$$

where $W(\langle \mathbf{f}, \mathbf{a}_{\mu(0)}\rangle,\ldots,\langle \mathbf{f}, \mathbf{a}_{\mu(n)}\rangle)$ denotes the Wronskian of the functions

$\langle \mathbf{f}, \mathbf{a}_{\mu(0)}\rangle, \ldots, \langle \mathbf{f}, \mathbf{a}_{\mu(n)}\rangle)$. In the last line of equation (3.11), we use the property of Wronskians that

$$|W(f_0, \ldots, f_n)| = |W(\langle f, \mathbf{a}_{\mu(0)}\rangle, \ldots, \langle f, \mathbf{a}_{\mu(n)}\rangle)| \cdot C,$$

where $C$ is a constant. Now we will estimate the first term on the right-hand side of equation (3.11). Let

$$g_{\mu(l)} = \frac{\langle \mathbf{f}, \mathbf{a}_{\mu(l)}\rangle}{\langle \mathbf{f}, \mathbf{a}_{\mu(0)}\rangle}, 0 \leq l \leq n.$$

Then $T_{g_{\mu(l)}}(r) \leq T_f(r) + O(1)$ for $0 \leq l \leq n$. Hence, by the Logarithmic Derivative Lemma,

$$\begin{aligned}
&\int_0^{2\pi} \log \left\{ \sum_{\mu \in T} \frac{|W(\langle \mathbf{f}, \mathbf{a}_{\mu(0)}\rangle, \ldots, \langle \mathbf{f}, \mathbf{a}_{\mu(n)}\rangle)(re^{i\theta})}{\prod_{j=0}^{n} |\langle \mathbf{f}(re^{i\theta}), \mathbf{a}_{\mu(j)}\rangle|} \right\} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log^+ \sum_{\mu \in T} \left( \frac{|W(1, g_{\mu(1)}, \ldots, g_{\mu(n)})|}{|g_{\mu(1)}, \ldots, g_{\mu(n)}|}(re^{i\theta}) \right) \frac{d\theta}{2\pi} + O(1) \\
&\leq O(\log T_f(r)) + \delta \log r + O(1) ||_{E_\delta}.
\end{aligned} \tag{3.12}$$

Now

$$\begin{aligned}
&\int_0^{2\pi} \log \left\{ ||\mathbf{f}||^{n+1} / |W(f_0, \ldots, f_n)(re^{i\theta})| \right\} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log ||\mathbf{f}||^{n+1} \frac{d\theta}{2\pi} + \int_0^{2\pi} \log \frac{1}{|W(f_0, \ldots, f_n)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&= (n+1) T_f(r) - N_W(0, r).
\end{aligned} \tag{3.13}$$

Combining (3.11), (3.12), and (3.13) concludes the proof. □

In order to deduce the SMT from the general SMT, we will prove the following lemma.

**Lemma 3.11.** *Let $H_1, \ldots, H_q$ be hyper planes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int_0^{2\pi} \max_{\mu \in T} \sum_{i=0}^{n} \lambda_{H_{\mu(i)}}(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1).$$

*Proof.* Let $\mathbf{a}_{\mu(i)}$ be the coefficient vectors of $H_j, 1 \leq j \leq q$. By definition,

$$\langle \mathbf{f}, \mathbf{a}_{\mu(i)} \rangle = a_0^{\mu(i)} f_0 + \cdots + a_n^{\mu(i)} f_n, 0 \leq i \leq n,$$

where $\mathbf{a}_{\mu(i)} = (a_0^{\mu(i)}, \ldots, a_n^{\mu(i)})$. By solving the system of linear equations above,

$$f_i = \widetilde{a}_0^{\mu(i)} \langle \mathbf{f}, \mathbf{a}_{\mu(0)} \rangle + \cdots + \widetilde{a}_n^{\mu(i)} \langle \mathbf{f}, \mathbf{a}_{\mu(n)} \rangle, 0 \leq i \leq n,$$

where $(\widetilde{a}_j^{\mu(i)})$ is the inverse matrix of $a_j^{\mu(i)}$. Thus for any $\mu \in T$,

$$||\mathbf{f}(z)|| \leq C \max_{0 \leq i \leq n} \{ \langle \mathbf{f}, \mathbf{a}_{\mu(i)} \rangle | \}. \tag{3.14}$$

For a given $z \in \mathbb{C}$, there exists a $\mu \in T$ such that

$$0 < |\langle \mathbf{f}(z), \mathbf{a}_{\mu(0)} \rangle| \leq \cdots \leq |\langle \mathbf{f}(z), \mathbf{a}_{\mu(n)} \rangle| \leq |\langle \mathbf{f}(z), \mathbf{a}_j \rangle|,$$

for $j \neq \mu(i)$, $i = 0, 1, \ldots, n$. Hence by (3.14)

$$\prod_{j=1}^{q} \frac{||\mathbf{f}(z)||}{|\langle \mathbf{f}(z), \mathbf{a}_j \rangle|} \leq C \max_{\mu \in T} \frac{||\mathbf{f}(z)||}{|\langle \mathbf{f}(z), \mathbf{a}_{\mu(i)} \rangle|}, \tag{3.15}$$

and the lemma is proved. □

We will in fact be able to further generalize this theorem to the setting of a complex projective variety. In this setting the theorem appears as follows.

**Theorem 3.12** ([RV17], Theorem 2.8)**.** *Let $X$ be a complex projective variety, and let $D$ be an effective Cartier divisor on $X$. Let $V$ be a nonzero linear subspace of $H^0(X, \mathcal{O}(D))$, and let $s_1, \ldots, s_q$ be nonzero elements of $V$. For each $i = 1, \ldots, q$, let $D_j$ be the Cartier divisor $(s_j)$. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{D_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \leq (dimV + \epsilon) T_{f,D}(r) \|,$$

*where the set $J$ ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent.*

*Proof.* Let $d = \dim V$. We may assume that $d > 1$ since otherwise all $D_j$ are the same divisor, the sets $J$ have at most one element each, and the theorem follows from the First Main Theorem.

Let $\Phi : X \to \mathbb{P}^{d-1}$ be the rational map associated to the linear system $V$. Let $X'$ be the closure of the graph of $\Phi$, and let $p : X' \to X$ and $\phi : X' \to \mathbb{P}^{d-1}$ be the projection morphisms. Let $\hat{f} : \mathbb{C} \to X'$ be the lifting of $f$.

Note that, even though $\Phi$ extends to the morphism $\phi : X' \to \mathbb{P}^{d-1}$, the linear system of $H^0(X', p^*\mathcal{O}(D))$ corresponding to $V$ may still have base points. However, there is an effective Cartier divisor $B$ on $X'$ such that, for each nonzero $s \in V$, there is a hyperplane $H$ in $\mathbb{P}^{d-1}$ such that $p^*(s) - B = \phi^* H$. More precisely, $\phi^*\mathcal{O}(1) \cong$

$\mathcal{O}(p^*D - B)$. The map

$$\alpha : H^0(X', \mathcal{O}(p^*D - B)) \to H^0(X, \mathcal{O}(p^*D))$$

defined by tensoring with the canonical global sections $s_B$ of $\mathcal{O}(B)$ is injective, and its image contains $p^*(V)$. The preimage $\alpha^{-1}(p^*(V))$ corresponds to a base-point free linear system for the divisor $p^*D - B$.

For each $j = 1, \ldots, q$, let $H_j$ be the hyperplane in $\mathbb{P}^{d-1}$ for which $p^*(s_j) - B = \phi^* H_j$. Then

$$\lambda_{p^*D_j} = \lambda_{\phi^* H_j} + \lambda_B + O(1). \tag{3.16}$$

By the functoriality of Weil functions, $\lambda_{p^*H_j}(\hat{f}(z)) = \lambda_{D_j}(f(z))$. Therefore, it will suffice to prove the inequality

$$\begin{aligned} & \int_0^{2\pi} \left( \max_J \sum_{j \in J} \lambda_{H_j}(\phi(\hat{f})(re^{i\theta})) + \lambda_B(\hat{f}(re^{i\theta})) \right) \frac{d\theta}{2\pi} \\ \leq \quad & (\dim V + \epsilon) T_{f,D}(r) \quad \|. \end{aligned}$$

For any subset $J$ of $1, \ldots, q$, the sections $s_j$, $j \in J$, are linearly independent elements of $V$ if and only if the hyperplanes $H_j$, $j \in J$, lie in general position in $\mathbb{P}^{d-1}$. Thus we may apply Cartan's Theorem from above to obtain that

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{H_j}(\phi(\hat{f})(re^{i\theta})) \frac{d\theta}{2\pi} \leq (\dim V + \epsilon) T_{\phi(\hat{f})}(r).$$

From (3.16), we get $T_{\phi(\hat{f})}(r) = T_{f,D}(r) - T_{\hat{f},B}(r) + O(1)$. On the other hand, since each set $J$ has at most $\dim V$ elements, and $B$ is effective, we get

$$(\#J)\lambda_B(x) \leq (\dim V)\lambda_B(x) + O(1)$$

for all $x \in X'$. Hence

$$\begin{aligned} & \int_0^{2\pi} \left( \max_J \sum_{j \in J} \lambda_{H_j}(\phi(\hat{f})(re^{i\theta})) + \lambda_B(\hat{f}(re^{i\theta})) \right) \frac{d\theta}{2\pi} \\ \leq \; & (\dim V + \epsilon)T_{f,D}(r) - (\dim V + \epsilon)T_{\hat{f},B}(r) + (\dim V) m_{\hat{f}}(r, B) \quad \| \\ \leq \; & (\dim V + \epsilon)T_{f,D}(r) \quad \|, \end{aligned}$$

where in the last inequality we used from the first main theorem that $m_{\hat{f}}(r, B) \leq T_{\hat{f},B}(r) + O(1)$, and the theorem is proved. $\square$

The basic theorem above motivates the notation of the Nevanlinna constant, which will be key to proving our result. Let $X$ be a normal projective variety, and $D$ be an effective divisor on $X$. For any section $s \in H^0(X, \mathcal{O}(D))$, we use $\mathrm{ord}_E(s)$ to denote the coefficients of $(s)$ in $E$, where $(s)$ is the divisor on $X$ associated to $s$. We will not recall the definition of normal projective variety here (see [Laz04], page 15 for the precise definition), but the condition of normality of $X$ is assumed so that $\mathrm{ord}_E D$ is defined for any prime divisor $E$, and any effective Cartier divisor $D$ on $X$ ([Laz04], Remark 1.1.4). We then have the following definition of the Nevanlinna constant.

**Definition 3.4.1** ([Ru15])**.** Let $X$ be a normal complex projective variety, and $D$ be an effective divisor on $X$. The *Nevanlinna constant* of $D$, denoted by *Nev*($D$), is given by

$$Nev(D) := \inf_N \left( \inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right), \tag{3.17}$$

where the infimum "$\inf_N$" is taken over all positive integers $N$, and the infimum "$\inf_{\{\mu_N, V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$ where $\mu_N$ is a positive real number, and $V_N \subset H^0(X, \mathcal{O}(ND))$ is a linear subspace with $\dim V_N \geq 2$ such that, for all $P \in \operatorname{supp} D$, there exists a basis $B$ of $V_N$ with

$$\sum_{s \in B} \operatorname{ord}_E(s) \geq \mu_N \operatorname{ord}_E(ND)$$

for all irreducible components $E$ of $D$ passing through $P$. If $\dim H^0(X, \mathcal{O}(ND)) \leq 1$ for all positive integers $N$, then we define $Nev(D) = +\infty$.

With this notation we have the following key theorem.

**Theorem 3.13** ([Ru15]). *Let $X$ be a complex normal projective variety and $D$ be an effective Cartier divisor on $X$. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq (Nev(D) + \epsilon)\, T_{f,D}(r) \quad \|_E$$

*holds for any Zariski dense holomorphic mapping $f : \mathbb{C} \to X$.*

*Proof.* Let $\sigma_0$ denote the set of all prime divisors occurring in $D$, so we can write

$$D = \sum_{E \in \sigma_0} \operatorname{ord}_E(D) E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 | \cap_{E \in \sigma} E \neq 0\}.$$

For an arbitrary $x \in X$, pick $\sigma \in \Sigma$ (depents on x) for which

$$\lambda_D(x) \leq \lambda_{D_{\sigma,1}}(x),$$

where $D_{\sigma,1} := \sum_{E\in\sigma} \mathrm{ord}_E(s)E$. Now for each $\sigma \in \Sigma$, by definition, there is a basis $B_\sigma$ of $V_N \subset H^0(X, ND)$ such that

$$\sum_{s\in B_\sigma} \mathrm{ord}_E(s) \geq \mu_N \mathrm{ord}_E(ND)$$

at all points $P \in \cap_{E\in\sigma} E$. Since $\Sigma$ is finite, $\{B_\sigma \mid \sigma \in \Sigma\}$ is a finite collection of bases of $V_N$. Thus, we have, using the property of Weil functions that, if $D_1 \geq D_2$, then $\lambda_{D_1} \geq \lambda_{D_2}$, we obtain that,

$$\lambda_{ND}(x) \leq \frac{1}{\mu_N} \max_{\sigma\in\Sigma} \sum_{s\in B_\sigma} \lambda_s(x).$$

The theorem is obtained by taking $x = f(re^{i\theta})$, integrating, and applying Theorem 3.12. □

**Definition 3.4.2.** Define $\delta_f(D)$, the *Nevanlinna defect* of $f$ with respect to $D$, by

$$\delta_f(D) := \lim\inf_{r\to+\infty} \frac{m_f(r, D)}{T_{f,D}(r)}.$$

Then we have the two following key corollaries.

**Corollary 3.14.** *Let $D$ be an effective Cartier divisor on a smooth complex projective variety $X$. Then*

$$\delta_f(D) \leq Nev(D)$$

*for any Zariski dense holomorphic map $f : \mathbb{C} \to X$.*

**Corollary 3.15.** *Let $D$ be an effective Cartier divisor on a complex normal projective variety $X$. If $Nev(D) < 1$, then every holomorphic map $f : \mathbb{C} \to X \backslash D$ is not Zariski dense, i.e., the image of $f$ must be contained in a proper subvariety of $X$.*

*Proof.* Note that $f : \mathbb{C} \to X\backslash D$ implies that $m_f(r, D) = T_{f,D}(r)+O(1)$. So $\delta_f(D) = 1$. Assume that $f$ is Zariski dense, then the above Corollary implies that

$$1 = \delta_f(D) \leq Nev(D) < 1$$

which is a contradiction. Thus, $f$ is not Zariski dense. □

This corollary will be the key to proving our version of the Second Main Theorem as it reduces the problem to just finding an upper bound for $Nev(D)$. Previous results can also be obtained by computing this Nevanlinna constant as exhibited in the following example.

**Example 3.16.** Let $X = \mathbb{P}^n$ and $D = H_1+\cdots+H_q$ where $H_1, \ldots, H_q$ are hyperplanes in $\mathbb{P}^n$ in general position. We take $N = 1$ and consider $V_1 := H^0(\mathbb{P}^n, \mathcal{O}(D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then $\dim V_1 = \binom{q+n}{n}$. For each $P \in \text{Supp} D$, since $H_1, \ldots, H_q$ are in general position, $P \in H_{i_1} \cap\cdots\cap H_{i_l}$ with $\{i_1, \ldots, i_l\} \subset \{1, \ldots, q\}$ and $l \leq n$. Without loss of generality, we can assume $H_{i_1} = \{z_1 = 0\}, \ldots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbb{P}^n$. Now we take the basis $B = \{z_0^{i_0} \ldots z_n^{i_n} \mid i_0 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then for each irreducible component $E$ of $D$ containing $P$, say $E = \{z_{j_0} = 0\}$ with $1 \leq j_0 \leq l$, we have $\text{ord}_E\{z_j = 0\} = 0$ for $j \neq j_0$,

$\text{ord}_E\{z_{j_0} = 0\} = 1$, and thus $\text{ord}_E D = 1$. On the other hand,

$$\sum_{s\in B} \text{ord}_E s = \sum_{\vec{i}} i_{j_0} = \frac{1}{n+1} \sum_{\vec{i}} (i_0 + \cdots + i_n) = \frac{q}{n+1} \binom{q+n}{n} = \frac{q}{n+1} \text{dim} V_1,$$

where the sum is taken for all vectors $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$, and we used the fact that the number of choices of $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$ is $\binom{q+n}{n}$. Thus we can take $\mu_1 = \frac{q}{n+1} \text{dim} V_1$, and hence,

$$Nev(D) \leq \frac{\text{dim} V_1}{\mu_1} = \frac{n+1}{q}.$$

# Chapter 4

# Main Results

## 4.1 Statement of the Main Results

We are now ready to state and prove the main results of this dissertation. Let $X$ be a complex projective variety of dimension $n$, and $D$ be a Cartier divisor on $X$. We will use the notation $D^n$ to denote the $n$-fold intersection of $D$ with itself. Following Aaron Levin [Lev09], let the divisor $D := \sum_{i=1}^{q}$ be a divisor on $X$ with $D_1, \ldots, D_q$ effective. $D$ is said to *have equidegree respect to* $D_1, \ldots, D_q$ if

$$D_i.D^{n-1} = \frac{1}{q} D^n$$

for $1 \leq i \leq q$. We also recall that a Cartier divisor $D$ (or the line sheaf $\mathcal{O}_X(D)$) on $X$ is said to be *numerically effective*, or *nef*, if $D.C \geq 0$ for for any closed integral curve $C$ on $X$ as mentioned in Chapter 2.

**Lemma 4.1** ([Lev09], Lemma 9.7)**.** *Let $X$ be a projective variety of dimension $n$. If $D_j, 1 \leq j \leq q$, are big and nef, then there exist positive real numbers $r_j$ such that*

$D = \sum_{j=1}^{q} r_j D_j$ *is of equidegree.*

*Proof.* We follow the simple proof given by Autissier [Aut1]. Let

$$\triangle := \{(t_1, \ldots, t_q) \in \mathbb{R}_+^q \mid t_1 + \cdots + t_q = 1\}.$$

Define a map $g : \triangle \to \triangle$ by letting, for $t = (t_1, \ldots, t_q) \in \triangle$,

$$g(t) = \left( \frac{\phi(t)}{(\sum_{j=1}^{q} t_j D_j)^{n-1}.D_1}, \cdots \frac{\phi(t)}{(\sum_{j=1}^{q} t_j D_j)^{n-1}.D_q} \right),$$

where $\phi(t) := \left( \sum_{i=1}^{q} \frac{1}{(\sum_{j=1}^{q} t_j D_j)^{n-1}.D_i} \right)^{-1}$. By the Brouwer's fixed point theorem, there exists a point $x = (x_1, \ldots, x_q) \in \triangle$ such that $g(x) = x$, i.e.

$$\phi(x) = (\sum_{j=1}^{q} x_j D_j)^{n-1}.(x_i D_i) \text{ for } i = 1, \ldots, q.$$

This implies, by summing up, that $q\phi(x) = (\sum_{j=1}^{q} x_j D_j)^n$. Thus

$$\frac{1}{q} \left( \sum_{j=1}^{q} x_j D_j \right)^n = \phi(x) = (x_i D_i). \left( \sum_{j=1}^{q} x_j D_j \right)^{n-1}$$

which proves the lemma. □

Recall that the divisors $D_1, \ldots, D_q$ on $X$ with $q > l$ are said to be in *$l$-subgeneral position* if, for any subset of $l+1$ elements $\{i_0, \ldots, i_l\}, \subset \{1, \ldots, q\}$,

$$\mathrm{supp} D_{i_0} \cap \cdots \cap \mathrm{supp} D_{i_l} = \emptyset.$$

When $l = \dim X$, then we say that the divisors $D_1, \ldots, D_q$ are *in general position* on

$X$. Now we are ready to state Ru's theorem.

**Theorem A** ([Ru15], Theorem 5.6)**.** Let $X$ be a complex normal projective variety of dimension $\geq 2$, and $D = D_1 + \cdots + D_q$ be a sum of effective big and nef Cartier divisors, in $l$-subgeneral position on $X$. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^{q} r_i D_i$ is of equidegree (such numbers exist due to Lemma 4.1). We further assume that there exists an integer $N_0 > 0$ such that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. Let $f : \mathbb{C} \to X$ be a Zariski dense holomorphic map. Then, for $\epsilon > 0$ small enough,

$$\sum_{j=1}^{q} r_j m_f(r, D_j) < \left(\frac{2l \dim X}{q} - \epsilon\right) \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r)\right) \|_E,$$

where $\|_E$ means the inequality holds for all r $\in (0, \infty)$ except for a possible set $E$ with finite Lebesgue measure.

In this dissertation, we improve the above theorem with an additional assumption that "$D_1, \ldots, D_q$ have no irreducible components in common". The following is the precise statement.

**Main Theorem** (Complex Part)**.** *Let $X$ be a complex normal projective variety of dimension $n \geq 2$. Let $D_1, \ldots, D_q$ be effective, big and nef Cartier divisors on $X$, and that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. We further assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in $l$-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^{q} r_i D_i$ is equidegree (such numbers exist due to Lemma 4.1). Let $f : \mathbb{C} \to X$ be holomorphic* *and Zariski dense. Then*

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \left(\frac{2n[(l+1)/2]}{q(1+\alpha)}\right)\left(\sum_{j=1}^{q} r_j T_{f,D_j}(r)\right) \qquad \|_E$$

*with*

$$\alpha = \frac{2^{-3n-2}\min_{1\leq i,j\leq q}(r_i^{n-2}r_j^2(D_i^{n-2}.D_j^2))\min_{1\leq i,j\leq q}(r_i^{n-1}r_j(D_i^{n-1}.D_j))}{(nD^n)^2} > 0,$$

*where* $[x]$ *denotes the smallest integer greater than* $x$*.*

Under the assumptions in the Main Theorem, we have the following defect relation

$$\delta_f(D) \leq \frac{2n[(l+1)/2]}{q(1+\alpha)} \tag{4.1}$$

for $D := \sum_{i=1}^{q} r_i D_i$. We note that, in the case when we study $f : \mathbb{C} \to X \backslash D$ (i.e. the image of $f$ omits $D$), by doing a blowing up, the smoothness condition (or the normal condition) of $X$, as well as the nefness condition of $D_j, 1 \leq j \leq q$, can all be removed by a lemma from Aaron Levin. The following is the exact statement of Levin's lemma.

**Lemma 4.2** (Lemma 9.10 in [Lev09])**.** *Let* $X$ *be a complex projective variety. Let* $D = \sum_{j=1}^{q} D_j$ *be a sum of effective Cartier divisors on* $X$*. Then there exists a nonsingular projective variety* $X'$*, a birational morphism* $\pi : X' \to X$*, and a divisor* $D' = \sum_{j=1}^{q} D'_j$ *on* $X'$ *such that* $suppD'_j \subset suppD_j$ *for all* $j$*, every irreducible component of* $D'$ *is nonsingular,* $|D'_j|$ *is base-point free for all* $j$ *(in particular* $D'_j$ *is nef), and* $\kappa(D'_j) = \kappa(D_j) = \dim \Phi_{D'_j}(X')$ *for all* $j$ *(where* $\kappa(D_j)$ *is the Kodaira dimension of* $D_j$*).*

Thus the defect relation (Cor. 3.15), together with Lemma 4.1, implies the following result.

**Corollary 4.3.** *Let $X$ be a complex projective variety of dimension $\geq 2$, and $D = D_1 + \cdots + D_q$ be a sum of big Cartier divisors, in $l$-subgeneral position on $X$. Assume that $D_1, \ldots, D_q$ have no irreducible components in common. If $q \geq 2n[(l+1)/2]$, then every holomorphic mapping $f : \mathbb{C} \to X \setminus \cup_{j=1}^{q} D_j$ must be degenerate.*

On the arithmetic side, similar to the analytic case, we can prove the following improvement of Ru's result ([Ru15a], Theorem 4.1).

**Main Theorem** (Arithmetic Part)**.** *Let $k$ be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let $X$ be a normal projective variety of dimension $n \geq 2$, and let $D_1, \ldots, D_q$ be effective, big and nef Cartier divisors on $X$, both defined over $k$, and that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. We further assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in $l$-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^{q} r_i D_i$ is equidegree (such numbers exist due to Lemma 4.1). Then*

$$\sum_{j=1}^{q} r_j m_S(x, D_j) \leq \left( \frac{2n[(l+1)/2]}{q(1+\alpha)} \right) \left( \sum_{j=1}^{q} r_j h_{D_j}(x) \right),$$

*holds for all $x \in X(k)$ outside a Zariski closed subset $Z$ of $X$, where*

$$\alpha = \frac{2^{-3n-2} \min_{1 \leq i,j \leq q}(r_i^{n-2} r_j^2 (D_i^{n-2}.D_j^2)) \min_{1 \leq i,j \leq q}(r_i^{n-1} r_j (D_i^{n-1}.D_j))}{(nD^n)^2} > 0.$$

The proof of this arithmetic result can be done in a similar way (see [[Ru15a]), so we omit the arithmetic proof here.

## 4.2 Proof of the Main Theorem

The proof of the Main Theorem relies on the notion of Nevanlinna constant $Nev(D)$, and the defect relation (as well as the Second Main Theorem) in terms of $Nev(D)$ from Chapter 3. Let $X$ be a normal projective variety, and $D$ be an effective Cartier divisor on $X$. As mentioned in chapter 3, the condition of normality of $X$ is assumed so that $\mathrm{ord}_E D$ (called the *coefficient of $D$ in $E$*) is defined for any prime divisor $E$ and any effective Cartier divisor $D$ on $X$ ([Laz04], Remark 1.1.4). For any section $s \in H^0(X, \mathcal{O}_X(D))$, we use $\mathrm{ord}_E\, s$ or $\mathrm{ord}_E(s)$ to denote the coefficients of $(s)$ in $E$ where $(s)$ is the divisor on $X$ associated to $s$. This assumption along with the following key lemmas will be enough to prove our main theorem.

**Lemma 4.4** ([Laz04], Corollary1.4.41). *Suppose $D$ is a nef Cartier divisor on a projective variety $X$ with $\dim X = n$. Then*

$$h^0(ND) = \frac{D^n}{n!}N^n + O(N^{n-1}). \tag{4.2}$$

*In particular, $D^n > 0$ if and only if $D$ is big.*

**Lemma 4.5** ([Aut1], Lemma 4.2). *Suppose $E$ is a big and base-point free Cartier divisor on a projective variety $X$, and $F$ is a nef Cartier divisor on $X$ such that $F - E$ is also nef. Let $\beta > 0$ be a positive real number. Then for any positive integers $N, m$ with $1 \le m \le \beta N$, we have*

$$\begin{aligned} h^0(NF - mE) \ \geq\ & \frac{F^n}{n!}N^n - \frac{F^{n-1}.E}{(n-1)!}N^{n-1}m \\ & + \frac{(n-1)F^{n-2}.E^2}{n!}N^{n-2}\min\{m^2, N^2\} + O(N^{n-1}), \end{aligned}$$

*where $O$ depends on $\beta$.*

*Proof.* Case $m \leq N$: The Riemann-Roch Theorem (see [Laz04]) tells us that

$$\chi(X, NF - mE) = \frac{1}{n!}(NF - mE)^n + O(N^{n-1}).$$

From this we obtain that

$$h^i(X, NF - mE) = O(N^{n-i})$$

for all $i$ since $F$ and $F - E$ are nef ([Laz04] p. 69), and $h^0(X, aD) - h^1(X, aD) = \chi(X, aD) + O(a^{n-1})$ if $D$ is nef by definition. By direct computation, we have that

$$\begin{aligned}(NF - mE)^n &= F^n N^n - nF^{n-1}EN^{n-1}m \\ &+ \sum_{i=2}^{n}(i-1)F^{i-2}(NF - mE)^{n-i}E^2N^{i-2}m^2.\end{aligned}$$

Combining these proves this case.

Case $m > N$: Let $N \leq i \leq \beta N$, then we have a short exact sequence

$$0 \to \mathcal{O}_X(NF - (i+1)E) \to \mathcal{O}_X(NF - iE) \to \mathcal{O}_Z((NF - iE)|_Z) \to 0,$$

where $Z = div(s)$ for some $s \in \Gamma(X, E)$. Then the long exact sequence of cohomology implies that

$$h^0(\mathcal{O}_X(NF - (i+1)E)) \geq h^0(\mathcal{O}_X(NF - iE)) - h^0(\mathcal{O}_Z((NF - iE)|_Z)).$$

Since

$$h^0(\mathcal{O}_Z((NF-iE)|_Z)) \leq h^0(\mathcal{O}_Z((NF|_Z)) = \frac{F^{n-1}E}{(n-1)!}N^{n-1} + O(N^{n-2}),$$

we have that

$$\begin{aligned} h^0(X, NF-mE) &\geq h^0(X, NL-NE) - \sum_{i=N}^{m-1} h^0(Z, (NF-iE)|_Z) \\ &\geq \frac{F^n}{n!}N^n - \frac{F^{n-1}E}{(n-1)!}N^{n-1}m + \frac{n-1}{n!}F^{n-2}E^2N^n - O(N^{n-1}), \end{aligned}$$

where the lower bound for the $h^0(X, NL-NE)$ is obtained from the previous case when $m \leq N$. □

**Lemma 4.6** ([Lev09], Lemma 10.1)**.** *Let $V$ be a vector space of finite dimension $d$ over a field $k$. Let $V = W_1 \supset W_2 \supset W_3 \supset \ldots \supset W_h$ and $V = W_1' \supset W_2' \supset W_3' \supset \ldots \supset W_{h'}'$ be two filtrations on $V$. Then there exists a basis $v_1, v_2, .., v_d$ of $V$ that contains a basis of each $W_j$ and $W_j'$.*

*Proof.* The proof will use induction on the dimension $d$. When $d = 1$ the result is trivial. By refining the first filtration, we may assume, without loss of generality, that $W_2$ is a hyperplane in $V$. Let $W_i^* = W_i' \cap W_2$ for $i = 1. \ldots, h'$. By the inductive hypothesis, there exists a basis $v_1, \ldots, V_{d-1}$ of $W_2$ containing a basis of each of $W_3, \ldots, W_h$ and $W_1^*, \ldots, W_h^*$. Let $l$ be the maximal index with $W_l' \not\subset W_2$, and let $v_d \in W_l' \backslash W_l^*$. We claim that $B = \{v_1, \ldots, v_d\}$ is a basis with the required property. It clearly contains a basis of $W_i$ for each $i$. Let $i \in \{1, \ldots, h'\}$. If $i > l$, then $W_i' = W_i^*$, and so by construction, $B$ contains a basis of $W_i'$. If $i \leq l$, then $v_d \in W_l' \backslash W_l^* \subset W_i' \backslash W_i^*$. Since $B$ contains a basis $B_i^*$ of $W_i^*$, and $W_i^*$ is a hyperplane in $W_i'$, we see that $B_i^* \cup \{v_d\}$

is a basis of $W_i'$. $\square$

*Proof of the Main Theorem.* We are now ready to prove our main theorem (the complex case only). By replacing $D$ with $N_0D$ if necessary, we can assume that the linear systems $|D_j|, 1 \leq j \leq q$, are base-point free. We first look at the special case when $r_j, 1 \leq j \leq q$, are all rational numbers.

For $P \in \mathrm{supp}D$, let $D_P := \sum_{i: P \in \mathrm{supp} D_i} r_i D_i$. Since intersection of any $l+1$ distinct $D_j$ is empty and no two of $D_1, \ldots, D_q$ have common components, we can write

$$D_P := D_{P,1} + D_{P,2},$$

where $D_{P,1}$ and $D_{P,2}$ are effective divisors with no irreducible components in common, and each $D_{P,i}$ is a sum of at most $[(l+1)/2]$ of the $r_1D_1, \ldots, r_qD_q$ for $i = 1, 2$. To compute the Nevanlinna constant for $D$, we let $N$ be a sufficiently large positive integer, which is divisible by the common denominators of $r_j, 1 \leq j \leq q$, and let $V_N = H^0(X, ND)$. We consider the two filtrations for $V_N$:

$$W_j := H^0(X, ND - jD_{P,1}), \quad \text{and} \quad W_j' := H^0(X, ND - jD_{P,2}),$$

and we use Lemma 4.6 above to construct a basis $B$ for $V_N$ which contains a basis for each $W_j$ and $W_j'$. Notice that, for $s \in H^0(X, ND - mD_{P,i})/H^0(X, ND - (m+1)D_{P,i})$ with $i = 1, 2$, we have $\frac{1}{\mathrm{ord}_E(D)} \mathrm{ord}_E\, s \geq m$ for any irreducible component $E$ of $D$ which contains $P$. Hence,

$$\frac{1}{\mathrm{ord}_E(ND)} \sum_{s \in B} \mathrm{ord}_E\, s \tag{4.3}$$

$$
\begin{aligned}
&\geq \frac{1}{N}\min_{i=1,2}\sum_{m=0}^{\infty} m(h^0(ND - mD_{P,i})) - h^0(ND-(m+1)D_{P,i})) \\
&= \frac{1}{N}\min_{i=1,2}\sum_{m=1}^{\infty} h^0(ND - mD_{P,i}).
\end{aligned}
$$

Now, for each $i = 1, 2$, we apply Lemma 4.5 with $F = D$, $E = D_{P,i}$ and $\beta_i = \frac{D^n}{nD^{n-1}.D_{P,i}}$, and denote $A_i := (n-1)D^{n-2}.D_{P,i}^2$, it yields

$$
\begin{aligned}
&\sum_{m=1}^{\infty} h^0(ND - mD_{P_i}) \qquad (4.4)\\
\geq &\sum_{m=1}^{[\beta_i N]}\left(\frac{D^n}{n!}N^n - \frac{D^{n-1}.D_{P,i}}{(n-1)!}N^{n-1}m + \frac{A_i}{n!}N^{n-2}\min\{m^2, N^2\}\right) + O(N^n)\\
= &\left(\frac{D^n}{n!}\beta_i - \frac{D^{n-1}.D_{P,i}}{(n-1)!}\frac{\beta_i^2}{2} + \frac{A_i}{n!}g(\beta)\right)N^{n+1} + O(N^n)\\
= &\left(\frac{\beta_i}{2} + \frac{A_i}{D^n}g(\beta_i)\right)D^n\frac{N^{n+1}}{n!} + O(N^n)\\
= &\frac{\beta_i}{2}\left(1 + \frac{2A_i}{\beta_i D^n}g(\beta_i)\right)D^n\frac{N^{n+1}}{n!} + O(N^n),
\end{aligned}
$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$if $x \leq 1$ and $g(x) = x - \frac{2}{3}$ for $x \geq 1$. From the assumption that $D$ is of equidegree with respect to $r_1D_1, \ldots, r_qD_q$ , we have, for $j = 1, \ldots, q$,

$$
(r_jD_j).D^{n-1} = \frac{1}{q}D^n
$$

which implies that, for $1 \leq i \leq 2$,

$$
D^{n-1}.D_{P,i} \leq \frac{[(l+1)/2]}{q}D^n.
$$

Hence

$$
\beta_i = \frac{D^n}{nD^{n-1}.D_{P,i}} \geq \frac{q}{n[(l+1)/2]} \geq \frac{1}{n}. \qquad (4.5)
$$

Here, above, we can assume that $q \geq [(l+1)/2]$ for otherwise the theorem trivially holds by the FMT. Hence $g(\beta_i) \geq \frac{1}{n^3}$. On the other hand, we have

$$A_i = (n-1)D^{n-2}.D_{P,i}^2 \geq \min_{1\leq i,j\leq q}(r_iD_i)^{n-2}.(r_jD_j)^2 = \min_{1\leq i,j\leq q}(r_i^{n-2}r_j^2(D_i^{n-2}.D_j^2)),$$

and

$$\beta_i \leq \frac{D^n}{n\min_{1\leq i,j\leq q}(r_iD_i)^{n-1}.(r_jD_j)} = \frac{D^n}{n\min_{1\leq i,j\leq q}(r_i^{n-1}r_j(D_i^{n-1}.D_j))}.$$

Hence, by combining (4.3) and (4.6), we have

$$\frac{1}{\mathrm{ord}_E(ND)}\sum_{s\in B}\mathrm{ord}_E\, s \geq \frac{q}{2n[(l+1)/2]}(1+2C_1)D^n\frac{N^n}{n!} + O(N^{n-1}),$$

where

$$C_1 = \frac{\min_{1\leq i,j\leq q}(r_i^{n-2}r_j^2(D_i^{n-2}.D_j^2))\min_{1\leq i,j\leq q}(r_i^{n-1}r_j(D_i^{n-1}.D_j))}{(nD^n)^2} \tag{4.6}$$

and thus, together with Lemma 4.4,

$$\frac{1}{\mathrm{ord}_E(ND)}\sum_{s\in B}\mathrm{ord}_E\, s \quad \geq \quad \frac{q}{2n[(l+1)/2]}(1+2C_1)h^0(ND) + o(h^0(ND)).$$

Therefore, from the definition of $Nev(D)$, we have

$$\begin{aligned} Nev(D) &\leq \liminf_{N\to+\infty}\frac{h^0(ND)}{\frac{q}{2n[(l+1)/2]}(1+2C_1)h^0(ND)+o(h^0(ND))} \\ &= \frac{2n[(l+1)/2]}{q(1+2C_1)}. \end{aligned}$$

Applying Theorem 3.13 with $\epsilon = \frac{2n[(l+1)/2]}{q}\frac{C_1}{(1+C_1)(1+2C_1)}$, for $D = \sum_{j=1}^q r_jD_j$, it gives

$$m_f(r, D) \leq \frac{2n[(l+1)/2]}{q(1+C_1)} T_{f,D}(r) \quad \|_E,$$

where $C_1$ is given in (4.6). This proves the case when $r_j, 1 \leq j \leq q$, are rational numbers.

We now prove the case that not all of $r_j, 1 \leq j \leq q$, are rational numbers. By the assumption that $D$ has equidegree with respect to $r_1D_1, \ldots, r_qD_q$, we have, for $1 \leq j \leq q$,

$$(r_jD_j).\left(\sum_{j=1}^q r_jD_j\right)^{n-1} = \frac{1}{q}\left(\sum_{j=1}^q r_jD_j\right)^n.$$

Let $C_1$ be the constant in (4.6) and fix

$$\delta_0 = \frac{2^{-4n-1}C_1D^n}{q(1+2^{-3n-1}C_1)}. \tag{4.7}$$

By the continuity, we can choose rational numbers $a_j, 1 \leq j \leq q$, which are close $r_j$ with

$$|a_j - r_j| \leq \min\left\{\frac{\epsilon_0}{4}(\min_{1\leq i\leq q} r_j), \frac{\epsilon_0(\min_{1\leq i\leq q} r_j)}{4\left(\frac{2n[(l+1)/2]}{q(1+2^{-3n-1}C_1)}\right)}\right\}, \tag{4.8}$$

where

$$\epsilon_0 = \min\left\{1, \frac{2n[(l+1)/2]2^{-3n-2}C_1}{q(1+2^{-3n-1}C_1)(1+2^{-3n-2}C_1)}\right\}, \tag{4.9}$$

and such that

$$(a_iD_i).\left(\sum_{j=1}^q a_jD_j\right)^{n-1} < \frac{1}{q}\left(\sum_{j=1}^q a_jD_j\right)^n + \delta_0. \tag{4.10}$$

Consider $D' := \sum_{j=1}^q a_jD_j$, and write $D'_P := D'_{P,1} + D'_{P,2}$ where $D'_{P,1}$ and $D'_{P,2}$ are effective divisors with no irreducible components in common, and each $D'_{P,j}$ is a sum of at most $[(l+1)/2]$ of the $a_1D_1, \ldots, a_qD_q$. Similar to the above, let $N$ be a positive integer, sufficiently large enough, which is divisible by the common denominators of $a_j, 1 \leq j \leq q$, by the same argument as deriving (4.3) and (4.4), there is a basis of $H^0(X, ND')$ such that

$$\frac{1}{\mathrm{ord}_E(ND')}\sum_{s\in B}\mathrm{ord}_E\, s \geq \min_{1\leq i\leq 2}\frac{\beta_i}{2}\left(1+\frac{2A_i}{\beta_i D^n}g(\beta_i)\right)D'^n\frac{N^n}{n!}+O(N^{n-1}), \tag{4.11}$$

where $\beta_i = \frac{D'^n}{nD'^{n-1}.D'_{P,i}}$, $A_i = (n-1)D'^{n-2}.{D'_{P,i}}^2$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$if $x \leq 1$ and $g(x) = x - \frac{2}{3}$ for $x \geq 1$. Now, from (4.10), we have, for $i = 1, 2$,

$$(D'^{n-1}.D'_{P,i}) \leq [(l+1)/2]\left(\frac{1}{q}D'^n+\delta_0\right) = \frac{[(l+1)/2]}{q}D'^n\left(1+\frac{q\delta_0}{D'^n}\right),$$

so, noticing that $D'^n \geq \frac{1}{2^n}D^n$, we have

$$\beta_i = \frac{D'^n}{nD'^{n-1}.D'_{P,i}} \geq \frac{q}{n[(l+1)/2]}\frac{1}{\left(1+\frac{q\delta_0}{D'^n}\right)} \geq \frac{q}{n[(l+1)/2]}\frac{1}{\left(1+\frac{2^nq\delta_0}{D^n}\right)}. \tag{4.12}$$

For same reason (i.e. we can assume that $q \leq [(l+1)/2](1+\frac{2^nq\delta_0}{D^2})$ for otherwise the theorem would automatically hold by the FMT), we get $\beta_i \geq \frac{1}{n}$ and thus

$$g(\beta_i) \geq \frac{1}{n^3}. \tag{4.13}$$

Also, noticing that

$$(D'^{n-2}.{D'_{P,i}}^2) \geq \min_{1\leq i,j\leq q}((a_iD_i)^{n-2}.(a_jD_j)) \geq \frac{1}{2^n}\min_{1\leq i,j\leq q}((r_iD_i)^{n-2}.(r_jD_j)^2),$$

and, similarly,

$$(D'^{n-1}.D'_{P,i}) \geq \frac{1}{2^n} \min_{1\leq i,j\leq q}((r_iD_i)^{n-1}.(r_jD_j)),$$

we have, also using the similar inequality $D'^n \leq 2^n D^n$,

$$A_i = (n-1)D'^{n-2}.{D'_{P,i}}^2 \geq \frac{1}{2^n} \min_{1\leq i,j\leq q}(r_i^{n-2}r_j^2(D_i^{n-2}.D_j^2)), \tag{4.14}$$

and

$$\beta_i = \frac{D'^n}{nD'^{n-1}.D'_{P,i}} \leq \frac{4^nD^n}{n\min_{1\leq i,j\leq q}(r_i^{n-1}r_j(D_i^{n-1}.D_j))}. \tag{4.15}$$

By combining (4.11), (4.12), (4.13), (4.14), (4.15), and Lemma 4.4, we obtain

$$\begin{aligned} &\frac{1}{\mathrm{ord}_E(ND')}\sum_{s\in B}\mathrm{ord}_E\, s \\ \geq\ &\left(\frac{q(1+2^{1-3n}C_1)}{2n[(l+1)/2](1+\frac{2^nq\delta_0}{D^n})}h^0(ND')+o(h^0(ND'))\right)\mathrm{ord}_E(ND'), \end{aligned} \tag{4.16}$$

where $C_1$ is given in (4.6). Hence, from the definition of the Nevanlina constant, we get

$$Nev(D') \leq \frac{2n[(l+1)/2](1+\frac{2^nq\delta_0}{D^n})}{q(1+2^{1-3n}C_1)}.$$

Applying Theorem 3.13 with

$$\epsilon = \frac{2n[(l+1)/2](1+\frac{2^nq\delta_0}{D^n})2^{-3n}C_1}{q(1+2^{-3n}C_1)((1+2^{1-3n}C_1)},$$

we get

$$\sum_{j=1}^{q} a_jm_f(r,D_j) \leq \left(\frac{2n[(l+1)/2](1+\frac{2^nq\delta_0}{D^n})}{q(1+2^{-3n}C_1)}\right)\left(\sum_{j=1}^{q}a_jT_{f,D_j}(r)\right)\ \|_E.$$

From (4.7),

$$\delta_0 = \frac{2^{-4n-1} C_1 D^n}{q(1 + 2^{-3n-1} C_1)},$$

so

$$\sum_{j=1}^{q} a_j m_f(r, D_j) \le \left( \frac{2n[(l+1)/2]}{q(1 + 2^{-3n-1} C_1)} \right) \left( \sum_{j=1}^{q} a_j T_{f,D_j}(r) \right) \quad \|_E.$$

Now, from (4.8),

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \le \sum_{j=1}^{q} a_j m_f(r, D_j) + \sum_{j=1}^{q} \frac{(\min r_j)\epsilon_0}{4} m_f(r, D_j) \ \|_E,$$

so, together with the First Main Theorem, we get

$$\begin{aligned}
&\sum_{j=1}^{q} r_j m_f(r, D_j) \\
\le\ & \left( \frac{2n[(l+1)/2]}{q(1 + 2^{-3n-1} C_1)} \right) \left( \sum_{j=1}^{q} a_j T_{f,D_j}(r) \right) + \frac{(\min r_j)\epsilon_0}{4} \left( \sum_{j=1}^{q} T_{f,D_j}(r) \right) \quad \|_E \\
=\ & \left( \frac{2n[(l+1)/2]}{q(1 + 2^{-3n-1} C_1)} \right) \left( \sum_{j=1}^{q} a_j T_{f,D_j}(r) \right) + \frac{\epsilon_0}{4} \left( \sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) \quad \|_E \\
\le\ & \left( \frac{2n[(l+1)/2]}{q(1 + 2^{-3n-1} C_1)} + \frac{\epsilon_0}{4} \right) \left( \sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) + \frac{\epsilon_0}{4} (\min r_j) \left( \sum_{j=1}^{q} T_{f,D_j}(r) \right) \\
& + \frac{\epsilon_0}{4} \sum_{j=1}^{q} r_j T_{f,D_j}(r) \quad \|_E \\
\le\ & \left( \frac{2n[(l+1)/2]}{q(1 + 2^{-3n-1} C_1)} + \epsilon_0 \right) \left( \sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) \\
\le\ & \left( \frac{2n[(l+1)/2]}{q(1 + \alpha)} \right) \left( \sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) \quad \|_E
\end{aligned}$$

where, by (4.9), we get

$$\alpha = \frac{2^{-3n-2} \min_{1 \leq i,j \leq q}(r_i^{n-2} r_j^2 (D_i^{n-2}.D_j^2)) \min_{1 \leq i,j \leq q}(r_i^{n-1} r_j (D_i^{n-1}.D_j))}{(nD^n)^2}.$$

# Bibliography


[Ahl39] L. Ahlfors. *Theory of Meromorphic Curves.* Ann Acad. Sci. Fenn, 1939.

[Aut1] P. Autissier. Géométries, points entiers et courbes entières. (French) [Geometry, integral points and integral curves] *Ann. Sci. Éc. Norm Supér*, (4) 42, 221-239, 2009.

[BS95] M. Beltrametti and A. Sommese *The Adjunction Theory of Complex Projective Varieties.* Walter de Gruyter, Berlin, New York, 1995.

[Car] H. Cartan. Sur les zéros des combinaisions linéarires de $p$ fonctions holomorpes données. *Mathematica(Cluj)* 7: 5-29, 1933.

[Ful98] W. Fulton. *Intersection Theory.* Second ed., Ergeb. Math. Grenzgeb. 2, Springer-Verlag, New York, 1998.

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons, New York, 1978.

[GK06] R. Greene and S. Krantz. *Function Theory of One Complex Variable.* 322, American Mathematical Society, Rhode Island, 2006.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HR12] G. Heier and M. Ru. Essentially large divisors and their arithmetic and function-theoretic inequalities. *Asian J. of Math.*, 16: 387-407, 2012.

[Kra04] S. Krantz. *Complex Analysis: The Geometric Viewpoint, Second Edition.* The Mathematical Association of America, 2004.

[Lan87] S. Lang. *Fundamentals of Diophantine Geometry.* Springer-Verlag, New York, 1983.

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry* I. Springer-Verlag, Berlin, 2004.

[Lev09] A. Levin. Generalizations of Siegel's and Picard's theorems. *Ann. of Math. (2)*, 170(2):609–655, 2009.

[Liao] H. Liao Quantitative Geometric and Arithmetic Results on Projective Surfaces. Proc. of AMS to appear.

[MR17] C. Mills and M. Ru. An Improved Defect Relation for Holomorphic Curves in Projective Varieties. Complex Analysis and Dynamical Systems VII, the Israel Mathematics Conference Proceedings (IMCP), Contemporary Mathematics (AMS), to appear.

[Nev29] R. Nevanlinna. *Le théoreme de Picard-Borel et la théorie de fonctions méromorphes.* Gauthier-Villars, Paris. 1929

[Ru01] M. Ru. *Nevanlinna Theory and Its Relation to Diophantine Approximation. World Scientific.* 2001.

[Ru04] M. Ru. A defect relation for holomorphic curves intersecting hypersurfaces. *Amer. J. Math.*, 126(1): 215–226, 2004.

[Ru09] M. Ru. Holomorphic curves into algebraic varieties. *Ann. of Math. (2)*, 169(1): 255–267, 2009.

[Ru15] M. Ru. A defect relation for holomorphic curves intersecting general divisors in projective varieties. *J. of Geometric Analysis*, 26: 2751-2776, 2016.

[Ru15a] M. Ru. A general Diophantine inequality. *Functiones et Approximatio*, to appear.

[RV17] M. Ru and P. Vojta. A Birational Nevanlinna Constant and its Consequences. Submitted.

[Voj87] P. Vojta. *Diophantine approximations and value distribution theory.* Volume 1239 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1987.

[Vojcm] P. Vojta. *Diophantine Approximation and Nevanlinna Theory. CIME notes*, 231 pages, 2007.

[YY03] C. Yang and H. Yi. *Uniqueness Theory of Meromorphic Functions.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.