A GENERALIZATION OF QUASI-CONTINUOUS FUNCTIONS AND RIEMANN INTEGRATION IN REFINEMENT SPACES

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Larry Gene Schoonover

May, 1972

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ABSTRACT

Two new properties are proven to be equivalent to the definition of quasi-continuous function. One of which is suitable for generalizing the definition to abstract spaces. Several properties of quasi-continuity in topological spaces are presented. Axioms are presented defining a refinement space in which several more properties of quasi-continuous functions are proven, and a Riemann type integral is defined in this space.

TABLE OF CONTENTS

INTRODUCTION

Many of the definitions by which we classify functions use properties inherent to the real numbers but absent even in the real plane. It is the prime purpose of this paper to extend the definitions of quasi-continuous functions and other related concepts to abstract topological spaces.

The real numbers provide a set of points in which the natural order is a total ordering of the space which agrees with the concept of limit point. Upon this rests the definition of quasi-continuous functions. The equivalence of this class of functions to the collection of uniform limits of sequences of step functions provides an approach to generalization. For example, a step function may be defined in the plane to be constant over the interiors of each of a finite number of rectangles covering the domain of the function, and assume arbitrary values at the boundaries of these rectangles. This definition, however, leads to a collection of "undesirable" functions. The property that each region has at most two boundary points is not true in the real plane. There are other ways that we may define step functions that prove to conform more closely to our conception of step functions in spaces other than the real numbers. We measure the suitability of an alternative definition by the properties which are preserved and by the number and importance of the spaces into which they fit.

Two important properties of the real numbers that are absent in the plane have been mentioned above. The space of real numbers with region meaning segment also has the properties that each region is connected, each bounded open, connected set is a region, each bounded set is compact and others that often arise but are only alluded to in proofs. It is upon these properties that we may generalize the concept of quasi-continuous functions and still maintain most of the properties which we found useful and important in the real numbers.

In this paper, we introduce a new property and prove its equivalence to the usual definition of quasi-continuous function. The new property uses the concept of connected set rather than "right" and "left", and thus is more suitable for generalization. In addition, we describe a space more general than the real numbers that preserves the properties necessary to define a Riemann type integral. In this setting we present many properties of quasi-continuous functions. This paper provides a rigorous support of our claim to a more general definition of quasi-continuous function.

NOTATION

The following notation and terminology will be used throughout this paper. R will denote the set of real numbers. R^n where n is a natural number, will denote the set $\sum_{i=1}^{n} \oplus S_i$, where $S_i = R$ for each $1 \leq i \leq n$.

The statement that s is a <u>segment</u> means s is a bounded, open, connected subset of R. The statement that I is an <u>interval</u> means I is a bounded, connected subset of R. We use this definition rather than one which requires intervals to be non-degenerate and closed. The statement that r is a <u>region in R</u> means r is a segment.

If $R = \{s_\lambda | \lambda \varepsilon \Lambda$ an indexing set$\}$ is a collection of sets then R* means the set $\{x | x \varepsilon s_\lambda$ for some $\lambda \varepsilon \Lambda\}$.

The statement that the point set M is <u>compact</u> in the topological space S means every infinite subset of M has a limit point in S.

The statement that the collection D is a <u>subdivision</u> of the point set M means D is finite, D* equals M, and if d and g are distinct elements of D, then d and g are disjoint.

The statement that the collection R is a <u>refinement</u> of the finite cover C of the point set M means R is a subdivision of M, and if r is an element of R and c is an element of C, then either r and c are disjoint or $r \subseteq c$.

In this paper, all functions will be considered to be real valued.

If f is a function defined on the topological space S, then

1. $D(f)$ denotes the collection of all points x in S such that f is discontinuous at x, and
2. if s is a segment, $S(s,f)$ denotes the collection of components of $\{x | f(x) \varepsilon s\}$.

   Note: If there is no such x with f(x) in s, then each of $S(s,f)$ and $\{x | f(x) \varepsilon s\}$ denotes the empty set. For convenience we consider the empty set to be both finite and connected throughout this paper.

Suppose f is a function defined on the topological space S. Let $x \varepsilon S$, and X be the set so that $y \varepsilon X$ if and only if there is a sequence of points of S, x_1, x_2, x_3, ... converging to x such that $f(x_1)$, $f(x_2)$, $f(x_3)$, ... converges to y. Then the symbol $\overline{\omega}(x)$ denotes the least upper bound of X, $\underline{\omega}(x)$ denotes the greatest lower bound of X, and $\omega(x) = \overline{\omega}(x) - \underline{\omega}(x)$.
Note: If X is empty, then each of $\underline{\omega}(x)$ and $\overline{\omega}(x)$ denotes 0.

QUASI-CONTINUOUS FUNCTIONS IN R

We introduce a series of four equivalent statements in theorem 3.3, each of which could serve as a definition for quasi-continuity. With the assurance that we will not contradict the traditional definition, we may use any one of the statements as a definition in other spaces and investigate its properties.

DEFINITION 3.1. The statement that the function f defined on the interval $[u,v]$ in R is quasi-continuous at the point x in $[u,v]$ means if x_1, x_2, x_3, ... is a monotonic sequence of points of $[u,v]$, converging to x, then $f(x_1)$, $f(x_2)$, $f(x_3)$, ... converges. If f is quasi-continuous at each point x in $[u,v]$, then f is said to be quasi-continuous.

Lemma 3.2 introduces a property of the real numbers that we will use to prove Theorem 3.3.

LEMMA 3.2. If M is a bounded subset of R which has infinitely many distinct components $\{m_\lambda | \lambda \varepsilon \Lambda$ an indexing set$\}$, then

(1) M^c (the complement of M) has infinitely many components $\{n_\lambda | \lambda \varepsilon \Gamma$ an indexing set$\}$, and

(2) If x is a point and $\{x_\lambda | \lambda \varepsilon \Lambda,\ x_\lambda \varepsilon m_\lambda\}$ has a countable subset x_1, x_2, x_3, ... that converges to x from the left (respectively right), then there exists a countable subset y_1, y_2, y_3, ...

of the set $\{y|\gamma\varepsilon\Gamma,\ y_\gamma\varepsilon n_\gamma\}$ that converges to x from the left (respectively right).

Proof: Let $x_1, x_2, x_3, \ldots$ be a subset of $\{x_\lambda|\lambda\varepsilon\Lambda$ and $x_\lambda\varepsilon m_\lambda\}$. The sequence $x_1, x_2, x_3, \ldots$ has a limit point x since M is bounded, and a bounded subset of R is compact. Thus (without loss of generality (w.l.o.g.)) we may suppose that some subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ converges to x from the left where n_i is a natural number for each natural number i. Let $z_1=x_{n_1}$, and R_1 denote a segment (r_1,x) so that z_1 is not an element of R_1. Since x is a limit point of $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ from the left, R_1 contains infinitely many points of the sequence, hence infinitely many components of M. The set $R_1 \cap M$ is not connected so there is a point y_1 in M^c and in R_1. Again since the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ converges to x from the left, there is a $z_2=x_{n_j}$ for some natural number j so that $y_1<z_2$. If we let R_2 be a region (r_2,x) not containing z_2, we may in a similar manner construct the sequences $z_1, z_2, z_3, \ldots$ and $y_1, y_2, y_3, \ldots$ with the properties that $z_i<y_i<z_{i+1}$, $y_i\varepsilon M^c$ for each natural number i. Since (z_i, z_{i+1}) and (z_j, z_{j+1}) are mutually separated if $i\neq j$, y_i and y_j must be in different components of M^c, hence M^c has infinitely many components.

The sequence $y_1, y_2, y_3, \ldots$ converges to x from the left and the second part of the lemma is also established.

THEOREM 3.3. If f is a bounded function defined on the interval $[u,v]$ the following are equivalent:

(1) The function f is quasi-continuous.

(2) If b is a number, $\varepsilon>0$, and $p\varepsilon[u,v]$, then

there is a number a, $b-\varepsilon<a<b$ and a region R containing p such that $S=\{x|x\varepsilon R,\ f(x)>a\}$ has only finitely many components.

(3) The function f is the uniform limit of a sequence of step functions, i.e. if $\varepsilon>0$, there is a step function g such that if $x\varepsilon[u,v]$, then $|f(x)-g(x)|<\varepsilon$.

(4) If I_1 and I_2 are segments in R and $\bar{I}_1\subset I_2$, then there is a finite subset R of $S(I_2,f)$ such that $S^*(I_1,f)\subseteq R^*$.

Proof: <u>(1) implies (2).</u>

Suppose (1) is true and that (2) is not true. Let p be an element of the interval $[u,v]$, and let b be a number and $\varepsilon>0$ so that for each number a, $b-\varepsilon<a<b$ and each region R containing p, the set $S=\{x|x\varepsilon R,\ f(x)>a\}$ has infinitely many components. Let a_1, a_2, and a_3 be numbers such that $b-\varepsilon<a_3<a_2<a_1<b$, R be a region containing p, and define $S_i=\{x|x\varepsilon R,\ f(x)>a_i\}$ for $1\leq i\leq 3$. Each S_i has infinitely many distinct components $\{m_{i,\lambda}|\lambda\varepsilon\Lambda$ an indexing set$\}$. There is a subset $x_{i,1}$, $x_{i,2}$, $x_{i,3}$, ... of the set $\{x_{i,\lambda}|\lambda\varepsilon\Lambda$, and $x_{i,\lambda}\varepsilon m_{i,\lambda}\}$ for $1\leq i\leq 3$, such that $x_{i,1}$, $x_{i,2}$, $x_{i,3}$, ... converges to p. At least two of these have subsequences converging to p from the same side. We may suppose (w.l.o.g.) that they be sequences of points in S_1^* and S_2^* and that their subsequences $y_{1,1}$, $y_{1,2}$, $y_{1,3}$, ... and $y_{2,1}$, $y_{2,2}$, $y_{2,3}$, ... converge from the left.

Since S_2 has infinitely many components, by lemma 3.2 $S_2{}^c$ also has infinitely many components and a sequence $z_1, z_2, z_3, \ldots$ of points of $S_2{}^c$ converging from the left to p. Since for each natural number j, $f(y_{1,j})>a_1$, and for each point z_j, $f(zj)<a_2$, the sequence $y_{1,1}, z_1, y_{1,2}, z_2, \ldots$ converges to p, but $f(y_{1,1}), f(z_1), f(y_{1,2}), f(z_2), \ldots$ does not converge, a contradiction.

<u>(2) implies (1).</u>

Suppose (2) is true, and $p\varepsilon[u,v]$. Let $x_1, x_2, x_3, \ldots$ be a sequence of points converging from the left to p. Then some subsequence of $f(x_1), f(x_2), f(x_3), \ldots$ converges since f is bounded. Let this sequential limit be y.

Let α and β be numbers so that $\beta<y<\alpha$. It is only necessary to show that there is a point r such that if $x\varepsilon(r,p)$ then $\beta<f(x)<\alpha$, since a similar argument may be used for sequences converging to p from the right.

There are numbers α' and β', and regions R_a and R_b such that $\beta<\beta'<y<\alpha'<\alpha$, and the sets $S_a = \{x|x\varepsilon R_a, f(x)>\alpha'\}$, and $S_b = \{x|x\varepsilon R_b, f(x)>\beta'\}$ have only finitely many components, as does $S_a\cap(C,p)$ and $S_b\cap(C,p)$ for each $C\varepsilon[u,p]$.

Let the finite collection of components of $S_a\cap(c,p)$ and $S_b\cap(c,p)$ be $\{A_1, A_2, A_3, \ldots, A_{n_a}\}$ and $\{B_1, B_2, B_3, \ldots, B_{nb}\}$ respectively for $c\varepsilon[u,p]$, and natural numbers n_a and n_b. Let $r_a\varepsilon\overline{A}_k$ be the right endpoint of the set $(\bigcup_{i=1}^{n_a}\overline{A}_i)$. Suppose (by way of contradiction (b.w.o.c.)) $r_a=p$. Since $f(x_1), f(x_2), f(x_3), \ldots$ converges to y, there is an $x_i \varepsilon A_k$ such that $f(x_i) < \alpha'$, which is a contradiction, and thus $r_a \neq p$.

Let $r_b \varepsilon \overline{B}_j$ be the right endpoint of the set $(\bigcup_{i=1}^{n_b} \overline{B}_i)$. Suppose (b.w.o.c.) that $r_b \neq p$. There is an x_i not in B_j such that $f(x_i) > \beta'$, a contradiction, and $r_b = p$.

Let r be a point in R_a and R_b such that r is in B_j, and $r > r_a$. If $x \varepsilon (r,p)$, then x is not in S_a but is in S_b, and we get $\beta < \beta' \leq f(x) \leq \alpha' < \alpha$, and f is quasi-continuous at p.

(1) implies (3).

Suppose (1) is true. At each point $p \varepsilon [u,v]$ either f is continuous or discontinuous. Let $p \varepsilon [u,v]$, and $\varepsilon > 0$.

Case 1. If f is continuous at p, then there is a segment s_p containing p such that if $x \varepsilon s_p$, then $f(x) \varepsilon (f(p) - \varepsilon/2, f(p) + \varepsilon/2)$.

Case 2. If f is discontinuous at p, then by (1), there is a number y_r such that if $r_1, r_2, r_3, \ldots$ converges to p from the right, then $f(r_1), f(r_2), f(r_3), \ldots$ converges to y_r. Thus there is a segment $s_{p,r} = (p, b_p)$ for some number b_p such that if $x \varepsilon s_{p,r}$ then $f(x) \varepsilon (y_r - \varepsilon/2, y_r + \varepsilon/2)$. Similarly there exists a number y_ℓ that is the limit of function values of points converging to p from the left, and a segment $s_{p,\ell} = (a_p, p)$ for some number a_p such that if $x \varepsilon s_{p,\ell}$ then $f(x) \varepsilon (y_\ell - \varepsilon/2, y_\ell + \varepsilon/2)$. Let $s_p = (a_p, b_p)$.

We have a collection of open sets $S = \{s_p \mid p \varepsilon [u,v]\}$ covering the closed, compact set $[u,v]$, thus there is a finite subcollection of S, $s_{p_1}, s_{p_2}, s_{p_3}, \ldots, s_{p_n}$ where $p_1, p_2, p_3, \ldots, p_n$ are points of $[u,v]$ and n is a natural number, covering $[u,v]$. Let $t_1, t_2, t_3, \ldots, t_m$ be a collection of segments such that for each $1 \leq i \leq n$, $s_{p_i} = t_k$ for some k if f is continuous at p_i, and $(a_{p_i}, p_i) = t_j$, $(p_i, b_{p_i}) = t_h$ for some natural numbers j and h if f is discontinuous at p_i. There is a collection of segments $\{u_1, u_2, u_3, \ldots, u_v\}$ such that u_i and u_j are disjoint whenever $i \neq j$, $\bigcup_{i=1}^{v} \bar{u}_i = [u,v]$, and if $1 \leq i \leq v$, $u_i \subseteq t_j$ for some natural number $j \leq m$. This statement will be proven later in a more general setting.

Associate with each segment u_i, $1 \leq i \leq v$ a point $z_i \varepsilon u_i$. Let f_ε denote the step function defined over the collection of segments $U = \{u_1, u_2, u_3, \ldots, u_v\}$ so that if $x \varepsilon u_i$, then $f_\varepsilon(x) = f(z_i)$. If x is a boundary point of some element of U, then $f_\varepsilon(x) = f(x)$. Since for each i, $1 \leq i \leq v$, $u_i \subseteq t_j$ for some natural number j, if $x \varepsilon u_i$, $|f(x) - f_\varepsilon(x)| = |f(x) - f(z_i)| \leq \varepsilon$, and f is the uniform limit of a sequence of step functions.

<u>(3) implies (4).</u>

Suppose (3) is true. Let $f_1, f_2, f_3, \ldots$ be a sequence of step functions such that for each $x \varepsilon [u,v]$, and natural number i, $|f_i(x) - f(x)| < 1/i$. Let each of I_1 and I_2 be segments so that $\bar{I}_1 \subseteq I_2$, and $d(I_1, I_2) = \varepsilon$. Let n denote a natural number such that $1/n < \varepsilon/2$. Let $D = \{d_1, d_2, d_3, \ldots, d_m\}$ be the collection of intervals of the defining subdivision of the step function f_n, and $x_1, x_2, x_3, \ldots, x_m$ be a sequence of real numbers such that if $1 \leq i \leq m$, then $f_n(x) = x_i$ whenever $x \varepsilon d_i$.

Let I_1 be denoted by (c,d), and $D'=\{d_{n_1}, d_{n_2}, d_{n_3}, \ldots, d_{n_j}\}$ where $n_1, n_2, n_3, \ldots, n_j$ are natural numbers less than or equal to m, be a subcollection of D so that $d_i \varepsilon D'$ whenever $x_i \varepsilon (c-\varepsilon/2, d+\varepsilon/2)$. Thus $S^*(I_1,f) \subseteq \bigcup_{i=1}^{j} d_{n_i} \subseteq S^*(I_2,f)$.

Since the elements of $S(I_2,f)$ are components of $S^*(I_2,f)$, and since each element of D is connected, there is a single element of $S(I_2,f)$ containing d_{n_i} for each n_i, $1 \leq i \leq j$, thus there is a finite collection of elements of $S(I_2,f)$ that contains all of the points of $S^*(I_1,f)$.

<u>(4) implies (1).</u>

Suppose (b.w.o.c.) f is not quasi-continuous at the point p. Then (w.l.o.g.) we may suppose that there are sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ both converging to p from the left with $f(x_1), f(x_2), f(x_3), \ldots$ converging to z_x, and $f(y_1), f(y_2), f(y_3), \ldots$ converging to z_y, $z_y \neq z_x$. Let $\varepsilon = |z_x - z_y|$.

Let $I_1 = (z_x - \varepsilon/4, z_x + \varepsilon/4)$, and $I_2 = (z_x - \varepsilon/3, z_x + \varepsilon/3)$. By property (4), there exists a finite collection $T=\{t_1, t_2, t_3, \ldots t_n\}$ of elements of $S(I_2,f)$ such that $S^*(I_1,f) \subseteq T^*$. Suppose t_j is the right most element of T that is to the left or contains p, and suppose (b.w.o.c.) that $p \not\varepsilon \bar{t}_j$. Since $x_1, x_2, x_3, \ldots$ converges to p from the left, and $f(x_1), f(x_2), f(x_3), \ldots$ converges to z_x, there is a natural number k such that $x_k \not\varepsilon t_j$ and $x_k \varepsilon S^*(I_1,f)$, a contradiction since $S^*(I_1,f) \subset T^*$. Thus $\bar{t}_j$ contains p. Since $y_1, y_2, y_3, \ldots$ converges to p from the left, there is a natural number i such that $y_k \varepsilon t_j$ if $k > i$. Since

$f(y_1)$, $f(y_2)$, $f(y_3)$, ... converges to $z_y \notin S^*(I_2,f)$, there is a y_h, $h>i$ so that $y_h \notin S^*(I_2,f)$. But each element of t_j is in $S^*(I_1,f)$, and $y_h \varepsilon t_j$. This is a contradiction and (4) implies (1).

This theorem provides several alternatives for generalization. We intend to show that property (4) provides the better definition. Consider the following examples.

Example 1: Let f be a function defined over the real plane, R^2, such that $f(x,y)=0$ if $x<0$, $f(x,y)=1$ if $x>0$. When $x=0$ then $f(x,y)= 3/2$ if y is rational and 0 otherwise.

Example 2: This example is the same as example 1 reflected through the x-y plane. Let f be a function defined over R^2 such that $f(x,y)=0$ if $x<0$, and $f(x,y)=-1$ if $x>0$. When $x=0$, then $f(x,y)=-3/2$ if y is rational and 0 otherwise.

Example 3: The set of rational numbers Q is countable, thus we may order them such that $Q=\{r_1, r_2, r_3, \ldots\}$. Let f be defined over R^2 such that $f(x,y)=0$ if $x<0$, $f(x,y)=1$ if $x>0$, $f(0,y)=1/i$ if $y=r_i$ for some natural number i, and 0 otherwise.

Example 3 has both property (2) and (4) of theorem 3.3. Examples 1 and 2 both fail to have property (4). The first two examples differ only slightly in definition and do not appear to have changed any basic properties from one definition to the other, but example 2 has property (2) while example 1 does not. Furthermore, property (2) is "one-sided" and a similar property could be defined with respect to the set $\{x|f(x)<b\}$. This property would hold for example 1,

thus we suspect there is a basic deficiency in property (2) as a definition for quasi-continuous functions.

PROPERTIES OF QUASI-CONTINUOUS FUNCTIONS IN ABSTRACT SPACES

Theorem 3.3 assures us that property (4) may be used as a definition of quasi-continuous functions without risk of contradicting our previous definitions.

DEFINITION 4.1. The statement that the function f defined on the topological space S is quasi-continuous means f is bounded, and whenever I_1 and I_2 are segments such that $\overline{I}_1 \subseteq I_2$, then there is a finite subset R of $S(I_2,f)$ so that $S^*(I_1,f) \subseteq R^*$.

DEFINITION 4.2. The statement that the function f defined on the topological space S is a step function means there is a subdivision of connected sets of S, $C = \{c_1, c_2, c_3, \ldots, c_n\}$ and a finite collection of real numbers $x_1, x_2, x_3, \ldots, x_n$ such that if $1 \leq i \leq n$ and $x \in c_i$, then $f(x) = x_i$.

The following theorems and propositions will elucidate some of the properties of quasi-continuous functions over spaces other than the real numbers.

THEOREM 4.3. If f is a function defined on the topological space S, and f is the uniform limit of a sequence of step functions, then f is quasi-continuous.

The proof of this theorem is similar to the proof that (3) implies (4) of theorem 3.3, and will not be repeated here.

<u>THEOREM 4.4.</u> If f is a quasi-continuous function defined on the topological space S, there exists a countable collection of connected sets C such that each point of D(f) is a boundary point of some element of C.

Proof: Since f is quasi-continuous, f is bounded. Suppose f is bounded above by v and below by u.

Let $\varepsilon>0$, and $M_\varepsilon=\{x|\omega(x)\geq\varepsilon\}$. Let $I_1, I_2, I_3, \ldots I_n$ be a finite collection of segments covering $[u,v]$, such that if $I_j=(a_j,b_j)$, then $|a_j-b_j|<\varepsilon/2$. Let $x\varepsilon M_\varepsilon$. There is a natural number j such that $f(x)\varepsilon I_j$. Let I_j' be a segment such that $\overline{I}_j \subseteq I_j'$, and $d(I_j,I_j')<\varepsilon/2$. There is a finite collection of connected sets $C_{j,\varepsilon}$ of $S(I_j',f)$ such that if y is in $S(I_j,f)$, then y is in some element of $C_{j,\varepsilon}$. Suppose $x\varepsilon M_\varepsilon$. Since $\omega(x)\geq\varepsilon$, any region containing x contains a point y such that $f(y)\not\varepsilon I_j'$ and hence y is not in any element of $C_{j,\varepsilon}$. Thus x is a boundary point of some element of the finite collection $C_{j,\varepsilon}$.

The collection of sets M_1, $M_{\frac{1}{2}}$, $M_{\frac{1}{4}}$, ... contains all of the points of D(f). Thus if $x\varepsilon D(f)$, then x is a boundary point of some element of the countable collection of connected sets $\{c_{i,j,e}|c_{i,j,e}\varepsilon C_{j,e}$ for i and j natural numbers and $e\varepsilon\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ }.

<u>COROLLARY 4.5.</u> If f is a quasi-continuous function in R defined over the interval $[u,v]$, then the set D(f) is countable.

Proof: In R each connected set has at most two boundary points. The corollary trivially follows.

<u>CONJECTURE 4.6.</u> The space of step functions with the usual definition of addition and scalar multiplication is a linear space.

Counterexample. Consider the following two step functions in R^2. Define $f_1(x,y)=1$ if $x \cdot \sin(1/x)-y=0$, and 0 otherwise. Define $f_2(x,y)= -1$ if $y=0$ and 0 otherwise. Then f_1+f_2 is not a step function.

<u>CONJECTURE 4.7.</u> Every continuous function is quasi-continuous.

Counterexample. Let S be the space of real numbers where the statement that s is a region means s is a segment or s is the point set $R=\{1 \cup [0,1/2] \cup [3/4,7/8] \cup \ldots\}$. Let f be defined such that $f(x)=1$ if $x \varepsilon R$, and 0 otherwise. First note that f is not quasi-continuous. The following shows that f is continuous. If $x \varepsilon [0,1]$ such that $x \notin R$, then $f(x)=0$. For $0<\varepsilon<1$ let $(-\varepsilon,\varepsilon)$ be an interval containing $f(x)$ The set R is closed and there exists a region r such that $x \varepsilon r$ and r does not intersect R. Thus if $y \varepsilon r$, $f(y) \varepsilon (-\varepsilon,\varepsilon)$, and f is continuous at x. If $x \varepsilon R$ and I is a segment containing 1, then if $y \varepsilon R$, $f(y) \varepsilon I$, and f is continuous at x.

The above counterexamples serve to point out two of the properties which appear to be basic to the concept of quasi-continuity if we are to provide any meaningful generalization. First, the difference of two regions is at most finitely many regions, and each region is connected.

These properties are also basic to Riemann integration. It is desirable therefore to impose certain restrictions on the topological spaces in which we work, to guarantee that we have the above mentioned properties.

REFINEMENT SPACES

We intend to define a Riemann-type integral in the spaces for which we define quasi-continuous functions, thus we require a space which will allow us to construct subdivisions and refinements of connected sets. We borrow and modify the concept of a σ-ring for this purpose.

DEFINITION 5.1. The statement that the collection of sets R is a partial ring means the empty set is in R, and if $r_1, r_2, r_3, \ldots, r_n$ are elements of R, there exists $t_1, t_2, t_3, \ldots, t_m$ elements of R for some natural number m such that $(\bigcup_{i=1}^{n} r_i)^c = (\bigcup_{i=1}^{m} t_i)$, and t_i and t_j are disjoint if $i \neq j$.

DEFINITION 5.2. The statement that the ordered triple (S, R, M) is a refinement space means the following axioms are satisfied.

Axiom 1. The space S is a Moore 1_3 space, i.e.

a. Every region is a point set.

b. 1) For each natural number n, G_n is a collection of regions covering S.

2) For each natural number n, $G_n \subseteq G_{n+1}$.

3) If a and b are points of the open set M, then there exists a natural number n such that if $g \in G_n$ and $a \in g$, then $b \notin g$ and $\overline{g} \subseteq M$.

Axiom 2. The collection of sets R satisfies the following.

a. If $r \varepsilon R$, then r is connected.

b. The collection R is a partial ring.

c. If s is a region in S containing the point x, then there is an open set $r \varepsilon R$, $x \varepsilon r$ such that $r \subseteq s$.

Axiom 3. The function M is a non-negative set function defined on the elements of R such that if a and b are disjoint elements of R, then $M(a \cup b) = M(a) + M(b)$.

Note that because R is a partial ring the following are true in the refinement space (S, R, M).

If $a_1, a_2, a_3, \ldots, a_n$ are elements of R, then there exists elements $t_1, t_2, t_3, \ldots, t_m$ of R such that $(a_1 \cap a_2 \cap a_3 \cap \ldots \cap a_n)^c = t_1 \cup t_2 \cup t_3 \cup \ldots \cup t_m$.

If $b_1, b_2, b_3, \ldots, b_p$ are elements of R then there exists a finite collection of elements $s_1, s_2, s_3, \ldots, s_q$ of R such that s_i and s_j are disjoint if $i \neq j$, and

$$(a_1 \cap a_2 \cap a_3 \cap \ldots \cap a_n) - (b_1 \cup b_2 \cup b_3 \cup \ldots \cup b_p)$$
$$= \{(a_1 \cap a_2 \cap a_3 \cap \ldots \cap a_n)^c \cup (b_1 \cup b_2 \cup b_3 \cup \ldots \cup b_p)\}^c$$
$$= \{(t_1 \cup t_2 \cup t_3 \cup \ldots \cup t_m \cup b_1 \cup b_2 \cup b_3 \cup \ldots \cup b_p)\}^c$$
$$= (s_1 \cup s_2 \cup s_3 \cup \ldots \cup s_q)$$

LEMMA 5.3. If C is a finite collection of refinement elements covering the refinement space (S, R, M), then there is a subdivision D of S of refinement elements so that if $d \varepsilon D$, then for each c in C, c and d are disjoint or $d \subseteq c$.

Proof. Let $d_1, d_2, d_3, \ldots, d_n$ be refinement elements covering the space S. Let A be the collection of all subcollections of $\{d_1, d_2, d_3, \ldots, d_n\}$. For $a = \{d_{a_1}, d_{a_2}, d_{a_3}, \ldots, d_{a_k}\} \varepsilon A$, let $b_a = (\bigcap d_i \mid d_i \varepsilon a) - (\bigcup d_j \mid d_j \not\varepsilon a)$.

Since by the note above, b_a is the union of finitely many disjoint refinement elements for each $a \varepsilon A$, and since A is finite, it is sufficient to show that if a and c are distinct elements of A, then b_a and b_c are disjoint, and that $(\bigcup b_a \mid a \varepsilon A)$ covers S.

Let $x \varepsilon S$. Then there exists $a \varepsilon A$ such that for the natural numbers $j_1, j_2, j_3, \ldots, j_m$, $a = \{d_{j_1}, d_{j_2}, d_{j_3}, \ldots, d_{j_m}\}$, and $x \varepsilon d_{j_i}$ for $1 \leq i \leq m$, but if $d_k \not\varepsilon a$, then $x \not\varepsilon d_k$. Thus $x \varepsilon (\bigcap d_i \mid d_i \varepsilon a)$ and $x \not\varepsilon (\bigcup d_j \mid d_j \not\varepsilon a)$ hence $x \varepsilon b_a$. Thus $\{b_a \mid a \varepsilon A\}$ covers S.

In order to prove that for a and c elements of A, b_a and b_c are disjoint, suppose $x \varepsilon b_{a_1}$ and $x \varepsilon b_{a_2}$ where a_1 and a_2 are distinct elements of A. Then there exists (w.l.o.g.) $d_i \varepsilon a_1$ but $d_i \not\varepsilon a_2$ since $a_1 \neq a_2$. If $x \varepsilon d_i$ then $x \not\varepsilon b_{a_2}$. If $x \not\varepsilon d_i$ then $x \not\varepsilon b_{a_1}$, hence b_{a_1} and b_{a_2} are disjoint for each distinct pair a_1 and a_2 of elements of A.

Furthermore, if d is an element of the cover, then for $a \varepsilon A$, either b_a and d are disjoint or $b_a \subseteq d$, and the proof of the lemma is complete.

<u>CORROLARY 5.4.</u> If R_1 and R_2 are subdivisions of S composed of refinement elements, then there exists a

subdivision R of S, composed of refinement elements, that is a common refinement of R_1 and R_2.

Proof. R_1+R_2 is a cover of S hence by the previous lemma there exists a subdivision R which is a refinement of both R_1 and R_2 and is composed of refinement elements.

The following example demonstrates that such spaces do exist and are more general in nature then R.

Example 4: Let S be a rectangular subspace of the real plane with sides parallel to the axes. If R is defined to be the empty set plus the set of rectangles with sides parallel to the axes, or line segments parallel to one or the other axes or single points, and M to be the area in the ordinary sense, clearly (S,R,M) is a refinement space.

More complex refinement spaces may be defined which lead to better generalizations of quasi-continuous functions, and it follows by induction that R^n with an appropriate collection of connected sets and an appropriate set function is a refinement space.

DEFINITION 5.5. The statement that the function f defined on the refinement space (S,R,M) is quasi-continuous means if I_1 and I_2 are segments such that $\bar{I}_1 \subseteq I_2$, then there is a finite collection of refinement elements R such that $S^*(I_1,f) \subseteq R^* \subseteq S^*(I_2,f)$.

DEFINITION 5.6. The statement that the function f defined on the refinement space (S,R,M) is a step function means there is a subdivision $D=\{d_1, d_2, d_3, \ldots, d_n\}$ of refinement elements and a finite sequence of real numbers $x_1, x_2, x_3, \ldots, x_n$ such that if $x \in d_i$, then $f(x)=x_i$.

It is easily seen that in R, definition 5.5 is equivalent to definition 4.1, and in refinement spaces, the proofs of theorems 4.3 and 4.4 are valid, using definition 5.5. Let Theorems 4.3' and 4.4' denote the theorems analogous to theorems 4.3 and 4.4.

The following theorems are a series of properties about quasi-continuous functions in refinement spaces.

THEOREM 5.7. A function f defined on the refinement space (S,R,M) is quasi-continuous if and only if it is the uniform limit of a sequence of step functions.

Proof. The uniform limit of a sequence of step functions is quasi-continuous is true by theorem 4.3'. Thus it remains only to be shown that if f is quasi-continuous then f is the uniform limit of a sequence of step functions. Let $\varepsilon>0$. Let $[u,v]$ denote an interval containing the range of f. We may cover $[u,v]$ with segments $s_1, s_2, s_3, \ldots, s_n$ such that the length of each $s_i<\varepsilon/2$. For each s_i, $1\leq i\leq n$, there is a segment s_i' so that $\bar{s}_i \;\; s_i'$ and $d(s_i,s_i')<\varepsilon/2$, and there is a finite collection R_i of refinement elements such that $S(s_i,f)^*\subseteq R_i^*\subseteq S(s_i',f)^*$. The collection $R = \{R_1, R_2, R_3, \ldots, R_n\}^*$ covers S, so there is a subdivision $D = \{d_1, d_2, d_3, \ldots, d_m\}$ that covers S and is a refinement of R. Define f_ε so that for each $x\varepsilon d_i$, there is an $R_j\varepsilon R$ such that $d_i \subseteq R_j^*$, $x\varepsilon R_j^*$. Let k be the least integer for which $x\varepsilon R_k^* \subseteq R_j^*$ and $f_\varepsilon(x)=c_k$ where $s_k=(c_k,b_k)$.

Now if $x\varepsilon d_i$ and $y\varepsilon d_i$, $d_i\varepsilon D$, then $f_\varepsilon(x)=f_\varepsilon(y)$ and f_ε is a step function. If $x\varepsilon d_i$ and $f_\varepsilon(x)=c_j$, $x\varepsilon R_j$, so

$|f(x)-f_\varepsilon(x)| = |f(x)-c_j|<\varepsilon$. Thus f is the uniform limit of a sequence of step functions.

PROPOSITION 5.8. The space of step functions over a refinement space (S,R,M) is linear.

Proof: Clearly if f is a step function and α is a scalar, then αf is a step function. Let f_1 and f_2 be step functions. Let R_1, $\{x_1, x_2, x_3, \ldots, x_n\}$ and R_2 $\{y_1, y_2, y_3, \ldots, y_m\}$ be the defining subdivisions and sequences of f_1 and f_2 respectively. Let $R=\{r_1, r_2, r_3, \ldots, r_p\}$ be a common refinement of R_1 and R_2. For each $r_i \varepsilon R$ there is exactly one $s_j \varepsilon R_1$ and $t_k \varepsilon R_2$ such that $r_i \subseteq s_j$ and $r_i \subseteq t_k$. Thus if x r_i define $f(x)=x_j+y_k$, and f(x) is a step function. Furthermore, $f(x)=x_j+y_k=f_1(x)+f_2(x)$. The other properties of linear spaces are trivially true since f is real valued.

PROPOSITION 5.9. If f is a continuous function defined over the compact refinement space (S,R,M), then f is quasi-continuous.

Proof: Let f be a continuous function defined over the compact refinement space (S,R,M), and let $\varepsilon>0$. The space S is closed. At each point x in S there exists a region R_x such that if $y \varepsilon R_x$, then $|f(y)-f(x)|<\varepsilon/2$. There exists an open refinement element S_x containing x such that $S_x \subseteq R_x$. Since S is closed and compact, let $S'=\{S_{x_1}, S_{x_2}, S_{x_3}, \ldots, S_{x_n}\}$ for $x_1, x_2, x_3, \ldots, x_n$ points of S, be a finite subcover of S. Let $R_1, R_2, R_3, \ldots, R_n$ be a subdivision of elements of R refining S'. If $y \varepsilon R_j \subseteq S_{x_i}$ for some natural numbers j and i, then let $f_\varepsilon(y)=f(x_i)$.

This f_ε is clearly a step function and f is the uniform limit of a sequence of step functions, hence quasi-continuous.

DEFINITION 5.10. The statement that the function f defined over the refinement space (S,R,M) is R-integrable means there is a real number J such that if $\varepsilon>0$, there is a subdivision of refinement elements D of the space S so that if $R = \{r_1, r_2, r_3, \ldots, r_n\}$ is a finite refinement of D, and $x_1, x_2, x_3, \ldots, x_n$ are points such that $x_i \varepsilon r_i$ for $1 \leq i \leq n$, then $|J - \sum_{i=1}^{n} f(x_i)M(r_i)| < \varepsilon$.

Since in a refinement space a common refinement of two subdivisions of refinement elements always exists, the Cauchy condition follows from the definition of R-integrable. We will use the Cauchy condition for our proof of Theorem 5.12. We state the Cauchy condition without proof.

THEOREM 5.11. Suppose f is a function defined over the refinement space (S,R,M). Then the following are equivalent.

(1) The function f is R-integrable.

(2) If $\varepsilon>0$, there is a subdivision $D = \{d_1, d_2, d_3, \ldots, d_n\}$ such that if $R = \{r_1, r_2, r_3, \ldots, r_m\}$ is a refinement of D, and $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_m$ are points such that $x_i \varepsilon d_i$, $1 \leq i \leq n$, and $y_i \varepsilon r_i$, $1 \leq i \leq m$, then

$$\left|\sum_{i=1}^{n} f(x_i)M(d_i) - \sum_{i=1}^{m} f(y_i)M(r_i)\right| < \varepsilon$$

<u>THEOREM 5.12.</u> If f is a bounded function defined over the compact refinement space (S,R,M), and for each $\varepsilon>0$, there is a countable collection $r_1, r_2, r_3, \ldots$ of open refinement elements covering $D(f)$ such that for each n, $\sum_{i=1}^{n} M(r_i)<\varepsilon$, then f is R-integrable.

Proof: If $\varepsilon>0$, define $M_\varepsilon=\{x|\omega(x)\geq\varepsilon\}$. Let $\varepsilon>0$, and suppose M_ε is infinite. If M_ε has no limit point then M_ε is closed. Suppose p is a limit point of M_ε, and suppose $\omega(p)<\varepsilon$. Denote $(\varepsilon-\omega(p))/2$ by δ. There is a region R containing p such that if $x\varepsilon R$, $f(x)<\overline{\omega}(p) + \delta/2$. But R must contain a point $m\varepsilon M_\varepsilon$. The following is a proof that either $\overline{\omega}(m)-\overline{\omega}(p)>\delta$, or $\underline{\omega}(p)-\underline{\omega}(m)>\delta$.

If $\overline{\omega}(m)\geq\overline{\omega}(p)>\underline{\omega}(p)\geq \underline{\omega}(m)$, then

$$\overline{\omega}(m)-\underline{\omega}(m) = (\overline{\omega}(m)-\overline{\omega}(p)) + (\overline{\omega}(p)-\underline{\omega}(p)) + (\underline{\omega}(p)-\underline{\omega}(m)) > \varepsilon.$$

$$\begin{aligned} \text{Thus } \overline{\omega}(m)-\overline{\omega}(p) &\geq \varepsilon-\omega(p) - (\underline{\omega}(p)-\underline{\omega}(m)) \\ &\geq 2\delta - (\underline{\omega}(p)-\underline{\omega}(m)). \end{aligned}$$

If $\underline{\omega}(p)-\underline{\omega}(m)<\delta$, then $\overline{\omega}(m)-\overline{\omega}(p)\geq\delta$. Similarly, if $\overline{\omega}(m)-\overline{\omega}(p)<\delta$, then $\underline{\omega}(p)-\underline{\omega}(m)\geq\delta$. If $\underline{\omega}(p)<\underline{\omega}(m)$, then $\overline{\omega}(m)-\overline{\omega}(p)>2\delta>\delta$, and if $\overline{\omega}(p)>\overline{\omega}(m)$, then $\underline{\omega}(p)-\underline{\omega}(m)>2\delta>\delta$. Hence we may suppose (w.l.o.g.) that $\overline{\omega}(m)-\overline{\omega}(p)<\delta$, since we may argue similarly for $\underline{\omega}(p)-\underline{\omega}(m)<\delta$. Since $\omega(m)\geq\varepsilon$, R must contain a point $x\varepsilon R$ such that $\overline{\omega}(m)-\delta/2<f(x)$. Thus $f(x) > \overline{\omega}(m) - \delta/2 > \overline{\omega}(p) + \delta/2$, and $f(x) > \overline{\omega}(p) + \delta/2$, but $x\varepsilon R$, and $f(x)<\overline{\omega}(p) + \delta/2$. This is a contradiction, hence $\omega(p)\geq\varepsilon$, and $p\varepsilon M_\varepsilon$. Therefore M_ε is closed.

Since f is bounded, there are real numbers a and b such that $a \leq f(x) \leq b$, and $|b-a|>1$ for each $x \in \mathcal{S}$. The number $M(\mathcal{S})$ is defined since the empty set is in $\mathcal{R}$, and $\mathcal{S}$ is the complement of the empty set. If $M(\mathcal{S})=0$, the theorem is trivial. If $M(\mathcal{S})>0$, let K be a natural number such that $K \cdot M(\mathcal{S})>1$. Let $\varepsilon>0$, and define $\delta=\varepsilon/(4 \cdot K \cdot M(\mathcal{S}) \cdot (b-a))$.

By the hypothesis there is a countable collection $R=\{s_1, s_2, s_3, \ldots\}$, of open refinement elements such that for each natural number n, $\sum_{i=1}^{n} M(s_i)<\delta$, and R^* covers $D(f)$. Since M_δ is closed and $\mathcal{S}$ is compact, there is a finite subcollection of R, $R'=\{r_1, r_2, r_3, \ldots, r_n\}$, that covers M_δ. The set $N=\mathcal{S}-(R')^*$ is closed since $\mathcal{S}$ is closed, and R'^* is open. For each $x \in N$, let S_x be a region containing x, such that if $y \in S_x$, then $|f(x)-f(y)|<\delta$. Such a region exists since $x \notin M_\delta$. Let $t_x \subseteq S_x$ be an open refinement element containing x. For $T=\{t_x | x \in N\}$, T^* covers N, thus some finite subcollection of T, $T'=\{t_1, t_2, t_3, \ldots, t_m\}$ covers N.

The collection $R'+T'$ covers $\mathcal{S}$. So there is a refinement of $\mathcal{S}$ consisting of refinement elements. Denote the elements of the refinement that are subsets of R'^* by $R''=\{r_1', r_2', r_3', \ldots, r_u'\}$, and those which are not by $T''=\{t_1', t_2', t_3', \ldots, t_v'\}$. Thus R'' subdivides R'^*, and T'' subdivides N.

Let $V=\{v_1, v_2, v_3, \ldots, v_j\}$ be a refinement of $R''+T''$. Let the elements of V be ordered such that $\bigcup_{i=1}^{w} v_i=N$ for some natural number w. Let $x_1, x_2, x_3, \ldots, x_u$ be points such that $x_i \varepsilon r_i'$, $y_1, y_2, y_3, \ldots, y_v$ be points such that $y_i \varepsilon t_i'$, and $z_1, z_2, z_3, \ldots, z_j$ be points such that $z_i \varepsilon v_i$. For each $v_i \subseteq r_k'$, let $q_i = x_k$, and for each $v_i \subseteq t_k'$, let $q_i = y_k$, for each natural number i, $1 \le i \le j$. Then

$$\left| \sum_{i=1}^{u} f(x_i) M(r_i') + \sum_{i=1}^{v} f(y_i) M(t_i') - \sum_{i=1}^{j} f(z_i) M(v_i) \right|$$
$$= \left| \sum_{i=1}^{j} f(q_i) M(v_i) - \sum_{i=1}^{j} f(z_i) M(v_i) \right|$$
$$= \left| \sum_{i=1}^{j} (f(q_i) - f(z_i)) M(v_i) \right|$$
$$= \left| \sum_{i=1}^{w} (f(q_i) - f(z_i)) M(v_i) + \sum_{i=w+1}^{j} (f(q_i) - f(z_i)) M(v_i) \right|.$$

Since $a \le f(x) \le b$ for each x is S,

$$\left| \sum_{i=w+1}^{j} (f(q_i) - f(z_i)) M(v_i) \right|$$
$$\le \sum_{i=w+1}^{j} (b-a) M(v_i)$$
$$= (b-1) \sum_{i=w+1}^{j} M(v_i).$$

Since $\bigcup_{i=w+1}^{j} v_i = R''*$,

$$(b-a) \sum_{i=w+1}^{j} M(v_i) = (b-a) \cdot M(R''*) \le (b-a) \cdot \varepsilon / (4 \cdot K \cdot M(S) \cdot (b-a))$$
$$= \varepsilon / (4 \cdot K \cdot M(S) \le \varepsilon/2.$$

Thus $\left| \sum_{i=w+1}^{j} (f(q_i) - f(z_i)) M(v_i) \right| < \varepsilon/2$.

Also since $\bigcup_{i=1}^{w} v_i = N$,

$$\sum_{i=1}^{w} (f(q_i) - f(z_i)) M(v_i)$$

$$\leq \sum_{i=1}^{w} 2\delta \cdot M(v_i) = 2\delta \cdot \sum_{i=1}^{w} M(v_i) = 2\delta \cdot M(N).$$

Since $\delta=\varepsilon/(4\cdot K\cdot M(S)\cdot(b-a))$, and $M(N)\leq M(S)$, we have

$$2\delta \cdot M(N) \leq \varepsilon/(2\cdot K\cdot(b-a)) \leq \varepsilon/2. \quad \text{Hence}$$

$$\left|\sum_{i=1}^{w}(f(q_i)-f(z_i))M(v_i) + \sum_{i=w+1}^{j}(f(q_i)-f(z_i)M(v_i)\right|$$

$$<|\varepsilon/2 + \varepsilon/2| = \varepsilon.$$

Thus f is R-integrable.

DEFINITION 5.13. The statement that the subset M of the refinement space (S,R,M) has measure 0 means for each $\varepsilon>0$, there is a countable collection of open refinement elements r_1, r_2, r_3, ... covering M so that for each natural number n, $\sum_{i=1}^{n} M(r_i)<\varepsilon$.

COROLLARY 5.14. If the quasi-continuous function f is defined over the refinement space (S,R,M), and the boundary of each element of R has measure 0, then f is R-integrable.

Proof: The set D(f) is a subset of the boundaries of countably many refinement elements $\{r_1, r_2, r_3, \ldots\}$. Let $\varepsilon>0$. For each natural number i with $\delta_i=\varepsilon/2^i$, let $\{c_{i,1}, c_{i,2}, c_{i,3}, \ldots\}$ be a countable collection of open refinement elements covering the boundary of r_i, so that for each natural number m, $\sum_{i=1}^{m} M(c_{i,j})<\delta_i$. Let $C=\{c|c=c_{i,j}$ for some natural numbers i and j$\}$, then C is a countable collection of open refinement elements covering D(f), and

if m is a natural number, and for $1 \leq i \leq m$, $c_i \varepsilon C$, $\sum_{i=1}^{m} M(c_i) < \varepsilon$.
Therefore D(f) has measure 0, and by theorem 5.12 is R-integrable.

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