WELL-POSEDNESS FOR WEAK SOLUTIONS OF AXISYMMETRIC DIV-CURL SYSTEMS

A Dissertation Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By

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Abstract

We study the axisymmetrc div-curl system on bounded volumes of revolution with normal and tangential boundary conditions. This vector system of equations arises in classical field theories. In particular, the electrostatic and magnetostatic axisymmetric Maxwell equations are axisymmetric div-curl systems. The analysis is based on orthogonal decompositions of axisymmetric vector fields.

The characterization of the scalar potentials and stream functions in the orthogonal decompositions leads to the analysis of axisymmetric Laplacian boundary value problems. Axisymmetric Laplacian eigenproblems give rise to natural bases for special gradient and curl subspaces for the orthogonal decompositions, and the eigenvalues appear as best constants in energy estimates for solutions of the axisymmetric Laplacian boundary value problems and in energy estimates for the axisymmetric div-curl system.

The results presented are valid for a general class of bounded C^2 volumes of revolution with a nonempty and connected intersection with the axis of symmetry. We allow the domain to contain toroidal holes.

Contents

Chapter 1

Introduction

Vector systems of equations are common and fundamental models in applications, e.g., Maxwell's equations in electromagnetism, Navier-Stokes equations in fluid dynamics, the Lamé system in elasticity. The div-curl system is a vector system of equations arising in classical field theories. For example, the electrostatic and magnetostatic Maxwell's equations have the form of a div-curl system. The primary focus of this thesis is the analysis of axisymmetric div-curl systems on bounded regions of revolution.

Axisymmetric vector fields can be represented using scalar potentials and it is important to carefully describe how these potentials should be chosen for specific fields. These representations give rise to linear axisymmetric Laplacian boundary value problems which characterize the scalar potentials and stream functions. Linear axisymmetric laplacian eigenvalue problems provide sharp estimates for some solutions of the aforementioned boundary value problems in terms of the interior and boundary sources, as well as bases for special gradient and curl subspaces of axisymmetric vector fields. Finally, well-posedness results for weak solutions of the axisymmetric div-curl system with normal or tangential boundary conditions are obtained using the scalar potentials and stream functions in the decomposition theorems.

Auchmuty and Alexander in [5] study the planar div-curl system with normal, tangential, and mixed boundary conditions using orthogonal decompositions with scalar potentials and stream functions, and in [6] carry out similar analysis for fully 3D div-curl systems with normal or tangential boundary conditions; [7] concerns the case of 3D div-curl systems with mixed boundary conditions. [9] by Bernardi, Dauge, and Maday is an exhaustive reference for analytic and numerical results on partial differential equations in axisymmetric domains with polygonal cross-sections. Our analysis differs by considering domains with multiply-connected boundaries. Analytic results on axisymmetric Maxwell's equations on domains with polygonal cross-sections are described in [1] by Assous, Ciarlet Jr., and Labrunie with a follow up [11] by Ciarlet Jr. and Labrunie describing some numerical results on the same problem. In [16] Mercier and Raugel analyze finite element methods for second order elliptic Dirichlet boundary value problems on axisymmetric domains with simply-connected cross-sections. Oh in [17] presents a theoretical framework for the analysis of axisymmetric problems using differential forms and exterior calculus. Other references on mathematical studies of vector systems are [2], [14], [12], [10], and [13] among others.

Chapter 2 describes the basic geometric setup and types of functions and vector fields used in our analysis. The axisymmetric domains considered here are specified by their cross-sections. We prove some analytic results on the functions and vector fields used in later chapters.

Chapter 3 is on linear axisymmetric Laplacian eigenproblems. These eigenproblems are used to derive sharp estimates for weak solutions of Laplacian boundary value problems studied in Chapter 4. The eigenproblems also give rise to natural bases for special gradient and curl subspaces appearing in the orthogonal decompositions studied in Chapter 5.

Chapter 4 is on linear axisymmetric Laplacian boundary value problems. These boundary value problems arise in the characterization of the scalar potentials and stream functions used in the orthogonal decompositions studied in Chapter 5, and in the well-posedness results for the div-curl systems studied in Chapter 6.

Chapter 5 in on orthogonal decomposition results for axisymmetric vector fields, in particular for axisymmetric poloidal fields. The results from Chapters 3 and 4 are used to exhibit bases for special subspaces appearing in the decompositions, and for characterizations of the scalar potentials and stream functions. A characterization of special harmonic fields determined by the topology of the cross-section is presented.

Chapter 6 is on well-posedness results for weak solutions of axisymmetric div-curl systems with normal or tangential boundary conditions. The decompositions from Chapter 5 are used to establish existence and uniqueness results. The estimates from Chapter 4 are used to derive energy estimates for weak solutions of the div-curl systems.

Chapter 2

Spaces of Axisymmetric Functions and Vector Fields

2.1 Geometrical Preliminaries

We use Cartesian coordinates $x = (x_1, x_2, x_3)$ to denote a generic point $x \in \mathbb{R}^3$. The axisymmetric domains we consider here are bounded regions of revolution in $\mathbb{R}^3$ whose axis of revolution is the x_3-axis and which intersect the axis of revolution. Let Ω_A denote such a volume. Let $\overline{\Omega}$ be the closed cross section of Ω_A in the x_1x_3-plane. We identify the open cross section Ω as a subset of the right half plane $\mathbb{R}^2_+ = \{(r, z) \in \mathbb{R}^2 : r > 0\}$. r and z are the cylindrical radius and height from the cylindrical coordinate system: $r = \sqrt{x_1^2 + x_2^2}, z = x_3$. Let $\partial\Omega$ denote the boundary of Ω in $\overline{\mathbb{R}^2_+}$, and let $\Gamma = \{(r, z) \in \partial\Omega : r > 0\}$. We assume that Γ consists of a single component $\Gamma = \Gamma_0$ or multiple components $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$. Ω_A and Ω are constrained to obey the following conditions:

(i) Ω_A is a bounded, connected, volume of revolution about the x_3-axis in $\mathbb{R}^3$ with C^2 boundary $\partial\Omega_A$, and $\Omega_A \cap \{(0,0,x_3) : x_3 \in \mathbb{R}\}$ is a connected subset of the x_3-axis.

(ii) Ω contained in the open region interior to $\Gamma_0 \cup \{r = 0\}$;

(iii) $\Gamma_1, \ldots, \Gamma_m$ are closed C^2 loops contained in the region interior to $\Gamma_0 \cup \{r = 0\}$;

(iv) or $\Gamma_1 = \cdots = \Gamma_m = \emptyset$, in which case $\Gamma = \Gamma_0$.

(v) the distance from (r, z) to $\partial\Omega \cap \{r = 0\}$ is r for all $(r, z) \in \Omega$.

Prototypical examples of Ω are shown in Figure 2.1The cross-section Ω of Ω_A when Γ has many components.figure.caption.2 and Figure 2.2The cross-section Ω of the volume of revolution Ω_A when $\Gamma = \Gamma_0$.figure.caption.3.

Henceforth, the domain Ω_A and the cross-section Ω are always assumed to satisfy the conditions (i) – (v) above.

The functions and vector fields we study will be essentially determined by their values in Ω and Γ.

2.2 Axisymmetric Functions

All functions considered here have range in $[-\infty, \infty]$ unless otherwise noted.

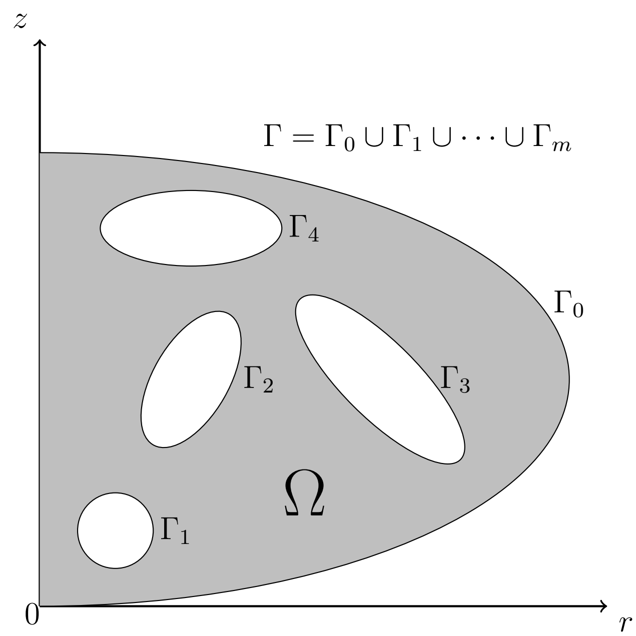

Figure 2.1: The cross-section Ω of Ω_A when Γ has many components.

Definition. Let F be a Lebesgue measurable function on the volume of revolution Ω_A. We say that F is *axisymmetric* if there is a Lebesgue measurable function f on Ω such that $F(x) = F(x_1, x_2, x_3) = f(\sqrt{x_1^2 + x_2^2}, x_3)$ almost everywhere on Ω_A. We call F an *axisymmetric lifting of* f.

Let r, θ, z be the cylindrical coordinates in $\mathbb{R}^3$ defined by

$$\begin{aligned} x_1 &= r\cos(\theta) \\ x_2 &= r\sin(\theta) \\ x_3 &= z. \end{aligned} \tag{2.1}$$

If $F \in L^1(\Omega_A)$ is axisymmetric with $F(x) = f(\sqrt{x_1^2 + x_2^2}, x_3)$ then the change of

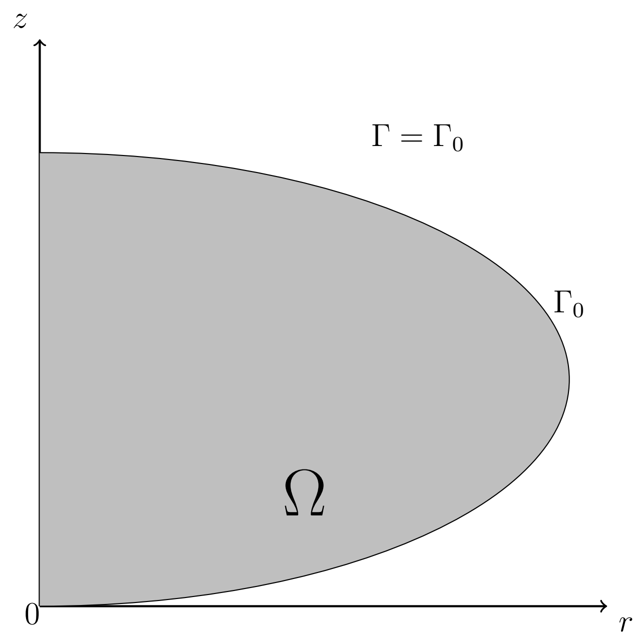

Figure 2.2: The cross-section Ω of the volume of revolution Ω_A when $\Gamma = \Gamma_0$.

variables theorem for Lebesgue integrals says that

$$\int_{\Omega_A} F(x)\, dx = 2\pi \int_{\Omega} f(r,z)\, r\, dr\, dz. \tag{2.2}$$

The function space $L_r^2(\Omega)$ is defined as all Lebesgue measurable functions $f(r,z)$ on Ω such that

$$\int_{\Omega} |f(r,z)|^2\, r\, dr\, dz < \infty. \tag{2.3}$$

$L_r^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{L_r^2} = \int_{\Omega} fg\, r\, dr\, dz. \tag{2.4}$$

For $f : \Omega \to \mathbb{R}$

$$\nabla f := \left(\frac{\partial f}{\partial r}, 0, \frac{\partial f}{\partial z}\right) \tag{2.5}$$

will denote the gradient of f. Let $C^\infty(\overline{\Omega}) = \{f|_{\overline{\Omega}} : f \in C^\infty(\mathbb{R}^2)\}$. The Sobolev space $H^1_r(\Omega)$ is defined the closure of $C^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{H^1_r}$ defined by

$$\|f\|^2_{H^1_r} = \int_\Omega \left(|f|^2 + |\nabla f|^2\right)\, r\,dr\,dz = \int_\Omega \left(|f|^2 + \left|\frac{\partial f}{\partial r}\right|^2 + \left|\frac{\partial f}{\partial z}\right|^2\right) r\,dr\,dz. \tag{2.6}$$

$H^1_r(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{H^1_r} = \int_\Omega \left(fg + \nabla f \cdot \nabla g\right)\, r\,dr\,dz. \tag{2.7}$$

Item (v) in Section 2.1 implies that $H^1_r(\Omega)$ coincides with the subspace of functions in $L^2_r(\Omega)$ whose weak derivatives with respect to r, z are also functions in $L^2_r(\Omega)$; see Remark 7.5 and Proposition 7.6 in [15]. Let $C^\infty_{z0}(\overline{\Omega}) = \{f|_{\overline{\Omega}} : f \in C^\infty_c(\mathbb{R}^2_+)\}$ where $\mathbb{R}^2_+ = \{(r,z) \in \mathbb{R}^2 : r > 0\}$ is the right-half plane of $\mathbb{R}^2$. $C^\infty_{z0}(\overline{\Omega}) \subset C^\infty(\overline{\Omega})$ and $C^\infty_{z0}(\overline{\Omega})$ consists of functions in $C^\infty(\overline{\Omega})$ with support away from the z-axis $\{r = 0\}$. The Sobolev space $V^1_r(\Omega)$ is defined as the closure of $C^\infty_{z0}(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{V^1_r}$ defined by

$$\|f\|^2_{V^1_r} = \|f\|^2_{H^1_r} + \int_\Omega \frac{|f|^2}{r}\,dr\,dz = \int_\Omega \left(|f|^2 + |\nabla f|^2 + \frac{1}{r^2}|f|^2\right) r\,dr\,dz \tag{2.8}$$

$V^1_r(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{V^1_r} = \langle f, g\rangle_{H^1_r} + \int_\Omega \frac{fg}{r}\,dr\,dz = \int_\Omega \left(fg + \nabla f \cdot \nabla g + \frac{fg}{r^2}\right) r\,dr\,dz \tag{2.9}$$

Let $C^\infty_{\Gamma 0}(\Omega)$ be the set of all smooth functions $f \in C^\infty(\overline{\Omega})$ such that $\mathrm{supp}(f) \cap \Gamma = \emptyset$, and let $H^1_{r,0}(\Omega)$ denote the closure of $C^\infty_{\Gamma 0}(\Omega)$ with respect to the H^1_r-norm. Let $V^1_{r,0}(\Omega)$ denote the closure of $C^\infty_c(\Omega)$ with respect to the V^1_r-norm.

Remark 2.2.1. *A standard argument shows that if $f \in H^1_r(\Omega)$ then the axisymmetric lifting $F(x) := f\left(\sqrt{x_1^2 + x_2^2}, x_3\right)$ belongs to $H^1_A(\Omega_A)$. This is the justification for* *introducing the weighted space $H^1_r(\Omega)$. Similarly, if $f \in H^1_{r,0}(\Omega)$, then a standard argument shows that F belongs to $H^1_A(\Omega_A) \cap H^1_0(\Omega_A)$.*

The Sobolev space $H^1_r(\Omega)$ is larger than the more common space $H^1(\Omega)$ since $L^2(\Omega) \subset L^2_r(\Omega)$. Therefore it is not immediately clear that a trace mapping onto Γ exists for functions in $H^1_r(\Omega)$. We also want the trace map to be compact and for the embedding $H^1_r(\Omega)$ into $L^2_r(\Omega)$ to be compact. Let $L^2_A(\Omega_A), H^1_A(\Omega_A)$ be the subspaces of $L^2(\Omega_A), H^1(\Omega_A)$ (resp.) consisting of axisymmetric functions. Note that these are closed subspaces of $L^2(\Omega_A), H^1(\Omega_A)$ respectively.

Lemma 2.2.2. *Let Ω_A, Ω satisfy conditions (i) – (v) in Section 2.1. Then the embeddings $H_r(\Omega) \hookrightarrow L^2_r(\Omega), H^1_{r,0}(\Omega) \hookrightarrow L^2_r(\Omega)$ are compact.*

Proof. The embedding of $H^1_r(\Omega)$ into $L^2_r(\Omega_A)$ is compact if and only if the embedding $H^1_A(\Omega_A)$ into $L^2_A(\Omega_A)$ is compact, as seen by identifying a function $F \in L^2_A(\Omega_A)$ with a representative $f \in L^2_r(\Omega)$. Our assumptions on Ω_A allow the use of Rellich's theorem for the embedding $H^1(\Omega_A) \hookrightarrow L^2(\Omega_A)$. Hence compactness of $H^1_A(\Omega_A) \hookrightarrow L^2_A(\Omega_A)$ follows since a bounded sequence in $H^1_A(\Omega_A)$ is bounded in $H^1(\Omega_A)$, and therefore contains a subsequence which is Cauchy in $L^2_A(\Omega_A)$. A similar argument shows that the embedding $H^1_{r,0}(\Omega) \hookrightarrow L^2_r(\Omega)$ is compact since the domain Ω_A is bounded and the embedding $H^1_0(\Omega_A) \hookrightarrow L^2(\Omega_A)$ is compact. □

Definition. Let $F : \partial\Omega_A \to \mathbb{R}$ be a measurable function with respect to the two-dimensional Hausdorff measure on $\partial\Omega_A$. We say F is *axisymmetric* if there is a function $f : \Gamma \to \mathbb{R}$ measurable with respect to the one-dimensional Hausdorff measure on Γ such that

$$F(x) = f\left(\sqrt{x_1^2 + x_2^2}, x_3\right) \quad \text{for all } x \in \partial\Omega_A. \tag{2.10}$$

Let $L^2_A(\partial\Omega_A)$ denote the subspace of $L^2(\partial\Omega_A)$ consisting of axisymmetric functions. We observe that $L^2_A(\partial\Omega_A)$ is a closed subspace of $L^2_A(\partial\Omega_A)$ just as $L^2_A(\Omega_A)$ is a closed subspace of $L^2(\Omega_A)$. Let $\gamma_A : H^1(\Omega_A) \to L^2(\partial\Omega_A)$ denote the trace map. Our conditions on $\partial\Omega_A$ imply that γ_A is compact.

Lemma 2.2.3. *Let Ω_A, Ω satisfy conditions (i) – (v) in Section 2.1. Then $\gamma_A : H^1_A(\Omega_A) \to L^2_A(\partial\Omega_A)$ is compact.*

Proof. Let $F \in H^1_A(\Omega_A)$ and let $f \in H^1_r(\Omega)$ be a representative on Ω. Let $f_n \in C^\infty(\overline{\Omega}), n \in \mathbb{N}$ be a sequence of smooth functions such that $f_n \to f$ in $H^1_r(\Omega)$. If $F_n, n \in \mathbb{N}$ are axisymmetric liftings to Ω_A of the f_n, then $F_n \in C^\infty(\overline{\Omega_A})$ and $F_n \to F$ in $H^1_A(\Omega_A)$. Therefore $\gamma_A F_n \to \gamma_A F$ in $L^2(\partial\Omega_A)$. $\gamma_A F_n = F_n|_{\partial\Omega_A} \in L^2_A(\partial\Omega_A)$ and $L^2_A(\partial\Omega_A)$ is closed, so $\gamma_A F \in L^2_A(\partial\Omega_A)$. $\square$

The preceding proof shows how to define a natural trace map $\gamma : H^1_r(\Omega) \to L^2_r(\Gamma)$ that is compact. Given $f \in H^1_r(\Omega)$, let $f_n \in C^\infty(\overline{\Omega}), n \in \mathbb{N}$ be a sequence of smooth functions such that $f_n \to f$ in $H^1_r(\Omega)$. For each f_n, let F_n denote its axisymmetric lifting to Ω_A. Then $F_n \to F$ in $H^1_A(\Omega_A)$ for some $F \in H^1_A(\Omega_A)$, and $\gamma_A F_n \to \gamma_A F$ in $L^2(\partial\Omega_A)$. $\gamma_A F_n \in L^2_A(\partial\Omega_A)$ and $L^2_A(\partial\Omega_A)$ is closed so $\gamma_A F \in L^2_A(\partial\Omega_A)$. We define γf to be the representative of $\gamma_A F$ on Γ such that (2.10equation.2.2.10) holds. Then if $f \in C^\infty(\overline{\Omega})$ and F is an axisymmetric lifting of f to Ω_A

$$\int_\Gamma |f|^2 \, r \, ds = \frac{1}{2\pi} \int_{\partial\Omega_A} |F|^2 \, d\sigma \leq C\|F\|^2_{H^1} = C\|f\|^2_{H^1_r} \tag{2.11}$$

so γ is continuous from $H^1_r(\Omega)$ to $L^2_r(\Gamma)$.

Corollary 2.2.4. *The trace* $\gamma : H^1_r(\Omega) \to L^2_r(\Gamma)$ *is compact.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_r(\Omega)$ and let F_n denote the axisymmetric lifting of f_n to Ω_A. Then $\{F_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1_A(\Omega_A)$ so a subsequence $\{\gamma_A F_{n_j}\}_{j\in\mathbb{N}}$ is Cauchy in $L^2_A(\partial\Omega_A)$ by compactness of γ_A. Therefore $\{\gamma f_{n_j}\}_{j\in\mathbb{N}}$ is Cauchy in $L^2_r(\Gamma)$ since

$$\int_\Gamma |\gamma f_n - \gamma f_k|^2 \, r \, ds = \frac{1}{2\pi} \int_{\partial\Omega_A} |F_n - F_k|^2 \, d\sigma. \tag{2.12}$$

$\square$

It follows that $V^1_{r,0}(\Omega)$ and $V^1_r(\Omega)$ are also compactly embedded in $L^2_r(\Omega)$ and that the trace mapping $\gamma|_{V^1_r(\Omega)} : V^1_r(\Omega) \subset H^1_r(\Omega) \to L^2_r(\Gamma)$ is compact.

2.3 Axisymmetric Vector Fields

Let $\{e_1, e_2, e_3\}$ be the standard Euclidean frame fields in $\mathbb{R}^3$. Let

$$u(x) = (u_1(x), u_2(x), u_3(x)) = u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3 \tag{2.13}$$

be a vector field on Ω_A. The cylindrical components u_r, u_θ, u_z are defined by

$$\begin{aligned} u_r(x) &= \frac{x_1}{\sqrt{x_1^2+x_2^2}} u_1(x) + \frac{x_2}{\sqrt{x_1^2+x_2^2}} u_2(x) \\ u_\theta(x) &= -\frac{x_2}{\sqrt{x_1^2+x_2^2}} u_1(x) + \frac{x_1}{\sqrt{x_1^2+x_2^2}} u_2(x) \\ u_z(x) &= u_3(x). \end{aligned} \tag{2.14}$$

We say that u is *axisymmetric* if the cylindrical components are axisymmetric functions.

An axisymmetric vector field u_A on Ω_A is thus identified with a vector field $u(r,z) = (u_r(r,z), u_\theta(r,z), u_z(r,z))$ on Ω by its cylindrical components, and conversely a vector field $u(r,z)$ on Ω defines an axisymmetric vector field on Ω_A. Let R_θ be the rotation matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.15}$$

Then u is axisymmetric if and only if $R_\theta^{-1} \circ u \circ R_\theta = u$ on Ω_A for all $\theta \in [-\pi, \pi)$.

Let $\{e_r, e_\theta, e_z\}$ be the cylindrical frame fields in $\mathbb{R}^3$.

Definition. If $u = (u_r, u_\theta, u_z) = u_r e_r + u_\theta e_\theta + u_z e_z$ is a vector field on Ω, then we call the vector field $U = (U_1, U_2, U_3)$ on Ω_A with components defined by

$$\begin{aligned} U_1(x) &= \frac{x_1}{\sqrt{x_1^2+x_2^2}} u_r\left(\sqrt{x_1^2+x_2^2}, x_3\right) - \frac{x_2}{\sqrt{x_1^2+x_2^2}} u_\theta\left(\sqrt{x_1^2+x_2^2}, x_3\right) \\ U_2(x) &= \frac{x_2}{\sqrt{x_1^2+x_2^2}} u_r\left(\sqrt{x_1^2+x_2^2}, x_3\right) + \frac{x_1}{\sqrt{x_1^2+x_2^2}} u_\theta\left(\sqrt{x_1^2+x_2^2}, x_3\right) \\ U_3(x) &= u_z\left(\sqrt{x_1^2+x_2^2}, x_3\right) \end{aligned} \tag{2.16}$$

an *axisymmetric lifting of* u.

Hence we restrict our attention to vector fields $u = (u_r, u_\theta, u_z)$ on Ω as our means to study axisymmetric vector fields on Ω_A. The space $L^2_r(\Omega; \mathbb{R}^3)$ is defined as the Hilbert space of vector fields $u = (u_r, u_\theta, u_z)$ on Ω with $u_r, u_\theta, u_z \in L^2_r(\Omega)$. $L^2_r(\Omega; \mathbb{R}^3)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{L^2_r} = \int_\Omega u \cdot v \, r \, dr \, dz = \int_\Omega u_r v_r + u_\theta v_\theta + u_z v_z \; r \, dr \, dz. \tag{2.17}$$

We will also denote the gradient of a function f on Ω using e_r, e_z by

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{\partial f}{\partial z} e_z. \tag{2.18}$$

Remark 2.3.1. *If $u_r, u_\theta \in V^1_r(\Omega)$ and $u_z \in H^1_r(\Omega)$, then a standard argument shows that the axisymmetric lifting U of u defined by* (2.16equation.2.3.16) *belongs to $H^1_A(\Omega_A; \mathbb{R}^3)$. This is the justification for introducing the weighted space $V^1_r(\Omega)$. Similarly, if $u_r, u_z \in V^1_{r,0}(\Omega)$ and $u_z \in H^1_{r,0}(\Omega)$, then a standard argument shows that U belongs to $H^1_A(\Omega; \mathbb{R}^3) \cap H^1_0(\Omega_A; \mathbb{R}^3)$.*

Definition. Let $u = (u_r, u_\theta, u_z)$ be a vector field on Ω. $(u_r, 0, u_z) = u_r e_r + u_z e_z$ is the *poloidal component* of u and $(0, u_\theta, 0) = u_\theta e_\theta$ is the *toroidal component* of u. The poloidal component of u is denoted u_P and the toroidal component is denoted u_T. u is called a *poloidal vector field* if $u_\theta = 0$, and u is called a *toroidal vector field* if $u_r = u_z = 0$.

The classical vector operators div and curl for vector fields $u(r, z)$ on Ω considered here are given in cylindrical coordinates by

$$\begin{aligned} \operatorname{div}(u) &= \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} \\ \operatorname{curl}(u) &= -\frac{\partial u_\theta}{\partial z} e_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) e_\theta + \frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} e_z. \end{aligned} \tag{2.19}$$

These definitions continue to hold for vector fields $u = (u_r, u_\theta, u_z)$ provided $u_r, u_\theta \in V_r^1(\Omega), u_z \in H_r^1(\Omega)$. We make note of the identities

$$
\begin{aligned}
\operatorname{div}(u) &= \operatorname{div}(u_P) \\
\operatorname{div}(u_T) &= 0 \\
\operatorname{curl}(u_P) &= (\operatorname{curl}(u))_T \\
\operatorname{curl}(u_T) &= (\operatorname{curl}(u))_P
\end{aligned}
\tag{2.20}
$$

and $\operatorname{div}(\operatorname{curl}(u)) = 0$ as usual. The divergence and curl for vector fields in $L_r^2(\Omega; \mathbb{R}^3)$ are defined by duality.

Definition. Let $u \in L_r^2(\Omega; \mathbb{R}^3)$. $\operatorname{div}(u) \in (H_{r,0}^1(\Omega))^*$ is defined by

$$
\langle \operatorname{div}(u), \phi \rangle = -\int_\Omega u \cdot \nabla \phi \, r \, dr \, dz \quad \forall \phi \in H_{r,0}^1(\Omega) \tag{2.21}
$$

and $\operatorname{curl}(u) \in (V_{r,0}^1(\Omega) \times V_{r,0}^1(\Omega) \times H_{r,0}^1(\Omega))^*$ is defined by

$$
\langle \operatorname{curl}(u), (v_r, v_\theta, v_z) \rangle = \int_\Omega u \cdot \operatorname{curl}(v_r e_r + v_\theta e_\theta + v_z e_z) \, r \, dr \, dz \tag{2.22}
$$

for all $(v_r, v_\theta, v_z) \in V_{r,0}^1(\Omega) \times V_{r,0}^1(\Omega) \times H_{r,0}^1(\Omega)$.

The identities in (2.20Axisymmetric Vector Fieldsequation.2.3.20) hold for definitions (2.21equation.2.3.21), (2.22equation.2.3.22) as well. Our conditions on Γ imply that a unit outward normal ν is defined a.e. on Γ. If $\operatorname{div}(u) \in L_r^2(\Omega)$ or $\operatorname{curl}(u) \in L_r^2(\Omega; \mathbb{R}^3)$, then the normal trace $u \cdot \nu$ or tangential trace $u \wedge \nu$ (resp.) is also defined by duality. $H_r^{1/2}(\Gamma)$ denotes the range of the trace $\gamma : H_r^1(\Omega) \to L_r^2(\Gamma)$, and $V_r^{1/2}(\Gamma)$ denotes the range of γ restricted to $V_r^1(\Omega)$.

Definition. Let $u \in L^2_r(\Omega;\mathbb{R}^3)$. If $\operatorname{div}(u) \in L^2_r(\Omega)$, then the *normal trace* $u \cdot \nu \in (H^{1/2}_r(\Gamma))^*$ is defined by

$$\langle u \cdot \nu, \gamma\phi \rangle = \int_\Omega u \cdot \nabla\phi \, r\, dr\, dz + \int_\Omega \phi \operatorname{div}(u)\, r\, dr\, dz \tag{2.23}$$

for all $\phi \in H^1_r(\Omega)$. If $\operatorname{curl}(u) \in L^2_r(\Omega;\mathbb{R}^3)$, then the *tangential trace* $u \wedge \nu \in (V^{1/2}_r(\Gamma) \times V^{1/2}_r(\Gamma) \times H^{1/2}_r(\Gamma))^*$ is defined by

$$\langle u \wedge \nu, (\gamma v_r, \gamma v_\theta, \gamma v_z) \rangle = \int_\Omega u \cdot \operatorname{curl}(v)\, r\, dr\, dz - \int_\Omega \operatorname{curl}(u) \cdot v\, r\, dr\, dz \tag{2.24}$$

for all $v = (v_r, v_\theta, v_z) \in V^1_r(\Omega) \times V^1_r(\Omega) \times H^1_r(\Omega)$.

Observe that the definition of $u \cdot \nu$ implies that $u_T \cdot \nu = 0$. This coincides with the geometric result that the unit outward normal of a smooth surface of revolution is poloidal.

2.4 Poincaré Inequalities

We will use Poincaré inequalities for functions in $H^1_r(\Omega)$ to prove various coercivity results. The following two versions hold by taking axisymmetric liftings of functions in Ω to the volume of revolution Ω_A, changing variables in the integrals, and then applying the Poincaré inequalities for $H^1_0(\Omega_A)$ and $H^1(\Omega_A)$. Denote

$$|\Omega| = \frac{\operatorname{vol}(\Omega_A)}{2\pi}. \tag{2.25}$$

This is the cross-sectional area of Ω.

Theorem 2.4.1. *There is a constant $C > 0$ such that*

$$\int_\Omega |f|^2\, r\, dr\, dz \le C \int_\Omega |\nabla f|^2\, r\, dr\, dz \quad \textit{for all } f \in H^1_{r,0}(\Omega). \tag{2.26}$$

Theorem 2.4.2. *There is a constant $C > 0$ such that*

$$\int_\Omega |f - \langle f \rangle|^2 \, r \, dr \, dz \le C \int_\Omega |\nabla f|^2 \, r \, dr \, dz \quad \text{for all } f \in H^1_r(\Omega) \tag{2.27}$$

where

$$\langle f \rangle = \frac{\int_\Omega f \, r \, dr \, dz}{|\Omega|}. \tag{2.28}$$

We will also need the following variant of the Poincaré inequality for functions in $V^1_r(\Omega)$. Recall

$$\operatorname{curl}(\psi e_\theta) = -\frac{\partial \psi}{\partial z} e_r + \frac{1}{r} \frac{\partial (r\psi)}{\partial r} e_z \tag{2.29}$$

for functions $\psi \in V^1_r(\Omega)$.

Theorem 2.4.3. *There is a constant $C > 0$ such that*

$$\int_\Omega |\psi|^2 \, r \, dr \, dz \le C \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz, \quad \forall \psi \in V^1_r(\Omega). \tag{2.30}$$

Proof. Our approach is to appeal to an existing estimate for three-dimensional fields to the axisymmetric lifting of ψe_θ for $\psi \in V^1_r(\Omega)$. Let Ω_A denote the C^2 volume of revolution obtained by rotating Ω about the z axis. Then our assumptions on Ω and $\partial\Omega$ imply that Theorem 5.1 from [3] is applicable to Ω_A. This theorem says that there is a $C > 0$ depending only on Ω_A such that

$$\int_{\Omega_A} |A|^2 \, dx \le C \int_{\Omega_A} |\operatorname{curl}(A)|^2 dx \tag{2.31}$$

for all $A \in H^1(\Omega_A; \mathbb{R}^3)$ such that:

1. $\operatorname{div}(A) = 0$ in Ω_A;

2. $A \cdot \nu = 0$ on $\partial\Omega_A$;

3. $A \perp \mathcal{H}^1(\Omega_A)$

where

$$\mathcal{H}^1(\Omega_A) =$$
$$\{h \in L^2(\Omega_A;\mathbb{R}^3) : \operatorname{div}(h) = 0 \text{ in } \Omega_A, \operatorname{curl}(h) = 0 \text{ in } \Omega_A, h\cdot\nu = 0 \text{ on } \partial\Omega_A\}. \tag{2.32}$$

It suffices to prove the estimate for $\psi \in C^\infty_{z0}(\overline{\Omega})$ by density, hence suppose $\psi \in C^\infty_{z0}(\overline{\Omega})$. Let B denote the axisymmetric lifting of ψe_θ, so that

$$B(x) = -\frac{x_2}{\sqrt{x_1^2+x_2^2}}\psi\left(\sqrt{x_1^2+x_2^2}, x_3\right)e_1 + \frac{x_1}{\sqrt{x_1^2+x_2^2}}\psi\left(\sqrt{x_1^2+x_2^2}, x_3\right)e_2 \tag{2.33}$$

where $x = (x_1, x_2, x_3)$. A direct calculation shows that

$$\begin{aligned}\sum_{i,j=1}^{3}\left|\frac{\partial B_i}{\partial x_j}\right|^2 &= \\ &\left|\frac{\partial\psi}{\partial r}\left(\sqrt{x_1^2+x_2^2}, x_3\right)\right|^2 + \left|\frac{\partial\psi}{\partial z}\left(\sqrt{x_1^2+x_2^2}, x_3\right)\right|^2 \\ &+ \frac{1}{x_1^2+x_2^2}\left|\psi\left(\sqrt{x_1^2+x_2^2}, x_3\right)\right|^2.\end{aligned} \tag{2.34}$$

Then $\psi \in C^\infty_{z0}(\overline{\Omega}) \subset V^1_r(\Omega)$ implies that

$$\int_{\Omega_A} |B|^2 + \sum_{i,j=1}^{3}\left|\frac{\partial B_i}{\partial x_j}\right|^2\,dx = 2\pi\int_{\Omega}|\psi|^2 + |\nabla\psi|^2 + \frac{|\psi|^2}{r^2}\,r\,dr\,dz < \infty, \tag{2.35}$$

hence $B \in H^1(\Omega_A;\mathbb{R}^3)$ The axisymmetric lifting preserves the divergence and Euclidean dot product so $\operatorname{div}(\psi e_\theta) = 0$ in Ω and $\psi e_\theta \cdot \nu = 0$ on Γ imply $\operatorname{div}(B) = 0$ in Ω_A and $B\cdot\nu = 0$ on $\partial\Omega_A$ respectively. ψe_θ is toroidal, so it suffices to check that B is orthogonal to every $h \in \mathcal{H}^1(\Omega_A)$ with zero poloidal component. If $h \in \mathcal{H}^1(\Omega_A)$ and h has no poloidal component, then $\operatorname{div}(h) = 0$ means

$$\frac{1}{r}\frac{\partial h_\theta}{\partial\theta} = 0, \tag{2.36}$$

therefore h_θ is independent of θ, hence $h_\theta e_\theta$ is an axisymmetric harmonic toroidal field. This means $h_\theta \equiv 0$ since Ω_A contains its axis of revolution. Then $\langle B, h\rangle_{L^2} = 0$ trivially. Now we apply Theorem 5.1 to obtain that

$$\int_{\Omega_A} |B|^2 \, dx \leq C \int_{\Omega_A} |\operatorname{curl}(B)|^2 \, dx,$$

and upon changing variables back to the cylindrical coordinates we get

$$\int_{\Omega} |\psi|^2 \, r \, dr \, dz \leq C \int_{\Omega} |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz.$$

This holds for arbitrary $\psi \in C^\infty_{z0}(\Omega)$ so we conclude that the estimate holds for all $\psi \in V^1_r(\Omega)$. □

Definition. We call the estimate (2.30equation.2.4.30) the *curl-Poincaré inequality for $V^1_r(\Omega)$.*

Corollary 2.4.4. $\|\operatorname{curl}(\psi e_\theta)\|_{L^2_r}$ *defines an equivalent norm on $V^1_r(\Omega)$.*

Proof. Again, the method of proof is to apply an existing result to the axisymmetric lifting v of ψe_θ for $\psi \in C^\infty_{z0}(\overline{\Omega})$. If v is such an axisymmetric lifting, then Corollary 1 on p.212 of [12] Ch. IX shows that there is a constant $C > 0$ independent of v such that

$$\|v\|^2_{H^1} \leq C(\|v\|^2_{L^2} + \|\operatorname{curl}(v)\|^2_{L^2}) \tag{2.37}$$

due to our conditions on $\partial\Omega_A$. Then changing variables back to cylindrical coordinates yields the estimate

$$\|\psi\|^2_{V^1_r} \leq C(\|\psi\|^2_{L^2_r} + \|\operatorname{curl}(\psi e_\theta)\|^2_{L^2_r}) \tag{2.38}$$

since $\|v\|_{H^1}^2 = 2\pi\|\psi\|_{V_r^1}^2$. Then we apply (2.30equation.2.4.30) to see that that there is $C > 0$ such that

$$\|\psi\|_{V_r^1}^2 \leq C(\|\psi\|_{L_r^2}^2 + \|\operatorname{curl}(\psi e_\theta)\|_{L_r^2}^2) \leq C\|\operatorname{curl}(\psi e_\theta)\|_{L_r^2}^2. \tag{2.39}$$

On the other hand, we may apply Young's inequality to see that

$$\begin{aligned}
\|\operatorname{curl}(\psi e_\theta)\|_{V_r^1}^2 &= \int_\Omega |\nabla\psi|^2 + \frac{|\psi|^2}{r^2} + \frac{2\psi}{r}\frac{\partial\psi}{\partial r}\, r\,dr\,dz \\
&\leq \int_\Omega |\nabla\psi|^2 + \frac{|\psi|^2}{r^2}\, r\,dr\,dz + 2\int_\Omega \frac{|\psi|^2}{r^2}\, r\,dr\,dz + \frac{1}{2}\int_\Omega \left|\frac{\partial\psi}{\partial r}\right|^2 r\,dr\,dz \\
&\leq C\int_\Omega |\nabla\psi|^2 + \frac{|\psi|^2}{r^2}\, r\,dr\,dz \\
&\leq C\|\psi\|_{V_r^1}^2
\end{aligned} \tag{2.40}$$

which proves the claim. □

Theorem 2.4.5. *There is a constant $C > 0$ such that*

$$\int_\Omega |f_r|^2 + |f_z|^2\, r\,dr\,dz \leq C\int_\Omega |\operatorname{curl}(f_r e_r + f_z e_z)|^2\, r\,dr\,dz \tag{2.41}$$

for all $f_r \in V_r^1(\Omega)$, $f_z \in H_r^1(\Omega)$ such that $\operatorname{div}(f_r e_r + f_z e_z) = 0$ in Ω, $(\gamma f_r e_r + \gamma f_z e_z)\cdot\nu = 0$ on Γ, and $(f_r e_r + f_z e_z) \in (\mathcal{H}_{\nu 0}(\Omega))^\perp$.

Proof. The argument here again relies on appealing to an existing result for fully three-dimensional fields in $L^2(\Omega_A;\mathbb{R}^3)$. Suppose that $f_r \in C_{z0}^\infty(\overline{\Omega})$, $f_z \in C^\infty(\overline{\Omega})$ such that $\operatorname{div}(f_r e_r + f_z e_z) = 0$ in Ω, $(f_r e_r + f_z e_z)\cdot\nu = 0$ on Γ, and $(f_r e_r + f_z e_z) \in (\mathcal{H}_{\nu 0}(\Omega))^\perp$. Let $F : \Omega_A \to \mathbb{R}^3$ be the axisymmetric lifting of $f_r e_r + f_z e_z$ to all of Ω_A. Then $\operatorname{div}(F) = 0$ in Ω_A and $F\cdot\nu = 0$ on $\partial\Omega_A$. Let $\mathcal{H}^1(\Omega_A)$ be as in the proof of the curl-Poincaré inequality for $V_r^1(\Omega)$. Now the estimate (2.41equation.2.4.41) is verified if

$F \in (\mathcal{H}^1(\Omega))^\perp$. $L^2_A(\Omega_A;\mathbb{R}^3)$ is a closed subspace of $L^2(\Omega_A;\mathbb{R}^3)$ so $F \in (\mathcal{H}^1(\Omega_A))^\perp$ if and only if F is orthogonal to $L^2_A(\Omega_A;\mathbb{R}^3)\cap\mathcal{H}^1(\Omega)$. Moreover $f_r e_r + f_z e_z$ is poloidal, so it suffices to check that F is orthogonal to every field in $L^2_A(\Omega_A;\mathbb{R}^3)\cap\mathcal{H}^1(\Omega_A)$ with zero toroidal component. Let $h \in L^2_A(\Omega_A;\mathbb{R}^3) \cap \mathcal{H}^1(\Omega_A)$ with $h_\theta = 0$. Then $h|_\Omega$ is well-defined since h is actually smooth by Weyl's lemma. Moreover $h|_\Omega$ is a poloidal field since $h_\theta = 0$ and $h|_\Omega \in \mathcal{H}_{\nu 0}(\Omega)$. Therefore $F \perp \mathcal{H}^1(\Omega_A)$ by the condition that $(f_r e_r + f_z e_z) \perp \mathcal{H}_{\nu 0}(\Omega)$. Then Theorem 5.1 from [3] and the density of $C^\infty_{z0}(\overline{\Omega})$ in $V^1_r(\Omega)$ and $C^\infty(\overline{\Omega})$ imply the estimate (2.41equation.2.4.41). $\square$

Chapter 3

Linear Axisymmetric Laplacian Eigenproblems

3.1 Introduction

Let Δ denote the axisymmetric Laplacian in cylindrical coordinates, i.e.

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.1}$$

This chapter will study eigenproblems and boundary value problems for the operators $-\Delta$ and $-\Delta + r^{-2}$. If e_θ is the azimuthal unit vector in cylindrical coordinates and ψ is a smooth function then, in cylindrical coordinates

$$\operatorname{curl}(\operatorname{curl}(\psi e_\theta)) = \left(-\Delta\psi + \frac{1}{r^2}\psi\right) e_\theta. \tag{3.2}$$

3.2 Eigenproblems for $-\Delta$

3.2.1 The Dirichlet Eigenvalue Problem

Consider the eigenvalue problem of finding a real-valued function $\phi \neq 0$ and real number λ such that

$$\begin{cases} -\Delta\phi = \lambda\phi \quad \text{in } \Omega, \\ \\ \quad \phi = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.3}$$

If such a pair (ϕ, λ) exists and ϕ is smooth, then we may integrate by parts to obtain that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, r \, dr \, dz = \lambda \int_\Omega \phi\psi \, r \, dr \, dz \quad \text{for all } \psi \in C^\infty_{\Gamma 0}(\Omega). \tag{3.4}$$

Both sides of (3.4The Dirichlet Eigenvalue Problemequation.3.2.4) are well-defined if $\phi, \psi \in H^1_{r,0}(\Omega)$, hence we consider the problem of finding nontrivial $(\phi, \lambda) \in H^1_{r,0}(\Omega) \times \mathbb{R}$ such that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, r \, dr \, dz = \lambda \int_\Omega \phi\psi \, r \, dr \, dz \quad \text{for all } \psi \in H^1_{r,0}(\Omega) \tag{3.5}$$

Definition. If $(\phi, \lambda) \in H^1_{r,0}(\Omega) \times \mathbb{R}$ is a nontrivial solution of (3.5The Dirichlet Eigenvalue Probleme then ϕ is a *Dirichlet eigenfunction of* $-\Delta$ *on* Ω *corresponding to the Laplacian Dirichlet eigenvalue* λ.

Let $a : H^1_{r,0}(\Omega) \times H^1_{r,0}(\Omega) \to \mathbb{R}$ be the bilinear form

$$a(\phi, \psi) = \int_\Omega \nabla\phi \cdot \nabla\psi \, r \, dr \, dz, \tag{3.6}$$

and let $\mathcal{A} : H^1_{r,0}(\Omega) \to \mathbb{R}$ denote the quadratic form associated to a

$$\mathcal{A}(\phi) = a(\phi, \phi) = \int_\Omega |\nabla\phi|^2 \, r \, dr \, dz. \tag{3.7}$$

so (3.5The Dirichlet Eigenvalue Problemequation.3.2.5) says

$$a(\phi, \psi) = \lambda \langle \phi, \psi \rangle_{L^2_r} \quad \text{for all } \psi \in H^1_{r,0}(\Omega). \tag{3.8}$$

Theorem 3.2.1. *The smallest Dirichlet eigenvalue λ_1 of $-\Delta$ is strictly positive and is characterized by the variational principle*

$$\frac{1}{\lambda_1} = \sup \int_\Omega |\phi|^2 \, r \, dr \, dz \quad s.t. \ \int_\Omega |\nabla \phi|^2 \, r \, dr \, dz = 1, \phi \in H^1_{r,0}(\Omega). \tag{3.9}$$

Proof. Let $C_1 = \{\phi \in H^1_{r,0}(\Omega) : \mathcal{A}(\phi) = 1\}$ and consider the problem of finding

$$\beta_1 = \sup_{\phi \in C_1} \|\phi\|_{L^2_r}. \tag{3.10}$$

The Cauchy-Schwarz inequality implies that $|a(\phi, \psi)| \leq \|\phi\|_{H^1_r} \|\psi\|_{H^1_r}$ for all $\phi, \psi \in H^1_{r,0}(\Omega)$ so a is continuous on $H^1_{r,0}(\Omega) \times H^1_{r,0}(\Omega)$. The Poincaré inequality for $H^1_{r,0}(\Omega)$ implies that there is a $C > 0$ such that $C\|\phi\|^2_{H^1_r} \leq \mathcal{A}(\phi)$ for all $\phi \in H^1_{r,0}(\Omega)$ so a is also coercive on $H^1_{r,0}(\Omega)$. $\|\phi\|_{L^2_r}$ is a norm on $H^1_{r,0}(\Omega) \subset L^2_r(\Omega)$ so $\|\phi\|^2_{L^2_r} > 0$ for all nonzero ϕ in $H^1_{r,0}(\Omega)$ and $\|\phi\|^2_{L^2_r} = 0$ only if $\phi = 0$. Moreover the embedding $H^1_r(\Omega) \to L^2_r(\Omega)$ is compact. Then we may apply Theorem 3.1 in [4] to conclude that:

(i) $\beta_1 > 0$ is finite;

(ii) there are maximizers $\pm\hat{\phi}_1$ of $\|\cdot\|^2_{L^2_r}$ on C_1 where β_1 is attained;

(iii) $\hat{\phi}_1$ is a Dirichlet eigenfunction of $-\Delta$ corresponding to the eigenvalue $\lambda_1 := 1/\beta_1$, i.e.

$$\int_\Omega \nabla \hat{\phi}_1 \cdot \nabla \psi \, r \, dr \, dz = \lambda_1 \int_\Omega \hat{\phi}_1 \psi \, r \, dr \, dz, \quad \forall \psi \in H^1_{r,0}(\Omega);$$

(iv) λ_1 is the smallest eigenvalue and

$$\int_\Omega |\phi|^2 \, r\,dr\,dz \le \frac{1}{\lambda_1} \int_\Omega |\nabla\phi|^2 \, r\,dr\,dz, \quad \forall \phi \in H^1_{r,0}(\Omega).$$

$\square$

The above proof shows that $a(\cdot,\cdot)$ defines an inner product on $H^1_{r,0}(\Omega)$. The variational principle for λ_1 may be iterated to generate a sequence of eigenfunctions that are orthonormal with respect to the bilinear form a. Let $\{\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_{k-1}\}$ be $k-1$ eigenfunctions corresponding to the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{k-1}$.

Theorem 3.2.2. *(i) The kth Dirichlet eigenvalue λ_k of $-\Delta$ is characterized by the variational principle*

$$\frac{1}{\lambda_k} = \sup \int_\Omega |\phi|^2 \, r\,dr\,dz \tag{3.11}$$

for all $\phi \in H^1_{r,0}(\Omega)$ such that $\int_\Omega |\nabla\phi|^2 \, r\,dr\,dz = 1$ and $\int_\Omega \nabla\phi \cdot \nabla\hat{\phi}_j \, r\,dr\,dz = 0$ for $j = 1, \dots, k-1$. We also have $\lambda_k \ge \lambda_{k-1}$ and

$$\int_\Omega |\phi|^2 \, r\,dr\,dz \le \frac{1}{\lambda_k} \int_\Omega |\nabla\phi|^2 \, r\,dr\,dz \tag{3.12}$$

for all $\phi \in H^1_{r,0}(\Omega)$ such that $\int_\Omega \nabla\phi \cdot \nabla\hat{\phi}_j \, r\,dr\,dz = 0$ for $j = 1, \dots, k-1$.

(ii) $\lambda_k \to \infty$ as $k \to \infty$ and the set of eigenfunctions $\mathcal{E} := \{\hat{\phi}_k : k \in \mathbb{N}\}$ is an orthonormal basis of $H^1_{r,0}(\Omega)$ with respect to the inner product $a(\phi,\psi) = \int_\Omega \nabla\phi \cdot \nabla\psi \, r\,dr\,dz$.

(iii) The normalized eigenfunctions $\tilde{\mathcal{E}} := \{\tilde{\phi}_k := \lambda_k^{-1/2}\hat{\phi}_k : k \in \mathbb{N}\}$ form an orthonormal basis of $L^2_r(\Omega)$ with respect to the standard inner product.

Proof. The bilinear form a is continuous and coercive, the inner product $\langle \phi, \psi \rangle_{L^2_r}$ is continuous, and $\|\phi\|^2_{L^2_r} > 0$ for all $\phi \in H^1_{r,0}(\Omega)$ so we may apply Theorem 4.2 of [4] to obtain the aforementioned variational characterization of λ_k. The embedding $H^1_{r,0}(\Omega) \hookrightarrow L^2_r(\Omega)$ is compact with dense range since Ω is bounded, therefore Theorem 4.3 and Theorem 4.6 of [4], respectively, imply that $\mathcal{E}$ is an orthonormal basis of $H^1_{r,0}(\Omega)$ with respect to the inner product a and that $\tilde{\mathcal{E}}$ is an orthonormal basis of $L^2_r(\Omega)$ with respect to the standard inner product. $\square$

Example. Let $\Omega_A = B_R(0) = \{x \in \mathbb{R}^3 : |x| < R\}$ be the ball of radius R centered at the origin. Then $\Omega = \{(r,z) \in \mathbb{R}^2_+ : r^2 + z^2 < R\}$. The Dirichlet eigenfunctions of $-\Delta$ on Ω are

$$\begin{aligned} \phi_{\ell,n}(r,z) = \left(\frac{R}{\tilde{j}_{\ell,n}\sqrt{r^2+z^2}}\right) J_{\ell+1/2}\left(\frac{\tilde{j}_{\ell,n}\sqrt{r^2+z^2}}{R}\right) P_\ell\left(\frac{r}{\sqrt{r^2+z^2}}\right), \\ \text{for } \ell = 0,1,2,\dots \quad \text{and } n = 1,2,3,\dots \end{aligned} \tag{3.13}$$

where $J_{\ell+1/2}$ is the half-integer Bessel function of the first kind, $\tilde{j}_{\ell,n}$ is the nth positive root of $J_{\ell+1/2}$, and P_ℓ is the Legendre polynomial of degree ℓ. The Dirichlet eigenvalues are

$$\left(\frac{\tilde{j}_{\ell,n}}{R}\right)^2 \quad \text{for } \ell = 0,1,2,\dots \quad \text{and } n = 1,2,3,\dots. \tag{3.14}$$

3.2.2 The Neumann Eigenvalue Problem

Consider the eigenvalue problem of finding a real-valued function $\phi \neq 0$ and real number λ such that

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega, \\ D_\nu \phi = 0 & \text{on } \Gamma. \end{cases} \tag{3.15}$$

Here $D_\nu \phi = \nabla\phi \cdot \nu$ is the normal derivative of ϕ. If such a pair (ϕ, λ) exists and ϕ is smooth, then we may integrate by parts to obtain that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, r\, dr\, dz = \lambda \int_\Omega \phi\psi \, r\, dr\, dz \quad \text{for all } \psi \in C^\infty(\overline{\Omega}). \tag{3.16}$$

Both sides of (3.16The Neumann Eigenvalue Problemequation.3.2.16) are well-defined if $\phi, \psi \in H^1_r(\Omega)$, hence we consider the problem of finding nontrivial $(\phi, \lambda) \in H^1_r(\Omega) \times \mathbb{R}$ such that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, r\, dr\, dz = \lambda \int_\Omega \phi\psi \, r\, dr\, dz \quad \text{for all } \psi \in H^1_r(\Omega) \tag{3.17}$$

Definition. If $(\phi, \lambda) \in H^1_r(\Omega) \times \mathbb{R}$ is a nontrivial (3.17The Neumann Eigenvalue Problemequation.3. then ϕ is a *Neumann eigenfunction of* $-\Delta$ *on* Ω *corresponding to the Laplacian Neumann eigenvalue* λ.

We observe that $\phi_0^{(N)} \equiv 1$ is a Neumann eigenfunction of $-\Delta$ corresponding to the Neumann eigenvalue $\lambda_0 = 0$. Let $H^1_{r,m}(\Omega)$ be the subspace of $H^1_r(\Omega)$ consisting of functions f such that $\int_\Omega f \, r\, dr\, dz = 0$. The existence of Neumann eigenfunctions $\mathcal{E}^{(N)} := \{\hat{\phi}_k^{(N)} : k \in \mathbb{N}\}$ and a nondecreasing sequence of strictly positive Neumann eigenvalues $\{\lambda_k^{(N)} : k \in \mathbb{N}\}$ such that $\mathcal{E}^{(N)}$ is a orthonormal basis of $H^1_{r,m}(\Omega)$ with respect to the inner product $a(\phi, \psi)$, and the normalized eigenfunctions $\tilde{\mathcal{E}}^{(N)} := \{\tilde{\phi}_k^{(N)} := (\lambda_k^{(N)})^{-1/2} \hat{\phi}_k^{(N)} : k \in \mathbb{N}\}$ form an orthonormal basis of $L^2_r(\Omega)$ with respect to the standard inner product is proved very similarly as the case for the Dirichlet eigenproblem.

Theorem 3.2.3. *The smallest nonzero eigenvalue* $\lambda_1^{(N)}$ *of* (3.17The Neumann Eigenvalue Probleme *is strictly positive and is characterized by the variational principle*

$$\frac{1}{\lambda_1^{(N)}} = \sup \int_\Omega |\phi|^2 \, r\, dr\, dz \quad s.t. \int_\Omega |\nabla\phi|^2 \, r\, dr\, dz = 1, \phi \in H^1_{r,m}(\Omega). \tag{3.18}$$

Proof. Let $C_1^{(N)} = \{\phi \in H^1_{r,m}(\Omega) : \mathcal{A}(\phi) = 1\}$ and consider the problem of finding

$$\beta_1^{(N)} = \sup_{\phi \in C_1^{(N)}} \|\phi\|_{L^2_r}. \tag{3.19}$$

The Cauchy-Schwarz inequality implies that $|a(\phi,\psi)| \leq \|\phi\|_{H^1_r}\|\psi\|_{H^1_r}$ for all $\phi,\psi \in H^1_{r,m}(\Omega)$ so a is continuous on $H^1_{r,m}(\Omega)\times H^1_{r,m}(\Omega)$. The Poincaré inequality for $H^1_r(\Omega)$ implies that there is a $C > 0$ such that $C\|\phi\|^2_{H^1_r} \leq \mathcal{A}(\phi)$ for all $\phi \in H^1_{r,m}(\Omega)$ so a is also coercive on $H^1_{r,m}(\Omega)$. $\|\phi\|_{L^2_r}$ is a norm on $H^1_{r,m}(\Omega) \subset L^2_r(\Omega)$ so $\|\phi\|^2_{L^2_r} > 0$ for all nonzero ϕ in $H^1_{r,m}(\Omega)$ and $\|\phi\|^2_{L^2_r} = 0$ only if $\phi = 0$. Then we may apply Theorem 3.1 in [4] to conclude that:

(i) $\beta_1^{(N)} > 0$ is finite;

(ii) there are maximizers $\pm\hat{\phi}_1^{(N)}$ of $\|\cdot\|^2_{L^2_r}$ on $C_1^{(N)}$ where $\beta_1^{(N)}$ is attained;

(iii) $\hat{\phi}_1^{(N)}$ is a Neumann eigenfunction of $-\Delta$ corresponding to the eigenvalue $\lambda_1^{(N)} := 1/\beta_1^{(N)}$, i.e.

$$\int_\Omega \nabla\hat{\phi}_1^{(N)}\cdot\nabla\psi\, r\,dr\,dz = \lambda_1\int_\Omega \hat{\phi}_1^{(N)}\psi\, r\,dr\,dz, \quad \forall\psi \in H^1_{r,m}(\Omega);$$

(iv) $\lambda_1^{(N)}$ is the smallest nonzero Neumann eigenvalue of $-\Delta$ and

$$\int_\Omega |\phi|^2\, r\,dr\,dz \leq \frac{1}{\lambda_1^{(N)}}\int_\Omega |\nabla\phi|^2\, r\,dr\,dz, \quad \forall\phi \in H^1_{r,m}(\Omega).$$

□

Theorem 3.2.4. *(i) The kth eigenvalue $\lambda_k^{(N)}$ of* (3.17The Neumann Eigenvalue Problemequation *is characterized by the variational principle*

$$\frac{1}{\lambda_k^{(N)}} = \sup\int_\Omega |\phi|^2\, r\,dr\,dz \tag{3.20}$$

for all $\phi \in H^1_{r,m}(\Omega)$ *such that* $\int_\Omega |\nabla\phi|^2\, r\, dr\, dz = 1$ *and* $\int_\Omega \nabla\phi\cdot\nabla\hat{\phi}_j^{(N)}\, r\, dr\, dz = 0$ *for* $j = 1, \ldots, k-1$*. We also have* $\lambda_k^{(N)} \geq \lambda_{k-1}^{(N)}$ *and*

$$\int_\Omega |\phi|^2\, r\, dr\, dz \leq \frac{1}{\lambda_k^{(N)}} \int_\Omega |\nabla\phi|^2\, r\, dr\, dz \tag{3.21}$$

for all $\phi \in H^1_{r,m}(\Omega)$ *such that* $\int_\Omega \nabla\phi\cdot\nabla\hat{\phi}_j^{(N)}\, r\, dr\, dz = 0$ *for* $j = 1, \ldots, k-1$*.*

(ii) $\lambda_k^{(N)} \to \infty$ *as* $k \to \infty$ *and the set of eigenfunctions* $\mathcal{E}^{(N)} = \{\hat{\phi}_k^{(N)} : k \in \mathbb{N}\}$ *is an orthonormal basis of* $H^1_{r,m}(\Omega)$ *with respect to the inner product* $a(\phi, \psi) = \int_\Omega \nabla\phi\cdot\nabla\psi\, r\, dr\, dz$*.*

(iii) The normalized eigenfunctions $\tilde{\mathcal{E}}^{(N)} := \{\tilde{\phi}_k^{(N)} := (\lambda_k^{(N)})^{-1/2}\hat{\phi}_k^{(N)} : k \in \mathbb{N}\}$ *form an orthonormal basis of* $L^2_r(\Omega)$ *with respect to the standard inner product.*

3.2.3 The Harmonic Steklov Eigenvalue Problem

Consider the eigenvalue problem of finding a real-valued function $\phi \neq 0$ and real number δ such that

$$\begin{cases} \Delta\phi = 0 \quad \text{in } \Omega, \\ D_\nu\phi = \delta\phi \quad \text{on } \Gamma. \end{cases} \tag{3.22}$$

If such a pair (ϕ, δ) exists and ϕ is smooth on $\Omega \cup \Gamma$, then we may integrate by parts to obtain that

$$\int_\Omega \nabla\phi\cdot\nabla\psi\, r\, dr\, dz = \delta \int_\Gamma \phi\psi\, r\, ds \quad \text{for all } \psi \in C^\infty(\overline{\Omega}). \tag{3.23}$$

Both sides of (3.23The Harmonic Steklov Eigenvalue Problemequation.3.2.23) are well-defined if $\phi, \psi \in H^1_r(\Omega)$, hence we consider the problem of finding nontrivial

$(\phi, \delta) \in H^1_r(\Omega) \times \mathbb{R}$ such that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \; r \, dr \, dz = \delta \int_\Gamma \phi\psi \; r \, ds \quad \text{for all } \psi \in H^1_r(\Omega) \tag{3.24}$$

Definition. If $(\phi, \delta) \in H^1_r(\Omega) \times \mathbb{R}$ is a nontrivial solution of (3.24The Harmonic Steklov Eigenvalue I then ϕ is a *harmonic Steklov eigenfunction on Ω corresponding to the harmonic Steklov eigenvalue δ.*

We see that $\phi_0 \equiv$ const. is a harmonic Steklov eigenfunction corresponding to the harmonic Steklov eigenvalue $\delta_0 = 0$. The remaining harmonic Steklov eigenvalues are characterized by variational principles over $H^1_{r,m}(\Omega)$.

Theorem 3.2.5. *The smallest strictly positive harmonic Steklov eigenvalue δ_1 is characterized by the variational principle*

$$\frac{1}{\delta_1} = \sup \int_\Gamma |\phi|^2 \; r \, ds \quad s.t. \quad \int_\Omega |\nabla\phi|^2 \; r \, dr \, dz = 1, \phi \in H^1_{r,m}(\Omega). \tag{3.25}$$

Proof. Let $C_1^{(N)} = \{\phi \in H^1_{r,m}(\Omega) : \mathcal{A}(\phi) = 1\}$ as before and consider the problem of finding

$$\epsilon_1 = \sup_{\phi \in C_1^{(N)}} \|\phi\|^2_{L^2_r(\Gamma)}. \tag{3.26}$$

a is continuous on $H^1_{r,m}(\Omega) \times H^1_{r,m}(\Omega)$ and coercive on $H^1_{r,m}(\Omega)$. $\|\phi\|_{L^2_r(\Gamma)}$ is strictly positive for some $\psi \in H^1_{r,m}(\Omega)$ since $H_{r,m}(\Omega) \neq H^1_{r,0}(\Omega)$. Moreover, the trace $\gamma : H^1_r(\Omega) \to L^2_r(\Gamma)$ is compact. Then we may apply Theorem 3.1 in [4] to conclude that:

(i) $\epsilon_1 > 0$ is finite;

(ii) there are maximizers $\pm\hat{\chi}_1$ of $\|\cdot\|^2_{L^2_r}$ on $C_1^{(N)}$ where ϵ_1 is attained;

(iii) $\hat{\chi}_1$ is a harmonic Steklov eigenfunction corresponding to the eigenvalue $\delta_1 := 1/\epsilon_1$, i.e.

$$\int_\Omega \nabla\hat{\chi}_1 \cdot \nabla\psi \, r\,dr\,dz = \delta_1 \int_\Gamma \hat{\chi}_1 \psi \, r\,dr\,dz, \quad \forall \psi \in H^1_{r,m}(\Omega);$$

(iv) δ_1 is the smallest nonzero harmonic Steklov eigenvalue and

$$\int_\Gamma |\phi|^2 \, r\,ds \leq \frac{1}{\delta_1} \int_\Omega |\nabla\phi|^2 \, r\,dr\,dz, \quad \forall \phi \in H^1_{r,m}(\Omega).$$

□

The bilinear form a satisfies the conditions necessary to apply Theorem 4.2 of [4] to obtain the following result.

Theorem 3.2.6. *(i) The ℓth eigenvalue δ_ℓ of* (3.24The Harmonic Steklov Eigenvalue Problemequ *is characterized by the variational principle*

$$\frac{1}{\delta_\ell} = \sup \int_\Gamma |\phi|^2 \, r\,ds \tag{3.27}$$

for all $\phi \in H^1_{r,m}(\Omega)$ such that $\int_\Omega |\nabla\phi|^2 \, r\,dr\,dz = 1$ and $\int_\Omega \nabla\phi \cdot \nabla\hat{\chi}_j \, r\,dr\,dz = 0$ for $j = 1, \ldots, k-1$. We also have $\delta_\ell \geq \delta_{\ell-1}$ and

$$\int_\Gamma |\phi|^2 \, r\,ds \leq \frac{1}{\delta_\ell} \int_\Omega |\nabla\phi|^2 \, r\,dr\,dz \tag{3.28}$$

for all $\phi \in H^1_{r,m}(\Omega)$ such that $\int_\Omega \nabla\phi \cdot \nabla\hat{\chi}_j \, r\,dr\,dz = 0$ for $j = 1, \ldots, \ell - 1$.

3.3 Eigenproblems for $-\Delta + \frac{1}{r^2}$

3.3.1 The Dirichlet Eigenvalue Problem

Consider the eigenvalue problem of finding a real-valued function $\phi \neq 0$ and real number $\tilde{\lambda}$ such that

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = \tilde{\lambda}\psi \quad \text{in } \Omega, \\ \psi = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.29}$$

The weak formulation is obtained using the usual approach. The weak form of (3.29The Dirichlet Eigenvalue Problemequation.3.3.29) is to find nontrivial $(\psi, \tilde{\lambda}) \in V^1_{r,0}(\Omega) \times \mathbb{R}$ satisfying

$$\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\, dr\, dz = \tilde{\lambda} \int_\Omega \psi\chi\, r\, dr\, dz \quad \text{for all } \chi \in V^1_{r,0}(\Omega). \tag{3.30}$$

Definition. If $(\psi, \tilde{\lambda}) \in V^1_{r,0}(\Omega) \times \mathbb{R}$ is a nontrivial solution of (3.30The Dirichlet Eigenvalue Probleme then ψ is a *Dirichlet eigenfunction of* $-\Delta + r^{-2}$ *on* Ω *corresponding to the Dirichlet eigenvalue* $\tilde{\lambda}$.

Let $b : V^1_r(\Omega) \times V^1_r(\Omega) \to \mathbb{R}$ be the bilinear form

$$b(\psi, \chi) = \int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\, dr\, dz, \tag{3.31}$$

and let $\mathcal{B} : V^1_r(\Omega) \to \mathbb{R}$ denote the quadratic form associated to b

$$\mathcal{B}(\psi) = b(\psi, \psi) = \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2\, r\, dr\, dz. \tag{3.32}$$

so (3.30The Dirichlet Eigenvalue Problemequation.3.3.30) says

$$b(\psi, \chi) = \tilde{\lambda}\langle \psi, \chi \rangle_{L^2_r} \quad \text{for all } \chi \in V^1_r(\Omega). \tag{3.33}$$

Theorem 3.3.1. *The smallest eigenvalue* $\tilde{\lambda}_1$ *of* (3.30The Dirichlet Eigenvalue Problemequation.3.3 *is strictly positive and is characterized by the variational principle*

$$\frac{1}{\tilde{\lambda}_1} = \sup \int_\Omega |\psi|^2 \, r \, dr \, dz \quad s.t. \quad \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz = 1, \psi \in V^1_{r,0}(\Omega). \tag{3.34}$$

Proof. Denote $S_1 = \{\psi \in V^1_{r,0}(\Omega) : \mathcal{B}(\psi) = 1\}$ and consider the problem of finding

$$\tilde{\beta}_1 = \sup_{\psi \in S_1} \|\psi\|^2_{L^2_r} = \sup_{\psi \in S_1} \int_\Omega |\psi|^2 \, r \, dr \, dz. \tag{3.35}$$

If $\psi, \chi \in V^1_{r,0}(\Omega)$, then

$$\begin{aligned} b(\psi, \chi) &= \int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta) \, r \, dr \, dz \\ &= \int_\Omega \left(\nabla\psi \cdot \nabla\chi + \frac{\psi\chi}{r^2} \right) r \, dr \, dz \\ &\leq \|\nabla\psi\|_{L^2_r} \|\nabla\chi\|_{L^2_r} + \|\psi\|_{L^2_{-1}} \|\chi\|_{L^2_{-1}} \\ &\leq \|\psi\|_{V^1_r} \|\chi\|_{V^1_r} \end{aligned} \tag{3.36}$$

so b is a continuous bilinear form on $V^1_{r,0}(\Omega) \times V^1_{r,0}(\Omega)$. The Poincaré-curl inequality for $V^1_{r,0}(\Omega)$ asserts that $\mathcal{B}$ is coercive on $V^1_{r,0}(\Omega)$. $\|\cdot\|_{L^2_r}$ is a norm on $V^1_{r,0}(\Omega) \subset L^2_r(\Omega)$ so $\|\psi\|_{L^2_r} > 0$ for all nonzero $\psi \in V^1_{r,0}(\Omega)$ and $\|\psi\|_{L^2_r} = 0$ if and only if $\psi = 0$. An application of Theorem 3.1 in [4] shows that

(i) $\tilde{\beta}_1 > 0$ is finite;

(ii) there are maximizers $\pm\hat{\psi}_1$ of $\|\cdot\|^2_{L^2_r}$ on S_1 where $\tilde{\beta}_1$ is attained;

(iii) $\hat{\psi}_1$ is an eigenfunction corresponding to the eigenvalue $\tilde{\lambda}_1 := 1/\tilde{\beta}_1$, i.e.

$$\int_\Omega \operatorname{curl}(\hat{\psi}_1 e_\theta) \cdot \operatorname{curl}(\chi e_\theta) \, r \, dr \, dz = \tilde{\lambda}_1 \int_\Omega \hat{\psi}_1 \chi \, r \, dr \, dz, \quad \forall \chi \in V^1_{r,0}(\Omega);$$

(iv) $\tilde{\lambda}_1$ is the smallest eigenvalue and

$$\int_\Omega |\psi|^2 \, r \, dr \, dz \leq \frac{1}{\tilde{\lambda}_1} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz, \quad \forall \psi \in V^1_{r,0}(\Omega).$$

□

This variational principle may be iterated to generate a sequence of eigenfunctions that are orthonormal with respect to the bilinear form b. Let $\{\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_{k-1}\}$ be $k-1$ eigenfunctions corresponding to the eigenvalues $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_{k-1}$.

Theorem 3.3.2. *(i) The kth eigenvalue of* (3.30The Dirichlet Eigenvalue Problemequation.3.3.3 *is characterized by the variational principle*

$$\frac{1}{\tilde{\lambda}_k} = \sup \int_\Omega |\psi|^2 \, r \, dr \, dz \tag{3.37}$$

for all $\psi \in V^1_{r,0}(\Omega)$ such that $\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \; r \; dr \; dz = 1$ and $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\psi}_j e_\theta) \, r \, dr \, dz = 0$ for $j = 1, \ldots, k-1$. We also have $\tilde{\lambda}_k \geq \tilde{\lambda}_{k-1}$ and

$$\int_\Omega |\psi|^2 \, r \, dr \, dz \leq \frac{1}{\tilde{\lambda}_k} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz \tag{3.38}$$

for all $\psi \in V^1_{r,0}(\Omega)$ such that $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\psi}_j e_\theta) \; r \; dr \; dz = 0$ for $j = 1, \ldots, k-1$.

(ii) $\tilde{\lambda}_k \to \infty$ as $k \to \infty$ and the set of eigenfunctions $\mathcal{F} := \{\hat{\psi}_k : k \in \mathbb{N}\}$ is an orthonormal basis of $V^1_{r,0}(\Omega)$ with respect to the inner product $b(\psi, \chi) = \int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta) \, r \, dr \, dz$.

(iii) The normalized eigenfunctions $\tilde{\mathcal{F}} := \{\tilde{\psi}_k := \tilde{\lambda}_k^{-1/2} \hat{\psi}_k : k \in \mathbb{N}\}$ form an orthonormal basis of $L^2_r(\Omega)$ with respect to the standard inner product.

Proof. This is proved very similarly to the case of the Dirichlet eigenproblem for $-\Delta$, so we omit the details of this proof. $\square$

3.3.2 A Conormal Neumann Eigenvalue Problem

Consider the conormal Neumann eigenvalue problem of finding a nonzero function ψ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = \lambda\psi \quad \text{in } \Omega, \\ \operatorname{curl}(\psi e_\theta) \wedge \nu = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.39}$$

We call this a conormal Neumann eigenvalue problem since formally

$$\operatorname{curl}(\psi e_\theta) \wedge \nu = \frac{1}{r}\nabla(r\psi) \cdot \nu \tag{3.40}$$

so the boundary condition $\operatorname{curl}(\psi e_\theta) \wedge \nu = 0$ is equivalent to $\nabla(r\psi) \cdot \nu = 0$. The weak form of this eigenvalue problem is to find nontrivial $\psi \in V_r^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\, dr\, dz = \lambda \int_\Omega \psi\chi\, r\, dr\, dz \quad \text{for all } \chi \in V_r^1(\Omega). \tag{3.41}$$

The existence of eigenfunctions $\mathcal{F}^{(N)} := \{\hat{\psi}_k^{(N)} : k \in \mathbb{N}\}$ and a nondecreasing sequence of strictly positive eigenvalues $\{\tilde{\lambda}_k^{(N)} : k \in \mathbb{N}\}$ such that $\mathcal{F}^{(N)}$ is a orthonormal basis of $V_r^1(\Omega)$ with respect to the inner product $b(\psi, \chi)$, and the normalized eigenfunctions $\tilde{\mathcal{F}}^{(N)} := \{\tilde{\psi}_k^{(N)} := (\tilde{\lambda}_k^{(N)})^{-1/2}\hat{\psi}_k^{(N)} : k \in \mathbb{N}\}$ form an orthonormal basis of $L_r^2(\Omega)$ with respect to the standard inner product is proved very similarly as the case for the Dirichlet eigenproblem for $-\Delta + r^{-2}$. We state the results for clarity, but omit the proofs as they are very similar. Interestingly, this Neumann eigenproblem has no zero eigenvalue.

Theorem 3.3.3. *The smallest eigenvalue* $\tilde{\lambda}_1^{(N)}$ *of* (3.41A Conormal Neumann Eigenvalue Probleme
is strictly positive and is characterized by the variational principle

$$\frac{1}{\tilde{\lambda}_1^{(N)}} = \sup \int_\Omega |\psi|^2 \, r \, dr \, dz \quad s.t. \quad \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz = 1, \psi \in V_r^1(\Omega). \tag{3.42}$$

Theorem 3.3.4. *(i) The kth eigenvalue of* (3.41A Conormal Neumann Eigenvalue Problemequa
is characterized by the variational principle

$$\frac{1}{\tilde{\lambda}_k^{(N)}} = \sup \int_\Omega |\psi|^2 \, r \, dr \, dz \tag{3.43}$$

for all $\psi \in V_r^1(\Omega)$ *such that* $\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \; r \; dr \; dz = 1$ *and* $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\psi}_j^{(N)} e_\theta) \, r \, dr \, dz = 0$ *for* $j = 1, \ldots, k-1$. *We also have* $\tilde{\lambda}_k^{(N)} \geq \tilde{\lambda}_{k-1}^{(N)}$ *and*

$$\int_\Omega |\psi|^2 \, r \, dr \, dz \leq \frac{1}{\tilde{\lambda}_k^{(N)}} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz \tag{3.44}$$

for all $\psi \in V_r^1(\Omega)$ *such that* $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\psi}_j^{(N)} e_\theta) \; r \; dr \; dz = 0$ *for* $j = 1, \ldots, k-1$.

(ii) $\tilde{\lambda}_k^{(N)} \to \infty$ *as* $k \to \infty$ *and the set of eigenfunctions* $\mathcal{F}^{(N)} := \{\hat{\psi}_k^{(N)} : k \in \mathbb{N}\}$ *is an orthonormal basis of* $V_r^1(\Omega)$ *with respect to the inner product* $b(\psi, \chi) = \int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta) \, r \, dr \, dz$.

(iii) The normalized eigenfunctions $\tilde{\mathcal{F}}^{(N)} := \{\tilde{\psi}_k^{(N)} := (\tilde{\lambda}_k^{(N)})^{-1/2} \hat{\psi}_k^{(N)} : k \in \mathbb{N}\}$ *form an orthonormal basis of* $L_r^2(\Omega)$ *with respect to the standard inner product.*

3.3.3 The Curl-Harmonic Steklov Eigenvalue Problem

Consider the eigenvalue problem of finding a real-valued function $\psi \neq 0$ and real number $\tilde{\delta}$ such that

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = 0 \quad \text{in } \Omega, \\ \dfrac{1}{r}\dfrac{\partial(r\psi)}{\partial r}\nu_r + \dfrac{\partial\psi}{\partial z}\nu_z = \tilde{\delta}\psi \quad \text{on } \Gamma. \end{cases} \tag{3.45}$$

If such a pair $(\psi, \tilde{\delta})$ exists and ψ is smooth, then we may integrate by parts to obtain that

$$\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\, dr\, dz = \tilde{\delta} \int_\Gamma \psi\chi\, r\, ds \quad \text{for all } \chi \in C^\infty_{z0}(\overline{\Omega}). \tag{3.46}$$

Both sides of (3.46The Curl-Harmonic Steklov Eigenvalue Problemequation.3.3.46) are well-defined if $\psi, \chi \in V^1_r(\Omega)$, hence we consider the problem of finding nontrivial $(\psi, \tilde{\delta}) \in V^1_r(\Omega) \times \mathbb{R}$ such that

$$\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\, dr\, dz = \tilde{\delta} \int_\Gamma \psi\chi\, r\, ds \quad \text{for all } \chi \in V^1_r(\Omega). \tag{3.47}$$

Definition. If $(\psi, \tilde{\delta}) \in V^1_r(\Omega) \times \mathbb{R}$ is a nontrivial solution of (3.47The Curl-Harmonic Steklov Eigenva then ψ is a *curl-harmonic Steklov eigenfunction on Ω corresponding to the curl-harmonic Steklov eigenvalue $\tilde{\delta}$.*

Since $\sqrt{\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2\, r\, dr\, dz}$ defines a norm on $V^1_r(\Omega)$, we see that all curl-harmonic Steklov eigenvalues are strictly positive. The first curl-harmonic Steklov eigenvalue is characterized by a variational principle over $V^1_r(\Omega)$.

Theorem 3.3.5. *The smallest positive curl-harmonic Steklov eigenvalue $\tilde{\delta}_1$ is characterized by the variational principle*

$$\frac{1}{\tilde{\delta}_1} = \sup \int_\Gamma |\psi|^2\, r\, ds \quad s.t. \quad \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2\, r\, dr\, dz = 1, \psi \in V^1_r(\Omega). \tag{3.48}$$

Proof. Let $S_1^{(C)} = \{\psi \in V_r^1(\Omega) : \mathcal{B}(\phi) = 1\}$ where $\mathcal{B}$ is the quadratic form $\mathcal{B}(\psi) = b(\psi, \psi)$ as before and consider the problem of finding

$$\tilde{\epsilon}_1 = \sup_{\phi \in S_1^{(N)}} \|\psi\|^2_{L^2_r(\Gamma)}. \tag{3.49}$$

b is continuous on $V_r^1(\Omega) \times V_r^1(\Omega)$ and coercive on $V_r^1(\Omega)$ as $\sqrt{\mathcal{B}(\psi)}$ defines a norm on $V_r^1(\Omega)$. $\|\phi\|_{L^2_r(\Gamma)}$ is strictly positive for some $\psi \in V_r^1(\Omega)$ since $V_{r,0}^1(\Omega)$ is a strict subset of $V_r^1(\Omega)$. Moreover, the trace $\gamma : V_r^1(\Omega) \to L^2_r(\Gamma)$ is compact. Then we may apply Theorem 3.1 in [4] to conclude that:

(i) $\tilde{\epsilon}_1 > 0$ is finite;

(ii) there are maximizers $\pm\hat{\chi}_1^{(C)}$ of $\|\cdot\|^2_{L^2_r}$ on $S_1^{(C)}$ where $\tilde{\epsilon}_1$ is attained;

(iii) $\hat{\chi}_1^{(C)}$ is a curl-harmonic Steklov eigenfunction corresponding to the eigenvalue $\tilde{\delta}_1 := 1/\tilde{\epsilon}_1$, i.e.

$$\int_\Omega \operatorname{curl}(\hat{\chi}_1^{(C)} e_\theta) \cdot \operatorname{curl}(\psi e_\theta)\, r\, dr\, dz = \tilde{\delta}_1 \int_\Gamma \hat{\chi}_1^{(C)} \psi\, r\, dr\, dz, \quad \forall \psi \in V_r^1(\Omega);$$

(iv) $\tilde{\delta}_1$ is the smallest nonzero curl-harmonic Steklov eigenvalue and

$$\int_\Gamma |\psi|^2\, r\, ds \leq \frac{1}{\tilde{\delta}_1} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2\, r\, dr\, dz, \quad \forall \psi \in V_r^1(\Omega).$$

□

As before, we may again apply Theorem 4.2 of [4] to obtain the following.

Theorem 3.3.6. *(i) The ℓth eigenvalue $\tilde{\delta}_\ell$ of* (3.47The Curl-Harmonic Steklov Eigenvalue Proble *is characterized by the variational principle*

$$\frac{1}{\tilde{\delta}_\ell} = \sup \int_\Gamma |\psi|^2\, r\, ds \tag{3.50}$$

for all $\psi \in V_r^1(\Omega)$ *such that* $\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \; r \; dr \; dz = 1$ *and* $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\chi}_j^{(C)} e_\theta)\, r\, dr\, dz = 0$ *for* $j = 1, \ldots, k-1$*. We also have* $\tilde{\delta}_\ell \geq \tilde{\delta}_{\ell-1}$ *and*

$$\int_\Gamma |\psi|^2 \; r\, ds \leq \frac{1}{\tilde{\delta}_\ell} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \; r\, dr\, dz \tag{3.51}$$

for all $\psi \in V_r^1(\Omega)$ *such that* $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\hat{\chi}_j^{(C)}) \; r \; dr \; dz = 0$ *for* $j = 1, \ldots, \ell - 1$*.*

Chapter 4

Linear Axisymmetric Laplacian Boundary Value Problems

4.1 Introduction

This chapter is on boundary value problems for $-\Delta$ and $-\Delta + r^{-2}$ where Δ is the Laplacian in cylindrical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{4.1}$$

These boundary value problems play a role in the characterization of the scalar potential and stream function in the orthogonal decompositions in Chapter 5.

4.2 Boundary Value Problems for $-\Delta$

4.2.1 The Dirichlet Problem for $-\Delta$

Homogenenous Boundary Data

Given a function f on Ω, consider the problem of finding a function $\phi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

Let $f \in (H^1_{r,0}(\Omega))^*$. The weak form of (4.2Homogenenous Boundary Dataequation.4.2.2) is to find a function $\phi \in H^1_{r,0}(\Omega)$ satisfying

$$\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\, dr\, dz = \langle f, \psi\rangle \quad \forall \psi \in H^1_{r,0}(\Omega). \tag{4.3}$$

Theorem 4.2.1. *There is a unique $\phi \in H^1_{r,0}(\Omega)$ satisfying* (4.3Homogenenous Boundary Dataequati

Proof. The bilinear form $a(\phi, \psi) := \int_\Omega (\nabla\phi \cdot \nabla\psi)\ r\ dr\ dz$ is clearly continuous over $H^1_{r,0}(\Omega)$ and the Poincaré inequality for $H^1_{r,0}(\Omega)$ implies that a is coercive over $H^1_{r,0}(\Omega)$. Therefore there is a unique $\phi \in H^1_{r,0}(\Omega)$ satisfying (4.3Homogenenous Boundary Dataequation.4.2.3) by the Lax-Milgram theorem. $\square$

Corollary 4.2.2. *Let $f \in (H^1_{r,0}(\Omega))^*$ and $\phi \in H^1_{r,0}(\Omega)$ satisfy* (4.3Homogenenous Boundary Dataequ *Then*

$$\|\nabla\phi\|_{L^2_r} \leq \left(1 + \lambda_1^{-1}\right)^{1/2} \|f\|_{H^1_{r,0}(\Omega)^*}. \tag{4.4}$$

where λ_1 is the smallest Dirichlet eigenvalue of $-\Delta$.

Proof. Let $\phi \in H^1_{r,0}(\Omega)$ satisfy (4.3Homogenenous Boundary Dataequation.4.2.3). We apply item (iv) from the proof of Theorem 3.2.1 to get

$$\begin{aligned}\int_\Omega |\nabla\phi|^2\, r\,dr\,dz &= \langle f, \phi\rangle \\ &\le \|f\|_{H^1_{r,0}(\Omega)^*}\|\phi\|_{H^1_{r,0}(\Omega)} \\ &= \|f\|_{H^1_{r,0}(\Omega)^*}\left(\|\phi\|^2_{L^2_r} + \|\nabla\phi\|^2_{L^2_r}\right)^{1/2} \\ &\le \|f\|_{H^1_{r,0}(\Omega)^*}\left(1+\lambda_1^{-1}\right)^{1/2}\|\nabla\phi\|_{L^2_r}\end{aligned} \tag{4.5}$$

which proves the claim. □

Inhomogenenous boundary data

Given a function f on Ω and a function g on Γ, consider the problem of finding a function $\phi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega, \\ \phi = g & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Let $f \in (H^1_{r,0}(\Omega))^*$ and $g \in \gamma(H^1_r(\Omega)) = H^{1/2}_r(\Gamma)$. We transform the inhomogenous problem to a homogenenous one by finding $\phi_g \in H^1_r(\Omega)$ such that $\gamma\phi = g$ in $L^2_r(\Gamma)$, and then consider finding $\tilde{\phi} \in H^1_{r,0}(\Omega)$ satisfying

$$\int_\Omega (\nabla\tilde{\phi}\cdot\nabla\psi)\, r\,dr\,dz = \langle f, \psi\rangle - \int_\Omega (\nabla\phi_g\cdot\nabla\psi)\, r\,dr\,dz \quad \forall\psi \in H^1_{r,0}(\Omega). \tag{4.7}$$

Theorem 4.2.3. *Let $f \in (H^1_{r,0}(\Omega))^*$ and $g \in H^{1/2}_r(\Gamma)$. Let $\phi_g \in H^1_r(\Omega)$ such that $\gamma\phi_g = g$ in $L^2_r(\Gamma)$. Then there is a unique $\tilde{\phi} \in H^1_{r,0}(\Omega)$ satisfying* (4.7Inhomogenenous boundary data

Proof. For $\psi \in H^1_{r,0}(\Omega)$, there is a $C > 0$ independent of ψ such that

$$\begin{aligned} \left| \int_\Omega (\nabla \phi_g \cdot \nabla \psi) \, r \, dr \, dz \right| &\le \|\nabla \phi_g\|_{L^2_r} \|\nabla \psi\|_{L^2_r} \\ &\le C \|\nabla \phi_g\|_{L^2_r} \|\psi\|_{H^1_r} \end{aligned} \tag{4.8}$$

so the right-hand side of (4.7Inhomogenenous boundary dataequation.4.2.7) defines a continuous linear functional in $(H^1_{r,0}(\Omega))^*$. Then we may apply the Lax-Milgram theorem to obtain the conclusion. □

Corollary 4.2.4. *Let $f, g, \phi_g, \tilde{\phi}$ be as in the previous theorem. Set $\phi = \tilde{\phi} + \phi_g$. Then $\gamma\phi = g$ in $L^2_r(\Gamma)$ and $-\Delta\phi = f$ in $(H_{r,0}(\Omega))^*$, that is,*

$$\int_\Omega (\nabla \phi \cdot \nabla \psi) \, r \, dr \, dz = \langle f, \psi \rangle \quad \forall \psi \in H^1_{r,0}(\Omega). \tag{4.9}$$

Proof. If $\phi = \tilde{\phi} + \phi_g$ then $\gamma\phi = \gamma\tilde{\phi} + \gamma\phi_g = g$ in $L^2_r(\Gamma)$ since $\gamma\tilde{\phi} = 0$ as $\phi \in H^1_{r,0}(\Omega)$. (4.9equation.4.2.9) holds upon rearranging (4.7Inhomogenenous boundary dataequation.4.2.7). □

4.2.2 The Neumann Problem for $-\Delta$

Homogeneous Boundary Data

Given a function f on Ω, consider the problem of finding a function $\phi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta \phi = f & \text{in } \Omega, \\ D_\nu \phi = 0 & \text{on } \Gamma. \end{cases} \tag{4.10}$$

Let $f \in L_r^2(\Omega)$ such that $\int_\Omega f \; r\,dr\,dz = 0$. The weak form of (4.10Homogeneous Boundary Dataequat
is to find a function $\phi \in H_r^1(\Omega)$ satisfying

$$\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\,dr\,dz = \int_\Omega f\psi \; r\,dr\,dz \quad \forall\psi \in H_r^1(\Omega). \tag{4.11}$$

Let $H_{r,m}^1(\Omega) = \left\{\phi \in H_r^1(\Omega) : \int_\Omega \phi \; r\,dr\,dz = 0\right\}$.

Theorem 4.2.5. *Let $f \in L_r^2(\Omega)$ such that $\int_\Omega f \; r\,dr\,dz = 0$.*

There is a unique $\phi \in H_{r,m}^1(\Omega)$ satisfying (4.11Homogeneous Boundary Dataequation.4.2.11)*.*

Proof. It is clear to see that $\int_\Omega f\psi \; r\,dr\,dz$ defines a continuous linear functional in $(H_{r,m}^1(\Omega))^*$ and that $a(\phi, \psi) = \int_\Omega (\nabla\phi \cdot \nabla\psi)\; r\,dr\,dz$ is a continuous bilinear form on $H_{r,m}^1(\Omega)$. The Poincaré inequality for $H_r^1(\Omega)$ implies that a is coercive over $H_{r,m}^1(\Omega)$, so the Lax-Milgram theorem implies that there is a unique $\phi \in H_{r,m}^1(\Omega)$ satisfying

$$\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\,dr\,dz = \int_\Omega f\psi \; r\,dr\,dz \quad \forall\psi \in H_{r,m}^1(\Omega). \tag{4.12}$$

More generally, if $\psi \in H_r^1(\Omega)$, then we may write $\psi = \psi_0 + \langle\psi\rangle$ where

$$\langle\psi\rangle = \frac{\int_\Omega \psi \; r\,dr\,dz}{\int_\Omega 1 \; r\,dr\,dz} \tag{4.13}$$

and $\psi_0 = \psi - \langle\psi\rangle$. Then if $\psi \in H^1_r(\Omega)$,

$$\begin{aligned}
\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\,dr\,dz &= \int_\Omega (\nabla\phi \cdot \nabla(\psi_0 + \langle\psi\rangle))\, r\,dr\,dz \\
&= \int_\Omega (\nabla\phi \cdot \nabla\psi_0)\, r\,dr\,dz \\
&= \int_\Omega f\psi_0\; r\,dr\,dz \\
&= \int_\Omega f(\psi - \langle\psi\rangle)\, r\,dr\,dz \\
&= \int_\Omega f\psi\; r\,dr\,dz - \langle\psi\rangle \int_\Omega f\; r\,dr\,dz \\
&= \int_\Omega f\psi\; r\,dr\,dz.
\end{aligned} \tag{4.14}$$

□

Corollary 4.2.6. *Let $f \in L^2_r(\Omega)$ such that $\int_\Omega f\ r\ dr\ dz = 0$ and let $\phi \in H^1_{r,m}(\Omega)$ be the unique function satisfying* (4.11Homogeneous Boundary Dataequation.4.2.11)*. Then*

$$\|\nabla\phi\|_{L^2_r} \le \lambda_1^{(N)^{-1/2}} \|f\|_{L^2_r} \tag{4.15}$$

where $\lambda_1^{(N)}$ is the smallest strictly positive Neumann eigenvalue of $-\Delta$ on $H^1_r(\Omega)$.

Proof. Let f, ϕ be as prescribed. Then we may apply item (iv) in the proof of Theorem 3.2.3 to get

$$\begin{aligned}
\int_\Omega |\nabla\phi|^2\, r\,dr\,dz &= \int_\Omega f\phi\, r\,dr\,dz \\
&\le \|f\|_{L^2_r}\|\phi\|_{L^2_r} \\
&\le \lambda_1^{(N)^{-1/2}} \|f\|_{L^2_r}.
\end{aligned} \tag{4.16}$$

□

Inhomogeneous Boundary Data

Given a function f on Ω and g on Γ, consider the problem of finding a function $\phi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega, \\ D_\nu \phi = g & \text{on } \Gamma. \end{cases} \tag{4.17}$$

Let $f \in L^2_r(\Omega), g \in L^2_r(\Gamma)$ such that $\int_\Omega f \ r\,dr\,dz = -\int_\Gamma g \ r\,ds$. The weak form of (4.17Inhomogeneous Boundary Dataequation.4.2.17) is to find a function $\phi \in H^1_r(\Omega)$ satisfying

$$\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\,dr\,dz = \int_\Gamma g\gamma\psi\, r\,ds + \int_\Omega f\psi\, r\,dr\,dz \quad \forall \psi \in H^1_r(\Omega). \tag{4.18}$$

Theorem 4.2.7. *Let $f \in L^2_r(\Omega), g \in L^2_r(\Gamma)$ such that $\int_\Omega f \ r\,dr\,dz = -\int_\Gamma g \ r\,ds$. There is a unique $\phi \in H^1_{r,m}(\Omega)$ satisfying* (4.18Inhomogeneous Boundary Dataequation.4.2.18).

Proof. If $g \in L^2_r(\Gamma)$ and $\psi \in H^1_r(\Omega)$, then the continuity of the trace $\gamma : H^1_r(\Omega) \to L^2_r(\Gamma)$ implies that there is $C > 0$ such that

$$\begin{aligned} \left| \int_\Gamma g\gamma\psi \ r\,ds \right| &\le \|g\|_{L^2_r(\Gamma)} \|\gamma\psi\|_{L^2_r(\Gamma)} \\ &\le C\|g\|_{L^2_r(\Gamma)} \|\psi\|_{H^1_r(\Omega)}, \end{aligned} \tag{4.19}$$

therefore the right hand side of (4.18Inhomogeneous Boundary Dataequation.4.2.18) defines a continuous linear functional on $H^1_{r,m}(\Omega)$. $a(\phi,\psi) = \int_\Omega (\nabla\phi \cdot \nabla\psi) \ r\,dr\,dz$ is a continuous and coercive bilinear form on $H^1_{r,m}(\Omega)$, so the Lax-Milgram theorem implies that there is a unique $\phi \in H^1_{r,m}(\Omega)$ satisfying

$$\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\,dr\,dz = \int_\Gamma g\gamma\psi\, r\,ds + \int_\Omega f\psi\, r\,dr\,dz \quad \forall \psi \in H^1_{r,m}(\Omega). \tag{4.20}$$

For $\psi \in H_r^1(\Omega)$, then write $\psi = \psi_0 + \langle \psi \rangle$ as before. Then if $\psi \in H_r^1(\Omega)$,

$$\begin{aligned}
\int_\Omega (\nabla\phi \cdot \nabla\psi)\, r\, dr\, dz &= \int_\Omega (\nabla\phi \cdot \nabla(\psi_0 + \langle\psi\rangle))\, r\, dr\, dz \\
&= \int_\Omega (\nabla\phi \cdot \nabla\psi_0)\, r\, dr\, dz \\
&= \int_\Gamma g\gamma\psi_0\, r\, ds + \int_\Omega f\psi_0\, r\, dr\, dz \\
&= \int_\Gamma g\gamma(\psi - \langle\psi\rangle)\, r\, ds + \int_\Omega f(\psi - \langle\psi\rangle)\, r\, dr\, dz \\
&= \int_\Gamma g\gamma\psi\, r\, ds + \int_\Omega f\psi\, r\, dr\, dz - \langle\psi\rangle \left(\int_\Gamma g\, r\, ds + \int_\Omega f\, r\, dr\, dz \right) \\
&= \int_\Gamma g\gamma\psi\, r\, ds + \int_\Omega f\psi\, r\, dr\, dz.
\end{aligned} \tag{4.21}$$

□

Corollary 4.2.8. *Let $f \in L_r^2(\Omega), g \in L_r^2(\Gamma)$, and suppose that ϕ is the unique function in $H_{r,m}^1(\Omega)$ satisfying* (4.18Inhomogeneous Boundary Dataequation.4.2.18). *Then*

$$\|\nabla\phi\|_{L_r^2} \leq \delta_1^{-1/2} \|g\|_{L_r^2(\Gamma)} + \lambda_1^{(N)^{-1/2}} \|f\|_{L_r^2(\Omega)} \tag{4.22}$$

where δ_1 is the smallest strictly positive harmonic Steklov eigenvalue on $H_{r,m}^1(\Omega)$ and $\lambda_1^{(N)}$ is the smallest strictly positive Neumann eigenvalue for $-\Delta$ on $H_{r,m}^1(\Omega)$.

Proof. Let f, g, ϕ be as prescribed. We apply item (iv) of Theorem 3.2.3 and item (iv) of Theorem 3.2.5 to obtain

$$\begin{aligned}
\int_\Omega |\nabla\phi|^2\, r\, dr\, dz &= \int_\Gamma g\gamma\phi\, r\, ds + \int_\Omega f\phi\, r\, dr\, dz \\
&\leq \|g\|_{L_r^2(\Gamma)} \|\gamma\phi\|_{L_r^2(\Gamma)} + \|f\|_{L_r^2(\Omega)} \|\phi\|_{L_r^2(\Omega)} \\
&\leq \left(\delta_1^{-1/2} \|g\|_{L_r^2(\Gamma)} + \lambda_1^{(N)^{-1/2}} \|f\|_{L_r^2(\Omega)} \right) \|\nabla\phi\|_{L_r^2}.
\end{aligned} \tag{4.23}$$

□

4.3 Boundary Value Problems for $-\Delta + r^{-2}$

4.3.1 The Dirichlet Problem for $-\Delta + r^{-2}$

Homogeneous Boundary Data

Given a function f on Ω, consider the problem of finding a function $\psi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = f & \text{in } \Omega, \\ \qquad\qquad \psi = 0 & \text{on } \Gamma. \end{cases} \tag{4.24}$$

Let $f \in (V^1_{r,0}(\Omega))^*$. The weak form of (4.24Homogeneous Boundary Dataequation.4.3.24) is to find a function $\psi \in V^1_{r,0}(\Omega)$ satisfying

$$\int_\Omega (\operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta))\, r\, dr\, dz = \langle f, \chi\rangle \quad \forall \chi \in V^1_{r,0}(\Omega). \tag{4.25}$$

Theorem 4.3.1. *Let $f \in (V^1_{r,0}(\Omega))^*$. There is a unique $\psi \in V^1_{r,0}(\Omega)$ satisfying* (4.25Homogeneous Boundary Dataequation.4.3.25).

Proof. The bilinear form $b(\psi, \chi) = \int_\Omega (\operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta))\ r\ dr\ dz$ is continuous over $V^1_{r,0}(\Omega)$, and the curl-Poincaré inequality implies that b is coercive over $V^1_{r,0}(\Omega)$. Therefore there is a unique $\psi \in V^1_{r,0}(\Omega)$ satisfying (4.25Homogeneous Boundary Dataequation.4.3.25) by the Lax-Milgram theorem. $\square$

Corollary 4.3.2. *Let $f \in (V^1_{r,0}(\Omega))^*$ and $\psi \in V^1_{r,0}(\Omega)$ satisfy* (4.25Homogeneous Boundary Dataequa *Then there is constant $C > 0$ such that*

$$\|\operatorname{curl}(\psi e_\theta)\|_{L^2_r} \le C\left(1 + \tilde{\lambda}_1^{-1}\right)^{1/2} \|f\|_{V^1_{r,0}(\Omega)^*} \tag{4.26}$$

where $\tilde{\lambda}_1$ is the smallest Dirichlet eigenvalue of $-\Delta + r^{-2}$ on $V^1_{r,0}(\Omega)$.

Proof. Let $\psi \in V^1_{r,0}(\Omega)$ satisfy (4.25Homogeneous Boundary Dataequation.4.3.25). $(\|\psi\|^2_{L^2_r} + \|\operatorname{curl}(\psi e_\theta)\|^2_{L^2_r})^{1/2}$ is an equivalent norm on $V^1_{r,0}(\Omega)$ according to Corollary 2.4.4, so we apply item (iv) from the proof of Theorem 3.3.1 to get

$$
\begin{aligned}
\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\, dr\, dz = \langle f, \psi\rangle& \\
&\le \|f\|_{V^1_{r,0}(\Omega)^*}\|\psi\|_{V^1_r} \\
&\le C\|f\|_{V^1_{r,0}(\Omega)^*}\left(\|\psi\|^2_{L^2_r} + \|\operatorname{curl}(\psi e_\theta)\|^2_{L^2_r}\right)^{1/2} \\
&\le C\|f\|_{V^1_{r,0}(\Omega)^*}\left(1 + \tilde{\lambda}_1^{-1}\right)^{1/2}\|\operatorname{curl}(\psi e_\theta)\|_{L^2_r}
\end{aligned}
\tag{4.27}
$$

which proves the claim. □

Inhomogenenous Boundary Data

Given a function f on Ω and a function g on Γ, consider the problem of finding a function $\psi : \Omega \to \mathbb{R}$ satisfying

$$
\begin{cases}
-\Delta\psi + \dfrac{1}{r^2}\psi = f & \text{in } \Omega, \\
\psi = g & \text{on } \Gamma.
\end{cases}
\tag{4.28}
$$

Let $f \in (V^1_{r,0}(\Omega))^*$ and $g \in \gamma(V^1_r(\Omega)) = V^{1/2}_r(\Gamma)$. We transform the inhomogenenous problem to a homogeneous problem by finding $\psi_g \in V^1_r(\Omega)$ such that $\gamma\psi_g = g$ in $L^2_r(\Gamma)$, and then consider finding $\psi \in V^1_{r,0}(\Omega)$ satisfying

$$
\int_\Omega (\operatorname{curl}(\psi e_\theta)\cdot \operatorname{curl}(\chi e_\theta))\, r\, dr\, dz = \langle f, \chi\rangle - \int_\Omega (\operatorname{curl}(\psi_g e_\theta)\cdot\operatorname{curl}(\chi e_\theta))\, r\, dr\, dz \tag{4.29}
$$

for all $\chi \in V^1_{r,0}(\Omega)$. This is the weak form of (4.28Inhomogenenous Boundary Dataequation.4.3.28).

Theorem 4.3.3. *Let $f \in (V^1_{r,0}(\Omega))^*$ and $g \in V^{1/2}_r(\Gamma)$. Let $\psi_g \in V^1_r(\Omega)$ such that $\gamma\psi_g = g$ in $L^2_r(\Gamma)$. Then there is a unique $\tilde{\psi} \in V^1_{r,0}(\Omega)$ satisfying* (4.29Inhomogenenous Boundary Da *for all $\chi \in V^1_{r,0}(\Omega)$.*

Proof. For $\chi \in V^1_{r,0}(\Omega)$, the curl-Poincaré inequality for $V^1_r(\Omega)$ implies that there is a constant $C > 0$ independent of χ such that

$$\begin{aligned}\left|\int_\Omega (\operatorname{curl}(\psi_g e_\theta) \cdot \operatorname{curl}(\chi e_\theta))\, r\, dr\, dz\right| &\leq \|\operatorname{curl}(\psi_g e_\theta)\|_{L^2_r} \|\operatorname{curl}(\chi e_\theta)\|_{L^2_r} \\ &\leq C \|\operatorname{curl}(\psi_g e_\theta)\|_{L^2_r} \|\chi\|_{V^1_r}\end{aligned} \tag{4.30}$$

so the right-hand side of (4.29Inhomogenenous Boundary Dataequation.4.3.29) defines a continuous linear functional in $(V^1_{r,0}(\Omega))^*$. Then we may apply the Lax-Milgram theorem to obtain the conclusion. □

Corollary 4.3.4. *Let $f \in (V^1_{r,0}(\Omega))^*, g \in V^{1/2}_r(\Gamma), \psi_g \in V^1_r(\Omega)$ such that $\gamma\psi_g = g$ in $L^2_r(\Gamma)$, and let $\tilde{\psi}$ be the unique function in $V^1_{r,0}(\Omega)$ satisfying* (4.29Inhomogenenous Boundary Dataeq *Set $\psi = \tilde{\psi} + \psi_g$. Then $\gamma\psi = g$ in $L^2_r(\Gamma)$ and $(-\Delta + r^{-2})\psi = f$ in $(V^1_{r,0}(\Omega))^*$, that is,*

$$\int_\Omega (\operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta))\, r\, dr\, dz = \langle f, \chi \rangle \quad \forall \chi \in V^1_{r,0}(\Omega). \tag{4.31}$$

Proof. If $\psi = \tilde{\psi} + \psi_g$ then $\gamma\psi = \gamma\tilde{\psi} + \gamma\psi_g = g$ in $L^2_r(\Gamma)$ since $\gamma\tilde{\psi} = 0$ as $\tilde{\psi} \in V^1_{r,0}(\Omega)$. (4.31equation.4.3.31) holds upon rearranging (4.29Inhomogenenous Boundary Dataequation.4.3.29)

□

4.3.2 A Conormal Neumann Problem for $-\Delta + r^{-2}$

Homogenenous Boundary Data

Given a function f on Ω, consider the problem of finding a function $\psi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = f & \text{in } \Omega, \\ \operatorname{curl}(\psi e_\theta) \wedge \nu = 0 & \text{on } \Gamma. \end{cases} \tag{4.32}$$

We call this a conormal Neumann problem since we may formally express

$$\operatorname{curl}(\psi e_\theta) \wedge \nu = \frac{1}{r}\left(\nabla(r\psi) \cdot \nu\right) e_\theta \quad \text{on } \Gamma. \tag{4.33}$$

Let $f \in (V_r^1(\Omega))^*$. The weak form of (4.32Homogenenous Boundary Dataequation.4.3.32) is to find a function $\psi \in V_r^1(\Omega)$ satisfying

$$\int_\Omega \left(\operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\right)\ r\,dr\,dz = \langle f, \chi \rangle \tag{4.34}$$

for all $\chi \in V_r^1(\Omega)$. Note that unlike the homogenenous Neumann problem for $-\Delta$, this conormal Neumann problem has no compatibility condition relating the source function f and the homogenenous boundary data. Moreover, the source f may be a linear functional and not necessarily a measurable function.

Theorem 4.3.5. *Let $f \in (V_r^1(\Omega))^*$. There is a unique $\psi \in V_r^1(\Omega)$ satisfying* (4.34Homogenenous Boundary Dataequation.4.3.34) *for all $\chi \in V_r^1(\Omega)$.*

Proof. The bilinear form $b(\psi, \chi) = \int_\Omega (\operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta))\, r\,dr\,dz$ is continuous over $V_r^1(\Omega)$ and the curl-Poincaré inequality for $V_r^1(\Omega)$ implies that b is also coercive over $V_r^1(\Omega)$. Then the Lax-Milgram theorem implies that there is a unique $\psi \in V_r^1(\Omega)$

satisfying (4.34Homogenenous Boundary Dataequation.4.3.34) for all $\chi \in V_r^1(\Omega)$.

$\square$

Corollary 4.3.6. *Let $f \in L_r^2(\Omega)$ and $\psi \in V_r^1(\Omega)$ be the unique function satisfying* (4.34Homogenenous Boundary Dataequation.4.3.34) *for all $\chi \in V_r^1(\Omega)$. Then*

$$\|\operatorname{curl}(\psi e_\theta)\|_{L_r^2} \leq \frac{\|f\|_{L_r^2}}{\sqrt{\tilde{\lambda}_1^{(N)}}} \tag{4.35}$$

where $\tilde{\lambda}_1^{(N)}$ is the smallest positive conormal Neumann eigenvalue of $-\Delta + r^{-2}$ over $V_r^1(\Omega)$.

Proof. Let f, ψ be as prescribed. Then we may apply Theorem 3.3.3 to get

$$\begin{aligned} \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz &= \int_\Omega f\psi \, r \, dr \, dz \\ &\leq \|f\|_{L_r^2} \|\psi\|_{L_r^2} \\ &\leq \frac{\|f\|_{L_r^2}}{\sqrt{\tilde{\lambda}_1^{(N)}}} \|\operatorname{curl}(\psi e_\theta)\|_{L_r^2} \end{aligned} \tag{4.36}$$

which proves the claim. $\square$

Inhomogenenous Boundary Data

Given a function f on Ω and g on Γ, consider the problem of finding a function $\psi : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = f & \text{in } \Omega, \\ \operatorname{curl}(\psi e_\theta) \wedge \nu = g e_\theta & \text{on } \Gamma. \end{cases} \tag{4.37}$$

Note that the boundary condition is just an equivalence of toroidal fields, so it reduces to one scalar equation which may be formally stated as

$$\frac{1}{r}\left(\nabla(r\psi)\right)\cdot\nu = g \quad \text{on } \Gamma. \tag{4.38}$$

Let $f \in (V_r^1(\Omega))^*, g \in L_r^2(\Gamma)$. The weak form of (4.37Inhomogenenous Boundary Dataequation.4.3.3 is to find a function $\psi \in V_r^1(\Omega)$ satisfying

$$\int_\Omega \left(\operatorname{curl}(\psi e_\theta)\cdot\operatorname{curl}(\chi e_\theta)\right)\; r\,dr\,dz = \int_\Gamma g\gamma\chi\; r\,ds + \langle f,\chi\rangle \tag{4.39}$$

for all $\chi \in V_r^1(\Omega)$.

Theorem 4.3.7. *Let $f \in (V_r^1(\Omega))^*, g \in L_r^2(\Gamma)$. There is a unique $\psi \in V_r^1(\Omega)$ satisfying* (4.39Inhomogenenous Boundary Dataequation.4.3.39) *for all $\chi \in V_r^1(\Omega)$.*

Proof. Let $f \in (V_r^1(\Omega))^*, g \in L_r^2(\Gamma), \chi \in V_r^1(\Omega)$. We may apply the continuity of the trace $\gamma : V_r^1(\Omega) \to L_r^2(\Gamma)$ to get

$$\begin{aligned}\left|\int_\Gamma g\gamma\chi\; r\,ds + \langle f,\chi\rangle\right| &\le \|g\|_{L_r^2(\Gamma)}\|\gamma\chi\|_{L_r^2(\Gamma)} + \|f\|_{V_r^1(\Omega)^*}\|\chi\|_{V_r^1(\Omega)} \\ &\le C\left(\|g\|_{L_r^2(\Gamma)} + \|f\|_{V_r^1(\Omega)^*}\right)\|\chi\|_{V_r^1(\Omega)}\end{aligned} \tag{4.40}$$

for some constant $C > 0$ independent of χ. Therefore the right-hand side of (4.39Inhomogenenous Boundary Dataequation.4.3.39) defines a continuous linear functional in $(V_r^1(\Omega))^*$, and we may conclude, just as in the homogenenous case using the Lax-Milgram theorem, that there is a unique $\psi \in V_r^1(\Omega)$ satisfying (4.39Inhomogenenous Boundary I for all $\chi \in V_r^1(\Omega)$. $\square$

Corollary 4.3.8. *Let $f \in L_r^2(\Gamma), g \in L_r^2(\Gamma)$, and $\psi \in V_r^1(\Omega)$ be the unique function satisfying* (4.39Inhomogenenous Boundary Dataequation.4.3.39) *for all $\chi \in V_r^1(\Omega)$.*

Then

$$\| \operatorname{curl}(\psi e_\theta)\|_{L^2_r} \leq \frac{\|g\|_{L^2_r(\Gamma)}}{\sqrt{\tilde{\delta}_1}} + \frac{\|f\|_{L^2_r(\Omega)}}{\sqrt{\tilde{\lambda}_1^{(N)}}} \tag{4.41}$$

where $\tilde{\delta}_1$ *is the smallest positive curl-harmonic Steklov eigenvalue of* $-\Delta + r^{-2}$ *over* $V_r^1(\Omega)$ *and* $\tilde{\lambda}_1^{(N)}$ *is the smallest positive conormal Neumann eigenvalue of* $-\Delta + r^{-2}$ *over* $V_r^1(\Omega)$*.*

Proof. Let f, g, ψ be as prescribed. Then we may apply Theorem 3.3.3 and Theorem 3.3.5 to get

$$\begin{aligned}
\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\,dr\,dz &= \int_\Gamma g\gamma\psi \; r\,ds + \int_\Omega f\psi \; r\,dr\,dz \\
&\leq \|g\|_{L^2_r(\Gamma)}\|\gamma\psi\|_{L^2_r(\Gamma)} + \|f\|_{L^2_r(\Omega)}\|\psi\|_{L^2_r(\Omega)} \\
&\leq \left(\frac{\|g\|_{L^2_r(\Gamma)}}{\sqrt{\tilde{\delta}_1}} + \frac{\|f\|_{L^2_r(\Omega)}}{\sqrt{\tilde{\lambda}_1^{(N)}}} \right) \|\operatorname{curl}(\psi e_\theta)\|_{L^2_r}
\end{aligned} \tag{4.42}$$

which proves the claim. □

Chapter 5

Orthogonal Decompositions for Axisymmetric Vector Fields

5.1 Orthogonal Decompositions for Poloidal Fields

This chapter studies orthogonal decompositions for axisymmetric vector fields in $L^2_r(\Omega;\mathbb{R}^3)$. Classical results of this type for divergence-free fields are presented in [8]. The first step is a decomposition into poloidal and toroidal components. Poloidal and toroidal vector fields are pointwise orthogonal in Ω so they are also mutually orthogonal in $L^2_r(\Omega;\mathbb{R}^3)$:

$$L^2_r(\Omega;\mathbb{R}^3) = \mathrm{Pol}(\Omega) \oplus \mathrm{Tor}(\Omega) \tag{5.1}$$

where $\mathrm{Pol}(\Omega)$ is the subspace of axisymmetric poloidal fields and $\mathrm{Tor}(\Omega)$ is the subspace of axisymmetric toroidal fields with components in $L_r^2(\Omega)$. The gradient subspaces $\mathrm{Grad}_0(\Omega), \mathrm{Grad}(\Omega)$ of $\mathrm{Pol}(\Omega)$ are defined by

$$\begin{aligned} \mathrm{Grad}_0(\Omega) &= \left\{ \nabla\phi = \frac{\partial\phi}{\partial r}e_r + \frac{\partial\phi}{\partial z}e_z : \phi \in H_{r,0}^1(\Omega) \right\} \\ \mathrm{Grad}(\Omega) &= \left\{ \nabla\phi = \frac{\partial\phi}{\partial r}e_r + \frac{\partial\phi}{\partial z}e_z : \phi \in H_r^1(\Omega) \right\}, \end{aligned} \tag{5.2}$$

and the curl subspace $\mathrm{Curl}_0(\Omega), \mathrm{Curl}(\Omega)$ are defined by

$$\begin{aligned} \mathrm{Curl}_0(\Omega) &= \left\{ \mathrm{curl}(\psi e_\theta) = -\frac{\partial\psi}{\partial z}e_r + \frac{1}{r}\frac{\partial(r\psi)}{\partial r}e_z : \psi \in V_{r,0}^1(\Omega) \right\} \\ \mathrm{Curl}(\Omega) &= \left\{ \mathrm{curl}(\psi e_\theta) = -\frac{\partial\psi}{\partial z}e_r + \frac{1}{r}\frac{\partial(r\psi)}{\partial r}e_z : \psi \in V_r^1(\Omega) \right\} \end{aligned} \tag{5.3}$$

Theorem 5.1.1. *Let $u \in \mathrm{Pol}(\Omega)$. Then*

1. $u \perp \mathrm{Grad}_0(\Omega)$ *if and only if* $\mathrm{div}(u) = 0$
2. $u \perp \mathrm{Grad}(\Omega)$ *if and only if* $\mathrm{div}(u) = 0$ *and* $u \cdot \nu = 0$
3. $u \perp \mathrm{Curl}_0(\Omega)$ *if and only if* $\mathrm{curl}(u) = 0$
4. $u \perp \mathrm{Curl}(\Omega)$ *if and only if* $\mathrm{curl}(u) = 0$ *and* $u \wedge \nu = 0$

Proof. Let $u \in \mathrm{Pol}(\Omega)$

1.

$$u \perp \mathrm{Grad}_0(\Omega) \Leftrightarrow \int_\Omega u \cdot \nabla\phi \, r \, dr \, dz \, \forall\phi \in H_{r,0}^1(\Omega) \Leftrightarrow \mathrm{div}(u) = 0. \tag{5.4}$$

2. $u \perp \mathrm{Grad}(\Omega)$ implies $u \perp \mathrm{Grad}_0(\Omega)$ so $\mathrm{div}(u) = 0$. Then

$$\langle u \cdot \nu, \gamma\phi \rangle = \int_\Omega u \cdot \nabla\phi \, r \, dr \, dz = 0 \, \forall\phi \in H_r^1(\Omega) \tag{5.5}$$

so $u \cdot \nu = 0$. Conversely, if $\operatorname{div}(u) = 0$ and $u \cdot \nu = 0$, then we may substitute into (2.23equation.2.3.23) to see that $u \perp \operatorname{Grad}(\Omega)$.

3.

$$u \perp \operatorname{Curl}_0(\Omega) \Leftrightarrow \int_\Omega u \cdot \operatorname{curl}(\psi e_\theta)\, r\, dr\, dz = 0\, \forall \psi \in V^1_{r,0}(\Omega) \Leftrightarrow \operatorname{curl}(u) = 0. \tag{5.6}$$

4. $u \perp \operatorname{Curl}(\Omega)$ implies $u \perp \operatorname{Curl}_0(\Omega)$ so $\operatorname{curl}(u) = 0$. Then

$$\langle u \wedge \nu, \gamma\psi e_\theta \rangle = \int_\Omega u \cdot \operatorname{curl}(\psi e_\theta)\, r\, dr\, dz = 0\, \forall \psi \in V^1_r(\Omega) \tag{5.7}$$

so $u \wedge \nu = 0$. Conversely, if $\operatorname{curl}(u) = 0$ and $u \wedge \nu = 0$, then we may substitute into (2.24equation.2.3.24) to see that $u \perp \operatorname{Curl}(\Omega)$. □

Let $N(\operatorname{div})$ and $N(\operatorname{curl})$ denote the subspaces of $\operatorname{Pol}(\Omega)$ consisting of poloidal fields with zero divergence and zero curl respectively.

Theorem 5.1.2. $\operatorname{Curl}(\Omega) \subset N(\operatorname{div})$ *and* $\operatorname{Grad}(\Omega) \subset N(\operatorname{curl})$.

Proof. 1. Let $\operatorname{curl}(\psi e_\theta) \in \operatorname{Curl}(\Omega)$. Then if $\phi \in C^\infty_{\Gamma 0}(\Omega)$ we may integrate by parts to get

$$\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \nabla\phi\, r\, dr\, dz = \int_\Omega \psi e_\theta \cdot \operatorname{curl}(\nabla\phi)\, r\, dr\, dz = 0 \tag{5.8}$$

since clearly $\operatorname{curl}(\nabla\phi) = 0$ for smooth ϕ. By density, we see that $\int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \nabla\phi\, r\, dr\, dz = 0$ for all $\phi \in H^1_{r,0}(\Omega)$, therefore $\operatorname{div}(\operatorname{curl}(\psi e_\theta)) = 0$.

2. Let $\nabla\phi \in \operatorname{Grad}(\Omega)$. Then if $\psi \in C^\infty_c(\Omega)$ we may integrate by parts to get

$$\int_\Omega \nabla\phi \cdot \operatorname{curl}(\psi e_\theta)\, r\, dr\, dz = \int_\Omega \phi \operatorname{div}(\operatorname{curl}(\psi e_\theta))\, r\, dr\, dz = 0 \tag{5.9}$$

since clearly $\operatorname{div}(\operatorname{curl}(\psi e_\theta)) = 0$ for smooth ψ. By density we see that $\int_\Omega \nabla\phi \cdot \operatorname{curl}(\psi e_\theta)\, r\, dr\, dz = 0$ for all $\psi \in V^1_{r,0}(\Omega)$, therefore $\operatorname{curl}(\nabla\phi) = 0$. □

Next we show that the gradient and curl subspaces are in fact closed subspaces of $\mathrm{Pol}(\Omega)$ and exhibit bases for these subspaces.

Theorem 5.1.3. $\mathrm{Grad}_0(\Omega)$ *and* $\mathrm{Grad}(\Omega)$ *are closed subspaces of* $\mathrm{Pol}(\Omega)$.

Proof. Fix $v \in \mathrm{Pol}(\Omega)$ and consider finding the orthogonal projection of v onto $\mathrm{Grad}_0(\Omega)$. Let $\mathcal{G}_v : H^1_r(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{G}_v(\phi) = \int_\Omega |\nabla\phi|^2\, r\, dr\, dz - 2\int_\Omega \nabla\phi \cdot v\, r\, dr\, dz. \tag{5.10}$$

Riesz's theorem says that a minimizer of $\mathcal{G}_v$ defines the projection of v onto $\mathrm{Grad}(\Omega)$. First observe that $\mathcal{G}_v$ is bounded below since

$$\mathcal{G}_v(\phi) = \|v - \nabla\phi\|^2_{L^2_r} - \|v\|^2_{L^2_r} \geq -\|v\|^2_{L^2_r}. \tag{5.11}$$

The Poincaré inequality for $H^1_{r,0}(\Omega)$ implies that there is a constant C independent of $\phi \in H^1_{r,0}(\Omega)$ such that

$$\begin{aligned}
\mathcal{G}_v(\phi) &= \frac{1}{2}\int_\Omega |\nabla\phi|^2\, r\, dr\, dz + \frac{1}{2}\int_\Omega |\nabla\phi|^2\, r\, dr\, dz - 2\int_\Omega \nabla\phi \cdot v\, r\, dr\, dz \\
&\geq \frac{C}{2}\int_\Omega |\phi|^2\, r\, dr\, dz + \frac{1}{2}\int_\Omega |\nabla\phi|^2\, r\, dr\, dz - 2\|v\|_{L^2_r}\|\nabla\phi\|_{L^2_r} \\
&\geq \frac{\min(1, C)}{2}\|\phi\|^2_{H^1_r} - 2\|v\|_{L^2_r}\|\phi\|_{H^1_r},
\end{aligned} \tag{5.12}$$

therefore $\mathcal{G}_v$ is coercive on $H^1_{r,0}(\Omega)$. $\mathcal{G}_v$ is strictly convex and continuous on $H^1_r(\Omega)$, so $\mathcal{G}_v$ is w.l.s.c. on $H^1_{r,0}(\Omega)$ and therefore $\mathcal{G}_v$ has a unique minimizer on $H^1_{r,0}(\Omega)$. This holds for all $v \in \mathrm{Pol}(\Omega)$, so the projectional functional $\|v - \nabla\phi\|^2_{L^2_r} = \mathcal{G}_v(\phi) + \|v\|^2_{L^2_r}$

is minimized by a unique gradient in $\mathrm{Grad}_0(\Omega)$ given any $v \in \mathrm{Pol}(\Omega)$, and we may conclude that $\mathrm{Grad}_0(\Omega)$ is closed. The proof that $\mathrm{Grad}(\Omega)$ is closed follows from a similar argument by replacing $H^1_{r,0}(\Omega)$ with $H^1_{r,m}(\Omega)$, applying the Poincaré inequality for $H^1_{r,m}(\Omega)$, and then noting that a gradient in $\mathrm{Grad}(\Omega)$ has a unique representative given by a function in $H^1_{r,m}(\Omega)$. $\square$

The poloidal gradient subspaces $\mathrm{Grad}_0(\Omega), \mathrm{Grad}(\Omega)$ are spanned by gradients of eigenfunctions of $-\Delta$. Let $\tilde{\mathcal{E}} = \{\tilde{\phi}_\ell := \lambda^{-1/2}\hat{\phi}_\ell : \ell \in \mathbb{N}\}$ be a maximal sequence of normalized Dirichlet eigenfunctions of $-\Delta$ in $H^1_{r,0}(\Omega)$ and let $\tilde{\mathcal{E}}^{(N)} = \{\tilde{\phi}^{(N)}_\ell := (\lambda^{(N)}_\ell)^{-1/2}\hat{\phi}^{(N)}_\ell\}$ be a maximal sequence of normalized nonconstant Neumann eigenfunctions as in Theorem 3.2.2.

Corollary 5.1.4. *The gradients of the normalized Dirichlet eigenfunctions* $G\tilde{\mathcal{E}} := \{\nabla\tilde{\phi}_\ell : \ell \in \mathbb{N}\}$ *form an orthonormal basis of* $\mathrm{Grad}_0(\Omega)$ *in* $L^2_r(\Omega;\mathbb{R}^3)$*, and the gradients of the normalized nonconstant Neumann eigenfunctions* $G\tilde{\mathcal{E}}^{(N)} := \{\nabla\tilde{\phi}^{(N)}_\ell : \ell \in \mathbb{N}\}$ *form an orthonormal basis of* $\mathrm{Grad}(\Omega)$ *in* $L^2_r(\Omega;\mathbb{R}^3)$*.*

Proof. We will show that $G\tilde{\mathcal{E}}$ is an orthonormal basis of $\mathrm{Grad}_0(\Omega)$. The proof that $G\tilde{\mathcal{E}}^{(N)}$ is an orthonormal basis of $\mathrm{Grad}(\Omega)$ is a very similar argument. We have

$$\int_\Omega |\nabla\tilde{\phi}_k|^2 \, r\,dr\,dz = \lambda_k^{-1}\left(\lambda_k \int_\Omega |\hat{\phi}_k|^2 \, r\,dr\,dz\right) = 1 \tag{5.13}$$

as $\hat{\phi}_k \in C_k$, and the $\nabla\tilde{\phi}_k$ are orthogonal in $L^2_r(\Omega;\mathbb{R}^3)$ by construction. Let $\psi \in H^1_{r,0}(\Omega)$ and suppose that $\int_\Omega \nabla\psi\cdot\nabla\tilde{\phi}_k \, r\,dr\,dz = 0$ for all $k \in \mathbb{N}$. Then

$$\lambda_k \int_\Omega \psi\tilde{\phi}_k \, r\,dr\,dz = \int_\Omega \nabla\psi\cdot\nabla\tilde{\phi}_k \, r\,dr\,dz = 0 \tag{5.14}$$

for all $k \in \mathbb{N}$ since the $\tilde{\phi}_k$ are eigenfunctions. $\lambda_k > 0$ for all k so we must have $\int_\Omega \psi\tilde{\phi}_k \, r\,dr\,dz = 0$ for all k. $\{\tilde{\phi}_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2_r(\Omega)$ which implies $\psi = 0$ and therefore $\nabla\psi = 0$. $\square$

Theorem 5.1.5. $\mathrm{Curl}_0(\Omega)$ *and* $\mathrm{Curl}(\Omega)$ *are closed subspaces of* $\mathrm{Pol}(\Omega)$*.*

Proof. Fix $v \in \mathrm{Pol}(\Omega)$ and consider finding the orthogonal projection of v onto $\mathrm{Curl}_0(\Omega)$. Let $\mathcal{C}_v : V^1_r(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{C}_v(\psi) = \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\,dr\,dz - 2\int_\Omega \operatorname{curl}(\psi e_\theta)\cdot v \, r\,dr\,dz. \tag{5.15}$$

$\mathcal{C}_v$ is bounded below on $V^1_r(\Omega)$ since

$$\mathcal{C}_v(\psi) = \|v - \operatorname{curl}(\psi e_\theta)\|^2_{L^2_r} - \|v\|^2_{L^2_r} \geq -\|v\|^2_{L^2_r}. \tag{5.16}$$

The curl-Poincaré inequality implies that there is a constant $C > 0$ independent of $\psi \in V^1_r(\Omega)$ such that

$$\begin{aligned}\mathcal{C}_v(\psi) &= \frac{1}{2}\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\,dr\,dz + \frac{1}{2}\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\,dr\,dz - 2\int_\Omega \operatorname{curl}(\psi e_\theta)\cdot v \, r\,dr\,dz \\ &\geq \frac{C}{2}\int_\Omega |\psi|^2 \, r\,dr\,dz + \frac{1}{2}\int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r\,dr\,dz - 2\|v\|_{L^2_r}\|\operatorname{curl}(\psi e_\theta)\|_{L^2_r} \\ &\geq \frac{\min(1,C)}{2}(\|\psi\|^2_{L^2_r} + \|\operatorname{curl}(\psi e_\theta)\|^2_{L^2_r}) - 2\|v\|_{L^2_r}\|\operatorname{curl}(\psi e_\theta)\|_{L^2_r}.\end{aligned} \tag{5.17}$$

The above estimate together with (2.39Poincaré Inequalitiesequation.2.4.39) imply that $\mathcal{C}_v$ is coercive on $V^1_r(\Omega)$. $\mathcal{C}_v$ is strictly convex and continuous on $V^1_r(\Omega)$, so $\mathcal{C}_v$ is w.l.s.c. on $V^1_r(\Omega)$ and therefore $\mathcal{C}_v$ has a unique minimizer on $V^1_r(\Omega)$. This holds for all $v \in \mathrm{Pol}(\Omega)$, so the projectional functional $\|v - \operatorname{curl}(\psi e_\theta)\|^2_{L^2_r} = \mathcal{C}_v(\phi) + \|v\|^2_{L^2_r}$ is minimized by a unique curl in $\mathrm{Curl}(\Omega)$ given any $v \in \mathrm{Pol}(\Omega)$, and we may conclude that $\mathrm{Curl}(\Omega)$ is closed. We may apply the same argument to the functional $\mathcal{C}_v$ restricted to $V^1_{r,0}(\Omega)$ to show that $\mathrm{Curl}_0(\Omega)$ is closed. $\square$

Let $\tilde{\mathcal{F}} := \{\tilde{\psi}_k := \tilde{\lambda}_k^{-1/2}\hat{\psi}_k : k \in \mathbb{N}\}$ be a maximal sequence of normalized Dirichlet eigenfunctions of $-\Delta + r^{-2}$ in $V^1_{r,0}(\Omega)$, and let $\tilde{\mathcal{F}}^{(N)} := \{\tilde{\psi}_k^{(N)} := (\tilde{\lambda}_k^{(N)})^{-1/2}\hat{\psi}_k^{(N)} : k \in \mathbb{N}\}$ be a maximal sequence of normalized conormal Neumann eigenfunctions of $-\Delta + r^{-2}$ in $V^1_r(\Omega)$.

Corollary 5.1.6. $C\tilde{\mathcal{F}} := \{\mathrm{curl}(\tilde{\psi}_\ell e_\theta) : \ell \in \mathbb{N}\}$ *is an orthonormal basis of* $\mathrm{Curl}_0(\Omega)$ *in* $L^2_r(\Omega;\mathbb{R}^3)$*, and* $C\tilde{\mathcal{F}}^{(N)} := \{\mathrm{curl}(\tilde{\psi}_\ell^{(N)} e_\theta) : \ell \in \mathbb{N}\}$ *is an orthonormal basis of* $\mathrm{Curl}(\Omega)$ *in* $L^2_r(\Omega;\mathbb{R}^3)$.

Proof. We will show that $C\tilde{\mathcal{F}} := \{\mathrm{curl}(\tilde{\psi}_\ell e_\theta) : \ell \in \mathbb{N}\}$ is an orthonormal basis of $\mathrm{Curl}_0(\Omega)$. The proof that $C\tilde{\mathcal{F}}^{(N)} := \{\mathrm{curl}(\tilde{\psi}_\ell^{(N)} e_\theta) : \ell \in \mathbb{N}\}$ is an orthonormal basis of $\mathrm{Curl}(\Omega)$ is a very similar argument. We have

$$\int_\Omega |\,\mathrm{curl}(\tilde{\psi}_k e_\theta)|^2 \; r\,dr\,dz = \tilde{\lambda}_k^{-1}\left(\tilde{\lambda}_k \int_\Omega |\hat{\psi}_k|^2 \; r\,dr\,dz\right) = 1 \tag{5.18}$$

as $\hat{\psi}_k \in C_k$, and the $\mathrm{curl}(\tilde{\psi}_k e_\theta)$ are orthogonal in $L^2_r(\Omega;\mathbb{R}^3)$ by construction. Let $\chi \in V^1_{r,0}(\Omega)$ and suppose that $\int_\Omega \mathrm{curl}(\chi e_\theta)\cdot\mathrm{curl}(\tilde{\psi}_k e_\theta)\; r\,dr\,dz = 0$ for all $k \in \mathbb{N}$. Then

$$\tilde{\lambda}_k \int_\Omega \chi\tilde{\psi}_k \; r\,dr\,dz = \int_\Omega \mathrm{curl}(\chi e_\theta)\cdot\mathrm{curl}(\tilde{\psi}_k e_\theta)\; r\,dr\,dz = 0 \tag{5.19}$$

for all $k \in \mathbb{N}$ since the $\tilde{\psi}_k$ are eigenfunctions. $\tilde{\lambda}_k > 0$ for all k so we must have $\int_\Omega \chi\tilde{\psi}_k \; r\,dr\,dz = 0$ for all k. $\{\tilde{\psi}_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2_r(\Omega)$ which implies $\chi = 0$ and therefore $\mathrm{curl}(\chi e_\theta) = 0$. $\square$

Definition. Let

$$\begin{aligned} N_{\nu 0}(\mathrm{div}) &= \{u \in N(\mathrm{div}) : u \cdot \nu = 0\} \\ N_{\tau 0}(\mathrm{curl}) &= \{u \in N(\mathrm{curl}) : u \wedge \nu = 0\}. \end{aligned} \tag{5.20}$$

Corollary 5.1.7. $\mathrm{Pol}(\Omega)$ *has the following orthogonal decompositions:*

$$\begin{aligned} \mathrm{Pol}(\Omega) &= \mathrm{Grad}_0(\Omega) \oplus N(\mathrm{div}) \\ &= \mathrm{Grad}(\Omega) \oplus N_{\nu 0}(\mathrm{div}) \\ &= \mathrm{Curl}_0(\Omega) \oplus N(\mathrm{curl}) \\ &= \mathrm{Curl}(\Omega) \oplus N_{\tau 0}(\mathrm{curl}). \end{aligned} \tag{5.21}$$

Proof. Theorem 4.1.1. shows that $\mathrm{Pol}(\Omega) = \overline{\mathrm{Grad}_0(\Omega)} \oplus N(\mathrm{div})$ and Theorem 4.1.4. shows that $\overline{\mathrm{Grad}_0(\Omega)} = \mathrm{Grad}_0(\Omega)$. The other decomposition follow similarly. □

Theorem 4.1.2. let's us refine these decompositions since $\mathrm{Grad}(\Omega) \subset N(\mathrm{curl})$ and $\mathrm{Curl}(\Omega) \subset N(\mathrm{div})$.

Definition. Let $\mathcal{H}_{\nu 0}(\Omega)$ be the orthogonal complement of $\mathrm{Grad}(\Omega) \oplus \mathrm{Curl}_0(\Omega)$ in $\mathrm{Pol}(\Omega)$, and let $\mathcal{H}_{\tau 0}(\Omega)$ denote the orthogonal complement of $\mathrm{Grad}_0(\Omega) \oplus \mathrm{Curl}(\Omega)$ in $\mathrm{Pol}(\Omega)$.

Corollary 5.1.8. $\mathrm{Pol}(\Omega)$ *has the following orthogonal decompositions:*

$$\begin{aligned} \mathrm{Pol}(\Omega) &= \mathrm{Grad}_0(\Omega) \oplus \mathrm{Curl}(\Omega) \oplus \mathcal{H}_{\tau 0}(\Omega) \\ &= \mathrm{Grad}(\Omega) \oplus \mathrm{Curl}_0(\Omega) \oplus \mathcal{H}_{\nu 0}(\Omega). \end{aligned} \tag{5.22}$$

Definition. A vector field $u = (u_r, u_\theta, u_z)$ on Ω is called *harmonic* if $\mathrm{div}(u) = 0$ and $\mathrm{curl}(u) = 0$.

In particular, $\mathcal{H}_{\nu 0}(\Omega)$ and $\mathcal{H}_{\tau 0}(\Omega)$ are spaces of harmonic poloidal fields. We will show that these are special finite dimensional subspaces of harmonic fields determined by the topology of the cross section Ω. The description of these special harmonic fields begins with showing that gradients of axisymmetric scalar potentials and curls of axisymmetric stream functions are sufficient to characterize every poloidal field in $L^2_r(\Omega; \mathbb{R}^3)$.

Theorem 5.1.9. *Let* $u \in \mathrm{Pol}(\Omega)$*. If* $u \perp \mathrm{Grad}(\Omega)$*, then* $u \in \mathrm{Curl}(\Omega)$*. If* $u \perp \mathrm{Curl}(\Omega)$*, then* $u \in \mathrm{Grad}(\Omega)$*.*

Proof. Let $u \in \mathrm{Pol}(\Omega)$ and suppose that $u \perp \mathrm{Grad}(\Omega)$. Let $\tilde{u}$ be the zero extension of u to all of $\mathbb{R}^2_+$, i.e.

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2_+ \setminus \overline{\Omega}. \end{cases} \tag{5.23}$$

If $\phi \in C^\infty(\mathbb{R}^2)$ then

$$\int_{\mathbb{R}^2_+} \tilde{u} \cdot \nabla\phi \, r \, dr \, dz = \int_\Omega u \cdot \nabla\phi \, r \, dr \, dz = 0 \tag{5.24}$$

since $\nabla\phi|_\Omega \in \mathrm{Grad}(\Omega)$ and $u \perp \mathrm{Grad}(\Omega)$. Let $\tilde{U}$ be an axisymmetric lifting of $\tilde{u}$ to the whole space $\mathbb{R}^3$. Then (5.24Orthogonal Decompositions for Poloidal Fieldsequation.5.1.24) implies $\mathrm{div}(\tilde{U}) = 0$ in $\mathbb{R}^3$, where $\mathrm{div}(\tilde{U})$ is meant in the weak sense. Theorem 3.4 and Remark 3.7 in [14] show that there is a vector potential $A \in H^1(\mathbb{R}^3; \mathbb{R}^3)$ such that $\mathrm{div}(A) = 0$ and $\tilde{U} = \mathrm{curl}(A)$. The equation $\mathrm{curl}(A) = \tilde{U}$ reads in Fourier transform

$$\mathcal{F}\tilde{U}(\xi) = 2i\pi(\xi_2\mathcal{F}A_3 - \xi_3\mathcal{F}A_2, \xi_3\mathcal{F}A_1 - \xi_1\mathcal{F}A_3, \xi_1\mathcal{F}A_2, -\xi_2\mathcal{F}A_1). \tag{5.25}$$

$\mathrm{div}(A) = 0$ means that A is determined by the equation

$$(4i\pi|\xi|^2)\mathcal{F}A = \mathcal{F}(\mathrm{curl}(\tilde{U})). \tag{5.26}$$

$\tilde{U}$ is poloidal so curl($\tilde{U}$) is toroidal, in particular, $(\text{curl}(\tilde{U}))_3 = 0$. Then (5.26Orthogonal Decompositi implies that $(\mathcal{F}A)_3 = 0$ so $A_3 = 0$. When $A_3 = 0$ there is a unique solution of (5.25Orthogonal Decompositions for Poloidal Fieldsequation.5.1.25) given by

$$\mathcal{F}A_1 = \frac{\mathcal{F}\tilde{U}_2}{2i\pi\xi_3}, \quad \mathcal{F}A_2 = -\frac{\mathcal{F}\tilde{U}_1}{2i\pi\xi_3}. \tag{5.27}$$

$\tilde{U}$ is poloidal, so the condition $\tilde{U}_\theta = 0$ means.

$$-x_2\tilde{U}_1 + x_1\tilde{U}_2 = 0. \tag{5.28}$$

If we apply the Fourier transform to (5.28Orthogonal Decompositions for Poloidal Fieldsequation.5.1 we get

$$-\frac{\partial(\mathcal{F}\tilde{U}_1)}{\partial\xi_2} + \frac{\partial(\mathcal{F}\tilde{U}_2)}{\partial\xi_1} = 0. \tag{5.29}$$

$\tilde{U}$ has compact support so $\mathcal{F}\tilde{U}$ is smooth and we may differentiate $\mathcal{F}A_1, \mathcal{F}A_2$ in (5.27Orthogonal Decompositions for Poloidal Fieldsequation.5.1.27) to get

$$\frac{\partial(\mathcal{F}A_1)}{\partial\xi_1} + \frac{\partial(\mathcal{F}A_2)}{\partial\xi_2} = \left(\frac{1}{2i\pi\xi_3}\right)\left(\frac{\partial(\mathcal{F}\tilde{U}_2)}{\partial\xi_1} - \frac{\partial(\mathcal{F}\tilde{U}_1)}{\partial\xi_2}\right). \tag{5.30}$$

Then (5.29Orthogonal Decompositions for Poloidal Fieldsequation.5.1.29) implies that

$$\frac{\partial(\mathcal{F}A_1)}{\partial\xi_1} + \frac{\partial(\mathcal{F}A_2)}{\partial\xi_2} = 0. \tag{5.31}$$

Now apply the inverse Fourier transform to (5.31Orthogonal Decompositions for Poloidal Fieldsequa to get $x_1A_1 + x_2A_2 = 0$ which implies $A_r = 0$. Thus the vector potential A is toroidal so we may write $A = A_\theta e_\theta$. Now apply the condition $\text{div}(A) = 0$ in cylindrical coordinates to get

$$\text{div}(A) = \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} = 0 \tag{5.32}$$

therefore A_θ is independent of θ, so $A \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ is an axisymmetric toroidal vector field. Hence there is a stream function ψ on Ω such that

$$A(x) = \psi\left(\sqrt{x_1^2 + x_2^2}, x_3\right) e_\theta \quad x \in \Omega_A. \tag{5.33}$$

Our conditions on $\partial\Omega_A$ guarantee that the restriction $A|_{\Omega_A}$ is an H^1 vector field on Ω_A, therefore the stream function is in $V_r^1(\Omega)$ according to (2.35Poincaré Inequalitiesequation.2.4.35). Then taking the restriction to Ω shows that $u = \mathrm{curl}(\psi e_\theta)$ as desired. The case that $u \perp \mathrm{Curl}(\Omega)$ is proved similarly by extending u to an axisymmetric vector field on all of $\mathbb{R}^3$ and using Fourier transform to construct the scalar potential $\phi \in H_r^1(\Omega)$ such that $u = \nabla\phi$. □

5.2 Characterization of $\mathcal{H}_{\tau 0}(\Omega)$

Let $h \in \mathcal{H}_{\tau 0}(\Omega)$. Then Theorem 4.1.7. shows that $h = \nabla\phi$ for some $\phi \in H_r^1(\Omega)$. The conditions $\mathrm{div}(h) = 0$ and $h \wedge \nu = 0$ yield the boundary problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \nabla\phi \cdot \tau = 0 & \text{on } \Gamma \end{cases} \tag{5.34}$$

where $\tau = (-\nu_z, 0, \nu_r)$. Let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ where $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ are the connected components of Γ such that Ω contained in the region interior to $\Gamma_0 \cup \{r = 0\}$, $\Gamma_j \cap \Gamma_k \neq \emptyset \Leftrightarrow j = k$, and the $\Gamma_1, \ldots, \Gamma_m$ are closed C^2 loops. For $j = 1, \ldots, m$, let $f_j \in H_r^1(\Omega)$ be a function whose trace on Γ is

$$f_j = \begin{cases} 0 & \text{on } \Gamma_0, \\ \delta_{j\ell} & \text{on } \Gamma_\ell, \text{ for } \ell = 1, \ldots, m. \end{cases} \tag{5.35}$$

Let $G_j = \{\phi_0 + f_j : \phi_0 \in H^1_{r,0}(\Omega)\}$ for $j = 1, \dots, m$. If $\Gamma = \Gamma_0$ then we set f_0 to some $H^1_{r,0}(\Omega)$ function and $G_0 = H^1_{r,0}(\Omega)$.

Let $\mathcal{D} : H^1_r(\Omega) \to \mathbb{R}$ be the Dirichlet energy

$$\mathcal{D}(\phi) = \int_\Omega |\nabla \phi|^2 \, r \, dr \, dz. \tag{5.36}$$

If $\Gamma = \Gamma_0$, then $\phi \equiv 0$ is the unique minimizer of $\mathcal{D}$ over $G_0 = H^1_{r,0}(\Omega)$. Consider the problem of minimizing $\mathcal{D}$ over G_j for $j \geq 1$.

Theorem 5.2.1. *$\mathcal{D}$ has a unique minimizer $\hat{\phi}$ on G_j for $j = 1, \dots, m$. $\hat{\phi}$ is the unique weak solution in $H^1_r(\Omega)$ of*

$$\begin{cases} \Delta \phi = 0 & \text{in } \Omega, \\ \phi = f_j & \text{on } \Gamma. \end{cases} \tag{5.37}$$

Proof. G_j may be expressed as $G_j = f_j + H^1_{r,0}(\Omega) = \{f_j + \phi : \phi \in H^1_{r,0}(\Omega)\}$. Then minimizing (5.36Characterization of $\mathcal{H}_{\tau 0}(\Omega)$equation.5.2.36) over G_j is equivalent to minimizing $\mathcal{D}_j : H^1_{r,0}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{D}_j(\phi) = \int_\Omega |\nabla \phi + \nabla f_j|^2 \, r \, dr \, dz. \tag{5.38}$$

Clearly, $\mathcal{D}_j$ is continuous, strictly convex, and the Poincaré inequality for $H^1_{r,0}(\Omega)$ implies that $\mathcal{D}_j$ is coercive on $H^1_{r,0}(\Omega)$. Therefore $\mathcal{D}$ has a unique minimizer $\hat{\phi}_j$ on G_j. The functional D_j is Gateaux differentiable on $H^1_{r,0}(\Omega)$ and the first order condition for a minimizer reads

$$\int_\Omega \nabla(\phi + f_j) \cdot \nabla \psi \, r \, dr \, dz = 0 \quad \text{for all } \psi \in H^1_{r,0}(\Omega). \tag{5.39}$$

Plugging in $\hat{\phi}_j$ into this condition shows that $\hat{\phi}_j$ is the unique weak solution in $H^1_r(\Omega)$ of the harmonic boundary value problem (5.37equation.5.2.37). $\square$

For $j = 1, \ldots, m$, let $h_j = \nabla\hat{\phi}_j$ where $\hat{\phi}_j$ is the unique minimizer in G_j of $\mathcal{D}$.

Theorem 5.2.2. *If $\Gamma = \Gamma_0$, then $\mathcal{H}_{\tau 0}(\Omega) = \{0\}$. Otherwise if $m \geq 1$, then $\{h_j : j = 1, \ldots, m\}$ is a basis of $\mathcal{H}_{\tau 0}(\Omega)$.*

Proof. Let $h \in \mathcal{H}_{\tau 0}(\Omega)$. As mentioned earlier, there is a scalar potential $\phi \in H_r^1(\Omega)$ such that $h = \nabla\phi$ where ϕ is a solution of the harmonic boundary value problem (5.34Characterization of $\mathcal{H}_{\tau 0}(\Omega)$equation.5.2.34). The boundary condition $\nabla\phi \cdot \tau = 0$ and the regularity of Γ imply that ϕ is constant on each component of Γ. If $\Gamma = \Gamma_0$, then the only solution of (5.34Characterization of $\mathcal{H}_{\tau 0}(\Omega)$equation.5.2.34) is $\phi =$ const. and then $h = \nabla\phi = 0$, so $\mathcal{H}_{\tau 0}(\Omega) = \{0\}$ is a trivial subspace. Now consider the case that $m \geq 1$ and suppose that $c_j, j = 0, 1, \ldots, m$ are constants such that $\phi = c_j$ on $\Gamma_j, j = 0, 1, \ldots, m$. The function

$$\psi = \phi - c_0 - \sum_{j=1}^{m}(c_j - c_0)\hat{\phi}_j \tag{5.40}$$

satisfies $\Delta\psi = 0$ in Ω and $\psi = 0$ on Γ, which means $\psi \equiv 0$. Therefore

$$\phi = c_0 + \sum_{j=1}^{m}(c_j - c_0)\hat{\phi}_j. \tag{5.41}$$

Now take the gradient to get

$$\nabla\phi = \sum_{j=1}^{m}(c_j - c_0)\nabla\hat{\phi}_j = \sum_{j=1}^{m}(c_j - c_0)h_j \tag{5.42}$$

so $\{h_1, \ldots, h_m\}$ spans $\mathcal{H}_{\tau 0}(\Omega)$. Suppose that there are constants $a_1, \ldots, a_m$, not all zero, such that

$$\sum_{j=1}^{m} a_j h_j = 0. \tag{5.43}$$

This says that

$$\nabla\left(\sum_{j=1}^{m} a_j\hat{\phi}_j\right) = 0 \quad \text{in } \Omega. \tag{5.44}$$

Γ is connected so we must have that $\sum_{j=1}^{m} a_j\hat{\phi}_j$ is constant on Ω. Taking traces onto Γ implies

$$\sum_{j=1}^{m} a_j\hat{\phi}_j = \begin{cases} a_\ell & \text{on } \Gamma_\ell, \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{5.45}$$

As $\sum_{j=1}^{m} a_j\hat{\phi}_j$ is constant, we must have that $a_j = 0$ for all j, a contradiction. Therefore $\{h_1, \ldots, h_m\}$ is a maximal spanning set for $\mathcal{H}_{\tau 0}(\Omega)$. $\square$

This result implies that the dimension of $\mathcal{H}_{\tau 0}(\Omega)$ is equal to the number of internal loops comprising $\Gamma \setminus \Gamma_0$. These loops correspond to toroidal holes in the volume of revolution Ω_A. We may also characterize the projection of a poloidal field u using the gradient basis for $\mathcal{H}_{\tau 0}(\Omega)$. Let $u \in \mathrm{Pol}(\Omega)$ and let u_{d0} be the projection of u onto $N(\mathrm{div})$. We may write $u_0 = u - \nabla\phi$ where $\nabla\phi$ is the projection of u onto $\mathrm{Grad}_0(\Omega)$. Then if $\hat{\phi}_j$ is as above and using the definition of the linear functional $u_{d0}\cdot\nu$ we get

$$\begin{aligned} \int_\Omega u\cdot\nabla\hat{\phi}_j\, r\,dr\,dz &= \int_\Omega \nabla\phi\cdot\nabla\hat{\phi}_j\, r\,dr\,dz + \int_\Omega u_{d0}\cdot\nabla\hat{\phi}_j\, r\,dr\,dz \\ &= 0 + \langle u_{d0}\cdot\nu, \phi_j\rangle \\ &=: \langle u_{d0}\cdot\nu, 1\rangle_{\Gamma_j}. \end{aligned} \tag{5.46}$$

If u_{d0} is regular enough, then $\langle u_{d0}\cdot\nu, 1\rangle_{\Gamma_j}$ may be written as

$$\int_{\Gamma_j} u_{d0}\cdot\nu\, r\,ds. \tag{5.47}$$

This says that that the projection of u onto the harmonic subspace $\mathcal{H}_{\tau 0}(\Omega)$ is uniquely determined by the flux of the divergence-free component u_{d0} through each interior Γ_j for $j = 1, \ldots, m$.

5.3 Characterization of $\mathcal{H}_{\nu 0}(\Omega)$

We may describe the fields in $\mathcal{H}_{\nu 0}(\Omega)$ in a similar manner to those in $\mathcal{H}_{\tau 0}(\Omega)$. If $k \in \mathcal{H}_{\nu 0}(\Omega)$, then there is a stream function $\psi \in V_r^1(\Omega)$ such that $k = \operatorname{curl}(\psi e_\theta)$ according to Theorem 4.1.7. Then the conditions $\operatorname{curl}(k) = 0$ and $k \cdot \nu = 0$ yield the boundary value problem

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = 0 \quad \text{in } \Omega, \\ \quad \nabla(r\psi)\cdot\tau = 0 \quad \text{on } \Gamma. \end{cases} \tag{5.48}$$

For $j = 1, \ldots, m$ let $g_j \in V_r^1(\Omega)$ be a function whose trace on Γ is

$$g_j = \begin{cases} 0 & \text{on } \Gamma_0, \\ \dfrac{\delta_{j\ell}}{r} & \text{on } \Gamma_\ell, \ell = 1, \ldots, m. \end{cases} \tag{5.49}$$

Let $K_j = \{\psi_0 + g_j : \psi_0 \in V_r^1(\Omega)\}$ for $j = 1, \ldots, m$. If $\Gamma = \Gamma_0$, we set g_0 to some $V_{r,0}^1(\Omega)$ function and $K_0 = V_{r,0}^1(\Omega)$. Let $\mathcal{B} : V_r^1(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{B}(\psi) = \int_\Omega |\operatorname{curl}(\psi e_\theta)|^2 \, r \, dr \, dz. \tag{5.50}$$

If $\Gamma = \Gamma_0$, then $\psi \equiv 0$ is the unique minimizer of $\mathcal{B}$ over $K_0 = V_{r,0}^1(\Omega)$ since $\sqrt{\mathcal{B}(\psi)}$ defines a norm on $V_r^1(\Omega)$.

Theorem 5.3.1. *$\mathcal{B}$ has a unique minimizer $\hat{\psi}$ on K_j for $j = 1, \ldots, m$. $\hat{\psi}$ is the unique weak solution in $V_r^1(\Omega)$ of*

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = 0 \quad \text{in } \Omega, \\ \qquad\qquad \psi = g_j \quad \text{on } \Gamma. \end{cases} \tag{5.51}$$

Proof. $K_j = \{\psi = \psi_0 + g_j : \psi_0 \in V^1_{r,0}(\Omega)\}$ for $j = 1, \ldots, m$, so minimizing $\mathcal{B}$ over G_j is equivalent to minimizing $\mathcal{B}_j : V^1_{r,0}(\Omega) \to \mathbb{R}$ over where

$$\mathcal{B}_j(\psi) = \int_\Omega |\operatorname{curl}(\psi e_\theta) + \operatorname{curl}(g_j e_\theta)|^2 \, r \, dr \, dz. \tag{5.52}$$

Clearly $\mathcal{B}_j$ is continuous, strictly convex, and the curl-Poincaré inequality for $V^1_r(\Omega)$ implies that $\mathcal{B}_j$ is coercive on $V^1_{r,0}(\Omega)$. Therefore $\mathcal{B}$ has a unique minimizer $\hat{\psi}$ on K_j. The functional $\mathcal{B}_j$ is Gateaux differentiable and the first order condition for a minimizer reads

$$\int_\Omega \operatorname{curl}((\psi + g_j)e_\theta) \cdot \operatorname{curl}(\chi e_\theta) \, r \, dr \, dz = 0 \quad \text{for all } \chi \in V^1_{r,0}(\Omega). \tag{5.53}$$

Plugging $\hat{\psi}_j$ into this condition shows that $\hat{\psi}_j$ is the unique weak solution in $V^1_r(\Omega)$ of the boundary value problem (5.51equation.5.3.51). $\square$

For $j = 1, \ldots, m$, let $k_j = \operatorname{curl}(\hat{\psi}_j e_\theta)$ where $\hat{\psi}_j$ is the unique minimizer in K_j of $\mathcal{B}$.

Theorem 5.3.2. *If $\Gamma = \Gamma_0$, then $\mathcal{H}_{\nu 0}(\Omega) = \{0\}$. Otherwise, if $m \geq 1$, then $\{k_j : j = 1, \ldots, m\}$ is a basis of $\mathcal{H}_{\nu 0}(\Omega)$.*

Proof. Let $k \in \mathcal{H}_{\nu 0}(\Omega)$. Then there is a stream function $\psi \in V^1_r(\Omega)$ such that $k = \operatorname{curl}(\psi e_\theta)$ where ψ is a solution of the boundary value problem (5.48Characterization of $\mathcal{H}_{\nu 0}(\Omega)$equati
The boundary condition $\nabla(r\psi) \cdot \tau$ and the regularity of Γ imply that $r\psi$ is constant on each component of Γ. If $\Gamma = \Gamma_0$ and $\psi = \text{const.}/r$ on Γ, then the trace of ψ on Γ is in $L^2_r(\Gamma)$ if and only if const. $= 0$. Therefore $\psi = 0$ on Γ and the only weak solution in $V^1_r(\Omega)$ of $\Delta\psi + r^{-2}\psi = 0$ with the boundary condition $\psi = 0$ is $\psi \equiv 0$. Hence

in this case we have $\mathcal{H}_{\nu 0}(\Omega) = \{0\}$. Now consider the case that $m \geq 1$ and suppose that $c_j, j = 1, \ldots, m$ are constants such that $\psi = c_j/r$ on Γ_j for $j = 1, \ldots, m$; we already saw that $\psi = 0$ on Γ_0. Consider the function

$$\chi = \psi - \sum_{j=1}^{m} c_j \hat{\psi}_j. \tag{5.54}$$

Then χ obeys $-\Delta\chi + r^{-2}\chi = 0$ and $\chi|_\Gamma = 0$, so we must have $\chi = 0$. Therefore

$$\psi = \sum_{j=1}^{m} c_j \hat{\psi}_j. \tag{5.55}$$

Taking curls in the equation above we get

$$\operatorname{curl}(\psi e_\theta) = \sum_{j=1}^{m} c_j \operatorname{curl}(\hat{\psi}_j e_\theta) = \sum_{j=1}^{m} c_j k_j \tag{5.56}$$

so $\{k_1, \ldots, k_m\}$ spans $\mathcal{H}_{\nu 0}(\Omega)$. Suppose that there are constants $a_1, \ldots, a_m$, not all zero, such that

$$\sum_{j=1}^{m} a_j k_j = 0. \tag{5.57}$$

This says that

$$\operatorname{curl}\left(\sum_{j=1}^{m} a_j \hat{\psi}_j\right) = 0 \quad \text{in } \Omega \tag{5.58}$$

or equivalently,

$$\nabla\left(\sum_{j=1}^{m} a_j (r\hat{\psi}_j)\right) = 0. \tag{5.59}$$

Ω is connected so $\sum_{j=1}^{m} a_j(r\hat{\psi}_j)$ is constant on Ω. Then taking traces on $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ we get

$$\sum_{j=1}^{m} a_j(r\hat{\psi}_j) = \begin{cases} a_\ell & \text{on } \Gamma_\ell, \ell = 1, \ldots, m \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{5.60}$$

Then we must have $a_j = 0$ for all $j = 0, 1, \ldots, m$, a contradiction. Hence the k_j's are a maximal spanning set for $\mathcal{H}_{\nu 0}(\Omega)$. $\square$

This characterization of $\mathcal{H}_{\nu 0}(\Omega)$ and $\mathcal{H}_{\tau 0}(\Omega)$ have the same dimension: the number of internal loops in the cross-section Ω which is equal to the number of toroidal holes in the cross-section Ω_A. The projection of a poloidal field onto $\mathcal{H}_{\nu 0}(\Omega)$ has an interpretation using circulations, just as the projection onto $\mathcal{H}_{\tau 0}(\Omega)$ has an interpretation using fluxes. Let u be a poloidal field and let u_{c0} the projection onto $N(\text{curl})$. We may write $u_{c0} = u - \text{curl}(\psi e_\theta)$ where $\text{curl}(\psi e_\theta)$ is the projection of u onto $\text{Curl}_0(\Omega)$. Then if $\hat{\psi}_j$ is as above and using the definition of the linear functional $u_{c0} \wedge \nu$, we get

$$\begin{aligned}\int_\Omega u \cdot \text{curl}(\hat{\psi}_j e_\theta)\, r\, dr\, dz &= \int_\Omega \text{curl}(\psi e_\theta) \cdot \text{curl}(\hat{\psi}_j e_\theta)\, r\, dr\, dz + \int_\Omega u_{c0} \cdot \text{curl}(\hat{\psi} e_\theta)\, r\, dr\, dz \\ &= 0 + \langle u_{c0} \wedge \nu, \hat{\psi}_j e_\theta \rangle \\ &=: \left\langle u_{c0} \wedge \nu, \frac{1}{r} e_\theta \right\rangle_{\Gamma_j}. \end{aligned} \tag{5.61}$$

When u_{c0} is smooth enough, $\langle u_{c0} \wedge \nu, e_\theta / r \rangle_{\Gamma_j}$ may be expressed as

$$\int_{\Gamma_j} u_{c0} \cdot \tau \; ds. \tag{5.62}$$

This says that that the projection of u onto the harmonic subspace $\mathcal{H}_{\nu 0}(\Omega)$ is uniquely determined by the circulation of the curl-free component u_{c0} around each interior Γ_j for $j = 1, \ldots, m$.

Chapter 6

Axisymmetric Div-curl systems

This chapter will describe well-posedness results on axisymmetric div-curl systems with normal or tangantial boundary conditions. The axisymmetric div-curl systems arise in classical field theories when the domain has rotational symmetry and the data is axisymmetric. In particular, the results of this chapter may be applied to some forms of the quasi-static Maxwell equations on domains such as those described in Section 2.1 with axisymmetric data as in Sections 2.2 and 2.3.

6.1 The Normal Div-curl System

This section studies the well-posedness of the normal div-curl system

$$\begin{cases} \operatorname{div}(u) = \rho \quad \text{in } \Omega, \\ \operatorname{curl}(u) = \omega \quad \text{in } \Omega, \\ u \cdot \nu = \mu \quad \text{on } \Gamma. \end{cases} \tag{6.1}$$

Here ρ is a function on Ω, ω is a vector field on Ω, and μ is a function on Γ. The boundary condition $u \cdot \nu = \mu$ is a single scalar equation. We decompose the analysis of this problem into poloidal and toroidal parts. The *poloidal-normal div-curl system* is

$$\begin{cases} \operatorname{div}(u_P) = \rho \quad \text{in } \Omega, \\ \operatorname{curl}(u_P) = \omega_T \quad \text{in } \Omega, \\ u_P \cdot \nu = \mu \quad \text{on } \Gamma. \end{cases} \tag{6.2}$$

The *toroidal-normal div-curl system* is

$$\begin{cases} \operatorname{div}(u_T) = 0 \quad \text{in } \Omega, \\ \operatorname{curl}(u_T) = \omega_P \quad \text{in } \Omega, \\ u_T \cdot \nu = 0 \quad \text{on } \Gamma. \end{cases} \tag{6.3}$$

The following conditions are imposed on the data ρ, ω, μ for (6.1The Normal Div-curl Systemequa

(N1) $\rho \in L^2_r(\Omega)$;

(N2) $\omega \in L^2_r(\Omega; \mathbb{R}^3)$;

(N3) $\operatorname{div}(\omega_P) = 0$;

(N4) $\mu \in L^2_r(\Gamma)$;

(N5) $\int_\Omega \rho \, r \, dr \, dz = \int_\Gamma \mu \, r \, ds$;

(N6) $\omega_P \perp \mathcal{H}_{\tau 0}(\Omega)$.

Definition. Let ρ, ω, μ be given such that conditions (N1) through (N6) are satisfied. A vector field $u \in L^2_r(\Omega; \mathbb{R}^3)$ is a *weak solution* of (6.1The Normal Div-curl Systemequation.6.1.1) provided it satisfies

$$\int_\Omega u \cdot \nabla\phi \, r \, dr \, dz = -\int_\Omega \rho\phi \, r \, dr \, dz + \int_\Gamma \mu\gamma\phi \quad \text{for all } \phi \in H^1_1(\Omega) \tag{6.4}$$

and

$$\int_\Omega u \cdot \operatorname{curl}(F) \, r \, dr \, dz = \int_\Omega \omega \cdot F \, r \, dr \, dz \quad \text{for all } F \in V^1_{r,0}(\Omega) \times V^1_{r,0}(\Omega) \times H^1_{r,0}(\Omega). \tag{6.5}$$

By linearity, $u \in L^2_r(\Omega; \mathbb{R}^3)$ is a weak solution of (6.1The Normal Div-curl Systemequation.6.1.1) if and only if the poloidal and toroidal components u_P, u_T are weak solutions of (6.2The Normal Div-curl Systemequation.6.1.2) and (6.3The Normal Div-curl Systemequation.6.1.3) respectively. The conditions (N5) and (N6) are actually compatibility conditions that must hold.

Theorem 6.1.1. *Let ρ, ω, μ be given such that (N1) through (N4) are satisfied and suppose that $u \in L^2_r(\Omega; \mathbb{R}^3)$ is a weak solution of* (6.1The Normal Div-curl Systemequation.6.1.1). *Then ρ, ω, μ satisfy (N5) and (N6).*

Proof. Let $u \in L^2_r(\Omega; \mathbb{R}^3)$ solve (6.1The Normal Div-curl Systemequation.6.1.1). If

$\phi \equiv 1 \in H_r^1(\Omega)$, then $\gamma\phi \equiv 1$ on Γ so

$$
\begin{aligned}
\langle \mu, 1\rangle &= \langle u\cdot\nu, 1\rangle \\
&= \int_\Omega u\cdot\nabla(1)\, r\,dr\,dz + \int_\Omega \operatorname{div}(u)(1)\, r\,dr\,dz \\
&= \int_\Omega \operatorname{div}(u)\, r\,dr\,dz \\
&= \int_\Omega \rho\, r\,dr\,dz.
\end{aligned}
\tag{6.6}
$$

If u solves (6.1The Normal Div-curl Systemequation.6.1.1) then, in particular, $\operatorname{curl}(u_T) = \omega_P \in \operatorname{Pol}(\Omega)$. This says that $\omega_P \in \operatorname{Curl}(\Omega)$ which implies $\omega_P \perp (\operatorname{Grad}_0(\Omega) \oplus \mathcal{H}_{\tau 0}(\Omega))$. $\square$

Now we want to show that when (N1) through (N6) are satisfied, then the system (6.1The Normal Div-curl Systemequation.6.1.1) has a solution $u \in L_r^2(\Omega;\mathbb{R}^3)$. To this end, we construct a solution u_P of the poloidal-normal div-curl system using the decompositions from the previous chapter. Consider $u_P = -\nabla\phi + \operatorname{curl}(\psi e_\theta)$ with $\phi \in H_r^1(\Omega)$ a weak solution of

$$
\begin{cases}
-\Delta\phi = \rho \quad \text{in } \Omega, \\
D_\nu\phi = -\mu \quad \text{on } \Gamma.
\end{cases}
\tag{6.7}
$$

and $\psi \in V_{r,0}^1(\Omega)$ a weak solution of

$$
\begin{cases}
-\Delta\psi + \dfrac{1}{r^2}\psi = \omega_\theta \quad \text{in } \Omega, \\
\psi = 0 \quad \text{on } \Gamma.
\end{cases}
\tag{6.8}
$$

A weak solution of (6.7The Normal Div-curl Systemequation.6.1.7) exists when the conditions (N1), (N4), and (N5) are satisfied according to Theorem 4.2.3; a weak solution of (6.8The Normal Div-curl Systemequation.6.1.8) exists when ω satisfies (N2) according to Theorem 4.3.1.

Lemma 6.1.2. *Suppose that ρ, ω, μ are given such that conditions (N1) through (N6) are satisfied. Let $\phi \in H_r^1(\Omega)$ be a weak solution of* (6.7The Normal Div-curl Systemequation.6.1.7) *and $\psi \in V_{r,0}^1(\Omega)$ be a weak solution of* (6.8The Normal Div-curl Systemequation.6.1.8). *Then $u_p = -\nabla\phi + \mathrm{curl}(\psi e_\theta)$ is a weak solution of the poloidal-normal div-curl system.*

Proof. Let $\chi \in H_r^1(\Omega)$. Then

$$\begin{aligned}\int_\Omega u_P \cdot \nabla\chi \, r\, dr\, dz &= \int_\Omega (-\nabla\phi + \mathrm{curl}(\psi e_\theta)) \cdot \nabla\chi \, r\, dr\, dz \\ &= -\int_\Omega \nabla\phi \cdot \nabla\chi \, r\, dr\, dz + \int_\Omega \mathrm{curl}(\psi e_\theta) \cdot \nabla\chi \, r\, dr\, dz \\ &= -\int_\Omega \nabla\phi \cdot \nabla\chi \, r\, dr\, dz\end{aligned} \tag{6.9}$$

since $\mathrm{curl}(\psi e_\theta)$ and $\nabla\chi$ are orthogonal in $L_r^2(\Omega; \mathbb{R}^3)$. A weak solution $\phi \in H_r^1(\Omega)$ of (6.7The Normal Div-curl Systemequation.6.1.7) satisfies

$$\int_\Omega \nabla\phi \cdot \nabla\chi \, r\, dr\, dz = \int_\Omega \rho\chi \, r\, dr\, dz - \int_\Gamma \mu\gamma\chi \, r\, ds \tag{6.10}$$

so

$$\begin{aligned}\int_\Omega u_P \cdot \nabla\chi \, r\, dr\, dz &= -\int_\Omega \nabla\phi \cdot \nabla\chi \, r\, dr\, dz \\ &= -\int_\Omega \rho\chi \, r\, dr\, dz + \int_\Gamma \mu\gamma\chi \, r\, ds.\end{aligned} \tag{6.11}$$

Now if $\chi \in V_{r,0}^1(\Omega)$ and ψ is a weak solution of (6.8The Normal Div-curl Systemequation.6.1.8), then

$$\int_\Omega \mathrm{curl}(\psi e_\theta) \cdot \mathrm{curl}(\chi e_\theta) \, r\, dr\, dz = \int_\Omega \omega_\theta\chi \, r\, dr\, dz, \tag{6.12}$$

therefore we may argue similarly as before to show that

$$\int_\Omega u_P \cdot \mathrm{curl}(\chi e_\theta) \, r\, dr\, dz = \int_\Omega \omega_\theta\chi \, r\, dr\, dz. \tag{6.13}$$

Hence u_P is a weak solution of the poloidal-normal div-curl system (6.2The Normal Div-curl Systeme

$\square$

The existence of a solution to the toroidal-normal div-curl problem (6.3The Normal Div-curl Syst is proved using the decomposition

$$\mathrm{Pol}(\Omega) = \mathrm{Grad}_0(\Omega) \oplus \mathrm{Curl}(\Omega) \oplus \mathcal{H}_{\tau 0}(\Omega). \tag{6.14}$$

Lemma 6.1.3. *Suppose that condition (N1) through (N6) are satisfied. Then there is a weak solution $u_T \in L^2_r(\Omega; \mathbb{R}^3)$ of the toroidal-normal div-curl system.*

Proof. The conditions (N3) and (N6) say that the poloidal field $\omega_P \in L^2_r(\Omega; \mathbb{R}^3)$ satisfies $\mathrm{div}(\omega_P) = 0$ and $\omega_P \perp \mathcal{H}_{\tau 0}(\Omega)$. If $\mathrm{div}(\omega_P) = 0$ then $\omega_P \perp \mathrm{Grad}_0(\Omega)$, so $\omega_P \in (\mathrm{Grad}_0(\Omega) \oplus \mathcal{H}_{\tau 0}(\Omega))^\perp = \mathrm{Curl}(\Omega)$. Then there is a unique $\chi \in V^1_r(\Omega)$ such that $\omega_P = \mathrm{curl}(\chi e_\theta)$. Set $u_T = \chi e_\theta$. Then

$$\int_\Omega u_T \cdot \nabla \eta \, r \, dr \, dz = 0 \tag{6.15}$$

for all $\eta \in H^1_r(\Omega)$ since u_T is toroidal and $\nabla\chi$ is poloidal. Let $F_r \in V^1_{r,0}(\Omega), F_z \in H^1_r(\Omega)$. Then the definition of $\omega_P = \mathrm{curl}(\chi e_\theta)$ implies that

$$\begin{aligned} \int_\Omega u_T \cdot \mathrm{curl}(F_r e_r + F_z e_z) \, r \, dr \, dz &= \int_\Omega \chi e_\theta \cdot \mathrm{curl}(F_r e_r + F_z e_z) \, r \, dr \, dz \\ &= \int_\Omega \omega_P \cdot (F_r e_r + F_z e_z) \, r \, dr \, dz. \end{aligned} \tag{6.16}$$

Hence u_T is a weak solution of the toroidal-normal div-curl system. $\square$

Corollary 6.1.4. *Let ρ, ω, μ be given such that conditions (N1) through (N6) are satisfied. Then the normal div-curl system* (6.1The Normal Div-curl Systemequation.6.1.1) *has a weak solution.*

Proof. Let $u = -\nabla\phi + \operatorname{curl}(\psi e_\theta) + \chi e_\theta$ with ϕ, ψ as in Lemma 5.1.2 and χ as in the proof of Lemma 5.1.3. Then $u_P = -\nabla\phi + \operatorname{curl}(\psi e_\theta)$ solves the poloidal-normal div-curl system, $u_T = \chi e_\theta$ solves the toroidal-normal div-curl system, so $u = u_P + u_T$ solves the complete normal div-curl system. $\square$

This resolves the existence problem for the normal div-curl system. We now address the problem of uniqueness. The orthogonal decompositions in Chapter 4 suggest that the uniqueness problem for the poloidal part depends on the topology of Ω, since the space of harmonic fields $\mathcal{H}_{\nu 0}(\Omega)$ are in the null-space of the divergence, curl, and normal trace operators. On the other hand, the toroidal part of the problem has uniqueness guaranteed since there are no nontrivial harmonic toroidal fields in $L^2_r(\Omega; \mathbb{R}^3)$.

Theorem 6.1.5. *Let ρ, ω, μ be given such that conditions (N1) through (N6) are satisfied. If Γ has a single component $\Gamma = \Gamma_0$, then there is a unique weak solution in $L^2_r(\Omega; \mathbb{R}^3)$ to the normal div-curl system. If Γ has multiple components $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$, then the set of weak solutions in $L^2_r(\Omega; \mathbb{R}^3)$ of the normal div-curl system is an m-dimensional affine subspace.*

Proof. Let u, v be two weak solutions of the normal div-curl system and let $w = u - v$ be their difference. The toroidal part w_T is therefore a harmonic toroidal field so it must have the form $w_T = (C/r)e_\theta$ for some constant C. If $w_T \in L^2_r(\Omega; \mathbb{R}^3)$ then

$$\int_\Omega \frac{C^2}{r^2}\, r\, dr\, dz = \int_\Omega \frac{C^2}{r}\, dr\, dz < \infty \tag{6.17}$$

if and only if $C = 0$ since $\overline{\Omega}$ has nontrivial intersection with the z-axis. Therefore $w_T = 0$, so $u_T = v_T$. The poloidal component w_P is a harmonic poloidal field in

$L^2_r(\Omega;\mathbb{R}^3)$ with zero normal trace, i.e. $w_P \in \mathcal{H}_{\nu 0}(\Omega)$. If $\Gamma = \Gamma_0$, then Theorem 4.3.2 asserts that $w_P = 0$, so $u_P = v_P$ and consequently $u = v$. If $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$ with $\Gamma_1, \ldots, \Gamma_m$ all nonempty, then Theorem 4.3.2 asserts that $\mathcal{H}_{\nu 0}(\Omega)$ is a m-dimensional subspace of $\mathrm{Pol}(\Omega)$. Therefore $u = v + \sum_{j=1}^m c_j \,\mathrm{curl}(\hat{\psi}_j e_\theta)$, with $\mathrm{curl}(\hat{\psi}_j e_\theta) = k_j$ as in Theorem 4.3.2, for some constants $c_1, \ldots, c_m$. □

A unique weak solution of the normal div-curl system in the case that Γ has multiple components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ may be obtained by prescribing extra conditions. Namely, the projection of the solution onto the one-dimensional subspaces $\{a\,\mathrm{curl}(\hat{\psi}_j e_\theta) : a \in \mathbb{R}\}, j = 1, \ldots, m$ uniquely determines a weak solution.

Corollary 6.1.6. *Let ρ, ω, μ be given such that conditions (N1) through (N6) are satisfied and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ as before with $\Gamma_1, \ldots, \Gamma_m$ all nonempty. Let $\{\mathrm{curl}(\hat{\psi}_j e_\theta : j = 1, \ldots, m\}$ be a basis for $\mathcal{H}_{\nu 0}(\Omega)$ as in Theorem 4.3.2. Then the normal div-curl system has a unique weak solution if the m functionals*

$$\int_\Omega u \cdot \mathrm{curl}(\hat{\psi}_j e_\theta)\; r\,dr\,dz, \quad j = 1, \ldots, m \tag{6.18}$$

are also prescribed in addition to ρ, ω, μ satisfying conditions (N1) through (N6).

Proof. If ρ, σ, μ are given such that conditions (N1) through (N6) satisfied, then there is an m-dimensional affine subspace of solutions $u + \mathcal{H}_{\nu 0}(\Omega)$ where u is a particular solution. The m functionals (6.18equation.6.1.18) uniquely determine the projection of a solution onto the subspace $\mathcal{H}_{\nu 0}(\Omega)$, hence there is a unique solution when the m functionals in (6.18equation.6.1.18) are prescribed. □

The prescription of the functionals in (6.18equation.6.1.18) may be interpreted as prescribing m circulations of the curl-free part of the desired vector field. Lastly we present an energy estimate demonstrating the dependence of a solution on the data ρ, ω, μ.

Corollary 6.1.7. *Let ρ, ω, μ be given satisfying conditions (N1) - (N6). Suppose that Γ has multiple components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ with $\Gamma_1, \dots, \Gamma_m$ all nonempty. Let $\{\mathrm{curl}(\hat{\psi}_j e_\theta) : j = 1, \dots, m\}$ be a basis for $\mathcal{H}_{\nu 0}(\Omega)$. Let $u \in L^2_r(\Omega; \mathbb{R}^3)$ be a solution of the normal div-curl system with $\kappa_j, j = 1, \dots, m$ the values of the m functionals*

$$\kappa_j = \int_\Omega u \cdot \mathrm{curl}(\hat{\psi}_j e_\theta)\, r\, dr\, dz, \quad j = 1, \dots, m \tag{6.19}$$

and denote $\kappa = (\kappa_1, \dots, \kappa_m)$. Then there is a constant $C > 0$ such that

$$\int_\Omega |u|^2\, r\, dr\, dz \le C \left(\int_\Gamma |\mu|^2\, r\, ds + \int_\Omega |\rho|^2\, r\, dr\, dz + \int_\Omega |\omega|^2\, r\, dr\, dz + |\kappa| \right). \tag{6.20}$$

Proof. Let u_P be the poloidal part of u and write $u_P = -\nabla\phi + \mathrm{curl}(\psi e_\theta) + k$ where $\nabla\phi$ is the projection onto $\mathrm{Grad}(\Omega), \mathrm{curl}(\psi e_\theta)$ is the projection onto $\mathrm{Curl}_0(\Omega)$, and k is the projection onto $\mathcal{H}_{\nu 0}(\Omega)$. The characterizations of ϕ, ψ as weak solutions of boundary value problems let us apply Corollary 4.2.8 and Corollary 4.3.2 to derive a constant $C > 0$ such that

$$\int_\Omega |u_P|^2\, r\, dr\, dz \le C \left(\int_\Gamma |\mu|^2\, r\, ds + \int_\Omega |\rho|^2\, r\, dr\, dz + \int_\Omega |\omega_T|^2\, r\, dr\, dz + |\kappa| \right). \tag{6.21}$$

$\omega_P \in \mathrm{Curl}(\Omega)$ with $\omega_P = \mathrm{curl}(u_\theta e_\theta)$ and $u_\theta \in V^1_r(\Omega)$ by conditions (N2), (N3), and (N6), so we may apply the curl-Poincaré inequality for $V^1_r(\Omega)$ to obtain

$$\int_\Omega |u_\theta e_\theta|^2\, r\, dr\, dz \le C \int_\Omega |\mathrm{curl}(u_\theta e_\theta)|^2\, r\, dr\, dz = C \int_\Omega |\omega_P|^2\, r\, dr\, dz \tag{6.22}$$

for some constant $C > 0$. Combining (6.21The Normal Div-curl Systemequation.6.1.21) and (6.22The Normal Div-curl Systemequation.6.1.22) yields (6.20equation.6.1.20).

$\square$

6.2 The Tangential Div-curl System

This section studies the well-posedness of the tangential div-curl system

$$\begin{cases} \operatorname{div}(u) = \rho \quad \text{in } \Omega, \\ \operatorname{curl}(u) = \omega \quad \text{in } \Omega, \\ u \wedge \nu = \sigma \quad \text{on } \Gamma. \end{cases} \tag{6.23}$$

where ρ, ω are as in Section 6.1, and σ is a vector field on Γ. The boundary condition for the tangential div-curl system consists of three scalar equations. We again decompose the analysis of this problem into poloidal and toroidal parts. The *poloidal-tangential div-curl system* is

$$\begin{cases} \operatorname{div}(u_P) = \rho \quad \text{in } \Omega, \\ \operatorname{curl}(u_P) = \omega_T \quad \text{in } \Omega, \\ u_P \wedge \nu = \sigma_T \quad \text{on } \Gamma. \end{cases} \tag{6.24}$$

The *toroidal-normal div-curl system* is

$$\begin{cases} \operatorname{div}(u_T) = 0 \quad \text{in } \Omega, \\ \operatorname{curl}(u_T) = \omega_P \quad \text{in } \Omega, \\ u_T \wedge \nu = \sigma_P \quad \text{on } \Gamma. \end{cases} \tag{6.25}$$

The following conditions are imposed on the data ρ, ω, σ for (6.23The Tangential Div-curl System

(N1) $\rho \in L^2_r(\Omega)$

(N2) $\omega \in L^2_r(\Omega; \mathbb{R}^3)$

(N3) $\operatorname{div}(\omega_P) = 0$

(N6) $\omega_P \perp \mathcal{H}_{\tau 0}(\Omega)$

(N7) $\sigma \in L^2_r(\Gamma; \mathbb{R}^3)$

(N8)

$$\int_\Omega \omega_P \cdot v_P \, r \, dr \, dz = -\int_\Gamma \sigma_P \cdot \gamma v_P \, r \, ds$$

for all $v_P = v_r e_r + v_z e_z$ with $v_r \in V^1_r(\Omega), v_z \in H^1_r(\Omega), \operatorname{curl}(v_P) = 0$.

Definition. Let ρ, ω, σ be given such that conditions (N1) – (N3), (N6) – (N8) are satisfied. A vector field $u \in L^2_r(\Omega; \mathbb{R}^3)$ is a *weak solution* of (6.23The Tangential Div-curl Systemequa
provided it satisfies

$$\int_\Omega u \cdot \nabla\phi \, r \, dr \, dz = -\int_\Omega \rho\phi \, r \, dr \, dz \quad \text{for all } \phi \in H^1_{r,0}(\Omega) \tag{6.26}$$

and

$$\int_\Omega u \cdot \operatorname{curl}(v) \, r \, dr \, dz = \int_\Gamma \sigma \cdot \gamma v \, r \, ds + \int_\Omega \omega \cdot v \, r \, dr \, dz \tag{6.27}$$

for all $v = (v_r, v_\theta, v_z)$ with $v_r, v_\theta \in V^1_r(\Omega)$ and $v_z \in H^1_r(\Omega)$.

Just like the normal div-curl system, $u \in L^2_r(\Omega; \mathbb{R}^3)$ is a weak solution of (6.23The Tangential Div-
if and only if the poloidal and toroidal components u_P, u_T are weak solutions of (6.24The Tangential Div-curl Systemequation.6.2.24) and (6.25The Tangential Div-curl Systemequa
respectively.

We construct solutions of the poloidal-tangential div-curl system by formulating boundary value problems for the scalar potential and stream function. Consider $u_P = -\nabla\phi + \mathrm{curl}(\psi e_\theta)$ where $\phi \in H^1_{r,0}(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta\phi = \rho & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma, \end{cases} \tag{6.28}$$

and $\psi \in V^1_r(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = \omega_\theta & \text{in } \Omega, \\ \mathrm{curl}(\psi e_\theta) \wedge \nu = \sigma_T & \text{on } \Gamma. \end{cases} \tag{6.29}$$

A weak solution of (6.28The Tangential Div-curl Systemequation.6.2.28) exists when (N1) is satisfied and a weak solution of (6.29The Tangential Div-curl Systemequation.6.2.29) exists when (N2) and (N7) are satisfied.

Lemma 6.2.1. *Suppose that ρ, ω, σ are given such that (N1) – (N3), (N6) – (N8) are satisfied. Let $\phi \in H^1_{r,0}(\Omega)$ be a weak solution of* (6.28The Tangential Div-curl Systemequation.6.2.28 *and $\psi \in V^1_r(\Omega)$ be a weak solution of* (6.29The Tangential Div-curl Systemequation.6.2.29). *Then $u_P = -\nabla\phi + \mathrm{curl}(\psi e_\theta)$ is a weak solution of the poloidal-tangential div-curl system.*

Proof. Let $\chi \in H^1_{r,0}(\Omega)$ and $u_P = -\nabla\phi + \mathrm{curl}(\psi e_\theta)$ where ϕ, ψ are weak solutions of (6.28The Tangential Div-curl Systemequation.6.2.28), (6.29The Tangential Div-curl Systemequatio respectively. Then

$$\begin{aligned} \int_\Omega u_P \cdot \nabla\chi \, r\,dr\,dz &= \int_\Omega \left(-\nabla\phi + \mathrm{curl}(\psi e_\theta)\right) \, r\,dr\,dz \\ &= -\int_\Omega \nabla\phi \cdot \nabla\chi \, r\,dr\,dz - \int_\Omega \mathrm{curl}(\psi e_\theta) \cdot \nabla\chi \, r\,dr\,dz \\ &= -\int_\Omega \rho\chi \, r\,dr\,dz \end{aligned} \tag{6.30}$$

since $\operatorname{curl}(\psi e_\theta)$ and $\nabla\chi$ are orthogonal in $L^2_r(\Omega;\mathbb{R}^3)$. Now if $\chi \in V^1_r(\Omega)$ we have

$$\begin{aligned}\int_\Omega u_P \cdot \operatorname{curl}(\chi e_\theta)\, r\,dr\,dz &= \int_\Omega (-\nabla\phi + \operatorname{curl}(\psi e_\theta)) \cdot \operatorname{curl}(\chi e_\theta)\, r\,dr\,dz \\ &= \int_\Omega \operatorname{curl}(\psi e_\theta) \cdot \operatorname{curl}(\chi e_\theta)\, r\,dr\,dz \\ &= \int_\Gamma \sigma_\theta \gamma\chi\, r\,ds + \int_\Omega \omega_\theta \chi\, r\,dr\,dz\end{aligned} \tag{6.31}$$

since $\nabla\phi$ and $\operatorname{curl}(\chi e_\theta)$ are orthogonal in $L^2_r(\Omega;\mathbb{R}^3)$. Therefore u_P is a weak solution of the poloidal-tangential div-curl system. □

The problem of existence of a solution to the toroidal-tangential div-curl system is more subtle. Unlike the normal div-curl system, the boundary condition $u_T \wedge \nu = \sigma_P$ is not immediately satisfied by virtue of u_T being toroidal. We consider instead writing $u_T = \operatorname{curl}(v_P)$ for some poloidal field $v_p = (v_r, 0, v_z)$ with $v_r \in V^1_r(\Omega), v_z \in H^1_r(\Omega)$. Let $X_P(\Omega) = \{v_p \in \operatorname{Pol}(\Omega) : v_r \in V^1_r(\Omega), v_z \in H^1_r(\Omega), \operatorname{div}(v_P) = 0 \text{ in } \Omega, v_P \cdot \nu = 0 \text{ on } \Gamma\}$ and define the norm on $X_P(\Omega)$ to be $\|v_P\|_{X_P} := \left(\|v_r\|^2_{V^1_r} + \|v_z\|^2_{H^1_r}\right)^{1/2}$. Now consider the variational problem of finding $v_p \in X_p(\Omega)$ such that

$$\int_\Omega \operatorname{curl}(v_P) \cdot \operatorname{curl}(w_P)\, r\,dr\,dz = \int_\Gamma \sigma_P \cdot \gamma w_P\, r\,ds + \int_\Omega \omega_P \cdot w_P\, r\,dr\,dz \tag{6.32}$$

for all $w_P = (w_r, 0, w_z)$ with $w_r \in V^1_r(\Omega), w_z \in H^1_r(\Omega)$.

Lemma 6.2.2. *Let $\omega \in L^2_r(\Omega;\mathbb{R}^3), \sigma \in L^2_r(\Gamma;\mathbb{R}^3)$. Then there is a unique $v_P \in X_P(\Omega) \cap \mathcal{H}_{\nu 0}(\Omega)^\perp$ satisfying* (6.32The Tangential Div-curl Systemequation.6.2.32) *for all $w_P \in X_P(\Omega) \cap \mathcal{H}_{\nu 0}(\Omega)^\perp$.*

Proof. Clearly the right-hand side of (6.32The Tangential Div-curl Systemequation.6.2.32) defines a continuous linear functional on $X_P(\Omega) \cap \mathcal{H}_{\nu 0}(\Omega)^\perp$. $B(v_P, w_P) = \int_\Omega \operatorname{curl}(v_P)\cdot$

$\operatorname{curl}(w_P)\, r\, dr\, dz$ is a continuous bilinear form on $X_P(\Omega)$, and Theorem 2.4.5 implies that B is coercive on $X_P(\Omega)\cap\mathcal{H}_{\nu 0}(\Omega)^\perp$. Then we may apply the Lax-Milgram theorem to conclude that there is a unique $v_P \in X_P(\Omega)\cap\mathcal{H}_{\nu 0}(\Omega)^\perp$ satisfying (6.32The Tangential Div-curl for all $w_P \in X_P(\Omega)\cap\mathcal{H}_{\nu 0}(\Omega)^\perp$. □

Theorem 6.2.3. *Let* $\omega \in L^2_r(\Omega;\mathbb{R}^3), \sigma \in L^2_r(\Gamma;\mathbb{R}^3)$ *such that conditions (N2), (N3), (N6) - (N8) are satisfied. Suppose that* $v_P \in X_p(\Omega)\cap\mathcal{H}_{\nu 0}(\Omega)^\perp$ *satisfies* (6.32The Tangential Div-curl Systemequation.6.2.32) *for all* $w_P \in X_P(\Omega)\cap\mathcal{H}_{\nu 0}(\Omega)^\perp$. *Then* v_P *also satisfies* (6.32The Tangential Div-curl Systemequation.6.2.32) *for all* $w_P \in \operatorname{Pol}(\Omega)$ *with* $w_r \in V^1_r(\Omega), w_z \in H^1_r(\Omega)$.

Proof. Let $w_P \in \operatorname{Pol}(\Omega)$ with $w_r \in V^1_r(\Omega), w_z \in H^1_r(\Omega)$, and write $w_P = \nabla\phi + \operatorname{curl}(\psi e_\theta) + k$ where

$$\begin{aligned} \nabla\phi &\in \operatorname{Grad}(\Omega), \\ \operatorname{curl}(\psi e_\theta) &\in \operatorname{Curl}_0(\Omega), \\ k &\in \mathcal{H}_{\nu 0}(\Omega). \end{aligned} \tag{6.33}$$

Then $\operatorname{curl}(w_P) = \operatorname{curl}(\operatorname{curl}(\psi e_\theta))$ and $\operatorname{curl}(\psi e_\theta) \in \mathcal{H}_{\nu 0}(\Omega)^\perp$. We need to check that $\operatorname{curl}(\psi e_\theta)_r \in V^1_r(\Omega)$ and $\operatorname{curl}(\psi e_\theta)_z \in H^1_r(\Omega)$. To do this, note that ψ is characterized as the unique weak solution in $V^1_{r,0}(\Omega)$ of

$$\begin{cases} -\Delta\psi + \dfrac{1}{r^2}\psi = \operatorname{curl}(w_P)_\theta \quad \text{in } \Omega, \\ \psi = 0 \quad \text{on } \Gamma. \end{cases} \tag{6.34}$$

$\operatorname{curl}(w_P)_\theta \in L^2_r(\Omega)$ as $w_r \in V^1_r(\Omega), w_z \in H^1_r(\Omega)$ imply

$$\operatorname{curl}(w_P) = \left(\frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r}\right) e_\theta. \tag{6.35}$$

Therefore

$$\operatorname{curl}(\operatorname{curl}(\psi e_\theta)) = \left(-\Delta\psi + \frac{1}{r^2}\psi\right) e_\theta = \operatorname{curl}(w_P) \in L^2_r(\Omega;\mathbb{R}^3). \tag{6.36}$$

Now we reinterpret (6.34The Tangential Div-curl Systemequation.6.2.34) in the Cartesian setting using an axisymmetric lifting Ψ of ψ where

$$\Psi(x) = -\frac{x_2}{r}\psi(r,x_3)e_1 + \frac{x_1}{r}\psi(r,x_3)e_2 \tag{6.37}$$

with $r = \sqrt{x_1^2 + x_2^2}$. The axisymmetric lifting preserves divergence-free fields so $\operatorname{div}(\Psi) = \operatorname{div}(\psi e_\theta) = 0$. Hence $\operatorname{curl}(\operatorname{curl}(\Psi)) = -\Delta_3\Psi$ where Δ_3 is the Laplacian in Cartesian coordinates in $\mathbb{R}^3$, and (6.36The Tangential Div-curl Systemequation.6.2.36) then implies

$$-\Delta_3\Psi = \operatorname{curl}(W_P) \tag{6.38}$$

where W_P is the axisymmetric lifing of w_P. We have $\Psi \in H^1_A(\Omega_A;\mathbb{R}^3) \cap H^1_0(\Omega_A;\mathbb{R}^3)$ according to Remark 2.3.1. $\Delta_3\Psi$ coincides with $(\Delta_3\Psi_1, \Delta_3\Psi_2, \Delta_3\Psi_3)$ in Cartesian coordinates so each Cartesian component $\Psi_j \in H^1_0(\Omega_A)$ is the unique weak solution of the system

$$\begin{cases} \Delta_3\Psi_j = \operatorname{curl}(W_P)_j \quad \text{in } \Omega_A, \\ \Psi_j = 0 \quad \text{on } \partial\Omega_A. \end{cases} \tag{6.39}$$

Then standard elliptic regularity theory asserts that $\Psi \in H^2(\Omega_A;\mathbb{R}^3)$ as $\operatorname{curl}(W_P) \in L^2(\Omega_A;\mathbb{R}^3)$. In particular $\operatorname{curl}(\Psi) \in H^1(\Omega_A;\mathbb{R}^3)$. This implies that $\operatorname{curl}(\psi e_\theta)_r \in V^1_r(\Omega), \operatorname{curl}(\psi e_\theta)_z \in H^1_r(\Omega)$ upon changing back to cylindrical coordinates. Then

$$\begin{aligned} \int_\Omega \operatorname{curl}(v_P)\cdot\operatorname{curl}(w_P)\, r\,dr\,dz &= \int_\Omega \operatorname{curl}(v_P)\cdot\operatorname{curl}(\operatorname{curl}(\psi e_\theta))\, r\,dr\,dz \\ &= \int_\Gamma \sigma_P\cdot\gamma(\operatorname{curl}(\psi e_\theta))\, r\,ds + \int_\Omega \omega_P\cdot\operatorname{curl}(\psi e_\theta)\, r\,dr\,dz \end{aligned} \tag{6.40}$$

since $\operatorname{curl}(\psi e_\theta) \in X_P(\Omega) \cap \mathcal{H}_{\nu 0}(\Omega)^\perp$. For $\nabla\phi, k$, a similar argument appealing to elliptic regularity in the Cartesian case will show that their r, z components are in $V_r^1(\Omega), H_r^1(\Omega)$ respectively. Since $\operatorname{curl}(\nabla\phi) = \operatorname{curl}(k) = 0$, we now apply condition (N8) to obtain

$$\int_\Gamma \sigma_P \cdot \gamma(\nabla\phi + k)\, r\, ds + \int_\Omega \omega_P \cdot (\nabla\phi + k)\, r\, dr\, dz = 0. \tag{6.41}$$

Therefore we may combine (6.40The Tangential Div-curl Systemequation.6.2.40) and (6.41The Tangential Div-curl Systemequation.6.2.41) to get

$$\int_\Omega \operatorname{curl}(v_P) \cdot \operatorname{curl}(w_P)\, r\, dr\, dz = \int_\Gamma \sigma_P \cdot \gamma w_P\, r\, ds + \int_\Omega \omega_P \cdot w_P\, r\, dr\, dz \tag{6.42}$$

which proves the claim. □

Corollary 6.2.4. *Let ω, σ be given such that conditions (N2) – (N3), (N6) – (N8) are satisfied. Then the toroidal-tangential div-curl system has a weak solution in $L_r^2(\Omega; \mathbb{R}^3)$.*

Proof. Take v_P as in the conclusion of Lemma 6.2.2 and note that $\operatorname{curl}(v_P)$ is a toroidal field in $L_r^2(\Omega; \mathbb{R}^3)$. Then Theorem 6.2.3 asserts that $\operatorname{curl}(v_P)$ is a weak solution of the toroidal-tangential div-curl system. □

Lemma 6.2.1 and Corollary 6.2.4 together show that the tangential div-curl system has a solution when ρ, ω, σ are given satisfying conditions (N1) – (N3), (N6) – (N8). Just as the normal div-curl system, the nullspace of the tangential div-curl system depends on the topology of the cross-section Ω. In this case, the nullspace is $\mathcal{H}_{\tau 0}(\Omega)$.

Theorem 6.2.5. *Let ρ, ω, σ be given such that conditions (N1) – (N3), (N6) – (N8) are satisfied. If Γ has a single component $\Gamma = \Gamma_0$ then there is a unique weak solution in $L^2_r(\Omega;\mathbb{R}^3)$ of the tangential div-curl system. If Γ has multiple components $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$, then the set of weak solutions in $L^2_r(\Omega;\mathbb{R}^3)$ of the tangential div-curl system is an m-dimensional affine subspace.*

Proof. This is proved very similarly to the case of the normal div-curl system. A weak solution of the toroidal-tangential div-curl system is unique since the difference of any two weak solutions must be a harmonic toroidal field in $L^2_r(\Omega;\mathbb{R}^3)$, but such field must be zero. The difference of any two weak solutions of the poloidal-tangential div-curl system is a field in $\mathcal{H}_{\tau 0}(\Omega)$. Theorem 5.2.2 asserts that $\mathcal{H}_{\tau 0}(\Omega) = \{0\}$ if $\Gamma = \Gamma_0$, so a weak solution of the poloidal-tangential div-curl system is unique. If $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m$, then Theorem 5.2.2 asserts that $\dim(\mathcal{H}_{\tau 0}(\Omega) = m$, in which case the set of weak solutions is an m-dimensional affine subspace. □

Corollary 6.2.6. *Let ρ, ω, σ be given satisfying conditions (N1) – (N3), (N6) – (N8), and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ with $\Gamma_1, \cdots, \Gamma_m$ all nonempty. Let $\{\nabla\hat{\phi}_j : j = 1, \ldots, m\}$ be a basis for $\mathcal{H}_{\nu 0}(\Omega)$. Then the tangential div-curl system has a unique weak solution if the m functionals*

$$\int_\Omega u \cdot \nabla\hat{\phi}_j \, r \, dr \, dz, \quad j = 1, \ldots, m \tag{6.43}$$

are also prescribed in addition to $\rho, \omega\sigma$ satisfying conditions (N1) – (N3), (N6) – (N8).

Proof. The set of solutions of the tangential div-curl form an m-dimensional affine subspaces isomorphic to $\mathcal{H}_{\nu 0}(\Omega)$, and the prescription of the m functionals in (6.43equation.6.2.43) uniquely determines the projection of a solution onto $\mathcal{H}_{\nu 0}(\Omega)$. $\square$

The prescription of the functionals in (6.43equation.6.2.43) may be interpreted as prescribing m fluxes through each Γ_j of the divergence-free part of the desired vector field. We may derive a similar energy estimate as in the case of the normal div-curl system.

Corollary 6.2.7. *Let ρ, ω, σ be given satisfying conditions (N1) – (N3), (N6) – (N8). Suppose that Γ has multiple components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ with $\Gamma_1, \ldots, \Gamma_m$ all nonempty. Let $\{\nabla\hat{\phi}_j : j = 1, \ldots, m\}$ be a basis for $\mathcal{H}_{\tau 0}(\Omega)$. Let $u \in L^2_r(\Omega; \mathbb{R}^3)$ be a solution of the normal div-curl system with $\eta_j, j = 1, \ldots, m$ the values of the m functionals*

$$\eta_j = \int_\Omega u \cdot \nabla\hat{\phi}_j \, r\,dr\,dz, \quad j = 1, \ldots, m \tag{6.44}$$

and denote $\eta = (\eta_1, \ldots, \eta_m)$. Then there is a constant $C > 0$ such that

$$\int_\Omega |u|^2 \, r\,dr\,dz \leq C\left(\int_\Gamma |\sigma_T|^2 \, r\,ds + \int_\Omega |\rho|^2 \, r\,dr\,dz + \int_\Omega |\omega|^2 \, r\,dr\,dz + |\eta|\right). \tag{6.45}$$

Proof. Let u_P be the poloidal part of u and write $u_P = -\nabla\phi + \operatorname{curl}(\psi e_\theta) + h$ where $\nabla\phi$ is the projection onto $\operatorname{Grad}_0(\Omega), \operatorname{curl}(\psi e_\theta)$ is the projection onto $\operatorname{Curl}(\Omega)$, and h is the projection onto $\mathcal{H}_{\tau 0}(\Omega)$. The characterizations of ϕ, ψ as weak solutions of boundary value problems let us apply Corollary 4.2.2 and Corollary 4.3.8 to derive a constant $C > 0$ such that

$$\int_\Omega |u_P|^2 \, r\,dr\,dz \leq C\left(\int_\Gamma |\sigma_T|^2 \, r\,ds + \int_\Omega |\rho|^2 \, r\,dr\,dz + \int_\Omega |\omega_T|^2 \, r\,dr\,dz + |\eta|\right). \tag{6.46}$$

$\omega_P \in \text{Curl}(\Omega)$ with $\omega_P = \text{curl}(u_\theta e_\theta)$ and $u_\theta \in V_r^1(\Omega)$ by conditions (N2), (N3), and (N6), so we may apply the curl-Poincaré inequality for $V_r^1(\Omega)$ to obtain

$$\int_\Omega |u_\theta e_\theta|^2 \, r \, dr \, dz \le C \int_\Omega |\, \text{curl}(u_\theta e_\theta)|^2 \, r \, dr \, dz = C \int_\Omega |\omega_P|^2 \, r \, dr \, dz \tag{6.47}$$

for some constant $C > 0$. Combining (6.46The Tangential Div-curl Systemequation.6.2.46) and (6.47The Tangential Div-curl Systemequation.6.2.47) yields (6.45equation.6.2.45).

□

The interesting part of (6.45equation.6.2.45) is that the right-hand side is independent of σ_P. Thus the energy of the solution in Ω may be controlled independent of the energy of σ_P on Γ. The reason is that the energy of u_T is completely controlled by the prescribed curl ω_P via the curl-Poincaré inequality.

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