GENERALIZED VIETORIS-BEGLE THEOREMS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Evelyn E. Thornton

May 1973

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ABSTRACT

This dissertation investigates Vietoris-Begle type theorems for sheaf theoretic homology, and explores the possibility of establishing a Vietoris-Begle type theorem for a more general functor which is constructed on a category of inverse systems.

The first part (Chapter 2) of the dissertation is devoted to a brief study of two basic cone constructions, and the almost p-solid condition, both introduced in recent papers of D. G. Bourgin [4], [5]. It is demonstrated that the almost p-solid condition guarantees that certain topological properties are preserved under cone constructions. In the second section of this chapter it is proved that the two cone spaces are homeomorphic. Finally, a so-called generalized mapping cylinder is introduced and it is shown that the cone spaces are homeomorphic to a subspace of this mapping cylinder.

In Chapter 3, Vietoris-Begle type theorems and their inverses are constructed for locally compact spaces using sheaf theoretic homology. Applications are given to Wilder's monotone theorem [11], and to a generalization of the Vietoris-Begle theorem to triple spaces given by Bialynicki-Birula [2].

In the final chapter, a contravariant functor $\hat{H}$ is

constructed on a category of inverse systems. An underlying category $\mathcal{U}$ of topological pairs is proved to be admissible for a cohomology theory in the sense of Eilenberg and Steenrod [10]. It is shown that Vietoris-Begle type maps are admissible for the category $\mathcal{U}$. However, it does not appear that a Vietoris-Begle theorem of any generality can be exhibited for the original functor $\hat{H}$. For this reason the conditions on the construction of $\hat{H}$ are relaxed and Vietoris-Begle type theorems are proved.

CONTENTS

CHAPTER I

INTRODUCTION

In various discussions concerning mapping problems the fibers or point inverses are of fundamental importance. In 1927, Vietoris [12] invented a homology theory for compact metric spaces. Through this endeavor, he was able to prove that a continuous map f of X onto Y, whose fibers have vanishing Vietoris homology groups in all dimensions $\leq n$, induces homomorphisms f_i mapping $H_i(X)$ isomorphically onto $H_i(Y)$ in all dimensions $i \leq n$. Begle [1] generalized this result, in 1950, to compact Hausdorff spaces, and it is presently referred to as the Vietoris-Begle theorem. A survey of the literature makes it evident that the theorem can be obtained for more general spaces and omology theories.

Using compact spaces and cohomology theory, Bourgin [4] gave a striking generalization which has led to a variety of results involving mappings. He relaxed the conditions on the fibers by insisting that the requirement of vanishing need not encompass the zero dimensional groups. In [7] he extended the results of [4] to paracompact spaces using sheaf-theoretic cohomology. His methods involved the use of a certain cone construction, and in this connection he introduced an important notion which he referred to as the almost p-solid condition.

This dissertation investigates Vietoris-Begle type theorems for sheaf theoretic homology, and explores the possibility of establishing a Vietoris-Begle type theorem for a more general functor which is constructed on a category of inverse systems. The cone constructions and the almost p-solid condition will be used throughout.

The first part (Chapter 2) of the dissertation is devoted to a brief study of two basic cone constructions and the almost p-solid condition. It is demonstrated that the almost p-solid condition guarantees that certain topological properties are preserved under cone constructions. In the second section of this chapter it is proved that the two cone spaces are homeomorphic. Finally, a so-called generalized mapping cylinder is introduced and it is shown that the cone spaces are homeomorphic to a subspace of this mapping cylinder.

In Chapter 3, Vietoris-Begle type theorems and their inverses are constructed for locally compact spaces using sheaf theoretic homology. Applications are given to Wilder's monotone theorem [11], and to a generalization of the Vietoris-Begle theorem to triple spaces given by Bialynicki-Birula [2].

In the final chapter, a contravariant functor $\hat{H}$ is constructed on a category of inverse systems. An underlying category of topological pairs is proved to be admissible for a cohomology theory in the sense of Eilenberg and Steenrod [10]. It is shown that Vietoris-Begle type maps are admissible for the category. However, it does not appear that a Vietoris-Begle theorem of any generality can be exhibited for the original functor $\hat{H}$. For this reason the conditions on the construction of $\hat{H}$ are relaxed and Vietoris-Begle type theorems are proved.

CHAPTER II

THE ALMOST P-SOLID CONDITION

Definition 2.0 [4]. Let $f:X \to Y$ be a continuous surjection of topological spaces, and denote by $X(y)$ the set $f^{-1}(y)$ and by S_p a subset of Y called the singular set. The statement that the pair (X,f) is almost p-solid, written A_pS, means that for each x in X and neighborhood $N(x)$, there is a neighborhood $N'(x) \subset N(x)$ such that all but finitely many of the $X(y)$'s that meet $N'(x)$ are contained in $N(x)$ for y in S_p. That is, the set

$$\rho = \{y \in Y | X(y) \cap N'(x) \neq \phi \text{ and } X(y) \cap N^{\sim}(x) \neq \phi\} \cap S_p$$

is finite. If $S_p = Y$ we say that (X,f) is largely sectioned. In all applications,

$$S_p = \{y \in Y | X(y) \text{ is not acyclic in some dimension} \leq p\}.$$

Theorem 2.1 Suppose (X,f) is almost p-solid. Then if $x \in X$ and $N(x)$ is a neighborhood of x, there is an open set $\hat{N}$ such that $x \in \hat{N} \subset N$ and $\hat{N}$ contains all but finitely many of the $X(y)$'s that meet it.

Proof. Let $x \in X$ and $N(x)$ be given. By the almost p-solid condition there is an $N'(x) \subset N(x)$ such that the set

$$\rho = \{y | X(y) \cap N'(x) \neq \phi \text{ and } X(y) \cap N^{\sim}(x) \neq \phi\} \cap S_p$$

is finite. Define

$$Y_x = \{y \in S_p | X(y) \cap N \neq \phi,\ X(y) \cap N^{\sim} \neq \phi \text{ and } X(y) \cap N' = \phi\}$$

and let

$$X(Y_x) = \cup \{X(y) | y \in Y_x\}.$$

Claim: $X(Y_x) \cup \partial N$ is closed.

Suppose p is a limit-point of $X(Y_x)$ and $p \,\bar{\varepsilon}\, \partial N$, then $p \in X(y)$ for some $y \in Y_x$. For if not, $p \in N$ or $p \in N^{\sim}$. If $p \in N$, then for each neighborhood M of p, contained in N, there is a non-finite set H contained in Y_x such that if $y \in H$, $M(p) \cap X(y) \neq \phi$ and $X(y) \cap N^{\sim} \neq \phi$. This implies that M does not satisfy the A_pS condition, a contradiction.

If $p \in X - (N \cup \partial N)$, then any neighborhood $W(p)$ contained in $X - (N \cup \partial N)$ fails to satisfy the A_pS condition. Therefore all limit points of $X(Y_x)$ which are not in $X(Y_x)$ lie on the boundary, ∂N, of N. This proves the claim.

Define

$$\hat{N} = \{X - [X(Y_x) \cup \partial N]\} \cap N.$$

Clearly, $\hat{N}$ is an open set containing x. Moreover, $\hat{N}$ contains all but finitely many of the $X(y)$'s that meet it.

Corollary 2.2. If X is a Hausdorff space and (X,f) is A_pS, then for each pair of distinct points x_1 and x_2 in X there exists disjoint open sets U and V containing x_1 and x_2

respectively, each of which contains all but finitely many of the $X(y)$'s that meet it.

In case X is a metric space the A_pS condition can be interpreted as follows:

> For each $x \varepsilon X$ and positive number r, there is a number r_x such that $0 < r_x < r$, and a finite subset $Y_{r_x} \subset S_p$ with the property that if $y \bar{\varepsilon} Y_{r_x}$ and $d(x, X(y)) < r_x$ then diam $X(y) < r$.

<u>Theorem 2.3</u>. If X is a separable metric space, and (X,f) is A_pS, then S_p is countable.

<u>Proof</u>: Let n be a positive integer, $x \varepsilon X$ and $N(x;r_n(x))$ a neighborhood of x such that $r_n(x) \leq \frac{1}{n}$. Let $\hat{N}(x) \subset N(x;r_n(x))$ be the open set given by theorem 2.1 and observe that if $x^1 \varepsilon \hat{N}(x)$ then $d(x^1,x) < r_n(x)$.

For each n, let $\alpha_n^1 = \{\hat{N}(x)\}_{x\varepsilon X}$ be an open cover of X. Since X is separable there is for each n a countable subset $H_n \subset X$ such that $\alpha_n = \{\hat{N}(x)\}_{x\varepsilon H_n}$ covers X.

If $x \varepsilon H_n$ there is a finite subset $Y_n(x) \subset S_p$ such that if $y \varepsilon S_p$, $y \bar{\varepsilon} Y_n(x)$ and $d(x,X(y)) < r_n(x)$, then diam $X(y) < \frac{1}{n}$. Now $Y_n = \bigcup_{x\varepsilon H_n} Y_n(x)$ and $Y_1 = \bigcup_n Y_n$ are countable sets, and furthermore $Y_1 \subset S_p$.

We need only show that $S_p \subset Y$ to complete the proof. Suppose $y_0 \varepsilon S_p$ and $y_0 \bar{\varepsilon} Y_1$, then $y_0 \bar{\varepsilon} Y_n$ for any n. For each n let $x_n \varepsilon H_n$ then $y_0 \bar{\varepsilon} Y_n(x_n)$ and $d(x_n,X(y_0)) < \frac{1}{n}$. But the diam $(X(y_0)) < \frac{1}{n}$ for each n gives diam $X(y_0) = 0$. Either

$X(y_0) = \phi$ or $X(y_0)$ is a single point, but both yield a contradiction.

Remark: Theorem 2.3 generalizes a theorem of Bourgin [7] given for compact metric spaces.

HOMEOMORPHIC CONES

In this section it is shown that the two cone spaces introduced in recent papers of D. G. Bourgin [4], [5] are homeomorphic. We review the construction here.

Let $f:X\to Y$ be a continuous surjection between completely regular Hausdorff spaces. The following nomenclature will be used:

1) I_y is the unit interval
2) $\Pi = \prod_{y\varepsilon Y} I_y$ is the Tychonoff parallelotope whose points are the functions $\psi:Y\to I$, with the usual Tychonoff topology.
3) $1_y \varepsilon \Pi$ is the function defined by $1_y(y) = 1$, $1_y(y^1)=0$ for $y \neq y^1$.
4) $J(y) = \{s1_y | s \varepsilon [0,1]\}$
5) $B(y) = X(y) \times J(y) \subset X \times \Pi$
6) $B = \bigcup_{y\varepsilon Y} B(y) \subset X \times \Pi$.

<u>Definition of X^*</u> [4]. In each $B(y)$ identify $X(y) \times 1_y$ to a point and denote this point by x^*_y. The resulting cone is denoted by $X^*(y)$ and the total space by X^* where $X^* = \bigcup_{y\varepsilon Y} X^*(y)$. X^* is given the identification topology under the projection $g:B\to X^*$. If $S \subset Y$, substitute S for Y throughout the notation. Then

$$X^*(S) = \bigcup_{y\varepsilon S} X^*(y) \cup \bigcup_{y\bar{\varepsilon} S} X(y) \times *$$

where $*$ is the zero function.

<u>Definition of ${}_1X^*$</u> [5]. Since X is completely regular it can be imbedded in $\Pi_\Lambda I_\lambda$ and this imbedded image is denoted by ${}_1X$. A cone ${}_1X^*(y)$ over ${}_1X(y)$ is constructed by choosing an arbitrary point $x_0(y)$ in ${}_1X(y)$ and then taking the join of $x_0(y) \times 1_Y$ and ${}_1X(y)$ in $\Pi_\Lambda \times \Pi_Y$. Then ${}_1X^* = \bigcup_{y \varepsilon Y} {}_1X^*(y)$.

If $S \subset Y$, then ${}_1X^*(S) \subset \Pi_\Lambda \times \Pi_S$ and

$${}_1X^*(S) = \bigcup_{y \varepsilon S} {}_1X^*(y) \cup \bigcup_{y \bar{\varepsilon} S} {}_1X(y) \times *.$$

<u>Definition 2.4</u>. Let X and Y be topological spaces, M a set and F_M a family of pairs (X_m, f_m) satisfying the following properties:

1) $X_m \subset X$ and $f_m : X_m \longrightarrow Y$ is continuous for each m
2) $\cup X_m = X$ and $\cup f(X_m) = Y$
3) if $x \bar{\varepsilon} \cap X_m$, then $x \varepsilon X_m$ for exactly one $m \varepsilon M$

The family F_m is open if and only if for each m, X_m is open in X and f_m is an open map. F_m is injective, if for each m, f_m is injective and if $m \neq n$, $f_m(x) = f_n(y)$ implies $x = y \varepsilon \bigcap_M X_m$.

<u>Lemma 2.5</u>. Let F_M be an open family and suppose that $f_m(x) = f_n(x)$ for all $x \varepsilon \bigcap_m X_m$. Then

a) there is a unique continuous open surjection F extending each f_m, and
b) if F_M is injective then F is a homeomorphism.

Proof: Define $F:X \to Y$ by $F(x) = f_m(x)$. Clearly F extends each f_m and is unique. To show continuity, let U be open in Y and let $U_m = U \cap f_m(X_m)$ for each $m \in M$. U_m is open in Y and

$$\bigcup_m U_m = \bigcup_m U \cap f_m(X_m) = U \cap \bigcup_m f(X_m) = U \cap Y = U$$

and

$$F^{-1}(U) = F^{-1}(\cup U_m) = \cup F^{-1}(U_m) = \cup f_m^{-1}(U_m)$$

is open.

Let V be open in X then $V_m = V \cap X_m$ is open in X and $\cup V_m = \bigcup_m V \cap X_m = V \cap X = V$. Now

$$F(V) = F(\cup V_m) = \cup F(V_m) = \cup f_m(V_m)$$

is open. Hence F is a continuous open surjection, and b) follows from the definition of F_M injective.

Lemma 2.6. For each $x(y) \in X(y)$ there is a continuous bijection $\sigma_{x(y)}:I \to L_{x(y)} \subset X^*(y)$

Proof: The set $X(y) \times 1_y \cup x(y) \times J(y)$ is closed in B and the set $x^*(y) \cup \{[x(y),s1_y] | s \neq 1\}$ is closed in $X^*(y)$, where $x^*(y)$ is the vertex of $X^*(y)$.

Let $L_{x(y)} = x^*(y) \cup \{[x(y),s1_y] | s \neq 1\}$ and consider the following maps:

$$I \xrightarrow{F} X(y) \times 1_y \cup x(y) \times J(y) \xrightarrow{g_1} L_{x(y)}$$

where F is a set valued map defined by

$$F(t) = \begin{cases} (x(y), t1_y) & \text{if } t \neq 1 \\ X(y) \times 1_y & \text{if } t = 1 \end{cases}$$

and g_1 is the map $g:B \to X^*$ restricted to $X(y) \times 1_y \cup x(y) \times J(y)$.

F is upper semicontinuous. To see this, let C be closed in $X(y) \times 1_y \cup x(y) \times J(y)$. Then $C = A \cup B$ where A is closed in $X(y) \times 1_y$ and B is closed in $x(y) \times J(y)$.

$$\begin{aligned} F^{-1}(C) &= \{t | F(t) \cap C \neq \phi\} \\ &= \{t | F(t) \cap A \neq \phi\} \cup \{t | F(t) \cap B \neq \phi\} \\ &= 1 \cup F^{-1}(B) \end{aligned}$$

hence $F^{-1}(C)$ is closed.

Define $\sigma_{x(y)} = g_1 F: I \to L_{x(y)} \subset X^*(y)$. If C is closed in $L_{x(y)}$, $g_1^{-1}(C)$ is closed in $X(y) \times 1_y \cup x(y) \times J(y)$ and $F^{-1}g_1^{-1}(C)$ is closed in F. Therefore $\sigma_{x(y)}$ is continuous. Since $F(1) = X(y) \times 1_y$ and $gF(1) = x^*(y)$, $\sigma_{x(y)} = Fg$ is bijective.

Let ${}_1x^*(y)$ denote the vertex of ${}_1X^*(y)$ where, ${}_1x^*(y) = {}_1x'(y) \times 1_y$ for some fixed ${}_1x'(y) \in {}_1X(y)$. Also let $[{}_1x^*(y) : {}_1x(y)]$ denote the linear path in ${}_1X^*(y)$ from the vertex ${}_1x^*(y)$ to the point ${}_1x(y) \in {}_1X(y)$. That is,

$$[{}_1x^*(y) : {}_1x(y)] = \{t\,{}_1x^*(y) + (1-t)\,{}_1x(y) \times * \,|\, t \in [0,1]\}.$$

Then for each ${}_1x(y) \in {}_1X(y)$ there is a continuous bijection

$\tau_{{}_1x(y)}: I \to [{}_1x^*(y) : {}_1x(y)]$ where

$\tau_{{}_1x(y)}(t) = (t\,{}_1x^*(y) + (1-t)\,{}_1x(y),\ t1_y)$

<u>Lemma 2.7</u>. For each $x(y) \in X(y)$ there is a continuous bijection

$$h_{x(y)}: L_{x(y)} \to [{}_1x^*(y) : {}_1x(y)]$$

<u>Proof</u>: Let $j: X \to {}_1X$ be the homeomorphism gotten from the imbedding. Define

$$h_{x(y)} = \tau_{j(x(y))}\sigma^{-1}{}_{x(y)} = \tau_{{}_1x(y)}\sigma^{-1}_{x(y)}.$$

Actually $h_{x(y)}$ is a homeomorphism since $L_{x(y)}$ is compact and $[{}_1x^*(y) : {}_1x(y)]$ is Hausdorff. Now $L'_{x(y)} = (L_{x(y)} - x(y) \times *)$ is open in $X^*(y) - X(y)$, hence

$\mathcal{F}_{X(y)} = \{(L'_{x(y)}, h'_{x(y)})\}$ is an open injective family of pairs where

$\cup L'_{x(y)} = X^*(y) - X(y)$, $\cup h'_{x(y)}(L'_{x(y)}) = {}_1X^*(y) - {}_1X(y)$,

$h'_{x(y)} = h_{x(y)}|_{L'_{x(y)}}$ and $\cap L'_{x(y)} = \{x^*(y)\}$

hence the combined function

$$h_y: X^*(y) - X(y) \to {}_1X^*(y) - {}_1X(y)$$

is a homeomorphism.

<u>Theorem 2.8</u>. X^* and ${}_1X^*$ are homeomorphic.

Proof of Theorem 2.8. Define $h^* : X^* \to {}_1X^*$ in the following way: $h^*[x(y), s1_y] = s\,{}_1x^*(y) + (1-s)\,{}_1x(y) \times *$ clearly h^* is bijective and $h^*|_{X^*(y) - X(y)} = h_y$.

The family $F_Y = \{(X^*(y) - X(y),\ h_y)\}$ is open and injective, hence the combined function

$$h: \bigcup_{y \in Y} (X^*(y)) - X(y)) \longrightarrow \bigcup_{y \in Y} {}_1X^*(y) - {}_1X(y)$$

is a homeomorphism and $h^*|_{X^* - X} = h$.

Since $h: X^* - X \to {}_1X^* - {}_1X$ is a homeomorphism

$j: X \to {}_1X$ is homeomorphism

and $h^*|X = j$ and $h^*|X^* - X = h$

it follows that h^* is a homeomorphism.

A GENERALIZED MAPPING CYLINDER

In this section a mapping cylinder is constructed, and it is shown that the cone spaces of the previous section are homeomorphic to a subspace of this cylinder.

Lemma 2.9. If X and K are topological spaces and K is compact, then $p\colon X \times K \to X$ defined by $p(x,k) = x$ is a closed map.

Proof: Suppose F is a closed set in $X \times K$ and x_0 is a limit point of $p(F)$. Then there is a sequence of distinct points $\{x_n\}$ contained in $p(F)$ such that $x_n \to x_0$.

Let $f = p^{-1}:X \to X \times K$. As an inverse of a continuous map, f is an upper semicontinuous set valued map. That is, if $z_n \to z_0$ in X and $u_n \to u_0$ in $X \times K$ such that $u_n \in f(z_n)$, for each n, then $u_0 \in f(z_0)$. Therefore $p(u_n) \longrightarrow p(u_0) = z_0$. Hence $u_0 \in p^{-1}(z_0)$.

For each n choose $k_n \in K$ so that $(x_n,k_n) \in F$ then the sequence $\{y_n\} = \{(x_n,k_n)\}$ has the property that for each n

$$y_n \in p^{-1}(x_n) \cap F.$$

There is a subsequence $\{k_{n_j}\}$ of $\{k_n\}$ and a point $k_0 \in K$ such that $k_{n_j} \to k_0$. Now the sequence $\{y_{n_j}\} = \{(x_{n_j}, k_{n_j})\}$ converges to $(x_0,k_0) = y_0$ and by upper semi-continuity of p^{-1} we have that $y_0 \in p^{-1}(x_0)$, implying y_0 is a limit point of F and hence is in F. But $p(y_0) = p(x_0, k_0) = x_0 \varepsilon p(F)$ and $P(F)$ is closed.

<u>Definition 2.10</u> [8]. If ϕ: $X \to Y$ is an identification then $A \subset X$ is ϕ-saturated if it is the complete inverse image of some set in Y. The ϕ-load of A is the ϕ-saturated set $\phi^{-1}(\phi(A))$, containing A.

<u>Lemma 2.11</u>. If $\phi: X \to Y$ is an identification, then

1) $A \subset X$ is ϕ saturated iff $A = \phi^{-1}\phi(A)$
2) ϕ is open (closed) if the ϕ-load of each open (closed) set in X is open (closed)
3) If A and A´ are disjoint ϕ-saturated sets in X then $\phi(A)$ and $\phi(A')$ are disjoint.

Let $J(Y) = \bigcup_{y \varepsilon Y} J(y) \subset \Pi$ where $J(y)$ and Π are as defined previously on page 8.

<u>Lemma 2.12</u>. $J(Y)$ is a compact subset of Π.

<u>Proof</u>: Let $\psi \varepsilon \Pi$ and suppose $\psi \bar{\varepsilon} J(Y)$. Since every function in $J(Y)$ is zero except at a single point, there must exist at least two distinct points y_1 and y_2 in Y such that $\psi(y_1)$ and $\psi(y_2)$ are nonzero. Choose a number ε so that $0 < \varepsilon < \min\{\psi(y_1), \psi(y_2)\}$ and define

$$N(\psi) = \{\phi \varepsilon \Pi \big| \; |\phi(y_1) - \psi(y_1)| < \varepsilon \text{ and } |\phi(y_2) - \psi(y_2)| < \varepsilon\}.$$

N is open in Π containing ψ and $N(\psi) \cap J(Y) = \phi$, therefore $J(Y)$ is a closed subset of Π and hence compact.

<u>Definition 2.13</u>. Let $f:X \to Y$ be a continuous function from the topological space X to the topological space Y, and let $X \times J(Y) \cup Y \times I$ be the disjoint topological union of $X \times J(Y)$ and $Y \times I$. Then the quotient space

$$\frac{X \times J(Y) \cup Y \times I}{(x(y),1_y) \sim (f(x),0)}$$

will be denoted by $Z_f[J(Y)]$ and called a generalized mapping cylinder of f.

<u>Definition 2.14</u>. Let ϕ: $X \times J(Y) \cup Y \times I \to Z_f[J(Y)]$ be the natural projection defined in the following way:

$$\phi(x,s1_y) = [(x,s1_y)]$$

$$\phi(y,t) = [(y,t)]$$

$$\text{where } [x,s1_y] = \begin{cases} (x,s1_y) & \text{if } f(x) \neq y \text{ and } s \in [0,1] \\ & \text{or if } f(x) = y \text{ and } s \neq 1 \\ (f(x),0) & \text{if } f(x) = y \text{ and } s = 1. \end{cases}$$

Then Z_f $[J(Y)]$ is given the identification topology under ϕ.

<u>Lemma 2.15</u>. Let $f:X\to Y$ be a continuous map between Hausdorff spaces such that (X,f) is largely sectioned. Then the map

$$\phi: X \times J(Y) \cup Y \times I \to Z_f[J(Y)]$$

is closed, if f is surjective.

<u>Proof</u>. Let F be closed in $X \times J(Y) \cup Y \times I$, then $F = A \cup B$

where A is closed in $X \times J(Y)$ and B is closed in $Y \times I$. Since ϕ is the identity on $Y \times I$ we have

1) $\phi^{-1}\phi(F) = \phi^{-1}\phi(A) \cup B$

2) $\phi^{-1}(\phi(A)) = A \cup A_1 \cup Y_1$

where

$Y_1 = \{(y,0) | (x(y),1_y) \varepsilon A\}$ and

$A_1 = \{(x(y),1_y) | y \varepsilon Y_1 \text{ and } (x(y),1_y) \bar{\varepsilon} A\}$

We show first that Y_1 is closed. Suppose $(y_0,0)$ is a limit point of Y_1 which does not belong to Y_1 then there is an infinite sequence of points $(y_n,0)$ contained in Y_1 such that $(y_n,0)$ converges to $(y_0,0)$. Since ϕ is continuous the sequence $\phi(y_n,0) = [x(y_n),1_{y_n}]$ converges to $\phi(y_0,0) = [x(y_0),1_{y_0}]$. Let V be open in Y containing y_0. Then $f^{-1}(V) = U$ is open in X containing $X(y_0)$ and containing every $X(y)$ that meets it. Choose $N(1_{y_0}) = \{\psi | \psi(y_0) > \frac{1}{2}\}$ Then $O = (U \times N) \cap J(Y)$ is an open ϕ-saturated set, and $\phi(O)$ contains no point of the sequence $\phi(y_n,0)$. This involves a contradiction, hence Y_1 is closed.

Suppose $z = (x,s1_y)$ is a limit point of $A \cup A_1$ which does not belong to A. Since A is closed there is an open set U containing z such that $A \cap U$ is empty. But U contains infinitely many points of A_1 consequently, there is an infinite sequence $\{(x_n(y),1_y)\} \subset A_1$ and converging to the point z. If the corresponding subset Y_1' of Y_1 is finite then almost all the points of the sequence lie on a single set $X(y) \times 1_y$. Since $X(y) \times 1_y$ is closed z belongs to A_1.

Suppose Y_1' is infinite, then there are open sets $W(x)$ containing x and $N(sl_y)$ containing sl_y such that $U \supset (W(x) \times N(sl_y)) \cap J$. By the A_pS condition, $W^1(x)$ can be chosen so that $W^1(x)$ contains all but finitely many of the $X_n(y)$'s. Consequently, $(W^1(x) \times N(sl_y)) \cap J$ contains infinitely many points of A. This involves a contradiction, hence $A \cup A_1$ is closed.

We have shown that the ϕ-load of F is closed, therefore $\phi(F)$ is closed.

Theorem 2.16. If $f:X \to Y$ is a continuous surjection between Hausdorff spaces and (X,f) is largely sectioned, then $Z_f[J(Y)]$ is Hausdorff.

Proof: Let $z_1 = [(x_1, sl_{y_1})]$ and $z_2 = [(x_2, tl_{y_2}]$ be distinct points of $Z_f[J(Y)]$. Suppose $x_1 \neq x_2$ and $x_1, x_2 \in X(\bar{y})$ for some $\bar{y} \in Y$.

If $y_1 = y_2 = \bar{y}$, then s and t cannot both be 1.

a) Suppose $s \neq 1$ and $t \neq 1$. Let δ be a positive number such that $\delta + \max\{s,t\} < 1$. Since $x_1 \neq x_2$, there are open sets U_1 and U_2 containing x_1 and x_2 respectively such that each contains all but finitely many of the $X(y)$'s that meet it. Let ρ_1 and ρ_2 be the finite subsets of Y such that if $y \bar{\in} \rho_i$ and $X(y) \cap U_i \neq \phi$ then $X(y) \subset U_i$, $i=1,2$. Choose

$N = \{\psi | \psi(y) < \delta + \max\{s,t\},\ y \in \rho = \rho_1 \cup \rho_2\}$.

Now N is open in Π and contains sl_{y_1} and tl_{y_2}. The sets

$A = (U_1 \times N) \cap J$ and $B = (U_2 \times N) \cap J$ are disjoint open ϕ-saturated sets in $X \times J \cup Y \times I$ containing $(x_1, s1_{y_1})$ and $(x_1, t1_{y_1})$ respectively. Hence $\phi(A)$ and $\phi(B)$ are disjoint open sets containing z_1 and z_2 respectively.

b) If $s \neq 1, t=1$. Let $\varepsilon = d(s,1)$ and let U be an open set containing $\bar{y}$, then $V = f^{-1}(U)$ is open in X containing x_1 and x_2. Also V contains every $X(y)$ that meets it. Choose

$N_1(s1_{y_1}) = \{\psi | \psi(\bar{y}) < s + 1/4\ \varepsilon\}$ and

$N_2(1_{y_2}) = \{\psi | \psi(\bar{y}) > s + 3/4\ \varepsilon\}$

Then $A = (V \times N_1) \cap J$ and $B = (V \times N_2) \cap J$ are disjoint open ϕ-saturated sets in $X \times J \cup Y \times I$ so $\phi(A)$ and $\phi(B)$ are disjoint open sets, containing z_1 and z_2 respectively.

Suppose $y_1 \neq y_2$. Let V be as in b) above, and consider the crucial case when $s = t = 1$. Choose

$N_1(1_{y_1}) = \{\psi | \psi(y_1) > \frac{1}{2}\}$

and

$N_2(1_{y_2}) = \{\psi | \psi(y_2) > \frac{1}{2}\}$

then $A = (V \times N_1) \cap J$ and $B = (V \times N_2) \cap J$ are disjoint ϕ-saturated open sets in $X \times J \cup Y \times I$.

If $y_1 = y_2$, $s = t = 1$ and $y_1 \neq \bar{y}$. Let U_1, U_2 and ρ be as in a). If $y_1 \in \rho$ there exists open sets $\hat{U}_1$ and $\hat{U}_2$ such that $\hat{U}_1 \subset U_1$, $\hat{U}_2 \subset U_2$ and $\hat{U}_i \cap X(y_1) = \phi$, i=1,2.

Choose $N(1_{y_1}) = \{\psi | \psi(y_1) > \frac{1}{2}\}$ then $(\hat{U}_1 \times N) \cap J$ and $(\hat{U}_2 \times N) \cap J$ are disjoint open ϕ-saturated sets in $X \times J \cup Y \times I$.

Theorem 2.17. X^* is homeomorphic to a subspace of $Z_f[J(Y)]$.

Proof. Since $B = \bigcup_{y \in Y} X(y) \times J(y)$ is a subset of $X \times J(Y)$.

Consider the following diagram

$$\begin{array}{ccc} B & \xrightarrow{j} & X \times J(Y) \cup Y \times I \\ g\downarrow & & \downarrow\phi \\ X^* & \xrightarrow{h} & Z_f[J(Y)] \end{array}$$

where j is the inclusion map and h is defined by $h([x(y), s1_y]) = [x(y), s1_y]$. Clearly h is an open continuous injection.

CHAPTER III

VIETORIS-BEGLE TYPE THEOREMS FOR HOMOLOGY WITH COEFFICIENTS IN A SHEAF

Throughout this chapter the Borel-Moore homology theory for locally compact Hausdorff spaces will be used. Coefficients are taken in a sheaf, and support families are paracompactifying. All sheaves are assumed to be sheaves of L-modules, where the ground ring L is a principle ideal domain. Reference [8] is cited for convenience throughout the chapter but is almost a direct transcript of the fundamental paper of Borel and Moore.

<u>Lemma 3.0</u> [8;189]. Let $f:X\to Y$, $\mathcal{L}$ a sheaf on X and $\mathcal{A}$ a sheaf on Y, then the natural homomorphism

$$f_c(\mathcal{L}) \otimes \mathcal{A} \to f_c(\mathcal{L} \otimes f^*\mathcal{A})$$

is an isomorphism when $\mathcal{L}$ is c-soft and torsion free or when $\mathcal{A}$ is torsion free.

<u>Lemma 3.1</u>. Let $\mathcal{K}^*$ be an injective differential sheaf on X such that

$$0 \to \mathcal{L} \overset{\varepsilon}{\to} \mathcal{K}^0 \to \mathcal{K}^1 \to \ldots \to \mathcal{K}^n \to \ldots$$

is exact at $\mathcal{K}^i$, $i=0,1,\ldots,n$. Then there is an injective resolution $\mathcal{L}^*$ of L on X and a homomorphism $h: \mathcal{L}^* \to \mathcal{K}^*$ such that $h^i: \mathcal{L}^i \to \mathcal{K}^i$ is an isomorphism for $i \leq n+1$. (L - principle idea domain.)

<u>Proof</u>: 1) $0 \to L \to \mathcal{K}^0 \to \mathcal{K}^1 \to .. \to \mathcal{K}^n \to d^n\mathcal{K}^n \to 0$

is exact. K^{n+1} is injective and $d^nK^n \subset K^{n+1}$, this together with the fact that every sheaf on X is a subsheaf of some injective sheaf yields the following collection of exact sequences, where L^{n+i} is injective for each i.

$$
\begin{array}{llllll}
2) & 0 \to d^nK^n & \xrightarrow{\varepsilon_1} & K^{n+1} & \xrightarrow{n_1} & K^{n+1}/Im\varepsilon_1 \to 0 \\
 & 0 \to K^{n+1}/Im\varepsilon_1 & \xrightarrow{\varepsilon_2} & L^{n+2} & \xrightarrow{n_2} & L^{n+2}/Im\varepsilon_2 \to 0 \\
 & 0 \to L^{n+2}/Im\varepsilon_2 & \xrightarrow{\varepsilon_3} & L^{n+3} & \xrightarrow{n_3} & L^{n+3}/Im\varepsilon_3 \to 0 \\
 & \quad\vdots & & \vdots & & \vdots \\
 & 0 \to L^{n+i}/Im\varepsilon_i & \xrightarrow{\varepsilon_{i+1}} & L^{n+i+1} & \xrightarrow{n_{i+1}} & L^{n+i+1}/Im\varepsilon_{i+1} \to 0 \\
 & \quad\vdots & & \vdots & & \vdots
\end{array}
$$

3) Splicing the sequences in 1) and 2) together we have the desired resolution L^* of L on X where $L^i = K^i$ for $i \leq n+1$. Let $h: L^* \to K^*$ be the homomorphism lifting the identity on L. Clearly $h^i: L^i \to K^i$ is an isomorphism for $i \leq n+1$.

<u>Theorem 3.2</u>. Let $f: X \to Y$ be a proper surjection between locally compact spaces and suppose that each $X(y)$, $y \in Y$, is connected and $H^p(X(y);L) = 0$ for $0 < p \leq n$. Then

$$f_*: H_p^{f^{-1}\phi}(X;f^*\mathcal{B}) \to H_p^{\phi}(Y;\mathcal{B})$$

is an isomorphism for $p < n$ and an epimorphism for $p=n$, when $\mathcal{B}$ is any sheaf on Y and ϕ is a paracompactifying family of supports on Y.

<u>Proof</u>: Let $\mathcal{I}^* = \mathcal{I}^*(X,L)$ and $\mathcal{L}^* = \mathcal{L}^*(Y,L)$ denote the canonical injective resolutions of L on X and Y respectively. f_* is induced by the canonical f-cohomorphism $k:\mathcal{L}^* \to \mathcal{I}^*$ which factors uniquely according to the following diagram

1)
$$\begin{tikzcd} & f\mathcal{I}^* \arrow[dr, "f"] & \\ \mathcal{L}^* \arrow[ur, dashed, "h"] \arrow[rr, "k"] & & \mathcal{I}^* \end{tikzcd}$$

where h is unique depending on $\mathcal{L}^*$.

Let U be open in Y, and let c denote the family of compact supports. $f^{-1}(c) = c$, since f is proper, and

$$\begin{aligned} \mathcal{D}(f\mathcal{I}^*)\ (U) &= \mathcal{H}om(\Gamma_c f\mathcal{I}^*;L^*)(U) \\ &= \mathrm{Hom}(\Gamma_{c(U)} f\mathcal{I}^*(U);L^*) \\ &= \mathrm{Hom}(\Gamma_{f^{-1}(c(U))} \mathcal{I}^*(f^{-1}(U));L^*) \\ &= \mathcal{H}om(\Gamma_{f^{-1}c(U)} \mathcal{I}^*;L^*)(f^{-1}(U)) \\ &= f\mathcal{H}om(\Gamma_c \mathcal{I}^*;L^*)(U) \\ &= f\mathcal{D}(\mathcal{I}^*)(U) \end{aligned}$$

hence we have that

$$\mathcal{D}(f\mathcal{I}^*) = f\mathcal{D}(\mathcal{I}^*).$$

fI^* is an injective differential sheaf therefore $\mathcal{D}(fI^*)$ is flabby and torsion free and by lemma 3.0

$$\mathcal{D}(fI^*) \otimes \mathcal{B} \xleftarrow{\tilde{=}} f[\mathcal{D}(I^*) \otimes f^*\mathcal{B}] = fC_*(X,f^*\mathcal{B}) .$$

Applying the section functor we have

$$\Gamma_\phi \mathcal{D}(fI^*) \otimes \mathcal{B} \approx \Gamma_\phi f C_*(X,f^*\mathcal{B}) \approx \Gamma_{f^{-1}\phi} C_*(X,f^*\mathcal{B})$$

and finally

2) $H_p^\phi(\mathcal{D}(fI^*) \otimes \mathcal{B}) \xleftarrow{\tilde{=}} H_p^{f^{-1}\phi}(X;f^*\mathcal{B})$, for all p.

Now f_* is induced by way of 2) from the homomorphism h in diagram 1). Note that the derived sheaf $\mathcal{H}^*(fI^*)$, of the injective differential sheaf fI^*, is the Leray sheaf of the map f with coefficients in L. The fact that f is proper and the spaces are locally compact guarantees that X(y) is taut, for $y \in Y$. So that if $y \in Y$ and $0 < p \leq n$

$$\mathcal{H}^p(fI^*)_y \approx \varinjlim_{U \supset y} H^p(f^{-1}(U);L) \approx H^p(f^{-1}(y);L) = 0,$$

therefore

3) $\qquad \mathcal{H}^p(fI^*) = 0, \qquad\qquad 0 < p \leq n.$

Since $f^{-1}(y)$ is connected and compact

$$\mathcal{H}^0(fI^*)_y \approx H^0(f^{-1}(y);L) \tilde{=} L$$

and

4) $$H^0(fI^*) \approx L.$$

The statements 3) and 4) impose exactness on fI^* at fI^p, for $p \le n$. By lemma 1.1 there is an injective resolution ${}_1L^*$ of L on Y and a homomorphism ${}_1h: {}_1L^* \to fI^*$ such that ${}_1h^i: {}_1L^i \to fI^i$ is an isomorphism for each $i \le n+1$. Since the homology is independent of the injective resolution chosen we can replace L^* by ${}_1L^*$ without difficulty.

Now $h : L^* \to fI^*$ induces a homomorphism on the duals

5) $$\mathcal{D}(fI^*) \xrightarrow{h_*} \mathcal{D}(L^*)$$

which is an isomorphism for $p \le n$. Clearly no information concerning $h(n+1)$ is given since for each p

$$\mathcal{D}_p(fI^*)(U) = \mathrm{Hom}(\Gamma_c fI^p(U);L^0) \oplus \mathrm{Hom}(\Gamma_c fI^{p+1}(U);L^1)$$

where $L^0 = Q$ the field of quotients of L and $L^1 = Q/L$.

From 5)

$$\mathcal{D}_*(fI^*) \otimes \mathcal{B} \approx \mathcal{D}_*(L^*) \otimes \mathcal{B} \qquad \text{for } p \le n$$

applying Γ_ϕ we have for $p \le n$, ϕ-paracompactifying

$$D_p = \Gamma_\phi(\mathcal{D}_p(fI^*) \otimes \mathcal{B}) \xrightarrow{\approx} \Gamma_\phi(\mathcal{D}_p(L^*) \otimes \mathcal{B}) = C_p^\phi(Y,\mathcal{B})$$

Clearly,

$$H^\phi(\Gamma_\phi \mathcal{D}_p(fI^*) \otimes \mathcal{B})) \approx H_p^\phi(Y;\mathcal{B}), \qquad p < n$$

For p = n, consider the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(Y,\mathcal{B}) \arrow[r, "d_{n+1}"] & C_n(Y,\mathcal{B}) \arrow[r, "d_n"] & C_{n-1}(Y,\mathcal{B}) \arrow[r] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[u, "h_{n+1}"'] \arrow[r, "\delta_{n+1}"] & D_n \arrow[u, "h_n"', "\approx"] \arrow[r, "\delta_n"] & D_{n-1} \arrow[u, "h_{n-1}"', "\approx"] \arrow[r] & \cdots
\end{tikzcd}$$

Since h_n is an isomorphism

$$h_n\Big|_{\ker_{\delta_n}} : \ker \delta_n \to \ker d_n \qquad \text{is an isomorphism.}$$

But $h_n(\mathcal{I}m\ \delta_{n+1})$ is merely contained in $\mathcal{I}m\ d_{n+1}$ therefore the induced map on the quotient is only epimorphic and

6) $$H_p^{\phi}(\Gamma\mathcal{D}(f\mathcal{I}^*) \otimes \mathcal{B}) \xrightarrow{h} H_p^{\phi}(Y;\mathcal{B})$$

is isomorphism for $p < n$ and an epimorphism for $p = n$.

Combining 2) and 6) we have that

$$f_* : H_p^{f^{-1}\phi}(X;f^*\mathcal{B}) \to H_p^{\phi}(Y;\mathcal{B})$$

is an isomorphism for $p < n$ and an epimorphism for $p = n$.

<u>Lemma 3.3</u>. Let $X = \cup U_\alpha$, where the U_α's are pairwise disjoint open sets in X. Let $\mathcal{A}$ be a sheaf on X and ϕ a paracompactifying family of supports on X. Then

$$H_*^{\phi}(X;\mathcal{A}) \simeq \prod_\alpha H_*^{\phi\cap U_\alpha}(U_\alpha;\mathcal{A}|U_\alpha) .$$

(Compare Bourgin [7;Lemma 2.6].)

<u>Proof</u>: For each U_α denote by $\mathcal{A}_{U_\alpha}$ the sheaf which is 0 on

$X-U_\alpha$ and $A|U_\alpha$ on U_α. Since the U_α's are pairwise disjoint $A \cong \Pi_\alpha A_{U_\alpha}$

1) $$C_*(X,A_{U_\alpha}) = C_*(X,L) \otimes A_{U_\alpha}$$
$$= C_*(X,L) \otimes A \otimes L_{U_\alpha}$$
$$= C_*(X,A) \otimes L_{U_\alpha}$$
$$= C_*(X,A)_{U_\alpha}$$

U_α open implies

2) $$C_*(X,A)_{U_\alpha} = (C_*(X,A)|U_\alpha)^X = C_*(U_\alpha,A)^X$$

3) From 1)

$$\Pi C_*(X,A_{U_\alpha}) = \Pi C_*(X,A)_{U_\alpha} \approx C_*(X,A)$$

and

4) $$\Gamma_\phi C_*(X,A) \approx \Gamma_\phi \prod_\alpha C_*(X,A_{U_\alpha}) \approx \prod_\alpha \Gamma_\phi C_*(X,A_{U_\alpha})$$

the last isomorphism holds by considering the map

$$s \longrightarrow \{s(x)(\alpha)\} \qquad \text{where } s:X \longrightarrow \Pi C_*(X,A_{U_\alpha})$$

$$s(x) \in \prod_\alpha C_*(X,A_{U_\alpha}) \qquad \text{and} \quad s(x)(\alpha): X \longrightarrow C_*(X,A_{U_\alpha})$$

this is clearly a bijection.

Hence from 2)

5) $$\Gamma_\phi C_*(X,A) \approx \prod_\alpha \Gamma_\phi C_*(X,A_U) \cong \prod_\alpha \Gamma_\phi C_*(U_\alpha,A)^X$$

But

$$C_*(U_\alpha,A)^X = i_c C_*(U_\alpha,L) \otimes A$$

$$= i_c(C_*(U_\alpha,L) \otimes i^*A)$$

$$= i_c(C_*(U_\alpha,L) \otimes A|U_\alpha)$$

and

$$\Gamma_\phi C_*(U_\alpha,A)^X = \Gamma_\phi i_c(C_*(U_\alpha,L) \otimes A|U_\alpha)$$

$$= \Gamma_{\phi(c)} C_*(U_\alpha,A|U_\alpha)$$

$$= \Gamma_{\phi\cap U} C_*(U_\alpha,A|U_\alpha) \; .$$

Hence 5) becomes

$$\Gamma_\phi C_*(X,A) \approx \prod_\alpha \Gamma_{\phi\cap U_\alpha} C_*(U_\alpha,A|U_\alpha).$$

Applying the homology functor we have

$$H_*^\phi(X;A) \approx \prod_\alpha H^{\phi\cap U_\alpha}(U_\alpha;A|U_\alpha).$$

Lemma 3.4 [8;200]. If A is closed in X and A is a sheaf on X which is $\phi|A$ elementary on X (or A is elementary and $\dim_{\phi|A} X < \infty$), then

$$H_*^\phi(X,A;A) \approx H_*^{\phi\cap X-A}(X-A;A).$$

Theorem 3.5. Let $f:X\to Y$ be a proper surjection of locally compact and paracompact spaces and let $\mathcal{B}$ denote the simple sheaf on Y. Suppose $(X,f,\mathcal{B})$ is A_pS and that each $X(y)$, $y \in Y$, is connected and $H^m(X(y),\mathcal{B}_y) = 0$ for $m = p,\ p + 1$ where $p > 0$. Then

$$f_* : H_m^{f^{-1}\phi}(X;f^*\mathcal{B}) \to H_m^{\phi}(Y;\mathcal{B})$$

is an epimorphism for $m=p+1$ and a monomorphism for $m=p$, with ϕ any family of supports on Y.

<u>Proof</u>: Let $S_p = \{y|H^m(X(y);\mathcal{B}_y) \neq 0$ for some $0<m<p\}$.

Construct the space X^* with respect to S_p. Then X^* is locally compact and paracompact. Define the extension of f to $F:X^* \longrightarrow Y$ by

$$F(X^*(y)) = f(X(y)) = y.$$

It follows that F is a proper surjection. Now

$$H^m_{\psi|X^*(y)}(X^*(y);F^*\mathcal{B}) = 0$$

for $0 < m \leq p + 1$, hence by theorem 3.2 $F^*(m)$ is an isomorphism for $m \leq p$ and an epimorphism for $m = p + 1$. Here $\psi = F^{-1}(\phi)$ the extension family of $f^{-1}(\phi)$.

The exact sequence of the pair $(X^*(y),X(y))$ yields

$$H_{m+1}^{\psi|X^*(y)}(X^*(y),X(y);F^*\mathcal{B}) \approx H_m^{\psi|X(y)}(X(y);F^*\mathcal{B}),\ m > 0\ .$$

Since $H^p(X(y);\mathcal{B}_y) = H^{p+1}(X(y);\mathcal{B}_y) = 0$ it follows from the universal coefficient theorem that $H_p(X(y);\mathcal{B}_y) = 0$. Furthermore since X is closed in X^* and $\mathcal{B}$ is $\psi|X$ elementary, It follows from 3.4 that

$$H^{\psi}_{*}(X^*,X;F^*\mathcal{B}) \approx H^{\psi\cap X^*-X}_{*}(X^*-X;F^*\mathcal{B}).$$

Observe that

$$X^* - X = \bigcup_{y\,\varepsilon\, S_p} X^*(y) - X(y)$$

satisfies 3.3 and

$$\begin{aligned}
H^{\psi\cap X^*-X}_{m+1}(X^*-X;F^*\mathcal{B}) &\approx \prod_{y\,\varepsilon\, S_p} H^{\psi\cap X^*-X}_{m+1}(X^*(y)-X(y);F^*\mathcal{B}) \\
&\approx \prod_{y\,\varepsilon\, S_p} H^{\psi|X^*(y)}_{m+1}(X^*(y),X(y);F^*\mathcal{B}) \\
&\approx \prod_{y\,\varepsilon\, S_p} H^{\psi|X(y)}_{m}(X(y);F^*\mathcal{B}).
\end{aligned}$$

Hence it follows from the assumption that for m=p

$$H^{\psi}_{p+1}(X^*,X;F^*\mathcal{B}) = 0$$

The exact sequence of the pair (X^*,X) yields the following diagram:

$$\begin{tikzcd}
{} \arrow[r] & H^{\psi}_{m+1}(X^*,X;F^*\mathcal{B}) \arrow[r] & H^{\psi}_{m}(X;F^*\mathcal{B}) \arrow[r, "\alpha(m)"] \arrow[dr, "f_*"'] & H^{\psi}_{m}(X^*;F^*\mathcal{B}) \arrow[r] \arrow[d, "F_*"] & {} \\
 & & & H^{\phi}_{m}(Y;\mathcal{B}) &
\end{tikzcd}$$

where $f_*(m) = F_*(m)\alpha(m)$. If m=p, $\alpha(p)$ is a monomorphism and $\alpha(p+1)$ is an epimorphism. Also $F_*(p)$ is an isomorphism and

$F_*(p+1)$ is an epimorphism. Hence $f_*(p)$ is a monomorphism and $f_*(p+1)$ is an epimorphism.

Corollary 3.5a. If in addition to the hypothesis of the theorem, $H^m(X(y);\mathcal{B}_y) = 0$ for $0 < p \leq m \leq q$. Then $f_*(m)$ is an isomorphism for $p < m < q$, an epimorphism for $m=q$ and a monomorphism for $m=p$.

We proceed now to get a Vietoris-Begle type theorem for the condition $H_p(X(y),\mathcal{B}_y) = 0$ instead of the cohomology condition. The difficulty here arises from the fact that

$$H_{-1}(X(y);L) = \mathcal{E}xt(H^0(X(y);L),L)$$

may not be zero along with the fact that $H_0(X(y);L)$ is not necessarily isomorphic to L even though X(y) is compact and connected. However, in case L is a field $H_{-1}(X(y);L) = 0$ and $H_0(X(y);L) \approx L$. If $\tilde{H}^0(X;L)$ denotes the reduced homology groups of degree zero then clearly if X is compact and connected and L is a field $\tilde{H}^0(X;L) = 0$. In the following only the reduced groups will be used.

Definition 3.6 [8;205]. Let $f:X \longrightarrow Y$ be continuous, $\mathcal{A}$ a sheaf on X and ψ a family of supports on X. The homology sheaf of the map f with coefficients in $\mathcal{A}$ is the derived sheaf $\mathcal{H}^{\psi}_*(f;\mathcal{A})$ of the differential sheaf $f_\psi C_*(X,\mathcal{A})$ on Y. If $A \subset X$ is locally closed the derived sheaf of $f_\psi C_*(X,A,\mathcal{A})$ is denoted by $\mathcal{H}^{\psi}_{*}(f,f|A;\mathcal{A})$.

Here $\mathcal{H}^{\psi}_{*}(f,f|A,\mathcal{A})$ is the sheaf generated by the presheaf $U \longrightarrow H^{\psi \cap f^{-1}U}(f^{-1}(U), f^{-1}(U) \cap A;\mathcal{A})$ where U is open in Y

Let $G^{q} = f_{\psi}C_{-q}(X,A,\mathcal{A})$. Since $C_{*}(X,A,\mathcal{A})$ is a $C^{0}(X,L)$ module it follows that $f_{\psi}C_{*}(X,A,\mathcal{A})$ is a $C^{0}(Y,L)$ module, but $C^{0}(Y,L)$ is flabby, implying that $f_{\psi}C_{*}(X,A,\mathcal{A})$ is ϕ-fine and hence ϕ-soft, for ϕ- paracompactifying. Assume also that $\phi(\psi)$ is paracompactifying where ϕ is a family of supports on Y.

Now G^{*} is a differential sheaf and G^{p} is ϕ-acyclic for each p. Let $C^{*}_{\phi}(Y,G^{*})$ be the associated grating and let

$$
\begin{aligned}
{}_{1}K^{(p)} &= \bigoplus_{i \geq p} \oplus_{j}\, C^{i}(Y,G^{j}) \\
{}_{2}K^{(q)} &= \bigoplus_{j \geq q} \oplus_{i}\, C^{i}(Y,G^{j})
\end{aligned}
$$

denote the two filtrations with

$$
{}^{n}K = \bigoplus_{i+j=n} C^{i}(Y,G^{j})
$$

the total complex.

Since $C^{i}(Y,G^{j}) = 0$ for $i < 0$, if we require that the $\dim_{\phi} Y < \infty$, we have both filtrations strongly regular. The first filtration yields the spectral sequence

$$ {}_1E_1^{pq} = {}_1H_\psi^q(C_\phi^p(Y,G^*)) = C_\phi^p(Y,\mathcal{H}_\psi^q(G^*)) $$

$$ {}_1E_2^{pq} = {}_1H^p(C_\phi(Y,\mathcal{H}_\psi^q(G^*))) $$

$$ = H_\phi^p(Y,\mathcal{H}_\psi^q(G^*)) \quad . $$

From the second filtration we have the spectral sequence

$$ {}_2E_1^{qp} = {}_1H^p(C_\phi^*(Y,G^q)) = H_\phi^p(Y,G^q) = 0 $$

for $p > 0$, since G^q is ϕ-acyclic. Thus

$$ {}_2E_2^{qp} = H(H^p(Y,G^*)) = 0 \quad \text{for} \quad p > 0 \quad \text{and} $$

$$ {}_2E_2^{q0} = H_\psi^q(H_\phi^0(Y,G^*)) = H_\psi^q(\Gamma_\phi G^*) \; . $$

From [6:496]

$$ {}_2E_2^{q0} \approx {}^qH(C^*(Y,G^*)) $$

and the targets of ${}_1E$ and ${}_2E$ are equal. Hence we are left with the spectral sequence

$$ E_2^{pq} = H_\phi^p(Y,\mathcal{H}^q(G^*)) \Rightarrow H^{p+q}(\Gamma_\phi G^*) \; . $$

Intrepreting $G^* = f_\psi C_*(X,A;A)$ we have

$$ E_2^{pq} = H_\phi^p(Y,\mathcal{H}_{-q}^\psi(f,f|A;A)) \Rightarrow H_{-p-q}^{\phi(\psi)}(X,A;A) \qquad (3.6a) $$

Lemma 3.7. Let $f:X \to Y$ be a proper surjection between locally compact and paracompact spaces; let ψ, ϕ be support families on X and Y respectively. Let $A \subset X$ be locally closed and let $\mathcal{A}$ be a sheaf on X. Assume that $\dim_\phi Y < \infty$ and for each $y \in Y$ $H_q^{\psi|X(y)}(X(y), X(y) \cap A; \mathcal{A}) = 0$, for $0 < q \le n$ and $q < 0$. Then for $p \le n$

$$H^p(Y, \mathcal{H}_0(f, f|A; \mathcal{A})) \approx H_{-p}(X, A; \mathcal{A}).$$

Proof: Since f is proper X(y) is ψ-taut and

$$\mathcal{H}_q(f, f|A; \mathcal{A})_y \approx H_q^{\psi|X(y)}(X(y), X(y) \cap A; \mathcal{A}) = 0 \quad \text{for } 0 < q \le n.$$

It follows that $\mathcal{H}_q(f, f|A; \mathcal{A}) = 0$ for $0 < q \le n$ and by the notation convention we have $\mathcal{H}_{-q}(f, f|A; \mathcal{A}) = 0$ for q in the same range.

From the spectral sequence 3.6a we have that $E_2^{pq} = 0$ for $p < 0$, $q < 0$ and $0 < q \le n$, therefore $E_2^{p0} \approx {}^pH$, that is,

$$H^p(Y; \mathcal{H}_0(f, f|A); \mathcal{A})) \approx H_{-p}(X, A; \mathcal{A})$$

Theorem 3.8. Let $f:X \longrightarrow Y$ be a proper surjection between locally compact and paracompact spaces, $\mathcal{B}$ a locally constant sheaf on Y with stalk L (a field) and ψ, ϕ support families on X and Y respectively.

If a) $H_p^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0$ for $0 \le p \le n$ and

b) $\dim_\phi Y < \infty$.

Then $f_*(p)$ is an isomorphism for $p < n$ and an epimorphism for $p = n$.

Proof: Imbed X in the mapping cylinder Z_f. Z_f is paracompact and locally compact, and the retraction map $F:Z_f \longrightarrow Y$ is closed. Z_f is the same homotopy type as Y, and the homotopy is proper [8;203]. Thus, for each p

$$H_p^{\psi^*}(Z_f;F^*\mathcal{B}) \approx H_p^{\phi}(Y;\mathcal{B})$$

where ψ^* is the extension of the support family ψ on X.

1) Since $Z_f(y)$ is compact and connected, and L is a field, it follows that $H_p(Z_f(y);L) = 0$ for $p > -1$. Hence the exact sequence of the pair $(Z_f(y), Z_f(y) \cap X)$ yields

$$H_{p+1}^{\psi^*|Z_f(y)}(Z_f(y),Z_f(y)\cap X;F^*\mathcal{B}) \approx H_p^{\psi|Z(y)\cap X}(Z_f(y)\cap X;F^*\mathcal{B}),\ p \le n$$

But

$$H_{-1}^{\psi^*|Z_f(y)\cap X}(Z_f(y)\cap X;F^*\mathcal{B}) = 0$$

Therefore

$$H_{p+1}^{\psi^*}(Z_f(y),Z_f(y)\cap X;F^*\mathcal{B}) = 0, \qquad p \le n.$$

2) If $p = -1$, then $H_0^{\psi^*|Z_f(y)}(Z_f(y), Z_f(y) \cap X; F^*\mathcal{B}) = 0$,

implies that $\mathcal{H}_0(F, F|X; F^*\mathcal{B})_y = 0$, and it follows that

$\mathcal{H}_0(F, F|X; F^*\mathcal{B}) = 0$. Applying lemma 3.7 we have

$$H_p^{\psi^*}(Z_f, X; F^*\mathcal{B}) = 0, \qquad p \leq n. \tag{2.a}$$

3) The exact sequence of the pair (Z_f, X) gives the diagram

$$\begin{tikzcd}
{} \arrow[r] & H_{p+1}^{\psi^*}(Z_f X; F^*\mathcal{B}) \arrow[r] & H_p^{\psi}(X; f^*\mathcal{B}) \arrow[r, "\alpha(p)"] \arrow[dr, "f_*"'] & H_p^{\psi^*}(Z_f; F^*\mathcal{B}) \arrow[r] \arrow[d, "F_*", "\approx"'] & H_p^{\psi^*}(Z_f, X; F^*\mathcal{B}) \arrow[r] & {} \\
 & & & H_p^{\phi}(Y; \mathcal{B}) & &
\end{tikzcd}$$

Applying 2a in the diagram, we obtain the desired results.

The following theorem is a partial converse of theorem 3.8. For convenience let $X(U)$ denote $f^{-1}(U)$, where U is open in Y.

<u>Theorem 3.9</u>. Let $f: X \longrightarrow Y$ be a proper surjection of locally compact and paracompact spaces. Let $\mathcal{B}$ denote the constant sheaf with stalks L (a field) and suppose for each U open in Y,

$$f_*: H_p^{\psi \cap X(U)}(X(U); f^*\mathcal{B}) \longrightarrow H_p^{\phi \cap U}(U; \mathcal{B})$$

is an isomorphism for $p \leq n$. Then

$$H_p^{\psi|X(y)}(X(y); F^*\mathcal{B}) = 0, \qquad p \leq n,$$

with ψ and ϕ any support families on X and Y respectively.

Proof: Let y be fixed in Y and consider the following diagram:

$$\begin{array}{ccc}
\varinjlim H_p^{\psi\cap X(U)}(X(U);f^*\mathcal{B}) & \xrightarrow{\approx} & \varinjlim H_p^{\phi\cap U}(U;\mathcal{B}) \\
\approx\downarrow & & \downarrow\approx \\
H_p^{\psi|X(y)}(X(y);f^*\mathcal{B}) & \longrightarrow & H_p^{\phi|y}(y;\mathcal{B})
\end{array}$$

where the direct limits are taken over all open sets U containing y. Since X(y) is ψ-taut the vertical maps are isomorphisms, and the assumption guarantees that the top horizontal map is an isomorphism. Clearly the bottom map is an isomorphism and

$$H_p^{\psi|X(y)}(X(y);f^*\mathcal{B}) \approx H_p^{\phi|y}(y;\mathcal{B}) \approx 0, \qquad p \le n.$$

The task of establishing a Vietoris-Begle type theorem which puts the conditions of vanishing on the homology modules has been accomplished in theorem 3.8. However, a sacrifice was made in requiring that the cohomology dimension of the space Y was finite, i.e. $\dim_\phi Y < \infty$. This condition can be lifted if we restrict ourselves to sheaves with stalks isomorphic to a finite field.

Theorem 3.10. Let $f:X\longrightarrow Y$ be a proper surjection between locally compact and paracompact spaces, and let $\mathcal{B}$ be a constant sheaf on Y with stalks L (a finite field). If

$$H_p^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0 \text{ for } \quad 0 \le p \le n,$$

then $f_*(p)$ is an isomorphism for $p < n$ and an epimorphism for $p = n$.

Proof: We use the same construction in the proof of theorem 3.8, and apply lemmas 3.3 and 3.4 instead of the spectral sequence argument.

In the theorems that follow we assume that $f:X \longrightarrow Y$ is a continuous proper surjection of locally compact and paracompact spaces, $\mathcal{B}$ is a constant sheaf on Y with stalks L (a finite field). Also ψ and ϕ are arbitrary families of supports on X and Y respectively.

Theorem 3.11. If $(X,f,\mathcal{B})$ is A_pS and for each $y \in Y$, $X(y)$ is connected and $H_m^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0$, $m = p,\ p+1$. Then $f_*(m)$ is an epimorphism for $m = p + 1$ and a monomorphism for $m = p$.

Proof: Construct the space X^* with respect to the set S_p, and let ψ^* extend the support family ψ on X to X^*. Since $H_m^{\psi^*|X^*(y)}(X^*(y);\mathcal{B}_y) = 0$ for $m \le p + 1$, the map $F:X^* \longrightarrow Y$ extending f satisfies the conditions of theorem 3.10. Hence

1) $F_*(m)$ is an isomorphism for $m \le p$ and an epimorphism for $m = p + 1$.

Since $F^*\mathcal{B}$ is $\psi^*|X$ elementary and X is closed in X^* it follows from lemma 3.4 that

$$H_m^{\psi\cap X^*-X}(X^*-X;F^*\mathcal{B}) \approx H_m^{\psi^*}(X^*,X;F^*\mathcal{B}).$$

Here $X^*-X = \bigcup\limits_{y\varepsilon Y} X^*(y)-X(y)$, so applying lemma 3.3 and the usual exact sequence argument we have

2) $$H_{m+1}^{\psi^*\cap X^*-X}(X^*-X;F^*\mathcal{B}) \approx \prod_{y\varepsilon S_p} H_m^{\psi^*|X(y)}(X(y);F^*\mathcal{B}).$$

From the assumption of the theorem, the right side of 2) is isomorphic to zero, when $m = p,\ p+1$. Making use of 1) and 2) in the diagram below:

$$\begin{tikzcd}
\to H_{m+1}(X^*,X;F^*\mathcal{B}) \arrow[r] & H_m(X;f^*\mathcal{B}) \arrow[rr, "\alpha(m)"] \arrow[dr, "f_*"'] & & H_m(X^*;F^*\mathcal{B}) \arrow[r] \arrow[dl, "F_*"] & H_m(X^*,X;F^*\mathcal{B}) \to \\
& & H_m(Y;\mathcal{B}) & &
\end{tikzcd} \qquad (3.11a)$$

we get the desired results.

The proof of theorem 3.11 establishes the fact $H_{p+2}^{\psi}(X^*,X;F^*\ ) = 0$, yet no information is given concerning $f_*(p+1)$ or $f_*(p+2)$, in diagram (3.11a). Apparently, the vanishing conditions can be weakened, in some way. It is shown in the following theorem that this can be accomplished if the dimension of the almost p-solid condition is augmented. This is not true in case cohomology modules are used.

<u>Theorem 3.12</u>. If $(X,f,\mathcal{B})$ is $A_{p+2}S$ and for each $y \varepsilon Y$

$X(y)$ is connected and $H_p^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0$, then $f_*(m)$ is an epimorphism for $m = p + 1$ and a monomorphism for $m = p$.

Proof: $S_{p+2} = \{y|H_m^{\psi|X(y)}(X(y);\mathcal{B}_y) \neq 0 \quad \text{for some } m < p + 2\}$. Imbed X in the space X* constructed with respect to the set S_{p+2}. The map $F:X^* \longrightarrow Y$ extending f satisfies the conditions of theorem 3.10, and it follows that $F_*(m)$ is an isomorphism for $m < p + 1$ and an epimorphism for $m = p + 1$. From previous arguments we have that

$$H_{p+1}^{\psi^*}(X^*,X;F^*\mathcal{B}) = 0.$$

Making use of the diagram 3.11a when $m = p$, we get the desired results.

Corollary 3.13. If $(X,f,\mathcal{B})$ is $A_{q+2}S$ and for each y in Y $X(y)$ is connected and $H_m^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0$ for $p \leq m \leq q$, then $f_*(m)$ is a monomorphism for $m = p$, an isomorphism for $p < m \leq q$ and an epimorphism for $m = q + 1$.

The following theorem is a converse of theorem 3.12.

Theorem 3.14. If $f_*(p)$ is a monomorphism, $f_*(p+1)$ is an epimorphism and $(X,f,\mathcal{B})$ is $A_{p+2}S$, then $H_p^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0$.

Proof: Consider the diagram 3.11a. $F_*(p+1)$ is an epimorphism and $F_*(p)$ is an isomorphism. Since $f_*(p)$ is a monomorphism and $f_*(p) = F_*(p)\alpha(p)$ it follows that $\alpha(p)$ is a monomorphism. Accordingly $\alpha(p+1)$ is an epimorphism.

By exactness $H^{\psi^*}_{p+1}(X^*,X;F^*\mathcal{B}) = 0$. But

$$H^{\psi^*}_{p+1}(X^*,X;F^*\mathcal{B}) \approx \prod_{y \varepsilon S_{p+2}} H^{\psi}_{p}(X(y);\mathcal{B}_y)$$

hence $H^{\psi}_{p}(X(y);\mathcal{B}_y) = 0$ if $y \varepsilon S_{p+2}$. Clearly if $y \bar{\varepsilon} S_{p+2}$

$H^{\psi}_{p}(X(y);\mathcal{B}_y) = 0$.

Corollary 3.15. If $f_*(m)$ is an isomorphism for $p < m \leq q$ a monomorphism for $m = p$ and an epimorphism for $m = q + 1$ and if $(X,f,\mathcal{B})$ is $A_{q+2}S$, then

$$H^{\psi|X(y)}_{m}(X(y);\mathcal{B}_y) = 0, \qquad p \leq m \leq q .$$

Bialynicki-Birula [2] generalized the Vietoris-Begle theorem using cohomology to the case of triple spaces and D. G. Bourgin [4] extended this result using his more general Vietoris-Begle type theorems for cohomology. The following theorem gives this extension for sheaf theoretic homology.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$, $h = gf$ where X and Y are locally compact and paracompact and Z is arbitrary. Let f and g be continuous proper surjections, $\mathcal{B}$ the constant sheaf on Y with stalks L (a finite field).

<u>Theorem 3.16</u>. If $(X,f,\mathcal{B})$ is $A_{q+2}S$, each $X(y)$ is connected and for $z \in Z$

$$H_m^{\psi}(h^{-1}(z);\mathcal{B}) \approx H_m^{\psi}(g^{-1}(z);\mathcal{B}), \ p \le m \le q$$

then $f_*(m)$ is a monomorphism for $m = p$, an isomorphism for $p < m \le q$ and an epimorphism for $m = p + 1$.

<u>Proof</u>: Let z be fixed in Z and let

$$f_z = f\big|_{h^{-1}(z)}$$

then $f_z : h^{-1}(z) \longrightarrow g^{-1}(z)$ is a proper surjection between locally compact and paracompact spaces and the induced homomorphism f_{z*} satisfies the conditions of corollary 3.15. Hence, for each $y \in g^{-1}(z)$

$$H_m^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0 \ , \qquad p \le m \le q.$$

But z was chosen arbitrarily so for each $y \in Y$

$$H_m^{\psi|X(y)}(X(y);\mathcal{B}_y) = 0, \qquad p \le m \le q.$$

Making use of corollary 3.13 we get the desired results.

In the following it will be shown that the image of an orientable n-homology manifold under Vietoris type maps is an orientable n-homology manifold. This is essentially R. L. Wilder's monotone theorem [11], and is an application to theorem 3.2.

Definition 3.17 [8;209]. A locally compact space X will be called an (L,n) space if $\mathcal{H}_p(X,L)$ is zero for $p \neq n$ and torsion free for $p = n$. $\mathcal{H}_n(X,L)$ is called the orientation sheaf of X and will be denoted by $\mathcal{O} = \mathcal{O}_X$. An (L,n) space X is said to be an n-dimensional homology manifold over L(n-hm_L) if $\mathcal{O}$ is locally constant with stalks isomorphic to L, and if $\dim_L X < \infty$. Furthermore if X is orientable then $\mathcal{O}$ is constant with stalks isomorphic to L.

Lemma 3.18. If $f:X \longrightarrow Y$ is a continuous map of locally compact spaces and paracompact spaces, ψ is a family of supports on X and $\dim_L X < \infty$. Then there is a spectral sequence of sheaves with

$$E_2^{pq} = \mathcal{H}_\psi^p(f;\mathcal{H}_{-q}^\psi(X,L)) \Rightarrow \mathcal{H}_{-p-q}^\psi(f;L)$$

Proof: For each U open in Y, let $A^{**}(U) = C^*_{\psi \cap f^{-1}(U)}(f^{-1}(U),\mathcal{G}^*)$ be a doubly graded complex where $\mathcal{G}^q = \mathcal{C}_{-q}(X,L)$. Introduce the two filtrations as before. Since $A^{ij} = 0$ for $i < 0$ and $\dim_L f^{-1}(U) < \infty$ the filtrations are strongly regular. $\mathcal{G}^*$ is $\psi \cap f^{-1}(U)$-acyclic since $\mathcal{C}_*(X,L)$ is flabby, hence there is a spectral sequence

$$E_2^{pq}(U) = H^p_{\psi\cap f^{-1}(U)}(f^{-1}(U); \mathcal{H}^q(G^*)) \Rightarrow H^{p+q}(\Gamma_{\phi\cap f^{-1}(U)}G^*)$$

$$= H^p_{\psi\cap f^{-1}}(f^{-1}(U); \mathcal{H}_{-q}(X,L)) \Rightarrow H^{\psi\cap f^{-1}(U)}_{-p-q}(f^{-1}(U);L)$$

Now $U \longrightarrow E_2^{pq}(U)$ is a presheaf on Y, and on taking direct limits we have

$$E_2^{pq} = \mathcal{H}^p_\psi(f; \mathcal{H}_{-q}(X,L)) \Rightarrow \mathcal{H}^\psi_{-p-q}(f;L)$$

<u>Lemma 3.19</u>. If X is an (L,n) space with orientation sheaf $\mathcal{O}$ and $f:X \longrightarrow Y$ is continuous, X paracompact, Y locally compact. Then

$$\mathcal{H}^\psi_p(f;L) \approx \mathcal{H}^{n-p}_\psi(f;\mathcal{O})$$

<u>Proof</u>: The preceding lemma gives the spectral sequence of sheaves

$$E_2^{n-p\,q} = \mathcal{H}^{n-p}_\psi(f; \mathcal{H}^\psi_{-q}(X,L)) \Rightarrow \mathcal{H}^\psi_{-n+p-q}(f;L)$$

Since X is an (L,n) space

$$E_2^{n-p\,q} = 0 \text{ for } q \neq -n$$

hence

$$E_2^{n-p\,q} = \mathcal{H}^{n-p}_\psi(f; \mathcal{H}_n(X,L)) \approx \mathcal{H}^\psi_p(f;L)$$

and $\mathcal{H}_n(X,L) = \mathcal{O}$ the orientation sheaf.

Theorem 3.20. If X is a paracompact orientable n-hm_L space, $f:X \longrightarrow Y$ is a proper surjection with Y locally compact, X(y) is connected and $H^p(X(y),L) = 0$ for $p > 0$, then Y is an orientable n-hm_L space.

Proof: For each U open in Y, $H_p(f^{-1}(U);L) \approx H_p(U;L)$, therefore the sheaves generated by the presheaves

$$U \longrightarrow H_p(f^{-1}(U);L) \qquad \text{and} \qquad U \longrightarrow H_p(U;L)$$

are isomorphic. This, together with the results of lemma 3.19, yields

$$\mathcal{H}_p(Y;L) \approx \mathcal{H}_p(f;L) \approx \mathcal{H}^{n-p}(f;\mathcal{O})$$

Since X(y) is compact, connected and taut and $\mathcal{O}$ is constant with stalks L, we have

$$\mathcal{H}^{n-p}(f;\mathcal{O})_y \approx H^{n-p}(X(y);\mathcal{O}|X(y)) \approx H^{n-p}(X(y);L)$$

but $H^{n-p}(X(y);L) \approx 0 \quad p \neq n$

$\approx L \quad p = n$

Hence $\mathcal{H}^{n-p}(f;\mathcal{O}) = \begin{cases} 0 \text{ for } p \neq n \\ L \text{ for } p = n \end{cases}$.

CHAPTER IV

VIETORIS-BEGLE TYPE THEOREMS FOR A FUNCTOR $\hat{H}$ CONSTRUCTED ON A CATEGORY OF INVERSE SYSTEMS

Consider the category of paracompact Hausdorff spaces and a cohomology functor H satisfying all axioms of a cohomology theory in the sense of Eilenberg and Steenrod. Coefficients will be taken in Q the field of rational numbers, and $H^*(X)$ will mean $H^*(X;Q)$.

A closed subset A of X is said to be H-finite if $H^m(A)$ is finitely generated for each integer m.

<u>Definition 4.0</u>. A closed set A in X is said to be <u>unavoidable</u>, if A is the union of two H-finite subsets of X and no H-finite subset of X contains this union. The set A is <u>avoidable</u> if A is contained in some H-finite set.

For each space X denote by K(X) the collection of all H-finite subsets of X, and by $\mathcal{K}(X) = \{X_\alpha\}$ the subcollection of K(X) having the property that finite unions are avoidable. Partially order $\mathcal{K}(X)$ by inclusion to get $\{\mathcal{K}(X);\subset\}$ directed.

Assign a vector space to each space X in the following way. For $X_\alpha \subset X_\beta$, let $p^\alpha{}_\beta\colon X_\alpha \longrightarrow X_\beta$ denote the inclusion map, and $p_\alpha{}^\beta(m)\colon H^m(X_\beta) \longrightarrow H^m(X_\alpha)$ denote the induced homomorphism on the H-cohomology group.

Then

$$\Sigma^m(X) = \{H^m(X_\alpha;Q);\ p^\beta_\alpha(m):K(X)\}$$

is an inverse system of finite dimensional vector spaces. Write

$$\hat{H}^m(X;Q) = \varprojlim \Sigma^m(X).$$

$\hat{H}$ is an exact functor from the category of inverse systems of finite dimensional vector spaces to the category of vector spaces. If X is H-finite then $H^*(X) = \hat{H}^*(X)$.

We wish now to determine what axioms of a cohomology theory are satisfied by the functor $\hat{H}$.

<u>Definition 4.1</u>. $f:X \longrightarrow Y$ is an admissible map for $\hat{H}$ if f is continuous and $f^{-1}(K(Y))$ is cofinal in $K(X)$, or some cofinal subcollection of $K(Y)$ has this property.

Note: Constant maps are not admissible in general.

For A closed in X define:

$$K(A) = \{X_\alpha \in K(X) \mid X_\alpha \subset A\}$$
$$K(X) \cap A = \{X_\alpha \in K(X) \mid X_\alpha \cap A \in K(X)\}$$
$$K(X,A) = \{(X_\alpha, A_\alpha) \mid X_\alpha \in K(X) \text{ and } X_\alpha \cap A = A_\alpha\}.$$

These sets are partially ordered but need not be directed.

<u>Definition 4.2</u>. The pair (X,A) is admissible if

$K(X) \cap A$ is cofinal in $K(X)$.

Proposition 4.3. If (X,A) is an admissible pair, then $K(X,A)$ and $K(A)$ are directed.

Proof: Let (X_α, A_α) and $(X_\beta, A_\beta) \varepsilon K(X,A)$. Since $K(X)$ is directed there is an $X_{\gamma'}$ such that $X_{\gamma'} \supset X_\alpha \cup X_\beta$, but (X,A) admissible gives an $X_\gamma \supset X_{\gamma'}$ such that $X_\gamma \cap A \varepsilon K(X)$. Consider the pair $(X_\gamma, X_\gamma \cap A) = (X_\gamma, A_\gamma)$ then

1) $\quad X_\gamma \supset X_{\gamma'} \supset X_\alpha \cup X_\beta \quad$ and

2) $\quad A_\gamma = X_\gamma \cap A \supset X_{\gamma'} \cap A \supset (X_\alpha \cup X_\beta) \cap A = X_\alpha \cap A \cup X_\beta \cap A = A_\alpha \cup A_\beta$

Hence $(X_\gamma, A_\gamma) \supset (X_\alpha, A_\alpha) \cup (X_\beta, A_\beta)$ and $K(X,A)$ is directed.

Let X_α, $X_\beta \varepsilon K(A)$ then $X_\alpha \subset A$ and $X_\beta \subset A$. Since $K(X)$ is directed there is an $X_{\gamma'}$ such that $X_{\gamma'} \supset X_\alpha \cup X_\beta$, but (X,A) admissible gives an $X_\gamma \varepsilon K(X)$ containing $X_{\gamma'}$ such that $X_\gamma \cap A \varepsilon K(X)$. Hence $X_\tau = X_\gamma \cap A \varepsilon K(A)$ and $X_\tau \supset X_\alpha \cup X_\beta$ yields that $K(A)$ is directed.

Lemma 4.4. If (X,A) is an admissible pair then the lattice of (X,A) is admissible.

Proof: Consider the lattice of inclusion maps.

$$\begin{tikzcd}
 & & (X,0) \arrow[dr] & & \\
(0,0) \arrow[r] & (A,0) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X) \\
 & & (A,A) \arrow[ur] & &
\end{tikzcd}$$

Clearly all pairs in the lattice are admissible. This establishes the admissibility of the inclusion maps.

Lemma 4.5. If $(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$ are admissible maps of admissible pairs, then gf is admissible.

Proof: We need to show that $(gf)^{-1}K(Z,C)$ is cofinal in $K(X,A)$. Let $(X_\alpha,A_\alpha) \in K(X,A)$ then there is a $(Y_\beta,B_\beta) \in K(Y,B)$ such that $f^{-1}(Y_\beta,B_\beta) \supset (X_\alpha,A_\alpha)$, and there is (Z_γ,C_γ) in $K(Z,C)$ such that $g^{-1}(Z_\gamma,C_\gamma) \supset (Y_\beta,B_\beta)$. Now $(gf)^{-1}(Z_\gamma,C_\gamma) = f^{-1}g^{-1}(Z_\gamma,C_\gamma) \supset f^{-1}(Y_\beta,B_\beta) \supset (X_\alpha,A_\alpha)$

Theorem 4.6. If $I = [0,1]$ is the closed unit interval, and (X,A) is an admissible pair then the cartesian product

$$(X,A) \times I = (X \times I,\ A \times I)$$

is admissible and the maps

$$g_0, g_1 : \ (X,A) \to (X,A) \times I$$

given by

$$g_0(x) = (x,0),\ g_1(x) = (x,1)$$

are admissible.

Claim: The projection map $\pi: X \times I \to X$ is admissible. Clearly $\pi^{-1}(K(X)) \subset K(X\times I)$, so only a cofinality argument is necessary. The following lemmas will establish the claim and theorem 4.6.

Lemma 4.6a. If $A \in \mathrm{K}(X) - K(X)$, then $\pi^{-1}(A) \in \mathrm{K}(X\times I) - K(X\times I)$.

Proof: Suppose $A \varepsilon K(X) - \mathcal{K}(X)$ then there is an A' in $K(X)$ such that $A \cup A'$ is unavoidable. Let $B = \pi^{-1}(A)$ and let $B' = \pi^{-1}(A')$. Each of B and B' belongs to $K(X \times I)$ and

$$H^m(B \cup B') \approx H^m(A \cup A')$$

for each integer m. $H^m(B \cup B')$ is infinite for some integer m, and every closed set containing $B \cup B'$ contains a homeomorphic copy of $A \cup A$. Hence $B \cup B$ is unavoidable and it follows $\pi^{-1}(A) \varepsilon K(X \times I) - \mathcal{K}(X \times I)$.

Lemma 4.6b. If $A \varepsilon \mathcal{K}(X \times I)$, then $\pi(A)$ is avoidable.

Proof: Suppose $\pi(A)$ is not avoidable then one of the following hold:

a) $\pi(A) \varepsilon K(X) - \mathcal{K}(X)$

b) $\pi(A)$ is unavoidable.

If a) holds, there is a $B \varepsilon K(X)$ such that $B \cup \pi(A)$ is unavoidable. But $\pi^{-1}(B)$ belongs to $K(X \times I) - \mathcal{K}(X \times I)$, by the previous lemma, and $\pi^{-1}(B) \cup A$ is unavoidable. Hence A belongs to $K(X \times I) - \mathcal{K}(X \times I)$. This involves a contradiction therefore a) cannot hold.

Suppose b) holds, then evidently A winds around infinitely many holes in the space intersecting itself only finitely many times, or A is the union of finitely many disjoint sets satisfying a).

There exists sets B_1 and B_2 in $K(X) - \mathcal{K}(X)$ such that $B_1 \cup B_2 = \pi(A)$ [$B_1 \cap B_2$ contains boundary points of infinitely many non bounding cycles]. Now $\pi^{-1}(B_1) \in K(X\times I) - \mathcal{K}(X\times I)$ and $\pi^{-1}(B_1) \cup A$ is unavoidable hence $A \bar{\in} \mathcal{K}(X\times I)$, a contradiction. Therefore, $\pi(A) \in \mathcal{K}(X)$ or $\pi(A) \subset \bigcup_{\alpha\in\phi} X_\alpha$ where $X_\alpha \in \mathcal{K}(X)$ and ϕ is a finite set.

<u>Lemma 4.6c</u>. $\mathcal{K}'(X\times I) = \{X_\alpha\times I | X_\alpha \in \mathcal{K}(X)\}$ is cofinal in $\mathcal{K}(X\times I)$.

<u>Proof</u>: For each $A \in \mathcal{K}(X\times I)$, $\pi(A)$ is avoidable, hence there is an $A' \in \mathcal{K}(X)$ such that $A' \supset \pi(A)$ and $\pi^{-1}(A') \in \mathcal{K}'(X\times I)$ and contains A. Theorem 4.6 follows if we use $\mathcal{K}'(X\times I)$ instead of the full collection $\mathcal{K}(X\times I)$.

Let $\mathcal{U}$ be the category whose objects are the admissible pairs of paracompact Hausdorff spaces, and whose morphisms are the admissible maps described earlier. From the previous lemmas, we have in the sense of Eilenberg and Steenrod [10] that $\mathcal{U}$ is an admissible category for a cohomology theory.

<u>Induced Homomorphisms</u>

Let $f: X \longrightarrow Y$ be an admissible map, and let

$$f^{-1}: \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$$

be the associated map of the H-finite sets. For each $Y_\beta \in \mathcal{K}(Y)$ let $X_\beta = f^{-1}(Y_\beta)$ and let

$$f_\beta = f_{|X_\beta}: X_\beta \longrightarrow Y_\beta$$

denote the restriction of f to X_β. Then if $Y_\beta \supset Y_\alpha$ the following diagram is commutative,

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "p^\alpha_\beta"'] & Y_\alpha \arrow[d, "`p^\alpha_\beta"] \\ X_\beta \arrow[r, "f_\beta"] & Y_\beta \end{tikzcd}$$

since

$$`p^\alpha_\beta\ f_\alpha(\bar{x}) = f_\alpha(\bar{x}) = f_\beta(\bar{x}) = f_\beta p^\alpha_\beta(\bar{x}).$$

Therefore

$$\{f^m_\alpha\} : \textstyle\sum^m(Y) \longrightarrow \sum^m(X)$$

is a homomorphism, and since $f^{-1}(K(Y))$ is cofinal in $K(X)$ we have that

$$\hat{f}^m: \hat{H}^m(Y) \longrightarrow \hat{H}^m(X)$$

is a homomorphism. Call $\hat{f}^m$ the induced homomorphism of f in dimension m.

<u>Coboundary Homomorphism</u>

<u>Lemma 4.7</u>. There is a coboundary homomorphism $\hat{\delta}$ such that for each admissible pair (X,A) and integer m,

$$\hat{\delta}: \hat{H}^{m-1}(A) \longrightarrow \hat{H}^m(X,A).$$

Proof: Let (X,A) be an admissible pair. For (X_α,A_α), (X_β,A_β) in $K(X,A)$ such that $(X_\alpha,A_\alpha) \subset (X_\beta,A_\beta)$ the diagram

$$\begin{array}{ccc} H^{m-1}(A_\alpha) & \xrightarrow{\delta} & H^m(X_\alpha,A_\alpha) \\ {\scriptstyle p^{\beta}_{\alpha|A_\alpha}}\uparrow & & \uparrow{\scriptstyle p^{\beta}_{\alpha}} \\ H^{m-1}(A_\beta) & \xrightarrow{\delta} & H^m(X_\beta,A_\beta) \end{array}$$

is commutative, where δ is the coboundary homomorphism for the functor H, and $p^{\beta}_{\alpha|A_\alpha}$ is the map induced by $p^{\alpha}_{\beta|A_\alpha}$.

The collection

$$\{\delta\} : \textstyle\sum^{m-1}(A) \longrightarrow \sum^{m-1}(X,A)$$

is a homomorphism, hence

$$\hat{\delta} : \hat{H}^{m-1}(A) \longrightarrow \hat{H}^m(X,A)$$

exists.

Lemma 4.8. If (X,A) is an admissible pair and $i : A \to X$, $j : X \to (X,A)$ are inclusion maps, then the sequence

$$\cdots \to \hat{H}^{m-1}(A) \xrightarrow{\hat{\delta}} \hat{H}^m(X,A) \xrightarrow{\hat{j}} \hat{H}^m(X) \to \cdots$$

is exact.

Proof: For $(X_\alpha,A_\alpha) \subset (X_\beta,A_\beta)$ in $K(X,A)$ let S_α and S_β denote the corresponding exact sequences, that is,

$$\begin{tikzcd} S_\alpha \equiv \ldots \arrow[r] & H^{m-1}(A_\alpha) \arrow[r] & H^m(X_\alpha,A_\alpha) \arrow[r] & H^m(X_\alpha) \arrow[r] & \ldots \\ S_\beta \equiv \ldots \arrow[r] & H^{m-1}(A_\beta) \arrow[r] \arrow[u] & H^m(X_\beta,A_\beta) \arrow[r] \arrow[u] & H^m(X_\beta) \arrow[r] \arrow[u] & \ldots \end{tikzcd}$$

The collection of maps p_α^β and $p_{\alpha|A_\alpha}^\beta$ insures the commutativity of the squares in the above diagram, and $\sum(S_\alpha)$ is an inverse system of exact sequences. Hence

$$\underset{\leftarrow}{L}\ \sum(S_\alpha) \equiv \ldots \rightarrow \hat{H}^{m-1}(A) \longrightarrow \hat{H}^m(X,A) \longrightarrow \hat{H}^m(X) \rightarrow \ldots$$

is exact, for $\hat{H}$ is an exact functor.

<u>Lemma 4.9</u> (Excision). If U is an open set in X, and $\bar{U}$ is contained in $A°$ (the interior of A), then the inclusion map $i : (X-U,A-U) \longrightarrow (X,A)$, if admissible, induces isomorphisms:

$$\hat{H}^m(X,A) \approx \hat{H}^m(X-U,A-U).$$

<u>Proof</u>: Assume $i : (X-U,A-U) \longrightarrow (X,A)$ is admissible. For each pair (X_α,A_α) in $K(X,A)$ choose $U_\alpha = (\bar{U} \cap A_\alpha)°$.

1) If $U_\alpha = \phi$, then $\bar{U}_\alpha \subset A_\alpha°$ and $i_\alpha : (X_\alpha-U_\alpha,A_\alpha-U_\alpha) \longrightarrow (X_\alpha,A_\alpha)$ is the identity map.

2) If $U_\alpha \neq \phi$, then $\bar{U}_\alpha = \bar{U} \cap A_\alpha \subset A° \cap A_\alpha = A_\alpha°$.

Assume i_α is H-admissible for each pair (X_α,A_α) then $(X_\alpha-U_\alpha,A_\alpha-U_\alpha)$ is H-finite and the collection

$$\{i_\alpha{}^m\} : \textstyle\sum^m(X-U,A-U) \longrightarrow \sum^m(X,A)$$

is an isomorphism. Hence $\hat{i}^m$ is an isomorphism.

<u>Theorem 4.10</u>. If (X,A) is admissible and $g_0, g_1 : (X,A) \longrightarrow (X,A) \times I$ are defined by $g_0(x) = (x,0)$, $g_1(x) = (x,1)$, then $\hat{g}_0^* = \hat{g}_1^*$.

<u>Proof</u>: By theorem 4.6, g_0 and g_1 are admissible on the cofinal subcollection $K(X,A) \times I$ of $K((X,A)\times I)$. Consequently, $g_0^{-1}[(x_\alpha,A_\alpha)\times I] = g_1^{-1}[(x_\alpha,A_\alpha)\times I]$ for $(X_\alpha,A_\alpha) \varepsilon K(X,A)$. The maps

$$g_{0\alpha}, g_{1\alpha} : (X_\alpha,A_\alpha) \to (X_\alpha,A_\alpha) \times I$$

are homotopic with the identity map of $(X_\alpha,A_\alpha) \times I$ as homotopy. Hence $g_0^*{}_\alpha = g_1^*{}_\alpha$, and on taking inverse limits it follows that $\hat{g}_0^* = \hat{g}_1^*$.

<u>Remark 4.11</u>. Given an admissible homotopy $h: X \times I \longrightarrow Y$, the maps hg_0 and hg_1 are admissible if $h^{-1}(K(Y))$ is cofinal in $K(X) \times I$.

Making use of the Vietoris-Begle theorem and the Vietoris-Begle type theorems of Bourgin [4], we establish the existence of the admissible maps described in the category $\mathcal{U}$

<u>Lemma 4.12.</u> Let $f: X \longrightarrow Y$ be a continuous closed surjection such that $f : K(X) \longrightarrow K(Y)$, and suppose that for each $y \varepsilon Y$ $H^m(X(y)) = 0$, $m \geq 0$. Then f is an admissible map for $\hat{H}$ and $\hat{f}^*(m)$ is an isomorphism for $m \geq 0$.

<u>Proof</u>: Since the Vietoris-Begle theorem is satisfied, it follows that $f_\alpha^*(m)$ is an isomorphism for $m \geq 0$. The condition that $f: K(X) \longrightarrow K(Y)$ guarantees that $f^{-1}(K(Y))$ is cofinal in $K(X)$.

Hence the collection

$$\{f^*(m)\} : \sum^m(Y) \longrightarrow \sum^m(X)$$

gives the induced homomorphism $\hat{f}^*(m)$ which is an isomorphism for each m.

Lemma 4.13. Let $f: X \longrightarrow Y$ be a continuous closed surjection satisfying:

1) $H^m(f^{-1}(Y_\alpha))$ is finitely generated for $m > p + 1$ and $Y_\alpha \in K(Y)$

2) $f : K(X) \longrightarrow K(Y)$

3) $H^m(X(y)) = 0$ for $m \leq p$ and $y \in Y$.

Then f is an admissible map for $\hat{H}$, and $f(m)$ is an isomorphism for $m \leq p$ and a monomorphism for $m = p + 1$.

Proof: For each $Y_\alpha \in K(Y)$, $H^m(Y_\alpha) \approx H^m(f^{-1}(Y_\alpha))$ for $m \leq p$. Condition 1) insures that $H^m(f^{-1}(Y_\alpha))$ is finitely generated for each m. Therefore the sets pulled back by f^{-1} belong to $K(X)$. The cofinal argument is given by condition 2) and the conclusion follows.

Lemma 4.14. Let $f: X \longrightarrow Y$ be a continuous closed surjection satisfying:

1) $f : K(X) \longrightarrow K(Y)$

2) $H^m(f^{-1}(Y_\alpha))$ is finitely generated for $q < m < p$, $Y_\alpha \in K(Y)$

3) (X,f) is A_pS

4) $H^m(X(y) = 0, p \le m \le q$

Then f is an admissible map for $\hat{H}$ and $\hat{f}(m)$ is an isomorphism for $p < m \le q$ and a monomorphism for $m = q + 1$.

Theorem 4.15. Let $f: X \longrightarrow Y$ be an admissible map, and suppose that $H^m(X(y)) \neq 0$. Then $\hat{H}^m(X(y)) \neq 0$.

Proof: Evidently $X(y)$ belongs to $K(X)$, since the pair (Y,y) is admissible. Let $H(Y)$ be the cofinal subcollection of $K(Y)$ whose elements contain the point y. Since f is admissible, $H(X) = f^{-1}H(Y)$ is cofinal in $K(X)$. Clearly, each element of $H(X)$ contains $X(y)$, and it follows that $K(X(y)) = \{X(y)\}$. Hence $\hat{H}^m(X(y)) = H^m(X(y)) \neq 0$.

It is evident from theorem 4.15 that the conditions imposed in order to secure admissible maps are quite strong. In fact, they annihilate any possibility of getting a Vietoris-Begle theorem of any generality. However, there does exist a continuous surjection with $\hat{H}^m(X(y)) = 0$ and $H^m(X(y)) \neq 0$. We suspect that some information can be gotten concerning induced homomorphisms on $\hat{H}$ if we relax our conditions on K and define $\hat{f}$ differently. We do this in the following way.

Definition 4.16. $\phi \subset K(X)$ will denote an admissible subcollection of $K(X)$ if the following properties hold:

1) ϕ is directed

2) $E(\phi) = X$, where $E(\phi) = \cup\phi$

Define $\hat{H}(X)$ with respect to ϕ by substituting ϕ in the original definition for $K(X)$. Write $\hat{H}_\phi(X)$ for $\hat{H}$ with respect to ϕ.

Definition 4.17. Let ϕ be an admissible subcollection of $K(X)$. A map $f : X \longrightarrow Y$ is said to be ϕ-admissible if $f(\phi)$ is an admissible collection of $K(Y)$. For each $X_\alpha \in \phi$, let $f_\alpha: X_\alpha \longrightarrow f(X_\alpha)$ be the restriction of f to X_α. The collection $\{f^*_\alpha\} : \sum^*_{f(\phi)}(Y) \longrightarrow \sum^*_\phi(X)$ is a homomorphism, and the limit map $\check{f}$ will denote the induced homomorphism.

If each f_α in the definition above satisfies the Vietoris-Begle theorem, then the following result is immediate.

Theorem 4.18. Let $f: X \longrightarrow Y$ denote a ϕ-admissible map such that for each X_α in ϕ and y in $f(X_\alpha)$, $H^p(f^{-1}_\alpha(y)) = 0$ for $p \leq m$. Then the induced homomorphism $\check{f}(p): \hat{H}^p_{f(\phi)}(Y) \longrightarrow \hat{H}^p_\phi(X)$ is an isomorphism for $p \leq m$ and a monomorphism for $p = m + 1$.

The following example shows that it is possible to construct a ϕ-admissible map $f : X \longrightarrow Y$ such that:

a) f satisfies the conditions of theorem 4.18

b. $\hat{H}^m(X(y)) = 0$ for each $y \in Y$ and $m \geq 0$ while $H^m(X(y)) \neq 0$ for some m.

Example 4.19. Let S^1 denote the unit circle, and let $\{s_n\}$ denote a sequence of points on S^1 converging to the point $s_0 = (1,0)$. For each nonnegative integer n, attach an arc A_n to S^1 so that the endpoints of A_n coincide with s_n and s_{n+1}. For each n, require that the following properties hold:

1) $A_n \cap S^1 - \{s_n, s_{n+1}\} = \phi$

2) $A_n \cap A_{n+1} = \{s_{n+1}\}$ and $A_n \cap A_{n+k} = \phi$ for $k > 1$.

Let $S_1 = \bigcup_{i=0}^{\infty} A_i$, and define $X = S^1 \cup S_1$. Give X the subspace topology of the plane, and let R_{S^1} be the equivalence relation $S^1 \times S^1 \cup \{(x,x) \mid x \varepsilon X\}$ on X. Define $Y = X/R_{S^1}$. Give Y the identification topology determined by the natural projection $f: X \longrightarrow Y$.

Let $\phi = K(X)$. S^1 does not belong to $K(X)$ since S^1 is H-finite and $S^1 \cup S_1$ is unavoidable. Hence every $X_\alpha \varepsilon \phi$ that meets S^1 meets it in an acyclic set. This implies that the pair (X, S^1) is admissible, and $\hat{H}^1_\phi(S^1) = 0$. Now f is ϕ-admissible and satisfies the condition of theorem 4.18, therefore

$$\hat{H}^m_{f(\phi)}(Y) \approx \hat{H}^m_\phi(X), \quad m \geq 0.$$

There is exactly one point $\bar{y}$ in Y such that $X(\bar{y})$ is not H-acyclic. In this case $X(\bar{y}) = S^1$, and $H^1(X(\bar{y})) = H^1(S^1) \neq 0$. But $\hat{H}^1_\phi(X(\bar{y})) = \hat{H}^1_\phi(S^1) = 0$.

In the following theorem, X and Y will be compact Hausdorff spaces, and we take H to be the Alexander cohomology functor so that $H^*(X,A) \approx H^*(X-A)$. Let $f: X \longrightarrow Y$ be a ϕ-admissible surjection. Assume that for each $y \in Y$ the pair $(X,X(y))$ is ϕ-admissible, and the elements of $\phi(X(y))$ are connected. In addition, if $X(y)$ is acyclic we will understand that $X(y)$ is hereditarily acyclic, with respect to closed connected subsets. Under these assumptions we get the following result.

<u>Theorem 4.20</u>. Let $\hat{H}^m_\phi(X(y)) = 0$ for $m < p$ and all $y \in Y$. Suppose that (X,f) is A_pS and for each $X_\alpha \in \phi$ the set

$$\rho_\alpha = \{y | X_\alpha \cap X(y) \neq \phi \text{ and } X_\alpha \cap X(y) \text{ is not acyclic } \}$$

is finite. Then $\check{f}^*(m)$ is an isomorphism for $m < p$, and a monomorphism for $m = p$.

<u>Proof</u>: Let

$$S_p = \{y | X_\alpha \cap X(y) \neq \phi \text{ and } H^m(X_\alpha \cap X(y)) \neq 0 \text{ for some } m<p \text{ and } X_\alpha \varepsilon \phi\}.$$

Construct the space X^* with respect to S_p, and let $F: X^* \longrightarrow Y$ be the usual extension of f.

For each $X_\alpha \in \phi$, let $X^*_\alpha = X_\alpha \cup \bigcup_{y \in \rho_\alpha} X^*_\alpha(y)$ where $X^*_\alpha(y)$ is a cone over $X_\alpha(y) = X_\alpha \cap X(y)$. Clearly X^*_α is H-finite. Define $\phi^* = \{X_\alpha{}^*\}$, evidently ϕ^* is directed and $E(\phi^*) = X^*$. F is ϕ^*-admissible, and for each $X^*_\alpha \in \phi^*$

$$F_\alpha: X^*_\alpha \longrightarrow f(X_\alpha)$$

is a Vietoris map. That is, for $y \in f(X_\alpha)$

$$F_\alpha^{-1}(y) = \begin{cases} F^{-1}(y) \cap X_\alpha^* = X_\alpha^*(y), & \text{if } y \in S_p \\ X(y) \cap X_\alpha & \text{if } y \bar{\epsilon} S_p \end{cases}$$

where X_α^* (y) is a cone over $X_\alpha(y)$. By theorem 4.18, $\check{F}^*(m)$ is an isomorphism for $m < p$ and a monomorphism for $m = p$.

The pair (X^*,X) is ϕ^*-admissible since each $X_\alpha^* \in \phi$ meets X in the H-finite set X_α. It follows from the admissibility of $(X,X(y))$ that the pairs $(X^*,X^*(y)$ and $(X^*(y),X(y))$ are admissible. We need the following lemma.

<u>Lemma 4.21</u>. $\hat{H}^m_{\phi^*}(X^*,X) \approx \prod_{y \in S_p} \hat{H}^{m-1}_\phi(X(y))$

<u>Proof</u>: Since H was chosen so that $H(X,A) \approx H(X-A)$ we have that

$$\hat{H}^m_{\phi^*}(X^*,X) = \varinjlim H^m(X_\alpha^*,X_\alpha) \approx \varinjlim H^m(X_\alpha^*-X_\alpha) \approx \hat{H}^m_{\phi^*}(X^*-X).$$

Now

$$X_\alpha^* - X_\alpha = \bigcup_{y \in S_p(\alpha)} X_\alpha^*(y) - X_\alpha(y)$$

is H-finite, therefore $S_p(\alpha)$ must be finite. Hence

$$H^m(X_\alpha^* - X_\alpha) \approx \prod_{y \in S(\alpha)} H(X_\alpha^*(y),X_\alpha(y)).$$

Since $X_\alpha^*(y)$ is a cone over $X_\alpha(y)$, it follows from the exact sequence of the pair $(X_\alpha^*(y),X_\alpha(y))$ that

$$H^m(X_\alpha^*(y),X_\alpha(y)) \approx H^{m-1}(X_\alpha(y)), \quad m \geq 1$$

and on taking inverse limits we have

$$\hat{H}^m(X^*(y),X(y)) \approx \hat{H}^{m-1}(X(y)), \quad m \geq 1.$$

Consider the collection $\{\hat{H}(X^*(y),X(y))\}_{y\varepsilon S_p}$. Let M' denote the collection of all finite subsets of S_p, and M the collection of all finite subsets $S(\alpha)$ of S_p, where $S(\alpha)$ corresponds to $H(X^*_\alpha,X_\alpha)$. M' is directed by inclusion and M is cofinal in M'. To see this let $m \varepsilon M'$, then there is a set $Y_\beta = f(X_\beta) \varepsilon f(\phi)$ such that Y_β covers m. Hence $S(\beta)$ is the finite set associated with the pair (X^*_β,X_β), and contains the set m.

From the above it follows that

$$\begin{aligned}\prod_{y\varepsilon S_p} \hat{H}^{m-1}_{\rho}(X(y)) &\approx \prod_{y\varepsilon S_p} \hat{H}^m(X^*(y),X(y)) \approx \prod_{y\varepsilon S_p} \hat{H}^m(X^*(y)-X(y)) \\ &\approx \varprojlim \prod_{y\varepsilon S(\alpha)} \hat{H}^m(X^*(y)-X(y)) \\ &\approx \varprojlim_{M} \prod_{y\varepsilon S(\alpha)} \varprojlim_{\alpha} H^m(X^*_\alpha(y)-X_\alpha(y) \\ &\approx \varprojlim_{M} \varprojlim_{\alpha} \prod_{y\varepsilon S(\alpha)} H^m(X^*_\alpha(y)-X_\alpha(y)) \\ &\approx \varprojlim_{(M,\alpha)} H^m(\cup X^*_\alpha(y)-X_\alpha(y)) \\ &\approx \varprojlim_{\alpha} H^m(X^*_\alpha-X_\alpha) \\ &\approx \hat{H}^m_{\phi^*}(X^*,X)\end{aligned}$$

This proves the lemma. To complete the proof of the theorem we use the usual exact sequence diagram.

$$\begin{tikzcd}
{} \arrow[r] & \hat{H}^m_{\phi^*}(X^*,X) \arrow[r] & H^m_{\phi^*}(X^*) \arrow[rr, "\alpha(m)"] & & \hat{H}^m_{\phi}(X) \arrow[r] & \hat{H}^{m+1}_{\phi^*}(X^*,X) \arrow[r] & {} \\
& & & H^m_{f(\phi)}(Y) \arrow[ul, "\check{F}"] \arrow[ur, "\check{f}"'] & & &
\end{tikzcd}$$

From the lemma and the assumption on $\hat{H}^m_{\phi}(X(y))$ we have

$$\hat{H}^{m+1}_{\phi}(X^*,X) = 0, \quad m < p \ .$$

This implies that $\alpha(m)$ is an isomorphism for $m < p$ and a monomorphism for $m = p$. $\check{F}(m)$ is an isomorphism for $m < p$ and a monomorphism for $m = p$. Since $\check{f}(m) = \alpha(m)\ \check{F}(m)$, it follows that $\check{f}(m)$ is an isomorphism for $m < p$ and a monomorphism for $m = p$.

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