THE GEOMETRY OF STRUCTURED PARSEVAL FRAMES AND FRAME POTENTIALS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

John Isaac Haas IV

May 2015

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Abstract

In this dissertation, we study the geometric character of structured Parseval frames, which are families of vectors that provide perfect Hilbert space reconstruction. Equiangular Parseval frames (EPFs) satisfy that the magnitudes of the pairwise inner products between frame vectors are constant. These types of frames are useful in many applications. However, EPFs do not always exist and constructing them is often difficult.

To address this problem, we consider two generalizations of EPFs, equidistributed frames and Grassmannian equal-norm Parseval frames, which include EPFs when they exist. We provide several examples of each type of Parseval frame. To characterize and locate these classes of frames, we develop an optimization program involving families of real analytic frame potentials, which are real-valued functions of frames. With the help of the Łojasiewicz gradient inequality, we prove that the gradient descent of these functions on the manifold of Gram matrices of Parseval frames always converges to critical points. We then show that, under certain conditions, the frames corresponding to the Gram matrices of the critical points for different frame potentials possess desirable geometric properties. These properties include the equal-norm, equiangular, non-orthodecomposable, equidistributed and Grassmannian equal-norm cases.

We also discuss the history of EPFs and frame potentials and provide a new characterization of EPFs in terms of the Fourier transform. Using this characterization, we reprove a known result regarding cyclic EPFs and difference sets.

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To my late father, John Isaac Haas III, to my late grandmothers, May Cox and Irene Haas, to my late cousin, Jana Haas, and to my late friend, J.R. Dodd:

This is for you.

Contents

Chapter 1

Introduction

1.1 Frame theory

Orthonormal bases have a long history in pure and applied mathematics. For example, the orthonormal columns of discrete Fourier transform matrices are used in signal processing [83], partial differential equations [2], and the representation theory of finite abelian groups [90]. Although bases are essential tools in mathematics, linear dependence is sometimes useful. A *finite frame* is a spanning set for a finite dimensional Hilbert space that generalizes the notion of an orthonormal basis by relaxing the need for linear independence.

A common example of when frames are useful occurs in signal analysis. If a sender encodes a K-dimensional vector as its inner products with an orthonormal basis and transmits them across a channel, then the loss or corruption of a single coefficient means an entire dimension of the data is lost and there is no guarantee that the receiver can recover the signal. By encoding a signal with a well-conditioned frame instead, it is possible to guarantee recovery after such a loss by exploiting the redundant representation of data that a frame allows [30, 22, 53].

Duffin and Schaeffer introduced frames to address problems in nonharmonic Fourier series in 1952 [38]; however, the popularity of modern frame theory is generally attributed to Daubechies, Grossman, and Meyer's seminal paper [36], where they developed the class of tight frames for signal reconstruction. Today, the theory of frames has proven useful for problems in pure mathematics [25, 66], applied mathematics [49, 86], science [33], and engineering[65]. For more information on the current state of frame theory, we refer to [31].

1.2 Equiangular frames and frame potentials

Since the earliest works, frames equipped with additional properties have been emphasized in research and applications [38, 36]. Some frame properties are spectral in nature, like the Parseval property, which requires perfect reconstruction [10]. Sometimes they involve an underlying algebraic structure, for example, that the frame is generated by a group representation [51]. Still other properties manifest as geometric conditions on the frame's vectors, for instance, requiring that the vectors have the same norm [30].

A particularly interesting geometric property occurs when a frame is *equiangular*, which means the inner products between its vectors have the same magnitude. The study of these objects can be traced back to 1948, when Hanntjes posed the problem of packing equiangular lines in real Euclidean space [50], a problem which was more thoroughly investigated a few decades later by Lemmens, Seidel, and other collaborators [63]. The introduction of this class of frames to the frame theory community is commonly attributed to the works of Heath and Strohmer [86] and Holmes and Paulsen [53], where the authors highlighted a wide range of applications for equiangular frames. In particular, equiangular frames that are additionally *tight*, a slight generalization of the Parseval property, are valuable in numerous settings. In coding theory, they are optimally resilient to one or two erasures [53]. In physics, certain classes of complex equiangular tight frames are ideal models for quantum measurement devices [75, 96]. Further applications occur in combinatorial design theory [86], speech recognition [69], and many other areas. Besides their many applications, equiangular frames are also appealing to the author of this work as instances of beauty in mathematics.

The problem with equiangular tight frames is that they do not always exist, depending on the number of frame vectors. For example, an equiangular tight frame in $\mathbb{C}^3$ can only exist if it is composed of $3, 4, 6, 7$, or 9 vectors [58, 79, 86, 89]. Furthermore, while the case of $\mathbb{C}^3$ is well-understood thanks to a recent result [89], the questions of when equiangular tight frames exist and how to construct them in complex vector spaces of dimension greater than 3 are currently open problems and the focus of much recent research [58, 79, 86, 89, 45, 87, 39].

An increasingly popular approach to locating structured frames involves the use of *frame potentials*, which are real-valued functions of the vectors of frames. By carefully defining such functions, desirable frames can be characterized as their minimizers and then pursued with techniques from optimization theory. Benedetto and Fickus introduced this idea to characterize tight frames whose vectors are all of unit norm [14]. Subsequently, frame potentials have been used to characterize and locate frames with various other properties [14, 26, 75, 72, 15, 20, 41], sometimes addressing deeper questions as well [20].

Of particular interest, the authors of [75] and [72] used a particular frame potential to characterize equiangular tight frames as minimizers. Unfortunately, their characterization depends on the assumption of existence, so it provides little information about how to locate them. The author of [41] applied a gradient descent for this potential on a matrix manifold, but she found that, apart from equiangular tight frames, this potential allows undesirable frames as fixed points.

In this dissertation, we continue the study of geometric frame properties by way of optimization techniques, with an emphasis on Parseval frames. In order to address the problematic fixed points encountered in [41] and the broader issue of existence for equiangular Parseval frames, we introduce two generalizations of equiangularity, the *equidistributed* and *Grassmannian equal-norm Parseval* properties, and develop an optimization program based on frame potentials to characterize and locate them. In order to do this, we also prove that the trajectories corresponding to the gradient descent of an analytic function on a compact, real analytic Riemannian manifold are guaranteed to converge to critical points. Along the way, we explore the history and applications of both equiangular frames and frame potentials, and we provide a characterization of equiangular frames in terms of the Fourier transform.

1.3 Outline

The remaining chapters of this paper are structured as follows.

In Chapter 2, we present the basic notation, terminology, and facts of frame theory. Because this paper focuses on Parseval frames, we also provide a few fundamental facts about this class of frames.

In Chapter 3, we focus on equiangular Parseval frames (EPFs). Section 3.1 explores the history of the topic, beginning with its roots in the equiangular line problem. In Section 3.2, we discuss the problem of existence and outline many of the known construction principles for EPFs. In Section 3.3, we define the equidistributed and Grassmannian equal-norm Parseval properties, which generalize the equiangular property, and provide several examples of each. Finally, in Section 3.4, we view frames from an operator theoretic viewpoint and characterize EPFs in terms of the Fourier transform.

In Chapter 4, we introduce the topic of frame potentials and survey some of its history. We exhibit many instances from frame literature where they are used to characterize frames with desirable properties and conclude by describing how we will use real analytic frame potentials to characterize and locate structured frames on $\mathcal{M}_{N,K}$ (see Definition 2.2.6), the set of Gram matrices for Parseval frames, via gradient descent.

In Chapter 5, we analyze the gradient descent of real analytic functions on real analytic Riemannian manifolds. By using a classical result from [64], we prove that the flow induced by such a descent system is guaranteed to converge in Section 5.2. In order to use this result, we first prove that $\mathcal{M}_{N,K}$ is a real analytic Riemannian manifold in Section 5.1. Finally, in Section 5.3, we provide a characterization of fixed points in this setting.

In Chapter 6, we present the main results of this work. In Section 6.1, we develop the one parameter family of frame potentials, $\{\Phi^{\eta}_{\mathrm{od}}\}_{\eta>0}$ (see Definition 6.1.1), and use a limiting procedure on the global minimizers of this family's members to characterize Grassmannian equal-norm Parseval frames. In Section 6.3, we develop the four parameter family of frame potentials, $\{\Phi^{\alpha,\beta,\delta,\eta} = \Phi^{\eta}_{sum} + \Phi^{\delta}_{diag} +$

$\Phi_{ch}^{\alpha,\beta}\}_{\alpha,\beta,\delta,\eta\in(0,\infty)}$ (see Definitions 6.2.1, 6.3.5, 6.3.8 and 6.3.14). Each member of this family is a sum of the nonnegative potentials Φ_{sum}^{η}, Φ_{diag}^{δ}, and $\Phi_{ch}^{\alpha,\beta}$, and the parameters $\alpha, \beta,$ and δ induce weights that determine the proportionality of how the diagonal versus the off-diagonal entries of the Gram matrices contribute to the potential value. Certain results based on manifolds of equal-norm frames identify undesirable critical points for frame potentials, the so-called orthodecomposable frames [85, 84] (see Definition 6.3.3). In Section 6.3.1, we show that whenever the value of $\Phi_{sum}^{\eta}(G)$ is sufficiently low, then G cannot contain zero entries, thereby ruling out the orthodecomposable case. In Section 6.3.2, we see that whenever G contains no zero entries and $\nabla\Phi_{diag}^{\delta}(G) = 0$ for all δ in a positive open interval, then G is equal norm. In Section 6.3.3, we show that whenever G contains no zero entries, it is equal-norm, and $\nabla\Phi_{ch}^{\alpha,\beta}(G) = 0$ for all α, β in positive open intervals, then G must be what we call equidistributed. Combining these results in Section 6.3.4 leads to a theorem which states that whenever the value $\Phi^{\alpha,\beta,\delta,\eta}(G)$ is sufficiently low and $\nabla\Phi^{\alpha,\beta,\delta,\eta}(G) = 0$ for all α, β, δ in positive open intervals, then G is equidistributed. This is followed by Theorem 6.3.23, where we provide a characterization of equidistributed frames which do not exhibit orthogonality between any of the frame vectors. Finally, in Section 6.4, we see that another limiting procedure gives rise to frames which are both Grassmannian equal-norm Parseval and equidistributed. Along the way, we also provide a simple characterization of equiangular Parseval frames in terms of frame potentials in Section 6.2.

Some of the results from Chapters 3, 5, and 6 were recently published in [19].

Chapter 2

Preliminaries

2.1 Frame essentials

2.1.1 Definition. A family of vectors $\mathcal{F} = \{f_j\}_{j\in J}$ is a *frame* for a real or complex Hilbert space $\mathcal{H}$ if there are constants $0 < A \le B < \infty$ such that for all $x \in \mathcal{H}$,

$$A\|x\|^2 \le \sum_{j\in J} |\langle x, f_j\rangle|^2 \le B\|x\|^2 .$$

We refer to the largest such A and the smallest such B as the *lower* and *upper frame bounds*, respectively. In the case that $A = B$, we call $\mathcal{F}$ a *tight frame*, and whenever $A = B = 1$, then $\mathcal{F}$ is a *Parseval frame*. If $\|f_j\| = \|f_l\|$ for all $j, l \in J$, then $\mathcal{F}$ is an *equal-norm frame*. If $\mathcal{F}$ is a an equal-norm frame and there exists a $C \ge 0$ such that $|\langle f_j, f_l\rangle| = C$ for all $j, l \in J$ with $j \ne l$, then we say $\mathcal{F}$ is *equiangular*. Because they are given special emphasis in Chapter 3, we refer to equiangular Parseval frames by the acronym *EPF*.

The *analysis operator* of the frame is the map $V : \mathcal{H} \to l^2(J)$ given by $(Vx)_j =$

$\langle x, f_j \rangle$. Its adjoint, V^*, is the *synthesis operator*, which maps $a \in l^2(J)$ to $V^*(a) = \sum_{j \in J} a_j f_j$. The *frame operator* is the positive, self-adjoint invertible operator $S = V^*V$ on $\mathcal{H}$ and the *Gramian* is the operator $G = VV^*$ on $\ell^2(J)$.

We focus on the case that $\mathcal{H} = \mathbb{F}^K$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, K is a positive integer, and always choose the canonical sesquilinear inner product. Thus, K always denotes the dimension of $\mathcal{H}$ over the field $\mathbb{F}$. Furthermore, we restrict ourselves to finite frames indexed by $J = \mathbb{Z}_N$, where $N \geq K$, and reserve the letter N to refer to the number of frame vectors in the frame(s) under consideration. When the group structure of $\mathbb{Z}_N$ is not important, we also number the frame vectors with $\{1, 2, \ldots, N\}$, with the tacit understanding that $N \equiv 0 \pmod{N}$.

2.2 Parseval frames

Since this paper is mostly concerned with finite Parseval frames, we call a Parseval frame for $\mathbb{F}^K$ consisting of N vectors an (N, K)-*frame*. If $\mathcal{F}$ is an (N, K)-frame with analysis operator V, then V is an isometry, since $\|Vx\|_2^2 = \sum_{j=1}^N |\langle x, f_j \rangle|^2 = \|x\|^2$ holds for all $x \in \mathbb{F}^K$. Conversely, if an $N \times K$ matrix V is an isometry, then the same argument shows that it is the analysis operator of the (N, K)-frame obtained by taking the columns of V^* as the frame vectors. Hence, the (N, K)-frames are in one-to-one correspondence with $N \times K$ isometry matrices, and they satisfy the reconstruction identity $x = \sum_{j=1}^N \langle x, f_j \rangle f_j$, or in terms of the analysis and synthesis operators, $x = V^*Vx$, or in terms of the frame operator, $S = V^*V = I_K$.

An advantage to working with the space of Parseval frames is that every (N, K)-frame can be identified as the projection of an orthonormal basis in an ambient N-dimensional Hilbert space. This result is ascribed to Naimark [70] and a complete proof can be found in [76]. Subsequently, this result has been generalized to other settings, including Banach spaces [29] and non-Parseval frames[28].

2.2.1 Theorem. *(Naimark, [70, 76].) If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an (N, K)-frame for $\mathcal{H}$, then there exists an N-dimensional Hilbert space H' and an orthonormal basis $\{b_j\}_{j\in\mathbb{Z}_N} \subset \mathcal{H}'$ such that $\mathcal{H}$ is a linear subspace of $\mathcal{H}'$ and $f_j = P_{\mathcal{H}} b_j$ for all $j \in \mathbb{Z}_N$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{H}'$ onto $\mathcal{H}$.*

We prove the converse of this statement.

2.2.2 Theorem. *If $\{b_j\}_{j\in\mathbb{Z}_N}$ is an orthonormal basis for $\mathbb{F}^N$ and $\mathcal{H} \subset \mathbb{F}^N$ is any K-dimensional linear subspace, then $\mathcal{F} = \{P_{\mathcal{H}} b_j\}_{j\in\mathbb{Z}_N}$ is an (N, K)-frame for $\mathcal{H}$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathbb{F}^N$ onto $\mathcal{H}$.*

Proof. Let $x \in \mathcal{H}$ so that $x = P_{\mathcal{H}} x$, then

$$\|x\|^2 = \sum_{j\in\mathbb{Z}_N} |\langle b_j, x\rangle|^2 = \sum_{j\in\mathbb{Z}_N} |\langle b_j, P_{\mathcal{H}} x\rangle|^2 = \sum_{j\in\mathbb{Z}_N} |\langle P_{\mathcal{H}} b_j, x\rangle|^2 = \sum_{j\in\mathbb{Z}_N} |\langle f_j, x\rangle|^2,$$

which proves the claim. □

By interpreting two arbitrary frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ and $\mathcal{F}' = \{f'_j\}_{j\in\mathbb{Z}_N}$ for $\mathbb{F}^K$ as vector-valued functions of the index set $\mathbb{Z}_N$, we endow the set of all such frames with the l^2-*distance* defined by

$$\|\mathcal{F} - \mathcal{F}'\| = \sqrt{\sum_{j\in\mathbb{Z}_N} \|f_j - f'_j\|^2},$$

which we square and reexpress in terms of the respective analysis operators, V and V' as

$$\|\mathcal{F}-\mathcal{F}'\|^2 = \operatorname{tr}((V-V')(V-V')^*) = \operatorname{tr}(VV^*) + \operatorname{tr}(V'V'^*) - 2\Re\operatorname{tr}(VV'^*). \quad (2.1)$$

With respect to this metric, we identify the closest (N,K)-frame to an arbitrary frame.

2.2.3 Proposition. *[8, 20] If $\mathcal{F}=\{f_j\}_{j\in\mathbb{Z}_N}$ is a frame over $\mathbb{F}^K$ with analysis operator V, then $\{S^{-1/2}f_j\}_{j\in\mathbb{Z}_N}$ minimizes the l^2-distance between $\mathcal{F}$ and all possible choices of (N,K)-frames, where $S=V^*V$ is the frame operator of $\mathcal{F}$.*

Proof. Since choosing an (N,K)-frame $\mathcal{F}'$ is equivalent to choosing the corresponding $N\times K$ isometry V', we minimize (2.1) over all possible choices of V'. Noting that the first two terms in this expression are constant, we seek to maximize the third term. After re-expressing V in its unique polar form $V=UP$, where U is an isometry and $P=(V^*V)^{1/2}=S^{1/2}$, the strictly positivity of P (due to the frame property) allows us to interpret the term

$$\operatorname{tr}(VV'^*) = \operatorname{tr}(UPV'^*) = \operatorname{tr}(V'^*UP)$$

as an inner product between V'^* and U. By the Cauchy Schwarz inequality,

$$|\operatorname{tr}(V'^*UP)|^2 \leq |\operatorname{tr}(V'^*VP)||\operatorname{tr}(U^*UP)| = \operatorname{tr}(P)^2,$$

so $2\Re\operatorname{tr}(VV'^*)$ is maximal when $V'=U$. This implies $V=V'S^{1/2}$, which is equivalent to $V'^*=S^{-1/2}V^*$. The claim follows by taking the columns of V'^* as the frame vectors. $\square$

2.2.1 Gram matrices of Parseval frames

Many geometric properties of frames discussed in this paper only depend on the inner products between frame vectors and on their norms, which are collected in the Gramian. For this reason, most of the main results refer to equivalence classes of Parseval frames.

2.2.4 Definition. Two frames $\mathcal{F} = \{f_j\}_{j\in J}$ and $\mathcal{F}' = \{f'_j\}_{j\in J}$ for a real or complex Hilbert space $\mathcal{H}$ are called *unitarily equivalent* if there exists a unitary operator U on $\mathcal{H}$ such that $f_j = Uf'_j$ for all $j \in J$.

Each equivalence class of frames is characterized by the corresponding Gram matrix.

2.2.5 Proposition. *The Gramians of two frames $\mathcal{F} = \{f_j\}_{j\in J}$ and $\mathcal{F}' = \{f'_j\}_{j\in J}$ for a finite dimensional real or complex Hilbert space $\mathcal{H}$ are identical if and only if the frames are unitarily equivalent.*

Proof. Assuming G is the Gramian for the frame $\mathcal{F}$ as well as for the frame $\mathcal{F}'$, then $G = VV^* = V'(V')^*$, where V and V' are the analysis operators belonging to $\mathcal{F}$ and $\mathcal{F}'$, respectively. By the polar decomposition, $V = (VV^*)^{1/2}U = G^{1/2}U$ and $V' = (V'(V')^*)^{1/2}U' = G^{1/2}U'$ with isometries U and U' from $\mathcal{H}$ to $\ell^2(J)$, thus $V^* = U^*U'(V')^*$. By the frame property, the range of U is identical to that of U' and that of G, so $Q = U^*U'$ is unitary, which shows that $V^*e_j = Q(V')^*e_j$ for each canonical basis vector e_j in $\ell^2(J)$, or equivalently, $f_j = Qf'_j$ for all $j \in J$. Conversely, if $\mathcal{F}$ and $\mathcal{F}'$ are unitarily equivalent, then it follows directly that the Gramians of both frames are identical. □

Special emphasis is given to the Gram matrices of (N, K)-frames. If $G = VV^*$

is the Gramian of an (N, K)-frame with analysis operator V, then it is a rank-K orthogonal projection, because $G^*G = VV^*VV^* = VV^* = G$ and the rank of G equals the trace, $\mathrm{tr}(G) = K$. Conversely, if $P : \mathbb{F}^N \to \mathbb{F}^N$ is a rank-K orthogonal projection matrix, then, by the spectral theorem, it can be decomposed into block form as

$$P = U \begin{pmatrix} I_K & 0_{K\times N-K} \\ 0_{N-K\times K} & 0_{N-K\times N-K} \end{pmatrix} U^*,$$

where the top-left block is the $K \times K$ identity matrix and all other blocks are zero. Thus, the $N \times K$ matrix V obtained by deleting the last $N - K$ columns of U is an isometry, so it is the analysis operator of an (N, K)-frame, and since $P = VV^*$, it follows that P is the Gram matrix of a Parseval frame. Therefore, the set of Gramians of (N, K)-frames is precisely the set of rank-K orthogonal projections.

2.2.6 Definition. We define for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\mathcal{M}_{N,K} = \left\{G \in \mathbb{F}^{N\times N} : G = G^2 = G^*, \mathrm{tr}(G) = K\right\}.$$

2.2.7 Proposition. *$\mathcal{M}_{N,K}$ is compact with respect to the topology induced by the Hilbert-Schmidt norm on $\mathbb{F}^{N\times N}$.*

Proof. Since $\mathrm{tr}(G) = K$ for every $G \in \mathcal{M}_{N,K}$, it is a subset of the sphere with radius $\sqrt{K}$ in $\mathbb{F}^{N\times N}$, so it is bounded. If $G = (G_{j,l})_{j,l=1}^N$ is a limit point of $\mathcal{M}_{N,K}$, then there exists a sequence $\left\{G(m) = (G(m)_{j,l})_{j,l=1}^N\right\}_{m=1}^\infty \subset \mathcal{M}_{N,K}$ such that $\lim_{m\to\infty} G_m = G$. Since convergence in Hilbert-Schmidt norm implies convergence in the entries, the diagonal entries of G are real and its off-diagonal entries satisfy

$$\begin{aligned}
|G_{j,l} - G_{l,j}| &= \lim_{m\to\infty} |G_{j,l} - G_{l,j}| \\
&\leq \lim_{m\to\infty} |G_{j,l} - G(m)_{j,l}| + |G(m)_{j,l} - G_{l,j}| \\
&= \lim_{m\to\infty} |G_{j,l} - G(m)_{j,l}| + |G(m)_{l,j} - G_{l,j}| \\
&= 0,
\end{aligned}$$

so G is self-adjoint. The entry-wise convergence also implies

$$\mathrm{tr}(G) = \mathrm{tr}(\lim_{m\to\infty} G_m) = \lim_{m\to\infty} \mathrm{tr}(G_m) = K.$$

Furthermore, since the entries of G^2 are polynomial in the entries of of G, entry-wise convergence also implies

$$G^2 = (\lim_{m\to\infty} G(m))^2 = \lim_{m\to\infty} G(m)^2 = \lim_{m\to\infty} G(m) = G,$$

so G is idempotent. Thus, $G \in \mathcal{M}_{N,K}$, so $\mathcal{M}_{N,K}$ is closed and therefore compact by the Heine-Borel theorem. $\square$

This subset of the Hermitians $\mathcal{M}_{N,K}$ carries the structure of a real analytic submanifold, which is proved in Section 5.1. Because the Gram matrices of (N, K)-frames are the main focus of this paper, whenever an element $G \in \mathcal{M}_{N,K}$ corresponds to an equal-norm or equiangular frame, then we say that G is *equal-norm* or *equiangular*, respectively.

Chapter 3

Equiangular Parseval Frames

3.1 Historical overview

Near the middle of the twentieth century, the problem of determining the maximal number of equiangular lines through K-dimensional Euclidean space appeared in the mathematical literature. Although this classical problem was not originally interpreted from a frame-theoretic perspective, we rephrase it here as follows.

3.1.1 Problem. For each $K \in \mathbb{N}$, what is the maximal value $N \in \mathbb{N}$ for which there exists an equal-norm, equiangular frame $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$ over $\mathbb{F}^K$?

In the real case, by taking the norms of the frame vectors to be of unit length, the elementary formula

$$|\cos\theta| = \frac{|\langle x, y\rangle|}{\|y\|\|y\|},\ \theta \text{ is the acute angle between } x, y \in \mathbb{R}^K, \tag{3.1}$$

reveals that the challenge is to find the largest possible set of lines (spanned by the frame vectors) for which the (acute) angles between all pairs of distinct lines is constant. While this question is relatively simple to understand, it has proven to be difficult to answer, as a general solution remains unknown.

Its investigation began at least as early as 1948 for the real case, when Hanntjes determined the maximal number of equiangular lines through $\mathbb{R}^2$ and $\mathbb{R}^3$ [50]. A few decades later, work in $\mathbb{R}^K$ continued as Delsarte, Goethels, Lemmens, Seidel, van Lint, and other collaborators solved N for most of the values of K ranging between 4 and 23 (see [93, 63] and references therein). The following result is attributed to Gerzon in [63], which establishes a (rarely sharp) bound for N.

3.1.2 Theorem. *(Gerzon bound, [63].) If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equal-norm, equiangular frame for $\mathbb{R}^K$, then*

$$N \leq \frac{K(K+1)}{2}.$$

Around this time, results in the literature also began to reflect an interest in the problem of determining the maximal number of equiangular lines in $\mathbb{C}^K$. In [37, 60], a similar bound is provided for this case.

3.1.3 Theorem. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equal-norm, equiangular frame for $\mathbb{C}^K$, then*

$$N \leq K^2.$$

In [94], Welch established a lower bound on the maximal magnitude among pairwise inner products occurring among a set of lines, which characterizes equiangularity in the case of sharpness. For the purpose of future reference in this paper, we trivially rescale the frame vectors' squared norms in this result by a factor of $\frac{K}{N}$; however, it is usually formulated for the case where the frame's vectors are unit norm.

3.1.4 Theorem. *(Welch Bound, [94].) If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equal-norm frame with $\|f_j\|^2 = \frac{K}{N}$ for all $j \in \mathbb{Z}_N$ and $\mu_{N,K} := \max_{l,j\in\mathbb{Z}_N, l\neq j} |\langle f_j, f_l\rangle|$, then*

$$C_{N,K} \leq \mu_{N,K},$$

where $C_{N,K} = \sqrt{\frac{K(N-K)}{N^2(N-1)}}$. Moreover, equality holds if and only if $\mathcal{F}$ is equiangular.

In light of this result, if $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is a frame with $\|f_j\|^2 = \frac{K}{N}$ for all $j \in \mathbb{Z}_N$ that satisfies equality in this theorem, then

$$\sum_{j,l\in\mathbb{Z}_N} |\langle f_j, f_l\rangle|^2 = \sum_{j,l\in\mathbb{Z}_N, j\neq l} C_{N,K}^2 + \sum_{j\in\mathbb{Z}_N} (\frac{K}{N})^2 = K.$$

Combining this observation with Theorem 4.1.2 (see the next chapter for details), we see that $\mathcal{F}$ is Parseval. We record the constant $C_{N,K} = \sqrt{\frac{K(N-K)}{N^2(N-1)}}$ for future use.

This prompts the following definition.

3.1.5 Definition. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equiangular (N, K)-frame over $\mathbb{F}^K$ for which

$$N = \max\{N' \in \mathbb{N} : \text{an equiangular } (N', K)\text{-frame over } \mathbb{F}^K \text{ exists}\},$$

then we say that $\mathcal{F}$ is *maximal*.

In 1999, Zauner conjectured in his doctoral thesis that maximal EPFs over $\mathbb{C}^K$ always achieve the bound in Theorem 3.1.3 and therefore always consist of K^2 vectors[96, 97]. Moreover, he conjectured that one can always find a so-called fiducial vector whose orbit is a maximal EPF under the unitary action of the Weyl-Heisenberg group, after identifying vectors that are unimodular multiples of each other [96], see also [75, 79, 24]. These objects are appealing to many communities, ranging from quantum theory, where they are referred to as symmetric, informationally complete, positive operator valued measures (SIC-POVMs) [75, 79, 3, 4], to combinatorial design theory, where they correspond to complex projective 2-designs [81, 52, 75, 86]. For these and other reasons, the search for equiangular (K^2, K)-frames over $\mathbb{C}^K$ remains an important open problem in mathematics today.

Shortly after the turn of the millennium, equiangularity was investigated in a frame-theoretic context for the first time by Holmes and Paulsen in [53] and Heath and Strohmer in [86]. By relaxing the emphasis on maximality, frame researchers were able to focus more on the properties and potential applications of general EPFs. For example, they are shown to yield error correction codes which are optimally resilient against one or two channel erasures [53] and are useful for larger numbers of erasures as well [22]. In [86], their connections to several areas of mathematics, including Grassmanian line packings (see Section 6.1 for details), are studied, while, in [18, 9], their value is established for the problem of reconstructing signals when only the magnitudes of the frame coefficients are known. As the diverse applicability of EPFs has become clearer over the last decade, researchers have directed much attention to the question of when they exist and how to construct them. In the next section, we discuss the existence problem and describe many of the known tools for constructing equiangular Parseval frames.

3.2 Existence and constructions of equiangular Parseval frames

Although the study of EPFs has received increasing attention over the last fifteen years, their existence remains a topic about which we know very little. Even with the bounds on the number of frame vectors provided in the preceding section, there are gaps where EPFs do not exist which these bounds do not preclude.

To clarify what we mean by gaps, consider an equiangular (N, K)-frame over $\mathbb{F}^K$. In order to span $\mathbb{F}^K$, the number of frame vectors must satisfy $N \geq K$. If $\mathbb{F} = \mathbb{C}$, then $N \leq K^2$ by Theorem 3.1.3, or if $\mathbb{F} = \mathbb{R}$, then $N \leq \frac{K(K+1)}{2}$ by Theorem 3.1.2. Unfortunately, equiangular (N, K)-frames do not exist for all $K \leq N \leq K^2$ when $\mathbb{F} = \mathbb{C}$ nor for all $K \leq N \leq \frac{K(K+1)}{2}$ when $\mathbb{F} = \mathbb{R}$. For example, although $N = 5$ falls within these necessary bounds for both the real and complex fields when $K = 3$, equiangular $(5, 3)$-frames exist for neither case by considering the Naimark complement of the case where $K = 2$ and $N = 5$ (see Theorem 3.2.1 and the following example for the details of this argument). Our understanding of these gaps is still very primitive.

Indeed, until 2014, it was not completely known for which values $3 \leq N \leq 9$ that equiangular Parseval frames consisting of N vectors exist over $\mathbb{C}^3$. Due to various works, we knew that EPFs exist for $N = 3, 4, 6, 7, 9$ [58, 79, 86]; however, the case of $N = 8$ was only recently settled in [89], where the author used a computer-aided technique from algebraic geometry to prove that complex $(8, 3)$-frames are never equiangular.

Although this example illustrates that our knowledge of this topic is still in an early stage, researchers have made steady progress over the last fifteen years with the aid of tools from various branches of mathematics, including the theories of groups, graphs, and combinatorics. Some of these results manifest as proofs of nonexistence, identifying pairs (N, K) for which EPFs cannot exist, while other results do the opposite, providing algorithms for constructing EPFs.

Before we summarize these techniques, we note that the set equality $\mathbb{C}^K = \operatorname{span}\{cx : c \in \mathbb{C}, x \in \mathbb{R}^K\}$ implies that any real EPF can be viewed as a complex EPF over $\mathbb{C}^K$, while the converse is not true; similarly, a proof of nonexistence for a real, equiangular (N, K)-frame does not, in general, extend to the complex case. As we outline these results, we specify when a given method holds only in the complex setting.

Two examples of EPFs that always exist in the real case are orthonormal bases and those corresponding to regular simplices. Orthonormal bases are, by definition, equiangular (K, K)-frames. Applying the Gram-Schmidt algorithm to any basis yields such a frame, and this obviously works in the complex setting as well. Only slightly less trivially, one can always obtain an equiangular $(K + 1, K)$-frame over $\mathbb{R}^K$ by taking the frame vectors as the vertices of a regular K-simplex centered at the origin [86]. In fact, every equal-norm $(K + 1, K)$-frame is equiangular and unitarily equivalent to a version of this frame where the individual vertices of the simplex are allowed to vary by a factor of ± 1. [47].

Thanks to the work of early pioneers on the subject, the study of EPFs over $\mathbb{R}^K$ can be reformulated as a graph-theoretic problem. In [53, 86], the authors exploit results from [80] to show that the existence of a real EPF is equivalent to the existence of an object known as a *regular two-graph*, thereby converting the problem of locating real equiangular Parseval frames to the problem of finding such graphs.

Because many regular two graphs had already been established in the literature by this time [80], this characterization provided many immediate examples of pairs (N, K) for which real EPFs exist. Concrete examples for many of these frames can be found in [91].

Several nonexistence proofs for real EPFs have also emerged over the years. Some proofs manifest as improvements on Gerzon's necessary bound, including [73], which improves the bound when the equiangular constant satisfies certain conditions, and [13], where the authors use semidefinite programming to improve the bound for $24 \leq K \leq 136$. Other proofs of nonexistence include [87], where the authors assert necessary integrality conditions on the relationship between K and N, and [12], where the possibility of certain real EPFs are excluded due to their relationship with certain spherical designs.

One common approach to locating a complex EPF is to construct its corresponding Gramian. If G is the Gramian of an equiangular (N, K)-frame, then

$$G = \frac{K}{N} I + C_{N,K} Q,$$

where the Q is an $N \times N$ self-adjoint matrix with a vanishing diagonal and unimodular off-diagonal entries known as the *signature matrix* of G. In [53], the authors showed that an arbitrary $N \times N$ self-adjoint matrix with a vanishing diagonal and unimodular off-diagonal entries is the signature matrix for the Gramian of an EPF if and only if it has exactly 2 distinct eigenvalues. Motivated by this characterization, the community has produced a multitude of such matrices. In [82], the author used special subsets of N-element abelian groups which satisfy certain properties, called *signature sets*, to generate signature matrices by the action of their left regular representations on the corresponding free vector spaces. Sometimes signature matrices for EPFs are built up directly with roots of unity [21, 23, 39], while, in other cases, authors focus on their relationship to Hadamard and conference matrices to obtain them [42, 88, 86]. In [86], the authors used conference matrices to prove the existence of two infinite families of equiangular $(2K, K)$-frames.

On the other hand, principles from group representation theory and combinatorics have played an important role in constructing complex EPFs at the level of the individual frame vectors.

Given an abelian group Γ with $|\Gamma| = N$, a broad class of equal-norm (N, K)-frames, known as *harmonic frames*, can be obtained by choosing any K rows from Γ's character table, setting the remaining columns as frame vectors and rescaling appropriately [48, 31]. If these K rows can be chosen to correspond to a *difference set*, which is a subset of Γ satisfying certain combinatorial conditions, then the generated harmonic frame is equiangular [58, 95]. Because difference sets are well-studied [34], this construction gave rise to several infinite families of EPFs. In the next two sections, we discuss *harmonic frames* in further detail, focusing on the case where $\Gamma = \mathbb{Z}_N$.

Another construction that has led to several infinite families of EPFs is the Steiner construction, which is also rooted in principles of combinatorial design theory [45]. One can construct an equiangular $(K+1, K)$-frame with unimodular entries by setting the frame vectors as the columns of a $(K+1) \times (K+1)$ DFT matrix with a single row removed. Such a frame has the intriguing property that the magnitude of the inner products between its vectors is equal to the magnitude of the product of any two entries of its vectors. By exploiting this, the authors were able to perform a tensor-like product with combinatorial objects called Steiner systems to generate EPFs. Similar to the cases of regular two graphs and difference sets, this result produced extensive families of EPFs because Steiner systems are well-studied [34]. Recently, a certain class of Steiner EPFs, called Kirkman EPFs, were shown to be unitary equivalent to a certain subset of harmonic EPFs [56]. In the next section, we generalize the Steiner construction to generate infinite families of equidistributed frames

We remark once more that Zauner conjectured that maximal, complex EPFs can always be obtained by the representation induced by the Weyl-Heisenberg group [96, 24]. The challenge is that a generating vector, called a *fiducial vector*, cannot be arbitrarily chosen, but must instead be carefully selected. It is believed that such vectors exist for all K as an eigenvector of a specific element of the Clifford group, which is the Heisenberg group's normalizer. Fiducial vectors have been confirmed analytically for many values between $2 \leq K \leq 48$ and with high numerical precision for all $K \leq 67$ [79]. Their study remains an active topic of mathematical research [34, 7, 59, 6, 5].

We conclude this section with the following valuable tool, often referred to as the Naimark Complement, which, roughly speaking, halves the problem of studying EPFs.

3.2.1 Theorem. *(Naimark.) If $\mathcal{F}$ is an (N, K)-frame over $\mathbb{F}^K$, then $\mathcal{F}$ is equiangular if and only if there exists an $(N, N-K)$-frame $\mathcal{F}'$ over $\mathbb{F}^{N-K}$ such that $\mathcal{F}'$ is also equiangular.*

Proof. Since $\mathcal{F}$ is an (N, K)-frame, its Gram matrix G is an orthogonal projection onto some subspace of $\mathbb{F}^N$ and $I_N - G$ is an orthogonal projection onto the complementary subspace, which corresponds to a class of unitarily equivalent $(N, N-K)$-frames as described in Section 2.2.1. Since the magnitudes of the off-diagonal entries of G and $I_N - G$ are identical, the claim follows. $\square$

For example, since neither real nor complex, equiangular $(5, 2)$-frames can exist due to the bounds in Theorem 3.1.2 and Theorem 3.1.3, respectively, it follows from Theorem 3.2.1 that equiangular $(5, 3)$-frames do not exist in these settings either. Similarly, since complex, equiangular $(7, 3)$-frames exist due to a difference set construction [58], taking the Naimark complement shows that complex, equiangular $(7, 4)$-frames must also exist.

For a more thorough account of the current knowledge regarding the existence and construction of equiangular Parseval frames, we refer to [44].

3.3 Generalizations of equiangular Parseval frames

To address the issue of nonexistence of equiangular Parseval frames, this section presents two classes of equal-norm (N, K)-frames that include equiangular frames whenever they exist. Both classes are nonempty for all pairs of positive integers K and $N \geq K$.

3.3.1 Equidistributed frames

While studying frame potentials, it became apparent that a certain class of critical points possess what we call the *equidistributed* property. This class of frames includes many structured frames that have already appeared in the literature: equiangular Parseval frames, mutually unbiased bases, and group frames. Surprisingly, the numerical implementation of a relatively simple optimization problem based on frame potentials led to frames with special structures, including Examples 3.3.4 and 3.3.5, as well as Examples 3.3.21 and 3.3.22 further below.

3.3.1 Definition. Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be an (N, K)-frame and let G be its Gramian. The frame $\mathcal{F}$ is called *equidistributed* if for each pair $p, q \in \mathbb{Z}_N$, there exists a permutation π on $\mathbb{Z}_N$ such that $|G_{j,p}| = |G_{\pi(j),q}|$ for all $j \in \mathbb{Z}_N$. In this case, we also say that G is equidistributed.

In other words, $\mathcal{F}$ is equidistributed if and only if the magnitudes in any column of the Gram matrix repeat in any other column, up to a permutation of their position. For Parseval frames, equidistribution implies that all frame vectors have the same norm.

3.3.2 Proposition. *If $\mathcal{F}$ is an equidistributed (N, K)-frame, then $\|f_j\|^2 = K/N$ for each $j \in \mathbb{Z}_N$.*

Proof. By assumption, for each $p \in \mathbb{Z}_N$ there exists π such that $|G_{j,p}| = |G_{\pi(j),1}|$ holds for the entries of the associated Gram matrix G for all $j \in \mathbb{Z}_N$ and thus by the Parseval identity

$$\|f_p\|^2 = \sum_{j=1}^N |\langle f_p, f_j\rangle|^2 = \sum_{j=1}^N |G_{j,p}|^2 = \sum_{j=1}^N |G_{\pi(j),1}|^2 = \|f_1\|^2 .$$

The trace condition $\sum_{j=1}^{N} G_{j,j} = \sum_{j=1}^{N} \|f_j\|^2 = K$ for the Gram matrices of Parseval frames then implies that each vector has the claimed norm. $\square$

An equidistributed frame can be interpreted as a special case of an s-*distance set*, which is an equal-norm (N, K)-frame, $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$, for which $s = |\{|\langle f_j, f_l\rangle| : j, l \in \mathbb{Z}_N, j \neq l\}|$; that is, s is the number of distinct magnitudes of inner products occurring between the frame vectors. These objects are interesting when s is small relative to $\frac{N(N-1)}{2}$, because they are difficult to construct and often exhibit more geometric structure. If a frame is equidistributed, then it is an s-distance set with $1 \leq s \leq N-1$, because all possible magnitudes of inner products between its distinct vectors occur on the first row of the Gram matrix. Gerzon-like bounds for the maximal number of vectors that can be achieved for various values of s are provided in [37, 71], while the special case $s = 2$ is emphasized in the study of mutually unbiased bases [77, 54, 4], which we discuss more below.

To illustrate our definition of equidistributed frames, we provide several examples. To begin, every equiangular Parseval frame is equidistributed.

3.3.3 Example. Equiangular Parseval frames. Let G be the Gram matrix of an equiangular (N, K)-frame. Since the magnitudes of the entries of any column of G consist of $N-1$ instances of $C_{N,K}$ and one instance of $\frac{K}{N}$, G is equidistributed.

A set of *mutually unbiased bases* is union of orthonormal bases such that the modulus of the inner product between any two vectors from distinct bases is constant. Such frames share many properties with equiangular Parseval frames [78, 55]. A slightly more general class, *mutually unbiased basic sequences*, consists of frames which are unions of orthonormal sequences such that the modulus of the inner product between any two vectors from distinct sequences is constant. After an appropriate rescaling of the vectors' norms, every mutually unbiased basic sequence is equidistributed.

3.3.4 Example. Mutually Unbiased Basic Sequences. Let $N = ML$ and $G \in \mathcal{M}_{N,K}$ be such that the matrix Q whose entries are $Q_{j,l} = |G_{j,l}|$ is the sum of Kronecker products of the form $Q = bI_M \otimes I_L + c(J_M - I_M) \otimes J_L$, where $b > 0$, $c \geq 0$, the matrices I_M and I_L are the $M \times M$ and $L \times L$ identity matrices, and J_M and J_L are the matrices of corresponding size whose entries are all 1. Each row of G has one entry of magnitude b, $L-1$ vanishing entries and $(M-1)L$ entries of magnitude c, so G is equidistributed. We also provide a concrete nontrivial example of such a $(6,4)$-frame with $M = 3$ and $L = 2$.

Let $\omega = e^{2\pi i/8}$, a primitive 8-th root of unity, $\lambda = \sqrt{\frac{1}{18}}$ and let

$$G = \begin{pmatrix} \frac{2}{3} & 0 & \lambda & i\lambda & \lambda & \lambda \\ 0 & \frac{2}{3} & i\lambda & \lambda & -\lambda & \lambda \\ \lambda & -i\lambda & \frac{2}{3} & 0 & \lambda\omega^5 & \lambda\omega^3 \\ -i\lambda & \lambda & 0 & \frac{2}{3} & \lambda\omega & \lambda\omega^3 \\ \lambda & -\lambda & \lambda\omega^3 & \lambda\omega^7 & \frac{2}{3} & 0 \\ \lambda & \lambda & \lambda\omega^5 & \lambda\omega^5 & 0 & \frac{2}{3} \end{pmatrix}.$$

One can verify that $G = G^* = G^2$ and clearly $\mathrm{tr}(G) = 4$. Thus, $G \in \mathcal{M}_{6,4}$. Since the magnitudes of the entries of every column consist of one instance of 0, one instance of $\frac{2}{3}$, and four instances of λ, it follows that G is equidistributed.

Next, we present an example for a type of frame that generalizes mutually unbiased basic sequences and equiangular Parseval frames. We call it a *block-equiangular* Parseval frame.

3.3.5 Example. Block-equiangular Parseval frame. Let $N = ML$ and $G \in \mathcal{M}_{N,K}$ be such that the matrix Q whose entries are $Q_{j,l} = |G_{j,l}|$ is the sum of Kronecker products of the form $Q = bI_M \otimes I_L + cI_M \otimes (J_L - I_L) + d(J_M - I_M) \otimes J_L$, where $b > 0$, $c, d \geq 0$. Each row of G has one entry of magnitude b, $L-1$ entries of magnitude c, and $(M-1)L$ entries of magnitude d. We provide an example for the Gram matrix of such a $(8,3)$-frame with $M = 4$ and $L = 2$.

Let $\mu = \frac{\sqrt{7/3}}{8}$, $\rho = \frac{1}{24}(-3 + 2\sqrt{3}i)$, $\zeta = \arccos(\sqrt{3/7}/2)$, $\lambda = \frac{1}{120}(\sqrt{21} - 24e^{-i\zeta})$, $\nu = \frac{1}{120}(7 + (8 - 20e^{i\pi/3})e^{2i\zeta})$, $\kappa = \frac{1}{120}(7 + (8 + 20e^{2i\pi/3})e^{2i\zeta})$, then

$$G = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \mu & \mu & \mu & \mu & \mu & \mu \\ \frac{1}{8} & \frac{3}{8} & \overline{\lambda} & \lambda & \lambda & \overline{\lambda} & \lambda & \overline{\lambda} \\ \mu & \lambda & \frac{3}{8} & \overline{\kappa} & \overline{\rho} & \overline{\rho} & \overline{\nu} & \rho \\ \mu & \overline{\lambda} & \kappa & \frac{3}{8} & \overline{\rho} & \nu & \rho & \rho \\ \mu & \overline{\lambda} & \rho & \rho & \frac{3}{8} & \kappa & \overline{\rho} & \nu \\ \mu & \lambda & \rho & \overline{\nu} & \overline{\kappa} & \frac{3}{8} & \overline{\rho} & \overline{\rho} \\ \mu & \overline{\lambda} & \nu & \overline{\rho} & \rho & \rho & \frac{3}{8} & \kappa \\ \mu & \lambda & \overline{\rho} & \overline{\rho} & \overline{\nu} & \rho & \overline{\kappa} & \frac{3}{8} \end{pmatrix}$$

is seen to be equidistributed by checking $|\kappa| = \frac{1}{8}$ and $|\rho| = |\nu| = |\lambda| = \mu$.

3.3.6 Example. Group frames. Let Γ be a finite group of size $N = |\Gamma|$ and $\pi : \Gamma \to B(\mathcal{H})$ be an orthogonal or unitary representation of Γ on the real or complex K-dimensional Hilbert space $\mathcal{H}$, respectively. Consider the orbit $\mathcal{F} = \{f_g = \pi(g)f_e\}_{g\in\Gamma}$ generated by a vector f_e of norm $\|f_e\| = \sqrt{K/N}$, indexed by the unit e of the group. If $\mathcal{F}$ is a Parseval frame $\mathcal{F} = \{f_g\}_{g\in\Gamma}$, then $\mathcal{F}$ is equidistributed, because $\langle f_g, f_h\rangle = \langle \pi(h^{-1}g)f_e, f_e\rangle$ and left multiplication by h^{-1} acts as a permutation on the group elements.

To have the Parseval property, it is sufficient for the representation to be irreducible [92], but it is not necessary. For example, *cyclic* (N, K)*-frames* can be constructed with representations of $(\mathbb{Z}_N, +)$ on $\mathcal{H}$, although irreducible representations of abelian groups are always of degree 1.

Consider the Discrete Fourier Transform matrix, $D = \frac{1}{\sqrt{N}}(e^{2\pi ijl/N})_{j,l=1}^N$. It is a rescaled matrix representation of the character table for $\mathbb{Z}_N$ and its columns form an orthonormal basis for $\mathbb{C}^N$. If A is a $K \times N$ matrix obtained by deleting any choice of $N - K$ rows from D, then its columns form a Parseval frame for $\mathbb{C}^K$ by Theorem 2.2.2, because the deletion of rows is isomorphically equivalent to an application of a coordinate projection to the columns of D. This process produces a harmonic frame as defined in the preceding section.

3.3.7 Definition. Let $n_1, n_2, ..., n_K \in \{1, 2, ..., N\}$ be any choice of distinct integers. An (N, K)-frame $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$, where $f_j = \frac{1}{\sqrt{N}}(e^{2\pi ijn_l/N})_{l=1}^K$ for all $j \in \mathbb{Z}_N$, is called a *cyclic* (N, K)*-frame generated by the sequence* $\{n_1, n_2 ..., n_K\}$. When the sequence $\{n_1, n_2 ..., n_K\}$ is not important, we call $\mathcal{F}$ a *cyclic frame*.

If $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$ is a cyclic frame, then it is the orbit of the vector whose entries are all $\frac{1}{\sqrt{N}}$ under the unitary action of the representation for $(\mathbb{Z}_N, +)$ defined by

$$j \mapsto \mathrm{diag}(e^{2\pi ijn_l/N})_{j=1}^K.$$

Thus, every cyclic frame is equidistributed, and, because the construction described above works for every pair of positive integers K and N with $K < N$, the existence of equidistributed frames is guaranteed in the complex setting.

3.3.8 Theorem. *For every* $N > K$*, an equidistributed* (N, K)*-frame over* $\mathbb{C}^K$ *exists.*

In the last section, we described two classes of EPFs with constructions based on combinatorial design principles: harmonic EPFs generated by difference sets and Steiner EPFs. In the upcoming example, we combine these ideas to produce infinite families of equidistributed frames. To begin, we explain the difference set construction for EPFs from [58, 95].

3.3.9 Definition. A (K, λ) *-difference set for* $\mathbb{Z}_N$ is a subset of distinct elements $\{n_1, n_2, ..., n_K\} \subset \mathbb{Z}_N$ such that every nonzero element $x \in \mathbb{Z}_N$ can be expressed as $x = n_j - n_l$ in exactly λ ways, where λ is a positive integer.

3.3.10 Theorem. *(Cyclic EPFs, [58, 95].) If $\mathcal{F}$ is a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2 ..., n_K\}$, then $\mathcal{F}$ is equiangular if and only if $\{n_1, n_2, ..., n_K\}$ is a (K, λ) -difference set for $\mathbb{Z}_N$ where $\lambda = \frac{K(K-1)}{N-1}$ is a positive integer.*

Proof. This is proved at the end of Section 3.4.2 with an approach based on the Fourier transform. □

For example, the set $\{1, 2, 4\}$ is a $(3, 1)$-difference set of the additive group $\mathbb{Z}_7$, since $1 - 2 = 6, 2 - 1 = 1, 1 - 4 = 4, 4 - 1 = 3, 2 - 4 = 5$ and $4 - 2 = 2$, so the cyclic frame generated by this set is equiangular.

Now we turn to Steiner systems.

3.3.11 Definition. Let α and β be positive integers so that $\kappa := \frac{\alpha-1}{\beta-1}$ and $\frac{\alpha}{\beta}\kappa$ are also positive integers. An $\frac{\alpha\kappa}{\beta} \times \alpha$ matrix A whose entries consist entirely of zeros and ones is an (α, β)-*Steiner matrix* if

1. A has exactly β ones in each row,
2. A has exactly κ ones in each column, and

3. every two distinct columns of A have a dot product of one.

In [45], the authors used Steiner systems to generate EPFs. Here, we generalize their approach to generate equidistributed frames

3.3.12 Example. Steiner equidistributed frames. Let A be an (α, β)-Steiner matrix with $\kappa := \frac{\alpha-1}{\beta-1}$ as above, let W^* be the synthesis matrix for an equiangular cyclic (ν, κ)-frame $\mathcal{C}$ generated by a (κ, λ) -difference set, and let V be the $\alpha\nu \times \frac{\alpha\kappa}{\beta}$ matrix whose adjoint V^* is defined as follows:

1. For each $j \in \{1, .., \alpha\}$, let F_j be the $\frac{\alpha\kappa}{\beta} \times \nu$ matrix obtained by replacing each one in the jth column of A with a distinct row from W^* and each zero entry with a row of zeros.

2. Concatenate and rescale to obtain $V^* = \frac{1}{\sqrt{\beta}}[F_1 F_2 \cdots F_\alpha]$.

Since the columns of W^* are the frame vectors of $\mathcal{C}$, the entries of W^* are of constant magnitude $\frac{1}{\sqrt{\nu}}$ by the definition of a cyclic frame. Using this fact along with property (1) in the definition of a Steiner matrix, it is straightforward to verify that $V^*V = I_{\alpha\kappa/\beta}$, so the columns of V^* form an $(\alpha\nu, \frac{\alpha\kappa}{\beta})$-frame $\mathcal{F}$, and by using this fact with property (2), $\mathcal{F}$ is equal-norm. To see that $\mathcal{F}$ is equidistributed, let $j \in \{1, 2, ..., \alpha\}$, $s \in \mathbb{Z}_\nu$ and consider the frame vector $f_s^{(j)}$ corresponding to sth column of the block F_j. Since $\mathcal{C}$ is equiangular, it follows that

$$|\langle f_s^{(j)}, f_t^{(j)} \rangle| = \frac{1}{\beta} C_{\nu,\kappa}$$

for all $s \in \mathbb{Z}_\nu$ with $s \neq t$. On the other hand, by the fact that the magnitudes of the nonzero entries of W^* are $\frac{1}{\sqrt{\nu}}$ combined with property (3) in the definition of a

Steiner matrix, we have

$$|\langle f_s^{(j)}, f_t^{(l)} \rangle| = \frac{1}{\nu\beta}$$

for all $t \in \mathbb{Z}_\nu$ and $l \in \{1, 2, ..., \alpha\}$ with $j \neq l$. In terms of the Gramian, this gives $\nu - 1$ instances of $\frac{1}{\beta} C_{\nu,\kappa}$ and $\nu(\alpha - 1)$ instances of $\frac{1}{\nu\beta}$ occurring among the magnitudes of the off-diagonal entries of the column corresponding to the vector $f_s^{(j)}$. Since j and s were arbitrary, this shows that $\mathcal{F}$ is equidistributed.

For a concrete example, let $\omega_7 = e^{2\pi i/7}$ and let

$$W^* = \frac{1}{\sqrt{7}} \begin{pmatrix} \omega_7^{1(1)} & \omega_7^{2(1)} & \omega_7^{3(1)} & \omega_7^{4(1)} & \omega_7^{5(1)} & \omega_7^{6(1)} & \omega_7^{7(1)} \\ \omega_7^{1(2)} & \omega_7^{2(2)} & \omega_7^{3(2)} & \omega_7^{4(2)} & \omega_7^{5(2)} & \omega_7^{6(2)} & \omega_7^{7(2)} \\ \omega_7^{1(4)} & \omega_7^{2(4)} & \omega_7^{3(4)} & \omega_7^{4(4)} & \omega_7^{5(4)} & \omega_7^{6(4)} & \omega_7^{7(4)} \end{pmatrix}.$$

This is the synthesis matrix for the cyclic $(7, 3)$-frame $\mathcal{C}$ generated by the $(3, 1)$-difference set $\{1, 2, 4\}$ described above, so $\mathcal{C}$ is equiangular by Theorem 3.3.10. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which is a $(4, 2)$-Steiner matrix with $\kappa = 3$. If r_1, r_2, r_3 denote the rows of W^* and z

denotes the 1×7 zero matrix, then the above construction yields

$$V^* = \frac{1}{2} \begin{pmatrix} r_1 & r_1 & z & z \\ r_2 & z & r_1 & z \\ r_3 & z & z & r_1 \\ z & r_2 & r_2 & z \\ z & r_3 & z & r_2 \\ z & z & r_3 & r_3 \end{pmatrix},$$

which is the synthesis matrix for an equidistributed $(28, 6)$-frame with 6 instances of $\frac{1}{2}\sqrt{\frac{2}{49}}$ and 21 instances of $\frac{1}{14}$ occuring among the magnitudes of off-diagonal entries for any choice of column from its Gram matrix.

Numerous examples of difference sets and Steiner matrices can be found in combinatorial design literature [34], so this construction generates many infinite families of equidistributed frames.

In the complex case, the Steiner EPFs introduced in [45] can be viewed as special cases of Steiner equidistributed frames. It is easy to check that $\{1, 2, ..., \kappa\}$ always forms a $(\kappa, \kappa - 1)$-difference set for $\mathbb{Z}_{\kappa+1}$ and the cyclic EPF generated by this sequence has the equiangular constant $C_{\kappa+1,\kappa} = \frac{1}{\kappa+1}$. In light of this, if the equiangular cyclic frame $\mathcal{C}$ in Example 3.3.12 is generated by this sequence, then $\mathcal{F}$ is a Steiner EPF.

As a final example, tensor products of equidistributed frames are again equidistributed.

3.3.13 Example. Tensor Products of Equidistributed Frames. Let $1 \leq K_1 < N_1$ and $1 \leq K_2 < N_2$ be integers, let $G_1 \in \mathcal{M}_{N_1,K_1}$ and $G_2 \in \mathcal{M}_{N_2,K_2}$ be equidistributed, and consider the Kronecker product $G = G_1 \otimes G_2$. Then G is an $N_1N_2 \times N_1N_2$ Hermitian matrix such that $G^2 = (G_1 \otimes G_2)^2 = G_1^2 \otimes G_2^2 = G_1 \otimes G_2 = G$, so it is an orthogonal projection. Furthermore, $G_{j,j} = \frac{K_1K_2}{N_1N_2}$ for all $j \in \mathbb{Z}_{N_1N_2}$, so $\mathrm{tr}(G) = K_1K_2$. Therefore, $G \in \mathcal{M}_{N_1N_2,K_1K_2}$. Now let $p, q \in \mathbb{Z}_{N_1N_2}$ with $p = p_1N_1 + p_2$ and $q = q_1N_1 + q_2$, and let $Q, Q_1,$ and Q_2 denote the matrices whose entries are the absolute values of the entries of $G, G_1,$ and G_2, respectively. Since G_1 and G_2 are equidistributed, row p of Q is of the form

$$\rho_p = \begin{pmatrix} (Q_1)_{p_1,1}X & (Q_1)_{p_1,2}X & \cdots & (Q_1)_{p_1,N_1}X \end{pmatrix},$$

where X is row p_2 of Q_2 and row q of Q is of the form

$$\rho_q = \begin{pmatrix} (Q_1)_{q_1,1}Y & (Q_1)_{q_1,2}Y & \cdots & (Q_1)_{q_1,N_1}Y \end{pmatrix},$$

where Y is row q_2 of Q_2. Since G_1 and G_2 are equidistributed, there is π_1 such that $|(Q_1)_{q_1,j}| = |(Q_1)_{q_2,\pi_1(j)}|$ for each $j \in \mathbb{Z}_{N_1}$ and similarly, the magnitudes of the entries in Y are obtained from those in X by applying a permutation π_2 to the indices. Thus, the magnitudes of the entries of ρ_q are a permutation of those of ρ_p, so G is equidisributed.

In analogy to the Naimark complement for EPFs, a similar duality holds for equidistributed frames.

3.3.14 Proposition. *(Naimark complement for equidistributed frames.) If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an (N, K)-frame over $\mathbb{F}^K$, then $\mathcal{F}$ is equidistributed if and only if there exists an equidistributed $(N, N-K)$-frame $\mathcal{F}' = \{f'_j\}_{j\in\mathbb{Z}_N}$ over $\mathbb{F}^{N-K}$ which satisfies*

$$|\langle f_1, f_j\rangle| = |\langle f'_1, f'_j\rangle| \textit{ for all } j \in \mathbb{Z}_N.$$

Proof. The proof is identical to that of Theorem 3.2.1. □

To conclude this section, we characterize equidistributed frames in terms of families of sums of exponentials. This result is fundamental to the classification of critical points of the combined potential, $\Phi^{\alpha,\beta,\delta,\eta}$, in Theorem 6.3.16. To prepare this, we introduce the notion of a frame being α-equipartitioned.

3.3.15 Definition. Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be an (N, K)-frame and fix $\alpha \in (0, \infty)$. For any $x \in \mathbb{Z}_N$, define $A_x^\alpha := \sum_{j\in\mathbb{Z}_N} e^{\alpha|\langle f_j, f_x\rangle|^2}$. If $A_x^\alpha = A_y^\alpha$ for all $x, y \in \mathbb{Z}_N$, then we say that $\mathcal{F}$ is *α-equipartitioned*. If G is the Gram matrix of $\mathcal{F}$, then we also say that G is α-equipartitioned.

3.3.16 Proposition. *Let $G = (G_{j,l})_{j,l=1}^N$ be the Gramian of an (N, K)-frame $\mathcal{F}$, and let $I \subseteq (0, \infty)$ be any open interval, then G is equidistributed if and only G is α-equipartitioned for all $\alpha \in I$.*

Proof. If G is equidistributed, the magnitudes of every column are the same as those of any other column, up to permutation. Thus, by definition of α-equipartitioning, it is trivial to verify that then G is α-equipartitioned for all $\alpha \in I$.

Conversely, consider for each $x \in \mathbb{Z}_N$ the function $f_x : (0, \infty) \to \mathbb{R} : \alpha \mapsto \sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{x,j}|^2}$. Let $x, y \in \mathbb{Z}_N$ be arbitrary. If $f_x(\alpha) = f_y(\alpha)$ for all $\alpha \in \mathbb{R}$ then since f_x and f_y are both analytic functions which agree on an open interval, it follows by the principle of analytic continuation that they must agree on all of $(0, \infty)$. In particular, this means that $\sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{x,j}|^2} = \sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{y,j}|^2}$ for all $\alpha \in (0, \infty)$. Thus,

$$\lim_{\alpha\to\infty} \frac{1}{\alpha} \log \left(\sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{x,j}|^2} \right) = \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \left(\sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{y,j}|^2} \right) .$$

If the maximum magnitude in row x is not equal to the maximum magnitude of row y, then this equation cannot hold. Similarly, if these maximal magnitudes did not occur with the same multiplicity in each column, then again the equation would not be possible. Thus, we can remove the index sets M_x and M_y corresponding to the maximal magnitudes in rows x and y from the sum in the definition of f_x and f_y to obtain the new identity

$$\sum_{j\in\mathbb{Z}_N\setminus M_x} e^{\alpha|G_{x,j}|^2} = \sum_{j\in\mathbb{Z}_N\setminus M_y} e^{\alpha|G_{y,j}|^2}$$

for all $\alpha \in (0,\infty)$. Repeating the procedure of isolating the strongest growth rate shows that every possible magnitude that appears in row x must agree in multiplicity with every possible magnitude that appears in row y. In other words, the magnitudes in row x are just a permutation of those in row y. Since x and y were arbitrary, we conclude that G is equidistributed. $\square$

3.3.2 Grassmannian equal-norm Parseval frames

In [86], the authors coined the term *Grassmannian frame* to describe a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ over $\mathbb{R}^K$ which minimizes the maximal magnitude among inner products between distinct frame vectors subject to $\|f_j\| = 1$ for all $j \in \mathbb{Z}_N$. Although EPFs do not exist for all choices of K and $N \geq K$, a compactness argument shows that this class of frames is a useful generalization which is never empty. This section concerns an alternative class of frames: those which optimize this objective function with the additional constraint of Parsevality. Because there are non-equiangular solutions to both programs which coincide, we digress momentarily to discuss the first case and its relationship to the line packing problem.

The origin of the term *Grasmannian frame* is attributed to the problem of packing points into the *Grassmannian space* $\mathcal{G}(K, L)$, which is the compact Riemannian manifold consisting of all L-dimensional subspaces in $\mathbb{R}^K$ [68].

3.3.17 Problem. Given $N, K, M \in \mathbb{N}$ with $K \geq M$ and a continuous metric d defined on $\mathcal{G}(K, L)$, how can one arrange a set of N subspaces $\{S_j\}_{j \in \mathbb{Z}_N}$ in $\mathbb{R}^K$ so that it maximizes

$$\min_{j,l \in \mathbb{Z}_N, j \neq l} d(S_j, S_l)?$$

The compactness of $\mathcal{G}(K, L)$ and continuity of this objective function show that a maximizer for this problem, called a *Grassmannian packing*, always exists, so the problem is well-defined. In their landmark paper [35], Conway, Hardin, and Sloane chose this metric to be the chordal distance defined by

$$d_c(S, T) = \sqrt{\sin^2 \theta_1 + \cdots + \sin^2 \theta_L},$$

where $\{\theta_j\}_{j=1}^N$ are the principal angles between the subspaces S and T. By associating the subspaces in $\mathbb{R}^K$ to their corresponding orthogonal projection matrices, they embedded the metric space $(\mathcal{G}(K, L), d_c)$ into a Euclidean sphere.

3.3.18 Theorem. *(Spherical Embedding, [35].) The map*

$$S \mapsto \tilde{P}_S = P_S - \frac{L}{K} I_K$$

is an isometric embedding of $\mathcal{G}(K, L)$ into the sphere of radius $\sqrt{L/2}$ in $\mathbb{R}^D$ with $D = \frac{K(K+1)}{2} - 1$, where P_S denotes the orthogonal projection onto the subspace $S \in \mathcal{G}(K, L)$ and $d_c(S, T) = \frac{1}{\sqrt{2}} \|P_S - P_T\|_2$.

Proof. Suppose $L \leq K/2$.

By the requirements of symmetricity and that the trace is L for any matrix which projects orthogonally onto an element of $\mathcal{G}(K,L)$, it follows by dimension counting that $\mathrm{span}\{\tilde{P}_S \in M_K(\mathbb{R}) : P_S \text{ projects orthogonally onto } S \in \mathcal{G}(K,L)\}$ and $\mathbb{R}^D$ are isomorphic as vector spaces. Since every such orthogonal projection is the Gram matrix for a (K,L)-frame, it follows from Parsevality that $\|\tilde{P}_S\|_2^2 = \frac{L(K-L)}{K}$ for all $S \in \mathcal{G}(K,L)$. Therefore, by the equality of the L^2-norm and the Hilbert-Schmidt norm on $M_K(\mathbb{C})$, it is sufficient to show that $d_c(S,T) = \frac{1}{\sqrt{2}}\|P_S - P_T\|_{H.S.}$ for all $S,T \in \mathcal{G}(K,L)$.

Let $S,T \in \mathcal{G}(K,L)$ with principal angles $\{\theta_j\}_{j=1}^L$. With an appropriate choice of basis, we can assume S is spanned by the columns of the matrix $A_S = (e_j)_{j=1}^L$, where $\{e_k = (\delta_{j,k})_{j=1}^K\}_{k=1}^K$is the canonical basis for $\mathbb{R}^K$. Furthermore, after transformation by an appropriate orthogonal operator, we may assume that T is spanned by the columns of the matrix $A_T = (\cos\theta_j e_j + \sin\theta_j e_{j+L})_{j=1}^L$. Because the columns of A_S and A_T are orthonormal bases for S and T, respectively, their corresponding orthogonal projections are diagonal matrices given by $P_S = A_S A_S^* = \mathrm{diag}\,(1_L(j))_{j=1}^K$ and $P_T = A_T A_T^* = \mathrm{diag}\,(1_L(j)\cos^2\theta_j))_{j=1}^K$, where 1_L is the indicator function on the set $\{1,2,\cdots,L\}$. In particular, $\mathrm{tr}(P_S P_T) = \sum_{j=1}^L \cos^2\theta_j$, so

$$
\begin{aligned}
d_c(S,T)^2 &= \sum_{j=1}^L \sin^2\theta_j \\
&= L - \sum_{j=1}^L \cos^2\theta_j \\
&= L - \mathrm{tr}(P_S P_T) \\
&= \frac{1}{2}\|P_S - P_T\|_{H.S.}^2
\end{aligned}
$$

This proves the claim for $L \leq K/2$. The case $L > K/2$ follows by noting that the elements of $G(K, L)$ are just the orthogonal complements of elements in $G(K, K-L)$. □

This result allowed the authors of [35] to extrapolate an upper bound on the minimal positive distance between the subspaces of a packing from Rankin's results on spherical caps [74]. A *spherical cap centered at x with angle θ* is defined by $C_x(\theta) = \{y \in \mathbb{R}^D : \|y\|_2 = 1, \arccos(\langle x, y\rangle) < \theta\}$ and a *θ-packing* is a set of points $\{x_1, ..., x_t\}$ on the unit sphere in $\mathbb{R}^D$ such that $C_{x_j}(\theta) \cap C_{x_l}(\theta) = \emptyset$ for all $j \neq l$. In particular, if given a Grassmannian packing with a sufficiently large number of subspaces, then, after embedding it into the sphere, the minimal distance between distinct points cannot exceed the minimal distance between distinct vertices on a *regular orthoplex,* which is a generalization of the octahedron to higher dimensions given by the unit ball induced by the l^1 norm.

3.3.19 Theorem. *(Orthoplex Bound, [74, 35].) Let $N > \frac{K(K+1)}{2}$. If $\{S_j\}_{j\in\mathbb{Z}_N} \subset \mathcal{G}(K, L)$ is a Grassmannian packing, then*

$$\min_{j,l\in\mathbb{Z}_N, j\neq l} d_c(S_j, S_l)^2 \leq \frac{L(K-L)}{K} \text{ for all } j, l \in \mathbb{Z}_N, \tag{3.2}$$

and equality occurs if the N points correspond to a subset of the $(K-1)(K+2)$ vertices of a regular orthoplex.

Proof. By Theorem 3.3.18, we can identify $\{S_j\}_{j\in\mathbb{Z}_N}$ with points $\{x'_j\}_{j\in\mathbb{Z}_N}$ on the sphere with radius $\sqrt{\frac{L(K-L)}{K}}$ centered at the origin in $\mathbb{R}^D$, where $D = \frac{K(K+1)}{2} - 1$, and we have

$$d_c(S_j, S_l)^2 = \frac{1}{2}\|x'_j - x'_l\|_2^2 = \frac{L(K-L)}{K}(1 - \cos\phi_{j,l}), \tag{3.3}$$

where $0 \leq \phi_{j,l} \leq \pi$ is the angle between x'_j and x'_l.

Thus, (3.2) is equivalent to showing $\min_{j,l \in \mathbb{Z}_N, j \neq l}(1 - \cos\phi_{j,l}) \leq 1$, which is equivalent to showing

$$\phi_{min} := \min_{j,l \in \mathbb{Z}_N, j \neq l} \phi_{j,l} \leq \frac{\pi}{2}, \tag{3.4}$$

and because the angles between the vectors $\{x'_j\}_{j \in \mathbb{Z}_N}$ do not depend on their lengths, we re-identify them by their corresponding unit vectors $\{x_j = \sqrt{\frac{K}{L(K-L)}} x'_j : j \in \mathbb{Z}_N\}$. Since these points correspond to a Grassmannian packing, ϕ_{min} is maximal. Therefore, after observing that two arbitrary but distinct caps $C_{x_j}(\theta)$ and $C_{x_l}(\theta)$ are disjoint if and only if $\phi_{min} \geq 2\theta$, we see that (3.4) and hence (3.2) will follow if we can show that $\theta = \frac{\pi}{4}$ is the maximal value for which N points can form a θ-packing on unit sphere in $\mathbb{R}^D$.

To prove this, let $D \in \mathbb{N}$ with $D > 1$ and define $N_D(\theta)$ to be the maximum number of points that can form a θ-packing on the unit sphere in $\mathbb{R}^D$. Because $N_D(\theta)$ is clearly non-increasing in θ, (3.2) will follow by verifying the claim that $N_D(\theta) \leq D + 1$ for all $\theta > \frac{\pi}{4}$.

To prove this, let $\theta > \frac{\pi}{4}$ and induct on D. We have $N_2(\theta) \leq 3$, so suppose $N_{D-1}(\theta) \leq D$ and, by way of contradiction, suppose there exists a θ-packing consisting of $D + 2$ points $x_1, x_2, .., x_{D+2}$ on the unit sphere in $\mathbb{R}^D$. With no loss in generality, we can assume that that $x_{D+2} = e_D$, which implies

$$x_j(D) = \langle x_j, x_{D+2} \rangle = \cos\phi_{j,D+2} \leq \cos 2\theta < 0, \text{ for all } j \in \{1, 2, ..., D+1\}, \tag{3.5}$$

where $x_j(l) = \langle x_j, e_l \rangle$ denotes the lth coordinate of x_j. None of the remaining points can be antipodal to x_{D+2}, for if $x_a = -e_D$ for some $1 \leq a \leq D+1$, then

$$x_j(D) = -\langle x_j, x_a \rangle = -\cos\phi_{j,a} \geq \cos 2\theta > 0, \text{ for all } j \in \{1,2,...,D+1\} \text{ with } j \neq a,$$

which contradicts (3.5). Therefore,

$$-1 < x_j(D) < 0 \text{ for all } j \in \{1,2,...,D+1\}. \tag{3.6}$$

Next, define $D+1$ points $y_1,...,y_{D+1}$ on the unit sphere in $\mathbb{R}^{D-1}$ by deleting the Dth coordinates of $x_1, x_2, ..., x_{D+1}$ and normalizing; that is, let

$$y_j(l) = \frac{1}{\lambda_j} x_j(l), \text{ for } l \in \{1,2,...,D-1\} \text{ and } j \in \{1,2,...,D+1\},$$

where

$$\lambda_j = \sqrt{\sum_{l=1}^{D-1} (x_j(l))^2} = \sqrt{1-(x_j(D))^2},$$

which is well-defined since $0 < \lambda_j$ by (3.6).

Next, fix $j, l \in \{1,...,D+1\}$ with $j \neq l$ and observe that

$$\begin{aligned}
\cos\phi'_{j,l} &= \langle y_j, y_l \rangle \\
&= \frac{1}{\lambda_j \lambda_l} (\langle x_j, x_l \rangle - x_j(D) x_l(D)) \\
&\leq \frac{1}{\lambda_j \lambda_l} (\cos 2\theta - \cos^2 2\theta) \\
&\leq \psi,
\end{aligned}$$

where $\psi = \frac{1}{\lambda_j \lambda_l} \frac{\cos 2\theta}{1+\cos 2\theta}$ and $\phi'_{j,l}$ denotes the angle between y_j and y_l.

Because $\theta > \frac{\pi}{4}$, it follows that

$$-1 \leq \langle y_j, y_l \rangle \leq \psi < 0,$$

which implies

$$\varphi := \frac{\arccos(\psi)}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Thus,

$$\cos \phi'_{j,l} \leq \cos 2\varphi \text{ for all } j, l \in \{1, 2, ..., D+1\} \text{ with } j \neq l,$$

which contradicts the induction hypothesis, since $y_1, ..., y_{D+1}$ form a ψ-packing on the unit sphere in $\mathbb{R}^{D-1}$. Therefore, $N_D(\theta) \leq D + 1$ and the claim follows.

To see that equality in (3.2) can be achieved whenever the embedded points correspond to a regular orthoplex, let $V = \{\pm e_j : j \in \{1, 2, ..., D\}\}$. This set of $2D$ points corresponds to the vertices of a regular orthoplex, and it is straightforward to verify that V forms a $\frac{\pi}{4}$-packing. The claim follows since removing points from this set will not affect the $\frac{\pi}{4}$-packing property.

□

In the case $L = 1$, the chordal distance becomes $d_c(S, T) = \sin \theta_1$, so the problem simplifies to maximizing the smallest (acute) angle between any pair of lines through the origin in $\mathbb{R}^K$. If $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$ is a set of unit vectors that generates a set of N 1-dimensional subspaces in $\mathbb{R}^K$, then the Pythagorean theorem and Equation 3.1 show that maximizing the smallest angles between these lines is equivalent to minimizing the maximal magnitude of inner products between elements of $\mathcal{F}$. Because an optimal line packing will obviously span $\mathbb{R}^K$ whenever $N \geq K$, it follows that the set $\mathcal{F}$ is a frame in this case. Thus, the packing problem for $\mathcal{G}(K, 1)$

with $N \geq K$ is equivalent to the problem of locating Grassmanian frames consisting of N vectors over $\mathbb{R}^K$, so their existence and the orthoplex bound discussed above carry over.

Next, we consider a modified version of Grassmannian frames, which are optimizers for this problem when the additional restriction of Parsevality is imposed. Once again, we express this property in terms of the corresponding Gram matrices.

3.3.20 Definition. Given the Gram matrix $G = (G_{j,l})_{j,l=1}^N$ of a frame consisting of N vectors over $\mathbb{F}^K$, define its *worst-case coherence* as

$$\mu(G) = \max_{j,l \in \mathbb{Z}_N, j \neq l} |G_{j,l}|.$$

A frame $\mathcal{F}$ is called a *Grassmannian equal-norm Parseval frame* if it is an equal-norm (N, K)-frame and its Gram matrix G satisfies

$$\mu(G) = \min_{G' \in \mathcal{M}_{N,K} \cap \Omega_{N,K}} \mu(G'),$$

where $\Omega_{N,K}$ denotes the set of Gram matrices corresponding to equal-norm frames $\mathcal{F} = \{f_j\}_{j=1}^N$ over $\mathbb{F}^K$ with $\|f_j\|^2 = K/N$ for all $j \in \mathbb{Z}_N$.

An argument similar to the proof of Proposition 2.2.7 shows that $\mathcal{M}_{N,K} \cap \Omega_{N,K}$, the set of Gram matrices belonging to equal-norm (N, K)-frames , is compact, and it is nonempty, as shown in [32]. By the continuity of the maximal off-diagonal magnitude, minimizers always exist over this restricted space.

Since EPFs achieve the Welch bound, it follows from Theorem 3.1.4 that Grassmannian frames, Grassmanian equal-norm Parseval frames and EPFs are identical for all settings in which equiangular (N, K)-frames exist. More generally, the set containment $\mathcal{M}_{N,K} \cap \Omega_{N,K} \subset \Omega_{N,K}$ implies that a tight frame which is a Grassmannian frame is, after an appropriate rescaling of the vector norms, a Grassmannian equal-norm Parseval frame. To demonstrate that this new definition is not redundant, we note that examples of Grassmanian frames which are not tight are provided in [16].

In [53], Grassmannian equal-norm Parseval frames are shown to be the optimal frames when frames are used as analog codes and up to two frame coefficients are erased in the course of a transmission. Based on the numerical construction of optimal frames for $\mathbb{R}^3$, they did not seem to have a simple geometric structure, apart from the case of equiangular Parseval frames. Nevertheless, it is intriguing that there are other dimensions for which we can find Grassmanian equal-norm Parseval frames that are not equiangular, but equidistributed. We provide examples for the case where $\mathbb{F} = \mathbb{R}$.

3.3.21 Example. Let $K = 2$, $N > 3$, and consider the $(N, 2)$-frame with analysis operator V whose frame vectors are given by the columns of the synthesis matrix,

$$V^* = \sqrt{\frac{2}{N}} \left(\begin{pmatrix} \cos(\pi j/N) \\ \sin(\pi j/N) \end{pmatrix} \right)_{j=1}^{N} .$$

This frame is easily verified to be a nonequiangular and Parseval, but it is equidistributed because it is a group frame induced by unitary action under $(\mathbb{Z}_N)$. Furthermore, as shown in [16], this frame is a Grassmannian frame after scaling the frame vectors to unit length, so it must also be a Grassmannian equal-norm Parseval frame.

3.3.22 Example. Let $K = 4$ and $N = 12$. Consider the $(12, 4)$-frame $\mathcal{F}$ with analysis operator V whose vectors are given by the columns of the following synthesis matrix.

$$V^* = \begin{pmatrix} 2a & 0 & 0 & 0 & a & a & a & -a & a & a & a & -a \\ 0 & 2a & 0 & 0 & a & a & -a & a & a & -a & a & a \\ 0 & 0 & 2a & 0 & a & -a & -a & -a & a & -a & -a & -a \\ 0 & 0 & 0 & 2a & a & -a & a & a & -a & -a & a & -a \end{pmatrix},$$

where $a = \frac{1}{2\sqrt{3}}$. This is an equal-norm sequence of vectors which can be grouped into 3 sets of 4 orthogonal vectors, thus it is straightforward to verify that this is a Parseval frame for $\mathbb{R}^4$. In addition, inspecting inner products between the vectors shows that they form, up to an overall scaling of the norms, a mutually unbiased basis. Thus $\mathcal{F}$ is equidistributed, as in Example 3.3.4. To see that this is a Grassmannian equal-norm Parseval frame, we show that it corresponds to an optimal line packing. The absolute values of the sines of all possible angles between frame vectors belong to the set $\{1, \sqrt{3}/2\}$. By the orthoplex bound in Theorem 3.3.19, $\sqrt{3}/2$ is indeed the largest possible value that the sine of the smallest angle can achieve. Therefore, these vectors are spanned by the lines of an optimal packing in $\mathcal{G}(4,1)$, so they form a Grassmannian frame. As with the previous example, because $\mathcal{F}$ is a Parseval frame which is simultaneously a Grassmannian frame, we conclude that is also a Grassmannian equal-norm Parseval frame.

3.4 Equiangular Parseval frames and modulation operators

Although it is not central to the main results of this thesis, this section presents results involving the modulation operators of frames (see Definition 3.4.4), which is an approach based on the Fourier transform. In [11], the authors used modulation operators to prove that the maximal number of of mutually unbiased bases that can exist in $\mathbb{F}^K$ is $K+1$. In our case, we provide characterizations of EPFs and equiangular cyclic frames.

Given any frame, one can associate it to the discrete, operator-valued function that maps the indices of the frame vectors to their corresponding rank one Hermitian matrices.

3.4.1 Definition. The *operator-valued map* of a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ consisting of N vectors over $\mathbb{F}^K$ is the map

$$\Lambda_{\mathcal{F}} : \mathbb{Z}_N \to \mathbb{F}^{K\times K} : j \mapsto f_j \otimes f_j^*.$$

An advantage of thinking of a frame in terms of its operator-valued map is that, by using the cyclicity of the trace function, the magnitudes of the inner products between frame vectors of $\mathcal{F}$ are encoded in the Hilbert Schmidt inner products between values of $\Lambda_{\mathcal{F}}$ by

$$\langle \Lambda_{\mathcal{F}}(j), \Lambda_{\mathcal{F}}(l)\rangle\rangle_{H.S.} = \mathrm{tr}(f_j f_j^* f_l f_l^*) = \mathrm{tr}(f_j^* f_l f_l^* f_j) = |\langle f_j, f_l\rangle|^2.$$

In a similar fashion, the results of this section rely on inspecting the Hilbert Schmidt inner products between the values of $\widehat{\Lambda}_{\mathcal{F}}$, the Fourier transform of $\Lambda_{\mathcal{F}}$. In order to formally define the Fourier transform of such a function, we consider the larger space of functions

$$\mathcal{L}_{N,K} := \{\Lambda : \mathbb{Z}_N \to \mathbb{F}^{K\times K}\},$$

which is a vector space under point-wise addition and scalar multiplication, and equip it with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ defined by

$$\langle\Lambda_1, \Lambda_2\rangle_{\mathcal{L}} = \sum_{j\in\mathbb{Z}_N} \langle\Lambda_1(j), \Lambda_2(j)\rangle_{HS}.$$

Because the set $\{\Lambda_{a,b,c} : j \mapsto \delta_{j,c}E_{a,b} : l \in \mathbb{Z}_N, a, b \in \mathbb{Z}_K\}$ forms an orthonormal basis for $\mathcal{L}_{N,K}$, it is finite-dimensional, so we may regard it as a Hilbert space with respect to the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{L}}$.

3.4.2 Definition. The *Fourier transform on* $\mathcal{L}_{N,K}$ is the linear map

$$\mathcal{T}_{F.T.} : \mathcal{L}_{N,K} \to \mathcal{L}_{N,K} : \Lambda \mapsto \hat{\Lambda},$$

where $\hat{\Lambda}$ is defined by

$$\hat{\Lambda}(\xi) := \sum_{j\in\mathbb{Z}_N} \omega_N^{j\xi}\Lambda(j)$$

and ω_N is the primitive Nth root of unity, $e^{2\pi i/N}$.

As is customary, when given $\Lambda \in \mathcal{L}_{N,K}$, we denote $\mathcal{T}_{F.T.}\Lambda$ as $\hat{\Lambda}$ and refer to it as the *Fourier transform of* Λ .

We recall the familiar Fourier inversion formula.

3.4.3 Theorem. *The map $\mathcal{T}_{F.T.}$ is invertible and its inverse is given by the formula*

$\mathcal{T}_{F.T.}^{-1}\widehat{\Lambda} = \Lambda$*, where*

$$\Lambda(j) = \frac{1}{N}\sum_{\xi\in\mathbb{Z}_N} \omega_N^{-j\xi}\widehat{\Lambda}(\xi)$$

for any $\Lambda \in \mathcal{L}_{N,K}$ *and* $j \in \mathbb{Z}_N$*. In particular,* $\Lambda(j) = \frac{1}{N}\widehat{\widehat{\Lambda}}(-j)$ *for every* $\Lambda \in \mathcal{L}_{N,K}$ *and* $j \in \mathbb{Z}_N$*.*

3.4.4 Definition. If $\Lambda_\mathcal{F}$ is the operator-valued map for a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ over $\mathbb{F}^K$ and $\widehat{\Lambda}_\mathcal{F}$ is its Fourier transform, then the operator $\widehat{\Lambda}_\mathcal{F}(\xi)$ is called the *ξth modulation operator of $\mathcal{F}$*, for each $\xi \in \mathbb{Z}_N$.

For convenience, we record two formulae which relate the inner products between the vectors of a frame, $\mathcal{F}$, and Hilbert Schmidt inner products between its modulation operators in the following lemma.

3.4.5 Lemma. *If* $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ *is any frame consisting of* N *vectors over* $\mathbb{F}^K$*, then*

1. $N^2|\langle f_a, f_b\rangle|^2 = \sum\limits_{\xi,\eta\in\mathbb{Z}_N} \omega_N^{b\eta-a\xi}\langle\widehat{\Lambda}_\mathcal{F}(\xi), \widehat{\Lambda}_\mathcal{F}(\eta)\rangle$ *for all* $a, b \in \mathbb{Z}_N$*.*
2. $\langle\widehat{\Lambda}_\mathcal{F}(\xi), \widehat{\Lambda}_\mathcal{F}(\zeta)\rangle_{H.S.} = \sum_{j,l\in\mathbb{Z}_N} \omega_N^{j\xi-l\zeta}|\langle f_j, f_l\rangle|^2$ *for all* $\xi, \zeta \in \mathbb{Z}_N$*.*

Proof. To see the first formula, let $a, b \in \mathbb{Z}_N$ and apply the Fourier inversion formula to obtain

$$\begin{aligned}
N^2|\langle f_a, f_b\rangle|^2 &= N^2\langle\Lambda_\mathcal{F}(a), \Lambda_\mathcal{F}(b)\rangle_{H.S.} \\
&= N^2\langle\frac{1}{N}\sum_{\xi\in\mathbb{Z}_N}\omega_N^{-a\xi}\widehat{\Lambda}(\xi), \frac{1}{N}\sum_{\eta\in\mathbb{Z}_N}\omega_N^{-b\eta}\widehat{\Lambda}(\eta)\rangle_{H.S.} \\
&= \sum_{\xi,\eta\in\mathbb{Z}_N}\omega_N^{b\eta-a\xi}\langle\widehat{\Lambda}_\mathcal{F}(\xi), \widehat{\Lambda}_\mathcal{F}(\eta)\rangle.
\end{aligned}$$

To see the second formula, let $\xi, \zeta \in \mathbb{Z}_N$, then

$$\begin{aligned}\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta) \rangle_{H.S.} &= \operatorname{tr}\Big(\sum_{j \in \mathbb{Z}_N} \omega_N^{j\xi} \Lambda(j) \sum_{l \in \mathbb{Z}_N} \omega_N^{-l\zeta} \Lambda(l) \Big) \\ &= \sum_{j,l \in \mathbb{Z}_N} \omega_N^{j\xi - l\zeta} \operatorname{tr}(\Lambda(j)\Lambda(l)) \\ &= \sum_{j,l \in \mathbb{Z}_N} \omega_N^{j\xi - l\zeta} |\langle f_j, f_l \rangle|^2.\end{aligned}$$

□

3.4.1 A characterization of equiangular Parseval frames

In this section, we characterize EPFs in terms of their modulation operators. We begin by characterizing Parseval frames with a simple lemma.

3.4.6 Lemma. *A frame $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$ over $\mathbb{F}^K$ is a Parseval frame if and only if $\widehat{\Lambda}_{\mathcal{F}}(0) = I_K$.*

Proof. This follows by realizing that $\widehat{\Lambda}_{\mathcal{F}}(0)$ is the frame operator for $\mathcal{F}$, so it is equal to the identity operator if and only if $\mathcal{F}$ is an (N, K)-frame.

□

Next, we provide a necessary condition which fully describes the Hilbert Schmidt inner products between the modulation operators of an EPF.

3.4.7 Proposition. *If $\mathcal{F} = \{f_j\}_{j \in \mathbb{Z}_N}$ is an equiangular (N, K)-frame, then the following hold:*

1. *$\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} = 0$ for every $\xi, \zeta \in \mathbb{Z}_N$ with $l \neq j$,*

2. *$\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|^2_{H.S.} = \frac{K(K-1)}{N-1}$ for $\xi \in \{1, 2, ..., N-1\}$ and*

3. *$\|\widehat{\Lambda}_{\mathcal{F}}(0)\|^2_{H.S.} = K$.*

Proof. Given $\xi, \zeta \in \mathbb{Z}_N$ with $\xi \neq 0$ or $\zeta \neq 0$, we have

$$\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} = \sum_{j,l\in\mathbb{Z}_N} \omega_N^{j\xi - l\zeta} |\langle f_j, f_l\rangle|^2$$

by Lemma 3.4.5. Since $\mathcal{F}$ is equiangular and Parseval, we have $\|f_j\|^2 = \frac{K^2}{N^2}$ for all $j \in \mathbb{Z}_N$ and $|\langle f_j, f_l\rangle|^2 = C^2_{N,K}$ for $j \neq l$. Splitting the sum above according to these factors yields

$$\begin{aligned}
\sum_{j,l\in\mathbb{Z}_N} \omega_N^{j\xi - l\zeta} |\langle f_j, f_l\rangle|^2 &= C^2_{N,K} \sum_{j,l\in\mathbb{Z}_N, j\neq l} \omega_N^{j\xi - l\zeta} + \frac{K^2}{N^2} \sum_{q\in\mathbb{Z}_N} \omega_N^{q(\xi-\zeta)} \\
&= C^2_{N,K} \sum_{j\in\mathbb{Z}_N} \omega_N^{j\zeta} \sum_{l\in\mathbb{Z}_N} \omega_N^{-l\xi} - C^2_{N,K} \sum_{p\in Zn} \omega_N^{p(\xi-\zeta)} + \frac{K^2}{N^2} \sum_{q\in\mathbb{Z}_N} \omega_N^{q(\xi-\zeta)} \\
&= 0 - C^2_{N,K} \sum_{p\in\mathbb{Z}_N} \omega_N^{p(\xi-\zeta)} + \frac{K^2}{N^2} \sum_{q\in\mathbb{Z}_N} \omega_N^{q(\xi-\zeta)},
\end{aligned}$$

where the first term vanishes due to the summation of consecutive powers of a root of unity. If $\xi \neq \zeta$, then the remaining two terms vanish for the same reason, so $\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} = 0$. If $\xi = \zeta \in \{1, 2, ..., N-1\}$, then the remaining terms can be rewritten to yield

$$\begin{aligned}
-C^2_{N,K} \sum_{p\in\mathbb{Z}_N} \omega_N^{p(\xi-\xi)} + \frac{K^2}{N^2} \sum_{q\in\mathbb{Z}_N} \omega_N^{q(\xi-\xi)} &= -C^2_{N,K} N + \frac{K^2}{N^2} N \\
&= -\frac{K(N-K)}{N^2(N-1)} N + \frac{K^2}{N^2} N \\
&= \frac{K(K-1)}{N-1},
\end{aligned}$$

so that the value of $\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|_{H.S.}^2$ is as claimed. Finally, by Lemma 3.4.6, we have $\|\widehat{\Lambda}_{\mathcal{F}}(0)\|_{H.S.}^2 = K$. ☐

As a corollary, we know the dimension of the subspace spanned by the modulation operators.

3.4.8 Corollary. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equiangular (N, K)-frame, then*

$$\dim(\operatorname{span}\{\widehat{\Lambda}_{\mathcal{F}}(\xi) : \xi \in \mathbb{Z}_N\}) = N.$$

Proof. This follows immediately from Proposition 3.4.7. ☐

Because $\dim(\mathbb{C}^{K\times K}) \leq K^2$, this provides an alternative proof for the maximal number of equiangular lines that can be packed into $\mathbb{C}^K$.

3.4.9 Corollary. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an equiangular (N, K)-frame for $\mathbb{C}^K$, then $N \leq K^2$.*

Finally, we show that the necessary conditions of Proposition 3.4.7 are also sufficient.

3.4.10 Theorem. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an (N, K)-frame, then $\mathcal{F}$ is equiangular if and only if $\{\widehat{\Lambda}_{\mathcal{F}}(\xi)\}_{\xi\in\mathbb{Z}_N}$ forms an orthogonal set with respect to the Hilbert Schmidt inner product and $\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|_{H.S.}^2 = \frac{K(K-1)}{N-1}$ for $\xi \in \{1, 2, ..., N-1\}$.*

Proof. The sufficiency of equiangularity in this statement follows from Proposition 3.4.7. To see necessity, suppose that $\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|_{HS}^2 = \frac{K(K-1)}{N-1}$ for all $\xi \in \{1, 2, ..., N-1\}$ and let $a, b \in \mathbb{Z}_N$ with $a \neq b$. By the first formula in Lemma 3.4.5 and the fact that $\|\widehat{\Lambda}_{\mathcal{F}}(N)\|^2 = K$ from Lemma 3.4.6, we have

$$\begin{aligned}
N^2|\langle f_a, f_b\rangle|^2 &= \sum_{\xi=1}^{N} \omega_N^{2\pi i\xi(b-a)/N} \|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|^2 \\
&= \frac{K(K-1)}{N-1} \sum_{\xi=1}^{N} e^{2\pi i\xi(b-a)/N} + K - \frac{K(K-1)}{N-1} \\
&= \frac{K(K-1)}{N-1} \sum_{\xi=1}^{N} e^{2\pi i\xi(b-a)/N} + \frac{K(N-K)}{N-1}.
\end{aligned}$$

The first term vanishes due the summation of consecutive powers of roots of unity; therefore, after dividing both sides by N^2, we obtain

$$|\langle f_a, f_b\rangle|^2 = \frac{K(N-K)}{N^2(N-1)} = C_{N,K}^2$$

as desired.

□

As a corollary to Theorem 3.4.10, we have a characterization of maximal complex EPFs.

3.4.11 Corollary. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is an (N, K)-frame for $\mathbb{C}^K$ with $N = K^2$, then $\mathcal{F}$ is equiangular if and only if the set*

$$\mathcal{B} = \left\{\frac{1}{\sqrt{K}}\widehat{\Lambda}_{\mathcal{F}}(0)\right\} \bigcup \left\{\sqrt{\frac{N-1}{K(K-1)}}\widehat{\Lambda}_{\mathcal{F}}(\xi)\right\}_{\xi\in\mathbb{Z}_N, \xi\neq 0}$$

forms an orthonormal basis for $\mathbb{C}^{K\times K}$ with respect to the Hilbert Schmidt inner product.

Proof. Since the dimension of $\mathbb{C}^{K\times K}$ is K^2, $\mathcal{B}$ is an orthogonal basis by Theorem 3.4.10. The unity of the norms for $\sqrt{\frac{N-1}{K(K-1)}}\widehat{\Lambda}_{\mathcal{F}}(\xi)$ with $\xi \in \mathbb{Z}_N, \xi \neq 0$ also follows from Theorem 3.4.10 and that $\frac{1}{\sqrt{K}}\widehat{\Lambda}_{\mathcal{F}}(0)$ is unit norm follows from Lemma 3.4.6.

□

3.4.2 Modulation operators of cyclic frames

In Section 6.3, we saw how the authors of [58, 95] characterized equiangular cyclic (N, K)-frames with difference sets. In this section, we reprove this result with modulation operators. We begin by recording a basic fact about the inner products between a cyclic frame's vectors.

3.4.12 Lemma. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is a cyclic (N, K)-frame, then*

$$|\langle f_j, f_l\rangle|^2 = |\langle f_0, f_{l-j}\rangle|^2$$

for all $j, l \in \mathbb{Z}_N$.

Proof. Since $\mathcal{F}$ is a cyclic frame, it is the orbit of the vector f_0 under the action of some unitary representation π of the additive group $\mathbb{Z}_N$ acting on $\mathbb{C}^K$, so

$$|\langle f_j, f_l\rangle|^2 = |\langle f_0, \pi(j^{-1}l)f_0\rangle|^2 = |\langle f_0, f_{l-j}\rangle|^2$$

for all $j, l \in \mathbb{Z}_N$. □

As was the case with EPFs, the Hilbert Schmidt inner products between the distinct modulation operators of a harmonic cyclic frame are zero.

3.4.13 Proposition. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is a cyclic (N,K)-frame, then*

$$\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} = 0$$

for all $\xi, \zeta \in \mathbb{Z}_N$ with $\xi \neq \zeta$.

Proof. Let $\xi, \zeta \in \mathbb{Z}_N$ with $\xi \neq \zeta$.

By combining Lemma 3.4.12 with Lemma 3.4.5, we obtain

$$\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} = \sum_{j,l\in\mathbb{Z}_N} \omega_N^{j\xi - l\zeta} |\langle f_j, f_l\rangle|^2 = \sum_{j,l\in\mathbb{Z}_N} \omega_N^{j\xi - l\zeta} |\langle f_0, f_{l-j}\rangle|^2,$$

where $\omega_N = e^{2\pi i/N}$. Reindexing this summation with $t = l - j$ yields

$$\begin{aligned}\langle \widehat{\Lambda}_{\mathcal{F}}(\xi), \widehat{\Lambda}_{\mathcal{F}}(\zeta)\rangle_{H.S.} &= \sum_{l,t\in\mathbb{Z}_N} \omega_N^{(l-t)\xi - l\zeta} |\langle f_0, f_t\rangle|^2 \\ &= \sum_{l\in\mathbb{Z}_N} \omega_N^{l(\xi-\zeta)} \sum_{t\in\mathbb{Z}_N} |\langle f_0, f_t\rangle|^2 \omega_N^{-t\xi},\end{aligned}$$

which vanishes because of the summation of consecutive powers of a root of unity in the first factor of the last line. □

Since the modulation operators of a cyclic (N,K)-frame $\mathcal{F}$ are Hilbert-Schmidt orthogonal, it is natural to ask when they satisfy the norm requirements of an EPF described in Theorem 3.4.10. In order to compute their norms, we recall that every harmonic cyclic (N,K)-frame is generated by a sequence $\{n_1, n_2, ..., n_K\} \subset \mathbb{Z}_N$ and use this to compute the matrix entries of the modulation operators directly.

3.4.14 Proposition. *If $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ is a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2..., n_K\}$, then the entries of its modulation operators consist entirely of zeros and ones. In particular,*

$$(\widehat{\Lambda}_{\mathcal{F}}(\xi))_{a,b} = \begin{cases} 1, & n_b - n_a = \xi \\ 0, & \textit{otherwise} \end{cases},$$

for every $\xi \in \mathbb{Z}_N$, where $\omega_N = e^{2\pi i/N}$.

Proof. First, we compute the entries of $\mathcal{F}$'s operator-valued map. If $j \in \mathbb{Z}_N$, then, by Definition 3.4.1 and Definition 3.3.7, we have

$$\Lambda_{\mathcal{F}}(j) = \frac{1}{\sqrt{N}}(\omega_N^{jn_a})_{a=1}^K \otimes \frac{1}{\sqrt{N}}(\omega_N^{jn_b})_{b=1}^K{}^*,$$

so the (a, b) entry of $\Lambda_{\mathcal{F}}(j)$ is

$$(\Lambda_{\mathcal{F}}(j))_{a,b} = \frac{1}{N}\omega_N^{j(n_a-n_b)}.$$

If $\xi \in \mathbb{Z}_N$, then by Definition 3.4.4, the (a, b) entry of $\widehat{\Lambda}_{\mathcal{F}}(\xi)$ is

$$(\widehat{\Lambda}_{\mathcal{F}}(\xi))_{a,b} = \left(\sum_{j\in\mathbb{Z}_N} \omega_N^{j\xi}\Lambda_{\mathcal{F}}(j)\right)_{a,b} = \sum_{j\in\mathbb{Z}_N} \omega_N^{j\xi}(\Lambda_{\mathcal{F}}(j))_{a,b} = \frac{1}{N}\sum_{j\in\mathbb{Z}_N} \omega_N^{j(\xi+n_a-n_b)}.$$

If $n_b - n_a \neq \xi$, the entry vanishes as a summation of consecutive powers of a root of unity; otherwise, $\omega_N^{j(\xi+n_a-n_b)} = 1$ and the claim follows. $\square$

It is clear from this computation that the Hilbert Schmidt norms of the modulation operators for a cyclic frame depend heavily on how the generating sequence $\{n_j\}_{j=1}^K$ is selected. As a corollary, we characterize the subset of cyclic (N, K)-frames whose modulation operators form linearly independent subsets of $\mathbb{C}^{K\times K}$.

3.4.15 Corollary. *Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ be a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2..., n_K\}$. If for each $\xi \in \mathbb{Z}_N$, there is at least one pair $(a, b) \in \{1, 2, ..., K\}\times\{1, 2, ..., K\}$ such that $n_b - n_a = \xi$, then $\{\widehat{\Lambda}_{\mathcal{F}}(\xi)\}_{xi\in\mathbb{Z}_N}$ is an orthogonal subset of $\mathbb{C}^{K\times K}$ with respect to the Hilbert Schmidt norm.*

Proof. The modulation operators are pairwise Hilbert Schmidt orthogonal by Proposition 3.4.13. Combining the hypothesis condition with Proposition 3.4.14 shows that each modulation operator is nonzero, so the claim follows. □

Furthermore, we have an upper bound on the dimension of the subspace spanned by the modulation operators of a cyclic frame.

3.4.16 Corollary. *If $\mathcal{F}$ is a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2..., n_K\}$, then* $\dim(\operatorname{span}\{\widehat{\Lambda}_{\mathcal{F}}(\xi) : \xi \in \mathbb{Z}_N\}) \leq K^2 - K + 1$.

Proof. By Proposition 3.4.14, the entries and therefore the norms of $\mathcal{F}$'s modulation operators are completely determined by the set $A = \{n_1, n_2, ..., n_K\}\times\{n_1, n_2, ..., n_K\}$, which has cardinality K^2. In particular, the pairs $(x, y) \in A$ are in one-to-one correspondence with the ones occurring among the matrix entries of the N modulation operators of $\mathcal{F}$. Since $\widehat{\Lambda}_{\mathcal{F}}(0) = I_K$ by Lemma 3.4.6 (or Proposition 3.4.14), the pairs $(n_1, n_1), (n_2, n_2), ..., (n_K, n_K) \in A$ correspond to the diagonal entries of $\widehat{\Lambda}_{\mathcal{F}}(0)$, which leaves $K^2 - K$ pairs remaining. Therefore, at most $K^2 - K$ modulation operators besides $\widehat{\Lambda}_{\mathcal{F}}(0)$ can be nonzero and the claim follows. □

In particular, this bound in conjunction with Theorem 3.4.10 shows that equiangular cyclic frames are never maximal.

Finally, we reprove the characterization of equiangular cyclic (N, K)-frames in terms of difference sets from Theorem 3.3.10. We recall the statement:

If $\mathcal{F}$ is a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2..., n_K\}$, then $\mathcal{F}$ is equiangular if and only if $\{n_1, n_2, ..., n_K\}$ is a (K, λ) -difference set for $\mathbb{Z}_N$ where $\lambda = \frac{K(K-1)}{N-1}$ is a positive integer.

Proof. By Theorem 3.4.10, $\mathcal{F}$ is equiangular if and only if its modulation operators are Hilbert Schmidt orthogonal, $\|\widehat{\Lambda}_{\mathcal{F}}(0)\|^2_{H.S.} = K$ and $\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|^2_{H.S.} = \frac{K(K-1)}{N-1}$ for $\xi \in \{1, 2, ..., N-1\}$. Because $\mathcal{F}$ is a cyclic frame, the orthogonality is guaranteed by Proposition 3.4.13 and that $\|\widehat{\Lambda}_{\mathcal{F}}(0)\|^2_{H.S.} = K$ is guaranteed by Lemma 3.4.6. Finally, by Proposition 3.4.14, the entries of each modulation operator consists entirely of ones and zeros. Since the Hilbert Schmidt norm is equal to the Frobenius norm on $\mathbb{C}^{K\times K}$, the formula for the entries in Proposition 3.4.14 implies $\|\widehat{\Lambda}_{\mathcal{F}}(\xi)\|^2_{H.S.} = \frac{K(K-1)}{N-1}$ for $\xi \in \{1, 2, ..., N-1\}$ if and only if $\{n_1, n_2, ..., n_K\}$ is a (K, λ) -difference set for $\mathbb{Z}_N$ with $\lambda = \frac{K(K-1)}{N-1}$. $\square$

In light of this characterization, we characterize the subset of cyclic frames whose modulation operators form linearly independent sets and saturate the upper bound in Corollary 3.4.16.

3.4.17 Corollary. *If $\mathcal{F}$ is a cyclic (N, K)-frame generated by the sequence $\{n_1, n_2..., n_K\}$ and $N = K^2 - K + 1$, then $\dim(\mathrm{span}\{\widehat{\Lambda}_{\mathcal{F}}(\xi) : \xi \in \mathbb{Z}_N\}) = K^2 - K + 1$ if and only if $\mathcal{F}$ is equiangular and $\{n_1, n_2..., n_K\}$ is a $(K, 1)$ -difference set for $\mathbb{Z}_N$.*

Proof. By Theorem 3.3.10, if a cyclic EPF is generated by a $(K, 1)$ -difference set, then $1 = \frac{K(K-1)}{N-1}$ and this is equivalent to $N = K^2 - K + 1$. By Theorem 3.4.10, the modulation operators are nonzero and Hilbert Schmidt orthogonal, so

$$\dim(\operatorname{span}\{\widehat{\Lambda}_{\mathcal{F}}(\xi) : \xi \in \mathbb{Z}_N\}) = K^2 - K + 1.$$

Conversely, if $\dim(\operatorname{span}\{\widehat{\Lambda}_{\mathcal{F}}(\xi) : \xi \in \mathbb{Z}_N\}) = K^2 - K + 1$, then every modulation must be nonzero. Since the pairs of the set $A = \{n_1, n_2, ..., n_K\} \times \{n_1, n_2, ..., n_K\}$ are in one-to-one correspondence with the nonzero entries of the modulation operators by Proposition 3.4.14 and since $\widehat{\Lambda}_{\mathcal{F}}(0) = I_k$ exhausts K of these pairs for its diagonal entries, the remaining $K^2 - K$ modulation operators must have exactly one nonzero entry and this is if and only if $\{n_1, n_2 ..., n_K\}$ is a $(K, 1)$ -difference set for $\mathbb{Z}_N$, by the formula for the entries in Proposition 3.4.14. $\square$

Chapter 4

Frame Potentials

In the previous section, Grassmannian frames were defined as minimizers of an optimization problem. Here, we develop this notion into a strategy for finding other special types of frames. A *frame potential* is a real-valued function that depends on the inner products between a frame's vectors. By carefully constructing such functions, frames with desirable properties can be characterized as their minimizers.

The purpose of the next two sections is to provide an overview of the literature involving frame potentials. Because their optimization is typically executed over either the class of equal-norm frames or the class of Parseval frames, we address the two cases separately below. Since the history began with Benedetto and Fickus using a frame potential in the equal-norm setting [14], we address this case first.

4.1 Equal-norm frames

The equal-norm constraint is a geometric property that provides certain advantages when optimizing frame potentials. In particular, an equal-norm frame's vectors can be thought of as points on a sphere.

In 2003, Benedetto and Fickus studied the relationship between frames with unit norm vectors and the physical notion of point charges on a sphere in $\mathbb{R}^K$ [14]. Coulumb's Law dictates that a charge x will repel a charge y of equal magnitude with an electromagnetic force whose magnitude is inversely proportional to the square of the distance between them. With the charge x fixed, this force can be mathematically interpreted as a *conservative* vector field on $\mathbb{R}^K$, because it is the negative gradient of the *Coulumb potential energy*, which is a differentiable real-valued function of the difference between the charges' positions. In an ideal setting, when a finite set of such charges is constrained to a conductive sphere, the overall force causes them to seek out a state of equilibrium where the charges are as far away from each other as possible. Such a configuration can be characterized as a (local) minimizer to the Coulumb potential energy. Many of such minimizers correspond to the vertices of regular polyhedra, which, in turn, correspond to equal-norm tight frames. Motivated by this, the authors adapted these ideas in order to characterize equal-norm tight frames. They introduced the notion of *frame force*, a term which subsequent literature has generally used to refer to a conservative vector field defined on some constrained set of frames whose corresponding *frame potential* admits critical points (ie equilibria) with desirable properties. In the case of [14], they defined the frame force of a unit norm vector y acting on a unit norm vector x as

$$FF(x, y) = \langle x, y\rangle(x - y),$$

which is induced by the potential given by

$$FP(x, y) = \frac{1}{2}|\langle x, y\rangle|^2.$$

Summing $FP(f_j, f_l)$ over all vectors in a unit norm frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ led to the p-th frame potential where $p = 1$ [14].

4.1.1 Definition. The *p-th frame potential* of a frame $\mathcal{F} = \{f_j\}_{j=1}^N$ for a real or complex Hilbert space $\mathcal{H}$ is given by

$$\Phi_p(\mathcal{F}) = \sum_{j,l=1}^{N} |\langle f_j, f_l\rangle|^{2p}.$$

They provided a lower bound for Φ_1 when restricted to frames whose vectors all have unit norm and showed that equality is achieved only when the frame is tight [14]. We rescale the norms to obtain a characterization of equal-norm Parseval frames.

4.1.2 Theorem. *(Benedetto and Fickus, [14].) If $\mathcal{F} = \{f_j\}_{j=1}^N$ is a frame for $\mathbb{F}^K$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\|f_j\|^2 = K/N$ for each $j \in \mathbb{Z}_N$, then*

$$\Phi_1(\mathcal{F}) = \sum_{j,l=1}^{N} |\langle f_j, f_l\rangle|^2 \geq K$$

and equality holds if and only if $\mathcal{F}$ is Parseval.

Proof. The assumption on the norms is equivalent to the condition on the diagonal entries, $G_{j,j} = K/N$, of the Gram matrix $G = VV^*$ of the frame $\mathcal{F}$. By the Cauchy-Schwarz inequality with respect to the Hilbert-Schmidt inner product, $\Phi_1(\mathcal{F}) = \operatorname{tr}(G^2) \geq (\operatorname{tr}(GP))^2/\operatorname{tr}(P^2)$, where P is the orthogonal projection onto the range of G in $\ell^2(\mathbb{Z}_N)$. However $\operatorname{tr}(GP) = \operatorname{tr}(G) = K = \operatorname{tr}(P) = \operatorname{tr}(P^2)$, thus the claimed lower bound follows. The case of equality holds if and only if G and P are collinear, which means whenever $\mathcal{F}$ is Parseval. $\square$

As an aside from the strictly equal-norm setting, we remark that the authors of [26] extended this result by considering Φ_1 when it is restricted, not to the unit-norm frames, but to frames $\mathcal{F} = \{f_j\}_{j=1}^N$ for $\mathbb{F}^K$ whose vectors' norms satisfy $\|f_j\| = c_j$ for fixed, positive constants $c_1, ..., c_N \in \mathbb{R}$. Such frames can be interpreted as sets of vectors lying on concentric spheres with radii c_j. In this setting, they provided a lower bound analogous to that of Theorem 4.1.2, which characterizes tight frames with the prescribed lengths as minimizers whenever they exist.

In 2004, Blume-Kohout, Scott, Caves, and Renes were also prompted by a problem in physics when they characterized symmetric, informationally complete, positive operator valued measures (SIC-POVMs), or equivalently maximal complex EPFs. By considering the restricted p-th frame potential for the case $p > 1$, they showed that maximal complex EPFs are its minimizers, whenever they exist among unit-norm frames [75]. A few years later, Oktay generalized this result to the non-maximal case [72]. These results are presented in the following theorem, where, as before, the norms are trivially rescaled for Parsevality.

4.1.3 Theorem. *([75, 72].) Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for $\mathbb{F}^K$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$,* *and* $\|f_j\|^2 = K/N$ *for all* $j \in \mathbb{Z}_N$*, and let* $p > 1$*, then*

$$\Phi_p(\mathcal{F}) = \sum_{j,l=1}^{N} |\langle f_j, f_l\rangle|^{2p} \geq \frac{K^{2p}(N-1)^{p-1} + K^p(N-K)^p}{(N-1)^{p-1}N^{2p-1}}$$

and equality holds if and only if $\mathcal{F}$ *is an equiangular Parseval frame.*

Proof. With the elementary properties of equal-norm frames and Jensen's inequality, we obtain the bound

$$\Phi_p(\mathcal{F}) = \sum_{j=1}^{N} \|f_j\|^{4p} + \sum_{j\neq l} |\langle f_j, f_l\rangle|^{2p} \geq \frac{K^{2p}}{N^{2p-1}} + \frac{1}{N^{p-1}(N-1)^{p-1}}(\sum_{j\neq l} |\langle f_j, f_l\rangle|^2)^p \,.$$

Expressing this in terms of Φ_1 and using the preceding theorem then gives

$$\begin{aligned}\Phi_p(\mathcal{F}) &\geq \frac{K^{2p}}{N^{2p-1}} + \frac{1}{N^{p-1}(N-1)^{p-1}}(\Phi_1(\mathcal{F}) - \frac{K^2}{N})^p \\ &\geq \frac{K^{2p}}{N^{2p-1}} + \frac{1}{N^{p-1}(N-1)^{p-1}}\frac{K^p(N-K)^p}{N^p} \\ &= \frac{K^{2p}(N-1)^{p-1} + K^p(N-K)^p}{(N-1)^{p-1}N^{2p-1}}\,.\end{aligned}$$

Moreover, equality holds in the Cauchy-Schwarz and Jensen inequalities if and only if $\mathcal{F}$ is Parseval and if there is $C \geq 0$ such that $|\langle f_j, f_l\rangle| = C$ for all $j, l \in \mathbb{Z}_N$ with $j \neq l$. $\square$

If equality holds, then an inspection of the proof shows that the magnitudes of the off-diagonal entries of the Gram matrix is the constant

$$C_{N,K} = \sqrt{\frac{1}{N(N-1)}\Big(K - \frac{K^2}{N}\Big)} = \sqrt{\frac{K(N-K)}{N^2(N-1)}}\,.$$

from Theorem 3.1.4, see also [46], [53], and [86].

4.1.4 Corollary. *Let* $p > 1$. *If* $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ *is a frame for* $\mathbb{F}^K$ *with* $\|f_j\|^2 = K/N$ *for each* $j \in \mathbb{Z}_N$*, then* $\Phi_p(\mathcal{F})$ *achieves the lower bounds in Theorem 4.1.2 and Theorem 4.1.3 if and only if* $\mathcal{F}$ *is an* (N, K)*-frame with* $|\langle f_j, f_l\rangle| = C_{N,K}$ *for all* $j \neq l$.

Next, we consider frame potentials as restrictions to the space of (N, K)-frames instead. This will prepare us for the main results of this thesis.

4.2 Parseval frames

While the equal-norm constraint allows the interpretation of frames as sets of points on a sphere, the Parseval constraint allows frames to be thought of as projections of orthonomal bases, as described in Theorem 2.2.1.

Benedetto and Kebo used this result to minimize the probability of a detection error for quantum measurements [15]. In certain scenarios, the possible states of a physical system are modeled with a finite set of unit norm vectors $\{x_j\}_{j\in\mathbb{Z}_N} \subset \mathbb{F}^K$ which occur with corresponding probabilities $\{p_j\}_{j\in\mathbb{Z}_N}$ that sum to one. A measurement apparatus can be thought of as a device which returns some index $j \in \mathbb{Z}_N$ that hopefully reflects the true state of the system and it can be modeled with a Parseval frame. Given an (N, K)-frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ over $\mathbb{F}^K$, the map $\Pi : \mathcal{P}(\mathbb{Z}_N) \to M_K(\mathbb{F})$ defined by

$$\Pi(L) = \sum_{l\in L} f_l \otimes f_l^*$$

is what physicists refer to as a *positive operator-valued measure*, which can be used to determine the probability of whether the measurement device's output are correct. If the system is in a state x_j during a measurement, then the probability that the device outputs a measurement $l \in L \subset \mathbb{Z}_N$ is given by

$$P_\Pi(L) = \langle x_j, \Pi(L)x_j \rangle$$

and, in particular, the probability that it correctly outputs $\{j\}$ is

$$P_\Pi(\{j\}) = \langle x_j, \Pi(\{j\})x_j \rangle = \langle x_j, f_j \otimes f_j^* x_j \rangle = |\langle x_j, f_j \rangle|^2.$$

Thus, the average probability of a correct measurement is $\sum\limits_{j \in \mathbb{Z}_N} p_j |\langle x_j, f_j \rangle|^2$ and it is straightforward to verify that

$$P_e(\mathcal{F}) := 1 - \sum_{j \in \mathbb{Z}_N} p_j |\langle x_j, f_j \rangle|^2,$$

is the average probability of a detection error when modeling the apparatus with the frame $\mathcal{F}$. Seeking to minimize P_e over the space of all (N, K)-frames, they simplified the problem by applying Theorem 2.2.1 to show that minimizing P_e over the space of (N, K)-frames is equivalent to minimizing it over the space of orthonormal bases in an ambient N-dimensional vector space, which admits a parametrization in terms of of the coordinate charts of the Lie group of unitary (or orthogonal) matrices [15]. By showing that P_e corresponds to a frame force F_e, the authors were able to interpret the second order ordinary differential equation

$$\ddot{x}(t) = F_e(x(t)),$$

where $x(t)$ is a parametrization of the N-dimensional orthonormal bases, as a Newtonian equation of motion [15]. This allowed them to apply the physical law of conservation of energy to characterize the minimizers to P_e as the constant solutions for this ODE which minimize the energy

$$E(t) = \frac{1}{2}[\dot{x}(t)]^2 + P_e(x(t)).$$

In [20], Bodmann and Casazza addressed the Paulsen Problem, which is to determine how far in l^2 distance the closest equal-norm (N, K)-frame is from an arbitrary frame consisting of N vectors over $\mathbb{F}^K$.

4.2.1 Problem. Given a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{Z}_N}$ for $\mathbb{F}^K$ and $\epsilon > 0$, find the largest value $\delta > 0$ such that there exists an equal-norm (N, K)-frame $\mathcal{F}' = \{f'_j\}_{j\in\mathbb{Z}_N}$ which satisfies

$$\|\mathcal{F} - \mathcal{F}'\| = \left(\sum_{j\in\mathbb{Z}_N} \|f_j - f'_j\|^2\right)^{\frac{1}{2}} \leq \epsilon$$

whenever $\mathcal{F}$ is *ϵ-nearly equal-norm with constant C*, which means

$$(1-\epsilon)C \leq \|f_j\| \leq (1-\epsilon)C \text{ for all } j \in \mathbb{Z}_N,$$

and *ϵ-nearly Parseval*, which means

$$(1-\epsilon)\|x\|^2 \leq \sum_{j\in\mathbb{Z}_N} |\langle x, f_j\rangle\rangle|^2 \leq (1-\epsilon)\|x\|^2 \text{ for all } x \in \mathbb{F}^K.$$

They showed that the vector norms for the closest (N, K)-frame to an arbitrary δ-nearly equal-norm frame, as given in Theorem 2.2.3, are controlled by a pair of norm inequalities similar to the ϵ-nearly equal-norm definition, which allowed them to reduce the problem with the assumption that $\mathcal{F}$ is Parseval. By applying a Naimark-like argument, they showed that the solutions to a system of ODEs which governs the evolution of orthonormal bases in N-dimensional Euclidean space correspond to the solutions of a similar system over the space of Parseval frames, which converge to equal-norm frames under certain conditions. By exploiting this correspondence and intermittently switching between solutions of this ODE by multiplying frame vectors with carefully chosen unimodular constants, they obtained a sequence which converges rapidly to a necessarily equal-norm minimizer for the *frame energy*

$$U(\mathcal{F}) = \sum_{j,l \in \mathbb{Z}_N} (\|f_j\|^2 - \|f_l\|^2)^2.$$

This ultimately provided an upper bound for the distance between $\mathcal{F}$ and its nearest equal-norm Parseval frame in terms of $U(\mathcal{F})$, whenever K and N are relatively prime [20].

4.2.1 Gram matrices of Parseval frames

By definition, the value of a frame potential depends only on the entries of its Gram matrix, so another advantage of restricting frame potentials to the space of Parseval frames is that, up to unitary equivalence of frames, the optimization of any frame potential is equivalent to the optimization of the corresponding function on the manifold $\mathcal{M}_{N,K}$. Although the benefits of $\mathcal{M}_{N,K}$'s manifold structure are not clarified until the next chapter, from here on we view frame potentials as functions of the Gramians of Parseval frames.

In this setting, we have a result which is analogous to Theorem 4.1.2, characterizing equal-norm (N, K)-frames.

4.2.2 Theorem. *Let $G \in \mathcal{M}_{N,K}$, then*

$$\sum_{j=1}^{N} |G_{j,j}|^2 \geq \frac{K^2}{N}$$

and equality holds if and only if $G_{j,j} = \frac{K}{N}$ for each $j \in \mathbb{Z}_N$.

Proof. We know that $\sum_{j=1}^{N} G_{j,j} = K$, so the Cauchy-Schwarz inequality gives

$$\sum_{j=1}^{N} |G_{j,j}|^2 \geq \frac{1}{N}(\sum_{j=1}^{N} G_{j,j})^2 = K^2/N$$

and equality is achieved if and only if $G_{j,j} = G_{l,l}$ for all $j, l \in \mathbb{Z}_N$. By summing the diagonal entries of G, we then obtain $NG_{j,j} = K$ for each $j \in \mathbb{Z}_N$. $\square$

In [41], Elwood lifted Φ_2 to the space of Gram matrices for (N, K)-frames and provided the following characterization of equiangular Parseval frames.

4.2.3 Theorem. *(Elwood, [41].) Let $G \in \mathcal{M}_{N,K}$, then*

$$\sum_{j,l=1}^{N} |G_{j,l}|^4 \geq \frac{K^2(K^2 - 2K + N)}{N^2(N-1)}$$

and equality holds if and only if $G_{j,j} = K/N$ and $|G_{j,l}| = C_{N,K}$ for each $j \neq l$.

Proof. We recall that by the fact that G is an orthogonal rank-K projection, one has that $\sum\limits_{j,l=1}^{N} |G_{j,l}|^2 = \sum\limits_{j=1}^{N} G_{j,j} = K$. With the help of these identities, we express the difference between the two sides of the inequality as a sum of quadratic expressions,

$$\sum_{j,l=1}^{N} |G_{j,l}|^4 - \frac{K^2(K^2+N-2K)}{N^2(N-1)}$$
$$= \sum_{\substack{j,\, l\, =\, 1 \\ j\, \neq\, l}}^{N} \left(|G_{j,l}|^2 - C_{N,K}^2\right)^2 + \sum_{j=1}^{N} \left(G_{j,j}^2 - \frac{K^2}{N^2}\right)^2 + \frac{2K(K-1)}{N(N-1)} \sum_{j=1}^{N} \left(G_{j,j} - \frac{K}{N}\right)^2 .$$

In this form it is manifest that this quantity is non-negative and that it vanishes if and only if G is a rank-K orthogonal projection with $G_{j,j} = K/N$ for all j and with $|G_{j,l}| = C_{N,K}$ for all $j \neq l$. $\square$

With this characterization in mind, the author studied a gradient descent for this potential on $\mathcal{M}_{N,K}$ [41]. Although she did not prove convergence for the trajectories induced by the gradient flow, she used a switching argument similar to the one used by Bodmann and Casazza in [20] to obtain sequences which converge to fixed points, provided that the limits exists. Unfortunately, the corresponding fixed point equations allowed undesirable frames as possible critical points.

4.3 Summary

Since the introduction of frame potentials by Benedetto and Fickus in 2003 [14], researchers have used them to characterize frames with various properties as minimizers or critical points [14, 26, 75, 72, 15, 20, 41], see also [27, 84, 17, 43, 57]. As the literature has developed, a hierarchy of deeper questions has emerged, where each question depends on the answer to the one before it.

1. Given a desired frame property P, is there a frame potential F which characterizes frames with P as minimizers?
2. If F is such a potential, is there an algorithm for locating the minimizers of F starting with a well-conditioned initial point?
3. Given such an algorithm, does it lead to the closest frame with P? Can it be used to determine how far an arbitrary frame is from the nearest optimal configuration?

In this document, the property P that we are interested in is the equidistributed property. By restricting certain real analytic frame potentials to the space of Gramians for Parseval frames and applying a gradient descent, we address questions (1) and (2), but leave question (3) as an open problem.

In the next chapter, we prove that solutions to the first order ODE induced by the gradient descent of a real analytic frame potential on $\mathcal{M}_{N,K}$ are guaranteed to converge to fixed points. In Chapter 6, we define several families of frame potentials on $\mathcal{M}_{N,K}$ and use them to characterize and locate both equidistributed and Grassmannian equal-norm Parseval frames.

Chapter 5

The Gradient Descent on $\mathcal{M}_{N,K}$

In order to locate (N, K)-frames with desirable geometric properties, we characterize their Gram matrices as critical points for families of analytic frame potentials on $\mathcal{M}_{N,K}$ and pursue them via gradient descent. In this chapter, we show that this method works and develop tools for computing the gradient. By adapting a result of Łojasiewicz, we prove that if $\mathcal{M}$ is a compact, real analytic Riemannian manifold, then the solutions of the first order ODE induced by the gradient flow of a real-valued, real analytic map defined on $\mathcal{M}$ converge to critical points. To apply these results, we first show that $\mathcal{M}_{N,K}$ is a real analytic Riemmannian manifold.

5.1 $\mathcal{M}_{N,K}$ as a real analytic, Riemmannian manifold

If $\mathbb{F} = \mathbb{C}$ (respectively $\mathbb{F} = \mathbb{R}$), then $\mathcal{M}_{N,K}$ is a subset of the (linear) manifold of Hermitian (respectively symmetric) $N \times N$ matrices equipped with the Hilbert-Schmidt norm. This induces a topology on $\mathcal{M}_{N,K}$ generated by the open balls $B(X, \sigma) = \{Y \in \mathcal{M}_{N,K} : \|Y - X\|_{H.S.} < \sigma\}$ of radius $\sigma > 0$ centered at each

$X \in \mathcal{M}_{N,K}$. To see that $\mathcal{M}_{N,K}$ is a real analytic manifold, we show that it can be covered by real analytic charts with corresponding real analytic change of coordinates maps. Once this is established, the claim follows by considering the maximal atlas induced by these charts. See Appendix A.1 for details.

5.1.1 Theorem. *The manifold $\mathcal{M}_{N,K}$ is a real analytic submanifold of the (linear) manifold of matrices $\mathbb{F}^{N\times N}$. The dimension of $\mathcal{M}_{N,K}$ is $K(N-K)$ if $\mathbb{F}=\mathbb{R}$ and $2K(N-K)$ if $\mathbb{F}=\mathbb{C}$.*

Proof. As before, we define $\mathcal{M}_{N,K}$ as the set of $N \times N$ orthogonal projections with rank K. Given any $G_0 \in \mathcal{M}_{N,K}$, we can find a subset of indices, $J = \{n_1, n_2, ..., n_K\} \subset \mathbb{Z}_N$ of size $|J| = K$ such that the rows of G indexed by J are linearly independent. By the orthogonality of G, removing the rows and columns corresponding to the indices in $\mathbb{Z}_N \setminus J$ from G_0 then yields the Gramian $(G_0)^{J,J}$ of the row vectors indexed by J, which is invertible since the rows are linearly independent. By continuity of the determinant in the entries of a matrix, there exists $\epsilon > 0$ such that for any $G \in \mathcal{M}_{N,K} \cap B(G_0;\epsilon)$ the $K \times K$ submatrix $G^{J,J}$ consisting of the rows from G indexed by J is invertible. Now, for $G \in B(G_0;\epsilon) \cap \mathcal{M}_{N,K}$, consider the map

$$\tilde{\phi}_J : B(G_0;\epsilon) \cap \mathcal{M}_{N,K} \to \mathbb{F}^{N\times K}(\mathbb{C}), G \mapsto (G^{J,J})^{-1} G^{J,N} .$$

Noting that $\tilde{\phi}_J(G)$ contains a $K \times K$ identity submatrix, we define the chart $\phi_J(G)$ to be the $K \times (N-K)$ matrix given by $\tilde{\phi}_J(G) = (I_K \; \phi_J(G))$, thereby defining what will be our local coordinates in $\mathbb{F}^{K\times(N-K)}$. Then $\tilde{\phi}_J$ is analytic, since the inverse of $G^{J,J}$ is rational in its entries; hence, ϕ_J is also analytic, since there is no loss of analyticity in the removal of entries.

To see that ϕ_J has an analytic inverse, we show that we can reconstruct G

from $\phi(G)$ in an analytic fashion. First, we reinsert the $K \times K$ identity block in a way that corresponds to J so that we have recovered the $K \times N$ matrix $A := \tilde{\phi}_J(G) = (G^{J,J})^{-1} G^{J,N}$, as above. Next, we form the $K \times K$ Gram matrix $Q = AA^* = (G^{J,J})^{-1} G^{J,N} (G^{J,N})^* ((G^{J,J})^{-1})^*$. Since $G_{J,N}$ was extracted from an orthogonal projection, $G^{J,N}(G^{J,N})^* = G^{J,J}$, so that $Q = (G^{J,J})^{-1}$ is analytic in the coordinates. Next, we orthogonalize the rows of A to obtain $B := Q^{-1/2} A = (G^{J,J})^{1/2} A$. The negative square root of Q is seen to be analytic in Q via a convergent power series expansion of $(cI - (cI - Q))^{-1/2}$ in terms of the powers of $cI - Q$, where $c > \|Q\|$. The rows of B then provide an orthonormal basis with the same span as the rows of A and $BB^* = I$. Thus, B is the synthesis operator of a Parseval frame with the Gram matrix

$$B^* B = ((Q^{-\frac{1}{2}} A))^* Q^{-\frac{1}{2}} A = G^{N,J} (G^{J,J})^{-1} G^{J,N} = G.$$

We see that the entries of G are analytic in the coordinates if there is $c > 0$ such that the power series expansion of $(cI - (cI - Q))^{-1/2}$ converges, so ϕ_J^{-1} is analytic on the range $\phi_J(B(G_0; \epsilon))$.

Combining the analyticity of the charts and of their inverses, we conclude that $\mathcal{M}_{N,K}$ is a real analytic manifold because $\phi_J \circ \phi_L^{-1}$ is analytic on the image of the intersection of the domains of ϕ_J and ϕ_L for any subsets J and L of size $|J| = |L| = K$. The dimension of $\mathcal{M}_{N,K}$ is the real dimension of $\mathbb{F}^{K \times (N-K)}$, which is $K(N-K)$ if $\mathbb{F} = \mathbb{R}$ and $2K(N-K)$ if $\mathbb{F} = \mathbb{C}$. $\square$

Moreover, the Hilbert-Schmidt norm induces a Riemannian structure on the tangent space $T\mathcal{M}_{N,K}$, as described in Appendix A.1.1. Via the embedding, the tangent space $T_{G_0}\mathcal{M}_{N,K}$ at $G_0 \in \mathcal{M}_{N,K}$ is identified with a subspace of the Hermitian (respectively symmetric) matrices, and the Riemannian metric is the real inner product $(X,Y) \mapsto X \cdot Y \equiv \mathrm{tr}(XY) = \mathrm{tr}(XY^*)$ restricted to the tangent space.

5.2 Convergence of the gradient descent

Let d be a positive integer and recall the definition for the gradient of an analytic function F on the analytic Riemannian manifold $\mathbb{R}^d$ from Appendix A.1.11. The following classical theorem of Łojasiewicz provides an upper bound for the distance between a critical point of an analytic function on $\mathbb{R}^d$ and local points in terms of the gradient.

5.2.1 Theorem. *(Łojasiewicz, p. 61 - 67 of [64]; see also [61, 67].) Let Ω be an open subset of $\mathbb{R}^d$ and $F : \Omega \to \mathbb{R}$ real analytic. For any $x \in \Omega$ there exist $C, \sigma > 0$ and $\theta \in (0, 1/2]$ such that for all $y \in B(x,\sigma) \cap \Omega$,*

$$|F(y) - F(x)|^{1-\theta} \leq C\|\nababla F(y)\| \, .$$

This result can be extended to hold for real analytic functions defined on any real analytic, Riemannian manifold.

5.2.2 Corollary. *Let $\mathcal{M}$ be a d-dimensional real analytic Riemannian manifold. Let $G_0 \in \Omega \subset \mathcal{M}$ and let $W : \Omega \to \mathbb{R}$ be real analytic, then there exist an open neighborhood $\mathcal{U}$ of G_0 in Ω and constants $C > 0$ and $\theta \in (0, 1/2]$ such that for all $G \in \mathcal{U}$,*

$$|W(G) - W(G_0)|^{1-\theta} \leq C\|\nabla W(G)\| \, .$$

Proof. Since the manifold is real analytic, after choosing a chart $\Gamma : \mathcal{M} \to \mathbb{R}^d$, there exists a neighborhood U of $x = \Gamma(G_0)$ in $\mathbb{R}^d$ such that $F = W \circ \Gamma^{-1}$ is a real analytic function on U. Thus, the Łojasiewicz inequality gives a bound for the values of F in terms of the Euclidean gradient ∇F in a set $B(x, \sigma) \cap U$. However, Γ is a diffeomorphism, thus by the continuity of the matrix-valued function obtained from applying the Riemannian metric to pairs of the coordinate vector fields $\{\frac{\partial}{\partial x_j}\}_{j=1}^d$ and by the fact that $B(x, \sigma) \cap U$ is paracompact in $\mathbb{R}^d$, there exists $C' > 0$ such that $\|\nabla F(\Gamma(G)\| \leq C' \|\nabla W(G)\|$ if $\Gamma(G) \in B(x, \sigma) \cap U$. The combination of the Łojasiewicz inequality in local coordinates with this norm inequality gives the claimed bound, valid in the neighborhood $\mathcal{U} = \Gamma^{-1}(B(x, \sigma) \cap U)$ of G_0. □

It is well known that the Łojasiewicz inequality can be used to prove convergence of gradient flows induced by analytic functions on $\mathbb{R}^d$. Since the frame potentials we define on $\mathcal{M}_{N,K}$ in Section 6 are all real analytic functions of matrix entries, we provide a proof of convergence in our setting, adapted from [67].

5.2.3 Proposition. *Suppose that $W : \mathcal{M}_{N,K} \to \mathbb{R}$ is real analytic and let γ be a global solution of the descent system $\dot{\gamma} = -\nabla W(\gamma)$. Then there is an element $G_0 \in \mathcal{M}_{N,K}$ such that $\gamma(t) \to G_0$ as $t \to \infty$ and $\nabla W(G_0) = 0$.*

Proof. First, we observe that $W(\gamma(t))$ is a nonincreasing function, since

$$
\begin{aligned}
\frac{d}{dt} W(\gamma(t)) &= \nabla W(\gamma(t)) \cdot \dot{\gamma}(t), \\
&= -\nabla W(\gamma(t)) \cdot \nabla W(\gamma(t)) \\
&= -\|\nabla W(\gamma(t))\|^2 \\
&\leq 0.
\end{aligned}
$$

Furthermore, since $\mathcal{M}_{N,K}$ is compact, there must some point $G_0 \in \mathcal{M}_{N,K}$ along with an increasing sequence t_n in $\mathbb{R}$, $t_n \to \infty$, which satisfies that $\gamma(t_n) \to G_0$. Thus, the continuity of W together with the fact that $t \mapsto W(\gamma(t))$ is nonincreasing implies that $\lim_{t\to\infty} W(\gamma(t)) = W(G_0)$.

Since adding a constant to our energy function will not alter the gradient flow, let us assume without loss of generality that $W(G_0) = 0$ and $W(\gamma(t)) \geq 0$ for all $t \geq 0$.

If $W(\gamma(t)) = 0$ for some $t_0 \geq 0$, then it follows that $W(\gamma(t)) = 0$ for all $t \geq t_0$. In particular, since $\|\nabla W(\gamma(t))\|^2 = -\frac{d}{dt}W(\gamma(t)) = 0$, we have $\dot{\gamma}(t) = \nabla W(\gamma(t)) = 0$ for all $t \geq 0$. In this case, the proof is complete.

Henceforth, we will consider the case where $W(\gamma(t)) > 0$ for all $t \geq 0$. Due to Corollary 5.2.2, we know that since W is real analytic in some neighborhood of G_0, it follows that there exist $C, \sigma > 0$ and $\theta \in (0, 1/2]$ such that

$$|W(\gamma(t)) - W(G_0)|^{1-\theta} = |W(\gamma(t))|^{1-\theta} \leq C\|\nabla W(\gamma(t))\|$$

for all $t \geq 0$ where $\gamma(t) \in B(G_0; \sigma) \cap \mathcal{M}_{N,K}$. Let $\epsilon \in (0, \sigma)$. Then there exists a sufficiently large $t_0 \in \mathbb{R}_+$ that yields

$$\int_0^{W(\gamma(t_0))} \frac{C}{s^{1-\theta}} ds + \|\gamma(t_0) - G_0\| < \epsilon.$$

Setting $t_1 = \inf\{t \geq t_0 : \|\gamma(t) - G_0\| \geq \epsilon\}$, we note that the Łojasiewicz inequality is satisfied for $t \in [t_0, t_1)$, which gives us

$$-\frac{d}{dt}\int_0^{W(\gamma(t))} \frac{C}{s^{1-\theta}} ds = C\frac{-\frac{d}{dt}W(\gamma(t))}{|W(\gamma(t))|^{1-\theta}} = C\frac{\|\nabla W(\gamma(t))\|^2}{|W(\gamma(t))|^{1-\theta}} \geq \|\nabla W(\gamma(t))\| = \|\dot{\gamma}(t)\|.$$

Since this inequality holds for any $t \in [t_0, t_1)$, it follows by integrating both sides that for any $t \in [t_0, t_1]$ we have

$$\begin{aligned}
\|\gamma(t) - G_0\| &\leq \|\gamma(t) - \gamma(t_0)\| + \|\gamma(t_0) - G_0\| \\
&\leq \int_{t_0}^{t_1} \|\dot{\gamma}(s)\| ds + \|\gamma(t_0) - G_0\| \\
&\leq C\int_{t_0}^{t_1} \frac{\|\nabla W(\gamma(s))\|^2}{|W(\gamma(s))|^{1-\theta}} ds + \|\gamma(t_0) - G_0\| \\
&= -C\int_{W(\gamma(t_0))}^{W(\gamma(t_1))} \frac{dv}{v^{1-\theta}} + \|\gamma(t_0) - G_0\| \\
&\leq C\int_0^{W(\gamma(t_0))} \frac{dv}{v^{1-\theta}} + \|\gamma(t_0) - G_0\| \\
&< \epsilon\,.
\end{aligned}$$

This shows that $t_1 = +\infty$, so that

$$\int_0^\infty \|\dot{\gamma}(t)\| dt \leq C\int_0^\infty \frac{\|\nabla W(\gamma(t))\|^2}{|W(\gamma(t))|^{1-\theta}} dt = C\int_0^{W(\gamma(0))} \frac{dv}{v^{1-\theta}} < \infty\,.$$

Thus, we see that $\|\dot{\gamma}(t)\| \in L^1(\mathbb{R}_+)$, and conclude that $\gamma(t) \to G_0$ as $t \to \infty$. $\square$

5.3 Characterization of fixed points for the gradient flow

Recall that when $\mathbb{F} = \mathbb{C}$ (respectively $\mathbb{F} = \mathbb{R}$), the embedding of $\mathcal{M}_{N,K}$ into the real vector space of Hermitian (respectively symmetric) $N \times N$ matrices induces a similar embedding of the tangent space to $\mathcal{M}_{N,K}$ at G_0,

$$T_{G_0}\mathcal{M}_{N,K} = \{\dot{\gamma}(0) : \gamma \in C^1(\mathbb{R}, \mathcal{M}_{N,K}), \gamma(0) = G_0\} \subset \mathbb{F}^{N \times N}$$

where $\dot{\gamma}$ is the (matrix-valued) derivative of γ. We use this embedding to compute gradients and characterize where the gradient vanishes.

5.3.1 Lemma. *Let $G_0 \in \mathcal{M}_{N,K}$, then the real linear map*

$$\begin{aligned} P_{G_0} : \mathbb{F}^{N \times N} &\to \mathbb{F}^{N \times N} \\ X &\mapsto (I - G_0)XG_0 + G_0X^*(I - G_0) \end{aligned}$$

is the orthogonal projection onto $T_{G_0}\mathcal{M}_{N,K}$.

Proof. As a first step, we observe that because P_{G_0} is idempotent, its range is the real vector space

$$\mathcal{V}_{G_0} = \{X \in \mathbb{F}^{N \times N} : X = (I - G_0)XG_0 + G_0X^*(I - G_0)\}\,.$$

We show that this vector space contains each tangent vector at G_0. Let $\gamma : (a, b) \to \mathcal{M}_{N,K}$ be a smooth curve such that $0 \in (a, b)$ and $\gamma(0) = G_0$. Since $\gamma(t)$ is an orthogonal projection for all $t \in (a, b)$, one has that $\gamma(t)^* = \gamma(t)$ and $\gamma(t) = \gamma(t)^2 = \gamma(t)^3$

for all $t \in (a,b)$. Therefore, differentiating $\gamma(t)^2 - \gamma(t)^3 = 0$ yields $\gamma(t)\dot{\gamma}(t)\gamma(t) = 0$. If $X = \dot{\gamma}(0)$, then at $t = 0$ this gives

$$G_0 X G_0 = 0\,.$$

Similarly, if $\iota(t) = I$, then the equations for the complementary projection, $\iota(t) - \gamma(t) = (\iota(t) - \gamma(t))^2 = (\iota(t) - \gamma(t))^3$ result in the identity

$$(I - G_0)X(I - G_0) = 0$$

for $X = \dot{\gamma}(0)$. This, together with $\dot{\gamma}(0)^* = \dot{\gamma}(0)$ shows that each tangent vector is in $\mathcal{V}_{G_0}$.

Moreover, from Section 5.1.1, we know the dimension of $\mathcal{M}_{N,K}$ is $2K(N-K)$ when $\mathbb{F} = \mathbb{C}$ and $K(N-K)$ when $\mathbb{F} = \mathbb{R}$. If U is a unitary (respectively orthogonal) matrix whose columns are eigenvectors of G_0, the first K columns corresponding to eigenvalue one, then if $X = X^*$ and $X = (I - G_0)XG_0 + G_0X(I - G_0)$, we know

$$X = U \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix} U^*$$

with some $Y \in \mathbb{F}^{K \times N-K}$, so the real dimension of the space $\mathcal{V}_{G_0}$ is $2K(N-K)$ when $\mathbb{F} = \mathbb{C}$ and $K(N-K)$ when $\mathbb{F} = \mathbb{R}$. This is precisely the dimension of the real manifold $\mathcal{M}_{N,K}$, thus the vector space is the span of all the tangent vectors.

Finally, we note that the map P_{G_0} is idempotent and self-adjoint with respect to the (real) Hilbert-Schmidt inner product. Thus, it is an orthogonal projection onto its range, the tangent space of $\mathcal{M}_{N,K}$ at G_0. □

Since P_{G_0} is the orthogonal projection onto $T_{G_0}\mathcal{M}_{N,K}$, it can be used to construct Parseval frames for $T_{G_0}\mathcal{M}_{N,K}$ from suitable orthonormal sequences. We first discuss the complex case and then the real case. In the following, $\Delta_{a,b}$ with $a, b \in \mathbb{Z}_N$ denotes the matrix unit whose only non-vanishing entry is a 1 in the ath row and the bth column.

5.3.2 Theorem. *Suppose $\mathbb{F} = \mathbb{C}$ and let $\{S_{a,a} : a \in \mathbb{Z}_N\} \cup \{S_{a,b}, T_{a,b} : a, b \in \mathbb{Z}_N, a > b\}$ be the orthonormal basis for the real vector space of the anti-Hermitian $N \times N$ matrices given by $S_{a,a} = i\Delta_{a,a}$, $S_{a,b} = i(\Delta_{a,b} + \Delta_{b,a})/\sqrt{2}$ and $T_{a,b} = (\Delta_{a,b} - \Delta_{b,a})/\sqrt{2}$ for $a > b$, then $P_{G_0}(S_{a,b}) = S_{a,b}G_0 - G_0S_{a,b}$ and $P_{G_0}(T_{a,b}) = T_{a,b}G_0 - G_0T_{a,b}$ provides a Parseval frame $\{P_{G_0}(S_{a,b}), P_{G_0}(T_{a,b})\}_{a,b=1}^N$ for the tangent space $T_{G_0}\mathcal{M}_{N,K}$.*

Proof. We first note that because $S_{a,b}$ and $T_{a,b}$ are anti-Hermitian, $G_0S_{a,b}G_0 + G_0S_{a,b}^*G_0 = 0$ and $G_0T_{a,b}G_0 + G_0T_{a,b}^*G_0 = 0$, which shows the simplified expressions for the projections onto the tangent space. Next, we show the Parseval property. Since $\{S_{a,b}, T_{a,b}\}_{a,b=1}^N$ is an orthonormal basis, the orthogonal projection P_{G_0} maps it to a Parseval frame for its span. This means we only need to show that the span of the projected vectors is the space of all tangent vectors at G_0.

Conjugating the orthonormal basis vectors $\{S_{a,a} : a \in \mathbb{Z}_N\} \cup \{S_{a,b}, T_{a,b} : a, b \in \mathbb{Z}_N, a > b\}$ with a unitary U does not change the span. We choose U so that it diagonalizes G_0, with the first K columns of U belonging to eigenvectors of G of eigenvalue one. Thus

$$
\begin{aligned}
(I-G_0)US_{a,b}U^*G_0+G_0U^*S_{a,b}^*U(I-G_0) =&U\begin{pmatrix} 0 & 0 \\ 0 & I_{N-K} \end{pmatrix} S_{a,b} \begin{pmatrix} I_K & 0 \\ 0 & 0 \end{pmatrix} U^* \\
&- U\begin{pmatrix} I_K & 0 \\ 0 & 0 \end{pmatrix} S_{a,b} \begin{pmatrix} 0 & 0 \\ 0 & I_{N-K} \end{pmatrix} U^*
\end{aligned}
$$

where I_K and I_{N-K} are identity matrices of size $K \times K$ and $(N-K)\times(N-K)$. Inserting the definition of $S_{a,b}$ shows that this is zero unless $a > K$ and $b \leq K$. In that case,

$$(I-G_0)US_{a,b}U^*G_0 - G_0U^*S_{a,b}U(I-G_0) = U(i\Delta_{a,b} - i\Delta_{b,a})U^*/\sqrt{2} = iUT_{a,b}U^* \,.$$

Similarly, if $a > K$ and $b \leq K$, then

$$(I-G_0)UT_{a,b}U^*G_0 - G_0U^*T_{a,b}U(I-G_0) = -iUS_{a,b}U^* \,.$$

The set $\{iUT_{a,b}U^*, -iUS_{a,b}U^*\}_{a>K, b\leq K}$ is by inspection the orthonormal basis of a $2K(N-K)$-dimensional real vector space of Hermitian matrices. Since this is in the range of P_{G_0}, it is a subspace of the tangent space. Its dimension then shows that the set $\{iUT_{a,b}U^*, -iUS_{a,b}U^*\}_{a>K,b\leq K}$ spans the entire tangent space. Consequently, $\{P_{G_0}(S_{a,a}) : a \in \mathbb{Z}_N\} \cup \{P_{G_0}(S_{a,b}), P_{G_0}(T_{a,b}) : a, b \in \mathbb{Z}_N, a > b\}$ is a Parseval frame for the tangent space. $\square$

An analogous theorem holds for the real case.

5.3.3 Theorem. *Suppose $\mathbb{F} = \mathbb{R}$ and let $\{T_{a,b} : a, b \in \mathbb{Z}_N, a > b\}$ be the orthonormal basis for the real vector space of the anti-symmetric $N \times N$ matrices given by $T_{a,b} =$*

$(\Delta_{a,b} - \Delta_{b,a})/\sqrt{2}$ *for* $a > b$*, then* $P_{G_0}(T_{a,b}) = T_{a,b}G_0 - G_0T_{a,b}$ *provides a Parseval frame* $\{P_{G_0}(T_{a,b})\}_{a,b=1}^N$ *for the tangent space* $T_{G_0}\mathcal{M}_{N,K}$.

Proof. The proof follows verbatim the proof of the complex case, with $\{S_{a,b}\}_{a\geq b}$ omitted from the basis of the anti-Hermitian matrices. We note that after conjugating with a suitable orthogonal matrix U, the resulting projection of $T_{a,b}$, with $a > K$ and $b \leq K$, onto the tangent space is

$$(I - G_0)UT_{a,b}U^*G_0 - G_0U^*T_{a,b}U(I - G_0) = -iUS_{a,b}U^*$$

which is indeed a real symmetric matrix. Dimension counting then gives that the image of $\{T_{a,b}\}_{a>b}$ is a basis for the $K(N - K)$-dimensional space of tangent vectors at G_0. $\square$

The appearance of anti-Hermitian (respectively anti-symmetric) matrices is natural if one considers that selecting $G_0 \in \mathcal{M}_{N,K}$ and a differentiable function $u \in C^1(\mathbb{R}, U(N))$ with values in $U(N)$ (respectively $O(N)$) , the manifold of $N \times N$ unitary (respectively orthogonal) matrices [62], induces curves in $\mathcal{M}_{N,K}$ of the form

$$\gamma(t) = u(t)G_0u^*(t)\,.$$

If $u(0) = I$ then from $\frac{d}{dt}u(t)u^*(t) = 0$, we see that $\dot{u}(0) + \dot{u}^*(0) = 0$, so $A = \dot{u}(0)$ is anti-Hermitian (respectively anti-symmetric) and

$$\dot{\gamma}(0) = AG_0 + G_0A^* = AG_0 - G_0A\,.$$

We denote the underlying map by $\Pi_{G_0} : U(N) \to \mathcal{M}_{N,K}$, $\Pi_{G_0}(U) = UG_0U^*$ (respectively $\Pi_{G_0} : O(N) \to \mathcal{M}_{N,K}$, $\Pi_{G_0}(U) = UG_0U^*$) and consider the associated tangent map $D\Pi_{G_0}$.

We recall that the embedding of the tangent spaces $T_IU(N)$ (respectively $T_IO(N)$) and of $T_{G_0}\mathcal{M}_{N,K}$ in $\mathbb{F}^{N\times N}$ induces via the Hilbert-Schmidt inner product a Riemannian structure on the tangent spaces.

5.3.4 Corollary. *The tangent map $D\Pi_{G_0}$ from $T_IU(N) = \{A \in \mathbb{F}^{N\times N}, A = -A^*\}$ (respectively $T_IO(N) = \{A \in \mathbb{F}^{N\times N}, A = -A^\intercal\}$) to $T_{G_0}\mathcal{M}_{N,K}$ given by*

$$D\Pi_{G_0}(A) = AG_0 - G_0A$$

is a surjective partial isometry.

Proof. The preceding theorems show that the map $D\Pi_{G_0}$ is the synthesis operator of a Parseval frame, so it is a surjective partial isometry. □

In order to characterize fixed points of the gradient flows associated with each potential, we lift frame potentials and gradients to the manifold of unitary (respectively orthogonal) matrices.

Given a function $\Phi : \mathcal{M}_{N,K} \to \mathbb{R}$ and $G_0 \in \mathcal{M}_{N,K}$, we consider the lifted function

$$\widehat{\Phi}_{G_0} : U(N) \to \mathbb{R}, \quad \widehat{\Phi}(U) = \Phi \circ \Pi_{G_0}(U) = \Phi(UG_0U^*)$$

when $\mathbb{F} = \mathbb{C}$ or

$$\widehat{\Phi}_{G_0} : O(N) \to \mathbb{R}, \quad \widehat{\Phi}(U) = \Phi \circ \Pi_{G_0}(U) = \Phi(UG_0U^\intercal)$$

when $\mathbb{F} = \mathbb{R}$.

5.3.5 Corollary. *Let $\Phi : \mathcal{M}_{N,K} \to \mathbb{R}$ be differentiable, then the gradient $\nabla\Phi(G_0) = 0$ if and only if $\nabla\widehat{\Phi}_{G_0}(I) = 0$.*

Proof. We first cover the complex case. Letting $\gamma(t) = u(t)G_0u^*(t)$, $u(0) = I$ and $\dot{u}(0) = A = -A^*$, then the chain rule gives $\frac{d}{dt}|_{t=0}\Phi(\gamma(t)) = \nabla\Phi(G_0) \cdot (AG_0 - G_0A)$. On the other hand, $\frac{d}{dt}|_{t=0}\Phi(\gamma(t)) = \frac{d}{dt}|_{t=0}\widehat{\Phi}_{G_0}(u(t)) = \nabla\widehat{\Phi}(I) \cdot A$. We conclude

$$\nabla\widehat{\Phi}_{G_0}(I) \cdot A = \nabla\Phi(G_0) \cdot D\Pi_{G_0}(A)$$

for any anti-Hermitian A. Thus, if $\nabla\Phi(G_0) = 0$ then $\nabla\widehat{\Phi}_{G_0}(I) = 0$. Conversely, since $D\Pi_{G_0}$ is surjective, $\nabla\widehat{\Phi}_{G_0}(I) = 0$ implies that $\nabla\Phi(G_0) = 0$.

In the real case, A^* is simply the transpose of A, so $A = -A^*$ means that A is skew-symmetric rather than anti-Hermitian. The same argument as in the complex case applies. $\square$

Chapter 6

Locating Structured Frames on $\mathcal{M}_{N,K}$

This chapter presents the main results of this work. We define several parametric families of nonnegative, analytic frame potentials on $\mathcal{M}_{N,K}$ and use them to locate frames with geometric properties. We show that Grassmannian equal-norm Parseval frames can be obtained as limits of global minimizers for one of these families, and with the assistance of the machinery developed in the preceding chapter, we characterize various classes of equidistributed frames and equidistributed, Grassmannian equal-norm Parseval frames in terms of fixed point equations for the other families. We also provide a simple characterization of equiangular Parseval frames, similar to the one from Theorem 4.1.3. In order to formulate these families of frame potentials in a convenient manner, we use the fact that every potential is at most quadratic in the elementary one-parameter potential, $E^{\eta}_{x,y}$.

6.0.6 Definition. Let $G = (G_{a,b})_{a,b=1}^{N} \in \mathcal{M}_{N,K}$. Given $x, y \in \mathbb{Z}_N$ and $\eta > 0$, we define the *exponential potential* $E^{\eta}_{x,y} : \mathcal{M}_{N,K} \to \mathbb{R}$ by

$$E^{\eta}_{x,y}(G) = e^{\eta |G_{x,y}|^2}.$$

6.1 Grassmanian equal-norm Parseval frames

It is natural to ask whether a characterization of Grassmannian equal-norm, Parseval frames in terms of frame potentials exists. To address this question, we introduce the family of *off-diagonal sum potentials*.

6.1.1 Definition. Let $G = (G_{a,b})_{a,b=1}^N \in \mathcal{M}_{N,K}$. Given $\eta > 0$, the *off-diagonal sum potential* is the map $\Phi_{\text{od}}^\eta : \mathcal{M}_{N,K} \to \mathbb{R}$ defined by

$$\Phi_{\text{od}}^\eta(G) = \sum_{j,l=1}^{N} (1 - \delta_{j,l}) E_{j,l}^\eta(G)\,,$$

where the Kronecker symbol $\delta_{j,l}$ vanishes if $j \neq l$ and contributes $\delta_{j,j} = 1$ otherwise.

Although a Grassmannian equal-norm, Parseval frame may fail to be a minimizer for Φ_{od}^η for a fixed value η, the family of frame potentials $\{\Phi_{\text{od}}^\eta\}_{\eta>0}$ characterizes them.

6.1.2 Proposition. *Let $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$, then*

$$\mu(G) = \lim_{\eta\to\infty} \frac{1}{\eta} \ln(\Phi_{\text{od}}^\eta(G))\,.$$

Moreover, if G' belongs to a Grassmannian, equal-norm Parseval frame and $G'' \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ does not, then there exists an $\eta > 0$ such that $\Phi_{\text{od}}^{\eta'}(G') < \Phi_{\text{od}}^{\eta'}(G'')$ for each $\eta' > \eta$.

Proof. We have for any $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$

$$e^{\eta\mu(G)} \leq \Phi_{\text{od}}^\eta(G) \leq N(N-1)e^{\eta\mu(G)},$$

thus $\lim_{\eta\to\infty} \frac{1}{\eta}\ln(\Phi_{\mathrm{od}}^{\eta}(G)) = \mu(G)$. Moreover, if G' is the Gram matrix of a Grassmannian equal-norm, Parseval frame and G'' is not, then $\mu(G'') = \mu(G') + \epsilon$ for some $\epsilon > 0$ and if $\eta > \ln(N(N-1))/\epsilon$, then $\eta\mu(G') + \ln N(N-1) < \eta\mu(G') + \eta\epsilon = \eta\mu(G'')$ and consequently $\Phi_{\mathrm{od}}^{\eta}(G') \leq N(N-1)e^{\eta\mu(G')} < e^{\eta\mu(G'')} \leq \Phi_{\mathrm{od}}^{\eta}(G'')$. $\square$

Although μ is continuous on $\mathcal{M}_{N,K} \cap \Omega_{N,K}$, it is not globally differentiable. Thus, locating even local minima is difficult. Fortunately, we can reduce the minimization problem for μ to finding minimizers for a sequence of frame potentials.

6.1.3 Proposition. *Let $\{\eta_m\}_{m=1}^{\infty}$ be a positive, increasing sequence such that $\lim\limits_{m\to\infty} \eta_m = +\infty$. If $\left\{G(m) = (G(m)_{a,b})_{a,b=1}^{N}\right\}_{m=1}^{\infty} \subseteq \mathcal{M}_{N,K} \cap \Omega_{N,K}$ is a sequence such that the restricted potential $\Phi_{\mathrm{od}}^{\eta_m}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ achieves its absolute minimum at $G(m)$ for every $m \in \{1, 2, 3, ...\}$, then there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim\limits_{s\to\infty} G(m_s) = G$, where G is the Gramian of a Grassmannian equal-norm, Parseval frame.*

Proof. By the compactness of $\mathcal{M}_{N,K} \cap \Omega_{N,K}$, there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim\limits_{s\to\infty} G(m_s) = G$. Now let $\widehat{G} \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ correspond to any Grassmannian equal-norm, Parseval frame. By hypothesis, we have

$$\Phi_{\mathrm{od}}^{\eta_{m_s}}\left(G(m_s)\right) \leq \Phi_{\mathrm{od}}^{\eta_{m_s}}\left(\widehat{G}\right)$$

for every $s \in \{1, 2, 3, ...\}$. Furthermore, for every $s \in \{1, 2, 3, ...\}$, we have

$$e^{\eta_{m_s}\mu(G(m_s))} \leq \Phi_{\mathrm{od}}^{\eta_{m_s}}(G(m_s)) \leq N(N-1)e^{\eta_{m_s}\mu(G(m_s))}.$$

Thus

$$\mu(G) = \lim_{s\to\infty} \mu(G(m_s)) = \lim_{s\to\infty} \frac{1}{\eta_{m_s}} \ln\left(\Phi_{\text{od}}^{\eta_{m_s}}(G(m_s))\right) \le \lim_{s\to\infty} \frac{1}{\eta_{m_s}} \ln\left(\Phi_{\text{od}}^{\eta_{m_s}}(\widehat{G})\right) = \mu(\widehat{G}),$$

where the first equality follows from continuity of the $\max$ function. This shows that G belongs to a Grassmannian equal-norm, Parseval frame. $\square$

6.2 Equiangular frames

If the off-diagonal sum potential is properly complemented by terms for the diagonal entries of G, then a simple characterization of equiangular Parseval frames can be derived.

6.2.1 Definition. Let $G = (G_{a,b})_{a,b=1}^N \in \mathcal{M}_{N,K}$. Given $\eta > 0$, we define the *sum potential* of G as

$$\Phi_{\text{sum}}^{\eta}(G) = \Phi_{\text{od}}^{\eta}(G) + \sum_{j=1}^{N} e^{-\eta(K^2/N^2 - C_{N,K}^2)} E_{j,j}^{\eta}(G)\,.$$

6.2.2 Proposition. *Let $G \in \mathcal{M}_{N,K}$, then*

$$\Phi_{\text{sum}}^{\eta}(G) \ge N^2 e^{\eta(K/N^2 - K^2/N^3 + K(N-K))/N^3(N-1)}$$

and equality holds if and only if G belongs to an equiangular Parseval frame.

Proof. We use Jensen's inequality to obtain

$$\Phi_{\text{sum}}^{\eta}(G) = \sum_{j,l=1}^{N} e^{\eta|G_{j,l}|^2 - \delta_{j,l}(K^2/N^2 - C_{N,K}^2)} \ge N^2 e^{(\eta/N^2)\sum_{j,l=1}^{N}(|G_{j,l}|^2 - \eta\delta_{j,l}(K^2/N^2 - C_{N,K}^2))}\,.$$

Now using the Parseval property gives $\sum_{j,l=1}^{N} |G_{j,l}|^2 = K$ and thus

$$\Phi_{\text{sum}}^{\eta}(G) \geq N^2 e^{\eta(K/N^2 - K^2/N^3 + K(N-K))/N^3(N-1)} .$$

Equality holds in Jensen's equality if and only if the average is over a constant. This implies that the diagonal entries equal $G_{j,j} = K/N$ and the magnitude of the off-diagonal entries equals $C_{N,K}$. □

6.3 Equidistributed frames

In Chapter 5, we learned that the gradient descent for any real-analytic frame potential on $\mathcal{M}_{N,K}$ always approaches a critical point. With this in mind, we direct our attention to the geometric character of the critical points for several choices of frame potentials, which culminates with results concerning equidistributed frames. In order to compute the fixed point equations, we recall from Corollary 5.3.5 that $\nabla\Phi$ vanishes at G_0 if and only if $\nabla\hat{\Phi}_{G_0}$ vanishes at I.

We start with $\nabla(\widehat{E}_{x,y}^{\alpha})_G(I)$. From here on, when computing the gradient $\nabla\widehat{\Phi}_G(I)$ corresponding to any frame potential Φ, we suppress the subscript G and the argument I and simply write $\nabla\widehat{\Phi}$.

6.3.1 Lemma. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. If $G \in \mathcal{M}_{N,K}$, $\alpha \in (0, \infty)$, $x, y \in \{1, 2, ..., N\}$, and let $(\widehat{E}_{x,y}^{\alpha})(U) = E_{x,y}^{\alpha}(UGU^*)$, then the (a, b) entry of the gradient of $\widehat{E}_{x,y}^{\alpha}$ at I is given as follows:*

$$[\nabla\widehat{E}_{x,y}^{\alpha}]_{a,b} = \alpha e^{\alpha|G_{x,y}|^2}(G_{y,x}G_{x,b}\delta_{y,a} - G_{a,y}G_{y,x}\delta_{b,x} + G_{x,y}G_{y,b}\delta_{a,x} - G_{a,x}G_{x,y}\delta_{b,y}) .$$

In particular, if $x = y$, then

$$[\nabla \widehat{E}_{x,x}^{\alpha}]_{a,b} = 2\alpha G_{x,x} e^{\alpha |G_{x,x}|^2} (\delta_{a,x} G_{x,b} - \delta_{b,x} G_{a,x})\,.$$

Proof. Let $S_{a,b} = i(\Delta_{a,b} + \Delta_{b,a})/\sqrt{2}$ for $a \leq b$ and let $T_{a,b} = (\Delta_{a,b} - \Delta_{b,a})/\sqrt{2}$ for $a < b$ as before.

We first compute the entries of the matrices $S_{a,b}G - GS_{a,b}$ and $T_{a,b}G - GT_{a,b}$,

$$\begin{aligned}
[S_{a,b}G - GS_{a,b}]_{x,y} &= \frac{i}{\sqrt{2}}[\Delta_{a,b}G + \Delta_{b,a}G - G\Delta_{a,b} - G\Delta_{b,a}]_{x,y} \\
&= \frac{i}{\sqrt{2}}[\delta_{a,x}G_{b,y} + \delta_{b,x}G_{a,y} - G_{x,a}\delta_{b,y} - G_{x,b}\delta_{y,a}]\,.
\end{aligned}$$

and

$$\begin{aligned}
[T_{a,b}G - GT_{a,b}]_{x,y} &= \frac{1}{\sqrt{2}}[\Delta_{a,b}G - \Delta_{b,a}G - G\Delta_{a,b} + G\Delta_{b,a}]_{x,y} \\
&= \frac{1}{\sqrt{2}}[\delta_{a,x}G_{b,y} - \delta_{b,x}G_{a,y} - G_{x,a}\delta_{b,y} + G_{x,b}\delta_{a,y}]\,.
\end{aligned}$$

Let $x, y \in \mathbb{Z}_N$, then

$$\begin{aligned}
\nabla \widehat{E}_{x,y}^{\alpha} \cdot S_{a,b} &= \frac{d}{dt}|_{t=0} E_{x,y}^{\alpha}(G + t(S_{a,b}G - GS_{a,b})) \\
&= \frac{i}{\sqrt{2}}\alpha e^{a|G_{x,y}|^2}(G_{y,x}(\delta_{a,x}G_{b,y} + \delta_{b,x}G_{a,y} - G_{x,a}\delta_{b,y} - G_{x,b}\delta_{y,a}) \\
&\qquad - G_{x,y}(\delta_{a,x}G_{y,b} + \delta_{b,x}G_{y,a} - G_{a,x}\delta_{b,y} - G_{b,x}\delta_{y,a}))
\end{aligned}$$

and

$$
\begin{aligned}
\nabla \widehat{E}^{\alpha}_{x,y} \cdot T_{a,b} &= \frac{d}{dt}|_{t=0} E^{\alpha}_{x,y}(G_0 + t(T_{a,b}G_0 - G_0 T_{a,b})) \\
&= \frac{1}{\sqrt{2}} \alpha e^{a|G_{x,y}|^2} (G_{y,x}(\delta_{a,x}G_{b,y} - \delta_{b,x}G_{a,y} - G_{x,a}\delta_{b,y} + G_{x,b}\delta_{a,y}) \\
&\qquad + G_{x,y}(\delta_{a,x}G_{y,b} - \delta_{b,x}G_{y,a} - G_{a,x}\delta_{b,y} + G_{b,x}\delta_{a,y}))\,.
\end{aligned}
$$

Thus, when $\mathbb{F} = \mathbb{C}$, summing the components of the gradient gives

$$
\begin{aligned}
[\nabla \widehat{E}^{\alpha}_{x,y}]_{a,b} &= [(\nabla \widehat{E}^{\alpha}_{x,y} \cdot S_{a,b})S_{a,b} + (\nabla \widehat{E}^{\alpha}_{x,y} \cdot T_{a,b})T_{a,b}]_{a,b} \\
&= \alpha e^{\alpha|G_{x,y}|^2} (G_{y,x}G_{x,b}\delta_{y,a} - G_{a,y}G_{y,x}\delta_{b,x} + G_{x,y}G_{y,b}\delta_{a,x} - G_{a,x}G_{x,y}\delta_{b,y})\,.
\end{aligned}
$$

Using the fact that $G_{j,l} = G_{l,j}$ for all $j, l \in \mathbb{Z}_N$ when $\mathbb{F} = \mathbb{R}$, we obtain again

$$
\begin{aligned}
[\nabla \widehat{E}^{\alpha}_{x,y}]_{a,b} &= [(\nabla \widehat{E}^{\alpha}_{x,y} \cdot T_{a,b})T_{a,b}]_{a,b} \\
&= \alpha e^{\alpha|G_{x,y}|^2} (G_{y,x}G_{x,b}\delta_{y,a} - G_{a,y}G_{y,x}\delta_{b,x} + G_{x,y}G_{y,b}\delta_{a,x} - G_{a,x}G_{x,y}\delta_{b,y})\,.
\end{aligned}
$$

$\square$

Because the expression for $\nabla \widehat{E}^{\alpha}_{x,y}$ does not depend on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, we do not distinguish between the two cases for the remaining gradient computations.

6.3.1 The sum potential and the absence of orthogonal frame vectors

Next, we investigate the sum potential.

6.3.2 Proposition. *Let $G \in \mathcal{M}_{N,K}$, let $a, b \in \mathbb{Z}_N$, and let $\widehat{\Phi}^{\eta}_{sum}(U) = \Phi^{\eta}_{sum}(UGU^*)$, then the (a,b) entry of the gradient of $\widehat{\Phi}^{\eta}_{sum}$ is given as follows:*

$$\left[\nabla \widehat{\Phi}^{\eta}_{sum}\right]_{a,b} = 2 \sum_{j \in \mathbb{Z}_N, j \notin \{a,b\}} (e^{\eta |G_{a,j}|^2} - e^{\eta |G_{b,j}|^2}) G_{a,j} G_{j,b} + 2e^{\eta |G_{a,b}|^2} (G_{a,b} G_{b,b} - G_{a,a} G_{a,b})$$
$$+ 2e^{\eta (C^2_{N,K} - \frac{K^2}{N^2})} \left(e^{\eta G^2_{a,a}} G_{a,a} G_{a,b} - e^{\eta G^2_{b,b}} G_{a,b} G_{b,b} \right) .$$

Proof. By linearity of the gradient operator, we have

$$\nabla \widehat{\Phi}^{\eta}_{sum} = \frac{1}{\eta} \left[\sum_{\substack{l,\, j \,=\, 1 \\ l \,\neq\, j}}^{N} \nabla \widehat{E}^{\eta}_{l,j} + e^{\eta \left(C^2_{N,K} - \frac{K^2}{N^2} \right)} \sum_{s=1}^{N} \nabla \widehat{E}^{\eta}_{s,s} \right] .$$

By applying Lemma 6.3.1, the claim follows. $\square$

In the investigation of gradient descent for equal-norm frames, nontrivially orthodecomposable frames presented undesirable critical points [40]. We show that this class of frames does not pose problems for our optimization strategy when an initial condition is met.

6.3.3 Definition. A frame $\mathcal{F}$ for a Hilbert space $\mathcal{H}$ is called *orthodecomposable* if there are mutually disjoint subsets $J_1, J_2, \ldots, J_m$ partitioning $\mathbb{Z}_N$ and subspaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ of $\mathcal{H}$ such that $\{f_j\}_{j \in J_k}$ is a frame for $\mathcal{H}_k$ and $\mathcal{H}_k \perp \mathcal{H}_l$ for all $k \neq l$, so $\mathcal{H} = \bigoplus_{k=1}^{m} \mathcal{H}_k$.

In terms of its Gram matrix G, a frame $\mathcal{F}$ is nontrivially orthodecomposable if there is some permutation matrix P which makes G block diagonal,

$$G' = PGP^* = \begin{pmatrix} G'_{1,1} & 0 \\ 0 & G'_{2,2} \end{pmatrix}$$

where $G'_{1,1} \not\equiv 0$ and $G'_{2,2} \not\equiv 0$.

A sufficiently small initial value of the sum potential rules out that the gradient descent on $\mathcal{M}_{N,K}$ encounters such orthodecomoposable frames.

6.3.4 Proposition. *Let $G \in \mathcal{M}_{N,K}$ and $\eta > 0$. Suppose that*

$$\Phi^{\eta}_{sum}(G) < 2 + (N^2-2)e^{\eta(K/(N^2-2)-K^2/N(N^2-2)+K(N-K)/(N(N-1)(N^2-2))},$$

then G contains no zero entries.

Proof. We prove the contrapositive. Let $G_{j,l} = 0$. Without loss of generality, we can assume that $j \neq l$ because if a diagonal entry in G vanishes, then so do all entries in the corresponding row. Now we can perform Jensen's inequality for the entries other than $G_{j,l}$ and $G_{l,j}$ and obtain

$$\Phi^{\eta}_{sum}(G) \geq 2 + (N^2-2)e^{\eta/(N^2-2)\sum_{j,l=1}^{N}(|G_{j,l}|^2-\eta\delta_{j,l}(K^2/N^2-C^2_{N,K}))}.$$

Inserting the value for $C_{N,K}$ and using the Parseval property gives the claimed bound. □

6.3.2 The diagonal potential and equal-norm Parseval frames

Next, we show how equal-norm Parseval frames which exhibit no orthogonality among the frame vectors can be obtained.

6.3.5 Definition. Let $G = (G_{a,b})_{a,b=1}^{N} \in \mathcal{M}_{N,K}$. Given $\delta \in (0,\infty)$, we define the *diagonal potential* $\Phi^{\delta}_{diag} : \mathcal{M}_{N,K} \to \mathbb{R}$ by

$$\begin{aligned}\Phi^{\delta}_{diag}(G) &= \frac{1}{\delta}\sum_{j=1}^{N} E^{\delta}_{j,j}(G) - \frac{N}{\delta} e^{\delta \frac{K^2}{N^2}} \\ &= \frac{1}{\delta}\sum_{j=1}^{N} e^{\delta |G_{j,j}|^2} - \frac{N}{\delta} e^{\delta \frac{K^2}{N^2}} .\end{aligned}$$

6.3.6 Proposition. *Let $G \in \mathcal{M}_{N,K}$, let $a, b \in \mathbb{Z}_N$, and let $\widehat{\Phi}^{\delta}_{diag}(U) = \Phi^{\delta}_{diag}(UGU^*)$. Then the (a,b) entry of the gradient of $\widehat{\Phi}^{\delta}_{diag}$ is given as follows:*

$$\left[\nabla \widehat{\Phi}^{\delta}_{diag}\right]_{a,b} = 2G_{a,b}(G_{a,a}e^{\delta|G_{a,a}|^2} - G_{b,b}e^{\delta|G_{b,b}|^2}) .$$

Proof. Observe that by linearity of the gradient operator, we have

$$\nabla \widehat{\Phi}^{\delta}_{diag} = \nabla \left[\frac{1}{\delta} \sum_{x \,\in\, \mathbb{Z}_N} \widehat{E}^{\delta}_{x,x} \right] = \frac{1}{\delta} \sum_{x \,\in\, \mathbb{Z}_N} \nabla \widehat{E}^{\delta}_{x,x}.$$

By summing over the different cases for x and applying Lemma 6.3.1 , the claim follows. $\square$

6.3.7 Proposition. *Let $\Phi^{\delta} := \Psi + \Phi^{\delta}_{diag}$ be a function on $\mathcal{M}_{N,K}$, where $\Psi : \mathcal{M}_{N,K} \to [0,\infty)$ is any real analytic function that does not depend on the parameter δ. Let*

$G \in \mathcal{M}_{N,K}$ with no zero entries and suppose $\nabla\Phi^\delta(G) = 0$ for all $\delta \in I$, where $I \subseteq (0, \infty)$ is an open interval, then G is the Gram matrix of an equal-norm Parseval frame.

Proof. Recall from Proposition 6.3.6 that each (a, b) entry of $\nabla\widehat{\Phi}^\delta_{diag}$ is given by

$$[\nabla\widehat{\Phi}^\delta_{diag}]_{a,b} = 2G_{a,b}(G_{a,a}e^{\delta|G_{a,a}|^2} - G_{b,b}e^{\delta|G_{b,b}|^2}).$$

Thus, by hypothesis and Corollary 5.3.5, we have

$$[\nabla\widehat{\Phi}^\delta]_{a,b} = [\nabla\widehat{\Psi}]_{a,b} + 2G_{a,b}(G_{a,a}e^{\delta|G_{a,a}|^2} - G_{b,b}e^{\delta|G_{b,b}|^2}) = 0, \ \forall a, b \in \mathbb{Z}_N, \forall\delta \in I.$$

Since $[\nabla\widehat{\Phi}^\delta]_{a,b}$ is constant for all $\delta \in I$ and since Ψ does not depend on δ, taking the derivative of this expression with respect to δ yields

$$\begin{aligned}\frac{d}{d\delta}[\nabla\widehat{\Phi}^\delta]_{a,b} &= \frac{d}{d\delta}[\nabla\widehat{\Psi}]_{a,b} + \frac{d}{d\delta}2G_{a,b}(G_{a,a}e^{\delta|G_{a,a}|^2} - G_{b,b}e^{\delta|G_{b,b}|^2}) \\ &= \quad 0 \qquad\qquad + 2G_{a,b}(G^3_{a,a}e^{\delta|G_{a,a}|^2} - G^3_{b,b}e^{\delta|G_{b,b}|^2}) \\ &= 0\end{aligned}$$

for all $a, b \in \mathbb{Z}_N$ and for all $\delta \in I$. Since G contains no zero entries, we can cancel the factor $2G_{a,b}$ in these equations to obtain

$$G^3_{b,b}e^{\delta G^2_{b,b}} = G^3_{a,a}e^{\delta G^2_{a,a}},$$

for all $a, b \in \mathbb{Z}_N$ and all $\delta \in I$. By the strict monotonicity of the function $x \mapsto x^3 e^{\delta x^2}$ on $\mathbb{R}_+$, this implies $G_{a,a} = G_{b,b}$ for all a, b in $\mathbb{Z}_N$. This is only possible if G is equal-norm, so we are done. $\square$

6.3.3 The chain potential and equipartitioning

In this section, we show that α-equipartitioned frames can be obtained under certain conditions with the chain potential. This requires two auxiliary functions: the exponential row sum potential and the link potential.

6.3.8 Definition. Let $G = (G_{a,b})_{a,b=1}^N \in \mathcal{M}_{N,K}$. Given $x \in \mathbb{Z}_N$ and $\alpha, \beta \in (0, \infty)$, we define the *exponential row sum potential* $R_x^{\alpha,\beta} : \mathcal{M}_{N,K} \to \mathbb{R}$ by

$$\begin{aligned} R_x^{\alpha,\beta}(G) &= \frac{1}{\alpha} \sum_{j=1, \mathrm{j} \neq x}^{N} E_{x,j}^{\alpha}(G) + \beta E_{x,x}^{1}(G) \\ &= \frac{1}{\alpha} \sum_{j=1, \mathrm{j} \neq x}^{N} e^{\alpha |G_{x,j}|^2} + \beta e^{|G_{x,x}|^2} . \end{aligned}$$

We define the *link potential* $L_x^{\alpha,\beta} : \mathcal{M}_{N,K} \to \mathbb{R}$ by

$$L_x^{\alpha,\beta}(G) = (R_x^{\alpha,\beta}(G) - R_{x+1}^{\alpha,\beta}(G))^2$$

and the *chain potential* $\Phi_{ch}^{\alpha,\beta} : \mathcal{M}_{N,K} \to \mathbb{R}$ by

$$\Phi_{ch}^{\alpha,\beta}(G) = \sum_{j \in \mathbb{Z}_N} L_j^{\alpha,\beta}(G) .$$

Next, we compute the gradients of $\widehat{R}_x^{\alpha,\beta}$ and $\widehat{L}_x^{\alpha,\beta}$at I.

6.3.9 Lemma. *Let $G \in \mathcal{M}_{N,K}$, $\alpha \in (0, \infty)$ and $x \in \{1, 2, ..., N\}$, and let $\widehat{R}_x^{\alpha,\beta}(U) = R_x^{\alpha,\beta}(UGU^*)$, then the (a, b) entry of the gradient of $\widehat{R}_x^{\alpha}$ at I is given as follows:*

$$\left[\nabla \widehat{R}_x^{\alpha,\beta}\right]_{a,b} = \sum_{\substack{j\,\in\,\mathbb{Z}_N \\ j\,\neq\,a}} e^{\alpha|G_{x,j}|^2}\left[-G_{a,j}G_{j,x}\delta_{b,x} + G_{x,j}G_{j,b}\delta_{a,x} - G_{a,x}G_{x,j}\delta_{b,j}\right]$$
$$+\,2\beta e^{|G_{x,x}|^2}\left[G_{x,x}G_{x,b}\delta_{x,a} - G_{a,x}G_{x,x}\delta_{b,x}\right]\,.$$

Proof. This computation follows immediately from Lemma 6.3.1 by observing that, because of linearity of the gradient operator, we have

$$\nabla \widehat{R}_x^{\alpha,\beta} = \nabla\left[\frac{1}{\alpha}\sum_{\substack{j\,\in\,\mathbb{Z}_N \\ j\,\neq\,a}} \widehat{E}_{x,j}^{\alpha} + \beta \widehat{E}_{x,x}^{1}\right] = \frac{1}{\alpha}\sum_{\substack{j\,\in\,\mathbb{Z}_N \\ j\,\neq\,a}} \nabla \widehat{E}_{x,j}^{\alpha} + \beta\nabla \widehat{E}_{x,x}^{1}.$$

$\square$

6.3.10 Lemma. *Let* $G \in \mathcal{M}_{N,K}$ *and* $\alpha, \beta \in (0,\infty)$. *Furthermore, let* $x, a \in \mathbb{Z}_N$ *and set* $b = a+1$. *Let* $\widehat{L}_x^{\alpha,\beta}(U) = L_x^{\alpha,\beta}(UGU^*)$, *then the* (a,b)*-entry (ie, along the* *superdiagonal) of the gradient of $\widehat{L}_x^{\alpha,\beta}$ at I is given as:*

$$
\begin{aligned}
\left[\nabla \widehat{L}_x^{\alpha,\beta}\right]_{a,b} &= 2(\widehat{R}_x^{\alpha,\beta} - \widehat{R}_{x+1}^{\alpha,\beta}) \\
&\times \left\{ \sum_{\substack{j \in \mathbb{Z}_N \\ j \neq a}} \left[e^{\alpha|G_{x,j}|^2} \left(-G_{a,j}G_{j,x}\delta_{b,x} + G_{x,j}G_{j,b}\delta_{a,x} - G_{a,x}G_{x,j}\delta_{b,j}\right) \right. \right. \\
&\qquad \left. - \; e^{\alpha|G_{x+1,j}|^2} \left(-G_{a,j}G_{j,x+1}\delta_{b,x+1} + G_{x+1,j}G_{j,b}\delta_{a,x+1} - G_{a,x+1}G_{x+1,j}\delta_{b,j}\right) \right] \\
&+ 2\beta \left[e^{|G_{x,x}|^2} \left(G_{x,x}G_{x,b}\delta_{x,a} - G_{a,x}G_{x,x}\delta_{b,x}\right) \right. \\
&\qquad \left. \left. - e^{|G_{x+1,x+1}|^2} \left(G_{x+1,x+1}G_{x+1,b}\delta_{x+1,a} - G_{a,x+1}G_{x+1,x+1}\delta_{b,x+1}\right) \right] \right\}.
\end{aligned}
$$

Proof. If we let $h(t) = t^2$, then we see that

$$L_x^{\alpha,\beta}(G) = (R_x^{\alpha}(G) - R_{x+1}^{\alpha,\beta}(G))^2 = h(R_x^{\alpha}(G) - R_{x+1}^{\alpha,\beta}(G)).$$

Therefore, by applying the chain rule and linearity of the gradient operator, we see that

$$
\begin{aligned}
\nabla \widehat{L}_x^{\alpha,\beta} &= h'(R_x^{\alpha,\beta}(G) - R_{x+1}^{\alpha,\beta}(G))\nabla(\widehat{R}_x^{\alpha,\beta} - \widehat{R}_{x+1}^{\alpha,\beta}) \\
&= 2(R_x^{\alpha,\beta}(G) - R_{x+1}^{\alpha,\beta}(G))(\nabla\widehat{R}_x^{\alpha,\beta} - \nabla\widehat{R}_{x+1}^{\alpha,\beta})
\end{aligned}
$$

The rest follows by Lemma 6.3.9. $\square$

For notational convenience, we define $\varphi_\alpha : \mathbb{Z}_N^3 \times \mathcal{M}_{N,K} \to \mathbb{C}$ for $\alpha > 0$ by

$$\varphi_\alpha(a,b,x,G) = \sum_{\substack{j \in \mathbb{Z}_N \\ j \neq a}} \Big[e^{\alpha|G_{x,j}|^2} \left(-G_{a,j}G_{j,x}\delta_{b,x} + G_{x,j}G_{j,b}\delta_{a,x} - G_{a,x}G_{x,j}\delta_{b,j}\right) \\ - e^{\alpha|G_{x+1,j}|^2} \left(-G_{a,j}G_{j,x+1}\delta_{b,x+1} + G_{x+1,j}G_{j,b}\delta_{a,x+1} - G_{a,x+1}G_{x+1,j}\delta_{b,j}\right)\Big] .$$

6.3.11 Proposition. *Let $G \in \mathcal{M}_{N,K}$, $\alpha, \beta, \in (0,\infty)$. Let $a \in \mathbb{Z}_N$ and set $b = a+1$, so that (a,b) entry of G falls on the superdiagonal, and let $\widehat{\Phi}_{ch}^{\alpha,\beta}(U) = \Phi_{ch}^{\alpha,\beta}(UGU^*)$, then the (a,b) entry of the gradient of $\widehat{\Phi}_{ch}^{\alpha,\beta}$ is given as follows:*

$$\begin{aligned}\left[\nabla\widehat{\Phi}_{ch}^{\alpha,\beta}\right]_{a,b} &= 2\sum_{j \in \mathbb{Z}_N} \left(R_j^{\alpha,\beta}(G) - R_{j+1}^{\alpha,\beta}(G)\right)\varphi_\alpha(a,b,x,G) \\ &\quad + 4\beta\Big\{-G_{a,a}G_{a,b}e^{|G_{a,a}|^2}[R_{a-1}^{\alpha,\beta}(G) - R_a^{\alpha,\beta}(G)] \\ &\qquad + (G_{a,a}G_{a,b}e^{|G_{a,a}|^2} + G_{a,b}G_{b,b}e^{|G_{b,b}|^2})[R_a^{\alpha,\beta}(G) - R_b^{\alpha,\beta}(G)] \\ &\qquad - G_{a,b}G_{b,b}e^{|G_{b,b}|^2}[R_b^{\alpha,\beta}(G) - R_{b+1}^{\alpha,\beta}(G)]\Big\} .\end{aligned}$$

Proof. Observe that

$$\nabla\widehat{\Phi}_{ch}^{\alpha,\beta} = \nabla\left[\sum_{j \in \mathbb{Z}_N} \widehat{L}_j^{\alpha,\beta}\right] = \sum_{j \in \mathbb{Z}_N} \nabla\widehat{L}_j^{\alpha,\beta}.$$

By summing over the different cases for j and using Lemma 6.3.10, the claim follows, where we have isolated the nonzero terms which are multiplied by the parameter β (i.e., corresponding to $j = a-1$, $j = a$, and $j = a+1$). ☐

6.3.12 Proposition. *Let $\Phi^{\alpha,\beta} := \Psi + \Phi_{ch}^{\alpha,\beta}$ be a function on $\mathcal{M}_{N,K}$, where $\Psi : \mathcal{M}_{N,K} \to$*

$[0, \infty)$ *is any real analytic function that does not depend on the parameters* α *or* β*. Let* $G \in \mathcal{M}_{N,K}$ *be equal-norm with no zero entries, fix* $\alpha \in (0, \infty)$*, and suppose that* $\nabla \Phi_{ch}^{\alpha,\beta}(G) = 0$ *for all* $\beta \in J$*, where* $J \subseteq (0, \infty)$ *is an open interval, then* G *is* α*-equipartitioned.*

Proof. Since Ψ does not depend on β and since $\nabla \Phi_{ch}^{\alpha,\beta} = 0$ for all $\beta \in J$, then using corollary 5.3.5 and taking the partial derivative with respect to β of an (a, b) entry of $\nabla \widehat{\Phi}_{ch}^{\alpha,\beta}$ gives

$$\begin{aligned} \frac{d}{d\beta}[\nabla \widehat{\Phi}_{ch}^{\alpha,\beta}]_{a,b} &= \frac{d}{d\beta}[\nabla \widehat{\Psi}(G)]_{a,b} + \frac{d}{d\beta}[\nabla \widehat{\Phi}_{ch}]_{a,b} \\ &= \quad 0 \qquad\qquad + \frac{d}{d\beta}[\nabla \widehat{\Phi}_{ch}^{\alpha,\beta}]_{a,b} \\ &= 0 \end{aligned}$$

for all $\beta \in J$. In particular, we have $\frac{d}{d\beta}[\nabla \widehat{\Phi}_{ch}^{\alpha,\beta}]_{a,b} = 0$ for all $a, b \in \mathbb{Z}_N$ and for all $\beta \in J$.

Next, we compute $\frac{d}{d\beta}[\widehat{\Phi}_{ch}^{\alpha,\beta}]_{a,b}$ for the case where $b = a + 1$ (ie, along the super-diagonal), thereby inducing a set of equations which will lead to the desired result. So, from here on, we suppose that $b = a + 1$.

First, we observe a simplification that results from the assumption that G is equal-norm. Referring back to Proposition 6.3.11, we note that every additive term of $[\nabla \widehat{\Phi}_{ch}]_{a,b}$ has a factor of the form $(R_j^{\alpha,\beta}(G) - R_{j+1}^{\alpha,\beta}(G))$. However, since G is equal-norm, we can replace each of these factors with $(R_j^{\alpha}(G) - R_{j+1}^{\alpha}(G))$ to denote the fact that the β terms corresponding to the diagonal entries have canceled because of the equal-norm property. After doing this, we see that there are only three terms of $[\nabla \widehat{\Phi}_{ch}^{\alpha,\beta}]_{a,b}$ which still depend on β. Now the desired partial derivative is easy to compute, which yields

$$
\begin{aligned}
\frac{d}{d\beta}[\nabla\widehat{\Phi}_{ch}]_{a,b} &= -4G_{a,a}G_{a,b}e^{|G_{a,a}|^2}[R^{\alpha,\beta}_{a-1}(G) - R^{\alpha,\beta}_{a}(G)] \\
&\qquad + 4(G_{a,a}G_{a,b}e^{|G_{a,a}|^2} + G_{a,b}G_{b,b}e^{|G_{b,b}|^2})[R^{\alpha,\beta}_{a}(G) - R^{\alpha,\beta}_{b}(G)] \\
&\qquad - 4G_{a,b}G_{b,b}e^{|G_{b,b}|^2}[R^{\alpha,\beta}_{b}(G) - R^{\alpha,\beta}_{b+1}(G) \\
&= 0
\end{aligned}
$$

Once again, because G is equal-norm, it follows that $G_{a,a} = G_{b,b} = \frac{K}{N}$, so this equation can be rewritten as

$$
\begin{aligned}
\frac{d}{d\beta}[\nabla\widehat{\Phi}_{ch}]_{a,b} &= -4\frac{K}{N}G_{a,b}e^{|\frac{K}{N}|^2}\left(R^{\alpha}_{a-1}(G) - 3R^{\alpha}_{a}(G) + 3R^{\alpha}_{b}(G) - R^{\alpha}_{b+1}(G)\right) \\
&= 0\,.
\end{aligned}
$$

Since G contains no zero entries, we can cancel the factor(s) $-4\frac{K}{N}G_{a,b}e^{|\frac{K}{N}|^2}$ to obtain

$$
R^{\alpha}_{a-1}(G) - 3R^{\alpha}_{a}(G) + 3R^{\alpha}_{b}(G) - R^{\alpha}_{b+1}(G) = 0
$$

for all $a, b \in \mathbb{Z}_N$, where $b = a + 1$. This induces the linear system $Ax = 0$, where A is the $N \times N$ circulant matrix

$$
A = \begin{pmatrix}
-3 & 3 & -1 & 0 & \cdots & 0 & 1 \\
1 & -3 & 3 & -1 & \cdots & 0 & 0 \\
0 & 1 & -3 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
3 & -1 & 0 & 0 & \cdots & 1 & -3
\end{pmatrix}
$$

and

$$x = \begin{pmatrix} R_1^\alpha(G) \\ R_2^\alpha(G) \\ \vdots \\ R_N^\alpha(G) \end{pmatrix}.$$

The circulant matrix A is the polynomial $A = -3I + 3S - S^2 + S^{N-1}$ of the cyclic shift matrix S. Therefore, its eigenvectors coincide with those of S,

$$v_j = \begin{pmatrix} 1 \\ \omega_j \\ \vdots \\ \omega_j^{N-1} \end{pmatrix}$$

and by the spectral theorem the eigenvalues of A are then given by

$$\lambda_j = -3 + 3\omega_j - \omega_j^2 + \omega_j^{N-1},\ j \in \mathbb{Z}_N,$$

where $\omega_j = e^{\frac{2\pi i j}{N}}$, the Nth roots of unity, are the corresponding eigenvalues of S.

The system $Ax = 0$ is homogeneous, so we would like to obtain the zero eigenspace of A. By letting $j \in \mathbb{Z}_N$, setting $\lambda_j = 0$, and then factoring, we obtain

$$\begin{aligned} 0 &= -3 + 3\omega_j - \omega_j^2 + \omega_j^{N-1} \\ &= -(1-\omega_j)(2-\omega_j-\omega_j^{N-1}). \end{aligned}$$

Inspecting both factors, we see that $\lambda_j = 0$ iff $\omega_j = 1$ iff $j \equiv 0 \mod N$. Thus, the zero eigenspace is 1-dimensional and spanned by the vector of all ones. In particular, this shows that

$$R_1^\alpha(G) = R_2^\alpha(G) = \cdots = R_N^\alpha(G).$$

In other words, G is α-equipartitioned. $\square$

6.3.13 Corollary. *Let $\Phi^{\alpha,\beta} := \Psi + \Phi_{ch}^{\alpha,\beta}$ be a function on $\mathcal{M}_{N,K}$, where $\Psi : \mathcal{M}_{N,K} \to [0,\infty)$ is any real analytic function that does not depend on the parameters α or β. Let $G \in \mathcal{M}_{N,K}$ be equal-norm with no zero entries, and, furthermore, suppose that $\nabla\Phi_{ch}^{\alpha,\beta}(G) = 0$ for all $\alpha \in I$ and all $\beta \in J$, where $I, J \subseteq (0,\infty)$ are open intervals, then G is equidistributed.*

Proof. Since J is an open interval, it follows by proposition 6.3.12 that G is α-equipartitioned for every $\alpha \in I$. Therefore, by Proposition 3.3.16, G is equidistributed. $\square$

6.3.4 A characterization of equidistributed frames

Finally, we combine these definitions to obtain the family of potential functions which will yield our main theorem, as defined below.

6.3.14 Definition. Let $G = (G_{a,b})_{a,b=1}^N \in \mathcal{M}_{N,K}$. Given $\alpha, \beta, \delta, \eta \in (0,\infty)$, we define the *combined potential*, $\Phi^{\alpha,\beta,\delta,\eta} : \mathcal{M}_{N,K} \to \mathbb{R}$, by

$$\Phi^{\alpha,\beta,\delta,\eta}(G) = \Phi_{ch}^{\alpha,\beta}(G) + \Phi_{diag}^{\delta}(G) + \Phi_{sum}^{\eta}(G).$$

6.3.15 Definition. Let $G \in \mathcal{M}_{N,K}$, let $I, J, T \subseteq (0,\infty)$ be open intervals and $\eta > 0$, then we say that G is a *family-wise critical point* with respect to $\{\Phi^{\alpha,\beta,\delta,\eta}\}_{\alpha\in I,\beta\in J,\delta\in T}$

if $\nabla\Phi^{\alpha,\beta,\delta,\eta}(G) = 0$ for all $\alpha \in I$, $\beta \in J$, and $\delta \in T$.

6.3.16 Theorem. *Let $G \in \mathcal{M}_{N,K}$, let $I, J, T \subseteq (0, \infty)$ be open intervals and $\eta > 0$. If G is a family-wise critical point with respect to $\{\Phi^{\alpha,\beta,\delta,\eta}\}_{\alpha\in I,\beta\in J,\delta\in T}$ and*

$$\Phi^{\alpha',\beta',\delta',\eta}(G) < 2 + (N^2 - 2)e^{\eta(K/(N^2-2)-K^2/N(N^2-2)+K(N-K)/(N(N-1)(N^2-2))}$$

for some $\alpha', \beta', \delta' \in (0, \infty)$, then G is equidistributed.

Proof. Since G is a family-wise critical point, $\nabla\Phi^{\alpha,\beta,\delta,\eta}(G) = 0$ for all $\alpha \in I, \beta \in J$ and $\delta \in T$.

Since $\Phi^{\alpha',\beta',\delta',\eta}(G) = \Phi^{\alpha',\beta'}_{ch}(G) + \Phi^{\delta'}_{diag}(G) + \Phi^{\eta}_{sum}(G) < D$, with

$$D = 2 + (N^2 - 2)e^{\eta(K/(N^2-2)-K^2/N(N^2-2)+K(N-K)/(N(N-1)(N^2-2))}$$

then it must also be the case that $\Phi^{\eta}_{sum}(G) < D$. Hence, G contains no zero entries, by Proposition 6.3.4.

Now, with respect to Proposition 6.3.7, we can momentarily rewrite $\Phi^{\alpha,\beta,\delta,\eta}$ as $\Phi^{\delta} = \Psi + \Phi^{\delta}_{diag}$, where $\Psi = \Phi^{\alpha,\beta}_{ch} + \Phi^{\eta}_{sum}$. Since we have confirmed that there are no zero entries, since Ψ does not depend on δ and since $\nabla\Phi^{\delta}(G) = 0$ for all $\delta \in T$, it follows from Proposition 6.3.7 that G corresponds to an equal-norm Parseval frame.

Finally, with respect to Corollary 6.3.13, we can once again rewrite $\Phi^{\alpha,\beta,\delta,\eta}$ as $\Phi^{\alpha,\beta} = \Psi + \Phi^{\alpha,\beta}_{ch}$, where $\Psi = \Phi^{\eta}_{sum} + \Phi^{\delta}_{diag}$ this time. Since we have confirmed that G contains no zeros, since G is equal-norm, Ψ does not depend on α or β, and $\nabla\Phi^{\alpha,\beta}_{ch}(G) = 0$ for all $\alpha \in I$ and $\beta \in J$, it follows from Corollary 6.3.13 that G is equidistributed.

□

Alternatively, we can relax the requirement on the value of the potential and simply assume that our family-wise critical point contains no zero entries. It is clear from the preceding proof that this would also yield equidistributivity, as stated below.

6.3.17 Corollary. *Let $\eta > 0$. If $G \in \mathcal{M}_{N,K}$ is a family-wise critical point with respect to the family of frame potentials $\{\Phi^{\alpha,\beta,\delta,\eta}\}_{\alpha\in I,\beta\in J,\delta\in T}$ and G contains no zero entries, then G is equidistributed.*

6.3.18 Proposition. *Let $\eta > 0$ and $G \in \mathcal{M}_{N,K}$ be equidistributed. If $\nabla\Phi^{\eta}_{sum}(G) = 0$, then G is a family-wise critical point with respect to $\{\Phi^{\alpha,\beta,\delta,\eta}\}_{\alpha,\beta,\delta\in(0,\infty)}$.*

Proof. Since G is equidistributed, we see immediately that $\Phi^{\alpha,\beta}_{ch}(G) = 0$ for all $\alpha, \beta \in (0,\infty)$, so it follows that $\nabla\Phi^{\alpha,\beta}_{ch}(G) = 0$ for all $\alpha, \beta \in (0,\infty)$. Similarly, since G must also be equal-norm, we have $\Phi^{\delta}_{diag}(G) = 0$ for all $\delta \in (0,\infty)$, which implies$\nabla\Phi^{\delta}_{diag}(G) = 0$ for all $\delta \in (0,\infty)$. Thus, if $\nabla\Phi^{\eta}_{sum}(G) = 0$, then

$$\nabla\Phi^{\alpha,\beta,\delta,\eta}(G) = \nabla\Phi^{\alpha,\beta}_{ch}(G) + \Phi^{\delta}_{diag}(G) + \nabla\Phi^{\eta}_{sum}(G) = 0$$

for all $\alpha, \beta, \delta \in (0,\infty)$, so the claim follows. □

The assumption in the preceding proposition is met when the frame is generated with a group representation as specified below.

6.3.19 Proposition. *Suppose Γ is a finite group of size $N = |\Gamma|$ with a unitary representation $\pi : \Gamma \to B(\mathcal{H})$ on the complex K-dimensional Hilbert space $\mathcal{H}$ and $\{f_g = \pi(g)f_e\}$ is an (N,K)-frame. If the Gram matrix G satisfies $G_{g,h} = G_{h^{-1},g^{-1}}$ for all $g, h \in \Gamma$, then $\nabla\Phi^{\eta}_{sum}(G) = 0$ for all $\eta \in (0,\infty)$.*

Proof. Fix $\eta \in (0,\infty)$. Since G is equidistributed (see Example 3.3.6), it is equal-norm, so the last two additive two terms from the (a,b) gradient entry in Proposition 6.3.2 cancel. Therefore, to show that the gradient vanishes, it is sufficient to show that

$$\sum_{j\in\Gamma} e^{\eta|G_{a,j}|^2} G_{a,j}G_{j,b} = \sum_{j\in\Gamma} e^{\eta|G_{b,j}|^2} G_{a,j}G_{j,b}$$

for all $a,b\in\Gamma$. As a first step, we note that the group representation gives $G_{x,y} = \langle f_y, f_x\rangle = \langle \pi(x^{-1}y)f_e, f_e\rangle \equiv H(x^{-1}y)$. Thus, we can change the summation index and get

$$\sum_{j\in\Gamma} e^{\eta|G_{a,j}|^2} G_{a,j}G_{j,b} = \sum_{j\in\Gamma} e^{\eta|H(j^{-1}a)|^2} H(j^{-1}a)H(b^{-1}j) = \sum_{j\in\Gamma} e^{\eta|H(j^{-1})|^2} H(j^{-1})H(b^{-1}aj)\,.$$

We also note that $|H(g)| = |H(g^{-1})|$, so in combination with changing the summation index we obtain

$$\sum_{j\in\Gamma} e^{\eta|G_{a,j}|^2} G_{a,j}G_{j,b} = \sum_{j\in\Gamma} e^{\eta|H(j)|^2} H(j^{-1})H(b^{-1}aj) = \sum_{j\in\Gamma} e^{\eta|H(j^{-1}b)|^2} H(b^{-1}j)H(b^{-1}aj^{-1}b)\,.$$

Finally, using the fact that the Gram matrix has the assumed structure gives $H(h^{-1}g) = H(gh^{-1})$ for $h = a^{-1}b$ and $g = j^{-1}b$, which yields

$$\sum_{j\in\Gamma} e^{\eta|G_{a,j}|^2} G_{a,j}G_{j,b} = \sum_{j\in\Gamma} e^{\eta|H(j^{-1}b)|^2} H(b^{-1}j)H(j^{-1}a) = \sum_{j\in\Gamma} e^{\eta|G_{b,j}|^2} G_{a,j}G_{j,b}\,.$$

This completes the proof, since η was arbitrary.

□

The claimed property of the Gramian is true if Γ is abelian.

6.3.20 Corollary. *Suppose Γ is a finite abelian group of size $N = |\Gamma|$ with a unitary* *representation $\pi : \Gamma \to B(\mathcal{H})$ on a real or complex K-dimensional Hilbert space $\mathcal{H}$ and $\{f_g = \pi(g)f_e\}$ is an (N,K)-frame, then $\nabla\Phi^{\eta}_{sum}(G) = 0$ for every $\eta \in (0,\infty)$.*

There is an abundance of equidistributed Parseval frames obtained with representations of abelian groups, in particular the harmonic frames that exist for any combination of the number of frame vectors N and dimension $K \leq N$.

6.3.21 Corollary. *For every pair of integers $1 \leq K \leq N$ and for every $\eta > 0$, there exists a family-wise critical point with respect to $\{\Phi^{\alpha,\beta,\delta,\eta}\}_{\alpha,\beta,\delta\in(0,\infty)}$ on $\mathcal{M}_{N,K}$.*

The gradient of the sum energy also vanishes for any Gramian corresponding to a mutually unbiased basic sequence which has been rescaled to admit Parsevality.

6.3.22 Proposition. *If $G \in \mathcal{M}_{N,K}$ is the Gramian of a mutually unbiased basic sequence, then $\nabla\Phi^{\eta}_{sum}(G) = 0$ for all $\eta \in (0,\infty)$.*

Proof. Fix $\eta \in (0,\infty)$. We recall that the Gram matrix has diagonal entries that are equal to K/N, and the other entries either vanish in diagonal blocks or have the same magnitude in off-diagonal blocks. Assuming the frame vectors are grouped in M subsets of size L, then $N = ML$ and there are $M(M-1)L^2$ off-diagonal entries of the same magnitude $C_{M,L,K}$. Based on the Hilbert-Schmidt norm of G, we then have

$$C_{M,L,K} = \sqrt{\frac{K(ML-K)}{M^2(M-1)L^3}}\,.$$

In order to make the block structure apparent in the notation, we write the matrix G as $G = (G^{(p,q)}_{x,y})_{p,q\in\mathbb{Z}_M, x,y\in\mathbb{Z}_L}$, where the doubly-indexed superscript indicates in which block the entry is and the subscript indicates the position within the block.

The absolute value of any entry then satisfies

$$|G_{x,y}^{(p,q)}| = \begin{cases} \frac{K}{ML}, & x = y, p = q \\ 0, & x \neq y, p = q \\ C_{M,L,K}, & p \neq q\,. \end{cases}$$

To see the claim, we verify that every entry of $\nabla\widehat{\Phi}^{\eta}_{sum}$ vanishes. Since this is automatically true for the diagonal entries, let $(p, x), (q, y) \in \mathbb{Z}_S \times \mathbb{Z}_L$ with $(p, x) \neq (q, y)$. One has that either $p = q$ or $p \neq q$. If $p = q$, then re-expressing the identity in Proposition 6.3.2 in terms of block notation and noting that the last two terms on the right-hand side cancel due to the equal-norm property yields

$$\begin{aligned} \left[\nabla\widehat{\Phi}^{\eta}_{sum}\right]_{x,y}^{(p,p)} = 2 &\sum_{\substack{t=1 \\ t \neq x,y}}^{L} \left(e^{\eta|G_{x,t}^{(p,p)}|^2} - e^{\eta|G_{y,t}^{(p,p)}|^2}\right) G_{x,t}^{(p,p)} G_{t,y}^{(p,p)} \\ &+ 2 \sum_{\substack{s=1 \\ s \neq p}}^{M} \sum_{\substack{t=1 \\ t \neq x,y}}^{L} \left(e^{\eta|G_{x,t}^{(p,s)}|^2} - e^{\eta|G_{y,t}^{(p,s)}|^2}\right) G_{x,t}^{(p,s)} G_{t,y}^{(s,p)}\,. \end{aligned}$$

The first series on the right-hand side is zero because $G_{x,t}^{(p,p)} = G_{y,t}^{(p,p)} = 0$ since $t \notin \{x, y\}$. The second series also vanishes because when $s \neq p$, then $|G_{x,t}^{(p,s)}| = |G_{y,t}^{(p,s)}| = C_{M,L,K}$. Thus, the block diagonal entries of the gradient vanish.

On the other hand, if $p \neq q$, then we get

$$\begin{aligned}\left[\nabla\widehat{\Phi}^{\eta}_{sum}\right]^{(p,q)}_{x,y} &= 2 \sum_{\substack{s,\,t\,=\,1\\(s,\,t)\,\neq\,(p,\,x),\,(q,\,y)}}^{M,L} (e^{\eta|G^{(p,s)}_{x,t}|^2} - e^{\eta|G^{(q,s)}_{y,t}|^2})G^{(p,s)}_{x,t}G^{(s,q)}_{t,y}\\ &= 2 \sum_{\substack{t\,=\,1\\t\,\neq\,x}}^{L} (e^{\eta|G^{(p,p)}_{x,t}|^2} - e^{\eta|G^{(q,p)}_{y,t}|^2})G^{(p,p)}_{x,t}G^{(p,q)}_{t,y}\\ &\quad + 2 \sum_{\substack{t\,=\,1\\t\,\neq\,y}}^{L} (e^{\eta|G^{(p,q)}_{x,t}|^2} - e^{\eta|G^{(q,q)}_{y,t}|^2})G^{(p,q)}_{x,t}G^{(q,q)}_{t,y}\\ &\quad + 2 \sum_{\substack{s\,=\,1\\s\,\neq\,p,\,q}}^{M} \sum_{t\,=\,1}^{L} (e^{\eta|G^{(p,s)}_{x,t}|^2} - e^{\eta|G^{(q,s)}_{y,t}|^2})G^{(p,s)}_{x,t}G^{(s,q)}_{t,y}\,.\end{aligned}$$

The first series vanishes because $G^{(p,p)}_{x,t} = 0$, the second one because $G^{(q,q)}_{t,y} = 0$ and the last one because $|G^{(p,s)}_{x,t}| = |G^{(q,s)}_{y,t}| = C_{M,L,K}$. This confirms that $\nabla\widehat{\Phi}^{\eta}_{sum} = 0$ and, since η was arbitrary, the claim is proven. $\square$

As a consequence of this Proposition and of Proposition 6.3.18, we know that Examples 3.3.21 and 3.3.22 provide us with family-wise critical points.

If the Gramian does not contain vanishing entries, then we can characterize equidistributed frames, taking advantage of the fact that the term Φ^{η}_{sum} in $\Phi^{\alpha,\beta,\delta,\eta}$ is no longer needed in this case.

6.3.23 Theorem. *Let $G \in \mathcal{M}_{N,K}$ and suppose that G contains no zero entries. The Gramian G is equidistributed if and only if $\nabla[\Phi^{\delta}_{diag}+\Phi^{\alpha,\beta}_{ch}](G) = 0$ for all $\alpha \in I$, $\beta \in J$, and $\delta \in T$, where $I, J, T \subseteq (0,\infty)$ are open intervals.*

Proof. If G is equidistributed, then $\Phi^{\delta}_{diag}(G) = 0$ and $\Phi^{\alpha,\beta}_{ch}(G) = 0$ for all $\alpha, \beta, \delta \in$

$(0, \infty)$, so it follows that $\nabla \Phi_{diag}^{\delta}(G) = 0$ and $\nabla \Phi_{ch}^{\alpha,\beta}(G) = 0$ for all $\alpha, \beta, \delta \in (0, \infty)$. For the converse, since G contains no zero entries, it follows by Proposition 6.3.7 that G is equal-norm. With this established, it then follows from Corollary 6.3.13 that G is equidistributed.

□

6.4 Equidistributed Grassmannian equal-norm Parseval frames

We conclude the discussion of the relation between frame potentials and the structure of optimizers by showing how an equidistributed Grassmannian equal-norm Parseval frame can be obtained as the limit of minimizers to the sequence $\{\Phi_{sum}^{\eta_n}\}_{n=1}^{\infty}$, where $\eta_n \to \infty$.

6.4.1 Proposition. *Let $\{\eta_m\}_{m=1}^{\infty}$ be a positive, increasing sequence such that $\lim_{m\to\infty} \eta_m = +\infty$. If $\left\{G(m) = (G(m)_{a,b})_{a,b=1}^{N}\right\}_{m=1}^{\infty} \subseteq \mathcal{M}_{N,K} \cap \Omega_{N,K}$ is a sequence of Gram matrices such that the restricted potential $\Phi_{sum}^{\eta_m}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ achieves its absolute minimum on $\mathcal{M}_{N,K} \cap \Omega_{N,K}$ at $G(m)$ for every $m \in \{1, 2, 3, ...\}$, then there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K}\cap\Omega_{N,K}$ such that $\lim_{s\to\infty} G(m_s) = G$, where G is the Gramian of a Grassmannian equal-norm Parseval frame.*

Proof. For each $m \in \{1, 2, ...\}$, since $G(m)$ is equal-norm, the diagonal part of $\Phi_{sum}^{\eta_m}(G(m))$ simplifies so that we can write

$$\Phi_{\text{sum}}^{\eta_m}(G(m)) = \Phi_{\text{od}}^{\eta_m}(G) + \sum_{j=1}^{N} e^{\eta_m C_{N,K}^2} .$$

Furthermore, by Parsevality, since each $G(m)$ is equal-norm, there must always exist an off diagonal entry $G(m)_{a,b}$ such that $|G(m)_{a,b}|^2 \geq C_{N,K}^2$. Hence, $\mu(G(m)) = \max_{a,b\in\mathbb{Z}_N} |G(m)_{a,b}|$ for every m, which allows us to replace $\Phi_{od}^{\eta_m}$ with $\Phi_{sum}^{\eta_m}$ in the proof strategy from Proposition 6.1.3, which shows the claim. $\square$

If the sequence of minimizing equal-norm Parseval frames has the stronger property of being equidistributed, then the limit of the corresponding subsequence is equidistributed as well.

6.4.2 Proposition. *Let $\{\eta_m\}_{m=1}^{\infty}$ be a positive, increasing sequence such that $\lim_{m\to\infty} \eta_m = +\infty$. If $\left\{G(m) = (G(m)_{a,b})_{a,b=1}^N\right\}_{m=1}^{\infty} \subseteq \mathcal{M}_{N,K} \cap \Omega_{N,K}$ is a sequence of equidistributed Gramians such that $\Phi_{sum}^{\eta_m}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ achieves its absolute minimum on $\mathcal{M}_{N,K} \cap \Omega_{N,K}$ at $G(m)$ for every $m \in \{1, 2, 3, ...\}$, then there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim_{s\to\infty} G(m_s) = G$, where G corresponds to an equidistributed Grassmannian equal-norm Parseval frame.*

Proof. By compactness of $\mathcal{M}_{N,K} \cap \Omega_{N,K}$, there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim_{s\to\infty} G(m_s) = G$. Since each $G(m_s)$ is equidistributed, we can define for each $\alpha \in (0, \infty)$ the sequence $\left\{x^{\alpha}(m_s) := \sum_{j\in\mathbb{Z}_N} e^{\alpha|G(m_s)_{a,j}|^2}\right\}_{s=1}^{\infty}$, where our definition is independent of the choice of $a \in \mathbb{Z}_N$. Since the entries of $\{G(m_s)\}_{m=1}^{\infty}$ converge to those of G, we have by continuity

$$\sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{a,j}|^2} = \lim_{m\to\infty} x^{\alpha}(m_s) = \sum_{j\in\mathbb{Z}_N} e^{\alpha|G_{b,j}|^2}$$

for all $a, b \in \mathbb{Z}_N$. Since this is true for every $\alpha \in (0, \infty)$, G is equidistributed by Proposition 3.3.16. Additionally, it follows by applying Proposition 6.4.1 to the sequence $\{m_s\}_{s=1}^{\infty}$ that G is a Grassmannian equal-norm Parseval frame, because any subsequence of $\{G(m_s)\}_{s=1}^{\infty}$ must also converge to G. $\square$

If we know that each $G(m)$ is a family-wise critical point without vanishing entries, then we can characterize this limit in terms of the gradient of frame potentials.

6.4.3 Theorem. *Let I, J, T be open intervals in $(0,\infty)$ and let $\{\eta_m\}_{m=1}^{\infty}$ be a positive, increasing sequence such that $\lim\limits_{m\to\infty} \eta_m = +\infty$. If $\left\{G(m) = (G(m)_{a,b})_{a,b=1}^{N}\right\}_{m=1}^{\infty} \subseteq \mathcal{M}_{N,K} \cap \Omega_{N,K}$ is a sequence such that $\Phi^{\eta_m}_{sum}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ achieves its absolute minimum on $\mathcal{M}_{N,K} \cap \Omega_{N,K}$ at $G(m)$, each $G(m)$ contains no vanishing entries, and each $G(m)$ is a family-wise fixed point with respect to $\{\Phi^{\alpha,\beta,\delta,\eta_m}\}_{\alpha\in I,\beta\in J,\delta\in T}$, then there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim\limits_{s\to\infty} G(m_s) = G$, where G is the Gram matrix of an equidistributed Grassmannian equal-norm Parseval frame.*

Proof. By Corollary 6.3.17, it follows that each $G(m)$ is equidistributed. Therefore, by Proposition 6.4.3, there exists a subsequence $\{G(m_s)\}_{s=1}^{\infty}$ and $G \in \mathcal{M}_{N,K} \cap \Omega_{N,K}$ such that $\lim\limits_{s\to\infty} G(m_s) = G$, where G corresponds to an equidistributed Grassmannian equal-norm Parseval frame. $\square$

The existence of equiangular Parseval frames for certain pairs of N and K provides an abundance of examples for which this theorem holds; however, due to the difficulty in verifying when a non-equiangular critical point of $\Phi^{\eta}_{sum}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ is at an absolute minimum, it cannot be stated outright that non-equiangular, equidistributed frames exist which satisfy the conditions of Proposition 6.4.2 or Theorem 6.4.3. By considering the Welch bound over $(2,6)$-frames and then applying the Naimark compliment, it is seen that complex equiangular $(4,6)$ frames do not exist; however, based on numerical experiments, it is the author's conjecture that Example 3.3.4 is an absolute minimizer of $\Phi^{\eta}_{sum}|_{\mathcal{M}_{N,K}\cap\Omega_{N,K}}$ for all $\eta \in (0,\infty)$

and therefore corresponds to an equidistributed Grassmannian equal-norm Parseval frame in this setting by Proposition 6.4.2. Similarly, it was shown in [89] that complex equiangular $(3, 8)$-frames do not exist; however, further numerical experiments have led to the additional conjecture that Example 3.3.5 is an absolute minimizer of $\Phi^{\eta}_{sum}|_{\mathcal{M}_{N,K} \cap \Omega_{N,K}}$ for all $\eta \in (0, \infty)$ and therefore corresponds to an equidistributed Grassmannian equal-norm Parseval frame in this setting by either Proposition 6.4.2 or Theorem 6.4.3. In addition, we know at least that there are equidistributed Grassmannian equal-norm Parseval frames that are family-wise critical points: the real Examples 3.3.21 and 3.3.22. Although these do not satisfy the assumptions of Theorem 6.4.3 because their Gramians have vanishing entries, a final conjecture is that these minimize $\Phi^{\eta}_{sum}|_{\mathcal{M}_{N,K} \cap \Omega_{N,K}}$ for all $\eta \in (0, \infty)$ and hence satisfy the assumptions of Proposition 6.4.2.

Appendix A

Supplementary Material

A.1 The theory of real analytic manifolds

Many results in this paper depend on the ability to induce a gradient flow on a Riemannian manifold. This section presents the essential background material for understanding this concept, with the assumption that the reader is familiar with basic concepts from topology, multivariable calculus and analysis. After introducing the general theory, we outline the matrix manifold approach, as is described in [1].

Informally, a manifold is a geometric object upon which one can apply the techniques of calculus. This depends heavily of the fact that every point is contained in a neighborhood that can be identified with an open subset of Euclidean space.

A.1.1 Definition. Let $\mathcal{M}$ be a nonempty topological space and let $x \in \mathcal{M}$. We call a homeomorphism $\phi : U \to V \subset \mathbb{R}^d$ between open sets $U \subset \mathcal{M}$ and V a *d-dimensional chart for x*, and we sometimes refer to it by the pair (ϕ, U). If $x \in \mathcal{U}$, then $\phi(x) \in \mathbb{R}^d$ is the *coordinate expression* of x with respect to this chart. In this case, the chart is said to contain x.

To take advantage of this idea, we require that the domains of intersecting charts can be seamlessly patched together.

A.1.2 Definition. Let $\mathcal{M}$ be a nonempty topological space. A *real analytic atlas for $\mathcal{M}$ into $\mathbb{R}^d$* is a collection of charts $\mathcal{A} = \{\phi_j : U_j \to V_j\}_{j \in J}$ such that

1. $\mathcal{M} = \bigcup_{j \in J} U_j$
2. For every $j, l \in J$, the *change of coordinates map*, defined by $\phi'_j \circ \phi'^{-1}_l : \phi_l(U_j \cap U_l) \to \phi_j(U_j \cap U_l)$, where ϕ'_j and ϕ'_l denote the respective restrictions of ϕ_j and ϕ_l to $U_j \cap U_l$, is real analytic.

A real analytic atlas $\mathcal{A}$ is *maximal* if there is no larger real analytic atlas $\mathcal{A}'$ such that $\mathcal{A} \subset \mathcal{A}'$.

In order to avoid certain pathological issues, for example, sequences converging to multiple points, we additionally require that the topology be Hausdorff and second countable. Since the manifolds considered in this paper are either Euclidean spaces or compact (topological) subspaces of Euclidean spaces, these properties occur naturally.

A.1.3 Definition. A *d-dimensional real analytic manifold* is a pair $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M}$ is a nonempty, second countable, Hausdorff topological space and $\mathcal{A}$ is a real analytic maximal atlas for $\mathcal{M}$ into $\mathbb{R}^d$. When there is no confusion about the atlas structure, we refer to $\mathcal{M}$ as a real analytic manifold or just an analytic manifold. If $\mathcal{N} \subset \mathcal{M}$ is a real analytic manifold with the subspace topology induced by $\mathcal{M}$, then we say that $\mathcal{N}$ is a *submanifold* of $\mathcal{M}$.

We are interested in optimizing real-valued, real analytic maps defined on real analytic manifolds, which requires a formal definition.

A.1.4 Definition. A function $F : \mathcal{M} \to \mathcal{N}$ between a d_1-dimensional real analytic manifold $\mathcal{M}$ and a d_2-dimensional analytic manifold $\mathcal{N}$ is *real analytic* if for every $x \in \mathcal{M}$ and every choice of charts (ϕ, U) and (ψ, V) containing x and $F(x)$, respectively, the composition $\psi' \circ F' \circ \phi^{-1}$ is real analytic, where F' denotes the restriction of F to the domain of ϕ and ψ' denotes the restriction of ψ to the image of $F' \circ \phi^{-1}$.

In order to generalize the notion of a directional derivative for a real analytic function $F : \mathcal{M} \to \mathbb{R}$ at a point $x \in \mathcal{M}$, we first develop the abstract notion of ”direction” on a manifold. This requires the use of curves.

A.1.5 Definition. Let $\mathcal{M}$ be a d-dimensional real analytic manifold and $(a, b) \subset \mathbb{R}$ with $0 \in (a, b)$. A map $\gamma : (a, b) \to \mathcal{M}$ is a *curve* if for every $t \in (a, b)$ and every chart $\phi : U \to V \subset \mathbb{R}^d$ containing $\gamma(t)$, the composition $\phi \circ \gamma' : \gamma^{-1}(U) \to V$ is a C^1 function, where γ' denotes the restriction of γ to $\gamma^{-1}(U)$. In this case, we write $\gamma \in C^1(\mathbb{R}, \mathcal{M})$.

Consider the curves $\gamma \in C^1(\mathbb{R}, \mathcal{M})$ that satisfy $\gamma(0) = x$. Informally, every such curve resembles a straight line passing through x in a sufficiently small neighborhood, which we interpret as the ”direction” along which we want to differentiate F. By reading through a chart, it is simple to verify that the classical derivative $\frac{d}{dt} F(\gamma(t))|_{t=0}$ is well-defined. Realizing that these curves induce real-valued maps on the set of all such functions leads to the definition of the tangent vector to a curve.

A.1.6 Definition. Let $\mathcal{M}$ be a d-dimensional real analytic manifold and let $\mathcal{S}(\mathcal{M})$ denote the set of real-valued, real analytic functions defined on $\mathcal{M}$. If γ is a curve in

$\mathcal{M}$ with $\gamma(0) = x$, then the mapping $\dot{\gamma} : \mathcal{S}(\mathcal{M}) \to \mathbb{R}$ defined by $\dot{\gamma}(F) = \frac{d}{dt} F(\gamma(t))|_{t=0}$ is called the *tangent vector to the curve* γ *at* $t = 0$.

Given a tangent vector to a curve, there are infinitely many other curves which induce the same function. Thus, we quotient the set of curves passing through x at $t = 0$ by this equivalence relation, which leads to the formal definition of a tangent vector.

A.1.7 Definition. Let $\mathcal{M}$ be a d-dimensional real analytic manifold. A *tangent vector to a point* $x \in \mathcal{M}$ is a mapping $\xi_x : \mathcal{S}(\mathcal{M}) \to \mathbb{R}$ for which there exists a curve γ on $\mathcal{M}$ with $\gamma(0) = x$, satisfying

$$\xi_x(F) = \dot{\gamma}(F) = \frac{d}{dt} F(\gamma(t))|_{t=0}.$$

In this case, the curve γ is said to realize the tangent vector ξ_x, and we refer to it interchangeably as ξ_x or $\dot{\gamma}$. The *tangent space to* $\mathcal{M}$ *at* x, denoted by $T_x(\mathcal{M})$, is the set of all tangent vectors to x. The *tangent bundle* for $\mathcal{M}$ is the disjoint union of all tangent spaces to points in $\mathcal{M}$, denoted

$$TM := \bigcup_{x \in \mathcal{M}} T_x(\mathcal{M}).$$

If $a, b \in \mathbb{R}$ and $\dot{\gamma}_1, \dot{\gamma}_2 \in T_x(\mathcal{M})$ correspond to the curves $\gamma_1 : (a, b) \to \mathcal{M}$ and $\gamma_2 : (c, d) \to \mathcal{M}$, then, after an appropriate translation, we can find a chart $\phi : U \to \mathbb{R}$ for x satisfying $\phi(x) = 0 \in \mathbb{R}^d$. In this case, it is straightforward to verify that the map $\gamma : (a, b) \cap (c, d) \to \mathcal{M}$ defined by

$$\gamma(t) = \phi^{-1}(a\phi(\gamma_1(t)) + b\phi(\gamma_2(t)))$$

is a curve satisfying $\gamma(0) = x$. If $F \in \mathcal{S}(\mathcal{M})$, then after restricting the domain of F to U, the composition $\psi = F \circ \phi^{-1}$ is a real analytic map from an open subset of $\mathbb{R}^d$ to $\mathbb{R}$, so a straightforward application of the chain rule yields

$$\dot{\gamma}(F) = \frac{d}{dt} F(\phi^{-1}(a\phi(\gamma_1(t)) + b\phi(\gamma_2(t))))|_{t=0} = a\dot{\gamma}_1(F) + b\dot{\gamma}_1(F),$$

which shows that $T_x(\mathcal{M})$ is a vector space. In this sense, the tangent space provides the best linear approximation for a manifold near any point. Furthermore, if $\phi : U \to \mathbb{R}$ is a chart for x, then for every $\dot{\gamma} \in T_x(\mathcal{M})$ corresponding to some curve γ, one can verify that the mapping

$$\dot{\gamma} \mapsto \frac{d}{dt} \phi(\gamma(t))|_{t=0}$$

is a linear isomorphism [62], which shows that $\dim(T_x(\mathcal{M}) = d$.

Next, we generalize the total derivative from classical multivariable calculus.

A.1.8 Definition. Let $F : \mathcal{M} \to \mathcal{N}$ be an analytic map between a d_1-dimensional real analytic manifold $\mathcal{M}$ and a d_2-dimensional analytic manifold $\mathcal{N}$ and fix $x \in \mathcal{M}$. The *tangent map of F at* x, denoted DF, is the mapping from $T_x(\mathcal{M})$ to $T_{F(x)}(\mathcal{N})$ defined by

$$\xi_x \mapsto \zeta_{F(x)},$$

where, if $\gamma \in C^1(\mathbb{R}, \mathcal{M})$ is any curve that realizes ξ_x, then $\zeta_{F(x)}$ is the tangent vector at $F(x)$ realized by the curve $F \circ \gamma \in C^1(\mathbb{R}, \mathcal{N})$.

By choosing appropriate charts, it is straightforward that $F \circ \gamma$ in this definition is indeed a curve, so that DF is well-defined. Similarly, if $a, b \in \mathbb{R}$ and $\dot{\gamma}_1, \dot{\gamma}_2 \in T_x(\mathcal{M})$ correspond to the curves $\gamma_1 : (a, b) \to \mathcal{M}$ and $\gamma_2 : (c, d) \to \mathcal{M}$, then an appropriate choice of charts also shows that $DF(a\dot{\gamma}_1 + b\dot{\gamma}_2) = aDF(\dot{\gamma}_1) + bDF(\dot{\gamma}_2)$, so DF is linear.

The points of the tangent bundle TM can be partitioned by the quotient map $\pi : TM \to \mathcal{M}$, which identifies every point in TM with its corresponding tangent space via the mapping $\xi_x \in T_x(\mathcal{M}) \mapsto x$. If $\mathcal{A} = \{\phi_j : U_j \to V_j\}_{j \in J}$ is a real analytic atlas for $\mathcal{M}$, then the mappings $\phi'_j : \pi^{-1}(U_j) \to V_j \times \mathbb{R}^d$, $j \in J$, defined by

$$\xi_x \in T_x(\mathcal{M}) \mapsto \phi(\pi(\xi_x)) \times (\xi_x((\phi_j)_1), ..., \xi_x((\phi_j)_d))^t,$$

where $(\phi_j)_1, ..., (\phi_j)_d$ are the component functions of the chart ϕ_j, can be shown to form a real analytic atlas for TM, when TM is endowed with the topology generated by the sets $\{\pi^{-1}(U_j)\}_{j \in J}$. See [62] for details. For us, the manifold structure of TM is only necessary for the definition a vector field.

A.1.9 Definition. A *vector field* on a real analytic manifold $\mathcal{M}$ is a real analytic map $\xi : M \to TM$ such that $\xi(x) := \xi_x \in T_x(\mathcal{M})$ for every $x \in \mathcal{M}$.

Of particular interest, if $\xi : M \to TM$ is a vector field and (ϕ, U) is a chart for a point $\mathcal{M}$, then we obtain the first order ordinary differential equation,

$$\frac{d}{dt}\phi(x(t)) = \xi(x(t)).$$

By patching charts together, this generates a *flow* over the entire manifold.

With this in mind, we define a manifold with a Riemannian structure, which allows us to interpret familiar geometric notions like angles and distance on manifolds. In particular, it allows us to define the gradient vector field of a real-valued, real analytic map.

A.1.10 Definition. A real analytic manifold $\mathcal{M}$ is a *Riemannian manifold* if for every $x \in \mathcal{M}$, the tangent space $T_x(\mathcal{M})$ is equipped with a real inner product $g_x : T_x(\mathcal{M}) \times T_x(\mathcal{M}) \to \mathbb{R}$ such that the mapping

$$x \mapsto g_x(\xi(x), \zeta(x))$$

from $\mathcal{M}$ to $\mathbb{R}$ is smooth for all choices of vector fields ξ, ζ on $\mathcal{M}$. In this case, the mapping g which maps each $x \in \mathcal{M}$ to the corresponding positive, bilinear form g_x is called the *Riemannian metric*.

A.1.11 Definition. Let $\mathcal{M}$ be a Riemannian manifold with a Riemannian metric g and let $F : \mathcal{M} \to \mathbb{R}$ be a real analytic map. The gradient of F at $x \in \mathcal{M}$, denoted $\nabla F(x)$, is the unique tangent vector of $T_x(\mathcal{M})$ that satisfies

$$g_x(\nabla F(x), \xi) = \xi(F), \text{ for all } \xi \in T_x(\mathcal{M}).$$

The assignment

$$x \mapsto \nabla F(x)$$

induces a vector field, called the *gradient vector field*.

The uniqueness of the gradient of F at x follows from the Riesz representation theorem. The following property of the gradient characterizes it as the direction of steepest descent.

A.1.12 Proposition. *If $\mathcal{M}$ is a Riemannian manifold, $\|\cdot\|_x$ is the norm on $T_x(\mathcal{M})$ induced by the Riemannian metric, and $F : \mathcal{M} \to \mathbb{R}$ is a real analytic map, then*

$$\frac{\nabla F(x)}{\|\nabla F(x)\|_x} = \operatorname*{argmax}_{\xi \in T_x(\mathcal{M}), \|\xi\|_x = 1} \xi(F).$$

Proof. By definition of the gradient and the Cauchy-Schwarz inequality,

$$|\xi(F)| = |g_x(\nabla F(x), \xi)| \leq \|\nabla F(x)\|_x \|\xi\|_x$$

for all $\xi \in T_x(\mathcal{M})$. The claim follows from the norm constraint along with the fact that the Cauchy-Schwarz inequality is saturated if and only if the two vectors are collinear. $\square$

With this in mind, we see that a function's value decreases if we follow the trajectory induced by the flow of the negative of the gradient vector field.

A.1.1 Matrix manifolds

Let $\mathcal{M}$ be the real vector space $\mathbb{F}^{d\times d}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. With the identification $\mathbb{C} \cong \mathbb{R}^2$, the global identity chart on $\mathcal{M}$ shows that it is a real analytic manifold with dimension d^2 when $\mathbb{F} = \mathbb{R}$ and dimension d^4 when $\mathbb{F} = \mathbb{C}$. As mentioned in the preceding section, for every $x \in \mathcal{M}$, the tangent space $T_x(\mathcal{M})$ is linearly isomorphic to $\mathbb{F}^{d\times d}$. In this case, however, many details of the abstract theory about tangent spaces can be disregarded. Since the classical directional derivative for a function $F \in S(\mathcal{M})$ at the point $x \in \mathcal{M}$ is now well-defined, we can equivalently redefine the tangent map $DF : T_x(\mathcal{M}) \to \mathbb{R}$ as

$$DF(\xi) = \lim_{t\to 0} \frac{F(x + t\xi) - F(x)}{t}, \ \xi \in \mathbb{F}^{d\times d},$$

which allows us to identify the abstract tangent vectors in $T_x(\mathcal{M})$ as actual matrices in $\mathbb{F}^{d\times d}$. With this identification, $\mathcal{M}$ is equipped with the Riemannian metric determined by the familiar Hilbert Schmidt inner product on $\mathbb{F}^{d\times d}$.

Now consider a d'-dimensional submanifold $\mathcal{N} \subset \mathcal{M}$. If $\gamma \in C^1(\mathbb{R}, \mathcal{N})$ is a curve with $\gamma(0) = x$, then, due to the vector space inclusion, it is well-defined to write

$$\xi = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} \in \mathbb{F}^{d \times d},$$

which can be viewed as an element of the tangent space of matrices, $T_x(\mathcal{M})$. Because γ also realizes the tangent vector $\dot{\gamma}$ in $T_x(\mathcal{N})$, as defined in Definition A.1.7, we see that

$$\dot{\gamma}(F') = \frac{d}{dt} F'(\gamma(t))|_{t=0} = \frac{d}{dt} F(\gamma(t))|_{t=0} = DF(\xi),$$

where F' denotes the restriction of F to the submanifold $\mathcal{N}$ for every $F \in S(\mathcal{M})$. In light of this, the abstract tangent vectors of $T_x(\mathcal{N})$ can also be view as matrices, and, in particular, we interpret $T_x(\mathcal{N})$ as a d'-dimensional linear subspace of $T_x(\mathcal{M})$. By this inclusion, $\mathcal{N}$ becomes a Riemannian manifold, where the the Riemannian metric is the restriction of the Hilbert Schmidt inner product on $T_x(\mathcal{M})$ to $T_x(\mathcal{N})$, for each point $x \in \mathcal{N}$.

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