FLUID FLOW IN THE ENTRANCE REGION
OF A DUCT OF CIRCULAR SECTOR

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Richard Lawrence Wendt

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ABSTRACT

The laminar flow in the entrance region of a circular sector is investigated by linearizing the governing equation and considering the velocity as the fully developed velocity plus a difference velocity.

The non-linear transformation is determined "after the fact" in that the axial coordinate is stretched, the problem solved in the transformed coordinate system and the "stretching factor" then determined by equating two expressions for the pressure gradient.

There are no experimental results with which to compare, however, the results compare very favorably with similar analyses in a circular tube.

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CHAPTER I

INTRODUCTION

The laminar flow of an incompressible fluid through a duct will undergo a development from some initial profile at the entrance of the duct to a fully developed profile at some point far downstream. The length of the duct in which the velocity develops is defined as the entrance region.

The condition that a flow in a circular tube is considered to be laminar is that Reynolds number, $R_e = \frac{D\bar{w}}{\nu}$ where D is the diameter of the duct, $\bar{w}$ is the mean velocity of the flow and ν is the kinematic viscosity, is less than 2100. No fluid is completely incompressible, however, a fluid is termed incompressible if the density can be considered constant.

The equations of motion governing the velocity solution for laminar incompressible flow are non-linear, therefore, the velocity solution is approximate. The non-linearity of the equations of motion is attributed to the inertia terms.

The present method of analysis to be employed in constructing a velocity solution is due to Sparrow, et al [1][1]. The analysis is a linear model technique in which a transformation from the linear system to the non-linear system is achieved

[1]All numbers in brackets refer to correspondingly numbered references in the Bibliography.

by equating two expressions for the pressure gradient. The velocity solution which is determined from the linear model is continuous over the cross section and along the axial coordinate.

The application of the analysis by Sparrow, et al, to the circular tube and to the parallel plate channel compares favorably to the prior analysis and to experimental results. The present application will be made to a circular sector.

In the course of the analysis, reference is made repeatedly to various theorems and definitions which are now presented.

The first theorem is referred to as Green's formula in two dimensions [2]. If ϕ and ψ have continuous second partial derivatives, then

$$\int(\psi\nabla^2\phi - \phi\nabla^2\psi)\,dA = \oint_C\left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) ds \tag{1.0}$$

In the special case where $\psi = 1$, then

$$\int\nabla^2\phi \; dA = \oint_C \frac{\partial\phi}{\partial n}\,ds \; . \tag{1.1}$$

The second can be derived from Green's theorem by an appropriate substitution of variables [2].

$$\int_A\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right)dA = -\int_A w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dA + \oint_C w(u\,dy - v\,dx) \tag{1.2}$$

A further substitution yields [2]

$$\int_A w\nabla^2 w\,dA = -\int_A\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dA + \oint_C w\frac{\partial w}{\partial n}\,ds \tag{1.3}$$

The outward normal derivative of the function $w(x,y)$ is defined as [2]

$$\lim_{n\to 0} \frac{w(x,y) - w(x',y')}{\Delta n} = \frac{\partial w}{\partial n}$$

where (x,y) lies on a simple closed curve c, (x',y') lies interior to c on the normal to c at (x,y) and Δn is the distance from (x',y') to (x,y) measured along the normal.

A useful relation for the definition of the outward normal derivative is [2]

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds}$$

where dy/ds and dx/ds are computed with respect to c.

Theorem 1. If $\{\phi_n\}$ is an orthogonal sequence of functions on $[a,b]$ and

$$\sum_{n=1}^{\infty} K_n \phi_n = 0 \qquad (1.4)$$

where the convergence is uniform, then

$$K_n = 0$$

for each n.

Proof. Multiplying Eq. (1.4) by ϕ_m, integrating over $[a,b]$ and using the uniform convergence of $\{\phi_n\}$ yields

$$\sum_{n=1}^{\infty} K_n \int_a^b \phi_n \phi_m = 0$$

Now $\int_a^b \phi_n \phi_m = 0$ if $n \neq m$ hence $K_m \int_a^b \phi_m^2 = 0$ but $\int_a^b \phi_m^2 \neq 0$, therefore, $K_m = 0$ for each m.

CHAPTER II

STATEMENT OF THE PROBLEM

The laminar incompressible flow in a straight duct with an axially unchanging cross-section which is a circular sector is to be considered. The duct axis lies along the positive z direction with x and y the cross-sectional coordinates. The equations of motion governing the flow development can be written as [1]

$$\bar{v} \cdot \nabla w = -\frac{d(P/\bar{\rho})}{dz} + \nu \nabla_1{}^2 w \tag{2.1}$$

$$\nabla \cdot \bar{v} = 0 \tag{2.2}$$

where $\bar{v}$ is the velocity vector having components u, v, w in the x, y, z directions, respectively. The pressure, density and kinematic viscosity are denoted by P, $\bar{\rho}$ and ν, respectively. The symbol ∇_1 defined by $\partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two dimensional Laplacian operator.

The first equation represents the conservation of linear momentum with the assumption that the static pressure is uniform across each section and the component $\mu\partial^2 w/\partial z^2$ is negligible compared to the components $\mu\partial^2 w/\partial x^2$ and $\mu\partial^2 w/\partial y^2$. The second equation is a statement of the conservation of mass.

Equation (2.1) is replaced by the following linear equation:

$$\varepsilon(z)\bar{w}\,\frac{\partial w}{\partial z} = \Lambda(z) + \nu\nabla_1{}^2 w \qquad (2.3)$$

where $\varepsilon(z)$ is an undetermined function of z and $\Lambda(z)$ is an undetermined function which includes the pressure gradient and the residual of the inertia terms. This procedure is due to Sparrow, et al [1] and has been widely accepted as a reasonable approximation to the non-linear model.

Integration of Eq. (2.3) over the cross-sectional area yields

$$\Lambda(z) = -\frac{\nu}{A}\oint_c \frac{\partial w}{\partial N}\,dl \qquad (2.4)$$

where c is the contour described by the duct walls and $\partial w/\partial N$ is the outward normal derivative of the velocity at the duct wall. The details of the derivation of $\Lambda(z)$ are presented in Theorem 1 of the Appendix.

The transformation from the linear to the non-linear coordinate systems is defined using

$$dz = \varepsilon(z)\,dz^* \qquad (2.5)$$

The transformation could, of course, be viewed as a function of z^*.

The combination of Eqs. (2.3), (2.4) and (2.5) gives

$$\bar{w}\,\frac{\partial w}{\partial z^*} = \nu\nabla_1{}^2 w - \frac{\nu}{A}\oint_c \frac{\partial w}{\partial N}\,dl \qquad (2.6)$$

Equation (2.6) is transformed from rectangular coordinates to cylindrical coordinates and becomes

$$\bar{w} \frac{\partial w}{\partial z^*} = \nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2}\right] - \frac{\nu}{A} \oint_c \frac{\partial w}{\partial N}\, dl \,. \tag{2.7}$$

The introduction of the following dimensionless variables facilitates the analysis for the circular sector.

$$\phi = \frac{w}{\bar{w}} , \ \rho = \frac{r}{L} , \ \beta = \frac{1}{Re} \frac{z^*}{L} , \ L = \sqrt{A} , \ Re = \frac{L\bar{w}}{\nu}$$

$$N = \frac{n}{L} , \ s = \frac{l}{L} , \ R = \theta_o , \ B = \rho_o = \frac{r_o}{L} = \left(\frac{2}{R}\right)^{1/2}$$

where θ_o is the angle subtended by the duct opening, ρ_o is the radius of the duct and $\bar{w}$ is the average velocity.

The non-dimensional linear differential equation governing the flow in the duct then becomes

$$\frac{\partial \phi}{\partial \beta} = \frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \theta^2} - \oint_c \frac{\partial \phi}{\partial n}\, ds \tag{2.8}$$

The details of the derivation are given in Theorem 2 of the Appendix.

The boundary condition on ϕ is that $\phi = 0$ on the duct wall and the initial condition on ϕ is that $\phi = 1$ at $\beta = 0$.

CHAPTER III

ANALYTICAL SOLUTION

A solution to Eq. (2.8) is assumed of the form

$$\phi(\rho,\theta,\beta) = \phi_e(\rho,\theta,\beta) + \phi_{fd}(\rho,\theta) \qquad (3.1)$$

where ϕ_{fd} is the fully developed velocity distribution and ϕ_e is the entrance region velocity distribution equal to the difference between the local velocity and the fully developed velocity.

Substitution of Eq. (3.1) into Eq. (2.8) gives

$$[-\frac{\partial \phi_e}{\partial \beta} + \nabla^2\phi_e - \oint_c \frac{\partial \phi_e}{\partial n} ds] + [\nabla^2\phi_{fd} - \oint_c \frac{\partial \phi_{fd}}{\partial n} ds] = 0 \qquad (3.2)$$

The differential equation governing the fully developed flow in in rectangular coordinates is

$$\nabla^2 w_{fd}(x,y) = \frac{1}{\bar{\mu}} \frac{dP}{dz} \qquad (3.3)$$

where $\bar{\mu}$ is the dynamic viscosity equal to the product of the kinematic viscosity, ν , and the density, $\bar{\rho}$. Transformation of coördinates from rectangular to non-dimensional cylindrical coordinates gives

$$\frac{1}{A} \nabla^2 \phi_{fd} (\rho,\theta) = \frac{1}{\bar{\mu}} \frac{dP}{dz}$$

Integration of Eq. (3.3) over the duct cross-section yields the result that

$$\int_A \nabla^2 w_{fd}(x,y)\, dA = \int_A \frac{1}{\bar{\mu}} \frac{dP}{dz} dA = \frac{A}{\bar{\mu}}\frac{dP}{dz} = A\nabla^2 w_{fd}(x,y)$$

and by Green's formula Eq. (1.1)

$$\int_A \nabla^2 w_{fd}(x,y)\ dA = \oint_c \frac{\partial w_{fd}}{\partial n}\ ds\ ,$$

therefore,

$$\nabla^2 w_{fd}(x,y) = \frac{1}{A} \oint_c \frac{\partial w_{fd}}{\partial n}\ ds$$

The Jacobian of the transformation from rectangular to non-dimensional cylindrical coordinates is ρA so

$$\frac{1}{A} \int_{A_t} \nabla^2 \phi_{fd} \rho A dA_t = \frac{A}{A} \frac{1}{\bar{\mu}} \frac{dP}{dz} = \frac{1}{\bar{\mu}} \frac{dP}{dz} = \frac{1}{A} \nabla^2 \phi_{fd}$$

where A_t represents the cross-sectional area in the non-dimensional cylindrical coordinates (ρ,θ) . By Green's formula , Eq. (1.1),

$$\frac{1}{A} \int_{A_t} \nabla^2 \phi_{fd} \rho A dA_t = \frac{1}{A} \oint_c \frac{\partial \phi_{fd}}{\partial n}\ ds\ ,$$

therefore,

$$\nabla^2 \phi_{fd} = \oint_c \frac{\partial \phi_{fd}}{\partial n}\ ds$$

and the second bracket of Eq. (3.2) is zero.

The requirement that the velocity solution ϕ becomes ϕ_{fd} at some point in the duct results in

$$\lim_{\beta \to \infty} \phi_e(\rho,\theta,\beta) = 0 \qquad (3.5)$$

In view of Eq. (3.5), a solution ϕ_e of the form

$$\phi_e(\rho,\theta,\beta) = \sum_{i=1}^{\infty} c_i g_i(\rho,\theta) e^{-\alpha_i^2 \beta}$$

is assumed where the c_i's and the α_i's are to be determined from the boundary and initial conditions.

Substituting Eq. (3.6) into Eq. (3.2) gives

$$\sum_{i=1}^{\infty} c_i \left[\frac{\partial^2 g_i}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g_i}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 g_i}{\partial \theta^2} + \alpha_i g_i - \oint_C \frac{\partial g_i}{\partial n}\, dS\right] e^{-\alpha_i^2 \beta} = 0 \qquad (3.7)$$

and if there is a function g_i such that

$$\frac{\partial^2 g_i}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g_i}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 g_i}{\partial \theta^2} + \alpha_i^2 g_i - \oint_C \frac{\partial g_i}{\partial n}\, dS = 0 \qquad (3.8)$$

with the boundary condition $g_i = 0$ on the duct wall, then Eq. (3.7) will be satisfied.

The eigenfunctions $g_i(\rho,\theta)$ form an orthogonal sequence over the cross-sectional area A with a weight function ρ by Theorem 3 of the Appendix.

Using the orthogonality properties of the eigenfunctions g_i and the initial condition on ϕ_e that

$$\phi_e(\rho,\theta,0) = \phi_o(\rho,\theta) - \phi_{fd}(\rho,\theta) ,$$

then the Fourier coefficients of Eq. (3.6) are determined by

$$C_i = \frac{(\phi_e, g_i)}{(g_i, g_i)} = \frac{\int_A (\phi_o - \phi_{fd})\, g_i \rho\, dA}{\int_A g_i^2 \rho\, dA} = -\frac{\int_A \phi_{fd} g_i\, \rho\, dA}{||g_i||^2} \qquad (3.9)$$

It remains to solve Eq. (3.8) for g_i over the rectangular region $0 \le \theta \le R$, $0 \le \rho \le B$.

Since $\oint_C \frac{\partial g_i}{\partial n}\, dS$ is a constant, the contour integral can be expanded as a Fourier sine series over the interval $0 < \theta < R$.

$$\oint_C \frac{\partial g_i}{\partial n}\, dS = M_i = \frac{4M_i}{\pi} \sum_{K=1}^{\infty} \frac{1}{K} \sin\left(\frac{K\pi\theta}{R}\right) \tag{3.10}$$

where $K = 1, 3, 5, \ldots,\quad i = 1, 2, 3, \ldots$

A solution g_i for Eq. (3.8) of the form

$$g_i(\rho,\theta) = \sum_{K=1}^{\infty} \hat{S}_K(\rho) \sin\frac{K\pi\theta}{R} \qquad K = 1, 3, 5, \ldots$$

is assumed. Substituting this function into Eq. (3.8) and performing the required differentiations gives, after a suitable arrangement of terms,

$$\sum_{K=1}^{\infty} [\rho^2 S_K''(\rho) + \rho S_K'(\rho) + (\alpha_i^2 \rho^2 - (\frac{K\pi}{R})^2) S_K(\rho) - \frac{4M_i}{\pi K}\rho^2] \sin\frac{K\pi\theta}{R} = 0$$

By Theorem 1 of Chapter I the following differential equation results:

$$\rho^2 S_K''(\rho) + \rho S_K'(\rho) + (\alpha_i^2 \rho^2 - (\frac{K\pi}{R})^2) S_K(\rho) = \frac{4M_i}{\pi K}\rho^2 \quad K = 1,3,5,\ldots$$

The Sturm-Liouville form of this differential equation is

$$(\rho S_K'(\rho))' + (\alpha_i^2 \rho - (\frac{K\pi}{R})^2 \frac{1}{\rho}) S_K(\rho) = \frac{4M_i}{\pi K}\rho \qquad 0 < \rho \leq B \tag{3.12}$$

with boundary conditions

$$S_K(0) = 0$$

$$S_K(B) = 0$$

The solution for S_K is obtained by constructing a Green's function. The solution for S_K is

$$S_K(\rho) = \frac{2M_i}{\alpha_{iK}} \left[\frac{Y\mu(\alpha_i B)}{J\mu(\alpha_i B)} J\mu(\alpha_i\rho) \int_0^B t\, J\mu(\alpha_i t)\,dt\right.$$

$$- Y\mu(\alpha_i\rho) \int_0^\rho t\, J\mu(\alpha_i t)\,dt$$

$$\left. - J\mu(\alpha_i\rho) \int_\rho^B t\, Y\mu(\alpha_i t)\,dt\right] \qquad (3.13)$$

where $\mu = \frac{K\pi}{R}$, $K = 1, 3, 5, \ldots$. The details of the construction are presented in Theorem 4 of the Appendix.

A solution for $\phi_e(\rho,\theta,\beta)$ of Eq. (3.6) has now been constructed with the exception that the eigenvalues α_i are as yet unknown. The eigenvalues are determined by using the fact that the eigenfunctions are orthogonal to unity with weight function ρ . The expression for the determination of the eigenvalues is

$$\sum_{K=1}^{\infty} \frac{1}{K} \int_0^B S_K(\alpha_i\rho)\rho\,d\rho = 0 , \qquad K = 1, 3, 5, \ldots \qquad (3.14)$$

The derivation of Eq. (3.14) is achieved by integrating Eq. (3.11) over the cross-section which is reduced to a single numerical integration since the trignometric function $\sin \frac{K\pi\theta}{R}$ can be integrated in closed form. The determination of the eigenvalues is deferred until Chapter IV.

The velocity solution, ϕ , for the developing flow will be complete when the solution for the fully developed velocity, ϕ_{fd} , is known.

The equation governing the fully developed velocity flow is

$$\frac{\partial^2 \phi_{fd}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi_{fd}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \phi_{fd}}{\partial \theta^2} = -\frac{1}{\overline{w}\mu}\frac{dP}{dz} \tag{3.15}$$

The pressure gradient, $\frac{dP}{dz}$, is a constant depending on the geometry of the duct and the flow rate of the fluid in the duct.

It is proposed first to construct a solution to

$$\nabla^2 \hat{\phi}_{fd} = -1 \tag{3.16}$$

where $\hat{\phi}_{fd} = \phi_{fd}/(\overline{\mu w}\ dP/dz)$ and then to normalize ϕ_{fd} by the mean velocity with the result that normalized fully developed velocity solution is independent of $\frac{dP}{dz}$, the pressure gradient. A proof of this is presented in Theorem 6.

A solution to Eq. (3.16) of the form

$$\hat{\phi}_{fd} = \sum_{K=1}^{\infty} F_K(\rho) \sin \frac{K\pi\theta}{R} \tag{3.17}$$

is assumed. Substitution of this into Eq. (3.16) gives, after a suitable arrangement of terms,

$$\sum_{K=1}^{\infty} [\rho^2 F_K''(\rho) + \rho F_K'(\rho) - (\frac{K\pi}{R})^2 F_K(\rho) + \frac{4\rho^2}{\pi K}] \sin \frac{K\pi\theta}{R} = 0 ,$$

where $K = 1, 3, 5, \ldots$.

By Theorem 1 of Chapter I the following differential equation results:

$$\rho^2 F_K''(\rho) + \rho F_K'(\rho) - (\frac{K\pi}{R})^2 F_K(\rho) + \frac{4\rho^2}{\pi K} = 0$$

The Sturm-Liouville form of this equation is

$$(\rho\ F_K{}'(\rho))' - (\frac{K\pi}{R})^2 \frac{1}{\rho} F_K(\rho) = \frac{-4\rho}{\pi K} \tag{3.18}$$

with boundary conditions

$$F_K(0) = 0$$

and

$$F_K(B) = 0$$

The solution for F_K is obtained by constructing a Green's function. The solution for F_K is

$$F_K(\rho) = \frac{4}{\pi K}\left[\frac{\rho^2 - \rho^{\mu}B^{(2-\mu)}}{\mu^2 - 4}\right] \quad \text{where} \quad \mu = \frac{K\pi}{R}\ ,\ K = 1,\ 3,\ 5,\ \ldots \tag{3.19}$$

The details of the construction are presented in Theorem 7 of the Appendix.

The normalized solution to Eq. (3.15) can be written as

$$\phi_{fd} = \frac{4}{\pi\int_{A_t}\hat{\phi}_{fd}\rho\, dA_t} \sum_{K=1}^{\infty} \frac{1}{K}\left[\frac{\rho^2 - \rho^{\mu}B^{(2-\mu)}}{\mu^2 - 4}\right] \sin\frac{K\pi\theta}{R} \tag{3.20}$$

where $\mu = \frac{K\pi}{R}$ and $K = 1,\ 3,\ 5,\ \ldots$

The velocity solution, ϕ , is now complete and is obtained by summing the entrance region velocity and the fully developed velocity, i.e.

$$\phi = \phi_e + \phi_{fd} \tag{3.21}$$

A graph of the non-dimensional velocity distribution, ϕ , for a value of $\beta = .01$ is presented in Chapter IV.

CHAPTER IV

NUMERICAL PROCEDURES AND RESULTS

Let $f(\alpha_i)$ denote the left hand side of Eq. (3.14). The eigenvalues are the positive roots of $f(\alpha_i)$. A modified method of false position was used to obtain the roots because other well known root solving techniques involve the derivative of $f(\alpha_i)$.

The roots of $f(\alpha_i)$ can be predicted approximately by observing that $f(\alpha_i)$ has singularities whenever $J_\mu(\alpha_i B)$ has a zero.

The Bessel function J_μ of the first kind is continuous everywhere and the Bessel function Y_μ of the second kind is continuous for positive real numbers. Hence $f(\alpha_i)$ is continuous for positive real numbers whenever $J_\mu(\alpha_i B)$ is different from zero. Thus $f(\alpha_i)$ has a zero whenever, on any interval of the positive real numbers, a change in sign of $f(\alpha_i)$ occurs and $J_\mu(\alpha_i B)$ is different from zero.

Let a_o and a_1 denote two points on which $f(\alpha_i)$ has a change in sign. The algorithm used to determine the roots of $f(\alpha_i)$ is [3]

$$\alpha_i = \alpha_i - \left[\frac{a_1 - \alpha_i}{f(a_1) - f(\alpha_i)}\right] f(\alpha_i) \ . \tag{4.1}$$

It is shown in Theorem 8 that the previous algorithm reduces to the method of false position. The method of false position is used to generate the first approximation for α_i and the iteration is performed using Eq. (4.1).

The integrals involved in the expression for $f(\alpha_i)$ are evaluated using Gaussian quadrature on the interval [0, 1], the ideas of which were obtained from [3]. The integrals are evaluated by use of the following:

$$\int_a^b f(x)\,dx = (b-a) \sum_{i=1}^{n} A_i \;\psi(u_i) \tag{4.2}$$

where the A_i's are the integrals from zero to one of the Lagrangian coefficient functions, the u_i's are the zeroes of the Legendre polynomials on the interval [0, 1] and $\psi(u_i) = f[(b-a)u_i + a]$.

The irregular spacing of the eigenvalues can be explained by observing that infinite series are involved in the expression for $f(\alpha_i)$ and for $\mu = K\pi/R$, $J\mu(\alpha_i B)$ has singularities as a function of α_i and μ.

All computation was performed on a computer using double precision arithmetic throughout. The first fourteen eigenvalues are presented in Table I.

The norm of the eigenfunctions g_i is defined by

$$||g_i|| = [\int_A [g_i(\rho,\theta)]^2 \rho\, dA]^{1/2} .$$

The norm is calculated using the orthogonality properties of the trigonometric functions $\sin \frac{K\pi\theta}{R}$. Then

$$||g_i|| = [\int_A \sum_{K=1}^{\infty} S_K^2(\rho) \sin^2 \frac{K\pi\theta}{R} \rho dA]^{1/2} \tag{4.3}$$

The numerical calculation of the double integral in Eq. (4.3) can be accomplished by a single numerical integration with respect to ρ since the integral of $\sin^2 \frac{K\pi\theta}{R}$ can be evaluated in closed form. The integral with respect to ρ was evaluated using Gaussian quadrature on the interval [0,1]. The first fourteen norms are presented in Table I.

The integral of $\hat{\phi}_{fd}$ of Eq. (3.20) is evaluated in closed form by using the uniform convergence of $\hat{\phi}_{fd}$, thereby integrating the series termwise and then numerically evaluating the sum of

$$\frac{RB^4}{2\pi} \sum_{K=1}^{\infty} \frac{1}{K^2} [\frac{1}{(\mu+2)^2}] \quad \text{where} \quad \mu = \frac{K\pi}{R} , K = 1, 3, 5, \ldots \tag{4.4}$$

The integral of $\hat{\phi}_{fd}$ has been calculated to have an approximate value of 0.023.

The Fourier coefficients c_i of Eq. (3.6) which are defined in Eq. (3.9) are evaluated numerically due to the nature of the expression for the eigenfunctions g_i . The coefficients c_i are calculated using the orthogonality properties of the trignometric functions $\sin \frac{K\pi\theta}{R}$. Thus

$$c_i = - \frac{1}{(.023)||g_i||^2} \sum_{K=1}^{\infty} \int_A (S_K(\rho) F_K(\rho) \sin^2 \frac{K\pi\theta}{R} \rho dA \quad \text{for} \quad K = 1, 3, 5, \ldots \tag{4.5}$$

The integral of $\sin^2 \frac{K\pi\theta}{R}$ is evaluated in closed form thereby reducing the double integral of Eq. (4.5) to a single numerical integration with respect to ρ . The integral was evaluated using Gaussian quadrature on the interval [0,1]. The first fourteen Fourier coefficients are presented in Table I.

The non-dimensional velocity ϕ can now be completely determined and is constructed by summing ϕ_e and ϕ_{fd} . The magnitude of the velocity at any point (ρ,θ,β) in the duct can be determined by numerically evaluating Eq. (3.21) with the aid of the eigenvalues and Fourier coefficients of Table I.

The velocity solution w cannot be regarded as complete until the relationship between the linear and non-linear coordinate systems is determined.

The first step in the evaluation of this transformation is the determination of the function $\varepsilon(\beta)$ of Eq. (2.5). The relationship between β and z is found by calculating the pressure gradient $\partial P/\partial z$ in two ways. It is proposed to calculate the pressure gradient by integrating the momentum Eq. (2.1) and solving for $\partial P/\partial z$ and then multiplying the momentum Eq. (2.1) by the velocity w , integrating over the duct cross-section and solving for $\partial P/\partial z$. Since entrance region analyses are approximate, the pressure gradients calculated on these different bases need not necessarily be the same.

Integration of the momentum Eq. (2.1) across the duct cross-section yields

$$- \frac{d(P/\bar{\rho})}{dz} = \frac{1}{A} \frac{d}{dz} \int_A w^2 dA - \frac{\nu}{A} \oint_C \frac{\partial w}{\partial N} dl \; . \qquad (4.6)$$

A proof of the above is presented in Theorem 9 of the Appendix.

Multiplying the momentum Eq. (2.1) by the velocity w and integrating over the duct cross-section gives

$$- \frac{d(P/\bar{\rho})}{dz} = \frac{1}{\bar{w}A} \frac{d}{dz} \int_A \frac{w^3}{2} dA + \frac{\nu}{\bar{w}A} \int_A (\nabla_1 w) \cdot (\nabla_1 w) dA \qquad (4.7)$$

by Theorem 10 of the Appendix.

The pressure gradients of Eq. (4.6) and (4.7) are equated and after a suitable rearrangement of terms

$$\varepsilon(\beta) = \frac{\frac{d}{d\beta} \left[\int_A (\phi^2 - \frac{\phi^3}{2}) \rho dA \right]}{\oint_C \frac{\partial \phi}{\partial n} ds + \int_A (\nabla_1 \phi) \cdot (\nabla_1 \phi) \, dA} \qquad (4.8)$$

as a result of Theorem 11 of the Appendix.

The expression for $\varepsilon(\beta)$ can be simplified by using the orthonormality of the eigenfunctions, the orthogonality of the function $\sin \frac{K\pi\theta}{R}$ and from Eq. (3.10) since the eigenfunctions are normalized, then $\oint_C \frac{\partial g_i}{\partial n} ds = \frac{1}{||g_i||}$. The numerator of $\varepsilon(\beta)$ is independent of any term involving integrals in which the integrand is only a function of ϕ_{fd} since $\frac{d}{d\beta} \phi_{fd} = 0$. Let $w = w_f$, then Eq. (4.6) becomes

$$- \frac{d(P/\bar{\rho})}{dz} = - \frac{\nu}{A} \oint_C \frac{\partial w_f}{\partial N} dl \qquad (4.9)$$

and Eq. (4.7) becomes

$$-\frac{d(P/\bar{\rho})}{dz} = \frac{\nu}{\bar{w}A}\int_A (\nabla_1 w_f)\cdot(\nabla_1 w_f)\,dA \ , \tag{4.10}$$

therefore,

$$\frac{\nu}{\bar{w}A}\int_A (\nabla_1 w_f)\cdot(\nabla_1 w_f)\,dA - \frac{\nu}{A}\oint_C \frac{\partial w_f}{\partial N}\,dl = 0 \tag{4.11}$$

Hence the denominator of $\varepsilon(\beta)$ is independent of any term involving integrals in which the integrand is only a function of ϕ_{fd}. Thus

$$\varepsilon(\beta) = \frac{-2\sum_{i=1}^{\infty} c_i^2\alpha_i^2||g_i||^2 e^{-2\alpha_i^2\beta} + 2\sum_{i=1}^{\infty} c_i^2\alpha_i^2||g_i||^2 e^{-\alpha_i^2\beta}}{\sum_{i=1}^{\infty} c_i e^{-\alpha_i^2\beta} + \int_A [(\frac{\partial\phi_e}{\partial\rho})^2 + 2\frac{\partial\phi_e}{\partial\rho}\frac{\partial\phi_f}{\partial\rho} + (\frac{\partial\phi_e}{\partial\theta})^2 + 2\frac{\partial\phi_e}{\partial\theta}\frac{\partial\phi_f}{\partial\theta}]\rho\,dA}$$

$$\frac{-\int_A \frac{3}{2}\phi^2\frac{d\phi_e}{d\beta}\rho\,dA}{\sum_{i=1}^{\infty} c_i e^{-\alpha_i^2\beta} + \int_A [(\frac{\partial\phi_e}{\partial\rho})^2 + 2\frac{\partial\phi_e}{\partial\rho}\frac{\partial\phi_f}{\partial\rho} + (\frac{\partial\phi_e}{\partial\theta})^2 + 2\frac{\partial\phi_e}{\partial\theta}\frac{\partial\phi_f}{\partial\theta}]\rho\,dA} \tag{4.12}$$

The double integral in the numerator of $\varepsilon(\beta)$ was evaluated using Simpson's 1/3 rule on a rectangle and the algorithm is expressed as follows:

$$I_s = \frac{hK}{9}[u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1}$$

$$+ 4(u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j}) + 16\,u_{ij}] \quad [5] \tag{4.13}$$

where u_{ij} is the integrand to be evaluated over a rectangle of dimensions $2h$ by $2K$ centered at i,j.

The initial condition on ϕ is that $\phi(\rho,\theta,0) = 1$, hence $\phi^2(\rho,\theta,0) = 1$. Since the eigenfunctions are orthogonal with weight function ρ, then $\varepsilon(0) = 1$. A graph of ε is presented and in comparison to the results achieved by Sparrow, et al [1] the trend of ε for various values of β is quite reasonable. The graph of ε approaches a limiting value of 0.288.

From Eq. (2.5)

$$dz = \varepsilon(z^*)\ dz^* \tag{4.14}$$

and integrating Eq. (4.14) yields the result that

$$z = \int_0^{\beta} \varepsilon(t)dt \tag{4.15}$$

A graph of z versus β is presented and the velocity w is completely determined with the use of the graph presenting the non-dimensional velocity, ϕ, and the graph of z versus β.

TABLE I

i	α_i	$\|\|g_i\|\|^2$	c_i
1	6.9223	$.1132 \times 10^{-2}$	.594977
2	8.9142	$.1084 \times 10^{-3}$	$.563270 \times 10^{2}$
3	9.5474	$.3633 \times 10^{-5}$	$-.124202 \times 10^{3}$
4	10.6661	$.3026 \times 10^{-4}$	$-.294250 \times 10^{2}$
5	11.9286	$.2634 \times 10^{-5}$	$.166251 \times 10^{3}$
6	15.3404	$.7278 \times 10^{-3}$	$.103703 \times 10^{2}$
7	15.9121	$.1636 \times 10^{-3}$	$.742509 \times 10^{1}$
8	16.9504	$.2593 \times 10^{-4}$	$-.644379 \times 10^{2}$
9	17.3965	$.1030 \times 10^{-4}$	$.896926 \times 10^{2}$
10	18.8292	$.6009 \times 10^{-5}$	$.184589 \times 10^{3}$
11	19.6553	$.6274 \times 10^{-5}$	$.159520 \times 10^{3}$
12	20.2228	$.2636 \times 10^{-6}$	$-.380502 \times 10^{3}$
13	21.1401	$.7291 \times 10^{-5}$	$.186210 \times 10^{3}$
14	21.7530	$.6237 \times 10^{-5}$	$.653036 \times 10^{2}$

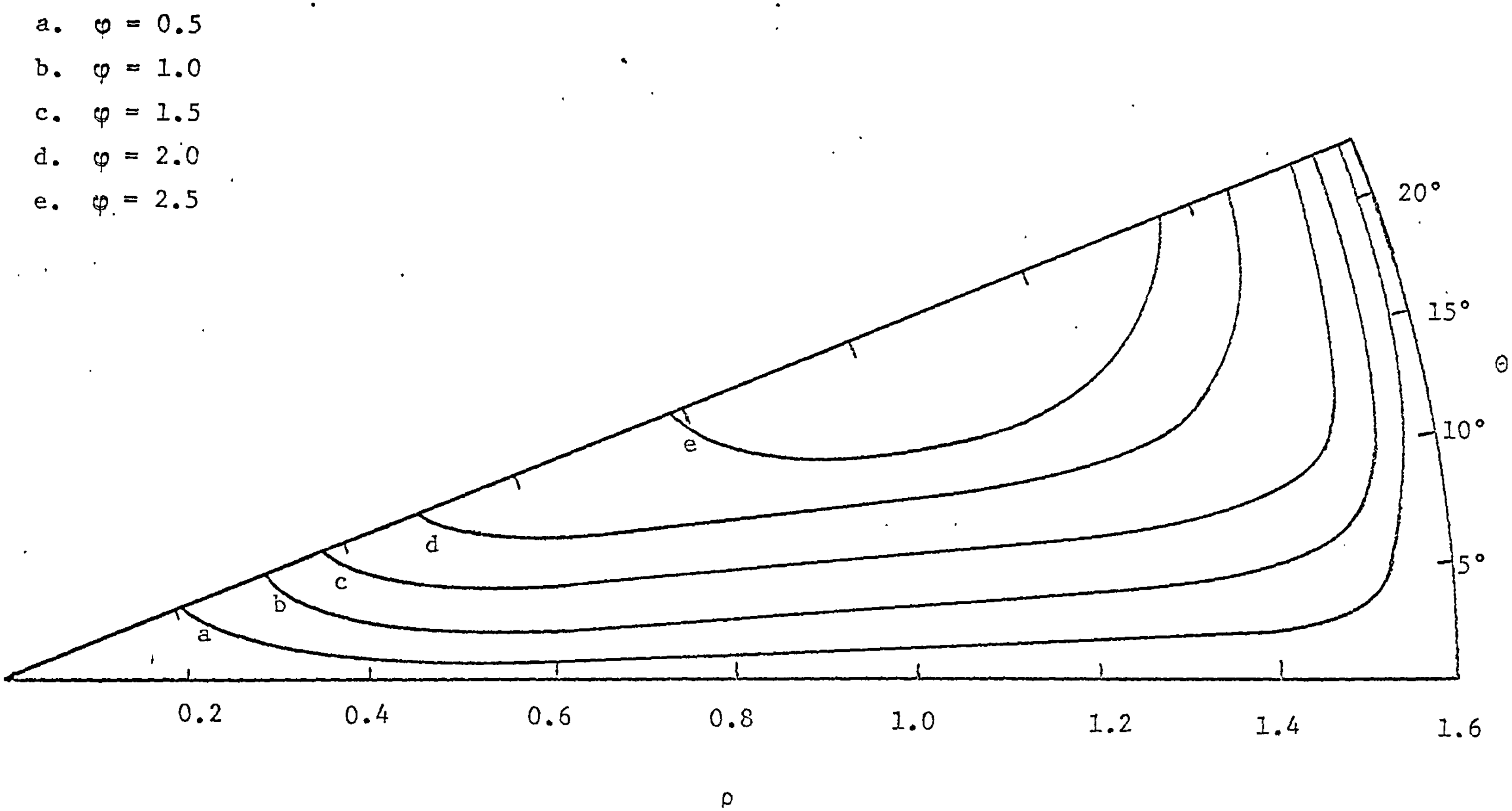

Figure 1. Velocity contour, $\beta = 0.001$

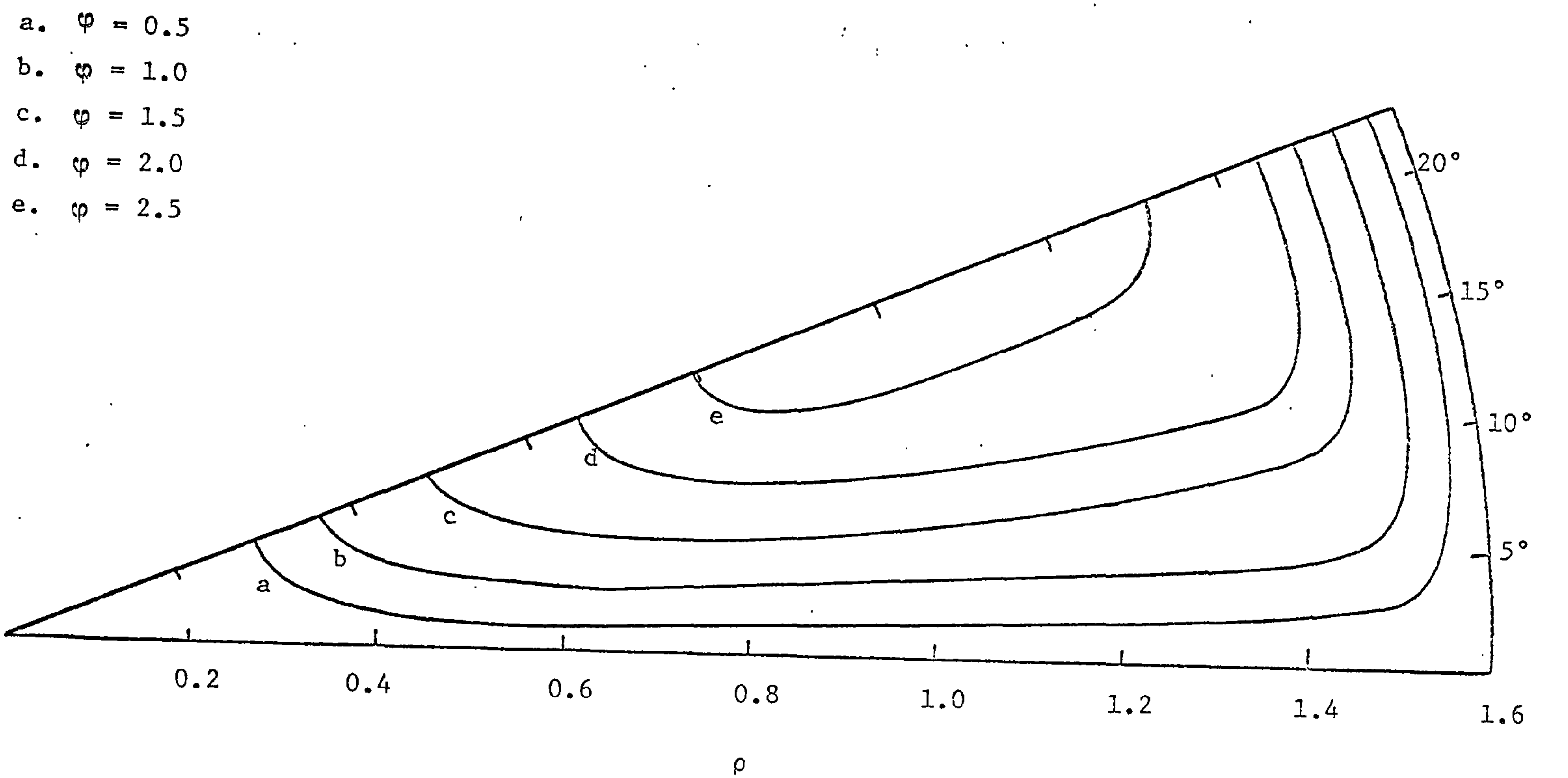

Figure 2. Velocity contour, $\beta = 0.01$

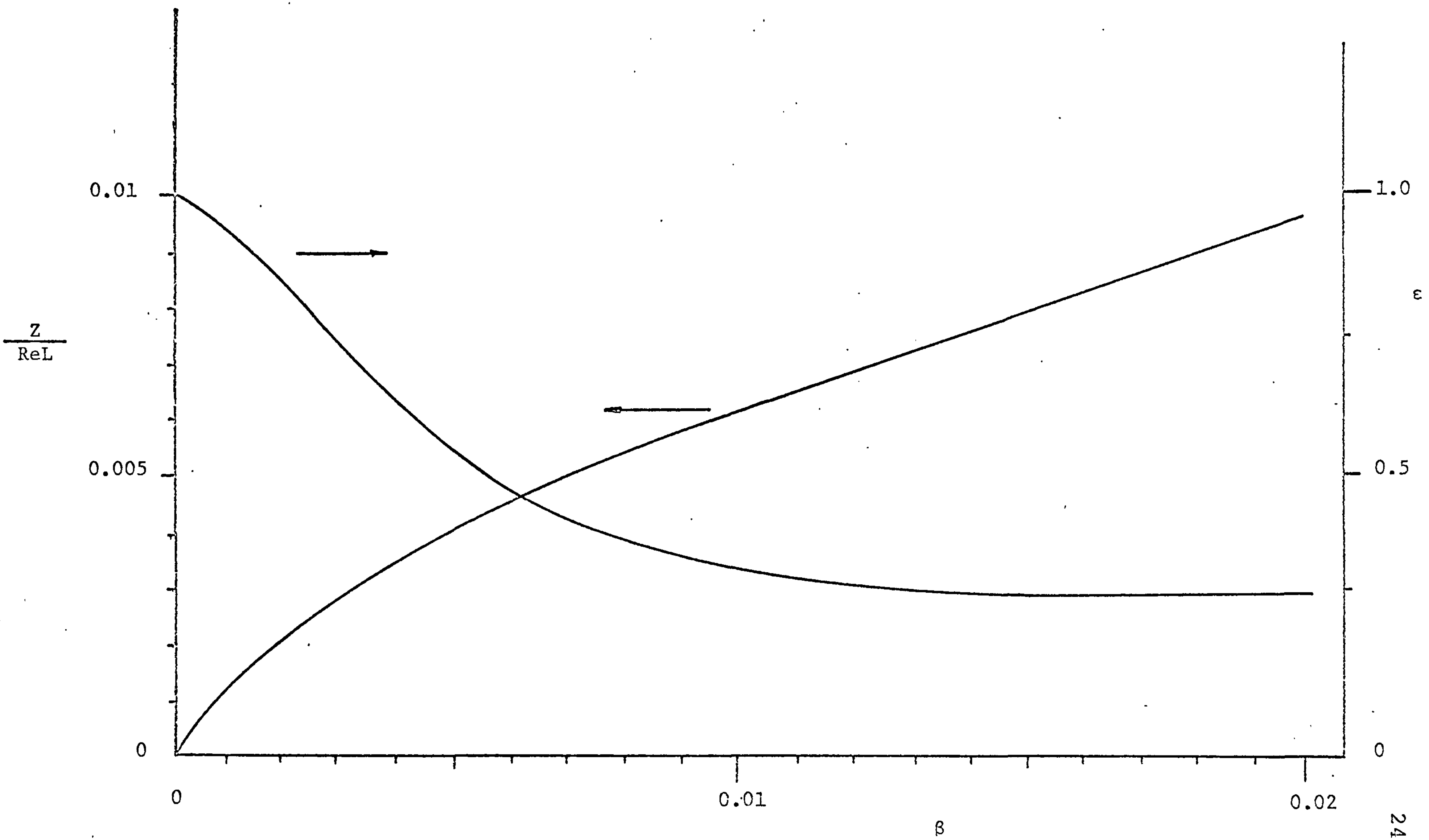

Figure 3. ε vs. β and β vs. Z/ReL

APPENDIX

Theorem 1:

$$\Lambda(z) = -\frac{\nu}{A}\oint_C \frac{\partial w}{\partial N}\, dl$$

Proof: The mean velocity is by definition

$$\bar{w} = \frac{1}{A}\int_A w\,dA$$

Equation (2.3) is integrated over the duct cross-sectional area in the following manner:

$$\bar{w}\int_A \varepsilon(z)\frac{\partial w}{\partial z}\,dA = \int_A \Lambda(z)\,dA + \nu\int_A \nabla_1^2 w\,dA$$

or

$$\bar{w}\,\varepsilon(z)\frac{d}{dz}\int_A w\,dA = \int_A \Lambda(z)\,dA + \nu\int_A \nabla_1^2 w\,dA$$

but $\int_A w\,dA = \bar{w}A$ is a constant, hence $\frac{d}{dz}\int_A w\,dA = 0$.

Then $\Lambda(z)\cdot A + \nu\oint_C \frac{\partial w}{\partial N}\,dl = 0$ where the equality of the area integral and the contour integral follows from Eq. (1.1).

Therefore, $\Lambda(z) = -\frac{\nu}{A}\oint_C \frac{\partial w}{\partial N}\,dl$

Theorem 2: The non-dimensional form of Eq. (2.7) is

$$\frac{\partial\phi}{\partial\beta} = \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} - \oint_C \frac{\partial\phi}{\partial n}\,ds$$

Proof: By definition $\phi = w/\bar{w}$, so $w = \bar{w}\phi$ and

$$\frac{\partial w}{\partial r} = \bar{w}\frac{\partial\phi}{\partial\rho}\frac{\partial\rho}{\partial r} = \frac{\bar{w}}{L}\frac{\partial\phi}{\partial\rho}\,, \quad \text{then}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{\bar{w}}{\rho A}\frac{\partial \phi}{\partial \rho} \tag{1}$$

$$\frac{\partial^2 w}{\partial r^2} = \bar{w}\frac{\partial^2 \phi}{\partial \rho^2}\left(\frac{\partial \rho}{\partial r}\right)^2 = \frac{\bar{w}}{A}\frac{\partial^2 \phi}{\partial \rho^2} \tag{2}$$

Since $\beta = z^*/ReL$, then

$$\bar{w}\frac{\partial w}{\partial z^*} = \bar{w}^2\frac{\partial \phi}{\partial \beta}\frac{\partial \beta}{\partial z^*} = \frac{\bar{w}^2}{ReL}\frac{\partial \phi}{\partial \beta} = \frac{\bar{w}\nu}{A}\frac{\partial \phi}{\partial \beta} \tag{3}$$

By definition $N = n/L$ so that

$$\frac{\partial w}{\partial N} = \bar{w}\frac{\partial \phi}{\partial n}\frac{\partial n}{\partial N} = \bar{w}L\frac{\partial w}{\partial n} \tag{4}$$

and $s = l/L$ so

$$ds = \frac{\partial s}{\partial L}\,dl = \frac{1}{L}\,dl \tag{5}$$

The combination of Eqs. (4) and (5) gives

$$\frac{\nu}{A}\oint_c \frac{\partial w}{\partial N}\,dl = \frac{\nu\bar{w}}{A}\oint_c \frac{\partial \phi}{\partial n}\,ds \tag{6}$$

The variable theta is non-dimensional so

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\bar{w}}{\rho^2 A}\frac{\partial^2 \phi}{\partial \theta^2} \tag{7}$$

Substitution of Eqs. (1) through (7) into Eq. (7) gives that

$$\frac{\bar{w}\nu}{A}\frac{\partial \phi}{\partial \beta} = \frac{\nu\bar{w}}{A}\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\nu\bar{w}}{\rho A}\frac{\partial \phi}{\partial \rho} + \frac{\nu\bar{w}}{\rho^2 A}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\nu\bar{w}}{A}\oint_c \frac{\partial \phi}{\partial n}\,ds$$

Hence

$$\frac{\partial \phi}{\partial \beta} = \frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \theta^2} - \oint_c \frac{\partial \phi}{\partial n}\,ds$$

<u>Theorem 3</u>: The sequence of functions $\{1, g_i\}_{i=1}^{\infty}$ forms an orthogonal sequence over the duct cross-section.

<u>Proof</u>: The Laplacian $\nabla^2 g_i$ in cylindrical coordinates is

$$\nabla^2 g_i = \frac{\partial^2 g_i}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g_i}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 g_i}{\partial \rho^2} \quad (1)$$

then Eq. (3.8) can be written as

$$\nabla^2 g_i + \alpha_i^2 g_i - \oint_c \frac{\partial g_i}{\partial n}\, ds = 0 \quad (2)$$

and

$$\nabla^2 g_j + \alpha_j^2 g_j - \oint_c \frac{\partial g_j}{\partial n}\, ds = 0 \quad (3)$$

Multiplying Eq. (2) by g_j and Eq. (3) by g_i gives

$$\alpha_i^2 g_i g_j = \oint_c g_j \frac{\partial g_i}{\partial n}\, ds - g_j \nabla^2 g_i \quad (4)$$

$$\alpha_j^2 g_i g_j = \oint_c g_i \frac{\partial g_j}{\partial n}\, ds - g_i \nabla^2 g_j \quad (5)$$

Subtracting Eq. (5) from Eq. (4) and integrating over the duct cross-section gives

$$(\alpha_i^2 - \alpha_j^2) \int_A g_i g_j\, dA = \int_A (g_i \nabla^2 g_j - g_j \nabla^2 g_i)\, dA + A \oint_c g_j \frac{\partial g_i}{\partial n}\, ds - A \oint_c g_i \frac{\partial g_j}{\partial n}\, ds$$

By Eq. (1.0)

$$\int_A (g_i \nabla^2 g_j - g_j \nabla^2 g_i)\, dA = \oint_c \left(g_i \frac{\partial g_j}{\partial n} - g_j \frac{\partial g_i}{\partial n}\right) ds\ , \text{ then}$$

$$(\alpha_i^2 - \alpha_j^2) \int_A g_i g_j\, dA = (1-A) \oint_c g_i \frac{\partial g_j}{\partial n}\, ds + (A-1) \oint_c g_j \frac{\partial g_i}{\partial n}\, ds$$

and $\oint_c g_i \frac{\partial g_j}{\partial n}\, ds = \oint_c g_j \frac{\partial g_i}{\partial n}\, ds = 0$ since

$g_i = g_j = 0$ on the duct wall. Therefore,

$(\alpha_i{}^2 - \alpha_j{}^2) \int_A g_i g_j dA = 0$. Hence

$$\int_A g_i g_j \, dA = 0 \quad \text{if} \quad i \neq j$$

which proves the orthogonality of any pair of eigenfunctions g_i and g_j .

It remains to be shown that the eigenfunctions are orthogonal with weight function ρ over the duct cross-section. In rectangular coordinates

$$\nabla^2 g_i + \alpha_i{}^2 g_i - \frac{1}{A} \oint_c \frac{\partial g_i}{\partial n} ds = 0 \qquad (6)$$

Integrating Eq. (6) over the duct cross-section gives

$$\int_A \nabla^2 g_i \, dA + \alpha_i{}^2 \int_A g_i \, dA - \frac{1}{A} \int_A \left(\oint_c \frac{\partial g_i}{\partial n} ds\right) dA = 0$$

but $\int_A \nabla^2 g_i \, dA = \oint_c \frac{\partial g_i}{\partial n} ds$

and $\frac{1}{A} \int_A \left(\oint_c \frac{\partial g_i}{\partial n} ds\right) dA = \oint_c \frac{\partial g_i}{\partial n} ds$

therefore,

$$\alpha_i{}^2 \int_A g_i \, dA = 0$$

which becomes

$$\int_{A_t} g_i \, \rho dA = 0$$

in the transformed coordinate system.

<u>Theorem 4</u>: The solution of Eq. (3.12) subject to the boundary conditions $S_K(0) = S_K(B) = 0$ is

$$S_K = \frac{2M_i}{\alpha_i K} \left[\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i \rho) \int_0^B t \, J_\mu(\alpha_i t) dt - Y_\mu(\alpha_i \rho) \int_0^\rho t \, J_\mu(\alpha_i t) dt \right.$$
$$\left. - J_\mu(\alpha_i \rho) \int_\rho^B t \, Y_\mu(\alpha_i t) dt\right]$$

Proof: The solution of Eq. (3.12) is achieved by constructing a Green's function as a solution to

$$(\rho\ S_K'(\rho))' + (\alpha_i^{\,2}\rho - (\frac{K\pi}{R})^2\ \frac{1}{\rho})\ S_K(\rho) = \frac{4M_i\rho}{\pi K} \qquad 0 \le \rho \le B \qquad (1)$$

with boundary conditions

$$S_K(0) = 0$$

$$S_K(B) = 0$$

The Green's function should satisfy

$$(t\ G'(\rho,t))' + \frac{1}{t}(\alpha_i^{\,2}t^2 - (\frac{K\pi}{R})^2)\ G(\rho,t) = 0 \qquad 0 \le t < \rho \qquad (2)$$

with $G(\rho,0) = 0$ and where $G'(\rho,t) = \frac{\partial G(\rho,t)}{\partial t}$

The general solution to Eq. (2) is

$$G(\rho,t) = A(\rho)\ J_\mu(\alpha_i t) + B(\rho)\ Y_\mu(\alpha_i t) \quad \text{where} \quad \mu = \frac{K\pi}{R} \qquad (3)$$

Applying the boundary conditions gives

$$G(\rho,0) = A(\rho)\ J_\mu(0) + B(\rho)\ Y_\mu(0) = 0$$

but $Y_\mu(0)$ is undefined so that $B(\rho) = 0$ and

$$G(\rho,t) = A(\rho)\ J_\mu(\alpha_i t) \qquad 0 \le t < \rho \qquad (4)$$

In the interval (ρ,B) the Green's function should satisfy the homogeneous differential Eq. (2) with the boundary condition that $G(\rho,B) = 0$. Hence

$$G(\rho,t) = C(\rho)\ J_\mu(\alpha_i t) + D(\rho)\ Y_\mu(\alpha_i t) \qquad \rho < t \le B \qquad (5)$$

Applying the boundary condition on G at $t = B$

$$G(\rho,B) = C(\rho)\ J_\mu(\alpha_i B) + D(\rho)\ Y_\mu(\alpha_i B) = 0 \qquad (6)$$

and solving for $C(\rho)$

$$C(\rho) = -D(\rho)\,\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} \tag{7}$$

Since G is continuous at $t = \rho$, then

$$A(\rho)\,J_\mu(\alpha_i\rho) = C(\rho)\,J_\mu(\alpha_i\rho) + D(\rho)\,Y_\mu(\alpha_i\rho) \quad .$$

Solving for $A(\rho)$ in terms of $D(\rho)$ gives

$$A(\rho) = -D(\rho)\,\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i\rho)} + D(\rho)\,\frac{Y_\mu(\alpha_i\rho)}{J_\mu(\alpha_i\rho)} \tag{8}$$

$G'(\rho,t)$ has a jump discontinuity equal to $-\frac{1}{\rho}$ at $t = \rho$, therefore,

$$C(\rho)\alpha_i\,J_\mu'(\alpha_i\rho) + D(\rho)\alpha_i\,Y_\mu'(\alpha_i\rho) - A(\rho)\alpha_i\,J_\mu'(\alpha_i\rho) = -\frac{1}{\rho} \tag{9}$$

Substituting Eqs. (7) and (8) into Eq. (9) and solving for $D(\rho)$ yields

$$-D(\rho)\,\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)}\,\alpha_i\,J_\mu'(\alpha_i\rho) + D(\rho)\,\alpha_i\,Y_\mu'(\alpha_i\rho)$$

$$+\left[D(\rho)\,\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} - D(\rho)\,\frac{Y_\mu(\alpha_i\rho)}{J_\mu(\alpha_i\rho)}\right]\alpha_i\,J_\mu'(\alpha_i\rho) = -\frac{1}{\rho}$$

Hence $D(\rho)\,[Y_\mu'(\alpha_i\rho)\,J_\mu(\alpha_i\rho) - Y_\mu(\alpha_i\rho)\,J_\mu'(\alpha_i\rho)] = -\dfrac{J_\mu(\alpha_i\rho)}{\rho\alpha_i}$

However, $\rho\,[J_\mu'(\alpha_i\rho)\,Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\,Y_\mu'(\alpha_i\rho)] = -\frac{2}{\pi}$ [5] (10)

by the derivation as presented in Theorem 5. Thus

$$D(\rho)(2/\pi) = -\frac{J_\mu(\alpha_i\rho)}{\alpha_i} \quad \text{or}$$

$$D(\rho) = -\frac{\pi\,J_\mu(\alpha_i\rho)}{2\alpha_i} \tag{11}$$

The construction of the Green's function is now complete and can be written as

$$G(\rho,t) = \frac{\pi}{2\alpha_i}\left[\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i\rho) - Y_\mu(\alpha_i\rho)\right] J_\mu(\alpha_i t) \qquad 0 \le t < \rho$$

$$G(\rho,t) = \frac{\pi}{2\alpha_i}\left[\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i\rho) J_\mu(\alpha_i t) - J_\mu(\alpha_i\rho) Y_\mu(\alpha_i t)\right] \qquad \rho < t \le B$$

The solution for S_K is obtained by integrating the product of the Green's function G and the non-homogeneous function $\frac{4M_i t}{\pi K}$ over the intervals $[0,\rho]$ and $[\rho,B]$ with respect to t.

$$S_K(\rho) = \frac{2M_i}{K\alpha_i}\left[\left(\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i\rho) - Y_\mu(\alpha_i\rho)\right) \int_0^\rho t\, J_\mu(\alpha_i t)\,dt\right.$$

$$\left. + \frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i\rho) \int_\rho^B t J_\mu(\alpha_i t)\,dt - J_\mu(\alpha_i\rho) \int_\rho^B t\, Y_\mu(\alpha_i t)\,dt\right] \quad (12)$$

Equation (12) can be simplified to

$$S_K(\rho) = \frac{2M_i}{\alpha_i K}\left[\frac{Y_\mu(\alpha_i B)}{J_\mu(\alpha_i B)} J_\mu(\alpha_i\rho) \int_0^B t\, J_\mu(\alpha_i t)\,dt\right.$$

$$\left. - Y_\mu(\alpha_i\rho) \int_0^\rho t\, J_\mu(\alpha_i t)\,dt - J_\mu(\alpha_i\rho) \int_\rho^B t\, Y_\mu(\alpha_i t)\,dt\right]$$

which was to be shown.

<u>Theorem 5</u>:

$$\rho[J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho)] = -\frac{2}{\pi} \qquad [4]$$

<u>Lemma 1</u>:

$$\rho[J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho)] = c$$

<u>Proof</u>: The Bessel function $J_\mu(\alpha_i\rho)$ of the first kind and $Y_\mu(\alpha_i\rho)$ of the second kind are solutions to Eq. (3.12).

Hence

$$\rho^2\alpha_i^2 J_\mu''(\alpha_i\rho) + \rho\alpha_i J_\mu'(\alpha_i\rho) + (\alpha_i^2\rho^2 - (\tfrac{K\pi}{R})^2) J_\mu(\alpha_i\rho) = 0 \quad (1)$$

$$\rho^2\alpha_i^2 Y_\mu''(\alpha_i\rho) + \rho\alpha_i Y_\mu'(\alpha_i\rho) + (\alpha_i^2\rho^2 - (\tfrac{K\pi}{R})^2) Y_\mu(\alpha_i\rho) = 0 \quad (2)$$

where $\mu = K\pi/R$. Multiplying Eq. (1) by $Y_\mu(\alpha_i\rho)$ and Eq. (2) by $J_\mu(\alpha_i\rho)$ and subtracting yields

$$\rho^2\alpha_i^2(J_\mu''(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu''(\alpha_i\rho))$$
$$+ \rho\alpha_i(J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho)) = 0 \quad (3)$$

Dividing Eq. (3) by $\rho\alpha_i$, then

$$\rho\alpha_i[J_\mu''(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu''(\alpha i\rho)]$$
$$+ [J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho)] = 0 \quad (4)$$

or that

$$\frac{d}{d\rho}[\rho\alpha_i(J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho))] = 0 \quad (5)$$

Hence

$$\rho[J_\mu'(\alpha_i\rho)\, Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho)\, Y_\mu'(\alpha_i\rho)] = c \quad (6)$$

<u>Lemma 2:</u>

$$J_{1/2}(\alpha_i\rho) = (\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \sin\alpha_i\rho$$

$$J_{-1/2}(\alpha_i\rho) = (\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \cos\alpha_i\rho$$

$$Y_{1/2}(\alpha_i\rho) = -(\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \cos\alpha_i\rho$$

$$Y_{-1/2}(\alpha_i\rho) = (\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \sin\alpha_i\rho$$

<u>Proof</u>: By definition

$$J_{1/2}(\alpha_i\rho) = (\frac{\alpha_i\rho}{2})^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{\alpha_i\rho}{2})^{2j}}{j!\ \Gamma(j+3/2)}$$

Recall that $\Gamma(\nu+1) = \nu\Gamma(\nu)$ and $\Gamma(1/2) = (\pi)^{\frac{1}{2}}$

$$J_{1/2}(\alpha_i\rho) = (\frac{\alpha_i\rho}{2})^{\frac{1}{2}} [\frac{1}{\Gamma(3/2)} - \frac{(\frac{\alpha_i\rho}{2})^2}{\Gamma(5/2)} + \frac{(\frac{\alpha_i\rho}{2})^4}{2!\Gamma(7/2)} - \frac{(\frac{\alpha_i\rho}{2})^6}{3!\Gamma(9/2)} + \ldots]$$

$$= (\frac{\alpha_i\pi}{2})^{\frac{1}{2}} [\frac{1}{1/2\Gamma(1/2)} - \frac{(\frac{\alpha_i\rho}{2})^2}{1/2\cdot 3/2\cdot\Gamma(1/2)}$$

$$+ \frac{(\frac{\alpha_i\rho}{2})^4}{2!\ 1/2\cdot 3/2\cdot 5/4\cdot\Gamma(1/2)} - \frac{(\frac{\alpha_i\rho}{2})^6}{3!\ 1/2\cdot 3/2\cdot 5/2\cdot 7/2\cdot\Gamma(1/2)} + \ldots]$$

$$= \frac{(\frac{\alpha_i\rho}{2})^{\frac{1}{2}}}{\Gamma(1/2)} [2 - \frac{(\alpha_i\rho)}{3} + \frac{(\alpha_i\rho)^4}{3\cdot 4\cdot 5} - \frac{(\alpha_i\rho)^6}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \ldots] \quad (7)$$

$$= \frac{(\frac{2}{\alpha_i\rho})^{\frac{1}{2}}}{\Gamma(1/2)} [\alpha_i\rho - \frac{(\alpha_i\rho)^3}{3!} + \frac{(\alpha_i\rho)^5}{5!} - \frac{(\alpha_i\rho)^7}{7!} + \ldots]$$

$$= (\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \sin\alpha_i\rho$$

$$J_{-1/2}(\alpha_i\rho) = (\frac{\alpha_i\rho}{2})^{\frac{1}{2}} [\frac{1}{\Gamma(1/2)} - \frac{(\frac{\alpha_i\rho}{2})^2}{\Gamma(3/2)} + \frac{(\frac{\alpha_i\rho}{2})^4}{2!\Gamma(5/2)} - \frac{(\frac{\alpha_i\rho}{2})^6}{3!\Gamma(7/2)} + \ldots]$$

$$= (\frac{\alpha_i\rho}{2})^{-\frac{1}{2}} [\frac{1}{\Gamma(1/2)} - \frac{(\frac{\alpha_i\rho}{2})^2}{1/2\Gamma(1/2)} + \frac{(\frac{\alpha_i\rho}{2})^4}{2!3/2\cdot 1/2\cdot\Gamma(1/2)}$$

$$- \frac{(\frac{\alpha_i\rho}{2})^6}{3!5/2\cdot 3/2\cdot 1/2\cdot\Gamma(1/2)} + \ldots]$$

$$= \frac{(\frac{\alpha_i\rho}{2})^{-\frac{1}{2}}}{\Gamma(1/2)} [1 - \frac{(\frac{\alpha_i\rho}{2})^2}{1/2} + \frac{(\frac{\alpha_i\rho}{2})^4}{2!3/2\cdot 1/2} - \frac{(\frac{\alpha_i\rho}{2})^6}{3!\cdot 5/2\cdot 3/2\cdot 1/2} + \ldots]$$

$$= \frac{\left(\frac{\alpha_i\rho}{2}\right)^{-\frac{1}{2}}}{\Gamma(1/2)} \left[1 - \frac{(\alpha_i\rho)^2}{2!} + \frac{(\alpha_i\rho)^4}{4!} - \frac{(\alpha_i\rho)^6}{6!} + \ldots\right]$$

$$= \left(\frac{2}{\pi\alpha_i\rho}\right)^{\frac{1}{2}} \cos \alpha_i\rho$$

By definition

$$Y_\mu(x) = \frac{J_\mu(x) \cos \mu\pi - J_{-\mu}(x)}{\sin \mu\pi}$$

therefore

$$Y_{1/2}(x) = \frac{J_{\frac{1}{2}}(x) \cos \pi/2 - J_{-\frac{1}{2}}(x)}{\sin \pi/2} = -J_{-\frac{1}{2}}(x) \qquad (8)$$

$$Y_{-1/2}(x) = \frac{J_{-\frac{1}{2}}(x) \cos -\pi/2 - J_{\frac{1}{2}}(x)}{\sin -\pi/2} = J_{\frac{1}{2}}(x)$$

<u>Proof (Theorem 5)</u>: Since Eq. (6) is true for all μ, let $\mu = 1/2$, then

$$J'_{1/2}(\alpha_i\rho) = -\frac{1}{2}\left(\frac{2}{\pi\alpha_i\rho}\right)^{-\frac{1}{2}} \frac{2}{\pi\alpha_i\rho^2} \sin \alpha_i\rho + \left(\frac{2}{\pi\alpha_i\rho}\right)^{\frac{1}{2}} \alpha_i \cos \alpha_i\rho \qquad (9)$$

and

$$Y'_{1/2}(\alpha_i\rho) = \frac{1}{2}\left(\frac{2}{\pi\alpha_i\rho}\right)^{-\frac{1}{2}} \frac{2}{\pi\alpha_i\rho^2} \cos \alpha_i\rho + \left(\frac{2}{\pi\alpha_i\rho}\right)^{\frac{1}{2}} \alpha_i \sin \alpha_i\rho \qquad (10)$$

Substitution of Eqs. (7), (8), (9), and (10) into Eq. (6) gives

$$J'_{1/2}(\alpha_i\rho)\, Y_{1/2}(\alpha_i\rho) - J_{1/2}(\alpha_i\rho)\, Y'_{1/2}(\alpha_i\rho)$$

$$= \left[-\frac{1}{2}\left(\frac{\pi\alpha_i\rho}{2}\right)^{\frac{1}{2}} \frac{2}{\pi\alpha_i\rho^2} \sin \alpha_i\rho + \left(\frac{2}{\pi\alpha_i\rho}\right)^{\frac{1}{2}} \alpha_i \cos \alpha_i\rho\right]$$

$$[-(\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \cos \alpha_i\rho] - [(\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \sin \alpha_i\rho] \; [\tfrac{1}{2} \; (\frac{\pi\alpha_i\rho}{2})^{\frac{1}{2}} \frac{2}{\pi\alpha_i\rho^2} \cos \alpha_i\rho$$

$$+ (\frac{2}{\pi\alpha_i\rho})^{\frac{1}{2}} \alpha_i \sin \alpha_i\rho] = \frac{1}{\pi\alpha_i\rho^2} \sin \alpha_i\rho \cos \alpha_i\rho - \frac{2}{\pi\alpha_i\rho} \alpha_i \cos^2\alpha_i\rho$$

$$- \frac{1}{\pi\alpha_i\rho^2} \sin \alpha_i\rho \cos \alpha_i\rho - \frac{2}{\pi\alpha_i\rho} \alpha_i \sin^2 \alpha_i\rho = - \frac{2}{\pi\rho}$$

Hence

$$\rho[J_\mu'(\alpha_i\rho) \; Y_\mu(\alpha_i\rho) - J_\mu(\alpha_i\rho) \; Y_\mu'(\alpha_i\rho)] = \frac{-2}{\pi}$$

Theorem 6: The normalized fully developed velocity, $\frac{\phi_{fd}}{\bar{\phi}_{fd}}$, is independent of the pressure gradient, dp/dz.

Proof: From Eq. (3.16)

$$\nabla^{2'} \left(\frac{\phi_{fd}}{\frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz}}\right) = -1 \qquad (1)$$

Denote the solution to Eq. (1) by $\psi(\rho,\theta)$. Now $\frac{\phi_{fd}}{\frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz}} = \psi(\rho,\theta)$ implies that $\phi_{fd} = \frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \psi(\rho,\theta)$. By definition $\bar{\phi}_{fd} = \int_{A_t} \phi_{fd} \, \rho dA_t$ so that by integrating $\phi_{fd} = \frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \psi(\rho,\theta)$ over the duct cross-section

$$\int_{A_t} \phi_{fd} \, \rho dA_t = \frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \int_{A_t} \psi(\rho,\theta) \rho dA_t$$

$$\bar{\phi}_{fd} = \frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \int_{A_t} \psi(\rho,\theta) \rho dA_t \qquad (2)$$

Now normalizing ϕ_{fd} by $\bar{\phi}_{fd}$ yields that

$$\frac{\phi_{fd}}{\bar{\phi}_{fd}} \quad \frac{\frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \psi(\rho,\theta)}{\frac{1}{\bar{\mu}\bar{w}} \frac{dP}{dz} \int_{A_t} \psi(\rho,\theta) \rho dA_t}$$

whence

$$\frac{\bar{\phi}_{fd}}{\phi_{fd}} = \frac{\psi(\rho,\theta)}{\int_{A_t} \psi(\rho,\theta)\rho dA_t} \quad \text{is independent of}$$

dP/dz .

<u>Theorem 7</u>: The solution of Eq. (3.18) subject to the boundary conditions $F_K(0) = 0$ and $F_K(B) = 0$ is

$$F_K(\rho) = \frac{4}{\pi} \sum_{K=1}^{\infty} \frac{1}{K}\left[\frac{\rho^2 - \rho^{\mu} B^{(2-\mu)}}{\mu^2 - 4}\right] \quad \text{where} \quad \mu = \frac{K\pi}{R} \quad K = 1, 3, 5, \ldots$$

<u>Proof</u>: The solution of Eq. (3.18) is achieved by constructing a Green's function. The Green's function should satisfy

$$(t\, G'(\rho,t))' - \frac{1}{t}\left(\frac{K\pi}{R}\right)^2 G(\rho,t) = 0 \qquad 0 \leq t < \rho \tag{1}$$

with $G(\rho,0) = 0$ and where $G'(\rho,t) = \frac{\partial G(\rho,t)}{\partial t}$.

The general solution to Eq. (1) is

$$G(\rho,t) = A(\rho)t^{\frac{K\pi}{R}} + B(\rho)\cdot t^{\frac{-K\pi}{R}} \qquad 0 \leq t < \rho \tag{2}$$

Applying the boundary condition gives

$$G(\rho,0) = A(\rho)\cdot 0 + B(\rho)\cdot 0 \quad , \text{but}$$

$\lim_{t \to 0} t^{-K\pi/R} = -\infty$, hence $B(\rho) = 0$ and

$$G(\rho,t) = A(\rho)t^{K\pi/R} \qquad 0 \leq t < \rho \tag{3}$$

In the interval (ρ,B) , the Green's function should satisfy the homogeneous differential equation (1) with the boundary condition that $G(\rho,B) = 0$.

$$G(\rho,t) = C(\rho)t^{K\pi/R} + D(\rho)t^{-K\pi/R} \qquad \rho < t \leq B \tag{4}$$

Applying the boundary condition on G at $t = B$

$$G(\rho,B) = C(\rho)B^{K\pi/R} + D(\rho)B^{-K\pi/R} = 0 \text{ , hence}$$

$$C(\rho) = \frac{D(\rho)}{B^{2\frac{K\pi}{R}}} \tag{5}$$

Since G is continuous at $t = \rho$, then

$A(\rho)\rho^{\frac{K\pi}{R}} = C(\rho)\rho^{\frac{K\pi}{R}} + D(\rho)\rho^{\frac{K\pi}{R}}$. Solving for $A(\rho)$ in terms of $D(\rho)$ gives

$$A(\rho) = -\frac{D(\rho)}{B^{2\frac{K\pi}{R}}} + D(\rho)\rho^{-2\frac{K\pi}{R}} \tag{6}$$

$G'(\rho,t)$ has a jump discontinuity equal to $-\frac{1}{\rho}$ at $t = \rho$, therefore,

$$C(\rho)\ \frac{K\pi}{R}\ \rho^{\frac{K\pi}{R}-1} - D(\rho)\ \frac{K\pi}{R}\ \rho^{-\frac{K\pi}{R}-1} - A(\rho)\ \frac{K\pi}{R}\ \rho^{\frac{K\pi}{R}-1} = -\frac{1}{\rho} \tag{7}$$

Substituting Eqs. (5) and (6) into Eq. (7) gives

$$-\frac{D(\rho)}{B^{2\frac{K\pi}{R}}}\ \frac{K\pi}{R}\rho^{\frac{K\pi}{R}-1} - D(\rho)\frac{K\pi}{R}\rho^{-\frac{K\pi}{R}-1} + [\frac{D(\rho)}{B^{2\frac{K\pi}{R}}} - D(\rho)\rho^{-\frac{2K\pi}{R}}]\ \frac{K\pi}{R}\rho^{\frac{K\pi}{R}-1} = -\frac{1}{\rho}$$

and solving for $D(\rho)$ yields

$$D(\rho) = \frac{\rho^{\frac{K\pi}{R}}}{2\ \frac{K\pi}{R}} \tag{8}$$

The construction of the Green's function is now complete and can be written as

$$G(\rho,t) = [\frac{-\rho^{\frac{K\pi}{R}} + B^{\frac{2K\pi}{R}}\rho^{\frac{-K\pi}{R}}}{2\ \frac{K\pi}{R}\ B^{\frac{2K\pi}{R}}}]t^{\frac{K\pi}{R}} \qquad 0 \le t < \rho$$

$$G(\rho,t) = \frac{-\rho^{\frac{K\pi}{R}}}{2\,\frac{K\pi}{R}\,B^{\frac{2K\pi}{R}}}\,t^{\frac{K\pi}{R}} + \frac{\rho^{\frac{K\pi}{R}}}{2\,\frac{K\pi}{R}}\,t^{-\frac{K\pi}{R}} \qquad \rho < t \leq B$$

The solution for F_K is obtained by integrating the product of the Green's function G and the non-homogeneous function $\frac{4t}{\pi K}$ over the intervals $[0,\rho]$ and $[\rho,B]$ with respect to t.

$$F_K(\rho) = [-\frac{\rho^{\frac{K\pi}{R}} + B^{\frac{2K\pi}{R}}\rho^{-\frac{K\pi}{R}}}{2\,\frac{K\pi}{R}\,B^{\frac{2K\pi}{R}}}]\frac{4}{\pi K}\int_0^\rho t^{\frac{K\pi}{R}+1}\,dt$$

$$- [\frac{\rho^{\frac{K\pi}{R}}}{2\,\frac{K\pi}{R}\,B^{\frac{2K\pi}{R}}}]\frac{4}{\pi K}\int_\rho^B t^{\frac{K\pi}{R}+1}\,dt + (\frac{\rho^{\frac{K\pi}{R}}}{2\frac{K\pi}{R}})\frac{4}{\pi K}\int_\rho^B t^{1-\frac{K\pi}{R}}\,dt \qquad (9)$$

After the integrals in Eq. (9) have been evaluated, then Eq. (9) can be simplified to

$$F_K(\rho) = \frac{4}{\pi K}\,[\frac{\rho^2 - \rho^{\mu}B^{(2-\mu)}}{\mu^2 - 4}] \quad \text{where} \quad \mu = \frac{K\pi}{R}\ ,\ K = 1,\ 3,\ 5,\ldots \qquad (10)$$

<u>Theorem 8</u>: The algorithm presented in Eq. (4.1) reduces to the method of false position.

<u>Proof</u>: The method of false position is derived in the following:

Let $[a,b]$ be an interval on which the expression $f(\alpha_i)$ of Eq. (4.1) has a change in sign. Suppose that $f(b)$ is greater than zero and that $f(a)$ is less than zero. The root

α_i is approximated by intersecting a straight through the points $(a,f(a))$ and $(b,f(b))$ with the axis.

The triangle described by the points $(a,f(a))$, $(b,f(a))$ and $(b,f(b))$ is similar to the triangle described by the points $(\alpha_i,0)$, $(b,0)$ and $(b,f(b))$.

Hence

$$\frac{f(b) - f(a)}{b - a} = \frac{f(b)}{b - \alpha_i}$$

Solving for α_i

$$\alpha_i = \frac{af(b) - bf(a)}{f(b) - f(a)} \tag{1}$$

The algorithm of Eq. (4.1) is

$$\alpha_i = \alpha_i - \left(\frac{(b - \alpha_i)}{f(b) - f(\alpha_i)}\right) f(\alpha_i) \tag{2}$$

Let α_i in the right hand side of Eq. (2) be replaced by a.

$$\alpha_i = a - \left(\frac{(b - a)}{f(b) - f(a)}\right) f(a)$$

$$\alpha_i = \frac{a(f(b) - f(a)) - (b-a)f(a)}{f(b) - f(a)}$$

Hence

$$\alpha_i = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

which was to be shown.

Theorem 9:

$$-\frac{d(P/\bar{\rho})}{dz} = \frac{1}{A}\frac{d}{dz}\int_A w^2 dA - \frac{\nu}{A}\oint_c \frac{\partial w}{\partial N}\, dl$$

Proof: Integrating Eq. (2.1) over the duct cross-section gives

$$\int_A -\frac{1}{\rho}\frac{\partial P}{\partial z}\, dA = \int_A \bar{v}\cdot\nabla w\, dA - \nu \int_A \nabla^2 w\, dA \tag{1}$$

Now $$\int_A -\frac{1}{\rho}\frac{\partial P}{\partial z}\, dA = -\frac{1}{\rho}\frac{\partial P}{\partial z} A \tag{2}$$

and by Eq.(1.1) of Chapter I

$$-\nu \int_A \nabla^2 w\, dA = -\nu \oint_c \frac{\partial w}{\partial N}\, dl \tag{3}$$

The inner product of the velocity vector $\bar{v}$ with the gradient of the axial component of the velocity is

$$\bar{v}\cdot\nabla w = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{4}$$

so that

$$\int_A \bar{v}\cdot\nabla w\, dA = \int_A \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) dA + \int_A w\frac{\partial w}{\partial z}\, dA$$

which becomes

$$\int_A \bar{v}\cdot\nabla w\, dA = \int_A \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) dA + \frac{d}{dz}\int_A \frac{w^2}{2}\, dA \tag{5}$$

From Eq. (1.2)

$$\int_A \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) dA = -\int_A w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dA + \oint_c w(udy - vdx) \tag{6}$$

but $w = 0$ on the duct wall hence

$$\oint_c w(udy - vdx) = 0 \tag{7}$$

and $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$, therefore (8)

$$-\int_A w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dA = -\int_A - w\frac{\partial w}{\partial z}\, dA = \frac{d}{dz}\int_A \frac{w^2}{2}\, dA \tag{9}$$

Combining Eqs. (2), (3), (5) and (9) yields

$$-\frac{1}{\rho}\frac{dP}{dz} = \frac{1}{A}\frac{d}{dz}\int_A w^2\, dA - \frac{\nu}{A}\oint_c \frac{\partial w}{\partial N}\, dl \tag{10}$$

which is the required result.

Theorem 10:

$$-\frac{d(P/\bar{\rho})}{dz} = \frac{1}{\bar{w}A}\frac{d}{dz}\int_A \frac{w^3}{2} + \frac{\nu}{\bar{w}A}\int_A (\nabla_1 w)\cdot(\nabla_1 w)\,dA$$

Proof: Multiplying Eq. (2.1) by the velocity w gives

$$-\frac{d(P/\bar{\rho})}{dz}\,w = w(\bar{v}\cdot\nabla w) - \nu w(\nabla^2 w) \tag{1}$$

Integrating over the duct cross-section yields

$$\int_A -\frac{d(P/\bar{\rho})}{dz}\,w\,dA = \int_A w(\bar{v}\cdot\nabla w)\,dA - \nu\int_A w(\nabla^2 w)\,dA \tag{2}$$

Recall that $\int_A w\,dA = \bar{w}A$ hence

$$\int_A -\frac{d(P/\bar{\rho})}{dz}\,w\,dA = -\frac{d(P/\bar{\rho})}{dz}\,\bar{w}A \tag{3}$$

Now

$$w(\bar{v}\cdot\nabla w) = wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y} + w^2\frac{\partial w}{\partial z} \tag{4}$$

and

$$\int_A\left(wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y} + w^2\frac{\partial w}{\partial z}\right)dA = \int_A\left(wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y}\right)dA = \frac{d}{dz}\int_A\frac{w^3}{3}\,dA \tag{5}$$

From Eq. (1.2)

$$\int_A\left(wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y}\right)dA = -\int_A w\left[\frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y}\right]dA + \oint_c w(wu\,dy - wv\,dx) \tag{6}$$

but $\oint_c w(wu\,dy - wv\,dx) = 0$ since $w = 0$ on the duct wall and

$$w\left[\frac{\partial(uw)}{\partial x} + \frac{\partial(vw)}{\partial y}\right] = w^2\frac{\partial u}{\partial x} + wu\frac{\partial w}{\partial x} + w^2\frac{\partial v}{\partial y} + wv\frac{\partial w}{\partial y}$$

hence Eq. (6) reduces to

$$\int_A\left(wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y}\right)dA = -\int w^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dA - \int_A\left(wu\frac{\partial w}{\partial x} + wv\frac{\partial w}{\partial y}\right)dA \tag{7}$$

which can be further simplified by recalling from Theorem 10 that

$$-\int_A w^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dA = \frac{d}{dz} \quad _A \frac{w^3}{3}\, dA \qquad (8)$$

hence

$$\int_A \left(wu\, \frac{\partial w}{\partial x} + wv\, \frac{\partial w}{\partial y}\right) dA = \frac{d}{dz} \int_A \frac{w^3}{6}\, dA \qquad (9)$$

The combination of Eqs. (5) and (9) yields

$$\int_A \left(wu\, \frac{\partial w}{\partial x} + wv\, \frac{\partial w}{\partial y} + w^2\, \frac{\partial w}{\partial z}\right) dA = \frac{d}{dz} \int_A \frac{w^3}{2}\, dA \qquad (10)$$

The remaining term of Eq. (2) is evaluated with the use of Eq. (1.3)

$$-\nu \int_A w\nabla^2 w dA = \nu \int_A \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dA - \nu \oint_C w\, \frac{\partial w}{\partial N}\, dl \qquad (11)$$

and $\oint_C w\, \frac{\partial w}{\partial N}\, dl = 0$ since $w = 0$ on the duct wall. Eq. (10) reduces to

$$-\nu \int_A w\nabla w^2 dA = \nu \int_A \nabla_1 w \cdot \nabla_1 w\, dA \qquad (12)$$

Combining Eqs. (3), (10) and (12) gives the result

$$-\frac{d(P/\bar{\rho})}{dz} = \frac{1}{\bar{w}A}\, \frac{d}{dz} \int_A \frac{w^3}{2}\, dA + \frac{\nu}{\bar{w}A} \int_A (\nabla_1 w)\cdot(\nabla_1 w) dA \qquad (13)$$

which completes the proof.

<u>Theorem 11</u>:

$$\varepsilon(\beta) = \frac{\frac{d}{d\beta}\left[\int_A \left(\phi^2 - \frac{\phi^3}{2}\right) \rho dA\right]}{\oint_C \frac{\partial \phi}{\partial n}\, ds + \int_A (\nabla_1 \phi)\cdot(\nabla_1 \phi) \rho dA}$$

<u>Proof</u>: Recall that $dz/dz^* = \varepsilon(z^*)$ from Eq. (2.5) and that the chain rule for derivatives is

$$d/dz = (d/dz^*)(dz^*/dz) = 1/\varepsilon(z^*)\ d/dz^*$$

Equating Eqs. (4.6) and (4.7) and with the use of the above

$$\frac{1}{A}\frac{d}{dz}\int_A \left(w^2 - \frac{w^3}{2\bar{w}}\right) dA = \frac{\nu}{A}\oint_C \frac{\partial w}{\partial N}\, dl + \frac{\nu}{\bar{w}A}\int_A \nabla_1 w \cdot \nabla_1 w\ dA \qquad (1)$$

becomes

$$\frac{1}{\varepsilon(z^*)}\frac{d}{dz^*}\int_A \left(w^2 - \frac{w^3}{\partial\bar{w}}\right) dA = \nu\oint_C \frac{\partial w}{\partial N}\, dl + \frac{\nu}{\bar{w}}\int_A \nabla_1 w \cdot \nabla_1 w\ dA \qquad (2)$$

and solving for $\varepsilon(z^*)$ yields

$$\varepsilon(z^*) = \frac{\frac{d}{dz^*}\int_A \left(w^2 - \frac{w^3}{2\bar{w}}\right) dA}{\nu\oint_C \frac{\partial w}{\partial N}\, dl + \frac{\nu}{\bar{w}}\int_A \nabla_1 w \cdot \nabla_1 w\ dA} \qquad (3)$$

By definition $\phi = w/\bar{w}$ or $w = \bar{w}\phi$ hence Eq. (3) becomes

$$\varepsilon(z^*) = \frac{\frac{d}{dz^*}\int_A \left(\phi^2\bar{w}^2 - \frac{\phi^3\bar{w}^3}{2\bar{w}}\right)\rho\, dA}{\nu\oint_C \frac{\partial\phi}{\partial n}\,\bar{w}\, ds + \frac{\nu}{\bar{w}}\int_A \bar{w}^2\ \nabla_1\phi\cdot\nabla_1\phi\,\rho\, dA} \qquad (4)$$

which reduces to

$$\varepsilon(z^*) = \frac{\bar{w}\,\frac{d}{dz^*}\int_A \left(\phi^2 - \frac{\phi^3}{2}\right)\rho\, dA}{\nu\oint_C \frac{\partial\phi}{\partial n}\, ds + \nu\int_A \nabla_1\phi\cdot\nabla_1\phi\ \rho\, dA} \qquad (5)$$

By definition $z^* = L\ Re\beta = A\,\frac{\bar{w}}{\nu}\,\beta$, hence $d/dz^* = (\nu/A\bar{w})\ d/d\beta$. Hence

$$\varepsilon(\beta) = \frac{\frac{d}{d\beta}\int_A \left(\phi^2 - \frac{\phi^3}{2}\right)\rho\, dA}{\oint_C \frac{\partial\phi}{\partial n}\, ds + \int_A \nabla_1\phi\cdot\nabla_1\phi\,\rho\, dA} \qquad (6)$$

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