THE POTENTIALS AND FIELDS OF AN ELECTROMAGNETIC DIPOLE IN ARBITRARY MOTION

A Thesis

Presented to

the Faculty of the Committee on Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Abdul Sattar Khan Lodhi

January 1969

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ABSTRACT

An electromagnetic dipole in motion constitutes a source and gives rise to an electromagnetic field. A covariant expression for the 4-vector current density is derived for such a source and the resulting field equations in terms of a 4-vector potential are solved using the Green's function. Potentials analogous to the Lienard-Weichert potentials are obtained for the electromagnetic dipole in motion. These potentials are used to calculate the general expressions for the electric and the magnetic fields due to a moving electric dipole. Using the expressions for the far-fields, a general expression for the Poynting vector is obtained. Finally, from the general expressions for the potentials, fields and the Poynting vector, some special cases are derived.

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CHAPTER I

INTRODUCTION

The basic laws of electromagnetism can be summarized in the differential form by the four Maxwell's equations:

$$\nabla \cdot \vec{E} = 4\pi \rho \tag{I-1}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \tag{I-2}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{I-3}$$

$$\nabla \cdot \vec{B} = 0 \ . \tag{I-4}$$

When combined with the Lorentz force law and Newton's second law of motion, these equations provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields. The solution of Maxwell's equations, which are coupled first order partial differential equations relating the various components of electric and magnetic fields, is simplified by the introduction of a scalar and a vector potential, ϕ and $\vec{A}$, defined by

$$\vec{B} = \nabla \times \vec{A} \tag{I-5}$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \phi \ , \tag{I-6}$$

and subject to the constraint Lorentz condition

$$\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \qquad \text{(I-7)}$$

Using these potentials, the Maxwell's equations lead to the following equations for $\vec{A}$ and ϕ

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = - \frac{4\pi}{c} \vec{J} \,, \qquad \text{(I-8)}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = - 4\pi \rho \ . \qquad \text{(I-9)}$$

These laws can easily be cast into covariant form by defining a 4-vector potential $A_\mu = (\vec{A}, i\phi)$ and a 4-vector current density $J_\mu = (\vec{J}, ic\rho)$. Using these two 4-vectors, equations (I-8) and (I-9) can be written as a single equation

$$\sum_\nu \frac{\partial^2 A_\mu}{\partial x_\nu} = - \frac{4\pi}{c} J_\mu \ . \qquad \text{(I-10)}$$

The Lorentz condition becomes

$$\sum_\nu \frac{\partial A_\nu}{\partial x_\nu} = 0 \,, . \qquad \text{(I-11)}$$

the first two Maxwell's equations take the form

$$\sum_\nu \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu \qquad \text{(I-12)}$$

and the last two equations reduce to

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \ . \qquad \text{(I-13)}$$

$F_{\mu\nu}$ is a completely antisymmetric second rank tensor known as the electromagnetic field tensor and is defined in terms of the 4-vector potential A_μ by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \ , \qquad \text{(I-14)}$$

its explicit form being[1]

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_z & -B_y & iE_x \\ -B_z & 0 & B_x & iE_y \\ B_y & -B_x & 0 & iE_z \\ -iE_x & -iE_y & -iE_z & 0 \end{bmatrix} . \qquad \text{(I-15)}$$

The 4-vector potential A_μ due to a given charge and current distribution can be evaluated by solving (I-10) and the fields can be obtained either by equations (I-5) and (I-6) or directly by the equation (I-14).

This procedure will be followed to derive the expressions for the electromagnetic potentials and fields of a particular system of charges and currents in motion. This system of charges and currents is assumed to be of such small dimension that it can be represented by Dirac delta function and the distribution of charges and currents within this system is such that the net charge is zero. Such a system will be termed an electromagnetic dipole and will be characterized by an electric dipole moment $\vec{p}$ and a magnetic dipole moment $\vec{m}$ in its rest frame.

The electromagnetic potentials for such a system were obtained by Bialas(2) using a technique originally due to Lorentz(3) and also by representing the sources by singular distributions of charges and current. His first method, however, is not rigorous and the calculations have been carried out only to first order of approximation. In the second method he uses the 4-dimensional formulation analogous to the one presented here to derive the potentials of a moving dipole. Later in the paper he derives expressions for certain properties of the radiation field for a restricted class of motions. An excellent account of this problem is given by Ellis(4). His approach is entirely different from the one presented here. He describes an electric dipole as two equal and opposite charges connected rigidly and calculates the potentials and fields due to the motion of such a system. The history of the dipole in 4-dimensions is described by a thin "ribbon" of constant width whose edges are the world lines of the two charges. Later in the paper, he discusses the radiation from the dipole for certain cases. He gives no discussion of the analogous magnetic dipole problem.

In the present treatment of the problem, equation (I-10) will be solved for general J_μ as an integral over a Green's function. The Lienard-Weichert potentials will then be derived for a moving charge and an electromagnetic dipole. These potentials will be used to derive the fields and the general expression for the Poynting vector will be calculated. Some special cases of this general problem will then be discussed.

CHAPTER II

SOLUTION OF MAXWELL'S EQUATIONS

The fields can be obtained by solving Maxwell's equations directly. However, instead of solving these coupled equations it is more convenient to obtain the potentials first and then obtain the fields from these potentials using (I-5) and (I-6) or (I-14). In terms of the 4-vector A_μ, Maxwell's equations lead to

$$\Box^2 A_\mu = -\frac{4\pi}{c} J_\mu \tag{II-1}$$

subject to the condition

$$\sum_\nu \frac{\partial A_\nu}{\partial x_\nu} = 0 .$$

Equation (II-1) can be solved either by direct integration[5] or by the more physical approach of constructing an appropriate Green's function and invoking the superposition principle. The latter method will be adopted here.

Assume that the required solution of (II-1) is the superposition integral

$$A_\mu(x) = \frac{1}{c}\int d^4x'\, G(x-x')\, J_\mu(x') \tag{II-2}$$

where $d^4x' = d^3\vec{x}'\,dx_0'$, $x_0' = ct'$ and $G(x-x')$ is a Green's function. Substituting this solution in equation (II-1), we find that $G(x-x')$ must satisfy

$$\Box^2 G(x-x') = -4\pi\,\delta^4(x-x')\,, \tag{II-3}$$

where $\delta^4(x-x') = \delta^3(\vec{x}-\vec{x}')\,\delta(x_0-x_0')$. Equations (II-2) and (II-3) insure that our proposed solution (II-2) satisfies the Lorentz condition (I-11). This follows since we have

$$\sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = \frac{1}{c}\sum_\mu \frac{\partial}{\partial x_\mu}\int d^4x'\,G(x-x')\,J_\mu(x')\,.$$

On the right hand side, the variable of integration is x' and hence the operator $\sum_\mu \frac{\partial}{\partial x_\mu}$ can be taken under the integral and the last equation can be written as

$$\sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = \frac{1}{c}\int d^4x' \sum_\mu \frac{\partial G}{\partial x_\mu} J_\mu(x')$$

since $J_\mu(x')$ does not depend on x and hence can be treated as constant with respect to the differentiation with respect to x . Now $G(x-x')$ is a function of $(x-x')$ and hence

$$\frac{\partial G}{\partial x_\mu} = -\frac{\partial G}{\partial x_\mu'}$$

Therefore the last expression can be written

$$\sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = -\frac{1}{c}\int d^4x' \sum_\mu \frac{\partial G}{\partial x_\mu'} J_\mu(x')\,.$$

Performing partial integration once, this becomes

$$\sum_{\mu} \frac{\partial A_\mu}{\partial x_\mu} = \frac{1}{c} \int d^4x' \, G(x-x') \sum_{\mu} \frac{\partial J_\mu(x')}{\partial x'_\mu} .$$

The right hand side is identically zero because of conservation of charge and hence the solution (II-2) satisfies the Lorentz condition regardless of the detailed form of $G(x-x')$.

The solutions of (II-3) satisfying causality conditions(6)

$$G(x-x') = 0 \quad \begin{cases} x_0 - x'_0 < 0 \\ \\ x_0 - x'_0 > 0 \end{cases} \tag{II-4}$$

are

$$G_\pm (x-x') = \frac{1}{R'} \delta(\xi_0 \pm R') , \tag{II-5}$$

where $R' = |\vec{R}'| = |\vec{x} - \vec{x}'|$ and $\xi_0 = x_0 - x'_0 = c(t-t')$.
Here $G_-(x-x')$ is the causal and G_+ the anticausal Green's function. For a general solution of (II-3), we can take a linear combination of the two solutions G_+ and G_- , i.e.

$$\begin{aligned} G(x-x') &= G_+ + G_- \\ &= \frac{1}{R'} \left[\delta(\xi_0 + R) + \delta(\xi_0 - R) \right] . \end{aligned}$$

The sum of two delta functions appearing on the right hand side can be written as a single delta function[7]

$$G(x-x') = 2\,\delta(\xi_0^2 - R'^2). \qquad \text{(II-6)}$$

This Green's function, when used with proper $J_\mu(x')$ in equation (II-2) gives the sum of the retarded and advanced potentials at once since all equations are linear. Thus

$$A_\mu(x) = \frac{2}{c}\int d^4x'\,\delta(\xi_0^2 - R'^2)\,J_\mu(x') = A^+_\mu(x) + A^-_\mu(x).$$

It should be understood that only the retarded potentials are to be retained after integration. The delta function appearing in the integral can be written in another convenient form by noting that

$$\xi_0^2 - R'^2 = c^2(t-t')^2 - |\vec{x}-\vec{x}'|^2 = -\sum_\mu (x_\mu - x'_\mu)^2 = -\sum_\mu R'^2_\mu .$$

We then have the simple result

$$A_\mu(x) = \frac{2}{c}\int d^4x'\,\delta\Big(\sum_\lambda R'^2_\lambda\Big)\,J_\mu(x') . \qquad \text{(II-7)}$$

Equation (II-7) is the general solution of (II-1) giving both the retarded and advanced potentials due to any charge and current distribution $J_\mu(x)$. To illustrate the techniques we shall use for the dipole problem, we will now obtain the fields of a moving point charge.

In the case of a moving point charge, the charge and current densities are given by

$$\rho(\vec{x}) = e\, \delta^3(\vec{x} - \vec{x}_p) \qquad \text{(II-8)}$$

$$\vec{J}(\vec{x}) = e\,\vec{v}\, \delta^3(\vec{x} - \vec{x}_p) \qquad \text{(II-9)}$$

$\vec{X}_p$ being the instantaneous position vector of the particle. We can write equations (II-8) and (II-9) in 4-vector form by recalling that the space part of the 4-vector current density is $\vec{J}$ and the time part is the charge density ρ i.e. $J_\mu(x) = (\vec{J}, ic\rho)$. With this definition the above equations lead to

$$J_\mu(x) = e(\vec{v},\ ic)\, \delta^3(\vec{x} - \vec{x}_p),$$

which can also be written as

$$J_\mu(x) = e\, U_\mu\, \delta^3(\vec{x} - \vec{x}_p)\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{(II-10)}$$

where $U_\mu = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(\vec{v},\ ic)$ is the 4-vector velocity of the particle. The last equation can be cast into another form that is more useful for our purposes. Further we shall explicitly demonstrate its covariance. We can write (II-10) as

$$J_\mu(x) = \int e\sqrt{1 - \frac{v^2}{c^2}}\; U_\mu\, \delta^3(\vec{x} - \vec{x}_p)\, \delta(t - t_p)\, dt_p,$$

where t_p is the time at which the particle is at the position $\vec{x}_p$. From the properties of delta functions we know that

$$\delta(t - t_p) = c\,\delta(ct - ct_p) = c\,\delta(x_0 - x_{0p}).$$

Using this result in the integral for J_μ , we have

$$J_\mu(x) = ec\int \sqrt{1-\frac{v^2}{c^2}}\; u_\mu\, \delta^3(\vec{x}-\vec{x}_p)\,\delta(x_0 - x_{0p})\, dt_p\ .$$

We now parameterize the motion by the particle's proper time, τ , and consider the 4-position to be an implicit function of τ . Then

$$J_\mu(x) = ec\int \sqrt{1-\frac{v^2}{c^2}}\; \delta^3(\vec{x}-\vec{x}_p)\,\delta(x_0 - x_{0p})\, u_\mu \frac{dt_p}{d\tau}\, d\tau\ .$$

But since $\dfrac{dt_p}{d\tau} = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, we have

$$J_\mu(x) = ec\int \delta^4(x - x_p)\, u_\mu\, d\tau\ . \tag{II-11}$$

From this last expression it is clear that $J_\mu(x)$ is a 4-vector.

The well-known Lienard-Weichert potentials due to a charge in motion can be obtained from (II-7) using (II-11). Substituting the right hand side of (II-11) for $J_\mu(x')$ in (II-7), we have

$$A_\mu(x) = \frac{2}{c}\int d^4x' \,\delta\left(\sum_\lambda R_\lambda'^2\right)\left[ec\int d\tau\; U_\mu(t_p)\,\delta^4(x - x_p(\tau))\right].$$

If we interchange the order of integrations, this becomes

$$A_\mu(x) = 2e\int d\tau\left[\int d^4x'\, U_\mu(t_p)\;\delta\left(\sum_\lambda R_\lambda'^2\right)\delta^4(x-x_p)\right].$$

We may now carry out the integration over the primed variables to obtain

$$A_\mu(x) = 2e\int d\tau\;U_\mu(t_p)\;\delta\left(\sum_\lambda R_\lambda^2\right) . \tag{II-12}$$

This integral is worked out in detail in the Appendix, and the final result is

$$A_\mu(x) = -e\left[\frac{U_\mu(t_p)}{\sum_\lambda R_\lambda U_\lambda}\right]_{s=0} \tag{II-13}$$

where $s = \sum_\lambda R_\lambda^2$.

In 3-vector notation, this expression for the potentials reduces to

$$\vec{A}(\vec{x}) = e\left[\frac{\vec{\beta}}{R(1-\vec{\beta}\cdot\vec{n})}\right]_{t_p = t - \frac{R}{c}} \tag{II-14}$$

$$\phi(\vec{x}) = e\left[\frac{1}{R(1-\vec{\beta}\cdot\vec{n})}\right]_{t_p = t - \frac{R}{c}} \tag{II-15}$$

where $\vec{\beta} = \frac{\vec{v}}{c}$, $\vec{n} = \frac{\vec{R}}{|\vec{R}|}$ = unit vector along $\vec{R}$, and $\vec{R} = (\vec{x} - \vec{x}_p)$. Equations (II-14) and (II-15) are the Lienart-Weichert potentials due to a moving charge e .

The corresponding fields can be obtained from these potentials by (I-5) and (I-6) or the electromagnetic field tensor can be obtained directly from (I-14). The latter method is simpler since we can avoid the algebraic complexities of differentiation with a constraint and, hence, will be followed here. We have

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

with A_μ given by (II-12). Substituting in the right hand side of the above equation yields

$$F_{\mu\nu} = 2e\left[\frac{\partial}{\partial x_\mu}\int d\tau\, u_\nu\, \delta(s) - \frac{\partial}{\partial x_\nu}\int d\tau\, u_\mu\, \delta(s)\right].$$

Assuming that we may interchange the order of integration and differentiation, this last equation becomes

$$F_{\mu\nu} = 2e\int d\tau\left[u_\nu \frac{\partial}{\partial x_\mu}\delta(s) - u_\mu \frac{\partial}{\partial x_\nu}\delta(s)\right].$$

But $\frac{\partial}{\partial x_\lambda}\delta(s) = \frac{\partial s}{\partial x_\lambda}\frac{d}{ds}\delta(s) = 2R_\lambda \frac{d}{ds}\delta(s)$

and hence the last integral becomes

$$F_{\mu\nu} = 2e\int d\tau\left[2\,u_\nu R_\mu - 2\,u_\mu R_\nu\right]\frac{d}{ds}\,\delta(s)\ .$$

Since s is an implicit function of τ we may write the above integral as

$$F_{\mu\nu} = 2e\int d\tau\left[2\left(u_\nu R_\mu - u_\mu R_\nu\right)\right]\frac{d\tau}{ds}\,\frac{d}{d\tau}\,\delta(s)\,.$$

However $\dfrac{d\tau}{ds} = -\dfrac{1}{2\sum_\lambda R_\lambda u_\lambda}$, and hence we have

$$F_{\mu\nu} = -2e\int d\tau\left[\frac{R_\mu u_\nu - R_\nu u_\mu}{\sum_\lambda R_\lambda u_\lambda}\right]\frac{d}{d\tau}\,\delta(s)\ .$$

Now performing partial integration once, this integral yields

$$F_{\mu\nu} = 2e\int d\tau\left[\frac{d}{d\tau}\left\{\frac{R_\mu u_\nu - R_\nu u_\mu}{\sum_\lambda R_\lambda u_\lambda}\right\}\right]\delta(s)$$

$$F_{\mu\nu} = 2e\int\frac{d}{d\tau}\left[\frac{R_\mu u_\nu - R_\nu u_\mu}{\sum_\lambda R_\lambda u_\lambda}\right]\delta(s)\ \frac{d\tau}{ds}\,ds$$

$$= -e\int\frac{d}{d\tau}\left[\frac{R_\mu u_\nu - R_\nu u_\mu}{\sum_\lambda R_\lambda u_\lambda}\right]\frac{\delta(s)}{\sum_\sigma R_\sigma u_\sigma}\ ds,$$

which can be integrated at once

$$F_{\mu\nu} = e\left[\frac{1}{\sum_\sigma R_\sigma u_\sigma}\ \frac{d}{d\tau}\left\{\frac{R_\nu u_\mu - R_\mu u_\nu}{\sum_\lambda R_\lambda u_\lambda}\right\}\right]_{s=0}\ .$$

Carrying out the indicated derivatives gives the covariant form of the field tensor, namely

$$F_{\mu\nu} = e\left[\frac{\dot{u}_\mu R_\nu - \dot{u}_\nu R_\mu}{\left(\sum_\lambda R_\lambda u_\lambda\right)^2} - \frac{(R_\nu u_\mu - R_\mu u_\nu)\,(c^2 + \sum_\lambda R_\lambda \dot{u}_\lambda)}{\left(\sum_\lambda R_\lambda u_\lambda\right)^3}\right]_{s=0} \quad \text{(II-16)}$$

We have adopted the notation of a dot representing differentiation with respect to proper time. (II-16) is the final expression for the electromagnetic field tensor due to a charge in motion. Using (I-15), this can be written in 3-vector form:

$$\vec{E}(\vec{x},t) = e\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{R^2(1-\vec{\beta}\cdot\vec{n})^3} - \frac{1}{Rc^2}\left\{\frac{\vec{a}}{(1-\vec{\beta}\cdot\vec{n})^2} - \frac{(\vec{n}\cdot\vec{a})(\vec{n}-\vec{\beta})}{(1-\vec{\beta}\cdot\vec{n})}\right\}\right]_{t_p = t - \frac{R}{c}}, \quad \text{(II-17)}$$

$$\vec{B}(\vec{x},t) = e\left[\frac{(\vec{\beta}\times\vec{n})(1-\beta^2)}{R^2(1-\vec{\beta}\cdot\vec{n})^3} + \frac{1}{Rc^2}\left\{\frac{\vec{a}\times\vec{n}}{(1-\vec{\beta}\cdot\vec{n})^2} + \frac{(\vec{n}\cdot\vec{a})(\vec{\beta}\times\vec{n})}{(1-\vec{\beta}\cdot\vec{n})^3}\right\}\right]_{t_p = t - \frac{R}{c}} \quad \text{(II-18)}$$

$$= \vec{n}\times\vec{E}$$

These are the fields due to a moving charge. It is interesting to note that in this particular case the magnetic field is everywhere perpendicular to the electric field and to the vector $\vec{R}$ extending from the particle location at time $t_p = t - \frac{R}{c}$ to the field point.

We shall use the same technique to obtain the potentials and fields due to a moving electromagnetic dipole. But first we need an expression for J_μ corresponding to a moving electromagnetic dipole. The next chapter deals with the derivation of a covariant expression for current density corresponding to dipole sources of the fields.

CHAPTER III

THE DIPOLE AS A FIELD SOURCE

In the rest frame of a particle having an electric dipole moment $\vec{p}$ and a magnetic dipole moment $\vec{m}$, the polarization and magnetization vectors $\vec{P}$ and $\vec{M}$ are given by

$$\vec{P}(\vec{x}) = \vec{p}\,\delta^3(\vec{x} - \vec{x}_p) \tag{III-1}$$

$$\vec{M}(\vec{x}) = \vec{m}\,\delta^3(\vec{x} - \vec{x}_p) \tag{III-2}$$

where $\vec{x}_p$ is the position vector of the particle. The two vectors $\vec{P}$ and $\vec{M}$ are equivalent to charge and current densities ρ and $\vec{J}$ given by

$$\rho(\vec{x}) = -\nabla\cdot\vec{P} = -\nabla\cdot\left[\vec{p}\,\delta^3(\vec{x} - \vec{x}_p')\right] \tag{III-3}$$

$$\vec{J}(\vec{x}) = c\nabla\times\vec{M} = c\,\nabla\times\left[\vec{m}\,\delta^3(\vec{x} - \vec{x}_p)\right]. \tag{III-4}$$

However, these are not written in covariant form, and hence are not convenient to use in their present form. In order to find the correct covariant form of (III-3) and

(III-4) we notice that, as is known from the macroscopic formulation of electrodynamics, the fields $\vec{P}$ and $\vec{M}$ form a tensor $M_{\mu\nu}$ given by[8]

$$M_{\mu\nu} = \begin{bmatrix} 0 & M_z & -M_y & iP_x \\ -M_z & 0 & M_x & iP_y \\ M_y & -M_x & 0 & iP_z \\ -iP_x & -iP_y & -iP_z & 0 \end{bmatrix}$$

This is a second rank completely antisymmetric tensor whose divergence gives the 4-vector

$$J_\mu(x) = c \sum_\nu \frac{\partial M_{\mu\nu}}{\partial x_\nu} . \tag{III-6}$$

Upon expansion in three vector form, this can be shown to be the covariant form of (III-3) and (III-4).

Now from the fields $\vec{P}$ and $\vec{M}$, we construct an array defined by (III-1) and (III-2), namely

$$M_{\mu\nu} = m_{\mu\nu} \, \delta^3(\vec{x}-\vec{x}_p) . \tag{III-7}$$

It is obvious that if $M_{\mu\nu}$ is a tensor, then $m_{\mu\nu}$ is not a tensor because $\delta^3(\vec{x}-\vec{x}_p)$ is not a scalar, or vice versa. However, $\sqrt{1-\beta^2}\,\delta^3(\vec{x}-\vec{x}_p)$ is a scalar, and if $M_{\mu\nu}$ is taken to be a tensor then

$$M_{\mu\nu} = m_{\mu\nu} \sqrt{1-\beta^2}\, \delta^3(\vec{x}-\vec{x}_p) \tag{III-8}$$

suggests that $m_{\mu\nu}$ is also a tensor. Hence if $m'_{\mu\nu}$ is the array constructed from $\vec{p}$ and $\vec{m}$ in a particular Lorentz frame, then in any other frame, $m_{\mu\nu}$ is given by the Lorentz transformation

$$m_{\mu\nu} = \sum_{\sigma}\sum_{\lambda} A_{\mu\sigma} A_{\nu\lambda} m'_{\sigma\lambda} \quad . \tag{III-9}$$

This array we shall call "the moments tensor." The tensor field $M_{\mu\nu}$ produced by $m_{\mu\nu}$ is then given by (III-8).

It is convenient to write (III-8) in a form in which it is immediately evident that $M_{\mu\nu}$ is indeed a tensor. To this end, consider the integral

$$I_{\mu\nu} = \int m_{\mu\nu}(t_p(\tau))\, \delta^3(\vec{x} - \vec{x}_p(\tau))\, \delta(t - t_p(\tau))\, d\tau \tag{III-10}$$

where again τ is the proper time of the particle, having an associated moments tensor $m_{\mu\nu}$, and $\vec{x}_p = \vec{x}_p(\tau)$; $t_p = t_p(\tau)$ is the parametric representation of the particle's world line. Now

$$\delta^3(\vec{x}-\vec{x}_p)\, \delta(t-t_p) = c\, \delta^3(\vec{x}-\vec{x}_p)\, \delta(ct-ct_p) = c\, \delta^4(x-x_p)$$

is certainly a scalar, $m_{\mu\nu}$ is by definition a tensor and $d\tau$ is a scalar. Hence the integrand in (III-10) is a tensor. But

$$I_{\mu\nu} = \int m_{\mu\nu}(t_p)\, \delta^3(\vec{x} - \vec{x}_p)\, \delta(t-t_p)\, d\tau$$

$$= \int m_{\mu\nu}(t_p)\, \delta^3(\vec{x} - \vec{x}_p)\, \delta(t - t_p)\, \frac{d\tau}{dt_p}\, dt_p \,,$$

and since

$$\frac{d\tau}{dt_p} = \sqrt{1 - \beta^2}$$

we have

$$I_{\mu\nu} = \int m_{\mu\nu}(t_p)\, \delta^3(\vec{x} - \vec{x}_p)\, \delta(t - t_p)\, \sqrt{1 - \beta^2}\, dt_p$$

$$= m_{\mu\nu}(t)\, \delta^3(\vec{x} - \vec{x}_p)\, \sqrt{1 - \beta^2} = M_{\mu\nu} \,.$$

Thus it is quite reasonable to take for the effective current of the particle in motion the expression

$$J_\mu(x) = c \sum_\nu \frac{\partial M_{\mu\nu}}{\partial x_\nu} = (\vec{J},\, ic\rho) \,.$$

But since

$$M_{\mu\nu} = \int m_{\mu\nu}(t_p)\, \delta^3(\vec{x} - \vec{x}_p)\, \delta(t - t_p)\, d\tau$$

$$= c \int m_{\mu\nu}(t_p)\, \delta^4(x - x_p)\, d\tau \,,$$

we have

$$J_\mu(x) = c^2 \sum_\nu \frac{\partial}{\partial x_\nu} \int m_{\mu\nu}(t_p)\, \delta^4(x - x_p)\, d\tau \,.$$

Now the variable of integration is τ and since the point of observation is independent of τ , the differentiation operator can be taken under the integral sign and we may write

$$J_\mu(x) = c^2 \sum_\nu \int m_{\mu\nu}(t_p) \frac{\partial}{\partial x_\nu}\left[\delta^4(x-x_p)\right] d\tau . \quad \text{(III-11)}$$

This is the covariant form of equations (III-3) and (III-4), representing the 4-vector current density due to a moving particle whose electromagnetic structure is characterized by the moments tensor $m_{\mu\nu}$.

The moments tensor $m_{\mu\nu}$ may be displayed in matrix form as

$$m_{\mu\nu} = \begin{bmatrix} 0 & m_z & -m_y & ip_x \\ -m_z & 0 & m_x & ip_y \\ m_y & -m_x & 0 & ip_z \\ -ip_x & -ip_y & -ip_z & 0 \end{bmatrix} \quad \text{(III-12)}$$

This choice of the tensor $m_{\mu\nu}$ characterizes both the electric and magnetic properties of the source. The kind of dipole depends on the form of the tensor $m_{\mu\nu}$. In the case of an electric dipole, having a dipole moment $\vec{p}$ in its rest frame and $\vec{m} = 0$, this tensor takes the form

$$m^{(0)}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & ip_x \\ 0 & 0 & 0 & ip_y \\ 0 & 0 & 0 & ip_z \\ -ip_x & -ip_y & -ip_z & 0 \end{bmatrix} \quad \text{(III-13)}$$

From the Lorentz transformation we may obtain the tensor $m_{\mu\nu}$ in any artibrary frame. In a frame relative to which the particle has velocity $\vec{v}$, the tensor $m_{\mu\nu}$ becomes

$$m_{\mu\nu} = \begin{bmatrix} 0 & \gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_z & \gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_y & i\gamma p_x \\ -\gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_z & 0 & \gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_x & i\gamma p_y \\ -\gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_y & -\gamma\left(\frac{\vec{p}\times\vec{v}}{c}\right)_x & 0 & i\gamma p_z \\ -i\gamma p_x & -i\gamma p_y & -i\gamma p_z & 0 \end{bmatrix}. \qquad \text{(III-14)}$$

The corresponding case of a particle with $\vec{p}=0$ in its rest frame gives

$$m^{(0)}_{\mu\nu} = \begin{bmatrix} 0 & m_z & -m_y & 0 \\ -m_z & 0 & m_x & 0 \\ m_y & -m_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(III-15)}$$

and in the frame in which the particle has velocity $\vec{v}$

$$m_{\mu\nu} = \begin{vmatrix} 0 & \gamma m_z & -\gamma m_y & -i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_x \\ -\gamma m_z & 0 & \gamma m_x & -i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_y \\ \gamma m_y & -\gamma m_x & 0 & -i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_z \\ i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_x & i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_y & i\gamma\left(\frac{\vec{m}\times\vec{v}}{c}\right)_z & 0 \end{vmatrix} \qquad \text{(III-16)}$$

The tensor $m_{\mu\nu}$ introduced above is a completely antisymmetric second rank tensor. The two invariants associated with this tensor are

$$\sum_{\mu}\sum_{\nu} m_{\mu\nu}\, m_{\mu\nu} \tag{III-17}$$

and

$$\sum_{\mu}\sum_{\nu}\sum_{\sigma}\sum_{\lambda} \epsilon_{\mu\nu\sigma\lambda}\, m_{\mu\nu}\, m_{\sigma\lambda} \,, \tag{III-18}$$

where $\epsilon_{\mu\nu\sigma\lambda}$ is a completely antisymmetric unit tensor of fourth rank defined by

$$\epsilon_{\mu\nu\sigma\lambda} = \begin{cases} 0 & \text{if any two indices are equal,} \\ \pm 1 & \text{for an even (odd) permutation of indices.} \end{cases}$$

The expressions (III-17) and (III-18) in the three vector form are

$$\vec{p}\cdot\vec{p} - \vec{m}\cdot\vec{m} \tag{III-19}$$

and

$$\vec{p}\cdot\vec{m} \tag{III-20}$$

From the transformation properties of the tensor $m_{\mu\nu}$ it can be seen that if a particle has an electric dipole moment p_o and a magnetic dipole moment m_o in its rest frame, then in another frame in which its velocity is $\vec{v}$, the electric and magnetic dipole moments are given by

$$\vec{p} = \gamma\left(\vec{p}_o - \frac{\vec{m}_o \times \vec{v}}{c}\right) + (1-\gamma^2)\,\frac{\vec{p}_o\cdot\vec{v}}{v^2}\,\vec{v} \,, \tag{III-21}$$

$$\vec{m} = \gamma\left(\vec{m}_0 + \frac{\vec{p}_0 \times \vec{V}}{c}\right) + (1-\gamma^2)\,\frac{\vec{m}_0\cdot\vec{V}}{V^2}\,\vec{V}. \qquad \text{(III-22)}$$

It is clear from these expressions that if a particle has only an electric dipole moment or only a magnetic dipole moment in its rest frame (i.e. $\vec{m}_0 = 0$ or $\vec{p}_0 = 0$), then in another inertial frame in which its velocity is $\vec{V}$, one generally observes both electric and magnetic dipole moments, the only exception being when the particle moves parallel to its axis in which case one only observes the electric or magnetic dipole moment, but when the motion is not along the dipole axis, then the electric and the magnetic dipole moments are mutually perpendicular. Further, if a particle has both electric dipole and magnetic dipole moments in its rest frame such that they are orthogonal, then $\vec{p}_0 \cdot \vec{m}_0$ vanishes and since $\vec{p}_0 \cdot \vec{m}_0$ is invariant under Lorentz transformation, $\vec{p}\cdot\vec{m}$ will be zero in any other Lorentz frame and it is possible to find a Lorentz frame in which such a particle will have either electric or magnetic moments according to whether $\vec{p}\cdot\vec{p} - \vec{m}\cdot\vec{m}$ is positive or negative. It is also seen that if $\vec{p}_0 = 0$, then $-\vec{m}_0\cdot\vec{m}_0 = \vec{p}\cdot\vec{p} - \vec{m}\cdot\vec{m}$ is an invariant which allows us to define unambiguously the numerical value of the magnetic moment of a particle. In particular this should be true for the elementary particles. Having obtained the expression for current density in the covariant form, we can now proceed to calculate the potential A_μ due to a relativistic electromagnetic dipole by solving (II-7) with J_μ given by (III-11).

CHAPTER IV

SOLUTION OF MAXWELL'S EQUATION WITH A RELATIVISTIC DIPOLE SOURCE

An electromagnetic dipole in motion constitutes a source of fields and the corresponding current density is given by (III-11). The general solution of Maxwell's equations in terms of potentials A_μ is given by (II-7), from which the potentials due to an electromagnetic dipole in motion can be calculated when (III-11) is substituted for J_μ , i.e.

$$A_\mu(x) = \frac{2}{c}\int d^4x'\,\delta\Big(\sum_\lambda R^2_\lambda\Big)\Big[c^2\sum_\nu \frac{\partial}{\partial x'_\nu}\int d\tau\; m_{\mu\nu}(t_p)\,\delta^4(x'-x_p)\Big]$$

$$= 2c\sum_\nu\int d^4x'\Big[\delta\Big(\sum_\lambda R'^2_\lambda\Big)\frac{\partial}{\partial x'_\nu}\int d\tau\; m_{\mu\nu}(t_p)\,\delta^4(x'-x_p)\Big].$$

Interchanging the order of differentiation and integration in this expression, we get

$$A_\mu(x) = 2c\sum_\nu\int d^4x'\Big[\delta\Big(\sum_\lambda R'^2_\lambda\Big)\int d\tau\; m_{\mu\nu}(t_p)\frac{\partial}{\partial x'_\nu}\Big\{\delta^4(x'-x_p)\Big\}\Big].$$

We may also interchange the order of integrations, so that the last equation can be written as

$$A_\mu(x) = 2c\sum_\nu \int d\tau \left[\int d^4x' \, \delta\left(\sum_\lambda R'^2_\lambda\right) m_{\mu\nu}(t_p) \frac{\partial}{\partial x'_\nu} \left\{ \delta^4(x'-x_p) \right\} \right].$$

Now considering the integral in the parenthesis, and performing partial integration once, we obtain

$$A_\mu(x) = -2c\sum_\nu \int d\tau \left[\int d^4x' \, m_{\mu\nu}(t_p) \, \delta^4(x'-x_p) \frac{\partial}{\partial x'_\nu} \left\{ \delta\left(\sum_\lambda R'^2_\lambda\right) \right\} \right]$$

which immediately leads to

$$A_\mu(x) = -2c\sum_\nu \int d\tau \, m_{\mu\nu}(t_p) \frac{\partial}{\partial x_\nu} \left\{ \delta\left(\sum_\lambda R^2_\lambda\right) \right\} \qquad \text{(IV-1)}$$

where $R_\lambda = (x_\lambda - x_{\lambda p})$. This integral is similar to (II-12) and may be evaluated using the substitution $s(\tau) = \sum_\lambda R^2_\lambda$, which leads to

$$A_\mu(x) = -2c\sum_\nu \int \frac{m_{\mu\nu}(t_p) R_\nu}{\sum_\lambda R_\lambda U_\lambda} \frac{d}{ds} \left\{ \delta(s) \right\} ds.$$

After performing partial integration once, we have

$$A_\mu(x) = 2c\sum_\nu \int \frac{d}{ds} \left[\frac{m_{\mu\nu}(t_p) R_\nu}{\sum_\lambda R_\lambda U_\lambda} \right] \delta(s) \, ds . \qquad \text{(IV-2)}$$

This integral can be evaluated immediately giving

$$A_\mu(x) = 2c \sum_\nu \left[\frac{d}{ds} \left\{ \frac{m_{\mu\nu}(t_p)\, R_\nu}{\sum_\lambda R_\lambda u_\lambda} \right\} \right]_{s=0} . \qquad \text{(IV-3)}$$

This can be written in an alternate form by noting

$$\frac{d}{ds} = \frac{d\tau}{ds} \cdot \frac{d}{d\tau} = - \frac{1}{2 \sum_\lambda R_\lambda u_\lambda} \frac{d}{d\tau} ,$$

and hence

$$A_\mu(x) = - c \sum_\nu \left[\frac{1}{\sum_\sigma R_\sigma u_\sigma} \frac{d}{d\tau} \left\{ \frac{m_{\mu\nu}(t_p)\, R_\nu}{\sum_\lambda R_\lambda u_\lambda} \right\} \right]_{s=0} . \qquad \text{(IV-4)}$$

Performing the differentiation, this can be written as

$$A_\mu(x) = -c \sum_\nu \left[\frac{\dot{m}_{\mu\nu} R_\nu + m_{\mu\nu} \dot{R}_\nu}{(\sum_\lambda R_\lambda u_\lambda)^2} - \frac{m_{\mu\nu} R_\nu \{ \sum_\lambda (R_\lambda \dot{u}_\lambda + \dot{R}_\lambda u_\lambda) \}}{(\sum_\lambda R_\lambda u_\lambda)^3} \right]_{s=0} , \qquad \text{(IV-5)}$$

where the dots denote derivatives with respect to τ .

But $\dot{R}_\nu = - u_\nu$, and hence (IV-5) finally becomes

$$A_\mu(x) = -c \sum_\nu \left[\frac{\dot{m}_{\mu\nu} R_\nu - m_{\mu\nu} u_\nu}{(\sum_\lambda R_\lambda u_\lambda)^2} - \frac{m_{\mu\nu} R_\nu \{ \sum_\lambda (R_\lambda \dot{u}_\lambda - u_\lambda^2) \}}{(\sum_\lambda R_\lambda u_\lambda)^3} \right]_{s=0} . \qquad \text{(IV-6)}$$

This general expression gives the potentials due to a moving dipole whose electromagnetic properties are characterized by the moments tensor $m_{\mu\nu}$ given by (III-12). The particular form of this expression depends on the form of the tensor $m_{\mu\nu}$. The individual cases of electric and magnetic dipoles can be deduced from this expression and the case of an electromagnetic dipole may be obtained by superposition. We shall now calculate the potentials of an electric and a magnetic dipole separately.

a) ELECTRIC DIPOLE

In the case of a moving electric dipole ($\vec{m} = 0$, $\vec{p} \neq 0$ in its rest frame), the form of the tensor $m_{\mu\nu}$ in any frame of reference relative to which the dipole has a velocity $\vec{v}$, is given by (III-14). The scalars appearing in the expression for $A_\mu(x)$ in (IV-6) can be easily written in a 3-vector form as

$$\sum_\lambda R_\lambda u_\lambda = -\gamma \ell c \tag{IV-7}$$

$$\sum_\lambda (R_\lambda \dot{u}_\lambda - u_\lambda^2) = c^2 + \gamma^2(\vec{R}\cdot\vec{a}) - \frac{\gamma^4}{c}\ell(\vec{v}\cdot\vec{a}), \tag{IV-8}$$

where $\ell = R(1-\vec{\beta}\cdot\vec{n})$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\vec{\beta} = \frac{\vec{v}}{c}$, $\vec{n} = \frac{\vec{R}}{|\vec{R}|}$

and $\vec{a}$ is the acceleration of the dipole. The constraint $\delta = 0$, when written in a three vector form leads to

$$t_p = t \pm \frac{R}{c}. \tag{IV-9}$$

The positive sign in the above relation corresponds to advanced potentials and the negative sign gives the retarded potentials. The causality conditions require that we retain only the retarded potentials.

Now the 4-vectors appearing in (IV-6) can also be cast into 3-vector form by using (III-14) for $m_{\mu\nu}$. There follows

$$\sum_{\nu} m_{\mu\nu} R_{\nu} = \left[\left\{ \vec{R} \times \left(\gamma \frac{\vec{p} \times \vec{v}}{c} \right) - (\gamma \vec{p}) R \right\}, \quad -i\, (\gamma \vec{p}) \cdot \vec{R} \right], \qquad \text{(IV-10)}$$

$$\sum_{\nu} \dot{m}_{\mu\nu} R_{\nu} = \left[\left\{ \vec{R} \times \left(\gamma \frac{\vec{p} \times \vec{v}}{c} \right)^{\bullet} - (\gamma \vec{p})^{\bullet} R \right\}, \quad -i (\gamma \vec{p})^{\bullet} \cdot \vec{R} \right], \qquad \text{(IV-11)}$$

$$\sum_{\nu} m_{\mu\nu} u_{\nu} = \left[\left\{ \vec{v} \times \left(\gamma \frac{\vec{p} \times \vec{v}}{c} \right) - \gamma^2 \vec{p} c \right\}, \quad -i\, \gamma^2 \vec{p} \cdot \vec{v} \right], \qquad \text{(IV-12)}$$

where the dots denote derivatives with respect to τ. These derivatives can be changed to derivatives with respect to t_p using

$$\frac{d}{d\tau} = \frac{dt_p}{d\tau} \frac{d}{dt_p} = \gamma \frac{d}{dt_p},$$

and hence

$$\left(\gamma \frac{\vec{p} \times \vec{v}}{c} \right)^{\bullet} = \frac{\gamma}{c} \left[\gamma^3 \frac{\vec{a} \cdot \vec{v}}{c^2} (\vec{p} \times \vec{v}) + \gamma (\dot{\vec{p}} \times \vec{v}) + \gamma (\vec{p} \times \vec{a}) \right], \qquad \text{(IV-13)}$$

$$(\gamma\vec{p})^{\cdot} = \gamma^4\left(\frac{\vec{a}\cdot\vec{v}}{c^2}\right)\vec{p} + \gamma^2\dot{\vec{p}}\,, \tag{IV-14}$$

where the dots on the right hand side of (IV-13) and (IV-14) represent derivatives with respect to t_p . Using the expressions (IV-7-14) in (IV-6), the potentials $A_\mu(x)$ can be written in an equivalent 3-vector form as

$$\vec{A}(\vec{x},t) = \Bigg[\left(\frac{1}{\gamma^2 l^2 c} + \frac{\vec{R}\cdot\vec{a}}{l^3c^3}\right)(\vec{p}\times\vec{v})\times\vec{R} + \left(\frac{R}{\gamma^2 l^3} - \frac{1}{l^2}\right.$$

$$\left. + \frac{(\vec{R}\cdot\vec{a})R}{l^3c^2}\right)\vec{p} + \Big\{(\vec{p}\times\vec{a})\times\vec{R} + (\dot{\vec{p}}\times\vec{v})\times\vec{R}$$

$$- (\vec{p}\times\vec{v})\times\vec{v} + Rc\dot{\vec{p}}\Big\}\frac{1}{l^2c^2}\Bigg]_{t_p = t - \frac{R}{c}}, \tag{IV-15}$$

and

$$\Phi(\vec{x},t) = \left[\left(\frac{1}{\gamma^2 l^3} + \frac{\vec{R}\cdot\vec{a}}{l^3c^2}\right)(\vec{p}\cdot\vec{R}) + \frac{1}{l^2c}\left(\dot{\vec{p}}\cdot\vec{R} - \vec{p}\cdot\vec{v}\right)\right]_{t_p = t - \frac{R}{c}}. \tag{IV-16}$$

Expressions (IV-15) and (IV-16) give the vector and scalar potentials due to a moving electric dipole with velocity $\vec{v}$ and acceleration $\vec{a}$, and also take into account the possible rotations or oscillations of the dipole during its motion.

b) MAGNETIC DIPOLE

In the case of a magnetic dipole ($\vec{p} = 0$, $\vec{m} \neq 0$ in its rest frame) the form of the tensor $m_{\mu\nu}$ in the rest frame of the particle is given by (III-15), its form in any frame relative to which the particle has velocity $\vec{V}$, being given by (III-16), so that the 4-vectors appearing in (IV-6) take the form given by

$$\sum_{\nu} m_{\mu\nu} R_{\nu} = \left[\left\{ \vec{R} \times (\gamma \vec{m}) + \left(\gamma \frac{\vec{m} \times \vec{V}}{c}\right) R \right\}, \; i\gamma\left(\frac{\vec{m} \times \vec{V}}{c}\right) \cdot \vec{R} \right], \quad \text{(IV-17)}$$

$$\sum_{\nu} \dot{m}_{\mu\nu} R_{\nu} = \left[\left\{ \vec{R} \times (\gamma \vec{m})^{\cdot} + \left(\gamma \frac{\vec{m} \times \vec{V}}{c}\right)^{\cdot} R \right\}, \; i\left(\gamma \frac{\vec{m} \times \vec{V}}{c}\right)^{\cdot} \cdot \vec{R} \right], \quad \text{(IV-18)}$$

$$\sum_{\nu} m_{\mu\nu} u_{\nu} = \left[\vec{0} \,, \; i\frac{\gamma^2}{c}(\vec{m} \times \vec{V}) \cdot \vec{V} \right] = \left[\vec{0} \,, \; i\,0 \right], \quad \text{(IV-19)}$$

whereas the scalars remain the same as in case of an electric dipole. The dots here represent the derivatives with respect to τ, and can be changed to derivatives with respect to t_p as was done in the case of the electric dipole. Using the above 3-vector forms, and changing the derivatives with respect to τ to derivatives with respect to t_p , the general expression (IV-6) in the case of a moving magnetic dipole gives

$$\vec{A}(\vec{x},t) = \left[\vec{m} \times \left\{ \frac{\vec{R}}{\gamma^2 \ell^3} + \frac{R\vec{a}}{\ell^2 c^2} - \frac{R\vec{v}}{\gamma^2 \ell^3 c} + \frac{(\vec{R}\cdot\vec{a})\vec{R}}{\ell^3 c^2} - \frac{R(\vec{R}\cdot\vec{a})\vec{v}}{\ell^3 c^3} \right\} + \dot{\vec{m}} \times \left\{ \frac{\vec{R}}{\ell^2 c} - \frac{R\vec{v}}{\ell^2 c^2} \right\} \right]_{t_p = t - \frac{R}{c}}, \quad \text{(IV-20)}$$

$$\phi(\vec{x},t) = \left[\left\{ \left(\frac{\vec{v}}{\gamma^2 \ell^3 c} + \frac{\vec{v}(\vec{R}\cdot\vec{a})}{\ell^3 c^3} + \frac{\vec{a}}{\ell^2 c^2} \right) \times \vec{m} + \frac{\vec{v} \times \dot{\vec{m}}}{\ell^2 c^2} \right\} \cdot \vec{R} \right]_{t_p = t - \frac{R}{c}} . \quad \text{(IV-21)}$$

These expressions represent the vector and scalar potentials due to a moving magnetic dipole. These potentials can be used to derive the electric and magnetic fields. We shall now calculate the electric and magnetic fields due to a moving electric dipole using the potentials (IV-15) and (IV-16) in equations (I-5) and (I-6).

CHAPTER V

THE ELECTROMAGNETIC FIELD

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ due to an electromagnetic dipole in motion can be calculated either by calculating the electromagnetic field tensor $F_{\mu\nu}$ directly using (I-14) as was done in the case of a moving point charge in Chapter II, or by using the appropriate potentials in equations (I-5) and (I-6). The latter method will be followed here to calculate these fields for the case of a moving dipole. The electric and magnetic fields due to a moving electric dipole are derived here in detail using the potentials (IV-15) and (IV-16) in equations (I-5) and (I-6). The fields due to a moving magnetic dipole can then be obtained from the dual of the field tensor corresponding to the case of the electric dipole and those of an electromagnetic dipole in motion can be obtained by superposition. Hence we only need to calculate the fields of a moving electric dipole. The prescribed procedure to obtain fields from the scalar and vector potentials is rendered somewhat complicated because of the constraint (IV-9) imposed on the potentials. This requires that the time t is involved in the potentials only through this constraint. Thus in general these potentials are functions of the form $\psi(x, y, z, t_p)$ and since t_p may be regarded a function of x, y, z, t, we have two types of partial derivatives of ψ to consider. In what follows, we shall use the following notation:

$\frac{d\psi}{dx}$ = derivative with respect to x when x contained implicitly in t_p as well as x appearing explicitly is varied, y, z, t being kept constant;

$\frac{\partial\psi}{\partial x}$ = derivative with respect to x when y, z, t_p are held constants and only the explicit x is varied;

$\frac{\partial\psi}{\partial t_p}$ = derivative of ψ with respect to t_p when explicit x, y, z are kept constant;

$\frac{d\psi}{dt}$ = derivative with respect to the t contained implicitly in t_p when both explicit and implicit x, y, z are kept constant.

With these definitions, we have the following results

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t_p} \cdot \frac{dt_p}{dt} , \qquad \text{(V-1)}$$

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial t_p} \frac{dt_p}{dx} , \qquad \text{(V-2)}$$

and similarly for derivatives with respect to y and z. Using these definitions and results, equations (I-5) and (I-6) can be rewritten as

$$\vec{E}(\vec{x},t) = -\frac{R}{c\ell}\frac{\partial \vec{A}}{\partial t_p} - \nabla\phi + \frac{\vec{R}}{c\ell}\frac{\partial \phi}{\partial t_p}, \qquad \text{(V-3)}$$

$$\vec{B}(\vec{x},t) = \nabla\times\vec{A} - \frac{\vec{R}}{c\ell}\times\frac{\partial \vec{A}}{\partial t_p}, \qquad \text{(V-4)}$$

where $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$, and hence the derivatives are all partial derivatives and can be evaluated using the expressions for $\vec{A}$ and $\vec{\phi}$.

In the case of a moving electric dipole, the vector and scalar potentials are given by (IV-15) and (IV-16) which can be rewritten as

$$\vec{A}(\vec{x},t) = \left[\left\{\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3 c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^3} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2 c^2} - \frac{\vec{p}\cdot\vec{v}}{\ell^2 c^2}\right\}\vec{v} + \frac{\vec{p}\cdot\vec{R}}{\ell^2 c^2}\vec{a} + \frac{\dot{\vec{p}}}{\ell c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-5)}$$

$$\phi(\vec{x},t) = \left[\left\{\frac{1}{\gamma^2\ell^3} + \frac{\vec{R}\cdot\vec{a}}{\ell^3 c^2}\right\}\vec{p}\cdot\vec{R} + \left\{\dot{\vec{p}}\cdot\vec{R} - \vec{p}\cdot\vec{v}\right\}\frac{1}{\ell^2 c}\right]_{t_p = t - \frac{R}{c}}. \qquad \text{(V-6)}$$

Using these expressions, the partial derivatives appearing in (V-3) and (V-4) may be evaluated. We first calculate $\frac{\partial \vec{A}}{\partial t_p}$ and get after simplification

$$\frac{\partial \vec{A}}{\partial t_p} = \Bigg[\Bigg\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^4 c^2} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^3 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 R c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^4}$$

$$- \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^3 R} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3 c} - \frac{2(\ddot{\vec{p}}\cdot\vec{R})}{\ell^2 R c} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^3} + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^3 c^3}$$

$$+ \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^3 c^3} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^4 c^4} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^2 c^2} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^3 c} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^2 R c}\Bigg\}\vec{v}$$

$$+ \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^3 c} - \frac{2(\vec{p}\cdot\vec{R})}{\ell^2 c R} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^3} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^2 c^2}$$

$$- \frac{2(\vec{p}\cdot\vec{v})}{\ell^2 c^2}\Bigg\}\vec{a} + \frac{(\vec{p}\cdot\vec{R})}{\ell^2 c^2}\dot{\vec{a}} + \frac{\ddot{\vec{p}}}{\ell c} + \frac{\dot{\vec{p}}}{\gamma^2 \ell^2} - \frac{\dddot{\vec{p}}}{R\ell}$$

$$+ \frac{\dot{(\vec{R}\cdot\vec{a})}}{\ell^2 c^2}\dot{\vec{p}}\Bigg]_{t_p = t - \frac{R}{c}} . \qquad \text{(V-7)}$$

Similarly $\frac{\partial \phi}{\partial t_p}$ gives

$$\frac{\partial \phi}{\partial t_p} = \Bigg[\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^4 c} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 R c} + \frac{3(\vec{p}\cdot\vec{R})c}{\gamma^4 \ell^4} - \frac{3(\vec{p}\cdot\vec{R})c}{\gamma^2 \ell^3 R}$$

$$- \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{v})}{\ell^4 c^2} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^3 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^4 c^3} + \frac{2(\vec{p}\cdot\vec{v})}{\ell^2 R}$$

$$- \frac{3(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^3} - \frac{3(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^3 c^2} + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2 \ell^3} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^2}$$

$$- \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^2 R} - \frac{2(\dot{\vec{p}}\cdot\vec{v})}{\ell^2 c} - \frac{\vec{p}\cdot\vec{a}}{\ell^2 c} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^2 c}\Bigg]_{t_p = t - \frac{R}{c}} . \qquad \text{(V-8)}$$

The evaluation of $\nabla\phi$ and $\nabla \times \vec{A}$ is simple because the derivatives involved are all partial derivatives, and hence $\vec{p}$, $\dot{\vec{p}}$, $\vec{a}$, $\dot{\vec{a}}$, and $\vec{v}$ are all constants with respect to these operations. Therefore we get

$$\nabla\phi = \Bigg[\left\{\frac{(\vec{p}\cdot\nabla)}{\gamma^2\ell^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{a}\cdot\nabla)}{\ell^3c^2} + \frac{(\vec{R}\cdot\vec{a})(\vec{p}\cdot\nabla)}{\ell^3c^2} + \frac{(\dot{\vec{p}}\cdot\nabla)}{\ell^2c}\right\}\vec{R}$$

$$-\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2\ell^4} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4c^2} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3c} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3c}\right\}\nabla\ell\Bigg]_{t_p=t-\frac{R}{c}}$$

which simplifies to

$$\nabla\phi = \Bigg[\left\{\frac{1}{\gamma^2\ell^3} + \frac{\vec{R}\cdot\vec{a}}{\ell^3c^2}\right\}\vec{p} + \frac{(\vec{p}\cdot\vec{R})}{\ell^3c^2}\vec{a} + \frac{\dot{\vec{p}}}{\ell^2c} - \Bigg\{\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4c^2}$$

$$+\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2\ell^4} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3c} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3c}\Bigg\}\left(\frac{\vec{R}}{R} - \frac{\vec{v}}{c}\right)\Bigg]_{t_p=t-\frac{R}{c}} \qquad \text{(V-9)}$$

and in the same manner we get

$$\nabla\times\vec{A} = \Bigg[\Bigg\{\frac{3(\vec{p}\cdot\vec{R})\vec{v}}{\gamma^2\ell^4Rc} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4Rc^3}\vec{v} + \frac{2(\dot{\vec{p}}\cdot\vec{R})\vec{v}}{\ell^3Rc^2} - \frac{2(\vec{p}\cdot\vec{v})\vec{v}}{\ell^3Rc^2} + \frac{\dot{\vec{p}}}{\ell^2Rc}$$

$$+\frac{2(\vec{p}\cdot\vec{R})\vec{a}}{\ell^3Rc^2}\Bigg\}\times\vec{R} + \left\{\frac{\vec{p}}{\gamma^2\ell^3c} + \frac{(\vec{R}\cdot\vec{a})\vec{p}}{\ell^3c^3} - \frac{(\vec{p}\cdot\vec{R})\vec{a}}{\ell^3c^3}\right\}\times\vec{v} - \frac{\vec{p}\times\vec{a}}{\ell^2c^2}\Bigg]_{t_p=t-\frac{R}{c}} \qquad \text{(V-10)}$$

Substituting expressions (V-7), (V-8), (V-9) and (V-10) in (V-3) and (V-4) we obtain the fields, namely

$$
\begin{aligned}
\vec{E}(\vec{x},t) = \Bigg[\Bigg\{ & \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^4 c^3} \\
& - \frac{3(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^4} - \frac{3(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^3} - \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^2} \\
& + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2 \ell^4 c} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^3} - \frac{2(\dot{\vec{p}}\cdot\vec{v})}{\ell^3 c^2} + \frac{(\ddot{\vec{p}}\cdot\vec{R})}{\ell^3 c^2} \Bigg\} \vec{R} \\
& - \Bigg\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\ddot{\vec{a}})}{\ell^4 c^4} \\
& + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2} + \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^3} \\
& + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2} - \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 R c^2} + \frac{(\ddot{\vec{p}}\cdot R)}{\ell^3 c^3} \Bigg\} R\vec{v} \\
& - \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} - \frac{(\vec{p}\cdot\vec{R})}{\ell^3 R c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3 c^3} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3} \Bigg\} R\vec{a} \\
& - \Bigg\{ \frac{1}{\gamma^2 \ell^3} + \frac{(\vec{R}\cdot\vec{a})}{\ell^3 c^2} \Bigg\} \vec{p} - \Bigg\{ \frac{R}{\gamma^2 \ell^3 c} + \frac{(\vec{R}\cdot\vec{a})R}{\ell^3 c^3} \Bigg\} \dot{\vec{p}} - \frac{(\vec{p}\cdot\vec{R})R\ddot{\vec{a}}}{\ell^3 c^3} - \frac{R\ddot{\vec{p}}}{\ell^2 c^2} \Bigg]_{t_p = t - \frac{R}{c}} ,
\end{aligned}
\tag{V-11}
$$

and

$$
\begin{aligned}
\vec{B}(\vec{x},t) = \Bigg[\Bigg\{ & \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2} \\
& + \frac{(\dot{\vec{p}}\cdot R)(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^3} \\
& + \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^3} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2} - \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 R c^2} \Bigg\} \vec{v}\times\vec{R} + \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} \\
& + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3 c^3} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3} \Bigg\} \vec{a}\times\vec{R} + \frac{(\vec{p}\cdot\vec{R})\dot{\vec{a}}\times\vec{R}}{\ell^3 c^3} + \frac{\ddot{\vec{p}}\times\vec{R}}{\ell^2 c^2} + \frac{\dot{\vec{p}}\times\vec{R}}{\gamma^2 \ell^3 c} + \frac{\vec{p}\times\vec{v}}{\gamma^2 \ell^3 c} \\
& - \frac{(\vec{p}\cdot\vec{R})(\vec{a}\times\vec{v})}{\ell^3 c^3} + \frac{(\vec{R}\cdot\vec{a})(\dot{\vec{p}}\times\vec{R})}{\ell^3 c^3} + \frac{(\vec{R}\cdot\vec{a})(\vec{p}\times\vec{v})}{\ell^3 c^3} + \frac{\vec{p}\times\vec{a}}{\ell^2 c^2} \Bigg]_{t_p = t - \frac{R}{c}} .
\end{aligned} \qquad \text{(V-12)}
$$

Expressions (V-11) and (V-12) represent the most general expressions for fields due to a moving electric dipole. It is apparent that these fields are not everywhere perpendicular to each other in contrast to the fields of a moving charge.

The radiation of electromagnetic waves is accompanied by radiation of energy. The energy flux is given by the Poynting vector $\vec{S}$ which is defined by

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) \qquad \text{(V-13)}$$

and represents the amount of field energy through unit area in unit time, assuming that at the given moment there are no charges on the surface itself. It must be remembered, however, that the above definition of Poynting vector is not a mandatory one. Since this vector is introduced in the electromagnetism only by way of its divergence, the curl of an arbitrary vector can be added to it without altering the physical facts of the case. However, this definition is adopted because of convenience, particularly in the electromagnetic theory of light. The total energy radiated can be calculated by integrating the normal component of $\vec{S}$ over a surface of a sphere of infinite radius surrounding the sources, and hence for this purpose we need to use only far-fields. For large values of $|\vec{R}|$, only terms proportional to $\frac{1}{R}$ are significant, as the other terms in the fields do not contribute to the surface integral. Thus the far-fields can be obtained from the general expressions (V-10) and (V-11) by retaining the terms proportional to $\frac{1}{R}$, that is

$$\vec{E}_f(\vec{x},t) = \Bigg[\left\{ \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{R^4 c^3} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5 c^4} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4 c^3} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{R^3 c^2} \right\} \vec{R}$$

$$- \left\{ \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{R^3 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^4 c^5} + \frac{(\dot{\vec{p}}\cdot R)(\vec{R}\cdot\vec{a})}{R^3 c^4} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{R^2 c^3} \right\} \vec{v}$$

$$- \left\{ \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^3 c^4} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{R^2 c^3} \right\} \vec{a} - \frac{(\vec{p}\cdot\vec{R})}{R^2 c^3} \dot{\vec{a}} - \frac{(\vec{R}\cdot\vec{a})\dot{\vec{p}}}{R^2 c^3} - \frac{\ddot{\vec{p}}}{R c^2} \Bigg]_{t_p = t - \frac{R}{c}}, \quad \text{(V-14)}$$

and

$$\vec{B}_f(\vec{x},t) = \Bigg[\left\{ \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{R^4c^4} + \frac{(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{R^4c^4} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5} + \frac{\dddot{\vec{p}}\cdot\vec{R}}{R^3c^3} \right\} \vec{v}\times\vec{R}$$

$$+ \left\{ \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4} + \frac{2(\ddot{\vec{p}}\cdot\vec{R})}{R^3c^3} \right\} \vec{a}\times\vec{R} + \frac{(\dot{\vec{p}}\cdot\vec{R})}{R^3c^3}\,\dot{\vec{a}}\times\vec{R}$$

$$+ \frac{\dddot{\vec{p}}\times\vec{R}}{R^2c^2} + \frac{(\vec{R}\cdot\vec{a})(\ddot{\vec{p}}\times R)}{R^3c^3} \Bigg]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-15)}$$

where we have used the fact that for large values of $|\vec{R}|$, $l \to R$. From the expressions (V-14) and (V-15) it is obvious that

$$\vec{B}_f = -\frac{\vec{E}_f \times \vec{R}}{|\vec{R}|} = \vec{n}\times\vec{E}_f, \qquad \text{(V-16)}$$

i.e. in the radiation zone, the fields $\vec{E}$ and $\vec{B}$ are perpendicular to each other and to the direction of propagation. Using (V-16), the Poynting vector can be written

$$\vec{S} = \frac{c}{4\pi}\left|\vec{E}_f\right|^2 \vec{n}. \qquad \text{(V-17)}$$

Since $\vec{E}_f$ is proportional to $\frac{1}{R}$, $\vec{S}$ is proportional to $\frac{1}{R^2}$, is in the radial direction, and falls off as $\frac{1}{R^2}$, in agreement with the inverse square law. Its explicit form is

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\Bigg[\bigg[\bigg\{\frac{(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\ddot{\vec{a}})^2}{R^6c^6} + \frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^4}{R^8c^8} + \frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4c^4}$$

$$+ \frac{6(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2(\vec{R}\cdot\ddot{\vec{a}})}{R^7c^7} + \frac{6(\ddot{\vec{p}}\cdot\vec{R})(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6c^6}\bigg\}\left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc}\right)$$

$$+ \bigg\{\frac{2(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5} + \frac{2(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})(\vec{R}\cdot\dot{\vec{a}})}{R^6c^6}\bigg\}\left(\frac{v^2}{c^2} - 3 - \frac{4(\vec{R}\cdot\vec{v})}{Rc}\right)$$

$$+ \frac{(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6c^6}\bigg\{\frac{v^2}{c^2} - 9 - \frac{6(\vec{R}\cdot\vec{v})}{Rc}\bigg\} + \frac{a^2}{c^2}\bigg\{\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6c^6} + \frac{4(\dot{\vec{p}}\cdot\vec{R})^2}{R^4c^4}$$

$$+ \frac{12(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5}\bigg\} + \frac{2(\vec{v}\cdot\vec{a})}{c^2}\bigg\{\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^3}{R^7c^7} + \frac{3(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\dot{\vec{a}})(\vec{R}\cdot\ddot{\vec{a}})}{R^6c^6}$$

$$+ \frac{9(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6c^6} + \frac{3(\vec{p}\cdot\vec{R})(\dddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5} + \frac{2(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\ddot{\vec{a}})}{R^5c^5}$$

$$+ \frac{2(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})}{R^5c^5} + \frac{2(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})}{R^4c^4}\bigg\} + \frac{2(\vec{v}\cdot\dot{\vec{a}})}{c^2}\bigg\{\frac{(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\dot{\vec{a}})}{R^5c^5} + \frac{(\ddot{\vec{p}}\cdot\vec{R})(\vec{p}\cdot\vec{R})}{R^4c^4}$$

$$+\frac{3(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6c^6}+\frac{(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5}\Bigg\}+\frac{2(\dot{\vec{p}}\cdot\vec{v})}{c^2}\Bigg\{\frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})(\vec{R}\cdot\dot{\vec{a}})}{R^5c^5}$$

$$+\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^3}{R^6c^6}+\frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5}+\frac{(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\Bigg\}+\frac{2(\ddot{\vec{p}}\cdot\vec{v})}{c^2}\Bigg\{\frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{R^4c^4}$$

$$+\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5}+\frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}+\frac{(\ddot{\vec{p}}\cdot\vec{R})}{R^3c^3}\Bigg\}+\frac{2(\vec{a}\cdot\dot{\vec{a}})}{c^2}\Bigg\{\frac{3(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})}{R^5c^5}$$

$$+\frac{2(\vec{p}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})}{R^4c^4}\Bigg\}+\frac{2(\dot{\vec{p}}\cdot\vec{a})}{c^2}\Bigg\{\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5}+\frac{2(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\Bigg\}+\frac{2(\ddot{\vec{p}}\cdot\vec{a})}{c^2}$$

$$\Bigg\{\frac{2(\dot{\vec{p}}\cdot\vec{R})}{R^3c^3}+\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\Bigg\}+\frac{(\vec{p}\cdot\vec{R})^2(\dot{\vec{a}}\cdot\dot{\vec{a}})}{R^4c^6}+\frac{(\vec{R}\cdot\vec{a})^2(\dot{\vec{p}}\cdot\dot{\vec{p}})}{R^4c^6}$$

$$+\frac{\ddot{\vec{p}}\cdot\ddot{\vec{p}}}{R^2c^4}+\frac{2(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\dot{\vec{a}})(\vec{R}\cdot\vec{a})}{R^4c^6}+\frac{2(\vec{p}\cdot\vec{R})(\ddot{\vec{p}}\cdot\dot{\vec{a}})}{R^3c^5}$$

$$+\frac{(\dot{\vec{p}}\cdot\dddot{\vec{p}})(\vec{R}\cdot\vec{a})}{R^3c^5}\Bigg]\vec{n}\,\Bigg]_{t_p=t-\frac{R}{c}}. \qquad \text{(V-18)}$$

This is the general expression for the Poynting vector $\vec{S}$. Having obtained the general expressions for potentials, fields and the Poynting vector, we can now discuss some special cases of this general problem.

SOME SPECIAL CASES

(1) Constant Electric Dipole at Rest:

The potentials and fields for this particular case can be obtained from the general expressions of the potentials and fields by letting $\ddot{\vec{\alpha}} = \dot{\vec{\alpha}} = \vec{V} = \dot{\vec{p}} = \dddot{\vec{p}} = 0$ and hence we obtain

$$\vec{A}(\vec{x},t) = 0 \,, \tag{V-19}$$

$$\Phi(\vec{x},t) = \frac{\vec{p}\cdot\vec{R}}{R^3} \,, \tag{V-20}$$

$$\vec{E}(\vec{x},t) = \frac{3(\vec{p}\cdot\vec{R})\vec{R}}{R^5} - \frac{\vec{p}}{R^3} \,, \tag{V-21}$$

$$\vec{B}(\vec{x},t) = 0 \,, \tag{V-22}$$

$$\vec{S}(\vec{x},t) = 0 \,. \tag{V-23}$$

These results are in agreement with the results obtained by elementary methods. The Poynting vector is identically zero everywhere, indicating that the constant electric dipole does not radiate energy.

(2) Oscillating Electric Dipole at Rest:

We can obtain the potentials and fields due to an oscillating electric dipole at rest from the general expressions by substituting $\dot{\vec{\alpha}} = \vec{\alpha} = \vec{V} = 0$ which then reduce to

$$\vec{A}(\vec{x},t) = \left[\frac{\dot{\vec{p}}}{Rc} \right]_{t_p = t - \frac{R}{c}} \,, \tag{V-24}$$

$$\Phi(\vec{x},t) = \left[\frac{\vec{p}\cdot\vec{R}}{R^3} + \frac{\dot{\vec{p}}\cdot\vec{R}}{R^2 c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-25)}$$

$$\vec{E}(\vec{x},t) = \left[-\frac{\vec{p}}{R^3} + \frac{3(\vec{p}\cdot\vec{R})}{R^5}\vec{R} - \frac{\dot{\vec{p}}}{R^2 c} + \frac{3(\dot{\vec{p}}\cdot\vec{R})\vec{R}}{R^4 c}\right.$$

$$\left. - \frac{\ddot{\vec{p}}}{R c^2} + \frac{(\ddot{\vec{p}}\cdot\vec{R})\vec{R}}{R^3 c^2}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-26)}$$

$$\vec{B}(\vec{x},t) = \left[\frac{\ddot{\vec{p}}\times\vec{R}}{R^2 c^2} + \frac{\dot{\vec{p}}\times\vec{R}}{R^3 c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-27)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\left[\left\{\frac{\ddot{\vec{p}}\cdot\ddot{\vec{p}}}{R^2 c^4} - \frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4 c^4}\right\}\vec{n}\right]_{t_p = t - \frac{R}{c}}. \qquad \text{(V-28)}$$

These results are also in agreement with other more elementary treatments. The Poynting vector is along the direction of observation, indicating that the energy flows along the radial direction.

(3) Constant Electric Dipole in Uniform Motion with Velocity Parallel to its Axis: $(\vec{p} \parallel \vec{v})$

The potentials and fields for a uniformly moving electric dipole are obtained from the general expressions by letting $\dot{\vec{p}} = \ddot{\vec{p}} = \dot{\vec{\alpha}} = \vec{\alpha} = 0$ and hence we have

$$\vec{A}(\vec{x},t) = \left[\left\{\frac{\vec{p}\cdot\vec{R}}{\gamma^2 \ell^3 c} - \frac{\vec{p}\cdot\vec{V}}{\ell^2 c^2}\right\}\vec{V}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-29)}$$

$$\phi(\vec{x},t) = \left[\frac{\vec{p}\cdot\vec{R}}{\gamma^2 \ell^3} - \frac{\vec{p}\cdot\vec{V}}{\ell^2 c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-30)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} - \frac{3(\vec{p}\cdot\vec{V})}{\gamma^2 \ell^4 c}\right\}\vec{R} + \left\{\frac{4(\vec{p}\cdot\vec{V})}{\ell^3 c^2} - \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^5 c}\right.\right.$$

$$\left.\left. - \frac{2R(\vec{p}\cdot\vec{V})}{\gamma^2 \ell^4 c^2}\right\}\vec{V} - \frac{\vec{p}}{\gamma^2 \ell^3}\right]_{t_p = t - \frac{R}{c}} \qquad \text{(V-31)}$$

$$\vec{B}(\vec{x},t) = \left[\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^5 c} - \frac{4(\vec{p}\cdot\vec{V})}{\ell^3 R c^2} + \frac{2(\vec{p}\cdot\vec{V})}{\gamma^2 \ell^4 c^2}\right\}\vec{V}\times\vec{R}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-32)}$$

$$\vec{S}(\vec{x},t) = 0 . \qquad \text{(V-33)}$$

In this case the Poynting vector vanishes and hence the uniformly moving electric dipole does not radiate energy just like the case of a uniformly moving point charge.

(4) Constant Electric Dipole Moving Uniformly Perpendicular to its Axis: $(\vec{p} \perp \vec{v})$.

In this case the expressions for potentials and fields reduce to

$$\vec{A}(\vec{x},t) = \left[\frac{(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^3 c} \vec{v} \right]_{t_p = t - \frac{R}{c}}, \qquad (V\text{-}34)$$

$$\Phi(\vec{x},t) = \left[\frac{\vec{p}\cdot\vec{R}}{\gamma^2 \ell^3} \right]_{t_p = t - \frac{R}{c}}, \qquad (V\text{-}35)$$

$$\vec{E}(\vec{x},t) = \left[\frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} \vec{R} - \frac{3(\vec{p}\cdot\vec{R})R}{\gamma^2 \ell^5 c} \vec{v} - \frac{\vec{p}}{\gamma^2 \ell^3} \right]_{t_p = t - \frac{R}{c}}, \qquad (V\text{-}36)$$

$$\vec{B}(\vec{x},t) = \left[\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^5 c} \vec{v}\times\vec{R} + \frac{\vec{p}\times\vec{v}}{\gamma^2 \ell^3 c} \right]_{t_p = t - \frac{R}{c}}, \qquad (V\text{-}37)$$

$$\vec{S}(\vec{x},t) = 0 . \qquad (V\text{-}38)$$

Here again the Poynting vector is zero and hence there is no energy radiated.

(5) Oscillating Electric Dipole Moving Uniformly Parallel to its Axis: $(\vec{p} \parallel \vec{v})$

For such a dipole the expressions for potentials and fields reduce to

$$\vec{A}(\vec{x},t) = \left[\left\{\frac{\vec{p}\cdot\vec{R}}{\gamma^2 \ell^3 c} + \frac{(\dot{\vec{p}}\cdot\vec{R})}{\ell^2 c^2} - \frac{\vec{p}\cdot\dot{\vec{V}}}{\ell^2 c^2}\right\}\vec{V} + \frac{\dot{\vec{p}}}{\ell c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-39)}$$

$$\Phi(\vec{x},t) = \left[\frac{(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^3} - \frac{(\vec{p}\cdot\vec{V})}{\ell^2 c} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2 c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-40)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} - \frac{3(\vec{p}\cdot\vec{V})}{\gamma^2 \ell^4 c} + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2 \ell^4 c} - \frac{2(\dot{\vec{p}}\cdot\vec{V})}{\ell^3 c^2} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^2}\right\}\vec{R}\right.$$

$$-\left\{\frac{3(\vec{p}\cdot\vec{R})R}{\gamma^2 \ell^5 c} + \frac{3\gamma^2 R(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2} - \frac{4(\vec{p}\cdot\vec{V})}{\ell^3 c^2} + \frac{2(\vec{p}\cdot\vec{V})R}{\gamma^2 \ell^4 c^2} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^3}R\right\}\vec{V}$$

$$\left. + \frac{\ddot{\vec{p}}R}{\ell^2 c^2} - \frac{\dot{\vec{p}}R}{\gamma^2 \ell^3 c} - \frac{\vec{p}}{\gamma^2 \ell^3}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-41)}$$

$$\vec{B}(\vec{x},t) = \left[\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} - \frac{4(\vec{p}\cdot\vec{V})}{\ell^3 R c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2} + \frac{2(\vec{p}\cdot\vec{V})}{\gamma^2 \ell^4 c^2}\right.\right.$$

$$\left. + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^2}\right\}\vec{V}\times\vec{R} + \frac{\ddot{\vec{p}}\times\vec{R}}{\ell^2 c^2} + \frac{\dot{\vec{p}}\times\vec{R}}{\gamma^2 \ell^3 c}\Bigg]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-42)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\left[\left[\left\{\frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4c^4}\left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc}\right) + \frac{2(\ddot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{v})}{R^3c^5}\right.\right.\right.$$

$$\left.\left. + \frac{\ddot{\vec{p}}\cdot\ddot{\vec{p}}}{R^2c^4}\right\}\vec{n}\right]_{t_p = t - \frac{R}{c}} . \qquad \text{(V-43)}$$

(6) Oscillating Electric Dipole Moving Uniformly Perpendicular to its Axis: $(\vec{p} \perp \vec{v})$

$$\vec{A}(\vec{x},t) = \left[\left\{\frac{(\vec{p}\cdot\vec{R})}{\gamma^2\ell^3c} + \frac{(\dot{\vec{p}}\cdot\vec{R})}{\ell^2c^2}\right\}\vec{v} + \frac{\dot{\vec{p}}}{\ell c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-44)}$$

$$\Phi(\vec{x},t) = \left[\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2c}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-45)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^4\ell^5} + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2\ell^4c} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3c^2}\right\}\vec{R} + \left\{\frac{3R(\vec{p}\cdot\vec{R})}{\gamma^2\ell^5c}\right.\right.$$

$$\left. - \frac{3\gamma^2R(\dot{\vec{p}}\cdot\vec{R})}{\ell^4c^2} - \frac{(\ddot{\vec{p}}\cdot\vec{R})R}{\ell^3c^3}\right\}\vec{v} - \frac{\ddot{\vec{p}}R}{\ell^2c^2} - \frac{\dot{\vec{p}}R}{\gamma^2\ell^3c}$$

$$\left. - \frac{\vec{p}}{\gamma^2\ell^3}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-46)}$$

$$\vec{B}(\vec{x},t) = \Bigg[\left\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^2} \right\} \vec{v}\times\vec{R}$$

$$+ \frac{\ddot{\vec{p}}\times\vec{R}}{\ell^2 c^2} + \frac{\dot{\vec{p}}\times\vec{R}}{\gamma^2 \ell^3 c} + \frac{\vec{p}\times\vec{v}}{\gamma^2 \ell^3 c} \Bigg]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-47)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi} \Bigg[\vec{n} \left\{ \frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4 c^4} \left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc} \right) \right.$$

$$+ \frac{\ddot{\vec{p}}\cdot\ddot{\vec{p}}}{R^2 c^4} \Bigg]_{t_p = t - \frac{R}{c}}. \qquad \text{(V-48)}$$

(7) Constant Electric Dipole Moving with Uniform Acceleration: $(\vec{p} \parallel \vec{v})$

$$\vec{A}(\vec{x},t) = \left[\left\{ \frac{\vec{p}\cdot\vec{R}}{\gamma^2 \ell^3 c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^3} - \frac{\vec{p}\cdot\vec{v}}{\ell^2 c^2} \right\} \vec{v} + \frac{\vec{p}\cdot\vec{R}}{\ell^2 c^2} \vec{a} \right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-49)}$$

$$\phi(\vec{x},t) = \left[\left\{ \frac{1}{\gamma^2 \ell^3} + \frac{\vec{R}\cdot\vec{a}}{\ell^3 c^2} \right\} (\vec{p}\cdot\vec{R}) - \frac{\vec{p}\cdot\vec{v}}{\ell^2 c} \right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-50)}$$

$$
\begin{aligned}
\vec{E}(\vec{x},t) = \Bigg[\Bigg\{ & \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^3} \\
& - \frac{3(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^4} - \frac{3(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^3} - \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^2} \Bigg\} \vec{R} \\
& - \Bigg\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} \\
& + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2} - \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 c^2 R} + \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^3} \Bigg\} R\vec{v} \\
& - \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} - \frac{\vec{p}\cdot\vec{R}}{\ell^3 R c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3} \Bigg\} R\vec{a} \\
& - \Bigg\{ \frac{1}{\gamma^2 \ell^3} + \frac{\vec{R}\cdot\vec{a}}{\ell^3 c^2} \Bigg\} \vec{p} \; \Bigg]_{t_p = t - \frac{R}{c}} ,
\end{aligned}
\qquad \text{(V-51)}
$$

$$
\begin{aligned}
\vec{B}(\vec{x},t) = \Bigg[\Bigg\{ & \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} \\
& + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2} \\
& + \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^3} - \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 R c^2} \Bigg\} \vec{v}\times\vec{R} + \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} \\
& + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3} \Bigg\} \vec{a}\times\vec{R} \; \Bigg]_{t_p = t - \frac{R}{c}} ,
\end{aligned}
\qquad \text{(V-52)}
$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\left[\vec{n}\left\{\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2\,\vec{a}\cdot\vec{a}}{R^6c^8} + \frac{18(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^3(\vec{v}\cdot\vec{a})}{R^7c^9}\right.\right.$$

$$\left.\left.+\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^4}{R^8c^8}\left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc}\right)\right\}\right]_{t_p=t-\frac{R}{c}} . \qquad \text{(V-53)}$$

(8) Constant Electric Dipole Moving with Uniform Acceleration: $(\vec{p} \perp \vec{v})$

$$\vec{A}(\vec{x},t) = \left[\left\{\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3c^3}\right\}\vec{v} + \frac{\vec{p}\cdot\vec{R}}{\ell^2c^2}\vec{a}\right]_{t_p=t-\frac{R}{c}} , \qquad \text{(V-54)}$$

$$\Phi(\vec{x},t) = \left[\left\{\frac{1}{\gamma^2\ell^3} + \frac{\vec{R}\cdot\vec{a}}{\ell^3c^2}\right\}(\vec{p}\cdot\vec{R})\right]_{t_p=t-\frac{R}{c}} , \qquad \text{(V-55)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2\ell^5c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4\ell^5} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4c^3}\right.\right.$$

$$\left. + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5c^4}\right\}\vec{R} - \left\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2\ell^5c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{}\right.$$

$$\left. + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4\ell^5c} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5c^5}\right\}R\vec{v} - \left\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2\ell^4c^2} - \frac{\vec{p}\cdot\vec{R}}{\ell^3Rc^2}\right.$$

$$\left.\left. + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4c^4}\right\}R\vec{a} - \left\{\frac{1}{\gamma^2\ell^3} - \frac{\vec{R}\cdot\vec{a}}{\ell^3c^2}\right\}\vec{p}\right]_{t_p=t-\frac{R}{c}} , \qquad \text{(V-56)}$$

$$\vec{B}(\vec{x},t) = \left[\left\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 l^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{V}\cdot\vec{a})}{c^4 l^4} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 l^5 c} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{l^5 c^5} \right\} \vec{V}\times\vec{R} \right.$$

$$\left. + \left\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 l^4 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^4 c^4} \right\} \vec{a}\times R + \frac{\vec{p}\times\vec{V}}{\gamma^2 l^3 c} - \frac{(\vec{R}\cdot\vec{a})(\vec{p}\times\vec{V})}{l^3 c^3} \right]_{t_p = t - \frac{R}{c}}, \quad \text{(V-57)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi} \left[\vec{n} \left\{ \frac{9a^2(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6 c^8} + \frac{2(\vec{V}\cdot\vec{a})}{c^2} \left(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^3}{R^7 c^7} \right) \right. \right.$$

$$\left. \left. + \frac{9(\vec{p}\cdot\vec{R})^2 (\vec{R}\cdot\vec{a})^4}{R^8 c^8} \left(\frac{v^2}{c^2} - 1 - \frac{2\vec{R}\cdot\vec{V}}{Rc} \right) \right\} \right]_{t_p = t - \frac{R}{c}}. \quad \text{(V-58)}$$

(9) Oscillating Electric Dipole Moving with Uniform Acceleration: $(\vec{p} \parallel \vec{v})$

$$\vec{A}(\vec{x},t) = \left[\left\{ \frac{(\vec{p}\cdot\vec{R})}{\gamma^2 l^3 c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^3 c^3} + \frac{(\dot{\vec{p}}\cdot\vec{R})}{l^2 c^2} - \frac{(\vec{p}\cdot\vec{v})}{l^2 c^2} \right\} \vec{v} \right.$$

$$\left. + \frac{(\vec{p}\cdot\vec{R})}{l^2 c^2} \vec{a} + \frac{\dot{\vec{p}}}{lc} \right]_{t_p = t - \frac{R}{c}}, \quad \text{(V-59)}$$

$$\Phi(\vec{x},t) = \left[\frac{(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3 c^2} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2 c} - \frac{\vec{p}\cdot\vec{v}}{\ell^2 c} \right]_{t_p = t - \frac{R}{c}} , \qquad \text{(V-60)}$$

$$\vec{E}(\vec{x},t) = \Bigg[\Bigg\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^3} - \frac{3(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c}$$

$$+ \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^4} - \frac{3(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^3} + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2 \ell^4 c} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^3}$$

$$- \frac{2(\dot{\vec{p}}\cdot\vec{v})}{\ell^3 c^2} - \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^2} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^2} \Bigg\} \vec{R} - \Bigg\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3}$$

$$- \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} + \frac{3\gamma(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2} + \frac{\vec{p}\cdot\vec{a}}{\ell^3 c^3}$$

$$+ \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2} - \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 R c^2} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^3} \Bigg\} R\vec{v}$$

$$- \Bigg\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} - \frac{\vec{p}\cdot\vec{R}}{\ell^3 c^2 R} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3 c^3} - \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3} \Bigg\} R\vec{a}$$

$$- \frac{\vec{p}}{\gamma^2 \ell^3} - \frac{(\vec{R}\cdot\vec{a})\vec{p}}{\ell^3 c^2} - \frac{R\dot{\vec{p}}}{\gamma^2 \ell^3 c} - \frac{(\vec{R}\cdot\vec{a})R\dot{\vec{p}}}{\ell^3 c^3} - \frac{R\dddot{\vec{p}}}{\ell^2 c^2} \Bigg]_{t_p = t - \frac{R}{c}} , \qquad \text{(V-61)}$$

$$\vec{B}(\vec{x},t) = \Bigg[\Bigg\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} - \frac{3(\vec{p}\cdot\vec{R})(\vec{v}\cdot\vec{a})}{\ell^4 c^4} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{\ell^4 c^2}$$

$$+ \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{(\vec{p}\cdot\vec{v})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{\ell^3 c^3} + \frac{2(\vec{p}\cdot\vec{v})}{\gamma^2 \ell^4 c^2}$$

$$- \frac{4(\vec{p}\cdot\vec{v})}{\ell^3 R c^2}\Bigg\}\vec{v}\times\vec{R} + \Bigg\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{\ell^3 c^3}$$

$$- \frac{2(\vec{p}\cdot\vec{v})}{\ell^3 c^3}\Bigg\}\vec{a}\times\vec{R} + \frac{\ddot{\vec{p}}\times\vec{R}}{\ell^2 c^2} + \frac{\dot{\vec{p}}\times\vec{R}}{\gamma^2 \ell^3 c} + \frac{(\vec{R}\cdot\vec{a})(\dot{\vec{p}}\times\vec{R})}{\ell^3 c^3}\Bigg]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-62)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\Bigg[\vec{n}\Bigg\{\Bigg(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^4}{R^8 c^8} + \frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4 c^4} + \frac{6(\vec{p}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6 c^6}\Bigg)\Bigg(\frac{v^2}{c^2} - 1$$

$$- \frac{2(\vec{R}\cdot\vec{v})}{Rc}\Bigg) + \frac{2(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5 c^5}\Bigg(\frac{v^2}{c^2} - 3 - \frac{4\vec{R}\cdot\vec{v}}{Rc}\Bigg) + \frac{(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6 c^6}$$

$$\Bigg(\frac{v^2}{c^2} - 9 - \frac{6(\vec{R}\cdot\vec{v})}{Rc}\Bigg) + \Bigg(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6 c^6} + \frac{4(\dot{\vec{p}}\cdot\vec{R})^2}{R^4 c^4} + \frac{12(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5 c^5}\Bigg)\frac{a^2}{c^2}$$

$$+ \frac{2(\vec{a}\cdot\vec{v})}{c^2}\Bigg(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^3}{R^7 c^7} + \frac{3(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6 c^6} + \frac{3(\vec{p}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5 c^5}$$

$$+ \frac{6(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6 c^6} + \frac{2(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})}{R^5 c^5} + \frac{2(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})}{R^4 c^4}\Bigg)$$

$$+ \frac{2(\dot{\vec{p}}\cdot\vec{v})}{c^2}\Bigg(\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^3}{R^6 c^6} + \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5 c^5} + \frac{(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4 c^4}\Bigg) + \frac{2(\ddot{\vec{p}}\cdot\vec{v})}{c^2}$$

$$\left(\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5} + \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{R^3c^3}\right) + \frac{2(\dot{\vec{p}}\cdot\vec{a})}{c^2}\left(\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^5}\right.$$

$$\left. + \frac{2(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\right) + \frac{2(\dddot{\vec{p}}\cdot\vec{a})}{c^2}\left(\frac{2(\dot{\vec{p}}\cdot\vec{R})}{R^3c^3} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\right) + \frac{(\vec{R}\cdot\vec{a})^2(\dot{\vec{p}}\cdot\dot{\vec{p}})}{R^4c^6}$$

$$\left. \left. + \frac{\ddot{\vec{p}}\cdot\ddot{\vec{p}}}{R^2c^4} + \frac{(\dot{\vec{p}}\cdot\dddot{\vec{p}})(\vec{R}\cdot\vec{a})}{R^3c^5}\right\}\right]_{t_p=t-\frac{R}{c}} . \qquad \text{(V-63)}$$

(10) Oscillating Electric Dipole Moving with Uniform Acceleration:

$$\vec{A}(\vec{x},t) = \left[\left(\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3c^3} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2c^2}\right)\vec{v} + \frac{(\vec{p}\cdot\vec{R})\vec{a}}{\ell^2c^2} + \frac{\dot{\vec{p}}}{\ell c}\right]_{t_p=t-\frac{R}{c}}, \qquad \text{(V-64)}$$

$$\Phi(\vec{x},t) = \left[\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3c^2} + \frac{\dot{\vec{p}}\cdot\vec{R}}{\ell^2c}\right]_{t_p=t-\frac{R}{c}}, \qquad \text{(V-65)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2\ell^5c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4\ell^5} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5c^4} + \frac{3(\dot{\vec{p}}\cdot\vec{R})}{\gamma^2\ell^4c}\right.\right.$$

$$\left. + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4c^3} + \frac{(\ddot{\vec{p}}\cdot\vec{R})}{\ell^3c^2}\right\}\vec{R} - \left\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2\ell^5c^3} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5c^5}\right.$$

$$+ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 l^5 c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{l^4 c^2} + \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^4 c^4} + \frac{(\dddot{\vec{p}}\cdot\vec{R})}{l^3 c^3}\Bigg\} R\vec{v} - \Bigg\{\frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^4 c^4}$$

$$+ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 l^4 c^2} - \frac{\dot{\vec{p}}\cdot\vec{R}}{l^3 R c^2} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{l^3 c^3}\Bigg\} R\vec{a} - \left(\frac{1}{\gamma^2 l^3} + \frac{(\vec{R}\cdot\vec{a})}{l^3 c^2}\right)\vec{p}$$

$$- \left(\frac{R}{\gamma^2 l^3 c} + \frac{(\vec{R}\cdot\vec{a})R}{l^3 c^3}\right)\dot{\vec{p}} - \frac{R}{l^2 c^2}\ddot{\vec{p}}\Bigg]_{t_p = t - \frac{R}{c}}, \qquad (V-66)$$

$$\vec{B}(\vec{x},t) = \Bigg[\Bigg\{\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 l^5 c^3} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 l^5 c} + \frac{3\gamma^2(\dot{\vec{p}}\cdot\vec{R})}{l^4 c^2}$$

$$+ \frac{(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{l^5 c^5} + \frac{\ddot{\vec{p}}\cdot\vec{R}}{l^2 c^3}\Bigg\}\vec{v}\times\vec{R}$$

$$+ \Bigg\{\frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 l^4 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{l^4 c^4} + \frac{2(\dot{\vec{p}}\cdot\vec{R})}{l^3 c^3}\Bigg\}\vec{a}\times\vec{R} + \frac{\ddot{\vec{p}}\times\vec{R}}{l^2 c^2}$$

$$+ \frac{\dot{\vec{p}}\times\vec{R}}{\gamma^2 l^3 c} + \frac{\vec{p}\times\vec{v}}{\gamma^2 l^3 c} + \frac{(\vec{R}\cdot\vec{a})(\vec{p}\times\vec{R})}{l^3 c^3}\Bigg]_{t_p = t - \frac{R}{c}}, \qquad (V-67)$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\Bigg[\vec{n}\Bigg\{\Bigg(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^4}{R^8 c^8} + \frac{(\ddot{\vec{p}}\cdot\vec{R})^2}{R^4 c^4} + \frac{6(\vec{p}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6 c^6}\Bigg)$$

$$\left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc}\right) - \frac{2(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5}\left(\frac{v^2}{c^2} - 3 - \frac{4(\vec{R}\cdot\vec{v})}{Rc}\right)$$

$$+ \frac{(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6c^6}\left(\frac{v^2}{c^2} - 9 - \frac{6(\vec{R}\cdot\vec{v})}{Rc}\right) + \frac{a^2}{c^2}\left(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2}{R^6c^6}\right.$$

$$\left. + \frac{4(\dot{\vec{p}}\cdot\vec{R})^2}{R^4c^4} + \frac{12(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5}\right) + \frac{2(\vec{v}\cdot\vec{a})}{c^2}\left(\frac{9(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^3}{R^7c^7}\right.$$

$$+ \frac{9(\vec{p}\cdot\vec{R})(\dot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^6c^6} + \frac{3(\dot{\vec{p}}\cdot\vec{R})(\ddot{\vec{p}}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^5} + \frac{2(\dot{\vec{p}}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})}{R^5c^5}$$

$$\left.\left. + \frac{2(\dot{\vec{p}}\cdot\vec{R})(\dddot{\vec{p}}\cdot\vec{R})}{R^4c^4}\right) + \frac{(\vec{R}\cdot\vec{a})^2(\dot{\vec{p}}\cdot\dot{\vec{p}})}{R^4c^6} + \frac{\dddot{\vec{p}}\cdot\dddot{\vec{p}}}{R^2c^4} + \frac{(\dot{\vec{p}}\cdot\dddot{\vec{p}})(\vec{R}\cdot\vec{a})}{R^3c^5}\right\}\Bigg]_{t_p=t-\frac{R}{c}} . \qquad \text{(V-68)}$$

(11) Constant Electric Dipole in a Circular Motion with its Axis Perpendicular to the Plane of Orbit:

$$\vec{A}(\vec{x},t) = \left[\left(\frac{\vec{p}\cdot\vec{R}}{\gamma^2\ell^3c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^3c^3}\right)\vec{v} + \frac{(\vec{p}\cdot\vec{R})}{\ell^2c^2}\vec{a}\right]_{t_p=t-\frac{R}{c}}, \qquad \text{(V-69)}$$

$$\Phi(\vec{x},t) = \left[\left(\frac{1}{\gamma^2 \ell^3} + \frac{(\vec{R}\cdot\vec{a})}{\ell^3 c^2} \right) (\vec{p}\cdot\vec{R}) \right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-70)}$$

$$\vec{E}(\vec{x},t) = \left[\left\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^2} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^4 c^3} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^4} \right\} \vec{R} \right.$$

$$- \left\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^4 c^4} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} \right\} R\vec{v}$$

$$- \left\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} - \frac{(\vec{p}\cdot\vec{R})}{\ell^3 R c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} \right\} R\vec{a} - \left\{ \frac{\vec{p}}{\gamma^2 \ell^3} + \frac{(\vec{R}\cdot\vec{a})\vec{p}}{\ell^3 c^2} \right\}$$

$$\left. - \frac{(\vec{p}\cdot\vec{R}) R \dot{\vec{a}}}{\ell^3 c^3} \right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-71)}$$

$$\vec{B}(\vec{x},t) = \left[(\vec{v}\times\vec{R}) \left\{ \frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\gamma^2 \ell^5 c^3} + \frac{(\vec{p}\cdot\vec{R})(\vec{R}\cdot\dot{\vec{a}})}{\ell^4 c^4} - \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{\ell^5 c^5} \right. \right.$$

$$\left. + \frac{3(\vec{p}\cdot\vec{R})}{\gamma^4 \ell^5 c} \right\} + (\vec{a}\times\vec{R}) \left\{ \frac{3(\vec{p}\cdot\vec{R})}{\gamma^2 \ell^4 c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{\ell^4 c^4} \right\} + \frac{(\vec{p}\cdot\vec{R})(\dot{\vec{a}}\times\vec{R})}{\ell^3 c^3}$$

$$\left. + \frac{(\vec{p}\times\vec{v})}{\gamma^2 \ell^3 c} - \frac{(\vec{p}\cdot\vec{R})}{\ell^3 c^3}(\vec{a}\times\vec{v}) + \frac{(\vec{R}\cdot\vec{a})}{\ell^3 c^3}(\vec{p}\times\vec{v}) + \frac{(\vec{p}\times\vec{a})}{\ell^2 c^2} \right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-72)}$$

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\left[\vec{n}\left\{\left(\frac{v^2}{c^2} - 1 - \frac{2(\vec{R}\cdot\vec{v})}{Rc}\right)\left(\frac{(\vec{p}\cdot\vec{R})^2\,(\vec{R}\cdot\dot{\vec{a}})^2}{R^6 c^6} + \frac{9(\vec{p}\cdot\vec{R})^2\,(\vec{R}\cdot\vec{a})^4}{R^8 c^8}\right.\right.\right.$$

$$\left. + \frac{6(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2\,(\vec{R}\cdot\dot{\vec{a}})}{R^7\, c^7}\right) + \frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2\, a^2}{R^6\, c^8} + \frac{(\vec{p}\cdot\vec{R})^2\,(\dot{\vec{a}}\cdot\dot{\vec{a}})}{R^4 c^8}$$

$$\left.\left. + \frac{2(\vec{v}\cdot\dot{\vec{a}})}{c^2}\left(\frac{(\vec{p}\cdot\vec{R})^2\,(\vec{R}\cdot\dot{\vec{a}})}{R^5 c^5} + \frac{3(\vec{p}\cdot\vec{R})^2\,(\vec{R}\cdot\vec{a})^2}{R^6\, c^6}\right)\right\}\right]_{t_p = t - \frac{R}{c}} . \qquad \text{(V-73)}$$

These expressions at the center of the orbit reduce to

$$\vec{A}(\vec{x},t) = 0\,, \qquad \text{(V-74)}$$

$$\phi(\vec{x},t) = 0\,, \qquad \text{(V-75)}$$

$$\vec{E}(\vec{x},t) = -\left[\left(\frac{1}{\gamma^2 R^3} + \frac{(\vec{R}\cdot\vec{a})}{R^3 c^3}\right)\vec{p}\right]_{t_p = t - \frac{R}{c}}, \qquad \text{(V-76)}$$

$$\vec{B}(\vec{x},t) = \left[\frac{\vec{p}\times\vec{v}}{\gamma^2 R^3 c} + \frac{(\vec{R}\cdot\vec{a})\,(\vec{p}\times\vec{v})}{R^3\, c^3} + \frac{\vec{p}\times\vec{a}}{\ell^2 c^2}\right]_{t_p = t - \frac{R}{c}} . \qquad \text{(V-77)}$$

(12) Non-Relativistic Approximation:

The potentials and fields in the rest frame of the dipole can also be derived from the general expressions by letting $\frac{v}{c} \longrightarrow 0$, and hence in the case of a constant electric dipole, the expressions for potentials and fields become

$$\vec{A}(\vec{x},t) = \left[\frac{(\vec{p}\cdot\vec{R})\vec{a}}{R^2c^2}\right]_{t_p=t-\frac{R}{c}}, \tag{V-78}$$

$$\phi(\vec{x},t) = \left[\frac{(\vec{p}\cdot\vec{R})}{R^3} + \frac{(\vec{R}\cdot\vec{a})(\vec{p}\cdot\vec{R})}{R^3c^2}\right]_{t_p=t-\frac{R}{c}}, \tag{V-79}$$

$$\vec{E}(\vec{x},t) = \left[\left(\frac{6(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^5c^2} + \frac{3(\vec{p}\cdot\vec{R})}{R^5} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})^2}{R^5c^4} - \frac{(\vec{p}\cdot\vec{a})}{R^3c^2}\right)\vec{R}\right.$$
$$\left. - \left(\frac{2(\vec{p}\cdot\vec{R})}{R^3c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^3c^4}\right)\vec{a} - \frac{\vec{p}}{R^3} - \frac{(\vec{R}\cdot\vec{a})\vec{p}}{R^3c^2}\right]_{t_p=t-\frac{R}{c}}, \tag{V-80}$$

$$\vec{B}(\vec{x},t) = \left[\left(\frac{3(\vec{p}\cdot\vec{R})}{R^4c^2} + \frac{3(\vec{p}\cdot\vec{R})(\vec{R}\cdot\vec{a})}{R^4c^4}\right)\vec{a}\times\vec{R} + \frac{\vec{p}\times\vec{a}}{R^2c^2}\right]_{t_p=t-\frac{R}{c}}, \tag{V-81}$$

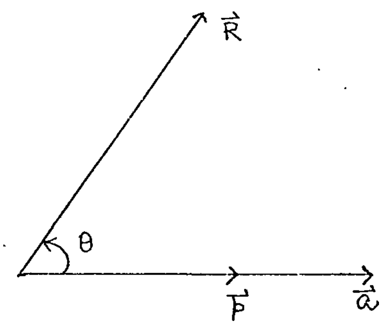
$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\left[\left(\frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^2a^2}{R^6c^8} - \frac{9(\vec{p}\cdot\vec{R})^2(\vec{R}\cdot\vec{a})^4}{R^8c^8}\right)\vec{n}\right]_{t_p=t-\frac{R}{c}}. \tag{V-82}$$

This vector has different values depending upon the orientation of $\vec{p}$ and the direction of $\vec{a}$. The total power radiated can be calculated as

$$\frac{dP}{d\Omega} = R^2\,\vec{S}\cdot\vec{n}\,, \tag{V-83}$$

and hence the total power radiated (as observed in the rest frame of the particle) can be obtained by integrating $\vec{S}\cdot\vec{n}$ over the surface of a sphere with center at the particle's position. We consider two cases:

(a) When $\vec{p} \parallel \vec{a}$:



In this case the Poynting vector becomes

$$\vec{S}(\vec{x},t) = \frac{q}{4\pi c^7}\left[\frac{p^2 a^4 \sin^2\theta \cos^4\theta}{R^2}\,\vec{n}\right]_{t_p = t-\frac{R}{c}} \tag{V-84}$$

and

$$\text{Power radiated} = P = \int R^2\,\vec{S}\cdot\vec{n}\,d\Omega$$

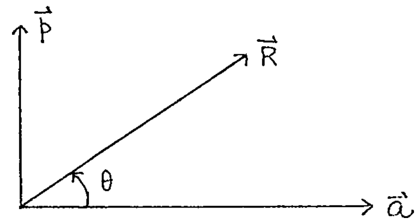
$$= \int_0^{2\pi}\int_0^{\pi} \frac{q\,p^2 a^4}{4\pi c^7} \cos^4\theta \sin^3\theta\,d\theta\,d\phi$$

which on performing integration gives

$$P = \frac{18 \dot{p}^2 a^4}{35 c^7} . \tag{V-85}$$

The Poynting vector itself is zero when $\theta = 0, \frac{\pi}{2}$ and π and has maximum value when $\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) \simeq \frac{\pi}{5}$.

(b) When $\vec{\dot{p}} \perp \vec{a}$:



The Poynting vector in this case reduces to

$$\vec{S}(\vec{x}, t) = \frac{9}{4\pi c^7} \left[\frac{\dot{p}^2 a^4 \sin^4\theta \cos^2\theta}{R^2} \vec{n} \right]_{t_p = t - \frac{R}{c}} \tag{V-86}$$

which again is zero when $\theta = 0, \frac{\pi}{2}$ and π and has maximum value when $\theta = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \simeq \frac{3\pi}{10}$.

The total power radiated is

$$P = \int R^2 \vec{S} \cdot \vec{n} \, d\Omega$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{9 \dot{p}^2 a^4 \sin^5\theta \cos^2\theta}{4\pi c^7} \, d\theta \, d\phi$$

which after integration becomes

$$P = \frac{24 \dot{p}^2 a^4}{35 c^7} . \tag{V-87}$$

CHAPTER VI

CONCLUSION

Considering the electromagnetic dipole to be of such small dimensions as to be represented by the Dirac delta function, the electromagnetic potentials and fields have been calculated. Throughout the treatment of the problem, it was assumed that the components of electric and magnetic dipole moments form a tensor $m_{\mu\nu}$. This assumption is further strengthened by considering the energy of an electromagnetic dipole in an external electromagnetic field. In the 3-vector form the energy is given by

$$\mathcal{E} = \vec{p}\cdot\vec{E} + \vec{m}\cdot\vec{B}, \tag{VI-1}$$

where $\vec{p}$ and $\vec{m}$ are the electric and magnetic dipole moments and $\vec{E}$ and $\vec{B}$ represent the external electric and magnetic fields. The covariant generalization of this energy term can be written as

$$\mathcal{E} = \frac{1}{2}\sum_{\mu}\sum_{\nu} m_{\mu\nu} F_{\mu\nu} \; . \tag{VI-2}$$

It is quite evident from (VI-2) that the energy of an electromagnetic dipole is a scalar quantity and hence is the same in all frames. Thus all experiments will produce the same measured value for the magnitude of the dipole moment independent of the frame of reference. Because the dipole moments are not scalars but 3-vectors in the rest frame of the particle, their apparent orientation will depend on the motion and in general will change along the trajectory. In particular, dipole moments are transformed into each other under Lorentz transformation. The invariants of the tensor $m_{\mu\nu}$ indicate that if $\vec{p}$ and $\vec{m}$ are orthogonal or have equal magnitudes in one frame of reference, they are orthogonal and have equal magnitudes in all frames of reference.

Considering the electric and magnetic fields due to a moving charge and a moving dipole, we notice that whereas in the former case the electric and magnetic fields are everywhere perpendicular to each other and to the direction of observation, in the latter case, however, this is not so, the fields being mutually perpendicular only in the radiation zone.

APPENDIX

EVALUATION OF $A_\mu(x) = 2e\int d\tau\, U_\mu(t_p)\, \delta\left(\sum_\lambda R_\lambda^2\right)$:

Let $s(\tau) = \sum_\lambda R_\lambda^2 = \sum_\lambda (x_\lambda - x_{\lambda p})^2$, and hence we may write the above integral as

$$A_\mu(x) = 2e\int ds\, \frac{d\tau}{ds}\, U_\mu(t_p)\, \delta(s) .$$

But

$$\frac{ds}{d\tau} = 2\sum_\lambda R_\lambda \frac{dR_\lambda}{d\tau} = -2\sum_\lambda R_\lambda u_\lambda ,$$

and hence

$$\frac{d\tau}{ds} = -\frac{1}{2\sum_\lambda R_\lambda U_\lambda} .$$

Substituting this in the above integral, we obtain

$$A_\mu(x) = -e\int \frac{U_\mu}{\sum_\lambda U_\lambda R_\lambda}\, \delta(s)\, ds .$$

This can be integrated immediately to give

$$A_\mu(x) = -e\left[\frac{U_\mu(t_p)}{\sum_\lambda R_\lambda u_\lambda}\right]_{s=0} ,$$

which is the expression we have in (II-12).

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