INTEGRAL TRANSFORMS, ANOMALOUS DIFFUSION, AND THE CENTRAL LIMIT THEOREM

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ABSTRACT

We present new connections among anomalous diffusion (AD), normal diffusion (ND) and the Central Limit Theorem (CLT). This is done by defining new canonical Cartesian-like position and Cartesian-like momentum variables and canonically quantizing these according to Dirac to define generalized negative semi-definite and self-adjoint Laplacian operators. These lead to new generalized Fourier transformations (GFT) and associated generalized probability distributions, which are form invariant under the corresponding transform. The new Laplacians also lead us to postulate generalized diffusion equations (GDE), which imply a connection to the CLT. We show that the derived diffusion equations have the O'Shaughnessy-Procaccia equations (OPE) as a special case. We also show that AD in the original, physical position is actually ND when viewed in terms of displacements in an appropriately transformed position variable. These tools allow us to prove the CLT for this class of diffusion equations.

THEORY

The physical interpretation of the CLT has its origins in Einstein's work on Brownian motion. In it he assumed the motion of particles to be random and modeled on discrete space-time, with the position being the sum of independent identically distributed (iid) variables and then taking the continuum limit, thus invoking the CLT, leading to a Gaussian distribution. An assumption that may be unwarranted was the independence of the variables and the equal weight given to various configurations, these assumptions becoming very relevant in the study of strongly correlated systems, open systems, long range systems, and in the subject of this poster, anomalous diffusion. Our method of attack uses generalized FTs and eigenfunctions suited to study these systems the main of which being the Φ_n transform [1]

$$\Phi_n g(\omega) = \int_{\mathbb{R}} \varphi_n(\omega t) g(t)\, dt$$

where $\varphi_n(\omega t) \equiv c_n(\omega t) + i s_n(\omega t)$ and the real and imaginary parts are given by

$$c_n(\eta) = \frac{1}{2}|\eta|^{n-\frac{1}{2}} J_{-1+\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right),$$

$$s_n(\eta) = -\frac{1}{2} sgn(\eta)|\eta|^{n-\frac{1}{2}} J_{1-\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν. In the case $\nu = 1$ we obtain the usual FT. What's useful is that the kernel of this integral transform give the eigenfunctions of the laplacian $-\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}$ which will be used to define the GDE. To fully prove the CLT though we will need a second family of transforms.

THEORY CONT.

The second family of transforms, or the canonical FT, was derived using a canonical transform to obtain new variables that were then used to build the transform. The integral transform is of the form

$$\mathcal{F}_W f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-iW(\omega)W(t)} W'(t)\, dt$$

with $W(x)$ being a canonical transform constructed as a bijective polynomial. When $W(x) = x$ this is again the standard FT.

In classical mechanics a standard technique to solving difficult problems is via the canonical transform in which variables are transformed such that $x \to W(x)$ and $p \to P_W$ subject to the constraint $\{W, P_W\} = 1$. Another way to think about such a procedure is that when dealing with anomalous phenomena it is better to look at the problem from the perspective of generalized variables, with integral transforms presented here doing just that.

Some of the properties of these families are as follows:

1. Both the Φ_n and canonical Fourier transforms are unitary thus preserving the L^2 norm.
2. $(\Phi_n)^4 f(t) = f(t)$ and $(\mathcal{F}_W)^4 f(t) = f(t)$ so that both transforms have eigenvalues $+1, +i, -1, -i$.
3. The eigenfunctions of the two transform form a complete basis and are dense in $L^2(\mathbb{R})$
4. Both families of transforms have uncertainty principles of the form $\Delta T \Delta \Omega \geq \frac{1}{2}$. Notably this also establishes previously conjectured results about the uncertainty principle of the Fourier-Bessel transform.

The two transforms together allowed us to solve the GDE $\frac{\partial}{\partial t} f(\eta, t) = -\frac{\partial}{\partial \eta}\frac{1}{\eta^{2n-2}}\frac{\partial}{\partial \eta} f(\eta, t)$ and prove properties about the solutions such as the CLT. They also provide the tools necessary to numerically evaluate nonlinear equations of the form $\frac{\partial}{\partial t} f(\eta, t) = -\frac{\partial}{\partial \eta}\frac{1}{\eta^{2n-2}}\frac{\partial}{\partial \eta} f(\eta, t)^\mu$. To see how this connects to more widely studied GDEs consider the OPE

$$\frac{\partial}{\partial t} f(r,t) = -\frac{1}{r^{d-1}}\frac{\partial}{\partial r} r^{d-1-\theta}\frac{\partial}{\partial r} f(r,t)$$

By substituting $W(r) = r^d$ we have

$$\frac{\partial}{\partial t}\tilde{f}(W(r),t) = -\frac{\partial}{\partial W}\frac{1}{W^{2\alpha-2}}\frac{\partial}{\partial W}\tilde{f}(W(r),t)$$

Where $\alpha = \frac{2+\theta}{2d}$. From here we are able to solve the OPE and use the canonical FT to establish the CLT for the attractor solution to this equation using standard fourier techniques [3][4].

APPLICATIONS AND FUTURE RESEARCH

Given the results presented we have a few major directions in which we want to generalized our work:

1. Extending the presented procedure to more general anomalous diffusion processes, including those with nonlinear characteristics.
2. Establishing the analogous thermodynamic results and interpretations for these processes (entropy, generalized statistics, and the like).
3. The exploration of the theory behind these families of transforms and their connection with quantum mechanics.

PLOTS

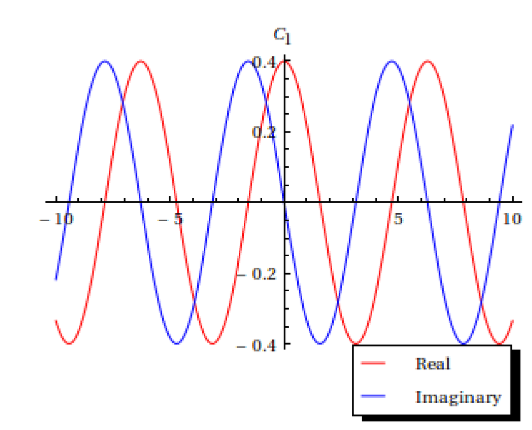

Figure 1: Plot of the real and imaginary parts of $e^{i\omega t}$ for $\omega = 1$.

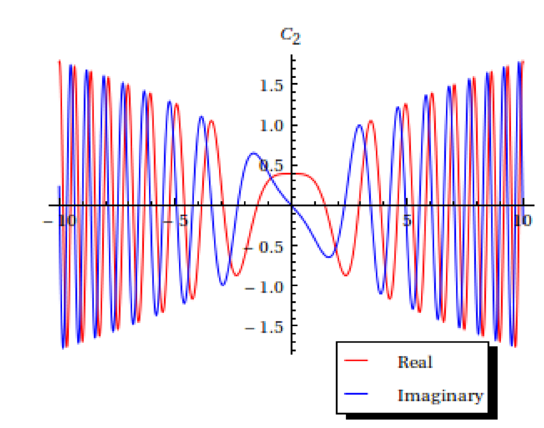

Figure 2: Plot of the real and imaginary parts of φ_2 for $\omega = 1$.

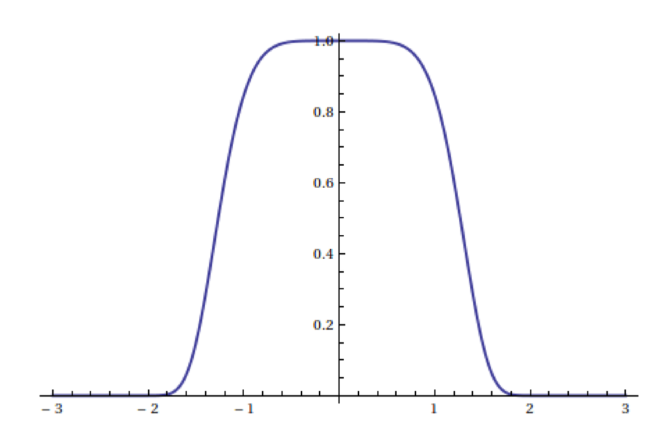
Figure 3: Plot of the function $\exp(-\frac{\omega^6}{6})$.

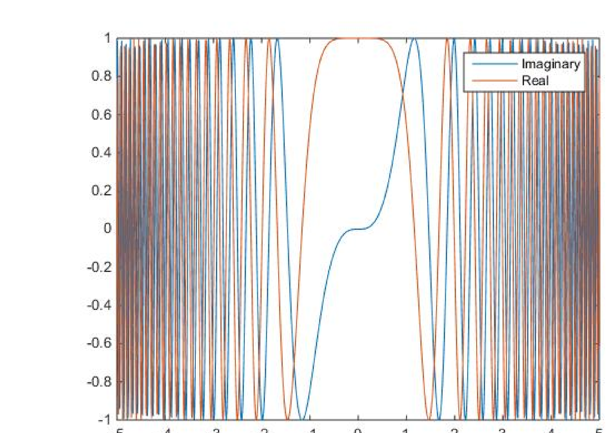

Figure 4: Plot of the real and imaginary parts of $\exp(-i\omega^3)$.

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