CARATHEODORY'S GENERAL OUTER MEASURE

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

William Ollie Alexander, Jr.

June 1955

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The present thesis is essentially an exposition of Caratheodory's general theories of outer measure of sets and set measurability; however some relations between measurability and additivity in regard to classes of point sets and functions defined over such classes have been briefly noted. The behavior, in the limit, of sequences of sets from additive classes has also been investigated superficially.

In gathering data, it was found helpful to obtain a great deal of information on various specific systems of outer measure. While not referred to explicitely in the thesis such additional data facilitated a presentation of the general theory as a system which lends itself implicitely to a classification of specific measures according to their generating functions.

While considerable work has been done toward linking measure theory to algebraic topology, discussions of these developments have been excluded for reasons of brevity and unity of approach.

It is proved in the thesis that several broad classes of sets are measurable for every set function which satisfies Caratheodory's definition of an outer measure function. It is further shown that measurability produces certain additivity conditions in sequences of measurable sets, and that monotonic sequences of sets taken from additive classes have definite additivity properties in the limit.

INTRODUCTION

The measure of a set of points is a generalization of the length, area, volume, or higher-dimensional extension of an interval, rectangle, or cell of three or more dimensions. The generalization arises in going from the definition of functions over such intervals, rectangles, or cells to the definition of functions over classes of point sets in n-space.

CARATHEODORY'S GENERAL OUTER MEASURE

Let R_n be the Euclidean n-space. We make some preliminary definitions.

D (1): A class $\mathfrak{a}$ of sets is said to be finitely additive if it is such that

I. $\phi \subset \mathfrak{a}$ where ϕ is the null set.

II. If $A, B \subset \mathfrak{a}$ then $(A - B) \subset \mathfrak{a}$.

III. If $A, B \subset \mathfrak{a}$ then $(A \cup B) \subset \mathfrak{a}$.

D (2): A class $\mathfrak{a}$ of sets is said to be completely additive if

I. $\phi \subset \mathfrak{a}$.

II. If $A \subset \mathfrak{a}$ then $C(A) \subset \mathfrak{a}$ where $C(A)$ is the complement of A with respect to R_n.

III. If $\{A_K\}$ is any sequence of sets from $\mathfrak{a}$ then

$$\bigcup_{K=1}^{\infty} A_K \subset \mathfrak{a}.$$

D (3): A set function ∇ is a real-valued function whose domain is a class of point sets.

D (4): ∇ is a completely additive set function if

I. The domain of ∇ is a completely additive class $\mathfrak{a}$ of sets.

II. If $\{E_k\}$ is a sequence of disjoint sets from $\mathfrak{a}$ then

$$\sum_{k=1}^{\infty} \tau(E_k)$$

is defined in the extended real number system; i.e., it converges to some finite or infinite value, and

$$\tau\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \tau(E_k).$$

III. $\tau(\phi) = 0.$

<u>D (5)</u>: A set function τ is said to be <u>non-decreasing</u> if, for A and B such that $A \supset B$

$$\tau(A) \geq \tau(B).$$

A <u>non-increasing set function</u> is defined similarly.

For τ non-decreasing and completely additive we have by the fact that ϕ is a subset of every set, that τ is everywhere non-negative.

<u>D (6)</u>: A <u>measure function</u> is a set function which is non-decreasing and completely additive.

<u>D (7)</u>: If $\rho(x, y)$ is the distance between the points x and y then the distance from a point x to a set A is defined by

$$\rho(x, A) = glb[\rho(x, y) \mid y \in A].$$

The distance from a set A to a set B is defined by
$$\rho(A, B) = \mathrm{glb}\,[\rho(x, y) \mid x \in A,\ y \in B].$$

We now define Caratheodory's postulates for an outer measure function.

<u>D (8)</u>: If τ is an extended real-valued function whose domain is the class of all subsets of the space R_n and τ is such that

C-I. τ is non-decreasing.

C-II. For any sequence $\{E_k\}$ of subsets of R_n
$$\tau\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \tau(E_k).$$

C-III. $\tau(\phi) = 0,\ \tau(A) \geq 0,\ A$ arbitrary.

C-IV. For $A, B \ni \rho(A, B) > 0$
$$\tau(A \cup B) = \tau(A) + \tau(B),$$

then τ is an <u>outer measure function</u> and is denoted by $\mu^{\star}$. The <u>outer measure</u> of a set A is denoted by $\mu^{\star}(A)$.

<u>D (9)</u>: If $\mu^{\star}$ is an outer measure function, and the set A is such that for any $W \subset R_n$
$$\mu^{\star}(W) = \mu^{\star}(W \cap A) + \mu^{\star}(W - A)$$

then A is said to be <u>measurable with respect to $\mu^{\star}$</u>. Its measure $\mu(A)$ is equal to its outer measure, $\mu^{\star}(A)$. We denote the fact that A is measurable with respect to μ^* by $A^{meas(\mu^*)}$.

<u>D (10)</u>: A set is said to be <u>Caratheodory measurable</u> if it is measurable for every outer measure function μ^*.

<u>T (1)</u>: If, for some $A \subset R_n$, $W \subset R_n$ arbitrary, and an outer measure function μ^*,

$$\mu^*(W) = \mu^*(W \cap A) + \mu^*(W - A \cap W),$$

then A is measurable μ^*.

<u>Proof</u>: We show that

$$\mu^*(W - A) = \mu^*(W - A \cap W).$$

Let $X \in (W - A)$. Then $X \in W$ and $X \notin A$. If $X \notin A$ then $X \notin (A \cap W)$. Thus $X \in (W - A \cap W)$ and

$$(W - A) \subset (W - A \cap W).$$

By C-I

(1) $$\mu^*(W - A) \leq \mu^*(W - A \cap W).$$

Now let $X \in (W - A \cap W)$. Then $X \in W$, $X \notin (A \cap W)$. If $X \notin (A \cap W)$ then $X \notin A$ or $X \notin W$. But $X \in W$, hence $X \notin A$ and we have that $X \in (W - A)$. Thus

$$(W - A \cap W) \subset (W - A)$$

and by C-I

(2) $$\mu^{*}(W - A \cap W) \leq \mu^{*}(W - A).$$

The theorem follows by (1) and (2).

At this point we prove a number of theorems which deal with characteristics of μ^{*}-measurable sets.

<u>T (2)</u>:
$$\left.\begin{array}{l} G \text{ open} \\ F \equiv C(G) \\ G_m \equiv \left[P \mid \rho(P, F) > \frac{1}{m}\right] \\ B \ni B \subset G \\ F \neq \phi,\ G \neq \phi \\ \mu^{*}(B) < +\infty \end{array}\right\}).$$

$$\mu^{*}(B) = \lim_{m \to \infty} \mu^{*}(B \cap G_m).$$

<u>Proof</u>: If $F_m \equiv \left[P \mid \rho(P, F) \leq \frac{1}{m}\right]$, then
$$F_m = C(G_m).$$

For if $X \in F_m$ then $\rho(X, F) \leq \frac{1}{m}$. Hence $X \notin G_m$ and so $X \in C(G_m)$. Now let $X \in C(G_m)$. Then $X \notin G_m$ and so $\rho(X, F) \leq \frac{1}{m}$. But now $X \in F_m$. We have that $F_m \subset C(G_m)$ and $C(G_m) \subset F_m$. Hence $F_m = C(G_m)$.

We show now that G_m is open by showing that F_m is closed.

Let P_0 be a limit point of F_m. Given $\epsilon > 0$ $\exists$ a neighborhood $N(P_0)$ of P_0 $\ni$ if $X \in N(P_0)$ then $\rho(X, P_0) < \epsilon$. Now $N(P_0)$ contains a point P distinct from P_0 $\ni$ $P \in F_m$. Then $\rho(P, F) \leq \frac{1}{m}$, and

$$\rho(P_0, F) \leq \rho(P_0, P) + \rho(P, F) < \epsilon + \frac{1}{m}.$$

Since ϵ was arbitrary,

$$\rho(P_0, F) \leq \frac{1}{m}.$$

Then $P_0 \in F_m$ and so F_m is closed. Since $F_m = C(G_m)$, G_m is open.

Now $F_m \supset F$. For $\rho(P, F) = 0 < \frac{1}{m}$ for any $P \in F$ $(m = 1, 2, \ldots)$. Also $F_1 \supset F_2 \supset \cdots$. For let $X \in F_{k+1}$. Then

$$\rho(X, F) \leq \frac{1}{k+1} < \frac{1}{k}$$

and $X \in F_k$.

Furthermore

(1) $$G_m \subset G \quad (m = 1, 2, \ldots):$$

Let $X \in G_m$. Then $X \notin C(G_m)$ and $X \notin F_m$. But $F \subset F_m$, hence $X \notin F$. Then $X \in G$.

And $G_1 \subset G_2 \subset \cdots$. For, let $X \in G_k$. Then $\rho(X, F) > \frac{1}{k} > \frac{1}{k+1}$; $X \in G_{k+1}$.

Now we have also that

(2) $$G = G_1 \cup (G_2 - G_1) \cup (G_3 - G_2) \cup \cdots \cup (G_m - G_{m-1}) \cup (G_{m+1} - G_m) \cup \cdots.$$

For let $X \in G_1 \cup (G_2 - G_1) \cup \cdots$. Then $X \in G_1$ or $X \in (G_k - G_{k-1})$ for some $k > 1$. If $X \in G_1$ then $X \in G$ by (1). If $X \in (G_k - G_{k-1})$ then $X \in G_k$ and $X \in G$.

For the converse, let $X \in G$. Then $\rho(X, F) > 0$, for if $\rho(X, F) = 0$ then $X \in F$. So $\exists$ an integer $k \ni \rho(X, F) > \frac{1}{k}$. Now $\exists$ a least integer $m \ni \rho(X, F) > \frac{1}{m}$. Then $X \in G_m$ and $X \notin G_{m-1}$ and so $X \in (G_m - G_{m-1})$. Thus $X \in G_1 \cup (G_2 - G_1) \cup \cdots$ and (2) follows.

We show now that, for $n > m$

(3)
$$\rho(F_n, G_m) > 0.$$

Let $P \in F_n$. Since F is closed $\exists$ $P_0 \in F$ $\ni \rho(P, P_0) \leq \frac{1}{n}$. Let $Q \in G_m$. Then

$$\rho(Q, P_0) \geq \rho(Q, F) > \frac{1}{m}.$$

We have then that

$$\rho(Q, P) + \rho(P, P_0) \geq \rho(Q, P_0),$$

or

$$\rho(P, Q) \geq \rho(Q, P_0) - \rho(P_0, P) > \frac{1}{m} - \frac{1}{n} > 0.$$

It is also true that

$$\mu^*(B \cap G_m) \leq \mu^*(B),$$

for $(B \cap G_m) \subset B$ and C-I holds.

Furthermore, it follows from $G_1 \subset G_2 \subset \cdots$ and C-I that

$$\mu^*(B\cap G_1) \leq \mu^*(B\cap G_2) \leq \cdots .$$

Now since $\mu^*(B) < +\infty$ we have that $\{\mu^*(B\cap G_m)\}$ is a bounded monotonic non-decreasing sequence. Then $\exists\ \lambda \ni$

$$\lambda = \lim_{m\to\infty} \mu^*(B\cap G_m)$$

and

(4) $$\lambda \leq \mu^*(B).$$

We show now that

$$B = (B\cap G_m) \cup (B\cap G_{m+1} - B\cap G_m) \cup (B\cap G_{m+2} - B\cap G_{m+1}) \cup \cdots .$$

First, let $x \in (B\cap G_m)\cup(B\cap G_{m+1} - B\cap G_m)\cup\ldots$. Then $x \in (B\cap G_m)$ or $x \in$ some

$$(B\cap G_{m+k} - B\cap G_{m+k-1}),\ k \geq 1.$$

If the former, $x \in B$. If the latter, then $x \in B\cap G_{m+k}$ and so $x \in B$. Thus

$$(B\cap G_m)\cup(B\cap G_{m+1} - B\cap G_m)\cup\cdots \subset B.$$

For the converse, let $x \in B$. Then $x \in G$ and $x \in G_1 \cup (G_2 - G_1) \cup \ldots$ by (2). Hence $x \in G_1$ or $x \in (G_k - G_{k-1})$ for some $k > 1$. If the former, $x \in G_1, G_2, \ldots$ and so $x \in G_m$ and $x \in (B\cap G_m)$. If the latter, and $k = m$, then $x \in (B\cap G_m)$. If $k < m$, $x \in (B\cap G_m)$ by $G_k \subset G_m$. If $k > m$, then $k = m + k_0$ for some k_0 and we have that $x \in (B\cap G_{m+k_0} - B\cap G_{m+k_0-1})$. Thus, in any instance

$$B \subset (B \cap G_m) \cup (B \cap G_{m+1} - B \cap G_m) \cup \cdots$$

and we have that

$$B = (B \cap G_m) \cup (B \cap G_{m+1} - B \cap G_m) \cup \cdots .$$

Then by C-II

(5) $$\mu^*(B) \leq \mu^*(B \cap G_m) + \mu^*(B \cap G_{m+1} - B \cap G_m) + \cdots .$$

Furthermore

(6) $$(B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_{p-1}) \subset$$
$$(B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_p) = (B \cap G_{p+1}).$$

For if $X \in (B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_{p-1})$ then either $X \in (B \cap G_{p+1} - B \cap G_p)$ and the first part of the statement follows by identity; or $X \in (B \cap G_{p-1}) \subset (B \cap G_p)$ and the first part of the statement again follows.

To show that

$$(B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_p) = (B \cap G_{p+1}).$$

Let X belong to the left-hand member. Then either $X \in (B \cap G_{p+1} - B \cap G_p)$ or $X \in (B \cap G_p)$. If the former, $X \in (B \cap G_{p+1})$ as desired. If the latter, then since $G_p \subset G_{p+1}$, $X \in (B \cap G_{p+1})$.

Conversely, let $X \in (B \cap G_{p+1})$. Either $X \in G_p$ or $X \notin G_p$. If the former, then since $X \in B$ we have $X \in (B \cap G_p)$. If the latter condition holds, we have that

$x \in (B \cap G_{p+1})$ and $x \notin (B \cap G_p)$ and so $x \in (B \cap G_{p+1} - B \cap G_p)$. In either case

$$x \in (B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_p).$$

So we have

$$(B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_p) = (B \cap G_{p+1}),$$

and (6) follows.

Then by C-I

(7) $$\mu^{\star}[(B \cap G_{p+1} - B \cap G_p) \cup (B \cap G_{p-1})] \leq \mu^{\star}(B \cap G_{p+1}).$$

Now we have that

$$(B \cap G_{p+1} - B \cap G_p) \subset F_p.$$

For if $x \in (B \cap G_{p+1} - B \cap G_p)$ then $x \in (B \cap G_{p+1})$ and we have $x \in B$. But $x \notin (B \cap G_p)$. Since $x \in B$ it must be that $x \notin G_p$. Then $x \in \mathcal{C}(G_p) = F_p$.

It is also true that

$$(B \cap G_{p-1}) \subset G_{p-1}.$$

Then by (3), since $p > p-1$

(8) $$\rho[(B \cap G_{p+1} - B \cap G_p), (B \cap G_{p-1})] \geq \rho(F_p, G_{p-1}) > 0.$$

It now follows from (7), (8), and C-IV that

$$\mu^{\star}(B \cap G_{p+1} - B \cap G_p) + \mu^{\star}(B \cap G_{p-1}) \leq \mu^{\star}(B \cap G_{p+1}).$$

Then

(9) $$\mu^{\star}(B \cap G_{p+1} - B \cap G_p) \leq \mu^{\star}(B \cap G_{p+1}) - \mu^{\star}(B \cap G_{p-1}).$$

From (5) and (9)

$$\mu^*(B) \leq \mu^*(B \cap G_m) + \sum_{p=1}^{\infty} [\mu^*(B \cap G_{m+p}) - \mu^*(B \cap G_{m+p-2})].$$

Since for any $k > 1$

$$\sum_{p=1}^{k} [\mu^*(B \cap G_{m+p}) - \mu^*(B \cap G_{m+p-2})] = [\mu^*(B \cap G_{m+k-1}) + \mu^*(B \cap G_{m+k})] - [\mu^*(B \cap G_m) + \mu^*(B \cap G_{m-1})],$$

we have that

$$\mu^*(B) \leq \lim_{p \to \infty} [\mu^*(B \cap G_{m+p-1}) + \mu^*(B \cap G_{m+p})] - \mu^*(B \cap G_{m-1})$$

$$= \lim_{p \to \infty} \mu^*(B \cap G_{m+p-1}) + \lim_{p \to \infty} \mu^*(B \cap G_{m+p}) - \mu^*(B \cap G_{m-1})$$

$$\leq \lambda + \lambda - \mu^*(B \cap G_{m-1}).$$

But

$$\lim_{m \to \infty} \mu^*(B \cap G_m) = \lambda.$$

Then in the limit

$$\mu^*(B) \leq (2\lambda - \lambda) = \lambda.$$

But by (4)

$$\mu^*(B) \geq \lambda.$$

Hence

$$\mu^*(B) = \lambda.$$

i.e.,

$$\mu^*(B) = \lim_{m \to \infty} \mu^*(B \cap G_m).$$

This completes the proof.

T (3): G open $\cdot) \cdot$ $G^{meas(\mu^*)}$.

Proof: Let W be arbitrary. It is easily shown that

$$W = (G \cap W) \cup (W - G \cap W).$$

Then by C-II

$$\mu^*(W) \leq \mu^*(G \cap W) + \mu^*(W - G \cap W).$$

It remains to show that

$$\mu^*(W) \geq \mu^*(G \cap W) + \mu^*(W - G \cap W).$$

Set $B \equiv (G \cap W)$. Then $B \subset G$, $B \subset W$. Now set $F \equiv C(G)$,

$$G_m \equiv [P \mid \rho(P, F) > \tfrac{1}{m}],$$

and $B_m \equiv (G_m \cap W)$. We show that

(1) $$\mu^*[(W - B) \cup B_m] = \mu^*(W - B) + \mu^*(B_m).$$

If either or both of the sets on the right are empty, the equality holds by C-III. If

$$(W - B) \neq \phi, \quad B_m \neq \phi$$

then we have

(2) $$(W - B) \subset F,$$

(3) $$B_m \subset G_m.$$

For if $X \in (W-B)$ then $X \notin B$ and $X \notin (G \cap W)$. Since $X \in W$ we have $X \notin G$ and so $X \in F$. This proves (2). Relation (3) follows from $B_m = (G_m \cap W)$. Thus

$$\rho[(W-B), B_m] \geq \rho(F, G_m) \geq \frac{1}{m} > 0.$$

Equation (1) now follows from C-IV. Now

$$W \supset (W-B) \cup B_m.$$

For if $X \in (W-B) \cup B_m$ and $X \in (W-B)$ then $X \in W$. If $X \in B_m$ then $X \in (G_m \cap W)$ and so $X \in W$.

Then by C-I and (1)

$$\mu^*(W) \geq \mu^*[(W-B) \cup B_m] = \mu^*(W-B) + \mu^*(B_m).$$

Now $B = (G \cap W)$. Furthermore

$$(G_m \cap B) = (G_m \cap G \cap W) = (B_m \cap G) \subset B_m.$$

Then by C-I

$$\mu^*(W) \geq \mu^*(W - G \cap W) + \mu^*(G_m \cap B).$$

Then from T (2) we have in the limit that

$$\mu^*(W) \geq \mu^*(W - G \cap W) + \mu^*(B)$$

$$= \mu^*(W - G \cap W) + \mu^*(G \cap W).$$

The theorem follows from T (1).

<u>T (4)</u>: $A^{\text{meas}(\mu^*)}$ $\cdot) \cdot$ $C(A)^{\text{meas}(\mu^*)}$

<u>Proof</u>: Since A is measurable μ^*,

(1) $$\mu^*(W) = \mu^*(W \cap A) + \mu^*(W - A).$$

We show that

(2) $$\mu^*(W) = \mu^*(W \cap C(A)) + \mu^*(W - C(A) \cap W).$$

Now

(3) $$(W - A) = (W \cap C(A)).$$

For let $X \in (W - A)$. Then $X \in W$, $X \notin A$. Thus $X \in C(A)$ and so $X \in (W \cap C(A))$. Conversely, if $X \in (W \cap C(A))$ then $X \in W$ and $X \in C(A)$, $X \notin A$. So $X \in (W - A)$ and (3) follows. Then

(4) $$\mu^*(W - A) = \mu^*(W \cap C(A)).$$

Also,

(5) $$\mu^*(W \cap A) = \mu^*(W - C(A) \cap W).$$

If $X \in (W \cap A)$ then $X \in W$, $X \notin C(A)$ and so $X \notin (C(A) \cap W)$. Thus $X \in (W - C(A) \cap W)$. If $X \in (W - C(A) \cap W)$ then $X \in W$, $X \notin (C(A) \cap W)$ and it must be that $X \notin C(A)$. So $X \in A$ and we have that $X \in (W \cap A)$. We have shown that

$$(W \cap A) = (W - C(A) \cap W),$$

and (5) follows.

Substitution of (4) and (5) into (1) yields (2), and the theorem follows.

We now have

C (1): F closed $\cdot)\cdot$ $F^{meas(\mu^*)}$.

Proof: $C(F)$ is open. The corollary follows from T (3) and T (4).

T (5): $A^{meas(\mu^*)}$, $B^{meas(\mu^*)}$ $\cdot)\cdot$ $(A\cap B)^{meas(\mu^*)}$.

Proof: Let W be arbitrary, and let $(A\cap W)$ correspond to the W of T (1). We may express the measurability of B by

(1) $$\mu^*(A\cap W) = \mu^*[B\cap(A\cap W)] + \mu^*[(A\cap W) - B\cap(A\cap W)].$$

Now let $(W - B\cap A\cap W)$ correspond to W. Since A is measurable we may write

(2) $$\mu^*(W - B\cap A\cap W) = \mu^*[A\cap(W - B\cap A\cap W)] + \mu^*[(W - B\cap A\cap W) - A\cap(W - B\cap A\cap W)].$$

We show that

(3) $$A\cap(W - B\cap A\cap W) = (A\cap W - B\cap A\cap W).$$

Let $X\in A\cap(W - B\cap A\cap W)$. Then $X\in A$, $X\in W$, and $X\notin(B\cap A\cap W)$. So $X\in(A\cap W - B\cap A\cap W)$.

Conversely, let $X\in(A\cap W - B\cap A\cap W)$. Then $X\in A$ and $X\in W$, $X\notin(B\cap A\cap W)$. Thus we have $X\in A$ and $X\in(W - B\cap A\cap W)$. Therefore $X\in A\cap(W - B\cap A\cap W)$, and (3) follows.

By (2) and (3) we have

(4) $$\mu^*(W - B\cap A\cap W) = \mu^*(A\cap W - B\cap A\cap W) + \mu^*[(W - B\cap A\cap W) - (A\cap W - B\cap A\cap W)].$$

It is easily shown that

$$(W - B \cap A \cap W) - (A \cap W - B \cap A \cap W) = (W - A \cap W).$$

Then (4) becomes

(5) $$\mu^*(W - B \cap A \cap W) = \mu^*(A \cap W - B \cap A \cap W) + \mu^*(W - A \cap W).$$

Now the measurability of A may be expressed by

(6) $$\mu^*(W) = \mu^*(A \cap W) + \mu^*(W - A \cap W).$$

From (1) and (6)

$$\mu^*(W) = \mu^*(B \cap A \cap W) + \mu^*(A \cap W - B \cap A \cap W) + \mu^*(W - A \cap W).$$

From (5)

$$\mu^*(W) = \mu^*(B \cap A \cap W) + \mu^*(W - B \cap A \cap W).$$

This says that $(A \cap B)$ is measurable.

<u>T (6)</u>: $A^{meas(\mu^*)}, B^{meas(\mu^*)} .\supset. (A \cup B)^{meas(\mu^*)}.$

<u>Proof</u>: We show that

(1) $$(A \cup B) = C(C(A) \cap C(B)).$$

Let $X \in (A \cup B)$. Then $X \in A$ and $X \notin C(A)$, or $X \in B$ and $X \notin C(B)$. In either case $X \notin C(A) \cap C(B)$. Then $X \in C(C(A) \cap C(B))$.

Now let $X \in C(C(A) \cap C(B))$. Then

$X \notin C(A) \cap C(B)$. Then $X \notin C(A)$ and $X \in A$ or $X \notin C(B)$ and $X \in B$. In either case $X \in (A \cup B)$ and (1) follows.

Now $C(A)$ and $C(B)$ are measurable by T (4), and the theorem follows by T (5) and T (4).

C (2): $A_k^{\text{meas}(\mu^*)}$, $(k = 1, 2, \cdot \cdot \cdot, n) \cdot) \cdot$

$$\left(\bigcup_{k=1}^{n} A_k\right)^{\text{meas}(\mu^*)}.$$

Proof: Follows from T (6) and induction.

T (7): $\left.\begin{array}{l} B_1 \supset B_2 \supset \cdot \cdot \cdot \\ B_k^{\text{meas}(\mu^*)}, \quad (k = 1, 2, \cdot \cdot \cdot) \end{array}\right\} \cdot) \cdot$

$$\left(\bigcap_{k=1}^{\infty} B_k\right)^{\text{meas}(\mu^*)}.$$

Proof: It is obvious that

(1) $$(W \cap B_1) \supset (W \cap B_2) \supset (W \cap B_3) \supset \cdot \cdot \cdot .$$

Also

(2) $$\left(W \cap \bigcap_{k=1}^{\infty} B_k\right) \subset (W \cap B_k).$$

For if $X \in \bigcap_{k=1}^{\infty} B_k$ then $X \in B_k$, $(k = 1, 2, \cdot \cdot \cdot)$.

From C-I and (1) the sequence $\{\mu^*(W \cap B_m)\}$ is monotonic non-increasing. By C-III it is bounded from below. Then $\exists\ \lambda \ni$

(3) $$\lim_{m \to \infty} \mu^*(W \cap B_m) = \lambda.$$

Since (2) holds $(k = 1, 2, \cdot \cdot \cdot)$ then

$$\mu^*(W \cap \bigcap_{k=1}^{\infty} B_k) \leq \mu^*(W \cap B_k), \quad (k = 1, 2, \cdots)$$

and so

(4) $$\mu^*(W \cap \bigcap_{k=1}^{\infty} B_k) \leq \lambda.$$

Now

(5) $$W = (W \cap \bigcap_{k=1}^{\infty} B_k) \cup (W - W \cap B_1) \cup (W \cap B_1 - W \cap B_2)$$
$$\cup \cdots \cup (W \cap B_{k-1} - W \cap B_k) \cup \cdots .$$

Certainly if x belongs to the term on the right, it belongs also to W. Suppose $x \in W$. Either $x \in \bigcap_{k=1}^{\infty} B_k$ (and (5) is proved), or $x \notin \bigcap_{k=1}^{\infty} B_k$. If the latter holds, $\exists$ a $k \ni x \notin B_k$. Let k_0 be the first such k. If $k_0 = 1$ then $x \in (W - W \cap B_1)$ and (5) follows. If $k_0 > 1$ then $x \in B_{k_0-1}$ and $x \in (W \cap B_{k_0-1} - W \cap B_{k_0})$. Thus we have (5).

Then by C-II

(6) $$\mu^*(W) \leq \mu^*(W \cap \bigcap_{k=1}^{\infty} B_k) + \mu^*(W - W \cap B_1)$$
$$+ \mu^*(W \cap B_1 - W \cap B_2) + \cdots .$$

Since B_1 is measurable

(7) $$\mu^*(W) = \mu^*(W \cap B_1) + \mu^*(W - W \cap B_1),$$

and since B_{k+1} is measurable and

$$(B_k \cap B_{k+1}) = B_{k+1}$$

we have

(8) $$\mu^*(W \cap B_k) = \mu^*(W \cap B_{k+1}) + \mu^*(W \cap B_k - W \cap B_{k+1}).$$

Substituting (7) and (8) into (6) we get

$$\mu^{\star}(W) \leq \mu^{\star}(W \cap \bigcap_{k=1}^{\infty} B_k) + [\mu^{\star}(W) - \mu^{\star}(W \cap B_1)]$$
$$+ [\mu^{\star}(W \cap B_1) - \mu^{\star}(W \cap B_2)]$$
$$+ [\mu^{\star}(W \cap B_2) - \mu^{\star}(W \cap B_3)] + \cdots$$

and so

$$\mu^{\star}(W) \leq \mu^{\star}(W \cap \bigcap_{k=1}^{\infty} B_k) + \mu^{\star}(W) - \lim_{m \to \infty} \mu^{\star}(W \cap B_m).$$

By (3)

$$\mu^{\star}(W) \leq \mu^{\star}(W \cap \bigcap_{k=1}^{\infty} B_k) + \mu^{\star}(W) - \lambda.$$

Thus

$$\lambda \leq \mu^{\star}(W \cap \bigcap_{k=1}^{\infty} B_k).$$

Then by (4)

(9) $$\lambda = \mu^{\star}(W \cap \bigcap_{k=1}^{\infty} B_k) = \lim_{m \to \infty} \mu^{\star}(W \cap B_m).$$

It is easily shown that (5) may be rewritten as

$$(W - W \cap \bigcap_{k=1}^{\infty} B_k) = (W - W \cap B_1) \cup (W \cap B_1$$
$$- W \cap B_2) \cup \cdots ,$$

From C-II, (7), and (8)

$$\mu^{\star}(W - W \cap \bigcap_{k=1}^{\infty} B_k) \leq \mu^{\star}(W - W \cap B_1)$$
$$+ \mu^{\star}(W \cap B_1 - W \cap B_2) + \cdots$$
$$= [\mu^{\star}(W) - \mu^{\star}(W \cap B_1)] + [\mu^{\star}(W \cap B_1)$$
$$- \mu^{\star}(W \cap B_2)] + \cdots$$
$$= \mu^{\star}(W) - \lim_{m \to \infty} \mu^{\star}(W \cap B_m).$$

We have thus from (9) that

$$\mu^*\left[W - W \cap \bigcap_{k=1}^{\infty} B_k\right] \leq \mu^*(W) - \mu^*\left[W \cap \bigcap_{k=1}^{\infty} B_k\right]$$

So

(10) $$\mu^*(W) \geq \mu^*\left[W \cap \bigcap_{k=1}^{\infty} B_k\right] + \mu^*\left[W - W \cap \bigcap_{k=1}^{\infty} B_k\right].$$

It is easily shown that

$$W = \left(W \cap \bigcap_{k=1}^{\infty} B_k\right) \cup \left(W - W \cap \bigcap_{k=1}^{\infty} B_k\right).$$

Then by C-II

$$\mu^*(W) \leq \mu^*\left[W \cap \bigcap_{k=1}^{\infty} B_k\right] + \mu^*\left[W - W \cap \bigcap_{k=1}^{\infty} B_k\right].$$

With (10) this says that

$$\mu^*(W) = \mu^*\left[W \cap \bigcap_{k=1}^{\infty} B_k\right] + \left[W - W \cap \bigcap_{k=1}^{\infty} B_k\right].$$

i.e.,

$$\bigcap_{k=1}^{\infty} B_k \quad \text{is measurable.}$$

<u>T (8)</u>: $A_k^{\text{meas}(\mu^*)}, \quad (k = 1, 2, \ldots) \,).$

$$\left(\bigcap_{k=1}^{\infty} A_k\right)^{\text{meas}(\mu^*)}.$$

<u>Proof</u>: Let

$$B_1 \equiv A_1, \quad B_2 \equiv A_1 \cap A_2, \quad \ldots,$$

$$B_m \equiv A_1 \cap A_2 \cap \ldots \cap A_m, \quad \ldots.$$

Now

$$(A_1 \cap A_2) \subset A_1, \quad (A_1 \cap A_2 \cap A_3) \subset (A_1 \cap A_2), \quad \ldots,$$

$$(A_1 \cap A_2 \cap \ldots \cap A_m) \subset (A_1 \cap A_2 \cap \ldots \cap A_{m-1}), \quad \ldots,$$

Furthermore, by T (5), $B_2 \equiv (A_1 \cap A_2)$ is measurable, $B_3 \equiv (A_1 \cap A_2) \cap A_3$ is measurable, ...

So we have that $B_1 \supset B_2 \supset \cdots$ with $B_K^{meas(\mu^*)}$, $(K = 1, 2, \cdots)$. The postulates of T (7) are now satisfied and so $\bigcap_{K=1}^{\infty} B_K$ is measurable.

Now

$$\bigcap_{K=1}^{\infty} A_K = \bigcap_{K=1}^{\infty} B_K.$$

For if $X \in \bigcap_{K=1}^{\infty} A_K$ then $X \in A_K$, $(K = 1, 2, \cdots)$. So $X \in A_1 \equiv B_1$, $X \in (A_1 \cap A_2) \equiv B_2, \cdots,$ and so $X \in B_K$, $(K = 1, 2, \cdots)$.

If $X \in \bigcap_{K=1}^{\infty} B_K$ then $X \in B_K$, $(K = 1, 2, \cdots)$. So $X \in B_1 \equiv A_1$, $X \in B_2 \equiv (A_1 \cap A_2), \cdots,$ $X \in B_m \equiv A_1 \cap A_2 \cap \cdots \cap A_m, \cdots$.

Then $X \in A_K$, $(K = 1, 2, \cdots)$.

Hence the theorem.

<u>T (9)</u>: $A^{meas(\mu^*)}$, $\mu^*(B) < +\infty$).

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) - \mu^*(A \cap B).$$

<u>Proof</u>: Let W correspond to $(A \cup B)$. Then since A is measurable

(1) $$\mu^*(A \cup B) = \mu^*[A \cap (A \cup B)] + \mu^*[(A \cup B) - A \cap (A \cup B)].$$

Now

(2) $$A = A \cap (A \cup B).$$

Also

(3) $$(A \cup B) - A \cap (A \cup B) = (B - A \cap B).$$

For, let $X \in (A \cup B) - A \cap (A \cup B)$. Then $X \in (A \cup B)$ and so $X \in A$ or $X \in B$. But we have also that $X \notin A \cap (A \cup B)$, and $X \notin A$ or $X \notin (A \cup B)$. But $X \in (A \cup B)$. Thus $X \notin A$ and so $X \notin (A \cap B)$. Now since $X \notin A$ and $X \in (A \cup B)$ we have $X \in B$. Thus it is that $X \in (B - A \cap B)$.

For the converse, let $X \in (B - A \cap B)$. Then $X \in B$ and $X \notin A$. Since $X \in B$, $X \in (A \cup B)$. Since $X \notin A$, $X \notin A \cap (A \cup B)$. Therefore $X \in (A \cup B) - A \cap (A \cup B)$ and (3) is verified.

Now from (2) and (3), (1) becomes

(4) $$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B - A \cap B).$$

By the measurability of A we may write also

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(B - A \cap B),$$

or

(5) $$\mu^*(B - A \cap B) = \mu^*(B) - \mu^*(A \cap B).$$

Equations (4) and (5) give

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) - \mu^*(A \cap B).$$

<u>C (3)</u>: $A_1^{\text{meas}(\mu^*)}, A_2^{\text{meas}(\mu^*)}, A_1 \cap A_2 = \phi$).

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2).$$

<u>Proof:</u> From T (9)

$$\mu^{*}(A_1 \cup A_2) = \mu^{*}(A_1) + \mu^{*}(A_2) - \mu^{*}(A_1 \cap A_2).$$

But $(A_1 \cap A_2) = \phi$. Hence by C-III, $\mu^{*}(A_1 \cap A_2) = 0$.

Thus

$$\mu^{*}(A_1 \cup A_2) = \mu^{*}(A_1) + \mu^{*}(A_2).$$

By T (6), $(A_1 \cup A_2)$ is measurable A_1 and A_2 are measurable, and so we have

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2).$$

by D (9).

<u>C (4):</u> $\left.\begin{array}{ll} A_k^{\text{meas}(\mu^{*})}, & (k = 1, 2, \cdots, n) \\ (A_m \cap A_n) = \phi & (m \neq n) \end{array}\right\} \cdot).$

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

<u>Proof:</u> Follows by C (3) and induction.

<u>T (10):</u> $\left.\begin{array}{ll} A_k^{\text{meas}(\mu^{*})}, & (k = 1, 2, \cdots) \\ (A_m \cap A_n) = \phi, & (m \neq n) \end{array}\right\} \cdot).$

$$\mu^{*}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu^{*}(A_k).$$

<u>Proof:</u> By C (4) and D (9)

(1) $$\mu^{*}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu^{*}(A_k).$$

for every n.

By C-II

(2) $$\mu^{\star}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu^{\star}(A_k).$$

Now

$$\bigcup_{k=1}^{\infty} A_k \supset \bigcup_{k=1}^{n} A_k .$$

Then by C-I

(3) $$\mu^{\star}\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \mu^{\star}\left(\bigcup_{k=1}^{n} A_k\right), \quad (n = 1, 2, \ldots).$$

In the limit, from (1) and (3),

(4) $$\mu^{\star}\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \mu^{\star}(A_k).$$

The theorem follows from (2) and (4).

<u>T (11)</u>: $\mu^{\star}(B) = 0 \;\cdot\supset\cdot\; B^{meas(\mu^{\star})}$.

<u>Proof</u>: Let W be arbitrary. Now

$$(B \cap W) \subset B .$$

Hence

(1) $$\mu^{\star}(B \cap W) \leq \mu^{\star}(B) = 0 .$$

We show that

(2) $$\mu^{\star}(W) = \mu^{\star}(B \cap W) + \mu^{\star}(W - B \cap W).$$

From (1)

$$\mu^{\star}(B \cap W) + \mu^{\star}(W - B \cap W) = \mu^{\star}(W - B \cap W).$$

Since

$$(W - B \cap W) \subset W$$

and

$$\mu^*(W - B\cap W) \leq \mu^*(W),$$

then

(3) $$\mu^*(B\cap W) + \mu^*(W - B\cap W) \leq \mu^*(W).$$

Now

(4) $$W \subset (B\cap W) \cup (W - B\cap W).$$

For, let $X \in W$. If $X \in (B\cap W)$, (4) is true. If not, then $X \in (W - B\cap W)$ and (4) follows.

We have then that

(5) $$\mu^*(W) \leq \mu^*[(B\cap W)\cup(W - B\cap W)]$$
$$\leq \mu^*(B\cap W) + \mu^*(W - B\cap W).$$

Equation (2) follows from (3) and (5).

T (12): $A^{meas(\mu^*)}, B^{meas(\mu^*)}). (A-B)^{meas(\mu^*)}$.

Proof: $(A-B) = (A\cap C(B))$. The theorem now follows from T (4) and T (5).

L (1): If $\{\Gamma_n\}$ is any sequence of sets, then $\exists$ a sequence $\{\Delta_n\}$ of disjoint sets $\ni$

$$\bigcup_{n=1}^{\infty} \Gamma_n = \bigcup_{n=1}^{\infty} \Delta_n .$$

Proof: Define a sequence $\{\Delta_n\} \ni$

$$\Delta_1 = \Gamma_1 ,$$

and for $n > 1$

$$\Delta_n = \left(\Gamma_n - \bigcup_{k=1}^{n-1} \Gamma_k\right).$$

We show first that

$$\bigcup_{n=1}^{\infty} \Gamma_n = \bigcup_{n=1}^{\infty} \Delta_n.$$

Let $x \in \bigcup_{n=1}^{\infty} \Gamma_n$. Then $x \in \Gamma_n$ for some n. Now there is a first such n, call it n_1. Then $x \notin \Gamma_n$ $(n = 1, 2, \ldots, n_1 - 1)$. Thus $x \in \left(\Gamma_{n_1} - \bigcup_{k=1}^{n_1 - 1} \Gamma_k\right)$ and $x \in \Delta_{n_1}$. Then $x \in \bigcup_{n=1}^{\infty} \Delta_n$.

Conversely, let $x \in \bigcup_{n=1}^{\infty} \Delta_n$. Then $x \in \Delta_n$ for some n, say n_0. So $x \in \left(\Gamma_{n_0} - \bigcup_{k=1}^{n_0 - 1} \Gamma_k\right)$. Thus $x \in \Gamma_{n_0}$ and so $x \in \bigcup_{n=1}^{\infty} \Gamma_n$.

We show now that the Δ_n are disjoint. Suppose $\exists$ an $x \ni x \in \Delta_m$ and $x \in \Delta_p$, $m \neq p$ and suppose $m > p$. (This is no restriction.)

Since $x \in \Delta_m$, $x \in \Gamma_m$ and $x \notin \bigcup_{k=1}^{m-1} \Gamma_k$. If $x \notin \bigcup_{k=1}^{m-1} \Gamma_k$ then $x \notin \Gamma_p$. Thus $x \notin \left(\Gamma_p - \bigcup_{k=1}^{p-1} \Gamma_k\right)$. So $x \notin \Delta_p$. This is a contradiction. Hence the Δ_n are disjoint and the lemma follows.

<u>L (2)</u>:
$$\left.\begin{array}{l} E_k^{\,meas(\mu^*)}, \quad (k = 1, 2, \ldots) \\ (E_i \cap E_j) = \phi, \quad (i \neq j) \\ S_n = \bigcup_{k=1}^{n} E_k, \quad (n = 1, 2, \ldots) \end{array}\right\} \supset .$$

$$\mu^*(A \cap S_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k),$$

A arbitrary, $(n = 1, 2, \ldots)$.

<u>Proof</u>: The proof is by induction. For $n = 1$ the lemma is true. We now assume the lemma true for $n = p$ and show that

$$\mu^*(A \cap S_{p+1}) = \sum_{k=1}^{p+1} \mu^*(A \cap E_k).$$

By C (2), S_p is measurable. Then with $(A \cap S_{p+1})$ corresponding to W we may write

(1) $$\mu^*(A \cap S_{p+1}) = \mu^*(A \cap S_{p+1} \cap S_p) + \mu^*[(A \cap S_{p+1}) - (A \cap S_{p+1} \cap S_p)].$$

It is true that

(2) $$(S_{p+1} \cap S_p) = S_p$$

and

(3) $$(A \cap S_{p+1}) - (A \cap S_{p+1} \cap S_p) = (A \cap E_{p+1}).$$

Equation (2) is obvious. For (3), let $x \in (A \cap S_{p+1}) - (A \cap S_{p+1} \cap S_p)$. Then $x \in A$, $x \in S_{p+1}$. Then $x \in E_{k_0}$ for some k_0, $(1 \leq k_0 \leq p+1)$. But $x \notin (A \cap S_{p+1} \cap S_p)$, and $x \in A$ and $x \in S_{p+1}$. This says that $x \notin S_p$. Thus $x \notin E_k$, $(k = 1, 2, \ldots, p)$, and so $x \in E_{p+1}$ and $x \in (A \cap E_{p+1})$.

For the converse, let $x \in (A \cap E_{p+1})$. Then $x \in A$ and $x \in E_{p+1}$. Thus $x \in \bigcup_{k=1}^{p+1} E_k = S_{p+1}$. But since the E_k are disjoint, $x \notin E_k$, $(k = 1, 2, \ldots, p)$ and so $x \notin \bigcup_{k=1}^{p} E_k = S_p$. Now $x \in (A \cap S_{p+1})$ and $x \notin S_p$. Hence

$$x \in (A \cap S_{p+1}) - (A \cap S_{p+1} \cap S_p)$$

and (3) follows. Substituting (2) and (3) into (1) we get

$$\mu^*(A \cap S_{p+1}) = \mu^*(A \cap S_p) + \mu^*(A \cap E_{p+1}).$$

But we have assumed that

$$\mu^*(A \cap S_p) = \sum_{k=1}^{p} \mu^*(A \cap E_k).$$

Hence we have

$$\mu^*(A \cap S_{p+1}) = \sum_{k=1}^{p} \mu^*(A \cap E_k) + \mu^*(A \cap E_{p+1})$$

$$= \sum_{k=1}^{p+1} \mu^*(A \cap E_k).$$

<u>L (3):</u> $\left.\begin{array}{l} E_k^{\text{meas}(\mu^*)},\ (k = 1, 2, \ldots) \\ (E_m \cap E_n) = \phi,\ (m \neq n) \\ S = \bigcup_{k=1}^{\infty} E_k \end{array}\right\} \cdot).$

$$\mu^*(A \cap S) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k),$$

A arbitrary.

<u>Proof:</u> It is true that

$$\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) \supset \left(A \cap \bigcup_{k=1}^{n} E_k\right)$$

i.e.,

$$(A \cap S) \supset (A \cap S_n)$$

Then

(1) $$\mu^*(A \cap S) \geq \mu^*(A \cap S_n).$$

By (1) and L (2)

$$\mu^{\star}(A \cap S) \geq \sum_{k=1}^{n} \mu^{\star}(A \cap E_k).$$

In the limit

(2) $$\mu^{\star}(A \cap S) \geq \sum_{k=1}^{\infty} \mu^{\star}(A \cap E_k).$$

By C-II and the fact that

$$(A \cap S) = \bigcup_{k=1}^{\infty}(A \cap E_k)$$

we have

(3) $$\mu^{\star}(A \cap S) \leq \sum_{k=1}^{\infty} \mu^{\star}(A \cap E_k).$$

The lemma follows from (3) and (2).

<u>L (4)</u>: $$\left.\begin{array}{l} E_k^{\text{meas}(\mu^{\star})}, \quad (k = 1, 2, \ldots) \\ (E_m \cap E_n) = \phi, \quad (m \neq n) \\ S = \bigcup_{k=1}^{\infty} E_k \end{array}\right\} \cdot) .$$

$$S^{\text{meas}(\mu^{\star})}.$$

<u>Proof</u>: Let A be arbitrary. Set $S_n = \bigcup_{k=1}^{n} E_k$. Then by C (2), S_n is measurable for each n and by D (9) we may write

$$\mu^{\star}(A) = \mu^{\star}(A \cap S_n) + \mu^{\star}(A - S_n).$$

Now

(1) $$(A - S_n) \supset (A - S).$$

For let $X \in (A-S)$. Then $X \in A$ and $X \notin S$. Then $X \notin E_k$, $(k = 1, 2, \ldots)$ and so $X \notin S_n$, $(n = 1, 2, \cdots)$. Thus $X \in (A-S_n)$ and (1) follows.

Then by L (2)

$$\mu^*(A) = \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A - S_n).$$

By C-I and (1)

$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A-S).$$

In the limit

(2) $$\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A-S).$$

Since $(A \cap S) = \bigcup_{k=1}^{\infty} (A \cap E_k)$ we have by C-II and (2)

(3) $$\mu^*(A) \geq \mu^*(A \cap S) + \mu^*(A-S).$$

Now

(4) $$(A \cap S) \cup (A-S) \supset A.$$

Then we have by C-II, (4), and C-I

(5) $$\mu^*(A \cap S) + \mu^*(A-S) \geq \mu^*[(A\cap S) \cup (A-S)] \geq \mu^*(A).$$

The lemma follows from (3) and (5).

We are now in a position to prove

$$\underline{\text{T (13)}}:\quad \left.\begin{array}{l} E_k^{meas(\mu^*)},\quad (k = 1, 2, \ldots) \\ S = \bigcup_{k=1}^{\infty} E_k \end{array}\right\} \supset.$$

$$S^{meas(\mu^*)}.$$

Proof: By L (1) $\exists$ a sequence $\{\Delta_k\}$ of disjoint sets $\ni$

$$S = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} \Delta_k$$

and, for each k

$$\Delta_k = \left(E_k - \bigcup_{n=1}^{k-1} E_n\right).$$

Now, by C (2) and T (12), Δ_k is measurable for every k. Since the Δ_k are disjoint it follows by L (4) that $\bigcup_{k=1}^{\infty} \Delta_k$ is measurable. Hence the theorem.

We now prove the theorem on the relation of measurability to measure functions.

T (14): Let μ^* be an outer measure function and $\mathfrak{M}$ the class of μ^*-measurable sets. If the domain of μ^* is $\mathfrak{M}$ then μ^* is a measure function.

Proof: μ^* satisfies C-I, C-II, C-III, and C-IV. By C-I μ^* is non-decreasing. It remains to show that μ^* is completely additive.

Now $\mathfrak{M}$ satisfies I, II, and III of D (2). Postulate I follows by C-III and T (11), II follows from T (4), and III follows by T (13). Thus $\mathfrak{M}$ is a completely additive class of sets.

Furthermore, I, II, III of D (4) are satisfied. For I follows from the complete additivity of $\mathfrak{M}$, II follows from T (10), and III follows from C-III. This says that μ^* is a completely additive set function on $\mathfrak{M}$ and the theorem follows by D (6).

We wish to prove now three theorems on relations between measure functions of limits of sequences of sets, and limits of sequences of measure functions of sets.

D (11): The limit superior and limit inferior of a sequence $\{u_n\}$ of real numbers are defined respectively by

$$\overline{\lim_{n \to \infty}}\, u_n = \operatorname*{glb}_{k \to \infty} \operatorname*{lub}_{n \geq k} u_n$$

and

$$\underline{\lim_{n \to \infty}}\, u_n = \operatorname*{lub}_{k \to \infty} \operatorname*{glb}_{n \geq k} u_n .$$

If $\{u_n\}$ is such that

$$\underline{\lim_{n \to \infty}}\, u_n = \overline{\lim_{n \to \infty}}\, u_n$$

we say that $\{u_n\}$ converges and denote the common limit by

$$\lim_{n \to \infty} u_n .$$

D (12): Let $\{\Gamma_n\}$ be a sequence of sets. The

<u>limit superior</u> and <u>limit inferior</u> of $\{\Gamma_n\}$ are defined respectively by

$$\overline{\lim_{n\to\infty}}\, \Gamma_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Gamma_n$$

and

$$\underline{\lim_{n\to\infty}}\, \Gamma_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Gamma_n .$$

If $\{\Gamma_n\}$ is such that

$$\underline{\lim_{n\to\infty}}\, \Gamma_n = \overline{\lim_{n\to\infty}}\, \Gamma_n$$

we say that $\{\Gamma_n\}$ <u>converges</u> and denote the common limit by

$$\lim_{n\to\infty} \Gamma_n .$$

<u>L (5)</u>: $\Gamma_1 \subset \Gamma_2 \subset \cdots \quad \cdot)\cdot$
$\{\Gamma_n\}$ converges, $\lim_{n\to\infty} \Gamma_n = \bigcup_{n=1}^{\infty} \Gamma_n$.

<u>Proof</u>: We show that

$$\underline{\lim_{n\to\infty}}\, \Gamma_n = \bigcup_{n=1}^{\infty} \Gamma_n \tag{1}$$

and that

$$\overline{\lim_{n\to\infty}}\, \Gamma_n = \bigcup_{n=1}^{\infty} \Gamma_n , \tag{2}$$

For (1):

Let $x \in \underline{\lim_{n\to\infty}}\, \Gamma_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Gamma_n$. Then $x \in \bigcap_{n=k}^{\infty} \Gamma_n$ for some $k = k_0$ and consequently $x \in \Gamma_{k_0}$. So $x \in \bigcup_{n=1}^{\infty} \Gamma_n$. Now let $x \in \bigcup_{n=1}^{\infty} \Gamma_n$. Then $x \in \Gamma_n$ for some $n = k'$. Then by hypothesis $x \in \Gamma_n$ $(n = k', k'+1, \ldots)$ and so $x \in \bigcap_{n=k'}^{\infty} \Gamma_n$. But since

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Gamma_n = \left(\bigcap_{n=1}^{\infty} \Gamma_n\right) \cup \left(\bigcap_{n=2}^{\infty} \Gamma_n\right) \cup \cdots \cup \left(\bigcap_{n=k'}^{\infty} \Gamma_n\right) \cup \cdots$$

it follows that $x \in \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\Gamma_n = \varliminf_{n\to\infty}\Gamma_n$. Thus (1) holds.

For (2), let $x \in \varlimsup_{n\to\infty}\Gamma_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Gamma_n$. Then $x \in \bigcup_{n=k}^{\infty}\Gamma_n$ for $k=1$ and so

$$x \in \bigcup_{n=1}^{\infty}\Gamma_n .$$

Conversely, let $x \in \bigcup_{n=1}^{\infty}\Gamma_n$. Then $x \in$ some Γ_{k_0} and

(3) $$x \in \bigcup_{n=k}^{\infty}\Gamma_n \quad (k = 1, 2, \ldots, k_0).$$

But by hypothesis $x \in \Gamma_n$ $(n = k_0+1, k_0+2, \ldots)$, hence

(4) $$x \in \bigcup_{n=k}^{\infty}\Gamma_n \quad (k = k_0+1, k_0+2, \ldots).$$

The relations (3) and (4) give $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Gamma_n$ and (2) is verified.

It can be shown by a similar process that

<u>L (6)</u>: $\Gamma_1 \supset \Gamma_2 \supset \cdots$).

$\{\Gamma_n\}$ converges, $\lim_{n\to\infty}\Gamma_n = \bigcap_{n=1}^{\infty}\Gamma_n$.

Let τ be a completely additive set function defined on a completely additive class $\mathfrak{a}$ of sets.

<u>T (15)</u>: $\{E_n\} \subset \mathfrak{a}$, $E_1 \subset E_2 \subset \cdots$).

$$\lim_{n\to\infty}\tau(E_n) = \tau(\lim_{n\to\infty}E_n).$$

<u>Proof</u>: Since $E_1 \subset E_2 \subset \cdots$ we have from

L (5) that $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$. Now let $E_0 = \phi$.
We show that

(1) $$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_n - E_{n-1}).$$

Let $x \in \bigcup_{n=1}^{\infty} E_n$. Then $x \in E_n$ for some $n = k$. $\exists$ a first such k, call it k_1. Then $x \notin E_{k_1-1}$ and so $x \in (E_{k_1} - E_{k_1-1})$. Then $x \in \bigcup_{n=1}^{\infty} (E_n - E_{n-1})$. Now let $x \in \bigcup_{n=1}^{\infty} (E_n - E_{n-1})$. Then $x \in (E_n - E_{n-1})$ for some $n = k$ and so $x \in E_k$. Then we have that $x \in \bigcup_{n=1}^{\infty} E_n$. This proves (1). Now we show that

(2) $$(E_n - E_{n-1}) \cap (E_m - E_{m-1}) = \phi, \quad (m \neq n).$$

There is no restriction in taking $m < n$. Let $x \in (E_n - E_{n-1})$ and $x \in (E_m - E_{m-1})$. Then $x \in E_n$, $x \notin E_{n-1}$, and $x \in E_m$. Since $x \notin E_{n-1}$ it follows by hypothesis that $x \notin E_k$ $(k < n)$. But $x \in E_m$. Since $m < n$ this is a contradiction and (2) follows.

We now have by (1), and II of D (4), that

$$\begin{aligned}
\tau(\lim_{n\to\infty} E_n) &= \tau\left(\bigcup_{n=1}^{\infty} E_n\right) = \tau\left(\bigcup_{n=1}^{\infty} (E_n - E_{n-1})\right) \\
&= \sum_{n=1}^{\infty} \tau(E_n - E_{n-1}) \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \tau(E_n - E_{n-1}) \\
&= \lim_{k\to\infty} \tau\left(\bigcup_{n=1}^{k} (E_n - E_{n-1})\right).
\end{aligned}$$

By the nature of the sequence $\{E_n\}$

$$\bigcup_{n=1}^{k} (E_n - E_{n-1}) = \bigcup_{n=1}^{k} E_n = E_k.$$

Thus

$$\tau(\lim_{n \to \infty} E_n) = \lim_{k \to \infty} \tau(E_k).$$

<u>L (7)</u>: $A \subset \mathfrak{a},\ B \subset \mathfrak{a} \ \cdot)\cdot\ (A-B) \subset \mathfrak{a}.$

<u>Proof</u>: Let $E_1 \subset \mathfrak{a}$, $E_2 \subset \mathfrak{a}$. By D (2), $\phi \in \mathfrak{a}$, and we may write

$$\bigcup_{n=1}^{\infty} E_n \subset \mathfrak{a}$$

where $E_n = \phi \quad (n > 2)$. This says that $(E_1 \cup E_2) \subset \mathfrak{a}$.

By D (2), $C(A) \subset \mathfrak{a}$ and so $B \cup C(A) \subset \mathfrak{a}$. Hence $C(B \cup C(A)) \subset \mathfrak{a}$. It is easily shown that

$$C(B \cup C(A)) = (A-B)$$

and the lemma follows.

<u>T (16)</u>: $\left.\begin{array}{l} \{E_n\} \subset \mathfrak{a},\quad E_1 \supset E_2 \supset \cdots \\ \tau(E_n) \text{ finite for some } n \end{array}\right\} \cdot)\cdot$

$$\lim_{n \to \infty} \tau(E_n) = \tau(\lim_{n \to \infty} E_n).$$

<u>Proof</u>: Let $A \subset \mathfrak{a}$, $B \subset \mathfrak{a}$. We show first that

(1) $$A \supset B,\ \tau(B) \text{ finite } \cdot)\cdot$$
$$\tau(A-B) = \tau(A) - \tau(B)$$

and

(2) $$A \supset B,\ \tau(A) \text{ finite } \cdot)\cdot$$
$$\tau(B) \text{ finite}.$$

For (1) we have that

$$A = (A-B) \cup B.$$

Since $(A-B) \cap B = \phi$ we have by II of D (4)

(3) $$\nabla(A) = \nabla(A-B) + \nabla(B).$$

Now $\nabla(B)$ is finite, so we may write

$$\nabla(A) - \nabla(B) = \nabla(A-B).$$

For (2): Since $\nabla(A)$ is finite, we have by (3) that $[\nabla(A-B) + \nabla(B)]$ is finite. If either $\nabla(A-B)$ or $\nabla(B)$ is infinite, then $\nabla(A)$ is infinite, contrary to the hypothesis of (2). If $\nabla(A-B)$ and $\nabla(B)$ are both infinite and (a): have opposite signs, or (b): have like signs, their sum is either (a): indeterminate, or (b): infinite. In either case the hypothesis of (2) is again contradicted. So both are finite.

We proceed with the proof of the theorem. Let n_0 be an n for which $\nabla(E_n)$ is finite. By L (6), $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$, and we have also that

(4) $$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n \subset E_{n_0}.$$

Since $\nabla(E_{n_0})$ is finite, we have by (4) and (2) that $\nabla(\bigcap_{n=1}^{\infty} E_n)$ is finite, and so $\nabla(\lim_{n\to\infty} E_n)$ is finite. Thus by (4) and (1) we have

(5) $$\nabla\left[\lim_{n\to\infty}(E_{n_0} - E_n)\right] = \nabla(E_{n_0} - \lim_{n\to\infty} E_n)$$

$$= \nabla(E_{n_0}) - \nabla(\lim_{n\to\infty} E_n).$$

Now for any K

(6) $$(E_{n_0} - E_K) \subset (E_{n_0} - E_{K+1}).$$

For if $X \in (E_{n_0} - E_K)$ then $X \in E_{n_0}$ and $X \notin E_K$. Since $E_K \supset E_{K+1}$, $X \notin E_{K+1}$ and (6) follows.

By L (7) and (6), T (15) applies and

(7) $$\nu\left[\lim_{n\to\infty} (E_{n_0} - E_n)\right] = \lim_{n\to\infty} \nu(E_{n_0} - E_n).$$

Then from (5) and (7)

(8) $$\nu(E_{n_0}) - \nu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \nu(E_{n_0} - E_n).$$

By the nature of the sequence $\{E_n\}$ we have that for $n \geq n_0$, $E_n \subset E_{n_0}$ and by (2), that E_n is finite $(n \geq n_0)$. Hence for any $n \geq n_0$, (1) holds and

(9) $$\lim_{n\to\infty} \nu(E_{n_0} - E_n) = \lim_{n\to\infty}\left[\nu(E_{n_0}) - \nu(E_n)\right].$$

Substituting (9) into (8), we have

$$\nu(E_{n_0}) - \nu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty}\left[\nu(E_{n_0}) - \nu(E_n)\right]$$

$$= \nu(E_{n_0}) - \lim_{n\to\infty} \nu(E_n).$$

The theorem follows by subtraction.

In T (14) it was shown that $\mathfrak{M}$ is a completely additive class of sets. Then with D (6), the hypotheses of T (15) and T (16) are satisfied for ν an outer measure function μ^*, and $\mathfrak{a} = \mathfrak{M}$; i.e., for μ^* and sequences $\{E_n\}$ composed of μ^*-measurable sets.

T (17): $\{E_n\} \subset \mathfrak{a}$, μ a measure function on $\mathfrak{a}$).

(1) $$\mu(\underline{\lim}_{n\to\infty} E_n) \leq \underline{\lim}_{n\to\infty} \mu(E_n),$$

and if $\{E_n\}$ is $\ni$ $\mu(\bigcup_{n=1}^{\infty} E_n) < \infty$ then

(2) $$\mu(\overline{\lim}_{n\to\infty} E_n) \geq \overline{\lim}_{n\to\infty} \mu(E_n).$$

Proof: We prove (1): For each n, let

$$A_n = \bigcap_{k=n}^{\infty} E_k.$$

Obviously

(3) $$A_n \subset E_n.$$

We show that for each n

(4) $$A_n \subset A_{n+1}.$$

Let $x \in A_n = \bigcap_{k=n}^{\infty} E_k$. Then $x \in E_n$ and also $x \in E_{n+1}, E_{n+2}, \ldots$. So $x \in \bigcap_{k=n+1}^{\infty} E_k = A_{n+1}$.

Thus by (3) and D (6)

$$\mu(A_n) \leq \mu(E_n)$$

and so

$$\underline{\lim}_{n\to\infty} \mu(A_n) \leq \underline{\lim}_{n\to\infty} \mu(E_n).$$

By (4) and D (6) we may apply T (15). Then by T (15), L (5), and D (12)

$$\underline{\lim}_{n\to\infty} \mu(E_n) \geq \underline{\lim}_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$$

$$= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k)$$

$$= \mu(\underline{\lim}_{n\to\infty} E_n).$$

The proof of (2) is similar to that of (1): For each n define

$$C(A_n) = C\left(\bigcap_{k=n}^{\infty} C(E_k)\right).$$

Now

(5) $$C(A_n) \supset E_n.$$

For, let $x \in E_n$. Then $x \notin C(E_n)$ and so $x \notin \bigcap_{k=n}^{\infty} C(E_k)$. Thus $x \in C\left(\bigcap_{k=n}^{\infty} C(E_k)\right) = C(A_n)$.

Also

(6) $$C(A_n) \supset C(A_{n+1}).$$

For if $x \in C(A_{n+1})$ then $x \notin A_{n+1} = \bigcap_{k=n+1}^{\infty} C(E_k)$ and so $x \notin$ some $C(E_{k_0})$ $(k_0 \geq n+1)$. But then $x \notin \bigcap_{k=n}^{\infty} C(E_k)$ and so $x \in C(A_n)$.

Finally, we show that for each n

(7) $$\mu(C(A_n)) < \infty.$$

To do this we show that

$$C(A_n) \subset \bigcup_{n=1}^{\infty} E_n.$$

Relation (7) will then follow from hypothesis and the fact that μ is non-decreasing.

Let $x \in C(A_n)$. Then $x \notin \bigcap_{k=n}^{\infty} C(E_k)$, and so $x \notin$ some $C(E_{k_0})$ $(k_0 \geq n)$. Then $x \in E_{k_0}$ and thus $x \in \bigcup_{n=1}^{\infty} E_n$.

Now μ is non-decreasing. Then by (5) and D (11)

(8) $$\overline{\lim_{n\to\infty}}\, \mu(E_n) \leq \overline{\lim_{n\to\infty}}\, \mu(C(A_n)).$$

By (6) and (7), T (16) holds for $\{C(A_n)\}$; i.e.,

(9) $$\lim_{n\to\infty} \mu(C(A_n)) = \mu(\lim_{n\to\infty} C(A_n)).$$

This says that $\lim_{n\to\infty} \mu(C(A_n))$ exists.

By D (11)

(10) $$\lim_{n\to\infty} \mu(C(A_n)) = \overline{\lim_{n\to\infty}} \mu(C(A_n)).$$

Substituting (10) into (9), and the result into (8), we get

$$\overline{\lim_{n\to\infty}} \mu(E_n) \leq \mu(\lim_{n\to\infty} C(A_n)).$$

By (6), L (6) applies. Then

(11) $$\overline{\lim_{n\to\infty}} \mu(E_n) \leq \mu(\lim_{n\to\infty} C(A_n)) = \mu(\bigcap_{n=1}^{\infty} C(A_n))$$
$$= \mu\left[\bigcap_{n=1}^{\infty} C(\bigcap_{k=n}^{\infty} C(E_k))\right].$$

But

(12) $$C(\bigcap_{k=n}^{\infty} C(E_k)) = \bigcup_{k=n}^{\infty} E_k.$$

For, let $x \in C(\bigcap_{k=n}^{\infty} C(E_k))$. Then $x \notin \bigcap_{k=n}^{\infty} C(E_k)$ and so $x \notin C(E_k)$ for some k_0 $(k_0 \geq n)$. Then $x \in E_{k_0}$ and $x \in \bigcup_{k=n}^{\infty} E_k$.

Conversely, let $x \in \bigcup_{k=n}^{\infty} E_k$. Then $x \in E_k$ for some k' $(k' \geq n)$. So $x \notin C(E_{k'})$ and thus $x \notin \bigcap_{k=n}^{\infty} C(E_k)$. Then $x \in C(\bigcap_{k=n}^{\infty} C(E_k))$.

By (11), (12), and D (12),

$$\overline{\lim_{n\to\infty}} \mu(E_n) \leq \mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = \mu(\overline{\lim_{n\to\infty}} E_n).$$

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