SAMPLED-DATA FILTERING AND CONTROL OF LINEAR PARAMETER VARYING SYSTEMS WITH DELAY

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Amin Ramezanifar

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Abstract

In this dissertation we address a variety of problems in filtering and control of dynamic systems with emphasis on their digital implementation. We focus on linear parameter varying (LPV) systems which have been widely utilized in engineering applications. LPV systems include a class of linear systems whose dynamic depends on time-varying parameters. These systems have resulted in significant improvements in the study of time-varying and nonlinear systems.

In a filtering problem, we aim to estimate the states of a dynamic system by utilizing the output measurement of the system. Applications of filters abound in practical and theoretical problems. In a control problem, the objective is to design a controller to ensure the closed-loop system stability and often to satisfy a prescribed level of performance. A main concern in the present study is the implementation of the controller or filter which is often fulfilled by means of a digital device operating in the discrete time domain. Due to the combination of the system continuous-time dynamics and the controller or filter discrete dynamics connected through analog to digital and digital to analog converter devices, the closed-loop system is a hybrid one and is difficult to analyze mathematically. The incorporation of continuous-time and discrete-time signals in a system is often referred to as sampled-data system. A particular difficulty in sampled-data systems is to ensure that the digital controller (filter) meets the design specifications in between the samples. In this dissertation, we develop new methods to take into account this requirement.

Two chapters of this dissertation are devoted to the design of filters for LPV systems. First, we design a continuous-time filter for a continuous-time state-delayed LPV system whose dynamics includes a time varying delay. Next, we address the sampled-data filter design problem for continuous-time LPV systems. In the second part of this dissertation, we investigate the control problem of LPV systems in the framework of sampled-data design. First we present a new approach for the sampled-data control of continuous-time LPV systems. Next, we extend the established results for LPV systems with internal delay.

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Chapter1

Introduction

1.1 Linear Parameter Varying system

Linear parameter varying systems (LPV) are a class of linear time-varying systems, whose dynamics depend on a time-varying vector of parameters. These parameters, also referred to as scheduling parameters, are assumed to be unknown in advanced when the system is modeled, but measurable in real-time [1, 2]. LPV system theory has led to significant improvements in the study of time-varying and nonlinear systems [3, 4]. Using a technique called quasi-LPV modeling, many nonlinear dynamic systems can be formulated in terms of LPV systems. This strategy suitably allows for hiding the nonlinearities of the system dynamic equations and converts the actual nonlinear model of the system into a simpler linear model effective for filtering and control design purposes. The employment of LPV approaches for practical applications have been successfully examined including engines and rotary machines, robotics, aeronautics and microelectrophoretical systems (MEMS) [5, 6, 7, 8, 9, 10].

Traditionally, in order to control a linear time varying or nonlinear system, the system is linearized at finite number of operating points and a set of local linear time invariant (LTI) controllers are designed corresponding to each point. Next, the controller is scheduled according to the operating point. The LPV gain scheduling approach has distinct advantage over the traditional gain scheduling methods in the sense that the synthesis of the controller is performed in a direct way such that the stability and performance specification of the closed loop system is guaranteed for the entire range of operation [11]. In this approach, the scheduling routine is carried out accurately based on the measurement or estimation of scheduling parameters.

For decades, linear matrix inequalities (LMIs) have proven useful in synthesis and analysis of control problems in a systematic way [12, 13]. These problems are formulated as optimization problems with a linear objective and a set of constraints that are affine in the decision variables. Once the synthesis conditions are presented as LMIs, they can be solved numerically using available software packages such as MATLAB LMI Control Toolbox [14], YALMIP [15], and SeDuMi [16]. The application of LMI approach in filtering and control problems has been considered in many studies and has been continuously extended for multi-objective problems, uncertain systems, discrete-time systems, reduced-order design, as well as analysis of LPV systems [1, 17, 18, 19, 20].

1.2 Sampled-Data Design Framework

In the past few decades, advances in the computing devices has led to efficient ways to digitally implement the controllers and filters for the physical systems, which function in continuous-time domain. Digital implementation of the controllers and filters results in the mixture of continuous-time and discrete-time systems and signals. Due to this hybrid nature, there is a need to adapt the continuous-time filtering theory to capture this level of complexity. The new branch of control theory that was born by paying attention to the problems related to the digital implementation of the controllers (filters) is called sampled-data control (filtering) theory, also known as hybrid systems theory. In a typical hybrid process the measurable output signals are periodically sampled with an analog to digital (A/D) converter. Then, the digitized inputs are processed using a digital device (controller or filter) and will be used either in a computer program or in real world by converting to analog values, i.e., a digital to analog (D/A) converter. The problem of digital design was studied primarily within the area of digital control theory [21, 22]. In contrast to the modern sampled-data theory, these approaches only approximately cope with the behavior of the continuous-time signals in the control system since the behavior of such systems can be captured and studied only at the sampling instants [23, 24]. Modern sampled-data control theory, on the other hand, provides an exact solution method for the analysis and synthesis of sampled-data control systems with inter-sample behavior taken completely into account [25].

An inordinate number of practical systems exist in continuous-time domain, for which it is more convenient to design a controller in the same domain. However, the implementation of the designed controller has to be often carried out in a digital device. To this aim, a designed continuous-time controller can be simply discretized using conventional methods, e.g., trapezoidal approximation, along with the use of sampling and holding devices cascaded with the plant [26, 27]. In this approach, known as the indirect method, the effect of sampling frequency does not affect the design procedure until the very last step, where the controller is discretized. Therefore, the need for redesigning the controller for different sampling rates is avoided. But since this method disregards the effect of sampling and holding devices in the design process, increasing the sampling period degrades the effectiveness of the control action, which may result in the instability of the closed-loop system. Conversely, the so-called direct sampled-data design always takes into account the effect of converter devices in the design process [25]. The main essence of all direct methods is to map the hybrid closed-loop system to either discrete-time or continuous-time domain and then using an existing design method in the associated time domain. While the former mapping is completed using lifting method, the latter one is carried out by means of input-delay approach.

Using the lifting technique for a control design problem, the plant is first augmented by the sampler and hold devices and is transformed to a system with a finite dimensional state space representation and with infinite dimensional input and output spaces. Next, an equivalent discrete-time system of the lifted augmented system is derived for which an $\mathcal{H}_\infty$ or $\mathcal{H}_2$ discrete-time controller is designed. The idea of lifting technique was extended to the LPV systems in [28, 29] to design state-feedback and output-feedback controllers. There have also been additional efforts on sampled-data control design for LPV systems, e.g., [30, 31]. In this dissertation, we employ the idea of lifting to synthesize a discrete-time filter for an LPV continues-time system. To this aim, we lift the continuous-time system and find its equivalent discrete-time representation. Then, we design a discrete-time filter to solve the energy-to-energy gain problem for the filtering error system. Alternatively, the input delay approach proposed in [32] and further developed in [33, 34, 35, 36] can be employed for direct sampled-data design. Utilizing a zero-order hold as the digital-to-analog converter, the same control command is used during a sampling period. Here, one can notice a delay in updating the control action which grows as time passes and then is reset to zero at the sampling instant. Using this fact, in the input-delay method, the piecewise-constant output of the zero-order-hold is mapped into a continuous-time signal with a time-varying delay. By this transformation in the sampled-data control and filter problems, we have a pure continuous-time system including delay in its model. The presence of delay in the system dynamics imposed by the input-delay approach, calls for a new design method to ensure the stability of the closed-loop system, as well as a desirable level of performance. The interested reader is referred to [37, 38] and numerous references therein on stability analysis and control of time delay systems. Utilizing this technique for LPV systems requires deep study of delayed LPV models which are more challenging than LTI systems to handle. There has been recent efforts devoted to the control design problem of time-delayed LPV systems [39, 40, 41, 42, 43]. The existing criteria for analysis of time delay systems are categorized into either delay-independent or delay-dependent approaches. In the delay-independent approach, a controller is designed such that the system remains stable regardless of the time delay magnitude. In contrast, by considering the information of the time delay, the delay-dependent approach leads to generally less conservative results specifically for smaller time delays. The key solution to delay-dependent synthesis methods is the appropriate usage of Lyapunov-Krasovskii functionals which takes into account information of the delay, including bounds on the delay, shape of the delay and rate of the delay.

1.3 Outline of the Thesis

In this dissertation we focus on two direct design approaches, that is, lifting method and input delay method, to address some open problems in system theory. In chapter 2, a filter design method for continuous-time LPV systems with time-varying delay is presented that aims to estimate some signals of the system. The design objective is to ensure the estimation error system stability and a prescribed level of performance defined to be the induced energy-to-energy gain (or $\mathcal{H}_\infty$-norm) from the disturbance input to the estimation error signal. In this chapter, the filter synthesis conditions are formulated in terms of LMI optimization problems. To show the capability of this study in estimation, the proposed design method is examined through a numerical example and the results are compared to a previously performed related study .

In chapter 3, we address the sampled-data filter design problem for continuous-time LPV systems. The filtering error system obtained from augmenting a continuous-time LPV system and the sampled-data filter is a hybrid system. The sampled-data filter design objective is to ensure the error system stability and a prescribed level of the induced energy-to-energy gain (or $\mathcal{H}_\infty$-norm) from the disturbance input to the estimation error. To this purpose, we employ a lifting method to design the filter in a direct way. The sampled-data filter synthesis conditions are formulated in terms of LMI optimization problems. The viability of the proposed design method to cope with variable sampling rates is illustrated through numerical examples, where reliable estimation of the LPV system outputs is achieved.

In chapter 4, we examine the sampled-data control problem for continuous-time LPV systems without and with internal delay. In order to analyze the sampled-data system, from stability and performance perspectives, first we use the input-delay approach to map the hybrid closed-loop system into the continuous-time domain with delay in the states. For an LPV system with internal time delay, this causes a closed-loop system with two types of delay, that is, the internal delay of the system and the one imposed by input-delay mapping. Next, for analysis of stability and performance of the system, we utilize Lyapunov-Krasovskii functionals which lead to delay-dependent matrix inequality conditions. Then, using a set of appropriately defined slack variables, we propose a sampled-data control synthesis condition based on the solution to an LMI optimization problem. To complete the discussion, we consider two structures for the sampled-data controller, namely full-order dynamic output-feedback and state-feedback controllers. In order to have an organized chapter, we present the results for the LPV systems without delay and with delay separately. Finally, several numerical examples will be presented for the sampled-data control of LPV systems with delay, without delay and for the dynamic output-feedback and state-feedback structures. The simulation result demonstrate the viability of the proposed control design method to satisfy the stability and performance objectives.

Chapter 5 summarizes the main contributions of this dissertation and provides remarks for future work in the area of sampled-data control and filtering.

The notation used throughout this dissertation is standard. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ is the set of non-negative real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the set of real vectors of dimension n and the set of real $n \times n$ matrices, respectively. In addition, $\mathbb{S}^{n\times n}_{++}$ denote the set of real symmetric positive definite matrices. The notation $(\cdot)^T$ denotes the transpose of a real matrix. Given a symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$, $X > 0$ ($X \geq 0$) denotes matrix positive definiteness (semi-definiteness). Given a matrix $Y \in \mathbb{R}^{n\times m}$ with rank r, the orthogonal complement $Y^{\perp}$ is defined as the $(n-r)\times n$ matrix that satisfies $Y^{\perp}Y=0$ and $Y^{\perp}Y^{\perp\top}>0$. In a symmetric matrix, the star $\star$ in the (i,j) element denotes transpose of the (j,i) element. The $\mathcal{L}_2$-norm of a vector valued function $f(t)$ is defined as $\|f\|_{\mathcal{L}_2}=\left\{\int_0^{\infty} f^T(t)f(t)dt\right\}^{1/2}$, which is indeed the energy of the signal $f(t)$. The $\mathcal{L}_2[a,b]$ norm of a continuous-time signal is defined as $\|f\|_{\mathcal{L}_2[a,b]}=(\int_b^a |f(t)|^2)^{\frac{1}{2}}$. The space of the time series with a finite $\mathcal{L}_2[a,b]$-norm is called the signal space $\mathcal{L}_2[a,b]$. The l_2-norm of a discrete signal is defined as $\|f\|_{l_2}=(\sum_{k=0}^{\infty}|f(k)|^2)^{\frac{1}{2}}$. Finally, $(\cdot)^*$ denotes the adjoint of an operator on the Hilbert space.

Chapter2

Filtering of Linear Parameter Varying Systems with Delay

2.1 Introduction

Filters utilize the output measurements of a dynamic system to estimate the states or a combination of the states of the system. From this point of view, an observer which estimates the non measurable states of a system can also be treated as a filter. Running in parallel, a filter is served to detect fault occurrence in dynamic systems by comparing some outputs of the physical system with the expected ones produced by the filter which represents a healthy system [44, 45]. The performance of a filter is often assessed in terms of a measure of the state estimation error which is the difference between the actual and the estimated state. In this context, specially when the statistical information is unknown, the $\mathcal{H}_\infty$ filtering method can be employed to minimize the energy of the estimation error signal for the worst bounded energy disturbance input [46, 47]. Other performance measures such as energy-to-peak gain, peak-to-peak gain or a combination of these objectives from the disturbance input to the estimation error signal can be also utilized for the $\mathcal{H}_\infty$ filter design as they are used in control design [48].

In this chapter, we are interested in the development of a method for the design of filters for linear parameter varying (LPV) systems including time-varying delay in the state vector. It is remarked that a system with delay in the input can readily be converted to the one with state delay. LPV systems constitute a class of linear systems whose dynamics depends on time-varying parameters, also known as scheduling parameters. Not only the time-varying model of the system, but also the inclusion of delay intensifies the complexity of the problem which calls for a state-of-the-art analysis method to handle this intricacy. The literature on stability analysis and control of time-delay systems is widespread (see [37, 38] for LTI and [49, 50, 51] for LPV delayed systems). The existing criteria for analysis of time delay systems are categorized into either delay-independent or delay-dependent approaches. In the delay-independent approach, a controller is designed such that the system remains stable regardless of the time delay magnitude. In contrast, by considering the information of the time delay, the delay-dependent approach leads to a generally less conservative results specifically for smaller time delays.

It is noted that there is a close connection between the solution of the $\mathcal{H}_\infty$ filtering problem and the $\mathcal{H}_\infty$ control problem. However, it is rewarding to derive an independent solution for the filtering problem which results in a simpler set of design conditions. In recent decades, the filtering problem of LPV systems with time delay has been studied widely [49, 50, 52, 53]. According to the literature, two class of filters are often utilized; the memoryless filter that is independent of delay and the irrational filter that includes the delay in its dynamics. Intuitively, the former structure leads in better performance since the structure of the filter coincides to that of the plant.

Contribution of this chapter is as following: We propose a method for the design of filters for LPV systems including state delay. The design should guarantee asymptotic stability of the estimation error system as well as providing a specified level of performance, namely the energy-to-energy gain from external disturbance to estimation error signal. The main result of this chapter is inspired by the method proposed in [54] for continuous-time controller synthesis of state-delayed LPV systems. The key solution to this problem is to find a parameter-dependent Lyapunov-Krasovskii functional that results in a delay-dependent synthesis method that can handle fast-varying time delay. To ensure that the solution to the synthesis problem is in the form of a linear matrix inequality (LMI) optimization problem, we introduce slack variables (see [55]) to relax the resulting condition in terms of an LMI problem. Using the derived formulation based on the slack variables, we then obtain the synthesis condition for the filtering design that is considerably simpler compared to the existing methods.

The chapter is organized as following: Section 2.2 presents the problem statement and preliminaries. In section 2.3, we analyze the stability and performance of LPV systems with time delay. In Section 2.4, the filter design formulation is presented. Section 2.5 examines the proposed design methods using a numerical example. The conclusion is presented in section 2.6.

2.2 Preliminaries and Problem Statement

We consider the following state-space representation for an LPV system

$$
\begin{aligned}
\dot{x}(t) &= A(\rho(t))x(t) + A_h(\rho(t))x(t - h(\rho(t))) + B_1(\rho(t))w(t) \\
z(t) &= C_1(\rho(t))x(t) + C_{1h}(\rho(t))x(t - h(\rho(t))) + D_{11}(\rho(t))w(t) \\
y(t) &= C_2(\rho(t))x(t) + C_{2h}(\rho(t))x(t - h(\rho(t))) + D_{21}(\rho(t))w(t) \\
x(\theta) &= \phi(\theta) \qquad \forall t \in [-h_m \quad 0],
\end{aligned}
\tag{2.1}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the vector of outputs to be estimated, $y(t) \in \mathbb{R}^{n_y}$ is the measurement vector, $w(t) \in \mathbb{R}^{n_w}$ is exogenous disturbance vector containing both process and measurement noise with finite energy. The system matrices $A(\cdot)$, $A_h(\cdot)$, $B_1(\cdot)$, $C_1(\cdot)$, $C_{1h}(\cdot)$, $D_{11}(\cdot)$, $C_2(\cdot)$, $C_{2h}(\cdot)$ and $D_{21}(\cdot)$ are real continuous functions of a time varying parameter vector $\rho(t)$ and of appropriate dimensions. In this model, $h(\cdot)$ is a differentiable scalar function denoting the parameter-dependent time delay and satisfies $0 \leq h(\cdot) \leq h_m$. Starting from $t = 0$, the initial condition $\phi(\cdot)$ determines the integral solution of (2.1) uniquely. The parameter vector $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$ is assumed to be measurable in real-time, where $\mathcal{F}_{\mathcal{P}}^{v}$ is the set of allowable parameter trajectories defined as

$$
\mathcal{F}_{\mathcal{P}}^{v} \equiv \{\rho(t) \in C(\mathbb{R}, \mathbb{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \leq v_i \quad i = 1, 2, ..., s \quad \forall t \in \mathbb{R}_+\}, \tag{2.2}
$$

where $C(\mathbb{R}, \mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathcal{P}$ is a compact set of $\mathbb{R}^s$, and $\{v_i\}_{i=1}^{s}$ are nonnegative numbers. The constraints in (2.2) imply that the parameter trajectories and their variations are bounded. Now consider an n^{th}-order discrete-time parameter-varying filter F represented by the following state-space description

$$
\begin{aligned}
\dot{x}_F(t) &= A_F(\rho(t))x_F(t) + A_{hF}(\rho(t))x_F(t - h(\rho(t))) + B_F(\rho(t))y(t) \\
\hat{z}(t) &= C_F(\rho(t))x_F(t) + C_{hF}(\rho(t))x_F(t - h(\rho(t))) + D_F(\rho(t))y(t),
\end{aligned}
\tag{2.3}
$$

where $x_F(t) \in \mathbb{R}^n$ is the state vector and $\hat{z}(t) \in \mathbb{R}^{n_z}$ is the estimation signal. The system matrices $A_F(\cdot)$, $A_{hF}(\cdot)$, $B_F(\cdot)$, $C_F(\cdot)$, $C_{hF}(\cdot)$ and $D_F(\cdot)$ are real continuous functions of the parameter vector $\rho(t)$ and of appropriate dimensions. We aim to design the aforementioned filter matrices so that $\hat{z}(t)$ be an estimation of the output signal $z(t)$. It is noted that in this chapter, we only consider the full-order filter design problem, where the filter has the same order as the plant. The results presented in this chapter can be extended to design reduced-order filters as well. The estimation is performed by feeding the information of the measurement signal $y(t)$ and the scheduling parameter $\rho(t)$ to the filter. To this purpose, we define the estimation error to be $e(t) = z(t) - \hat{z}(t)$. Figure 2.1 shows the estimation error system configuration. For this error system that relates the disturbance signal $w(t)$

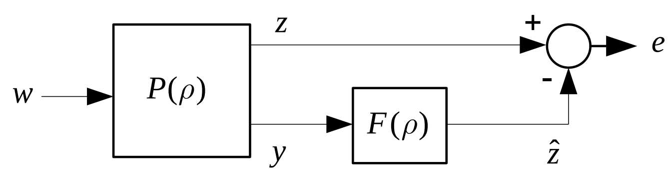

Figure 2.1: Estimation error system.

to the estimation error signal $e(t)$, the induced $\mathcal{L}_2$-norm ($\mathcal{H}_\infty$-norm) is defined as

$$\|T_{we}\|_{i,2} = \sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{v}} \sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}, \tag{2.4}$$

where T_{we} is the operator inducing the disturbance $w(t)$ to the estimation error $e(t)$. This value also known as energy-to-energy gain of the augmented system, indicates the worst case output energy $\|e\|_{\mathcal{L}_2}$ over all bounded energy disturbances $\|w\|_{\mathcal{L}_2}$ over all admissible parameter vector $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$. We aim to design the filter F so that the the filtering error system consists of system (2.1) and filter (2.3) is asymptotically stable, and also the corresponding energy-to-energy gain is minimized, i.e.,

$$\min_{F} \|T_{we}\|_{i,2}. \tag{2.5}$$

Instead of the optimal design problem (2.5), one can solve the γ-suboptimal energy-to-energy gain in which a filter F is sought such that

$$\|T_{we}\|_{i,2} < \gamma, \tag{2.6}$$

where γ is a given positive scalar. If the inequality (2.6) holds true, then the estimation error energy will be bounded by $\gamma\|w\|_{\mathcal{L}_2}$ for any nonzero disturbance $w(t)$ with bounded energy. That is, as long as $w(t) \in \mathcal{L}_2 - \{0\}$, regardless of its nature, the error does not exceed a specific bound.

Now we augment the plant (2.1) with the filter (2.3) to obtain the state-space representation of the error system. Defining

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix}, \tag{2.7}$$

we have

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{A}_h\bar{x}(t-h) + \bar{B}w(t) \\ z(t) &= \bar{C}\bar{x}(t) + \bar{C}_h\bar{x}(t-h) + \bar{D}w(t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}, \ \bar{A}_h = \begin{bmatrix} A_h & 0 \\ B_F C_{2h} & A_{hF} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B_1 \\ B_F D_{21} \end{bmatrix}, \\ &\bar{C} = \begin{bmatrix} C_1 - D_F C_2 & -C_F \end{bmatrix}, \ \bar{C}_h = \begin{bmatrix} C_{1h} - D_F C_{2h} & -C_{hF} \end{bmatrix}, \text{and} \\ &\bar{D} = \begin{bmatrix} D_{11} - D_F D_{21} \end{bmatrix}. \end{aligned} \tag{2.9}$$

Equation (2.8) is a continuous-time LPV system with state delay. For the sake of simplicity, during the discussion we may drop the dependency of the matrices to the parameter vector. Next, we provide some useful lemmas that will play a key role in the proofs of the main results of the chapter.

Lemma 2.1. (Cauchy-Schwarz Inequality): For any positive definite matrix P and any $v(\alpha) \in \mathbb{R}^n$

$$h\int_{t-h}^{t} v(\alpha)^T P v(\alpha) d\alpha \geq \left[\int_{t-h}^{t} v(\alpha) d\alpha\right]^T P \left[\int_{t-h}^{t} v(\alpha) d\alpha\right].$$

(See [56])

Lemma 2.2. (Projection Lemma): Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices Λ and Γ of appropriate dimensions, the linear matrix inequality

$$\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0 \tag{2.10}$$

is feasible in matrix Θ if and only if

$$\mathcal{N}_\Lambda^T\Psi\mathcal{N}_\Lambda < 0 \tag{2.11}$$

and

$$\mathcal{N}_\Gamma^T\Psi\mathcal{N}_\Gamma < 0, \tag{2.12}$$

where $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Gamma$ are any basis of the null-space of Λ and Γ. For a matrix $\Gamma \in \mathbb{R}^{n\times m}$ with rank r, $\mathcal{N}_\Gamma \in \mathbb{R}^{(n-r)\times n}$ and satisfies the two conditions $\mathcal{N}_\Gamma\Gamma = 0$ and $\mathcal{N}_\Gamma\mathcal{N}_\Gamma^T > 0$. (See [48])

2.3 Stability and Performance Analysis of Time Delay LPV Systems

In this section, we present stability and $\mathcal{H}_\infty$-norm performance analysis conditions for time delay LPV systems by deriving a set of linear matrix inequality (LMI) conditions. To this aim, we utilize delay-dependent Lyapunov Krasovskii functionals.

2.3.1 Stability Analysis

We first consider the unforced closed-loop LPV system (2.8), that is

$$\dot{\bar{x}}(t) \;\; = \;\; \bar{A}(\rho(t))\bar{x}(t) + \bar{A}_h(\rho(t))\bar{x}(t - h(\rho(t))). \tag{2.13}$$

Lyapunov-Krasovskii stability theory serves as a useful tool to achieve delay-dependent conditions for the stability analysis of the system represented by (2.13). To this aim, we need to find a positive definite functional with an infinitesimal upper bound, whose time derivative is negative. The interested reader is referred to [57, 56, 58, 37] for an extensive review of the theory and the Lyapunov-Krasovskii functional selection. As the first result of this chapter, we present the following theorem as a sufficient condition to ensure asymptotic stability of the LPV system (2.13).

Theorem 2.1. The time-delay LPV system (2.13) is asymptotically stable for all $h(\cdot) \leq h_m$, if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \rightarrow \mathbb{S}_{++}^{2n\times 2n}$ and constant matrices $R, Q \in \mathbb{S}_{++}^{2n\times 2n}$ such that for all $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & h_m \bar{A}^T R \\ \star & -(1-\dot{h})Q - R & h_m \bar{A}_h^T R \\ \star & \star & -R \end{bmatrix} < 0. \tag{2.14}$$

Proof: We consider the following Lyapunov-Krasovskii functional

$$V(\bar{x}_t, \rho) = V_1(x, \rho) + V_2(\bar{x}_t, \rho) + V_3(\bar{x}_t, \rho) \tag{2.15}$$

with

$$\begin{aligned} V_1(\bar{x}, \rho) &= \bar{x}^T(t) P(\rho(t)) \bar{x}(t), \\ V_2(\bar{x}_t, \rho) &= \int_{t-h(\rho(t))}^{t} \bar{x}^T(\xi) Q \bar{x}(\xi)\, d\xi, \text{ and} \\ V_2(\bar{x}_t, \rho) &= \int_{-h_m}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(\xi)\ h_m R\ \dot{\bar{x}}(\xi)\, d\xi\, d\theta, \end{aligned}$$

where the notation $\bar{x}_t(\theta)$ is used to represent $\bar{x}(t+\theta)$ for $\theta \in [-h_m \ \ 0]$. It can be shown that (2.15) is a positive definite with infinitesimal upper bound functional. It is noted that (2.15) is chosen to be dependent on the LPV parameter vector $\rho(t)$ and the maximum sampling interval h to result in less conservative stability conditions. In order for the system to be asymptotically stable, it suffices that time derivative of (2.15) along the trajectories of the system (2.13) is negative definite. One can readily conclude

$$\dot{V}_1(\bar{x}, \rho) = \dot{\bar{x}}^T(t) P(\rho) \bar{x}(t) + \bar{x}^T(t) P(\rho) \dot{\bar{x}}(t) + \bar{x}^T(t) \dot{P}(\rho) \bar{x}(t), \tag{2.16}$$

$$\dot{V}_2(\bar{x}_t, \rho) \quad = \quad \bar{x}^T(t)Q\bar{x}(t) + (1-\dot{h})\bar{x}^T(t-h)\ Q\ x(t-h), \tag{2.17}$$

and

$$\dot{V}_3(\bar{x}_t, \rho) = h_m^2 \dot{\bar{x}}^T(t) R\ \bar{x}(t) - \int_{t-h_m}^{t} \dot{\bar{x}}^T(\theta)\ h_m R\ \dot{\bar{x}}(\theta) d\theta. \tag{2.18}$$

Since $h \leq h_m$, the integral term in (2.18) satisfies

$$-\int_{t-h_m}^{t} \dot{\bar{x}}^T(\theta)\ h_m\ R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\int_{t-h}^{t} \dot{\bar{x}}^T(\theta)\ h_m\ R\ \dot{\bar{x}}(\theta) d\theta.$$

Employing Lemma 2.1, we can bound the right hand side of the above inequality by

$$\begin{aligned} -\int_{t-h}^{t} \dot{\bar{x}}^T(\theta)\ h_m\ R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\frac{h_m}{h}\left(\int_{t-h}^{t} \dot{\bar{x}}^T(\theta)\ d\theta\right)^T R \left(\int_{t-h}^{t} \dot{x}^T(\theta)\ d\theta\right) = \\ -\frac{h_m}{h}\left[\bar{x}(t) - \bar{x}(t-h)\right]^T R \left[\bar{x}(t) - \bar{x}(t-h)\right]. \end{aligned}$$

Since $-\frac{h_m}{h} \leq -1$, the following inequality is obtained

$$-\int_{t-h}^{t} \dot{\bar{x}}^T(\theta)\ h_m R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\left[\bar{x}(t) - \bar{x}(t-h)\right]^T R \left[\bar{x}(t) - \bar{x}(t-h)\right]. \tag{2.19}$$

Substituting (2.19) in (2.18), we obtain

$$\dot{V}_3(\bar{x}_t, \rho) \quad \leq \quad h_m^2 \dot{\bar{x}}^T(t) R \dot{\bar{x}}(t) - \left[\bar{x}(t) - \bar{x}(t-h)\right]^T R \left[\bar{x}(t) - \bar{x}(t-h)\right]. \tag{2.20}$$

Substituting for $\dot{\bar{x}}(t)$ in (2.16) and (2.20) and then collecting the terms yields

$$\begin{aligned} \dot{V}(\bar{x}_t, \rho) \quad &\leq \quad \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \end{bmatrix} h_m^2 R \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix} \\ &= \quad \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix} R^{-1} \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}, \end{aligned} \tag{2.21}$$

where

$$\mathcal{X} = \begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R \\ \star & -(1-\dot{h})Q - R \end{bmatrix}.$$

To ensure that $\dot{V}(\bar{x}_t, \rho) < 0$, it is sufficient that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R \\ \star & -(1-\dot{h})Q - R \end{bmatrix} + \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix} R^{-1} \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix}^T < 0.$$

Finally, applying Schur complement to the above LMI results in the condition (2.14), and this completes the proof.

2.3.2 Performance Analysis

Next, we present the performance analysis condition for the time-delay LPV system (2.8). The derived condition will be used in the next section for filter design.

Theorem 2.2. The LPV system (2.8) is asymptotically stable and the condition $\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$ holds true for $h(\cdot) \leq h_m$ and zero initial condition if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$, constant matrices $R, Q \in \mathbb{S}_{++}^{2n\times 2n}$ and a positive scalar γ such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & P\bar{B} & \bar{C}^T & h_m \bar{A}^T R \\ \star & -(1-\dot{h})Q - R & 0 & \bar{C}_h^T & h_m \bar{A}_h^T R \\ \star & \star & -\gamma I & \bar{D}^T & h_m \bar{B}^T R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{2.22}$$

Proof: We first define a Lyapunov-Krasovskii functional similar to the one introduced in Theorem 2.1. Next, we apply the following congruent transformation

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

to (2.22). In the obtained inequality, it can be observed that the negative definiteness of the upper left 3×3 block matrix, in light of Theorem 2.1, concludes the asymptotical stability of the system (2.8). Applying Schur complement to (2.22) two times results in

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & P\bar{B} \\ \star & -(1-\dot{h})Q - R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{D}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{D}^T \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{B}^T \end{bmatrix} h_m^2 R \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{B}^T \end{bmatrix}^T < 0.$$

Multiplying the above inequality from left and right by $[\bar{x}^T(t) \quad \bar{x}^T(t-\tau) \quad w^T(t)]^T$ and its transpose respectively, following by minor algebraic manipulations yields

$$\begin{aligned} &\dot{\bar{x}}^T(t)P\bar{x}(t) + \bar{x}^T(t)P\dot{\bar{x}}(t) + \bar{x}^T(t)\dot{P}\bar{x}(t) \\ &+\bar{x}^T(t)Q\bar{x}(t) + (1-\dot{h})\bar{x}^T(t-h)\ Q\ x(t-h) \\ &+h_m^2\dot{\bar{x}}^T(t)R\dot{\bar{x}}(t) - [\bar{x}(t) - \bar{x}(t-h)]^T R\,[\bar{x}(t) - \bar{x}(t-h)] \\ &-\gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0. \end{aligned}$$

Finally, employing (2.16), (2.17) and (2.20), we end up

$$\dot{V}(\bar{x}_t, \rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0. \tag{2.23}$$

Integrating both sides of the inequality (2.23) from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the asymptotical stability and zero initial condition), we arrive at

$$\|z\|_{\mathcal{L}_2} \le \gamma \|w\|_{\mathcal{L}_2},$$

and this completes the proof.

2.3.3 Introduction of slack variables

In order to establish a synthesis condition for the filter (2.3), the corresponding system matrices (2.9) are substituted in (2.22); this, however, results in a bilinear matrix inequality problem due to the byproduct of the Filter matrices with the unknown matrix function P and matrix R. Therefore, we will seek an alternative method based on the introduction of *slack variables* to reformulate the corresponding problem to ensure that an LMI is achieved. The following lemma provides an alternative way to deal with the matrix inequality (2.22).

Lemma 2.3. The LPV system (2.8) is asymptotically stable for all $h(\cdot) \leq h_m$ and satisfies $\|z\|_{\mathcal{L}_2} < \gamma\|w\|_{\mathcal{L}_2}$, if there exist a continuously differentiable matrix function $P(\rho) : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$ and parameter dependent matrices $V_1(\rho), V_2(\rho), V_3(\rho) : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$ and constant matrices $R, Q \in \mathbb{S}_+^{2n\times 2n}$ a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$, the following LMI problem has a feasible solution

$$\begin{bmatrix} -V_1 - V_1^T & P - V_2^T + V_1\bar{A} & -V_3^T + V_1\bar{A}_h & V_1\bar{B} & 0 & h_m R \\ \star & \dot{P} + Q - R + \bar{A}^T V_2^T + V_2\bar{A} & R + \bar{A}^T V_3^T + V_2\bar{A}_h & V_2\bar{B} & \bar{C}^T & 0 \\ \star & \star & -(1-\dot{h})Q - R + \bar{A}_h^T V_3^T + V_3\bar{A}_h & V_3\bar{B} & \bar{C}_h^T & 0 \\ \star & \star & \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{2.24}$$

Proof: We start with rewriting (2.24) as $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$, with

$$\Psi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & h_m R \\ \star & \dot{P} + Q - R & R & 0 & \bar{C}^T & 0 \\ \star & \star & -(1-\dot{h})Q - R & 0 & \bar{C}_h & 0 \\ \star & \star & \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} -I & \bar{A} & \bar{A}_h & \bar{B} & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}. \tag{2.25}$$

The matrix variables V_1, V_2 and V_3 are known as slack variables [55]. We next use Lemma 2.2 (Projection Lemma) by finding the bases for the null space of Λ and Γ as

$$\mathcal{N}_\Lambda = \begin{bmatrix} \bar{A} & \bar{A}_h & \bar{B} & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \qquad \mathcal{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We then substitute the two matrices above in the solvability conditions of Lemma 2.2. Using the solvability condition (2.11) results in the LMI condition (2.24). On the other hand, the solvability condition (2.12) leads to the following LMI

$$\begin{bmatrix} -\gamma I & \bar{D}^T & 0 \\ \star & -\gamma I & 0 \\ \star & \star & -R \end{bmatrix} < 0, \tag{2.26}$$

which is part of LMI (2.24) and is always satisfied as long as there is a feasible solution to (2.24). In summary, feasibility of the LMI condition (2.24) ensures that the LMI problem (2.22) is feasible and based on Theorem 2.2, the proof of Lemma 2.3 is complete.

Remark It is noted that choosing lower number of slack variables, *e.g.*, one or two, would be also possible but leads to more conservative results compared to those proposed here. In fact, using one or two slack variables in (2.25) results in a simpler form of LMI in (2.24) but the feasibility of the LMI problem is not guaranteed particularly for larger time delays. On the other hand, we found out that stacking more than three slack variables in (2.25) results in a complicated LMI problem, which adds to the design complexity.

2.4 Filter design for continuous LPV systems

In order to find the filter matrices, we employ lemma 2.3 for the closed loop system (2.8). In the matrix inequality (2.24), we substitute the closed loop matrices $\bar{A}$, $\bar{A}_h$, $\bar{B}_1$, $\bar{C}_1$ and $\bar{C}_{1h}$ from (2.9) and select the three slack to be as $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ for given scalar valued λ_1, λ_2 and λ_3. Next, we partition V into

$$V = \begin{bmatrix} X & N \\ N^T & E \end{bmatrix}. \tag{2.27}$$

We then define

$$V^{-1} = \begin{bmatrix} Y & M \\ M^T & F \end{bmatrix}, \tag{2.28}$$

where $N(\rho)$, $M(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n}$ and $X(\rho), Y(\rho), E(\rho)$, $F(\rho) : \mathbb{R}^s \to \mathbb{S}^{n\times n}_{++}$. Equations (2.27) and (2.28) together impose a set of constraints as

$$\begin{aligned} XY + NM^T &= I \quad \text{and} \\ YN + ME &= 0. \end{aligned}$$

Next, we perform the congruent transformation $\mathcal{T} = diag(Z^T, Z^T, Z^T, I, I, Z^T)$ on (2.24) with

$$Z = \begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix}. \tag{2.29}$$

Consequently, we get the following matrix inequality

$$\begin{bmatrix} -2\tilde{V} & \tilde{P}-\lambda_2\tilde{V}+\tilde{A} & -\lambda_3\tilde{V}+\tilde{A}_h & \tilde{B} & 0 & h_m\tilde{R} \\ \star & \dot{\tilde{P}}+\tilde{Q}-\tilde{R}+\lambda_2(\tilde{A}+\tilde{A}^T) & \tilde{R}+\lambda_3\tilde{A}^T+\lambda_2\tilde{A}_h & \lambda_2\tilde{B} & \tilde{C}^T & 0 \\ \star & \star & -(1-\dot{h})\tilde{Q}-\tilde{R}+\lambda_3(\tilde{A}_h+\tilde{A}_h^T) & \lambda_3\tilde{B} & \tilde{C}_h^T & 0 \\ \star & \star & \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} < 0, \tag{2.30}$$

where

$$\tilde{V} = Z^T V Z = \begin{bmatrix} Y & I \\ I & X \end{bmatrix},$$

and also

$$\tilde{P} = Z^T P Z, \quad \dot{\tilde{P}} = Z^T \dot{P} Z, \quad \tilde{R} = Z^T R Z, \quad \tilde{Q} = Z^T Q Z.$$

In addition the plant related matrices are obtained as

$$\begin{aligned}
\tilde{A} &= Z^T V \bar{A} Z = \begin{bmatrix} AY & A \\ XAY + NB_F C_2 Y + NA_F M^T & XA + NB_F C_2 \end{bmatrix} = \begin{bmatrix} AY & A \\ \hat{A} & XA + \hat{B}C_2 \end{bmatrix}, \\
\tilde{A}_h &= Z^T V \bar{A}_h Z = \begin{bmatrix} A_h Y & A_h \\ XA_h Y + NB_F C_{2h} Y + NA_{hF} M^T & XA_h + NB_F C_{2h} \end{bmatrix} \\
&= \begin{bmatrix} A_h Y & A_h \\ \hat{A}_h & XA_h + \hat{B}C_{2h} \end{bmatrix}, \\
\tilde{B} &= Z^T V \bar{B} = \begin{bmatrix} B_1 \\ XB_1 + NB_F D_{21} \end{bmatrix} = \begin{bmatrix} B_1 \\ XB_1 + \hat{B}D_{21} \end{bmatrix}, \\
\tilde{C} &= \bar{C} Z = \begin{bmatrix} C_1 Y - D_F C_2 Y - C_F M^T & C_1 - D_F C_2 \end{bmatrix} = \begin{bmatrix} C_1 Y - \hat{C} & C_1 - D_F C_2 \end{bmatrix}, \text{ and} \\
\tilde{C}_h &= \bar{C}_h Z = \begin{bmatrix} C_{1h} Y - D_F C_{2h} Y - C_{hF} M^T & C_{1h} - D_F C_{2h} \end{bmatrix} = \begin{bmatrix} C_{1h} Y - \hat{C}_h & C_{1h} - D_F C_{2h} \end{bmatrix}.
\end{aligned}$$

Here, we have used the change of variables as following:

$$\begin{aligned}
\hat{A} &= XAY + NB_F C_2 Y + NA_F M^T, \\
\hat{A}_h &= XA_h Y + NB_F C_{2h} Y + NA_{Fh} M^T, \\
\hat{B} &= NB_F, \\
\hat{C} &= D_F C_2 Y + C_F M^T, \text{ and} \\
\hat{C}_h &= D_F C_{2h} Y + C_{hF} M^T. \qquad (2.31)
\end{aligned}$$

Finally, by reversing the transformations in (2.31), the filter matrices are obtained as

$$\begin{aligned}
B_F &= N^{-1}\hat{B}, \\
A_F &= N^{-1}(\hat{A} - XAY - NB_F C_2 Y)M^{-T}, \\
A_{hF} &= N^{-1}(\hat{A}_h - XA_h Y - NB_F C_{2h} Y)M^{-T}, \\
C_F &= (\hat{C} - D_F C_2 Y)M^{-T}, \text{ and} \\
C_{hF} &= (\hat{C}_h - D_F C_{2h} Y)M^{-T}. \qquad (2.32)
\end{aligned}$$

The following theorem summarizes the discussion:

Theorem 2.3. If there exist a parameter-dependant continuously differentiable matrix $\tilde{P}(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$, parameter-dependant matrices $\tilde{R}(\rho), \tilde{Q}(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$ and $X(\rho)$, $Y(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{n\times n}$ and also $\hat{A}(\rho), \hat{A}_h(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n}$ and $\hat{B}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n_y}$ and $\hat{C}(\rho), \hat{C}_h(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_z\times n}$ and $D_F(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_z\times n_y}$, two given scalars λ_2, $\lambda_3 \in \mathbb{R}$ and a positive scalar γ such that the LMI condition (2.30) holds true for all admissible parameter $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$ and then there exist a filter in the form of (2.3) such that estimation error system is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} < \gamma\|w\|_{\mathcal{L}_2}$ for $h(\cdot) \leq h_m$. In addition, matrices of such a controller are obtained as following.
1- Solve M and N from the factorization problem

$$I - XY = NM^T.$$

2- Solve the filter matrices from (2.32).

Remark In LMI (2.30), the (2,2) entry includes a derivative term that can be replaced by $\dot{\tilde{P}} = \frac{\partial \tilde{P}}{\partial \rho}\dot{\rho}$. Due to the affine dependency of this matrix inequality on $\dot{\rho}$, it is only required to solve this feasibility problem at vertices of $\dot{\rho}$. Therefore, in this matrix inequality, one can replace the term $\dot{\tilde{P}}$ with $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial \tilde{P}}{\partial \rho} \right)$ [59]. The summation means that every combination of + and - should be included in the inequality. That is, the matrix inequality (2.30) actually represents 2^s different combinations in the summation.

Remark The LMI condition (2.30) contains the value of maximum delay h_m and this confirms that our developed results are delay-dependent. In addition, due to appearance of $\dot{h}$, the solution is also rate dependant.

Remark The LMI problem (2.30) is infinite-dimensional due to the dependency of the system matrices on LPV parameters continuously. A standard approach to solve the parameterized LMIs like (2.30) is to initially select some basis functions to represent the dependency of the matrix variables on the LPV parameters and then grid the parameter space. Finally, the obtained finite-dimensional LMI problem is solved at the grid points and then checked on a finer grid [1]. We may assign various structures to the LMI variables. One obvious choice is when they are taken constant that results in a simple implementation but literally ends up in a poor performance measure. Alternatively, a standard approach is to employ some basis functions to represent the dependency of the LMI variables on the LPV parameters, e.g.,

$$X = X_0 + \rho(t)X_1 + \frac{1}{2}\rho^2(t)X_2 + ... \tag{2.33}$$

and similarly for the other variables. The more elaborate the LMI variables are, the more satisfactory outcome is obtained at the price of computation effort.

Remark The optimal $\mathcal{H}_\infty$ performance can be obtained by solving the LMI optimization problem of minimizing γ subject to the convex constraint (2.30) by fixing λ_2 and λ_3. In addition, a line search may be performed to determine the maximum value of the time delay interval h_m.

2.5 Simulation Results

In this section, we present an illustrative example to show the effectiveness of the proposed filtering method. Also we compare the obtained results with a related work performed in the past (see [51]) which offers an accurate estimation for the case of time-varying delay. Consider a time-delayed LPV system defined by

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+0.2\sin(t) \\ -2 & -3+0.1\sin(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\sin(t) & 0.1 \\ -0.2+0.1\sin(t) & -0.3 \end{bmatrix} x(t-h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\
z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 \end{bmatrix} x(t-h(t)) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0.3w(t).
\end{aligned} \tag{2.34}
$$

We aim to estimate the output signal $z(t)$, provided the noisy signal $y(t)$ is measured. We assume that the sine term in the above model corresponds to the LPV parameter, i.e., $\rho(t) = \sin(t)$, whose functional representation is not known a priori but it can be measured in real time. It is apparent that the parameter space and its rate correspond to $[-1\ 1]$. Also the associated time-varying delay is considered to be parameter dependent satisfying $h(\rho(t)) = \lambda|\sin(t)|$, where λ is the magnitude of the periodic time delay. Next we employ the Theorem 2.3 to design the filter matrices. We consider a first order polynomial in terms of scheduling parameter for the LMI variables. We also grid the parameter space and solve the LMIs at the gridding points and then check them on a finer grid. Table 2.1 shows the worst case energy-to-energy gain of the estimation error system, versus different magnitude λ of the periodic time delay, using the proposed approach in this chapter and the one in [51]. It is apparent that the performance of the estimation has been improved. Also it is observed that for $\lambda \geq 1$, where $\dot{h} \geq 1$, the method in [51] fails, whereas, the current study can handle this case effectively.

Table 2.1: Sub-optimal values for γ for various delays regarding to different methods.

λ	0.1	0.5	0.9	1	2	2.5
Ref. [51]	0.29	0.3	0.45	-	-	-
Theorem 2.3	0.04	0.12	0.34	0.54	1.3	6.7

Shown in Figure 2.2 is the time simulation of the signal $z(t)$ along with the estimated signal corresponding to the filter obtained with Theorem 2.3 (signal z_1) and the method in [51] (signal z_2). The state initial condition for the plant (2.34) is assumed to be $[1, -2]$, the magnitude of time delay is $\lambda = 0.5$ and the disturbance signal is a pulse as indicated in

Figure 2.2. It is seen that the Theorem 2.3 provides better estimation of signal $z(t)$, given even circumstances.

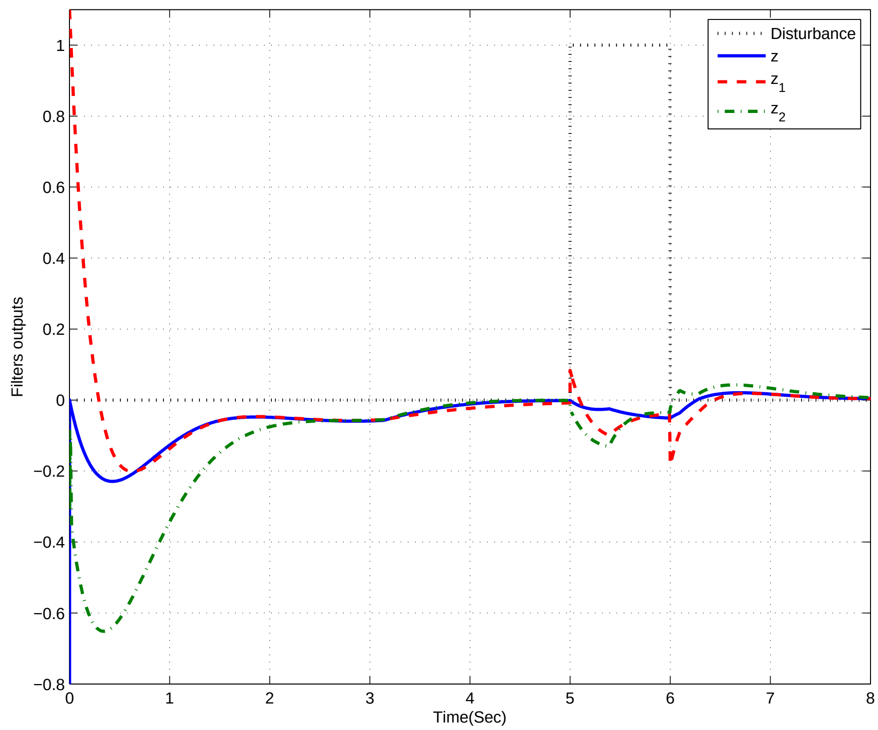

Figure 2.2: Comparison between two filters; $\hat{z}_1$ this study, $\hat{z}_2$ ref [51].

2.6 Chapter Conclusions

In this chapter, a new filtering design method for LPV systems with state delay was presented. With an appropriate choice of Lyapunov-Kravoskii functional, a synthesis condition was obtained that was delay and rate dependent. Utilizing slack variables, the synthesis conditions were expressed in terms of LMI optimization problems to be solved effectively using the existing tools. We examined the proposed results using a numerical example. We observed that the proposed method had less conservative results compared to the existing methods in the literature.

Chapter3

Sampled-Data Filtering Using Lifting Method

3.1 Introduction

Filters utilize the output measurements of a dynamic system to estimate the states or a linear combination of the states of the system. The performance of a filter is often assessed in terms of a measure of the state estimation error which is the difference between the actual and the estimated state. The literature on various versions of Kalman filtering technique is rich (see e.g., [60, 61]). Using the statistical information of the exogenous disturbance input of the system, the Kalman filter minimizes the variance of the state estimation error. In contrast, when the statistical information is unknown, the $\mathcal{H}_\infty$ filtering method can be proposed to minimize the energy of the estimation error signal for the worst bounded energy disturbance input [46, 47]. Other performance measures such as energy-to-peak gain, peak-to-peak gain or a combination of these objectives from the disturbance input to the estimation error signal can be also utilized for the filtering design problem [48].

Among several factors that affect the search for improved filter design strategies, one can mention the challenges posed by signal recovery and estimation under time-varying dynamics. Recently, linear parameter varying (LPV) systems theory has led to significant steps forward in the study of time-varying systems. LPV systems constitute a class of linear time-varying systems whose dynamics depends on time-varying parameters, also known as scheduling parameters. When such parameters are available in real-time, they can be employed for control and filter synthesis purposes resulting in less conservative conditions compared to fixed robust controllers and filters [62]. In addition, within the framework of quasi-LPV, we can model a large class of nonlinear systems as LPV systems. In a quasi-LPV system, the scheduling parameters are not only a function of exogenous signals but also of the system states. Some of the recent studies in this area have addressed the filter design problem in LPV systems, especially in the continuous-time domain. In [63], the $\mathcal{H}_\infty$ filtering problem for a class of polytopic LPV systems is considered. The design of fault detection and isolation filters for LPV systems has been another area of interest for researchers (see, e.g., [44, 64]). In addition, the authors in [51] addressed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter design for LPV systems, where the system contains delay in the states. While the aforementioned references examine the LPV filter design problem in continuous-time domain, our main concern in this chapter is to develop an LPV filter design method that is implemented in discrete time.

In the past few decades, advances in computing devices has led to efficient ways to digitally implement controllers and filters for continuous-time physical systems [65]. Digital implementation of the filters results in a mixture of continuous-time and discrete-time signals and systems forming a hybrid dynamical system. In a typical hybrid process, the measurable output signals are periodically sampled with an analog to digital (A/D) converter. Then, the digitized outputs are processed using a digital device (controller or filter) and fed to the plant after being converted to analog signals through a D/A converter. Due to the aforementioned hybrid nature of the system, there has been a need to adapt the continuous-time filtering theory to capture this level of complexity. The issue of digital implementation has been studied primarily within the area of digital control theory. Therein, the existing methods only approximately cope with the behavior of the continuous-time signals in the control system since the behavior of such systems can be captured and studied only at the sampling instants [23, 24]. In contrast to the traditional approaches, sampled-data control theory provides an exact solution method for the analysis and synthesis of sampled-data control systems with the inter-sample behavior taken into account [25]. Within this context, [66] presented a framework to design an $\mathcal{H}_\infty$ controller for sampled-data systems. Using a lifting technique, they solved the sampled-data control problem in terms of an equivalent discrete-time system, where the plant is augmented with the sampler and hold devices and is lifted to a system with a finite-dimensional state-space representation and with infinite-dimensional input and output spaces (see [24, 25]). The lifting technique was shown to preserve the input-output energy-to-energy gain of the closed-loop hybrid system. In [28, 29], the idea of lifting technique was applied to the LPV sampled-data systems, where they solved energy-to-energy and energy-to-peak gain problems to design state feedback and output feedback controllers. A benefit of this formulation is that the sampling interval can be varying as a function of the scheduling parameters. This is the case in event-sampling systems, such as engines where the sampling interval is a function of the engine speed. There have also been some additional recent efforts on sampled-data control design for LPV systems (see, e.g., [31, 30]).

The lifting method essentially maps the hybrid system to the discrete-time domain in an equivalent representation. As an alternative method, [32] introduced a sampled-data $\mathcal{H}_\infty$ control and filtering methodology that maps the hybrid system to the continuous-time domain (see also [33, 34, 67]). In this approach, the digital control law is represented as a delayed control and thus the augmentation of the plant and the controller (or the filter) leads to a state-delay system. Comparing the two approaches described above, the lifting method is more cumbersome but results in an improved performance, while the input delay approach is more conservative due to the introduction of delay to the system. In addition, the input delay method can be extended for systems with intrinsic delay, as well as uncertain sampling times or uncertain system matrices.

The contribution of this chapter is as follows. We employ the lifting method to synthesize a discrete-time filter for a continuous-time LPV system. In that aspect the obtained discrete-time filter captures the inter-sample behavior of the system. The corresponding synthesis conditions to guarantee energy-to-energy (or H_∞) performance objective on the filtering error are formulated in terms of linear matrix inequalities (LMIs). To this purpose, we assume that the scheduling parameters of the LPV system are piecewise constant. The results of this chapter has been published in [68, 69].

This chapter is organized as follows. Section 2 presents the problem statement. In Section 3, we present the lifting method employed to find an equivalent discrete-time LPV state-space representation of a continuous-time LPV system. Next, we propose a solution to the LPV sampled-data filtering problem by designing a discrete-time LPV filter for the discrete-time LPV system obtained using the lifting method. As an alternative solution, in Section 4, we describe the conventional procedure to first design a continuous-time filter and then discretize the designed filter using a discretization method. Section 5 illustrates the proposed LPV sampled-data filtering design using a numerical example. We also present the results of comparative studies between the LPV sampled-data design and the approximate discretization. Section 6 concludes the chapter.

3.2 Problem Statement

Consider a stable n^{th}-order LPV system with the following state-space representation

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t)\\ z(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t)\\ y(t) &= C_2(\rho(t))x(t),\end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated, $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector and $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector containing process noise. The system matrices $A(\cdot), B_1(\cdot), C_1(\cdot), D_{11}(\cdot)$ and $C_2(\cdot)$ are real continuous functions of a time-varying parameter vector $\rho(t)$ and of appropriate dimensions. It is assumed that the parameter vector is bounded piecewise constant. We first describe the sampling scenario we consider in this study. We assume time intervals $[0, t_1)$, $[t_1, t_2)$, $\ldots$, $[t_k, t_{k+1})$, $\ldots$ that are not necessarily equi-spaced with t_k's being the sampling instants. For the sake of brevity, throughout this chapter, k will be used to represent t_k, and the length of the k^{th} interval will be represented by τ_k, that is, $\tau_k = t_{k+1} - t_k$.

Next, we consider an n^{th}-order discrete-time parameter-varying filter F described by the following state-space representation

$$\begin{aligned}x_F(k+1) &= A_F(\rho(k))x_F(k) + B_F(\rho(k))y(k)\\ z_F(k) &= C_F(\rho(k))x_F(k) + D_F(\rho(k))y(k),\end{aligned} \tag{3.2}$$

where $x_F(k)$, $y(k)$ and $z_F(k)$ represent the discrete-time filter state vector, the discrete samples of measurement data, *i.e.,* $y(k) = y(t_k)$ and the filter output, respectively. All the system matrices are defined to be of appropriate dimensions. In the aforementioned filter structure, not only the measured output signal $y(t)$ is sampled, but also the parameter vector $\rho(t)$ is sampled synchronously at t_k (for $k = 0, 1, 2, \ldots$). Using $z_F(k)$, we build a continuous-time step-wise signal $\hat{z}(t)$ as $\hat{z}(t) = z_F(k)$ for $t_k \leq t < t_{k+1}$ in order to estimate the signal $z(t)$ in (3.1). The filter design problem described above is a hybrid filtering problem, where the physical system has a continuous dynamics, while the filter to estimate the plant output is implemented in a digital computer. Figure 3.1 shows the configuration of the hybrid system under study, the interconnection of the open-loop continuous-time system and the discrete-time filter, along with the signal conversion devices. We assume the A/D converter is an

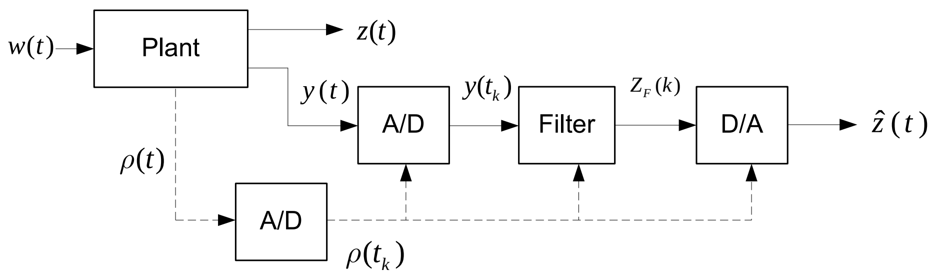

Figure 3.1: The block diagram of the hybrid system.

ideal sampler, the D/A converter is a zero-order hold and that the quantization errors are neglected. In Figure 3.1, the dependency of the converters on the parameter $\rho(t_k)$ remarks that sampling and holding frequency is not necessarily constant and may vary arbitrarily according to the exogenous parameter(s). It is noted that, in a typical LPV system, the parameter vector $\rho(t)$ varies continuously and is assumed to be measurable in real-time, that is the parameter space is

$$\mathcal{F}_{\mathcal{P}}^{v} \equiv \{\rho : \rho(t) \in C(\mathbb{R}, \mathbb{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \leq v_i \quad i = 1, 2, ..., s \quad \forall t \in \mathbb{R}_+\}, \tag{3.3}$$

where $C(\mathbb{R}, \mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathcal{P}$ is a compact set of $\mathbb{R}^s$, and $\{v_i\}_{i=1}^s$ are nonnegative numbers. However, according to the configuration in Figure 3.1, in the current study we can measure it only at sampling instants. Therefore, we assume that in the continuous-time system, the parameter vector does not change in between two consecutive samples. Hence, the set of all admissible trajectories for the parameter vector $\rho(t)$ in (3.1) is defined as

$$\begin{aligned}\mathcal{E}_{\mathcal{P}}^{v} \triangleq \{\rho : \rho(t) \in \mathcal{P},\ \rho(t_k + t) = \rho(t_k) \quad & , |\rho_i(t_{k+1}) - \rho_i(t_k)| \leq v_i, \\ k \quad \in \quad & \mathbb{Z}^+, \quad i = 1, 2, ..., s \quad \forall t \in [0, \tau_k)\}. \end{aligned} \tag{3.4}$$

Although this assumption seems restrictive, but it is valid in many practical systems, where during the sampling instants, the parameter changes are insignificant and without the loss of generality, it could be neglected.

Figure 3.2 shows the estimation error defined as $e(t) = z(t) - \hat{z}(t)$ along with the filtering problem configuration where P and F are the plant and filter, and S and H are sampling and holding devices, respectively. For the error system that relates the disturbance signal

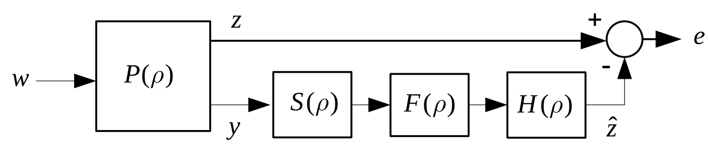

Figure 3.2: Estimation error system.

$w(t)$ to the estimation error signal $e(t)$, the induced $\mathcal{L}_2$-gain (or the $\mathcal{H}_\infty$-norm) is defined as

$$\|T_{we}\|_{i,2} = \sup_{\rho \in \mathcal{E}_{\mathcal{P}}^{v}} \sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}, \tag{3.5}$$

where T_{we} is the operator mapping the disturbance $w(t)$ to the estimation error $e(t)$. This quantity, also known as the energy-to-energy gain of the augmented system, indicates the worst case output energy $\|e\|_{\mathcal{L}_2}$ over all bounded energy disturbances $\|w\|_{\mathcal{L}_2}$ for all admissible parameter vectors $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$. In this chapter, we aim to design the filter F so that the following conditions are satisfied

- The filtering error system is asymptotically stable, and
- The energy-to-energy gain of the filtering error system is minimized, i.e.,

$$\min_{F} \|T_{we}\|_{i,2}. \tag{3.6}$$

Instead of the optimal design problem (3.6), one can solve the γ-suboptimal energy-to-energy gain problem, in which a filter F is sought such that

$$\|T_{we}\|_{i,2} < \gamma, \tag{3.7}$$

where γ is a given positive scalar. If the inequality (3.7) holds true, then the estimation error energy will be bounded by $\gamma\|w\|_{\mathcal{L}_2}$ for any nonzero disturbance $w(t)$ with bounded energy. That is, as long as $w(t) \in \mathcal{L}_2 - \{0\}$, regardless of its nature, the energy of the error signal does not exceed a specific bound.

Remark Here, we only consider the full-order filter design problem, where the filter has the same order as the plant. It is, however, noted that the results presented in this chapter can be extended to design reduced-order filters as well, using the approach in [70].

Remark It is noted that in (3.1), we assume there is no feed through matrix D_{21} influencing the measurement signal $y(t)$ in order for the sampling operator to be well defined [66]. This is not a restrictive assumption and when it is not the case, we can cascade $y(t)$ with a strictly proper filter to relax this requirement.

Preceding to the discussion and for further justification, we slightly change the aforementioned filtering problem, where we convert the configuration of the filtering problem to the well known control design problem. This is done so that we can benefit from the existing techniques developed for sampled-data control design. Figure 3.3(a) illustrates the new configuration. In this arrangement, we construct a new plant in which $e(t) = z(t) - \hat{z}(t)$

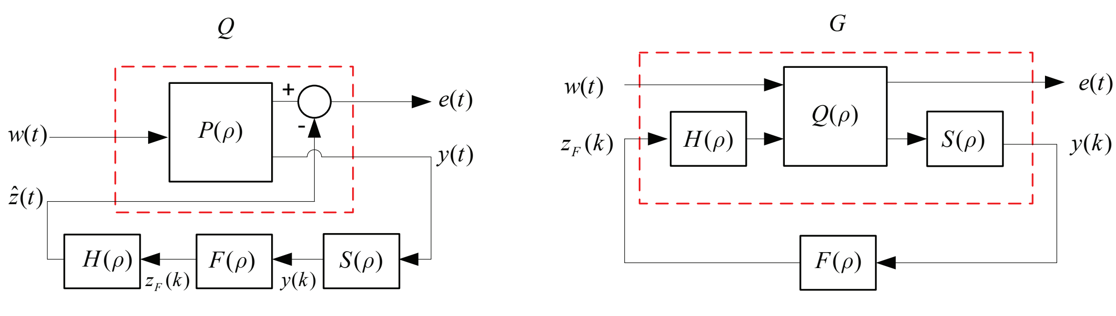

Figure 3.3: (a) Describing the filter in control configuration (b) Augmenting the sample and hold devices

is the signal to be controlled and $\hat{z}(t)$ is the control input. The state-space representation of the new augmented plant Q in Figure 3.3(a) is

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t) \\ e(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))\hat{z}(t) \\ y(t) &= C_2(\rho(t))x(t), \end{aligned} \tag{3.8}$$

where $D_{12} = -I$. The main reason for this rearrangement will be described in the following section. Next, we have to augment the sample and hold devices with the plant Q so that we can employ the so-called lifting technique. To this purpose, we form a new plant G by augmenting the system Q and the sample and hold devices as shown in Figure 3.3(b) described by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t) \\ e(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))z_F(k) \\ y(k) &= C_2(\rho(t_k))x(t_k). \end{aligned} \tag{3.9}$$

It is important to note that using this configuration the effect of both converters is directly taken into account within the design process. A simpler approach to solve the sampled-data filtering problem without casting it into the control design problem has been addressed by [68]; however, that is not as accurate as the one proposed in this chapter since only the sampling device is augmented with the plant before the lifting technique is applied.

3.3 Solution to the Hybrid Filtering Problem

In this section, we introduce the lifting method to address the problem of sampled-data filtering of continuous-time LPV systems. First we will describe the lifting technique in detail and apply that on the system (3.9) to map the augmented plant G to the discrete-time domain. Next, we design a discrete-time filter for the mapped system.

3.3.1 Discretizing Using Lifting Technique

We consider a signal $f(t) \in \mathcal{L}_2[0, \infty)$. By signal lifting, we mean breaking $f(t)$ into the intervals $[0, t_1), [t_1, t_2), \ldots, [t_k, t_{k+1}), \ldots$ and constructing a sequence of signals denoted by $f_k(t)$, whose elements are defined as $f_k(t) = f(t_k + t)$ for $0 \leq t < t_{k+1} - t_k$ or equivalently $0 \leq t < \tau_k$. Collecting all the elements in a vector, we define $\underline{f} = [\cdots, f_{-1}(t), f_0(t), f_1(t), \cdots]^T$. It is evident that each element $f_k(t)$ belongs to $\mathcal{L}_2[0, \tau_k]$. Next, we consider the continuous-time LPV system G as illustrated in Figure 3.3(b). One can think of G as an operator acting on the input pair $w(t)$ and $z_F(k)$ to provide the output pair $e(t)$ and $y(k)$. The lifting of the system G is the process of finding an operator denoted by $\underline{G}$ that maps the lifted signal

$[\underline{w}^T, z_F(k)^T]^T$ to the lifted signal $[\underline{e}^T, y(k)^T]^T$ as depicted in Figure 3.4, in the sense that both systems have equivalent closed-loop $\mathcal{H}_\infty$-norm. Next, we derive the state-space realization

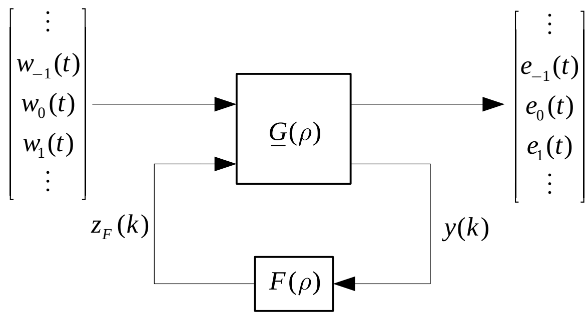

Figure 3.4: The lifted sampled-data system.

describing the lifted system $\underline{G}$. The integral solution to the state-space representation (3.9) is

$$x(t_k + t) \quad = \quad \Phi(t_k + t, t_k)x(t_k) + \int_{t_k}^{t_k+t} \Phi(t_k + t, s)B_1(\rho(s))w(s)ds, \tag{3.10}$$

for $t \in [0, \tau_k)$, where $\Phi(t_2, t_1) = \exp\left(\int_{t_1}^{t_2} A(\rho(\xi))d\xi\right)$ (for $0 \le t_1 \le t_2 < \tau_k$) is the corresponding state transition matrix. Since we have assumed the parameter space is piecewise constant, the state transition matrix becomes $\Phi(t_k + t, t_k) = \exp(A(\rho(t_k))\tau_k)$. Thus, if we change the integral variable and use the lifted signal definition, (3.10) can be simplified as

$$x_k(t) = e^{A(\rho(t_k))t}x_k(0) + \int_0^t e^{A(\rho(t_k))(t-s)}B_1(\rho(t_k))w_k(s)ds,$$

for $t \in [0, \tau_k)$. Note that $x_k(t) = x_{k+1}(0)$ for $t = \tau_k$. Similarly, the lifted output signal is

$$\begin{aligned} e_k(t) \quad = \quad & C_1(\rho(t_k))\left\{e^{A(\rho(t_k))t}x_k(0) + \int_0^t e^{A(\rho(t_k))(t-s)}B_1(\rho(t_k))w_k(s)ds\right\} \\ & +D_{11}(\rho(t_k))w_k(t) + D_{12}(\rho(t_k))z_F(k) \end{aligned}$$

and

$$y_k(0) \quad = \quad C_2(\rho(t_k))x_k(0).$$

Finally, we can represent the state-space realization of $\underline{G}$, i.e., the lifted version of G, as

$$\begin{aligned} x_{k+1}(0) &= A_d(\rho(k))x_k(0) + \underline{B}_1(\rho(k))w_k(s) \\ e_k(t) &= \underline{C}_1(\rho(k))x_k(0) + \underline{D}_{11}(\rho(k))w_k(s) + D_{12}(\rho(k))z_F(k) \\ y_k(0) &= C_2(\rho(k))x_k(0), \end{aligned} \tag{3.11}$$

where $A_d = e^{A(\rho(t_k))\tau_k}$ and

$$\begin{aligned} \underline{B}_1 &: \mathcal{L}_2[0,\tau_k] \to \mathbb{R}^n, \quad \underline{B}_1 w_k = \int_0^{\tau_k} e^{A(\rho(t_k))(\tau_k - s)} B_1(\rho(t_k)) w_k(s) ds, \\ \underline{C}_1 &: \mathbb{R}^n \to \mathcal{L}_2[0,\tau_k], \quad (\underline{C}_1 x_k)(t) = C_1(\rho(t_k)) e^{A(\rho(t_k))t} x_k, \text{ and} \\ \underline{D}_{11} &: \mathcal{L}_2[0,\tau_k] \to \mathcal{L}_2[0,\tau_k], \quad (\underline{D}_{11} w_k)(t) = C_1(\rho(t_k)) \int_0^t e^{A(\rho(t_k))(t-s)} B_1(\rho(t_k)) w_k(s) ds \\ &\qquad + D_{11}(\rho(t_k)) w_k(t). \end{aligned} \tag{3.12}$$

The lifted system (3.11) has infinite-dimensional input and output spaces but its state-space realization is finite-dimensional with the dimension equal to that of the original system. The question is now how to describe this system using a discrete-time LPV model such that the stability and an upper bound on the $\mathcal{H}_\infty$-norm of the closed-loop system is preserved. Indeed, at this stage we seek for the lifted system's state-space matrices that would be implemented in discrete-time by sampling the input vector and parameter signals at discrete time instants. This equivalent discrete-time system is determined to be

$$\begin{aligned} x_d(k+1) &= A_{dd}(\rho(k))x_d(k) + B_{1d}(\rho(k))w_d(k) + B_{2d}(\rho(k))z_F(k) \\ e_d(k) &= C_{1d}(\rho(k))x_d(k) + D_{12d}(\rho(k))z_F(k) \\ y_d(k) &= C_2(\rho(k))x_d(k), \end{aligned} \tag{3.13}$$

with the matrices A_{dd} and B_{1d} given by

$$\begin{aligned} A_{dd} &= A_d + \underline{B}_1 \underline{D}_{11}^* (\gamma^2 I - \underline{D}_{11}\underline{D}_{11}^*)^{-1} \underline{C}_1 \quad \text{and} \\ B_{2d} &= \underline{B}_1 \underline{D}_{11}^* (\gamma^2 I - \underline{D}_{11}\underline{D}_{11}^*)^{-1} \underline{D}_{12}. \end{aligned} \tag{3.14}$$

In addition, the matrices B_{1d}, C_{1d} and D_{12d} are given by

$$
\begin{aligned}
B_{1d}B_{1d}^T &= \gamma^2 \underline{B}_1(\gamma^2 I - \underline{D}_{11}^* \underline{D}_{11})^{-1} \underline{B}_1^* \quad \text{and} \\
\begin{bmatrix} C_{1d}^T \\ D_{12d}^T \end{bmatrix} \begin{bmatrix} C_{1d} & D_{12d} \end{bmatrix} &= \gamma^2 \begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix} (\gamma^2 I - \underline{D}_{11} \underline{D}_{11}^*)^{-1} \begin{bmatrix} \underline{C}_1 & \underline{D}_{12} \end{bmatrix}.
\end{aligned} \tag{3.15}
$$

The following theorem states the equivalence of the initial hybrid LPV system and the lifted discrete-time LPV system with respect to stability and energy-to-energy gain.

Theorem 3.1. Consider two dynamical systems, one of which is formed by the interconnection of the continuous-time system (3.9) with the sampled-data system (3.2), and second one is formed by interconnecting the discrete-time system (3.13) with (3.2). The following statements are equivalent provided that $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)} < \gamma$

- The former system is stable and has the energy-to-energy gain less than γ.
- The latter system is stable and has the energy-to-energy gain less than γ.

Proof: Please refer to [25].

In order to apply Theorem 3.1, the $\mathcal{L}_2[0, \tau_k)$ induced gain of $\underline{D}_{11}$, as well as several other operator compositions must be evaluated. For a complete discussion on the evaluation of these operators, the reader is referred to [25], where the design for the LTI case has been addressed. As a quick reference, the procedure for evaluating the aforementioned operators is presented in Appendix A.

Using the lifted LPV discrete-time system (3.13), the next step is to design a discrete-time parameter-dependent filter represented by (3.2). We discuss the design procedure in the following section.

3.3.2 Discrete-time Filter Design for Discrete-time LPV Systems

In this section, we consider an LPV system represented by (3.13), where the objective is to design a discrete-time system F described by the state-space representation (3.2) such that the energy-to-energy gain from the disturbance w_d to the estimation error e_d is less than γ with γ being a given positive scalar. First, we present preliminaries in the form of two lemmas that are required for the discussions in this section. For a proof of the two lemmas, the interested reader is referred to [48].

Lemma 3.1. Consider a stable discrete-time LPV system represented by

$$\begin{aligned} x_d(k+1) &= \mathfrak{A}_d(\rho(k))x_d(k) + \mathfrak{B}_d(\rho(k))w_d(k) \\ y_d(k) &= \mathfrak{C}_d(\rho(k))x_d(k), \end{aligned}$$

and let γ be a given positive scalar. Then, the energy-to-energy gain of the system from w_d to y_d is less than γ if and only if there exists a parameter-dependent symmetric positive definite matrix $P(\rho(k))$ such that

$$\begin{bmatrix} \mathfrak{A}_d(\rho(k)) & \mathfrak{B}_d(\rho(k)) \\ \mathfrak{C}_d(\rho(k)) & 0 \end{bmatrix}^T \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathfrak{A}_d(\rho(k)) & \mathfrak{B}_d(\rho(k)) \\ \mathfrak{C}_d(\rho(k)) & 0 \end{bmatrix} < \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Lemma 3.2. Consider the matrices Γ, Λ, Θ and a symmetric matrix R. There exists a matrix F such that the quadratic matrix inequality

$$(\Theta + \Gamma F \Lambda)^T R(\Theta + \Gamma F \Lambda) < Q \tag{3.16}$$

has a solution if and only if the following conditions hold true

$$\Lambda^{T\perp}(Q - \Theta^T R \Theta)\Lambda^{T\perp T} > 0 \tag{3.17}$$

and

$$\Gamma^{\perp}(R^{-1} - \Theta Q^{-1}\Theta^T)\Gamma^{\perp T} > 0. \tag{3.18}$$

In this case, all the possible solutions for the matrix F are parameterized by

$$F = -\Omega\Gamma^T R\Theta\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + \Psi^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}, \tag{3.19}$$

where L is an arbitrary matrix such that $\|L\| < 1$ and

$$\begin{aligned} \Phi &= (Q - \Theta^T R\Theta + \Theta^T R\Gamma\Omega\Gamma^T R\Theta)^{-1}, \\ \Psi &= \Omega - \Omega\Gamma^T R\Theta(\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi)\Theta^T R\Gamma\Omega, \text{ and} \\ \Omega &= (\Gamma^T R\Gamma)^{-1}. \end{aligned}$$

The following theorem gives the solution to the problem mentioned at the beginning of this section.

Theorem 3.2. For a given positive scalar γ, there exists an n^{th}-order system F represented in state-space form by (3.2) to make the energy-to-energy gain of the system (3.13) from w_d to e_d less than γ, if and only if there exist parameter-dependent matrices $X > 0$ and $Y > 0$ such that for all admissible parameters, there is a feasible solution to the set of LMIs

$$C_2^{T\perp}\left(Y(\rho(k-1)) - A_{dd}^T Y(\rho(k)) A_{dd} - C_{1d}^T C_{1d}\right) C_2^{T\perp T} > 0, \tag{3.20}$$

$$\begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp} \left(\begin{bmatrix} X(\rho(k)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A_{dd} & B_{1d} \\ C_{1d} & 0 \end{bmatrix}^T \begin{bmatrix} X(\rho(k-1)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{dd} & B_{1d} \\ C_{1d} & 0 \end{bmatrix} \right) \begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp T} > 0, \tag{3.21}$$

and

$$\begin{bmatrix} Y(\rho(k)) & \gamma I \\ \gamma I & X(\rho(k)) \end{bmatrix} \geq 0. \tag{3.22}$$

Proof: To solve the γ-suboptimal $\mathcal{H}_\infty$ design problem, we first examine the closed-loop representation of systems (3.13) and (3.2) (where $y(k)$ and $y_d(k)$ are the same). Defining $\bar{x}_d(k) = [x_d^T(k), x_F^T(k)]^T$, we have

$$\begin{aligned} \bar{x}_d(k+1) &= (\bar{A} + \bar{B}F\bar{M})\bar{x}_d(k) + \bar{D}w_d(k) \\ e_d(k) &= (\bar{C} + \bar{H}F\bar{M})\bar{x}_d(k), \end{aligned}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_{dd}(\rho(k)) & 0 \\ 0 & 0 \end{bmatrix}, & \bar{B} &= \begin{bmatrix} B_{2d}(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, \\ \bar{M} &= \begin{bmatrix} C_2(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, & \bar{D} &= \begin{bmatrix} B_{1d}(\rho(k)) \\ 0 \end{bmatrix}, \text{ and} \\ \bar{C} &= \begin{bmatrix} C_{1d}(\rho(k)) & 0 \end{bmatrix}, & \bar{H} &= \begin{bmatrix} D_{12d}(\rho(k)) & 0 \end{bmatrix}. \end{aligned}$$

In addition, the matrix F defined by

$$F = \begin{bmatrix} D_F(\rho(k)) & C_F(\rho(k)) \\ B_F(\rho(k)) & A_F(\rho(k)) \end{bmatrix}, \tag{3.23}$$

includes the unknown matrices corresponding to the filter state-space representation. Next, we use Lemma 3.1 as the LMI-based condition to ensure that there is a solution to the γ-suboptimal filtering problem. The lemma states that there exists a γ-suboptimal LPV filter if and only if there exists a parameter-dependent matrix $P(\rho(k)) > 0$ such that

$$\begin{bmatrix} \bar{A}+\bar{B}F\bar{M} & \bar{D} \\ \bar{C}+\bar{H}F\bar{M} & 0 \end{bmatrix}^T \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}+\bar{B}F\bar{M} & \bar{D} \\ \bar{C}+\bar{H}F\bar{M} & 0 \end{bmatrix} < \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Next, we use Lemma 3.2 to determine a set of LMI conditions to ensure the existence of the filter F. The associated matrices are determined to be

$$\Theta = \begin{bmatrix} \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \bar{B} \\ \bar{H} \end{bmatrix} = \begin{bmatrix} B_{2d} & 0 \\ 0 & I \\ D_{12d} & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \bar{M} & 0 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \tag{3.24}$$

and

$$R = \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, \quad Q = \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \tag{3.25}$$

It can be verified that

$$\Gamma^{\perp} = \begin{bmatrix} \begin{bmatrix} B_{2d} \\ D_{12d} \end{bmatrix}^{\perp} & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}, \quad \Lambda^{T\perp} = \begin{bmatrix} C_2^{T\perp} & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}. \tag{3.26}$$

Next, we partition P as

$$P(\rho(k)) = \begin{bmatrix} Y(\rho(k)) & Y_{12}(\rho(k)) \\ Y_{12}^T(\rho(k)) & Y_{22}(\rho(k)) \end{bmatrix}. \tag{3.27}$$

Then, the solvability condition (3.17) leads to the LMI (3.20). In addition, the solvability condition (3.18) leads to a matrix inequality problem, in which the matrix P^{-1} appears.

Then, applying the congruence transformation $\mathcal{T} = diag(\gamma I, \gamma I)$ and introducing

$$\gamma^2 P^{-1}(\rho(k)) = \begin{bmatrix} X(\rho(k)) & X_{12}(\rho(k)) \\ X_{12}^T(\rho(k)) & X_{22}(\rho(k)) \end{bmatrix}$$

leads to (3.21). The (1,1) entries of the two matrices P and P^{-1} are related through

$$Y - \gamma^2 X^{-1} = Y_{12} Y_{22}^{-1} Y_{12}^T \geq 0, \tag{3.28}$$

which implies that

$$Y(\rho(k)) - \gamma^2 X^{-1}(\rho(k)) \geq 0,$$

or equivalently the LMI (3.22).

Remark It is noted that the inequality conditions in Theorem 3.2 are parameterized LMIs. To solve this infinite-dimensional LMI problem, we initially pick some basis functions to represent the dependency of the matrix variables on the LPV parameters, e.g., by selecting first order polynomials as

$$X = X_0 + \rho X_1, \quad Y = Y_0 + \rho Y_1. \tag{3.29}$$

Next, the set of LMI problem is solved to determine X_0, X_1, Y_0 and X_1 at the selected grid points. The results are finally checked on a finer grid.

Remark It is important to note that in Theorem 3.2, the matrix inequalities (3.20), (3.21) and (3.22) are not linear in terms of γ, since the associated matrices $A_{dd}, B_{1d}, B_{2d}, C_{1d}$ and D_{12d} obtained from (3.14) and (3.15) are themselves dependent on γ. So, in order to find the optimum γ, we decrease γ successively in a loop and solve the feasibility problem of the LMIs (with $A_{dd}, B_{1d}, B_{2d}, C_{1d}$ and D_{12d} updated accordingly). The search is terminated as soon as the set of LMIs become infeasible.

Remark At each sampling instant, the set of LMIs depends on the LPV parameter vector at both k^{th} and $(k-1)^{th}$ samples. Hence, there is a need to store parameters during the design process. One can also replace $\rho(k-1)$ by $r(k)$ in the corresponding LMIs and treat it as a new parameter vector $r \in \mathcal{E}^v_{\mathcal{P}}$. In this case, the feasibility or optimization problem corresponding to the sampled-data filter design should be solved over the new parameter space $\mathcal{E}^v_{\mathcal{P}} \times \mathcal{E}^v_{\mathcal{P}}$.

After solving the LMIs associated with Theorem 3.2 offline, the filter matrices (3.23) are determined at each sampling instant as following:

Step 1: The scheduling parameter ρ is measured.

Step 2: For a predetermined value of γ and the current value of ρ, the discrete-time system matrices in (3.13) are updated, using the process given in Appendix A.

Step 3: The matrices Θ, Γ and Λ in (3.24) are determined. Using X_0, X_1, Y_0 and Y_1, from (3.29) X and Y are calculated. Once X and Y are calculated, Y_{12} and Y_{22} can be determined from the factorization problem (3.28) using singular value decomposition (SVD).

Step 4: Next, P is found from (3.27) and subsequently R and Q in (3.25) are determined.

Step 5: Finally, F is obtained from (3.19). By partitioning matrix F, the filter matrices A_F, B_F, C_F and D_F are then obtained from (3.23).

3.4 Continuous-Time LPV Filter Discretization

A conventional solution to the sampled-data filter problem is to design a continuous-time LPV filter for the continuous-time LPV plant and then apply a standard discretization method to find a discrete-time representation of the filter. In this section, we present the LMI-based conditions to design a continuous-time filter for a given continuous-time LPV system. Then, we discuss the use of trapezoidal approximation method to discretize the designed filter. We first present two lemmas that are important in the proof of the main results of this section. For more details, please refer to [48].

Lemma 3.3. Consider a stable continuous-time LPV system represented by

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}(\rho(t))x(t) + \mathfrak{B}(\rho(t))w(t) \\ y(t) &= \mathfrak{C}(\rho(t))x(t) + \mathfrak{D}(\rho(t))w(t), \end{aligned}$$

and let γ be a given positive scalar. Then energy-to-energy gain of the system from w to y is less than γ if and only if there exist a parameter-dependent symmetric positive definite matrix $P(\rho(t))$ that satisfies the following matrix inequality

$$\begin{bmatrix} \dot{P}(\rho) + P(\rho)\mathfrak{A}(\rho) + \mathfrak{A}^T(\rho)P(\rho) & P(\rho)\mathfrak{B}(\rho) & \mathfrak{C}^T(\rho) \\ \mathfrak{B}^T(\rho)P(\rho) & -\gamma^2 I & \mathfrak{D}^T(\rho) \\ \mathfrak{C}(\rho) & \mathfrak{D}(\rho) & -I \end{bmatrix} < 0.$$

Lemma 3.4. Consider the matrices Γ, Λ, Θ and a symmetric matrix R. There exists a matrix F such that the quadratic matrix inequality

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0 \tag{3.30}$$

has a solution if and only if the following conditions hold true

$$\Gamma^{\perp} \Theta \Gamma^{\perp T} < 0 \tag{3.31}$$

and

$$\Lambda^{T\perp} \Theta \Lambda^{T\perp T} < 0. \tag{3.32}$$

In this case, all the possible solutions for the matrix F are parameterized by

$$F = -R^{-1}\Gamma^T \Phi \Lambda^T \Psi + \Omega^{\frac{1}{2}} L \Psi^{\frac{1}{2}}, \tag{3.33}$$

where Φ, R and L are free parameters satisfying

$$\Phi = (\Gamma R^{-1} \Gamma^T - \Theta)^{-1} > 0, \quad R > 0, \quad \|L\| < 1$$

and Φ and Ψ are defined by

$$\begin{aligned} \Omega &= R^{-1} - R^{-1}\Gamma^T(\Phi - \Phi\Lambda^T\Psi\Lambda\Phi)\Gamma R^{-1} \\ \Psi &= (\Lambda\Phi\Lambda^T)^{-1}. \end{aligned}$$

3.4.1 Design of Continuous-Time Filters for LPV Systems

Considering the LPV system represented by (3.1), we define a continuous-time filter F described by the following state-space representation

$$\begin{aligned} \dot{x}_F(t) &= A_F(\rho(t))x_F(t) + B_F(\rho(t))y(t) \\ \hat{z}(t) &= C_F(\rho(t))x_F(t) + D_F(\rho(t))y(t). \end{aligned} \tag{3.34}$$

The design objective is to ensure that the energy-to-energy gain from the disturbance w to the estimation error e becomes less than γ, where γ is a given positive scalar and the estimation error is defined as $e(t) = z(t) - \hat{z}(t)$. The following theorem gives the solution to this problem.

Theorem 3.3. For a given positive scalar γ, there exists an n^{th}-order filter (3.34) to solve the γ-suboptimal continuous-time filtering problem if and only if there exist parameter-dependent matrices $X > 0$ and $Y > 0$ such that for all admissible parameters, there is a feasible solution to the set of LMIs

$$\begin{bmatrix} \dot{X}(\rho) + A(\rho)X(\rho) + X(\rho)A^T(\rho) & B_1(\rho) \\ B_1^T(\rho) & -\gamma^2 I \end{bmatrix} < 0, \tag{3.35}$$

$$\begin{bmatrix} C_2^{T\perp}(\rho) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{Y}(\rho) + Y(\rho)A(\rho) + A^T(\rho)Y(\rho) & C_1(\rho) \\ C_1^T(\rho) & -I \end{bmatrix} \begin{bmatrix} C_2^{T\perp}(\rho) & 0 \\ 0 & I \end{bmatrix}^T < 0, \tag{3.36}$$

and

$$Y(\rho(t)) \geq X(\rho(t)). \tag{3.37}$$

Proof: Defining $\bar{x}(t) = [x^T(t), x_F^T(t)]^T$, the estimation error dynamics is given by

$$\begin{aligned} \bar{x}(t) &= (\bar{A} + \bar{B}F\bar{M})\bar{x}(t) + \bar{D}w(t) \\ e(t) &= (\bar{C} + \bar{H}F\bar{M})\bar{x}(t) + \bar{E}w(t), \end{aligned} \tag{3.38}$$

where

$$\bar{A} = \begin{bmatrix} A(\rho(t)) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} C_2(\rho(t)) & 0 \\ 0 & I \end{bmatrix},$$

$$
\begin{aligned}
\bar{D} &= \begin{bmatrix} B_1(\rho(t)) \\ 0 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} C_1(\rho(t)) & 0 \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} -I & 0 \end{bmatrix}, \quad \text{and} \\
\bar{E} &= D_{11}.
\end{aligned}
$$

and

$$
F = \begin{bmatrix} D_F(\rho(t)) & C_F(\rho(t)) \\ B_F(\rho(t)) & A_F(\rho(t)) \end{bmatrix}. \tag{3.39}
$$

Next, we apply Lemma 3.3 to the augmented system (3.38) to obtain

$$
\begin{bmatrix} \dot{P} + P(\bar{A} + \bar{B}F\bar{M}) + (\bar{A} + \bar{B}F\bar{M})^T P & P\bar{D} & (\bar{C} + \bar{H}F\bar{M})^T \\ \bar{D}^T P & -\gamma^2 I & \bar{E}^T \\ \bar{C} + \bar{H}F\bar{M} & \bar{E} & -I \end{bmatrix} < 0.
$$

This matrix inequality can be cast in the form of (3.30) with

$$
\Gamma = \begin{bmatrix} P\bar{B} \\ 0 \\ \bar{H} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \bar{M} & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} \dot{P} + P\bar{A} + \bar{A}^T P & P\bar{D} & \bar{C}^T \\ \bar{D}^T P & -\gamma^2 I & \bar{E}^T \\ \bar{C} & \bar{E} & -I \end{bmatrix}. \tag{3.40}
$$

One can readily obtain that

$$
\Gamma^{\perp} = \begin{bmatrix} \begin{bmatrix} \bar{B} \\ \bar{H} \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}, \quad \Lambda^{T\perp} = \begin{bmatrix} \bar{M}^{T\perp} & 0 & 0 \\ 0 & 0 & I \end{bmatrix},
$$

where

$$
\begin{bmatrix} \bar{B} \\ \bar{H} \end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \quad \bar{M}^{T\perp} = \begin{bmatrix} C_2^{T\perp} & 0 \end{bmatrix}.
$$

Before we proceed to apply the solvability conditions in Lemma 3.4, we partition $P(\rho(t))$

and $P^{-1}(\rho(t))$ as

$$P(\rho(t)) = \begin{bmatrix} Y(\rho(t)) & Y_{12}(\rho(t)) \\ Y_{12}^T(\rho(t)) & Y_{22}(\rho(t)) \end{bmatrix}, \qquad P^{-1}(\rho(t)) = \begin{bmatrix} X(\rho(t)) & X_{12}(\rho(t)) \\ X_{12}^T(\rho(t)) & X_{22}(\rho(t)) \end{bmatrix}. \tag{3.41}$$

The solvability condition (3.31) becomes

$$\dot{X}(\rho) + A(\rho)X(\rho) + X(\rho)A^T(\rho) + \frac{1}{\gamma^2}B_1(\rho)B_1^T(\rho) < 0.$$

Applying the Schur complement on the above inequality yields (3.35). In addition, the $(1,1)$ entries of the two matrices P and P^{-1} are related through

$$Y - X = Y_{12}Y_{22}^{-1}Y_{12}^T \geq 0 \tag{3.42}$$

that yields (3.37). Once the matrices X and Y are found, the matrices Y_{12} and Y_{22} can be determined from the factorization problem (3.42). Subsequently the matrix P in (3.41) is calculated. Substituting the obtained matrices in (3.40), the filter state-space matrices in (3.39) are computed by (3.33).

Remark In the matrix inequalities (3.35) and (3.36), the (1,1) entries include a derivative term that can be replaced by $\dot{X} = \frac{\partial X}{\partial \rho}\dot{\rho}$ and $\dot{Y} = \frac{\partial Y}{\partial \rho}\dot{\rho}$, respectively. Due to the affine dependency of matrix inequalities on $\dot{\rho}$, it is only required to solve the feasibility problem at vertices of $\dot{\rho}$. Therefore, one can replace the term $\dot{X}$ with $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial X}{\partial \rho}\right)$ and $\dot{Y}$ with $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial Y}{\partial \rho}\right)$, where v_i is defined in (3.3) [59]. The summation means that every combination of + and - should be included in the inequality. That is, the corresponding inequalities actually represents 2^s different combinations in the summation.

3.4.2 Trapezoidal Discretization of the Continuous-time LPV Filter

Among various options for discretization of a continuous-time dynamic system, we employ the trapezoidal approximation that is a counterpart of the bilinear transformation for LPV systems. The proposed formulation in this section is adopted from the work of [27]. It is, however, slightly tailored for nonuniform sampling periods. This approach is moderately accurate and advantageously reduces the computational cost.

Considering the sampling interval $t_k \leq t < t_{k+1}$, we assume that for the continuous-time filter (3.34), the state vector $x_F(t_k)$ is known. Then, at the end of sampling interval, we have

$$\begin{aligned} x_F(t_{k+1}) &= x_F(t_k) + \int_{t_k}^{t_{k+1}} \left(A_F(\rho(\tau))x(\tau) + B_F(\rho(\tau))y(\tau)\right) d\tau \\ \hat{z}(t_k) &= C_F(\rho(t_k))x_F(t_k) + D_F(\rho(t_k))y(t_k). \end{aligned} \tag{3.43}$$

Using he trapezoidal approximation for the integral part in (3.43) and with a simplified notation, we obtain

$$\begin{aligned} x_F(k+1) \approx x_F(k) + \frac{\tau_k}{2} \big(A_F(\rho(k))x_F(k) + B_F(\rho(k))y(k) + A_F(\rho(k+1))x_F(k+1) \\ + B_F(\rho(k+1))y(k+1)\big) . \end{aligned}$$

Next, we gather all the terms corresponding to the sampling time t_{k+1} and rename them $x_d(k+1)$, that is

$$x_d(k+1) = (I - \frac{\tau_k}{2} A_F(\rho(k+1)))x_F(k+1) - \frac{\tau_k}{2} B_F(\rho(k+1))y(k+1), \tag{3.44}$$

which implies that

$$x_F(k) = (I - \frac{\tau_{k-1}}{2} A_F(\rho(k)))^{-1}(x_d(k) + \frac{\tau_{k-1}}{2} B_F(\rho(k))y(k)). \tag{3.45}$$

Finally, we substitute (3.44) and (3.45) into (3.43) to obtain a discrete-time representation of the filter designed in continuous-time. The following theorem characterizes the filter state-space matrices for the implementation purposes.

Theorem 3.4. Consider the LPV filter (3.34) designed in continuous-time. The sampled dynamics of this filter is represented by the following discrete-time state-space model

$$\begin{aligned} x_d(k+1) &= A_{Fd}(\rho_k)x_d(k) + B_{Fd}(\rho_k)y(k) \\ \hat{z}_d(k) &= C_{Fd}(\rho_k)x_d(k) + D_{Fd}(\rho_k)y(k), \end{aligned}$$

where

$$
\begin{aligned}
A_{Fd} &= (I + \frac{\tau_k}{2} A_F)(I - \frac{\tau_{k-1}}{2} A_F)^{-1}, \\
B_{Fd} &= \frac{\tau_k + \tau_{k-1}}{2}(I - \frac{\tau_{k-1}}{2} A_F)^{-1} B_F, \\
C_{Fd} &= C_F(I - \frac{\tau_{k-1}}{2} A_F)^{-1}, \text{and} \\
D_{Fd} &= \frac{\tau_{k-1}}{2} C_F(I - \frac{\tau_{k-1}}{2} A_F)^{-1} B_F + D_F.
\end{aligned}
$$

This discrete-time system is then placed in the Filter block in Figure 3.1. It is noted that the filter state-space matrices are functions of $\rho(k)$ and updated at each sampling instant.

3.5 Simulation Results

In this section, we present some numerical results obtained from applying the proposed sampled-data LPV filter design methods. We consider a forth-order resonant system corresponding to a double mass-spring-damper system with nonlinear springs as shown in Figure 3.5.

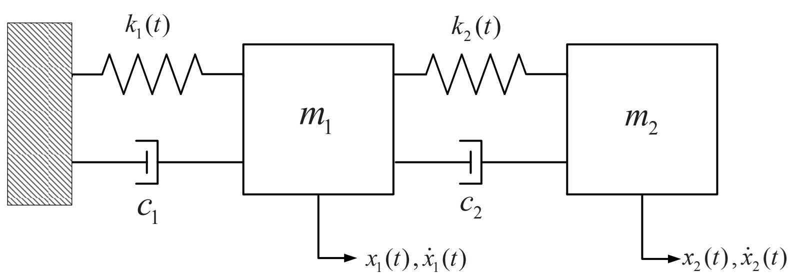

Figure 3.5: The double mass-spring-damper system.

The dynamic model of the system is described by

$$
m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 - c_2\dot{x}_2 = w_1(t)
$$
$$
m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = w_2(t),
$$

where m_1 and m_2 are masses, k_1 and k_2 are stiffness of the springs, c_1 and c_2 are the damping coefficients, and $w_1(t)$ and $w_2(t)$ are external force disturbances acting on the masses. The objective is to design a sampled-data filter to estimate the mass velocities using the measurement of the positions. The associated state-space model of the system is as follows

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1(t)+k_2(t)}{m_1} & -\frac{c_1+c_2}{m_1} & -\frac{k_2(t)}{m_1} & -\frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2(t)}{m_2} & \frac{c_2}{m_2} & -\frac{k_2(t)}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\
z(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}.
\end{aligned} \tag{3.46}
$$

The parameters are assumed to be

$$
\begin{aligned}
m_1 &= m_2 = \frac{1}{11}[kg], \\
k_1 &= k_2 = 8 + 2\sin(t)[N/m], \text{ and} \\
c_1 &= c_2 = 0.5[N.s/m].
\end{aligned} \tag{3.47}
$$

We assume that the *sine* term in (3.47) corresponds to the LPV parameter, i.e., $\rho(t) = \sin(t)$, whose functional representation is not known a priori but since it is a time-dependent variable, it can be produced in a digital device. We note that the parameter space is $[-1, 1]$. It is also assumed that the system is affected by an input disturbance signal $w_1(t) = 1$ for $t \in [0, 1]$ and $w_1(t) = 0$ otherwise. We consider three designs corresponding to different sampling rates. First, we design a discrete-time filter for the case of a constant sampling rate $\tau_k = 0.1$. Figures 3.6 and 3.7 illustrate the estimates of $z_1(t)$ and $z_2(t)$, respectively, using the proposed sampled-data method along with the actual outputs of the continuous-time system. For comparison purposes, we have also shown in these figures the trapezoidal approximation of an LPV filter designed in continuous-time. As observed, the estimation performance using the discretization of the continuous-time filter design is inferior to that of he proposed sampled-data design. It is emphasize that, if we use the rectangular approximation for discretization of the continuous-time filter, the estimation performance even worsens compared to the trapezoidal approximation. In this example the optimal value of γ is obtained to be 2.5. It is noted that the output tracking is even improved for lower sampling rates than $\tau_k = 0.1$. In the second scenario, we examine the case of a constant sampling rate $\tau_k = 0.2$ which is quite large with respect to the frequency of the output signals. Figures 3.8 and 3.9 show the estimation results using the sampled-data and trapezoidal approximation methods. While the latter method fails to provide a good estimate, the former provides reliable estimates of the output of the continuous-time LPV system. In this case, the optimal value of γ is determined to be 4.4. Finally, we consider a case with a variable sampling rate, in which the sampling rate changes according to the pattern

$$t_{k+1} = t_k + 0.2(1 + 0.5\sin(0.2t_k)).$$

Starting from $t_0 = 0$, the pattern above is associated with a time-varying sampling. Figures 3.10 and 3.11 show acceptable estimation results that the sampled-data LPV filter can provide. The optimal value of γ is obtained to be 5.3. Also shown in these figures are the results of trapezoidal approximation of an LPV filter designed in continuous-time. It is obvious that the sampled-data filter exhibits better performance.

3.6 Chapter Conclusions

In this chapter, we presented a sampled-data filter design method for stable continuous-time LPV systems using the lifting technique. The design method consisted of obtaining an equivalent discrete-time LPV system by employing the lifting method and subsequently design a discrete-time LPV filter for the lifted system. It was shown that the sampled-data approach effectively handles large and even variable sampling rates. Numerical results demonstrated the viability of the proposed sampled-data filtering method. In addition, to compare the results with conventional filtering methods, we designed a continuous-time

LPV filter and discretized it by means of trapezoidal approximation. This approach is a fast solution alternative for the sampled-data control and filtering problems that reduces the computational effort at the cost of accuracy. This method can give favorable results specially when the LMI optimization leads to a sufficiently small $\mathcal{H}_\infty$-norm for the estimation error system. The simulation results demonstrated the improved estimation performance achieved using the developed LPV sampled-data filter design method compared to the discretization of a filter designed in continuous-time. The improvement was observed for three cases with small, large and variable sampling rates.

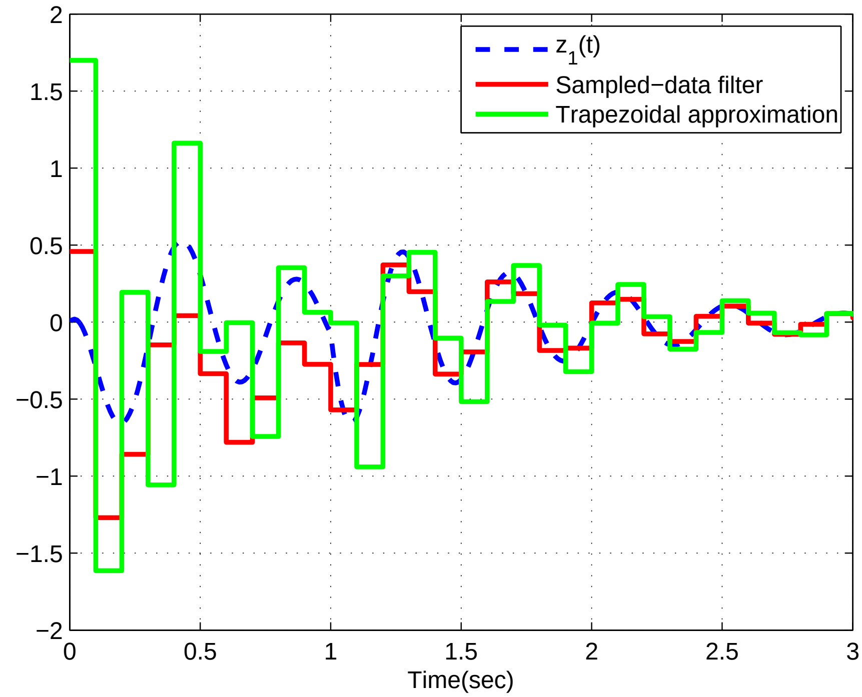

Figure 3.6: The estimation results of $z_1(t)$ for the sampling rate of 0.1 sec.

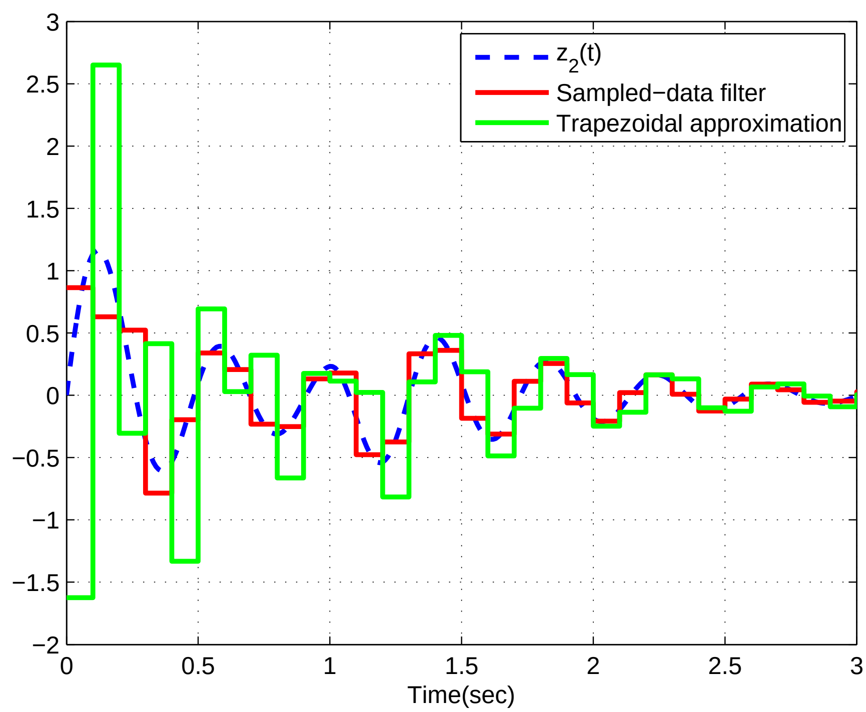

Figure 3.7: The estimation results of $z_2(t)$ for the sampling rate of 0.1 sec.

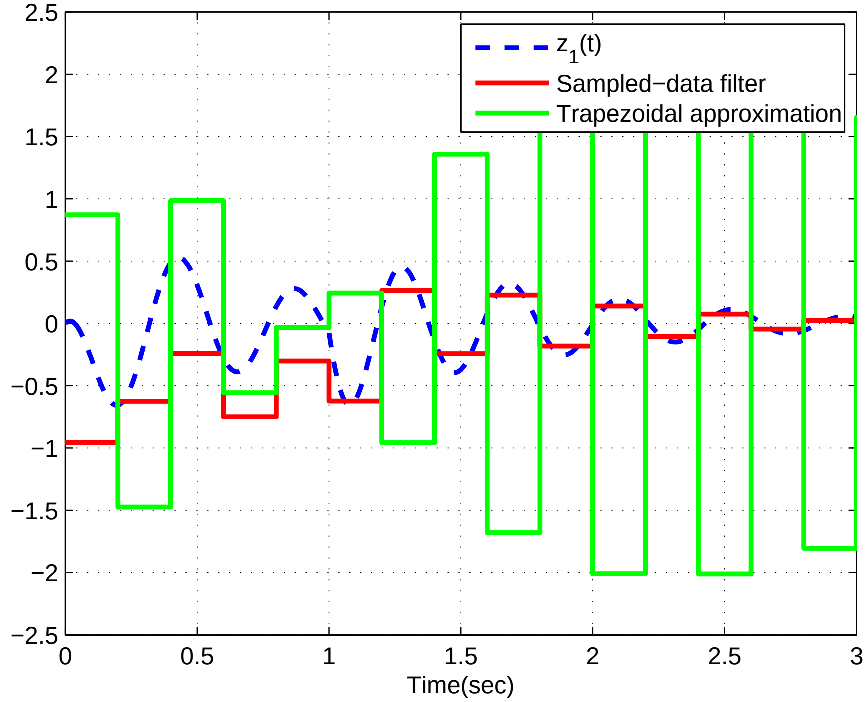

Figure 3.8: The estimation results of $z_1(t)$ for the sampling rate of 0.2 sec.

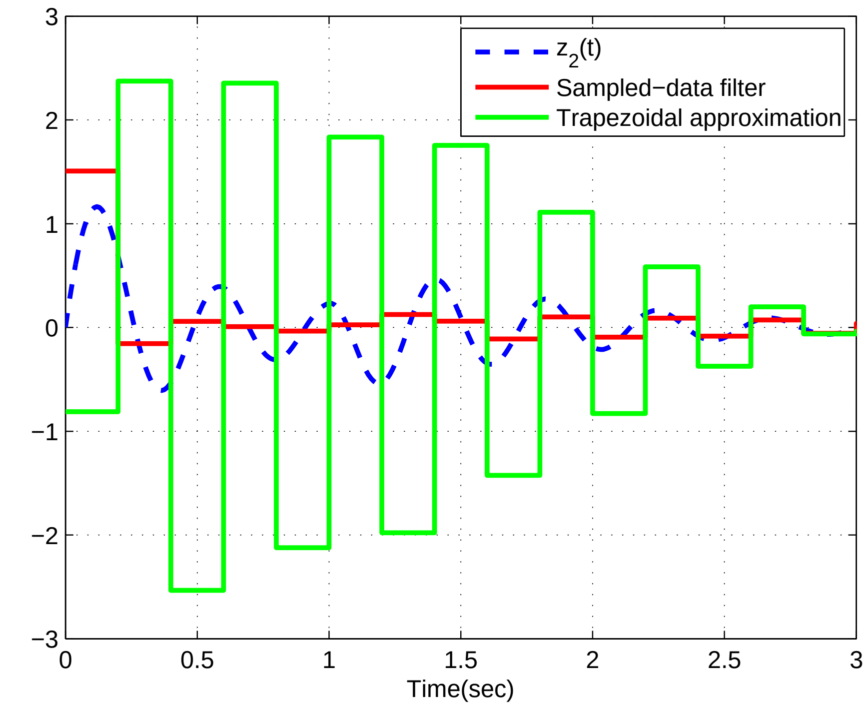

Figure 3.9: The estimation results of $z_2(t)$ for the sampling rate of 0.2 sec.

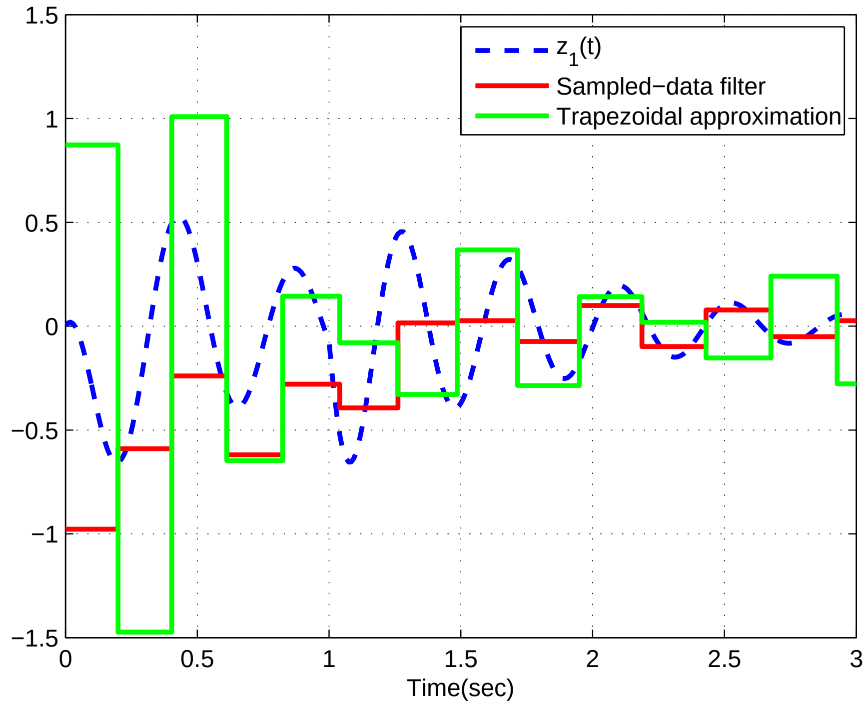

Figure 3.10: The estimation results of $z_1(t)$ with variable sampling rate.

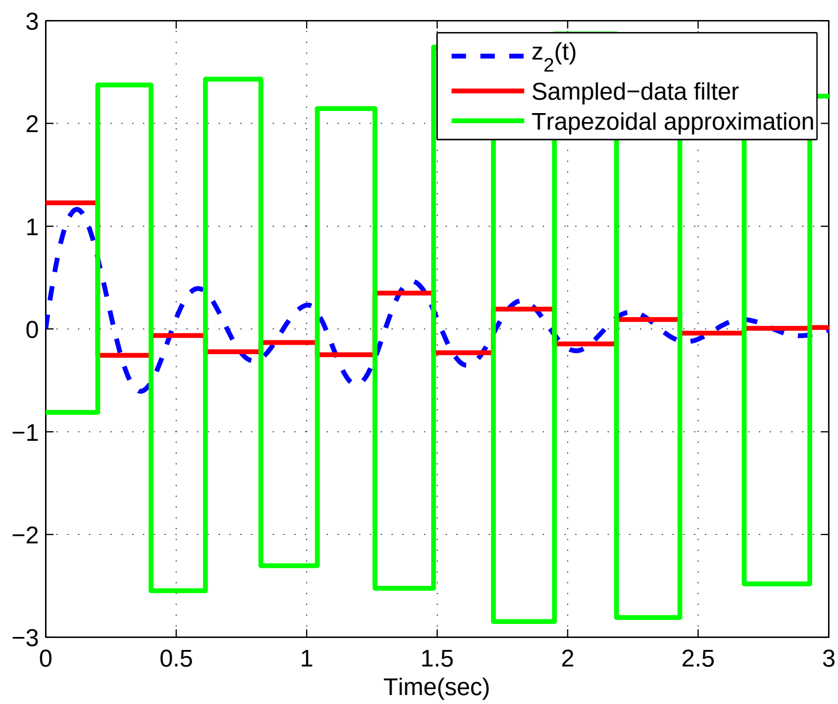

Figure 3.11: The estimation results of $z_2(t)$ with variable sampling rate.

Chapter4

Sampled-Data Control of LPV Systems with or without Time Delay

4.1 Introduction

The majority of physical systems function in continuous-time and significant amount of effort has been dedicated to the design of controllers for such systems. The implementation of a controller is often fulfilled by means of a digital instrument operating in discrete time. Among numerous benefits of digital rendering one can notify reliability, flexibility and cost efficiency [71]. The notion of the sampled-data system points out to the incorporation of continuous-time and discrete-time signals in a system. In a sampled-data control system, an analog to digital (A/D) device followed by a digital controller and a digital to analog (D/A) device along with the continuous-time plant form the closed-loop system. The corresponding sampled-data control design problem seeks to find a discrete-time controller that guarantees closed-loop stability and performance for the continuous-time plant. The sampled-data control design problem is more challenging for time-varying and nonlinear systems compared to the linear time invariant (LTI) case. The development of linear parameter varying (LPV) system theory has led to significant advances in the study of time-varying and nonlinear systems [2]. LPV systems include a class of linear systems, whose dynamics depends on time-varying parameters, known as scheduling variables. When such parameters are available in real-time, they can be employed for control synthesis purposes. For a comprehensive study on LPV systems and applications, see [4].

Digital control of analog systems yields a closed-loop system with hybrid configuration that is difficult to handle mathematically. A particular difficulty, that has been the main concern of researchers in this area, is to ensure that the digital controller meets the design specifications in between the samples. A general approach to address this is to first discretize the plant, then design a discrete-time controller and finally cascade the digitally designed controller with the plant using the converter interfaces. It has been shown that this simple approach, categorized as indirect design, guarantees stability and desired performance of inter-samples in the case of step inputs but not in general for other inputs [25]. As an alternative indirect method, one may begin by designing a continuous-time controller and then try to find a discrete-time representation by means of a conventional discretization method.

In [27], the discretization of an LPV continuous-time controller is discussed. For a survey on discretization methods of LPV systems, as well as the corresponding approximation error, refer to [72]. This approach benefits from the fact that the control design in continuous-time domain is well established and easier to do. In addition, as opposed to the former method, the sampling frequency does not influence the design until the last step where the controller is discretized. Thus the need for redesigning the controller for different sampling rates is avoided. But since this method disregards the effect of sample and hold devices in the design process, increasing the sampling period degrades the effectiveness of the control action that may result in instability of the closed-loop system. In contrast, the objective of the so-called direct sampled-data design is to guarantee the stability and performance of the closed-loop system [73, 74, 75, 76, 77]. One well-established approach for direct sampled-data control is the lifting method presented by [66]. This approach addresses the sampled-data problem in terms of an equivalent discrete-time system, where the plant is first augmented by the converter devices and then lifted to a system with a finite-dimensional state-space but infinite-dimensional input and output spaces. Using the lifting technique, the authors in [29] addressed the problem of sampled-data control design for LPV systems. Unfortunately, the lifting method is computationally complex because it requires the evaluation of some operators of the lifted system [34]. An alternative direct method proposed in [78] and further studied in [32, 33, 34] copes with the hybrid nature of sampled-data control systems by reformulating the digital control law as a delayed continuous-time system, that is

$$u(t_k) = u(t - (t - t_k)) = u(t - \tau_k(t)), \tag{4.1}$$

for $t_k \leq t < t_{k+1}$, where t_k $(k = 0, 1, 2, ...)$ signifies the sampling instant and $\tau_k(t) = t - t_k$ denotes a time-varying delay. By taking advantage of this idea, the closed-loop system is mapped to the continuous-time domain containing a delay in the states. Utilizing the input-delay method calls for a sophisticated analysis method to overcome the imposed delay. The literature on stability analysis and controller synthesis of time-delay systems is rich (see [37, 38] and numerous references therein). The existing criteria for analysis of time delay systems are categorized into either delay-independent or delay-dependent approaches.

In the delay-independent approach, a controller is designed such that the system remains stable regardless of the time delay magnitude. In contrast, by considering the information of the time delay, the delay-dependent approach leads to generally less conservative results specifically for smaller time delays. The analysis and control synthesis of LPV systems with time delay also has been of interest to many researchers in controls community in the past decade (see, e.g., [59, 40, 41, 79]). One of the deficiencies that the majority of these approaches suffer from is that they fail to handle fast varying time delay ($\dot{\tau_k} \geq 1$), which we face with in this study.

The contribution of this study is as following. We propose a method for the design of sampled-data controllers for LPV systems in two cases, namely LPV systems without delay and with internal delay. The proposed design guarantees asymptotic stability and a specified level of induced $\mathcal{L}_2$ gain performance on the closed-loop hybrid system. The formulation can address variable sampling rate cases that often appear in engineering applications, such as automotive engines and manufacturing processes with event-based coupling. The main results of this study are inspired by those employing the input delay method for sampled-data control design proposed by [33] and [54] for continuous-time controller synthesis of state-delayed LPV systems. In particular, we use a parameter-dependent Lyapunov-Krasovskii functional that results in a delay-dependent synthesis method to handle fast-varying time delay. To ensure that the solution to the synthesis problem is in the form of a linear matrix inequality (LMI) optimization problem, we introduce appropriate slack variables to relax the resulting condition in the form of an LMI problem. We consider two different structures for our discrete-time controller, that is, full-order dynamic output-feedback and state-feedback structures.

This chapter is organized as follows: Section 4.2 presents the analysis and synthesis conditions for sampled-data control of LPV Systems without delay. In Section 4.3 all the results are extended to LPV systems with internal delay. Section 4.4 illustrates the capability of the proposed design methods using multiple numerical examples in which different scenarios are examined. The concluding remarks are made in section 4.5.

4.2 Part I: Sampled-Data Control of LPV Systems without Delay

In the first part of this chapter we study an LPV system without internal delay in its model. We consider the following state-space representation for the LPV system

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t) + B_2(\rho(t))u(t)\\ z(t) &= C_1(\rho(t))x(t) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))u(t)\\ y(t) &= C_2(\rho(t))x(t),\end{aligned} \tag{4.2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the vector of controlled outputs, $y(t) \in \mathbb{R}^{n_y}$ is the measurement vector, $w(t) \in \mathbb{R}^{n_w}$ is exogenous disturbance vector with finite energy and $u(t) \in \mathbb{R}^{n_u}$ is the control input vector. The system matrices $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C_1(\cdot)$, $D_{11}(\cdot)$, $D_{12}(\cdot)$ and $C_2(\cdot)$ are real continuous functions of a time varying parameter vector $\rho(t)$ and of appropriate dimensions. The parameter vector $\rho(t)$ varies continuously and is assumed to be measurable in real-time, that is the parameter space is

$$\mathcal{F}_{\mathcal{P}}^{v} \equiv \{\rho : \rho(t) \in C(\mathbb{R}, \mathbb{R}^s), \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \le v_i \quad i = 1, 2, ..., s \quad \forall t \in \mathbb{R}_+\}, \tag{4.3}$$

where $C(\mathbb{R}, \mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathcal{P}$ is a compact set of $\mathbb{R}^s$, and $\{v_i\}_{i=1}^s$ are nonnegative numbers. It is noted that in (4.2), we assume there is no feed through matrix D_{21} influencing the measurement signal y(t) in order for the sampling operator to be well defined [66]. This is not a restrictive assumption and when it is not the case, we can cascade $y(t)$ with a strictly proper filter to relax this requirement.

Now we consider an n^{th}-order discrete-time parameter-varying controller K represented by the following state-space description

$$\begin{aligned}x_d(k+1) &= A_d(\rho(k))x_d(k) + B_d(\rho(k))y(k)\\ u_d(k) &= C_d(\rho(k))x_d(k) + D_d(\rho(k))y(k),\end{aligned} \tag{4.4}$$

where k stands for t_k, the sampling instant. In addition, $x_d(k)$, $y(k)$ and $u_d(k)$ represent the discrete-time controller state vector, the discrete samples of measurement data, *i.e.,*

$y(k) = y(t_k)$ and the discrete control input, respectively. Again all the system matrices are defined to be of appropriate dimensions. In this study we assume a potentially variable sampling scenario, where the time intervals $[0, t_1)$, $[t_1, t_2)$, ..., $[t_k, t_{k+1})$, ... are not necessarily equi-spaced. The sampling intervals are constrained to $0 < t_{k+1} - t_k \leq \tau_m$, where τ_m is a real positive scalar. From the fact that $\tau_k(t) = t - t_k$, we conclude that $\tau_k(t) \leq \tau_m$. From $u_d(k)$, we generate a continuous-time step-wise signal $u(t)$ as

$$u(t) = u_d(k) \qquad t_k \leq t < t_{k+1}.$$

The control design problem described above is a hybrid control problem, where the physical system has a continuous-time dynamics, while the controller is implemented in a digital computer. Shown in Figure 4.1 is the configuration of the closed-loop system, the interconnection of the open-loop continuous-time plant and the discrete-time controller, along with the signal conversion devices. In this structure, not only the measured output sig-

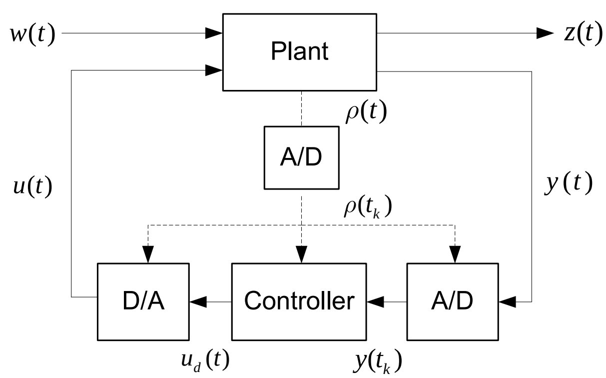

Figure 4.1: The block diagram of the control problem.

nal $y(t)$ is sampled, but also the parameter vector $\rho(t)$ is sampled synchronously at t_k (for $k = 0, 1, 2, \ldots$). However, according to the configuration in Figure 4.1, $\rho(t)$ is measured only at the sampling instants. Hence, we will assume that in the continuous-time system (4.2), the parameter vector remains constant in between two consecutive samples. Consequently, the set of all admissible trajectories for the parameter vector $\rho(t)$ in (4.2) is defined as

$$\mathcal{E}_{\mathcal{P}}^{v} \equiv \{\rho : \rho(t) \in \mathcal{P},\ \rho(t_k + t) = \rho(t_k), \qquad |\rho_i(t_{k+1}) - \rho_i(t_k)| \leq v_i,$$

$$k \in \mathbb{Z}^+, \quad i = 1, 2, ..., s \quad \forall t \in [0, \tau_k)\}. \tag{4.5}$$

Interconnecting the plant (4.2), the controller (4.4) and the converter devices, a closed-loop system T_{wz} is configured that relates the disturbance signal $w(t)$ to the controlled signal $z(t)$. For this closed-loop system the induced $\mathcal{L}_2$-norm ($\mathcal{H}_\infty$-norm) is defined as

$$\|T_{wz}\|_{i,2} = \sup_{\rho \in \mathcal{E}_{\mathcal{P}}^{v}} \sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}. \tag{4.6}$$

This quantity, also known as energy-to-energy gain of the closed-loop system, indicates the worst case output energy $\|z\|_{\mathcal{L}_2}$ over all bounded energy disturbances $\|w\|_{\mathcal{L}_2}$ for all admissible values of the parameter vector $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$. In this chapter, we aim to design the controller K so that the following conditions are satisfied

- The closed loop system is asymptotically stable, and
- The energy-to-energy gain of the closed-loop system is minimized, i.e.,

$$\min_{K} \|T_{wz}\|_{i,2}. \tag{4.7}$$

Instead of the optimal design problem (4.7), it is more convenient to consider the γ-suboptimal energy-to-energy gain in which a controller K is sought such that

$$\|T_{wz}\|_{i,2} < \gamma, \tag{4.8}$$

where γ is a given positive scalar. If the inequality (4.8) holds true, then the energy of the output signal will be bounded by $\gamma\|w\|_{\mathcal{L}_2}$ for any nonzero disturbance $w(t)$ with bounded energy.

Since we aim to establish a design scheme in the continuous-time framework, we begin with a continuous-time controller that takes into account the effect of sampling and hold devices, that is

$$\dot{x}_K(t) \quad = \quad A_K(\rho(t_k))x_K(t) + A_{K\tau}(\rho(t_k))x_K(t_k) + B_K(\rho(t_k))y(t_k)$$

$$
\begin{aligned}
u_K(t_k) &= C_K(\rho(t_k))x_K(t_k) + D_K(\rho(t_k))y(t_k) \\
u(t) &= u_K(t_k) \qquad t_k \le t < t_{k+1}.
\end{aligned} \tag{4.9}
$$

In this configuration, the input to the controller is sampled at t_k and the output of the controller is passed through a zero-order hold at each sampling time. With the assumption (4.5) for the parameter space, we can find an exact discrete-time representation of the controller above to be implemented in a digital computer. Such a conversion will be discussed later. It is noted that in (4.9), the controller includes an additional term $A_{K\tau}(\rho(t))x_K(t_k)$. For the simplicity of further derivations, in the controller above, we replace $\rho(t_k)$ with $\rho(t)$ which holds true for $t_k \le t < t_{k+1}$, due to the parameter space (4.5) assumption, i.e.,

$$
\begin{aligned}
\dot{x}_K(t) &= A_K(\rho(t))x_K(t) + A_{K\tau}(\rho(t))x_K(t_k) + B_K(\rho(t))y(t_k) \\
u_K(t_k) &= C_K(\rho(t))x_K(t_k) + D_K(\rho(t))y(t_k) \\
u(t) &= u_K(t_k) \qquad t_k \le t < t_{k+1}.
\end{aligned} \tag{4.10}
$$

Due to the presence of discrete terms in (4.10), it is difficult to augment the controller with the analog plant (4.2) to determine a unified state-space representation for the closed-loop system. Thus, the input delay approach introduced in (4.1) is employed to map this model to the continuous-time domain. Replacing for $y(t_k)$ from (4.2) and using the input delay representation for the terms $x_K(t_k)$ and $u_K(t_k)$, we rewrite the controller (4.10) as

$$
\begin{aligned}
\dot{x}_K(t) &= A_K(\rho(t))x_K(t) + A_{K\tau}(\rho(t))x_K(t-\tau_k) + B_K(\rho(t))C_2(\rho(t))x(t-\tau_k) \\
u(t) &= C_K(\rho(t))x_K(t-\tau_k) + D_K(\rho(t))C_2(\rho(t))x(t-\tau_k).
\end{aligned} \tag{4.11}
$$

Next we augment the plant (4.2) with the controller (4.11) to obtain the corresponding closed-loop system representation. Defining

$$
\bar{x}(t) = \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix},
$$

we have

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{A}_\tau \bar{x}(t-\tau_k) + \bar{B}w(t) \\
z(t) &= \bar{C}\bar{x}(t) + \bar{C}_\tau \bar{x}(t-\tau_k) + D_{11}w(t)
\end{aligned} \tag{4.12}
$$

where

$$
\begin{aligned}
&\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix}, \quad \bar{A}_\tau = \begin{bmatrix} B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_{K\tau} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \text{ and} \\
&\bar{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \bar{C}_\tau \begin{bmatrix} D_{12} D_K C_2 & D_{12} C_K \end{bmatrix}.
\end{aligned} \tag{4.13}
$$

Here, for the sake of simplicity, we have dropped the dependency of the matrices on the LPV parameter vector. In summary, the interconnection of the open-loop system (4.2) and the output-feedback controller (4.4) along with converter devices is represented as a continuous-time LPV state delay system using the input delay approach. In what follows, we develop the essential tools for stability and performance analysis of the time-delay system (4.12) and further utilize them to derive the synthesis conditions for the design of the sampled-data controller.

4.2.1 Stability Analysis

We first consider the unforced closed-loop LPV system (4.12), that is

$$
\dot{\bar{x}}(t) = \bar{A}(\rho(t))\bar{x}(t) + \bar{A}_\tau(\rho(t))\bar{x}(t-\tau_k(t)). \tag{4.14}
$$

Lyapunov-Krasovskii stability theory serves as a useful tool to achieve delay-dependent conditions for the stability analysis of the system represented by (4.14). To this aim, we need to find a positive definite functional with an infinitesimal upper bound, whose time derivative is negative. The interested reader is referred to ([37, 56, 57, 58]) for an extensive review of the theory and the Lyapunov-Krasovskii functional selection. As the first result of this chapter, we present the following theorem as a sufficient condition to ensure asymptotic stability of the LPV system (4.14).

Theorem 4.1. The time-delay LPV system (4.14) is asymptotically stable for all $\tau_k(t) \leq \tau_m$ if there exist matrices $P, R \in \mathbb{S}_{++}^{2n\times 2n}$ such that for all $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} - R & P\bar{A}_\tau + R & \tau_m \bar{A}^T R \\ \star & -R & \tau_m \bar{A}_\tau R \\ \star & \star & -R \end{bmatrix} < 0. \tag{4.15}$$

proof: We consider the following Lyapunov-Krasovskii functional

$$V(\bar{x}_t, \rho) = V_1(\bar{x}, \rho) + V_2(\bar{x}_t, \rho) \tag{4.16}$$

with

$$V_1(\bar{x}, \rho) = \bar{x}^T(t) P \bar{x}(t)$$

and

$$V_2(\bar{x}_t, \rho) = \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(\xi)\ \tau_m R\ \dot{\bar{x}}(\xi)\ d\xi\ d\theta,$$

where the notation $\bar{x}_t$ is used to represent $\bar{x}(t+\theta)$ for $\theta \in [-\tau_m, 0]$. It is noted that (4.16) is chosen to be dependent on the maximum sampling interval τ_m to result in less conservative stability conditions. In order for the system (4.14) to be asymptotically stable, it suffices that the time derivative of (4.16) along the trajectories of the system (4.14) is negative definite. We have

$$\dot{V}_1(\bar{x}_t, \rho) = \dot{\bar{x}}^T(t) P \bar{x}(t) + \bar{x}^T(t) P \dot{\bar{x}}(t)$$

and

$$\dot{V}_2(\bar{x}_t, \rho) = \tau_m^2 \dot{\bar{x}}^T(t) R\ \bar{x}(t) - \int_{t-\tau_m}^{t} \dot{\bar{x}}^T(\theta)\ \tau_m R\ \dot{\bar{x}}(\theta) d\theta. \tag{4.17}$$

Since $\tau_k(t) \leq \tau_m$, the second term of the right hand side in (4.17) satisfies

$$-\int_{t-\tau_m}^{t} \dot{\bar{x}}^T(\theta)\ \tau_m R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\int_{t-\tau_k}^{t} \dot{\bar{x}}^T(\theta)\ \tau_m R\ \dot{\bar{x}}(\theta) d\theta.$$

Employing Lemma 2.1, we can bound the right hand side of the above inequality by

$$\begin{aligned}-\int_{t-\tau_k}^{t} \dot{\bar{x}}^T(\theta)\ \tau_m R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\frac{\tau_m}{\tau_k}\left(\int_{t-\tau_k}^{t} \dot{\bar{x}}^T(\theta)\ d\theta\right)^T R \left(\int_{t-\tau_k}^{t} \dot{x}^T(\theta)\ d\theta\right) = \\ -\frac{\tau_m}{\tau_k}\left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right]^T R \left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right].\end{aligned}$$

Since $-\frac{\tau_m}{\tau_k} \leq -1$, the following inequality is obtained

$$-\int_{t-\tau_m}^{t} \dot{\bar{x}}^T(\theta)\ \tau_m R\ \dot{\bar{x}}(\theta)\ d\theta \leq -\left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right]^T R \left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right]. \tag{4.18}$$

Substituting (4.18) in (4.16), we obtain

$$\dot{V}(\bar{x}_t,\rho) \quad \leq \quad \dot{\bar{x}}^T P \bar{x} + \bar{x}^T P \dot{\bar{x}} + \tau_m^2 \dot{\bar{x}}^T R \dot{\bar{x}} - \left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right]^T R \left[\bar{x}(t)-\bar{x}(t-\tau_k(t))\right]. \tag{4.19}$$

Further simplification and collecting the terms in (4.19) yields

$$\begin{aligned}\dot{V}(x_t,\rho) \quad &\leq \quad \begin{bmatrix}\bar{x}(t)\\ \bar{x}(t-\tau_k)\end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix}\bar{A}^T\\ \bar{A}_\tau^T\end{bmatrix} \tau_m^2 R \begin{bmatrix}\bar{A}^T\\ \bar{A}_\tau^T\end{bmatrix}^T\right) \begin{bmatrix}\bar{x}(t)\\ \bar{x}(t-\tau_k)\end{bmatrix} \\ &= \quad \begin{bmatrix}\bar{x}(t)\\ \bar{x}(t-\tau_k)\end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix}\tau_m \bar{A}^T R\\ \tau_m \bar{A}_\tau^T R\end{bmatrix} R^{-1} \begin{bmatrix}\tau_m \bar{A}^T R\\ \tau_m \bar{A}_\tau^T R\end{bmatrix}^T\right) \begin{bmatrix}\bar{x}(t)\\ \bar{x}(t-\tau_k)\end{bmatrix},\end{aligned} \tag{4.20}$$

where

$$\mathcal{X} = \begin{bmatrix}\bar{A}^T P + P\bar{A} - R & P\bar{A}_\tau + R\\ \star & -R\end{bmatrix}.$$

To ensure that $\dot{V}(x_t,\rho) < 0$ using (4.20), it is sufficient that

$$\begin{bmatrix}\bar{A}^T P + P\bar{A} - R & P\bar{A}_\tau + R\\ \star & -R\end{bmatrix} - \begin{bmatrix}\tau_m \bar{A}^T R\\ \tau_m \bar{A}_\tau^T R\end{bmatrix} (-R^{-1}) \begin{bmatrix}\tau_m \bar{A}^T R\\ \tau_m \bar{A}_\tau^T R\end{bmatrix}^T < 0.$$

Finally, applying Schur complement to the above LMI results in condition (4.15), and this completes the proof.

4.2.2 Performance Analysis

We consider the closed-loop state-space representation for the LPV system (4.12), i.e.,

$$\begin{aligned}\dot{x}(t) &= \bar{A}(\rho(t))x(t) + \bar{A}_\tau(\rho(t))x(t-\tau_k(t))) + \bar{B}(\rho(t))w(t) \\ z(t) &= \bar{C}(\rho(t))x(t) + \bar{C}_\tau(\rho(t))x(t-\tau_k(t)) + D_{11}(\rho(t))w(t).\end{aligned} \tag{4.21}$$

Next, we present the performance analysis condition for the time-delay LPV system (4.21). The derived condition will be used in the next section for sampled-data control design.

Theorem 4.2. The LPV system (4.21) is asymptotically stable for all $\tau_k(t) \leq \tau_m$ and satisfies $\|z\|_{\mathcal{L}_2} \leq \gamma\|w\|_{\mathcal{L}_2}$ if there exist constant matrices $P, R \in \mathbb{S}^{2n\times 2n}_{++}$ and a positive scalar γ such that for all $\rho(t) \in \mathcal{E}^v_{\mathcal{P}}$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} - R & P\bar{A}_\tau + R & P\bar{B} & \bar{C}^T & \tau_m \bar{A}^T R \\ \star & -R & 0 & \bar{C}_\tau^T & \tau_m \bar{A}_\tau^T R \\ \star & \star & -\gamma I & D_{11}^T & \tau_m \bar{B}^T R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{4.22}$$

Proof: We use the Lyapunov-Krasovskii functional in Theorem 4.1. Next, we apply the following congruent transformation

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

to (4.22). In the obtained inequality, it can be observed that the negative definiteness of the upper left 3×3 block matrix, in light of Theorem 4.1, concludes the asymptotical stability of the system (4.21). Applying Schur complement to (4.22) twice, results in

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} - R & P\bar{A}_\tau + R & P\bar{B} \\ \star & -R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{C}_\tau^T \\ D_{11}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} \bar{C}_1^T \\ \bar{C}_\tau^T \\ D_{11}^T \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix} \tau_m^2 R \begin{bmatrix} \bar{A}^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix}^T < 0.$$

Multiplying the above inequality from left and right by $[\bar{x}^T(t) \quad \bar{x}^T(t-\tau_k) \quad w^T(t)]^T$ and its transpose, respectively, following by straightforward algebraic manipulations yields

$$\begin{aligned} & \dot{\bar{x}}^T P\bar{x} + \bar{x}^T P\dot{\bar{x}} \\ & + h^2 \dot{\bar{x}}^T R\dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-\tau_k(t))]^T R\,[\bar{x}(t) - \bar{x}(t-\tau_k(t))] \\ & -\gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0, \end{aligned}$$

and using (4.19), we have

$$\dot{V}(\bar{x}_t, \rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0. \tag{4.23}$$

Integrating both sides of the inequality (4.23) from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the asymptotic stability and zero initial conditions), we arrive at

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2},$$

and this completes the proof.

4.2.3 LMI Relaxation Using Slack Variables

For the design of the sampled-data controllers, our objective is to establish a synthesis condition to ensure that the closed-loop system (4.21) is stable and satisfies a prescribed level of $\mathcal{H}_\infty$ performance. To this purpose, the corresponding system matrices (4.12) are substituted in (4.22); this, however, results in a bilinear matrix inequality problem due to the product of the controller matrices $A_K, A_{K\tau}, B_K, C_K$ and D_K with the unknown matrix function P and matrix R. Therefore, we will seek an alternative method based on the introduction of slack variables to reformulate the corresponding problem to ensure that an

LMI problem is achieved. The following lemma provides an alternative way to represent the matrix inequality (4.22).

Lemma 4.1. The LPV system (4.21) is asymptotically stable for all $\tau_k(t) \le \tau_m$ and satisfies $\|z\|_{\mathcal{L}_2} \le \gamma \|w\|_{\mathcal{L}_2}$, if there exist constant matrices $P,\ R \in \mathbb{S}_{++}^{2n\times 2n}$, parameter-dependent matrices $V_1(\rho),\ V_2(\rho),\ V_3(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$, the following LMI problem has a feasible solution

$$\begin{bmatrix} -V_1 - V_1^T & P - V_2^T + V_1\bar{A} & -V_3^T + V_1\bar{A}_\tau & V_1\bar{B} & 0 & \tau_m R \\ \star & -R + \bar{A}^T V_2^T + V_2\bar{A} & R + \bar{A}^T V_3^T + V_2\bar{A}_\tau & V_2\bar{B} & \bar{C}^T & 0 \\ \star & \star & -R + \bar{A}_\tau^T V_3^T + V_3\bar{A}_\tau & V_3\bar{B} & \bar{C}_\tau^T & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{4.24}$$

proof: We start with rewriting (4.24) in the form of $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$, with

$$\Psi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & \tau_m R \\ \star & -R & R & 0 & \bar{C}^T & 0 \\ \star & \star & -R & 0 & \bar{C}_\tau & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} -I & \bar{A} & \bar{A}_\tau & \bar{B} & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \text{and} \quad \Theta = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{4.25}$$

The matrix variables V_1, V_2 and V_3 are known as slack variables [55]. We next use Lemma 2.2 by finding the bases for the null space of Λ and Γ as

$$\mathcal{N}_\Lambda = \begin{bmatrix} \bar{A} & \bar{A}_\tau & \bar{B} & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \text{ and } \quad \mathcal{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We then substitute the two matrices above in the solvability conditions of Lemma 2.2. Using the solvability condition (2.11) results in the LMI condition (4.24). On the other hand, the second solvability condition, *i.e.*, (2.12), leads to the following LMI

$$\begin{bmatrix} -\gamma I & D_{11}^T & 0 \\ \star & -\gamma I & 0 \\ \star & \star & -R \end{bmatrix} < 0, \tag{4.26}$$

which is part of LMI (4.24) and is always satisfied as long as there is a feasible solution to (4.24). In summary, feasibility of the LMI condition (4.24) ensures that the LMI problem (4.22) is feasible and based on Theorem 4.2, the proof of Lemma 4.1 is complete.

4.2.4 Dynamic Output-feedback Control Design

In this section, we employ the performance analysis condition presented in the previous section to develop an LMI-based procedure for the sampled-data control design.

Theorem 4.3. If there exist parameter-dependent matrices $X(\rho)$, $Y(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{n\times n}$, $\tilde{P}(\rho), \tilde{R}(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$, $\hat{A}(\rho), \hat{A}_\tau(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n}$, $\hat{B}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n_y}$, $\hat{C}(\rho), \hat{C}_\tau(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u\times n}$ and $D_K(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u\times n_y}$, two given scalars λ_2, $\lambda_3 \in \mathbb{R}$ and a positive scalar γ such that the LMI condition

$$
\begin{bmatrix}
-2\begin{bmatrix} Y & I \\ I & X \end{bmatrix} & \tilde{P} - \lambda_2 \begin{bmatrix} Y & I \\ I & X \end{bmatrix} + \begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix} & -\lambda_3 \begin{bmatrix} Y & I \\ I & X \end{bmatrix} + \begin{bmatrix} B_2\hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix} & \begin{bmatrix} B_1 \\ XB_1 \end{bmatrix} & 0 & \tau_m \tilde{R} \\
\star & -\tilde{R} + \lambda_2 \left(\begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix} + \begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix}^T \right) & \tilde{R} + \lambda_3 \begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix}^T + \lambda_2 \begin{bmatrix} B_2\hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix} & \lambda_2 \begin{bmatrix} B_1 \\ XB_1 \end{bmatrix} & \begin{bmatrix} Y^T C_1^T \\ C_1^T \end{bmatrix} & 0 \\
\star & \star & -\tilde{R} + \lambda_3 \left(\begin{bmatrix} B_2\hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix} + \begin{bmatrix} B_2\hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix}^T \right) & \lambda_3 \begin{bmatrix} B_1 \\ XB_1 \end{bmatrix} & \begin{bmatrix} \hat{C}^T D_{12}^T \\ C_2^T D_K^T D_{12}^T \end{bmatrix} & 0 \\
\star & \star & \star & -\gamma I & D_{11}^T & 0 \\
\star & \star & \star & \star & -\gamma I & 0 \\
\star & \star & \star & \star & \star & -\tilde{R}
\end{bmatrix} < 0, \tag{4.27}
$$

holds true for all admissible parameters $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$, then there exists a controller in the form of (4.11) such that the closed-loop system (4.12) is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$ for all $\tau_k(t) \leq \tau_m$. In addition, the controller state-space matrices are obtained as follows:
1- Find M and N from the factorization problem

$$I - XY = NM^T.$$

2- Find the controller matrices in the following order

$$
\begin{aligned}
A_K &= N^{-1}(\hat{A} - XAY)M^{-T} \\
B_K &= N^{-1}(\hat{B} - XB_2 D_K) \\
C_K &= (\hat{C} - D_K C_2 Y)M^{-T} \\
A_{K\tau} &= N^{-1}(\hat{A}_\tau - XB_2 D_K C_2 Y - NB_K C_2 Y - XB_2 C_K M^T)M^{-T}.
\end{aligned} \tag{4.28}
$$

Proof: In order to find the controller matrices, we apply Lemma 4.1 to the closed-loop system (4.12). In the matrix inequality (4.24), we substitute the closed-loop system matrices $\bar{A}$, $\bar{A}_\tau$, $\bar{B}$, $\bar{C}$ and $\bar{C}_\tau$ from (4.13) and select the three slack to be $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ with appropriately selected scalars λ_1, λ_2 and λ_3. Next, we partition V into

$$V = \begin{bmatrix} X & N \\ N^T & E \end{bmatrix}. \tag{4.29}$$

We then define

$$V^{-1} = \begin{bmatrix} Y & M \\ M^T & F \end{bmatrix}, \tag{4.30}$$

where $N(\rho),\ M(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n}$ and $E(\rho),\ F(\rho) : \mathbb{R}^s \to \mathbb{S}^{n\times n}_{++}$. Equations (4.29) and (4.30) together impose a set of constraints as

$$\begin{aligned} &XY + NM^T = I \quad \text{and} \\ &YN + ME = 0. \end{aligned}$$

Next, performing the congruent transformation $\mathcal{T} = diag(Z^T, Z^T, Z^T, I, I, Z^T)$ with

$$Z = \begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix}, \tag{4.31}$$

we have the following matrix inequality

$$\begin{bmatrix} -2\tilde{V} & \tilde{P} - \lambda_2\tilde{V} + \tilde{A} & -\lambda_3\tilde{V} + \tilde{A}_\tau & \tilde{B} & 0 & \tau_m\tilde{R} \\ \star & -\tilde{R} + \lambda_2(\tilde{A} + \tilde{A}^T) & \tilde{R} + \lambda_3\tilde{A}^T + \lambda_2\tilde{A}_\tau & \lambda_2\tilde{B} & \tilde{C}^T & 0 \\ \star & \star & -\tilde{R} + \lambda_3(\tilde{A}_\tau + \tilde{A}_\tau^T) & \lambda_3\tilde{B} & \tilde{C}_\tau^T & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} < 0, \tag{4.32}$$

where

$$\tilde{V} = Z^T V Z = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}$$

and $\tilde{P} = Z^T P Z$ and $\tilde{R} = Z^T R Z$. In addition, the plant related matrices $\tilde{A}$, $\tilde{A}_\tau$, $\tilde{B}$, $\tilde{C}$ and $\tilde{C}_\tau$ are defined as

$$\begin{aligned}
\tilde{A} &= Z^T V \bar{A} Z = \begin{bmatrix} AY & A \\ XAY + NA_K M^T & XA \end{bmatrix} = \begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix}, \\
\tilde{A}_\tau &= Z^T V \bar{A}_\tau Z = \begin{bmatrix} B_2(D_K C_2 Y + C_K M^T) & B_2 D_K C_2 \\ XB_2 D_K C_2 Y + NB_K C_2 Y + XB_2 C_K M^T + NA_{K\tau} M^T & (XB_2 D_K + NB_K) C_2 \end{bmatrix} \\
&= \begin{bmatrix} B_2 \hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix}, \\
\tilde{B} &= Z^T V \bar{B} = \begin{bmatrix} B_1 \\ XB_1 \end{bmatrix}, \\
\tilde{C}_\tau &= \bar{C}_\tau Z = \begin{bmatrix} D_{12}(D_K C_2 Y + C_K M^T) & D_{12} D_K C_2 \end{bmatrix} = \begin{bmatrix} D_{12} \hat{C} & D_{12} D_K C_2 \end{bmatrix}, \text{ and} \\
\tilde{C} &= \bar{C} Z = \begin{bmatrix} C_1 Y & C_1 \end{bmatrix}.
\end{aligned} \tag{4.33}$$

By substituting these matrix expressions in (4.32), we obtain the LMI in Theorem 4.3. In (4.33), we have used the change of variables as following

$$\begin{aligned}
\hat{A} &= XAY + NA_K M^T, \\
\hat{A}_\tau &= XB_2 D_K C_2 Y + NB_K C_2 Y + XB_2 C_K M^T + NA_{K\tau} M^T, \\
\hat{B} &= XB_2 D_K + NB_K, \quad \text{and} \\
\hat{C} &= D_K C_2 Y + C_K M^T.
\end{aligned}$$

Thus by reversing the transformations, the controller matrices are obtained as given in (4.28). This completes the proof.

Remark In Theorem 4.3, the LMI problem (4.27) is infinite-dimensional due to the dependency of the system matrices on LPV parameters. A standard approach to solve this parameterized LMI problem is to initially select some basis functions to represent the dependency of the matrix variables on the LPV parameters and then grid the parameter space. Finally, the obtained finite-dimensional LMI problem is solved at the grid points and then checked on a finer grid [1]. To this aim, a certain structure may be imposed on the decision variables. One obvious choice is to consider them parameter-independent (constant). Alternatively, a standard approach is to employ some basis functions to represent the dependency of the LMI variables on the LPV parameters, e.g.,

$$X = X_0 + \rho X_1 + \frac{1}{2}\rho^2 X_2 + ... \tag{4.34}$$

and similarly for the other variables. Additional terms could improve the closed-loop performance but at the expense of higher computational efforts.

Remark It is noted that if the there exist a feasible solution for the LMI condition (4.27) wherein the parameters $\hat{A}, \hat{A}_\tau, \hat{B}_K, \hat{C}_K$ and $\hat{D}_K$ are chosen to be independent of the scheduling parameter, then we can relax the assumption (4.5) and retain the original parameter space (4.3).

Remark The optimal $\mathcal{H}_\infty$ performance of the corresponding formulation can be obtained by solving the LMI optimization problem of minimizing γ subject to the convex constraint (4.27) by fixing λ_2 and λ_3. In addition, a line search may be performed to determine the maximum value of the sampling period τ_m. It is noted that the LMI condition (4.27) contains the value of maximum delay τ_m and this confirms that our developed results are delay-dependent.

4.2.5 Digital Equivalence of the Designed Analog Controller

Up to this point, we have designed a continuous-time controller (4.9) for the LPV system (4.2). To implement this controller, we have to find the corresponding discrete-time representation. The integral solution to the controller's state-space representation yields

$$\begin{aligned} x_K(t_{k+1}) = e^{A_K(\rho(t_k))\tau_k} x_K(t_k) &+ \left(\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-s)A_K(\rho(t_k))} ds \right) A_{K\tau}(\rho(t_k)) x_K(t_k) \\ &+ \left(\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-s)A_K(\rho(t_k))} ds \right) B_K(\rho(t_k)) y(t_k) \\ u_K(t_k) = C_K(\rho(t_k)) x_K(t_k) &+ D_K(\rho(t_k)) y(t_k). \end{aligned} \tag{4.35}$$

According to the assumption (4.5) concerning the piecewise constant nature of parameter space, the integrals associated with the above expression have a closed form solution. Therefore, we may find an equivalent discrete-time state-space representation that produces the exact values of the controller (4.9) described by

$$\begin{aligned} x_d(k+1) &= A_d(\rho(k)) x_d(k) + B_d(\rho(k)) y(k) \\ u_d(k) &= C_d(\rho(k)) x_d(k) + D_d(\rho(k)) y(k), \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} A_d &= e^{A_K(\rho(t_k))\tau_k} + \left(e^{A_K(\rho(t_k))\tau_k} - I \right) A_K^{-1}(\rho(t_k)) A_{K\tau}(\rho(t_k)), \\ B_d &= \left(e^{A_K(\rho(t_k))\tau_k} - I \right) A_K^{-1}(\rho(t_k)) B_K(\rho(t_k)), \\ C_d &= C_K(\rho(t_k)), \quad \text{and} \\ D_d &= D_K(\rho(t_k)). \end{aligned} \tag{4.37}$$

The discrete-time controller (4.36) is then used in the configuration shown in Figure 4.1.

Remark In order to solve the sampled-data control design problem, several intermediate steps are taken. Some of the steps are implemented offline and some in real-time. In the offline computation, LMI (4.27) in Theorem 4.3 is solved that provides the variables required for controller design, namely the basic functions associated with $X, Y, \hat{A}, \hat{A}_\tau, \hat{B}, \hat{C}$ and D_K. The variables are then stored for online computation as following. At each sampling instant, the scheduling parameter is measured and the aforementioned variables are updated accordingly. Then, steps 1 and 2 in Theorem 4.3 are evaluated. Once we have $A_K(\rho(t_k)), B_K(\rho(t_k)), C_K(\rho(t_k))$ and $A_{K\tau}(\rho(t_k))$, by utilizing (4.37), we determine the digital controller matrices at each sampling instants.

4.2.6 State-Feedback Control Problem

It is extremely beneficial to consider the case of state-feedback as an special case individually and state the design procedure for it. For the state feedback controller we assume that all the state variables are fully available, that is, $C_2 = I$ or $y(t) = x(t)$ in (4.2). The controller output at each sampling instant is updated based on the value of the state vector and corresponding LPV parameter vector, that is,

$$u_d(t_k) = K(\rho(t_k))x(t_k). \tag{4.38}$$

Using a zero-order hold, the controller output becomes a piecewise-constant signal and is fed to the continuous-time plant, i.e.,

$$u(t) = u_d(t_k) \quad t_k \leq t < t_{k+1}. \tag{4.39}$$

Therefore, the state feedback controller is characterized as $u(t) = K(\rho(t_k))x(t - \tau_k(t))$. According to the assumption (4.5), we would rather formulate (4.39) as following

$$u(t) = K(\rho(t))x(t - \tau_k(t)). \tag{4.40}$$

The closed-loop system of interconnection of (4.2) and (4.40) is expressed as (4.12) with $\bar{x} = x$, and also

$$\begin{aligned} &\bar{A} = A, \quad \bar{A}_\tau = B_2K, \quad \bar{B} = B_1 \quad \text{and} \\ &\bar{C} = C_1, \quad \bar{C}_\tau = D_{12}K. \end{aligned} \tag{4.41}$$

Similar to the output-control case, we aim to design the state-feedback controller matrix K, such that the closed-loop system (4.12) with matrices (4.41) is stable and its energy-to-energy gain satisfy (4.8). The stability and performance analysis procedure we proposed for closed-loop system (4.12) are still valid, however, simpler LMI condition is obtained in comparison with Theorem 4.3, as we will state in the following theorem.

Theorem 4.4. Consider the system (4.2). There exist a sampled-data controller of the form (4.38) such that the corresponding hybrid closed-loop system is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} \leq \gamma\|w\|_{\mathcal{L}_2}$ for all $\tau_k(t) \leq \tau_m$, if there exist a parameter-dependent matrices $\tilde{K}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u \times n}$ and $\tilde{P}(\rho),\ \tilde{R}(\rho), U(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{n\times n}$, a positive scalar γ and real scalars λ_2 and λ_3 such that the LMI condition

$$\begin{bmatrix} -2U & \tilde{P} - \lambda_2 U + AU & -\lambda_3 U + B_2\tilde{K} & B_1 & 0 & \tau_m R \\ \star & -\tilde{R} + \lambda_2(AU + UA^T) & \tilde{R} + \lambda_3 UA^T + \lambda_2 B_2 \tilde{K} & \lambda_2 B_1 & UC_1^T & 0 \\ \star & \star & -\tilde{R} + \lambda_3(B_2\tilde{K} + \tilde{K}^T B_2^T) & \lambda_3 B_1 & \tilde{K}^T D_{12}^T & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} < 0 \tag{4.42}$$

is feasible for any admissible trajectory $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$. Then, the corresponding controller gain in (4.38) is obtained by $K(\rho(t))|_{t=t_k} = \tilde{K}(\rho(t))U^{-1}(\rho(t))|_{t=t_k}$.

Proof: We first start from the matrix inequality (4.24) and substitute the system matrices from (4.41). Then, we place a constraint on the slack variables as $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$, where $V : \mathbb{R}^s \to \mathbb{S}_{++}^{n\times n}$ and λ_2, λ_3 are real scalars. Applying the congruent transformation $T = diag(U, U, U, I, I, U)$ with $U = V^{-1}$ and considering $U^T PU = \tilde{P}$, $U^T RU = \tilde{R}$ and $KU = \tilde{K}$, the LMI condition in (4.42) is obtained and this completes the proof. Interested readers are referred to [68] for more details on state-feedback design.

4.3 Part II: Sampled-Data Control of LPV Systems with Time Delay

So far we have proposed a sampled-data control design method for LPV systems without delay in the plant model. In this section we extend the established analysis and synthesis conditions for state-delayed LPV systems. We consider the following state-space representation for a time-delay LPV system

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + A_h(\rho(t))x(t - h(\rho(t))) + B_1(\rho(t))w(t) + B_2(\rho(t))u(t) \\ z(t) &= C_1(\rho(t))x(t) + C_{1h}(\rho(t))x(t - h(\rho(t))) + D_{11}(\rho(t))w(t) + D_{12}(\rho(t))u(t) \\ y(t) &= C_2(\rho(t))x(t) \end{aligned}$$

$$x(\theta) \quad = \quad \phi(\theta) \qquad \forall t \in [-h_m \quad 0], \tag{4.43}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the vector of controlled outputs, $w(t) \in \mathbb{R}^{n_w}$ is exogenous disturbance vector containing process noise with finite energy, $y(t) \in \mathbb{R}^{n_y}$ is the measurement output signal and $u(t) \in \mathbb{R}^{n_u}$ is the control input vector. The system matrices $A(\cdot)$, $A_h(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C_1(\cdot)$, $C_{1h}(\cdot)$, $D_{11}(\cdot)$, $D_{12}(\cdot)$ and $C_2(\cdot)$ are real continuous functions of a time varying parameter vector $\rho(t)$ and of appropriate dimensions. In this model, $h(\cdot)$ is a differentiable scalar function denoting the parameter-dependent time delay and satisfies $0 \leq h(\cdot) \leq h_m$. Starting from $t = 0$, the initial condition $\phi(\cdot)$ determines the integral solution of (4.43) uniquely. It is assumed that the vector of parameters is bounded piecewise-constant as (4.5).

First, we consider the full-order dynamic output-feedback structure and then we discuss about state-feedback structure. Consider an n^{th}-order discrete-time parameter-varying controller K represented by the following state-space description

$$\begin{aligned} x_d(k+1) &= A_d(\rho(k))x_d(k) + \sum_i A_i x_d(k-i) + B_d(\rho(k))y(k) \\ u_d(k) &= C_d(\rho(k))x_d(k) + \sum_i C_i x_d(k-i) + D_d(\rho(k))y(k), \end{aligned} \tag{4.44}$$

where $x_d(k)$, $y(k)$ and $u_d(k)$ represent the discrete-time filter state vector, the discrete samples of measurement data, *i.e.,* $y(k) = y(t_k)$ and the discrete control input, respectively. All the system matrices are defined to be of appropriate dimensions. Here, the controller has delay in its dynamics to get less conservative and improved performance compared to a memoryless structure. From $u_d(k)$ we build a continuous-time step-wise signal $u(t)$ as

$$u(t) \quad = \quad u_d(k) \qquad t_k \leq t < t_{k+1}.$$

The control design problem described above is a hybrid control problem, where the physical system has a continuous dynamics, while the controller is implemented in a digital computer. The configuration of the closed loop system is similar to that of Figure 4.1. The design should guarantee asymptotic stability and a specified level of performance on the closed-loop hybrid system, namely the energy-to-energy gain (or equivalently $\mathcal{H}_\infty$-norm) of the closed-loop system.

Since we aim to establish a design scheme in the continuous-time framework, we initiate with a controller which has a continuous state, that is

$$
\begin{aligned}
\dot{x}_K(t) &= A_K(\rho(t_k))x_K(t) + A_{Kh}(\rho(t_k))x_K(t-h(\rho(t_k))) + A_{K\tau}(\rho(t_k))x_K(t_k) + B_K(\rho(t_k))y(t_k) \\
u_K(t_k) &= C_K(\rho(t_k))x_K(t_k) + D_K(\rho(t_k))y(t_k) \\
u(t) &= u_K(t_k) \qquad t_k \le t < t_{k+1}.
\end{aligned}
\tag{4.45}
$$

After designing the controller matrices associated with this model, we will find an equivalent discrete model such that it nearly produces the output as (4.45). It is remarked, in contrast to the conventional methods where a continuous-time controller is discretized without taking the converter devices into account, in (4.45) the influence of sampling and holding devices are fully included. In this configuration, the input to the controller is sampled at t_k and the output of the controller is passed through a zero-order hold at each sampling time. The existence of discrete-time terms in (4.45) makes it difficult to augment the controller with the analog plant (4.43) and get a unified state space representation. Thus, the input delay approach introduced in (4.1) is employed to map this model to the continues-time domain. Replacing for $y(t_k)$ from (4.43) and using the input delay method for the terms $x_K(t_k)$ and $u(t_k)$, we rewrite the controller (4.45) as

$$
\begin{aligned}
\dot{x}_K(t) &= A_K(\rho(t))x_K(t) + A_{Kh}(\rho(t))x_K(t-h(\rho(t))) + A_{K\tau}(\rho(t))x_K(t-\tau_k) + B_K(\rho(t))C_2(\rho(t))x(t-\tau_k) \\
u(t) &= C_K(\rho(t))x_K(t-\tau_k) + D_K(\rho(t))C_2(\rho(t)x(t-\tau_k).
\end{aligned}
\tag{4.46}
$$

For the simplicity of further derivations, in the controller above, we have replaced $\rho(t_k)$ with $\rho(t)$ which is correct for $t_k \le t < t_{k+1}$, due to the parameter space (4.5) assumption. It is also stressed that from $\tau_k = t - t_k$ (for $t_k \le t < t_{k+1}$), we have $\tau_k \le t_{k+1} - t_k \le \tau_m$, where τ_m defined to be the maximum sampling interval. Now we augment the plant (4.43) with the controller (4.46) to obtain the corresponding closed loop system. Defining

$$\bar{x}(t) \quad = \quad \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix}$$

then

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{A}_h\bar{x}(t-h) + \bar{A}_\tau\bar{x}(t-\tau_k) + \bar{B}w(t) \\ z(t) &= \bar{C}\bar{x}(t) + \bar{C}_h\bar{x}(t-h) + \bar{C}_\tau\bar{x}(t-\tau_k) + D_{11}w(t) \end{aligned} \tag{4.47}$$

where

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix}, \bar{A}_h = \begin{bmatrix} A_h & 0 \\ 0 & A_{Kh} \end{bmatrix}, \bar{A}_\tau = \begin{bmatrix} B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_{K\tau} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \text{ and} \\ &\bar{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \bar{C}_h = \begin{bmatrix} C_{1h} & 0 \end{bmatrix}, \bar{C}_\tau = \begin{bmatrix} D_{12}D_KC_2 & D_{12}C_K \end{bmatrix}. \end{aligned} \tag{4.48}$$

In summary, the interconnection of the open-loop system (4.43) and the controller (4.46) is represented as a continuous-time LPV system including two delay terms, one due to the inherent system delay and the other as a result of the input delay method. It is noticed that, if the measurement signal $y(t)$ in (4.43) had included delayed states or external inputs, some extra delay terms would have appeared in the closed-loop system (4.47). The results of this section can be extended to cover these cases as well. It is also noted that in (4.45), the terms $A_{Kh}(\rho(t))x_K(t_h)$ and $A_{K\tau}(\rho(t))x_K(t_k)$ are added purposefully. These terms prevent $\bar{A}_h$ and $\bar{A}_\tau$ in (4.48) of being diagonally zero which is necessary for our optimization program in the future. In what follows, we propose analysis results for stability and performance of the delayed-state space system (4.47).

4.3.1 Stability Analysis

We first consider the following unforced LPV system obtained from (4.47)

$$\dot{\bar{x}}(t) = \bar{A}(\rho(t))\bar{x}(t) + \bar{A}_h(\rho(t))\bar{x}(t-h) + \bar{A}_\tau(\rho(t))\bar{x}(t-\tau_k). \tag{4.49}$$

Similar to section 4.2.1, we employ Lyapunov-Krasovskii stability theory to obtain delay-dependent conditions for the stability analysis of the system represented by (4.49). The following result present a sufficient condition to ensure asymptotic stability of the LPV system represented by (4.49).

Theorem 4.5. The time-delay LPV system (4.49) is asymptotically stable for all $0 < h(t) \leq h_m$ with $\tau_k(t) \leq \tau_m$ and $\dot{\tau}_k = 1$, if there exist matrices $P,\ Q_h,\ R_h,\ R_\tau \in \mathbb{S}_{++}^{2n\times 2n}$ such that for all $\rho(t) \in \mathcal{E}_{\mathcal{P}}^v$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} \Sigma_{1,1} & P\bar{A}_h + R_h & P\bar{A}_\tau + R_\tau & h_m\bar{A}^T R_h & \tau_m \bar{A}^T R_\tau \\ \star & -(1-\dot{h})Q_h - R_h & 0 & h_m \bar{A}_h^T R_h & \tau_m \bar{A}_h^T R_\tau \\ \star & \star & -R_\tau & h_m \bar{A}_\tau^T R_h & \tau_m \bar{A}_\tau^T R_\tau \\ \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & -R_\tau \end{bmatrix} < 0, \tag{4.50}$$

with $\Sigma_{1,1} = \bar{A}^T P + P\bar{A} + Q_h - R_h - R_\tau$.

Proof: We consider the following Lyapunov-Krasovskii functional

$$V(x_t, \rho) = V_1(x, \rho) + V_h(x_t, \rho) + V_\tau(x_t, \rho), \tag{4.51}$$

with

$$\begin{aligned} V_1(\bar{x}, \rho) &= \bar{x}^T(t) P \bar{x}(t) \\ V_h(\bar{x}_t, \rho) &= \int_{t-h(t)}^{t} \bar{x}^T(\xi) Q_h \bar{x}(\xi) d\xi + \int_{-h_m}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(\xi) h_m R_h \dot{\bar{x}}(\xi)\ d\xi d\theta \\ V_\tau(\bar{x}_t, \rho) &= \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(\xi) \tau_m R_\tau \dot{\bar{x}}(\xi)\ d\xi d\theta, \end{aligned}$$

where $\bar{x}_t(\theta)$ is used to represent $\bar{x}(t+\theta)$ for $\theta \in [-h_m, 0]$ or $\theta \in [-\tau_m, 0]$. It is noted that (4.51) is chosen to be dependent on the maximum value of delays to result in less conservative stability conditions. In order for the system (4.49) to be asymptotically stable, it suffices that the time derivative of (4.51) along the system trajectory (4.49) is negative definite. We have

$$\dot{V}_1(\bar{x}_t, \rho) = \dot{\bar{x}}^T P \bar{x} + \bar{x}^T P \dot{\bar{x}}$$

and

$$\dot{V}_h = \bar{x}^T(t)Q_h\bar{x}(t) - (1-\dot{h}(t))\bar{x}^T(t-h)Q_h\bar{x}(t-h) + h_m^2\ \dot{\bar{x}}^T(t)R_h\dot{\bar{x}}(t) - \int_{t-h_m}^{t} \dot{\bar{x}}^T(\theta)h_m R_h\dot{\bar{x}}(\theta)d\theta.$$

Similar to (4.18), $\dot{V}_h$ is bounded by

$$\begin{aligned}
\dot{V}_h &\leq \bar{x}^T Q_h \bar{x} - (1-\dot{h})\bar{x}^T(t-h)Q_h\bar{x}(t-h) \\
&+ h_m^2\ \dot{\bar{x}}^T R_h\ \dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-h)]^T R_h [\bar{x}(t) - \bar{x}(t-h)]
\end{aligned}$$

and also

$$\dot{V}_\tau \leq \tau_m^2\ \dot{\bar{x}}^T R_\tau\ \dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-\tau)]^T R_\tau [\bar{x}(t) - \bar{x}(t-\tau)].$$

Substituting in (4.51) for $\dot{V}_1$, $\dot{V}_h$ and $\dot{V}_\tau$, we obtain

$$\begin{aligned}
\dot{V}(\bar{x}_t,\rho) &\leq \dot{\bar{x}}^T P\bar{x} + \bar{x}^T P\dot{\bar{x}} \\
&+ \bar{x}^T\ Q_h\ \bar{x} - (1-\dot{h})\bar{x}^T(t-h)\ Q_h\ \bar{x}(t-h) \\
&+ h_m^2\ \dot{\bar{x}}^T R_h\ \dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-h)]^T R_h [\bar{x}(t) - \bar{x}(t-h)] \\
&+ \tau_m^2\ \dot{\bar{x}}^T R_\tau\ \dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-\tau)]^T R_\tau [\bar{x}(t) - \bar{x}(t-\tau)].
\end{aligned} \tag{4.52}$$

To derive an inequality condition in a matrix form, we replace for $\dot{\bar{x}}$ in (4.52) from (4.49) and gather the relevant terms as follows

$$\dot{V}(\bar{x}_t,\rho) \leq \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \\ \bar{x}(t-\tau_k) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \end{bmatrix} h_m^2 R_h \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \end{bmatrix} \tau_m^2 R_\tau \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \\ \bar{x}(t-\tau_k) \end{bmatrix}$$

$$= \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \\ \bar{x}(t-\tau_k) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} h_m\bar{A}^T R_h \\ h_m\bar{A}_h^T R_h \\ h_m\bar{A}_\tau^T R_h \end{bmatrix} R_h^{-1} \begin{bmatrix} h_m\bar{A}^T R_h \\ h_m\bar{A}_h^T R_h \\ h_m\bar{A}_\tau^T R_h \end{bmatrix}^T + \begin{bmatrix} \tau_m\bar{A}^T R_\tau \\ \tau_m\bar{A}_h^T R_\tau \\ \tau_m\bar{A}_\tau^T R_\tau \end{bmatrix} R_\tau^{-1} \begin{bmatrix} \tau_m\bar{A}^T R_\tau \\ \tau_m\bar{A}_h^T R_\tau \\ \tau_m\bar{A}_\tau^T R_\tau \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \\ \bar{x}(t-\tau_k) \end{bmatrix}, \tag{4.53}$$

where

$$\mathcal{X} = \begin{bmatrix} \Sigma_{1,1} & P\bar{A}_h + R_h & P\bar{A}_\tau + R_\tau \\ \star & -(1-\dot{h})Q_h - R_h & 0 \\ \star & \star & -R_\tau \end{bmatrix},$$

with $\Sigma_{1,1}$ as defined before. To ensure that $\dot{V}(x_t, \rho) < 0$ using (4.53), it is sufficient that

$$\mathcal{X} - \begin{bmatrix} h_m \bar{A}^T R_h \\ h_m \bar{A}_h^T R_h \\ h_m \bar{A}_\tau^T R_h \end{bmatrix} (-R_h^{-1}) \begin{bmatrix} h_m \bar{A}^T R_h \\ h_m \bar{A}_h^T R_h \\ h_m \bar{A}_\tau^T R_h \end{bmatrix}^T - \begin{bmatrix} \tau_m \bar{A}^T R_\tau \\ \tau_m \bar{A}_h^T R_\tau \\ \tau_m \bar{A}_\tau^T R_\tau \end{bmatrix} (-R_\tau^{-1}) \begin{bmatrix} \tau_m \bar{A}^T R_\tau \\ \tau_m \bar{A}_h^T R_\tau \\ \tau_m \bar{A}_\tau^T R_\tau \end{bmatrix}^T < 0.$$

Finally, applying Schur complement to the above inequality condition twice results in the condition (4.50), and this completes the proof.

4.3.2 Performance Analysis

Next, we consider the state-space model (4.47) and derive the corresponding performance analysis conditions.

Theorem 4.6. The LPV system represented by (4.47) is asymptotically stable and the condition $\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$ is satisfied for all $0 < h(t) \leq h_m$ with $\tau_k(t) \leq \tau_m$, $\dot{\tau}_k = 1$ and zero initial condition if there exist matrices $P,\ Q_h,\ R_h,\ R_\tau \in \mathbb{S}_{++}^{2n\times 2n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathcal{E}_{\mathcal{P}}^{v}$, the following matrix inequality condition holds true

$$\begin{bmatrix} \Sigma_{1,1} & P\bar{A}_h + R_h & P\bar{A}_\tau + R_\tau & P\bar{B} & \bar{C}^T & h_m\bar{A}^T R_h & \tau_m \bar{A}^T R_\tau \\ \star & \Sigma_{2,2} & 0 & 0 & \bar{C}_h^T & h_m \bar{A}_h^T R_h & \tau_m \bar{A}_h^T R_\tau \\ \star & \star & -R_\tau & 0 & \bar{C}_\tau^T & h_m \bar{A}_\tau^T R_h & \tau_m \bar{A}_\tau^T R_\tau \\ \star & \star & \star & -\gamma I & D_{11}^T & h_m \bar{B}^T R_h & \tau_m \bar{B}^T R_\tau \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & \star & \star & -R_\tau \end{bmatrix} < 0, \tag{4.54}$$

where $\Sigma_{1,1} = \bar{A}^T P + P\bar{A} + Q_h - R_h - R_\tau$ and $\Sigma_{2,2} = -(1-\dot{h})Q_h - R_h$.

Proof: We first consider a Lyapunov-Krasovskii functional similar to the one introduced in Theorem 4.5. Next, we apply the congruent transformation $\mathcal{T}$ to matrix inequality (4.54), where

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}.$$

In the obtained inequality, it can be observed that the negative definiteness of the upper left 4×4 block matrix, in light of Theorem 4.5, concludes the asymptotical stability of the system (4.47). Applying Schur complement to (4.54) three times results in

$$\begin{bmatrix} \Sigma_{1,1} & P\bar{A}_h + R_h & P\bar{A}_\tau + R_\tau & P\bar{B} \\ \star & \Sigma_{2,2} & 0 & 0 \\ \star & \star & -R_\tau & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{C}_\tau^T \\ D_{11}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{C}_\tau^T \\ D_{11}^T \end{bmatrix}^T$$
$$+ \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix} h_m^2 R_h \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix} \tau_m^2 R_\tau \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{A}_\tau^T \\ \bar{B}^T \end{bmatrix}^T < 0.$$

Multiplying the above inequality from left and right by $\zeta^T(t)$ and $\zeta(t)$, respectively, where

$$\zeta(t) = [\bar{x}^T(t) \quad \bar{x}^T(t-h) \quad \bar{x}^T(t-\tau_k) \quad w^T(t)]^T$$

and with further algebraic manipulations, we obtain

$$\dot{\bar{x}}^T P \bar{x} + \bar{x}^T P \dot{\bar{x}}$$
$$+\bar{x}^T Q_h \bar{x} - (1-\dot{h})\bar{x}^T(t-h) Q_h \bar{x}(t-h)$$
$$+h_m^2 \dot{\bar{x}}^T R_h \dot{\bar{x}} - [\bar{x}(t) - \bar{x}(t-h)]^T R_h [\bar{x}(t) - \bar{x}(t-h)]$$

$$+\tau_m^2 \dot{\bar{x}}^T R_\tau \ \dot{\bar{x}} - \left[\bar{x}(t) - \bar{x}(t-\tau_k)\right]^T R_\tau \left[\bar{x}(t) - \bar{x}(t-\tau_k)\right]$$
$$-\gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0$$

and using (4.52), we have

$$\dot{V}(\bar{x}_t, \rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0. \tag{4.55}$$

Integrating both sides of the inequality (4.55) from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the asymptotical stability and zero initial condition), we arrive at

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2},$$

and this completes the proof.

4.3.3 LMI Relaxation Using Slack Variables

For the purpose of control design, the system matrices (4.48) are substituted in (4.54); this, however, results in a bilinear matrix inequality problem due to the byproduct of the controller matrices with the unknown matrix function P and the matrices R_m and R_τ. similar to what we did in section 4.2.3, we introduce a set of slack variables to relax the matrix inequality (4.54) in terms of an LMI. The following lemma provides the solution.

Lemma 4.2. The LPV system represented by (4.47) is asymptotically stable for all $0 < h(t) \leq h_m$ and $\tau_k(t) \leq \tau_m$, $\dot{\tau} = 1$ and satisfies $\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$ if there exist matrices $P,\ Q_h,\ R_h,\ R_\tau \in \mathbb{S}_{++}^{2n\times 2n}$, parameter dependent matrices $V_1,\ V_2,\ V_3,\ V_4 : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathcal{E}_{\mathcal{P}}^v$, the following matrix inequality holds true

$$\left[\begin{array}{cccc}
-V_1 - V_1^T & P - V_2^T + V_1\bar{A} & -V_3^T + V_1\bar{A}_h & -V_4^T + V_1\bar{A}_\tau \\
\star & \Sigma_{2,2} & R_h + \bar{A}^T V_3^T + V_2\bar{A}_h & R_\tau + \bar{A}^T V_4^T + V_2\bar{A}_\tau \\
\star & \star & -(1-\dot{h})Q_h - R_h + \bar{A}_h^T V_3^T + V_3\bar{A}_h & \bar{A}_h^T V_4^T + V_3\bar{A}_\tau \\
\star & \star & \star & -R_\tau + \bar{A}_\tau^T V_4^T + V_4\bar{A}_\tau \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star
\end{array}\right.$$

$$
\begin{matrix}
V_1\bar{B} & 0 & h_m R_h & \tau_m \\
V_2\bar{B} & \bar{C}^T & 0 & 0 \\
V_3\bar{B} & \bar{C}_h^T & 0 & 0 \\
V_4\bar{B} & \bar{C}_\tau^T & 0 & 0 \\
-\gamma I & D_{11}^T & 0 & 0 \\
\star & -\gamma I & 0 & 0 \\
\star & \star & -R_h & 0 \\
\star & \star & \star & -R_\tau
\end{matrix}
\Bigg] , \tag{4.56}
$$

where $\Sigma_{2,2} = Q_h - R_h - R_\tau + \bar{A}^T V_2^T + V_2\bar{A}$.

Proof: We rewrite (4.56) as $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$ with

$$
\Psi = \begin{bmatrix}
0 & P & 0 & 0 & 0 & 0 & h_m R_h & \tau_m R_\tau \\
\star & \Delta_{2,2} & R_h & R_\tau & 0 & \bar{C}^T & 0 & 0 \\
\star & \star & \Delta_{3,3} & 0 & 0 & \bar{C}_h^T & 0 & 0 \\
\star & \star & \star & -R_\tau & 0 & \bar{C}_\tau^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & D_{11}^T & 0 & 0 \\
\star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -R_h & 0 \\
\star & \star & \star & \star & \star & \star & \star & -R_\tau
\end{bmatrix}, \tag{4.57}
$$

where

$$
\Delta_{2,2} = Q_h - R_h - R_\tau \ , \quad \Delta_{3,3} = -(1-\dot{h})Q_h - R_h,
$$

and

$$
\Lambda \quad = \quad \begin{bmatrix} -I & \bar{A} & \bar{A}_h & \bar{A}_\tau & \bar{B} & 0 & 0 & 0 \end{bmatrix},
$$

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix},$$

where the matrix variables V_i $(i = 1, \ldots, 4)$ are the slack variables. Next, we use Lemma 2.2 by finding the bases for the null space of Λ and Γ as

$$\mathcal{N}_\Lambda = \begin{bmatrix} \bar{A} & \bar{A}_h & \bar{A}_\tau & \bar{B} & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad \mathcal{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Using the solvability condition (2.11) results in the LMI condition (4.54) through some algebraic manipulations. On the other hand, the second solvability condition, i.e., (2.12), leads to the following LMI

$$\begin{bmatrix} -\gamma I & \bar{D}_1^T & 0 & 0 \\ \star & -\gamma I & 0 & 0 \\ \star & \star & -R_h & 0 \\ \star & \star & \star & -R_\tau \end{bmatrix} < 0, \tag{4.58}$$

which is part of the LMI (4.56) and is always satisfied as long as (4.56) holds true. In summary, feasibility of the LMI condition (4.56) ensures that the LMI problem (4.54) is feasible and hence the proof of Lemma 4.2 is complete.

4.3.4 Dynamic Output-feedback Control Design

In order to find the controller matrices, we employ Lemma 4.2 for the closed loop system (4.47). In the matrix inequality (4.56), we substitute the closed loop matrices $\bar{A}$, $\bar{A}_h$, $\bar{B}$, $\bar{C}$ and $\bar{C}_\tau$ from (4.48) and select the three slack to be $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ for given scalar valued λ_1, λ_2 and λ_3. Then we partition V, V^{-1} similar to (4.29) and (4.30) respectively, and define Z similar to (4.31) and perform the congruent transformation $\mathcal{T} = diag(Z^T, Z^T, Z^T, Z^T, I, I, Z^T, Z^T)$ on matrix inequality (4.56). Consequently, we have the following LMI

$$\left[\begin{array}{cccccccc}
-2\tilde{V} & \tilde{P} - \lambda_2\tilde{V} + \tilde{A} & -\lambda_3\tilde{V} + \tilde{A}_h & -\lambda_4\tilde{V} + \tilde{A}_\tau & \tilde{B} & 0 & h_m\tilde{R}_h & \tau_m\tilde{R}_\tau \\
\star & \tilde{\Sigma}_{2,2} & \tilde{R}_h + \lambda_3\tilde{A}^T + \lambda_2\tilde{A}_h & \tilde{R}_\tau + \lambda_4\tilde{A}^T + \lambda_2\tilde{A}_\tau & \lambda_2\tilde{B} & \tilde{C}^T & 0 & 0 \\
\star & \star & -(1-\dot{h})\tilde{Q}_h - \tilde{R}_h + \lambda_3(\tilde{A}_h^T + \tilde{A}_h) & \lambda_4\tilde{A}_h^T + \lambda_3\tilde{A}_\tau & \lambda_3\tilde{B} & \tilde{C}_h^T & 0 & 0 \\
\star & \star & \star & -\tilde{R}_\tau + \lambda_4(\tilde{A}_\tau^T + \tilde{A}_\tau) & \lambda_4\tilde{B} & \tilde{C}_\tau^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & D_{11}^T & 0 & 0 \\
\star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -\tilde{R}_h & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\tilde{R}_\tau
\end{array}\right] < 0, \tag{4.59}$$

with

$$
\begin{aligned}
&\tilde{V} = Z^T V Z = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}, \\
&\tilde{P} = Z^T P Z, \\
&\tilde{Q}_h = Z^T Q_h Z, \quad \tilde{R}_h = Z^T R_h Z, \quad \tilde{R}_\tau = Z^T R_\tau Z, \\
&\tilde{\Sigma}_{2,2} = \tilde{Q}_h - \tilde{R}_h - \tilde{R}_\tau + \lambda_2(\tilde{A}^T + \tilde{A}),
\end{aligned} \tag{4.60}
$$

and also

$$
\begin{aligned}
\tilde{A} = Z^T V \bar{A} Z, \quad \tilde{A}_h = Z^T V \bar{A}_h Z, \quad \tilde{A}_\tau = Z^T V \bar{A}_\tau Z, \quad \text{and} \\
\tilde{B} = Z^T V \bar{B}, \quad \tilde{C} = \bar{C} Z, \quad \tilde{C}_h = \bar{C}_h Z, \quad \tilde{C}_\tau = \bar{C}_\tau Z.
\end{aligned} \tag{4.61}
$$

Using the fact that

$$
Z^T V Z = \begin{bmatrix} Y & I \\ I & X \end{bmatrix} \text{ and } Z^T V = \begin{bmatrix} I & 0 \\ X & N \end{bmatrix},
$$

we can obtain the expression for the variables in (4.61) as following

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} AY & Y \\ XAY + NA_K M^T & XA \end{bmatrix} = \begin{bmatrix} AY & A \\ \hat{A} & XA \end{bmatrix}, \\
\tilde{A}_h &= \begin{bmatrix} A_h Y & Y \\ XA_h Y + NA_{Kh} M^T & XA_h \end{bmatrix} = \begin{bmatrix} A_h Y & A_h \\ \hat{A}_h & XA_h \end{bmatrix}, \\
\tilde{A}_\tau &= \begin{bmatrix} B_2(D_K C_2 Y + C_K M^T) & B_2 D_K C_2 \\ XB_2 D_K C_2 Y + NB_K C_2 Y + XB_2 C_K M^T + NA_{K\tau} M^T & (XB_2 D_K + NB_K) C_2 \end{bmatrix} \\
&= \begin{bmatrix} B_2 \hat{C} & B_2 D_K C_2 \\ \hat{A}_\tau & \hat{B} C_2 \end{bmatrix}, \\
\tilde{B} &= \begin{bmatrix} B_1 \\ XB_1 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C_1 Y & C_1 \end{bmatrix}, \tilde{C}_h = \begin{bmatrix} C_{1h} Y & C_{1h} \end{bmatrix}, \text{ and}
\end{aligned}
$$

$$\tilde{C}_\tau = \begin{bmatrix} D_{12}(D_K C_2 Y + C_k M^T) & D_{12} D_K C_2 \end{bmatrix} = \begin{bmatrix} D_{12}\hat{C} & D_{12} D_K C_2 \end{bmatrix}. \tag{4.62}$$

In the above, we have used the change of variables as following

$$\begin{aligned}
\hat{A} &= XAY + NA_K M^T, \\
\hat{A}_h &= XA_hY + NA_{Kh}M^T, \\
\hat{A}_\tau &= XB_2D_KC_2Y + NB_KC_2Y + XB_2C_KM^T + NA_{K\tau}M^T, \\
\hat{B} &= XB_2D_K + NB_K, \text{ and} \\
\hat{C} &= D_KC_2Y + C_KM^T.
\end{aligned} \tag{4.63}$$

Thus by reversing the transformations in (4.63), the controller matrices are obtained as

$$\begin{aligned}
A_K &= N^{-1}(\hat{A} - XAY)M^{-T}, \\
A_{Kh} &= N^{-1}(\hat{A}_h - XA_hY)M^{-T}, \\
B_K &= N^{-1}(\hat{B} - XB_2D_K), \\
C_K &= (\hat{C} - D_KC_2Y)M^{-T}, \text{ and} \\
A_{K\tau} &= N^{-1}(\hat{A}_\tau - XB_2D_KC_2Y - NB_KC_2Y - XB_2C_KM^T)M^{-T}.
\end{aligned} \tag{4.64}$$

The following theorem summarizes the discussion.

Theorem 4.7. Consider the time-delay LPV system represented by (4.43). If there exist parameter dependent matrices $\tilde{P}(\rho), \tilde{Q}_h(\rho), \tilde{R}_h(\rho), \tilde{R}_\tau(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{2n\times 2n}$ and $X(\rho), Y(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{n\times n}$ and also $\hat{A}(\rho), \hat{A}_h(\rho), \hat{A}_\tau(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n}$ and $\hat{B}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n\times n_u}$ and $\hat{C}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u\times n}$ and $D_K(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u\times n_y}$, a positive scalar γ and real scalars λ_2, λ_3 and λ_4 such that for all admissible parameter $\rho(t) \in \mathcal{E}_{\mathcal{P}}^v$, the LMI condition (4.59) holds true, then there exist a controller in the form of (4.45) such that the closed-loop system (4.47) is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} \le \gamma \|w\|_{\mathcal{L}_2}$ for all $0 < h(t) \le h_m$ and $\tau_k(t) \le \tau_m,\ \dot{\tau}_k = 1$. In addition, the controller matrices are obtained in the following procedure

1- Find M and N from the factorization problem

$$I - XY = NM^T.$$

2- Find the controller matrices from (4.64).

4.3.5 Digital Equivalence of the Designed Analog Controller

Up to this point, we have designed the controller (4.45) for the LPV system (4.43). For the implementation of this controller, we have to find the corresponding discrete representation. The integral solution to this state space representation yields

$$\begin{aligned}
x_K(t_{k+1}) = e^{(t_{k+1}-t_k)A_K(\rho(t_k))}x_K(t_k) \quad &+ \quad \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-s)A_K(\rho(t_k))}A_{Kh}(\rho(t_k))x_K(s-h(\rho(t_k)))\ ds \\
&+ \quad \left(\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-s)A_K(\rho(t_k))}ds\right) A_{K\tau}(\rho(t_k))x_K(t_k) \\
&+ \quad \left(\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-s)A_K(\rho(t_k))}ds\right) B_K(\rho(t_k))y(t_k) \\
u(t_k) = C_K(\rho(t_k))x_K(t_k) \quad &+ \quad D_K(\rho(t_k))y(t_k).
\end{aligned} \tag{4.65}$$

Among three integral terms in the above, the first one to which we refer as I_1, cannot be found precisely due to the fact that the continuous state $x_K(t)$ is not available in real time but only at sampling instants. In other hands, due to the assumption (4.5), the rest of the integrals above are computed precisely. Thus, I_1 is to be computed approximately. It is evident that the value of I_1 is determined by the controller state value in the span $t \in [t_k - h(\rho(t_k)), t_{k+1} - h(\rho(t_k))]$ (in short $t \in [t_k - h_k, t_{k+1} - h_k]$). Referring to Figure 4.2, there are possibly N samples associated with the controller states in the past in the aforementioned span. It is noted that $N = N(h_k)$, that is, since the sampling frequency is

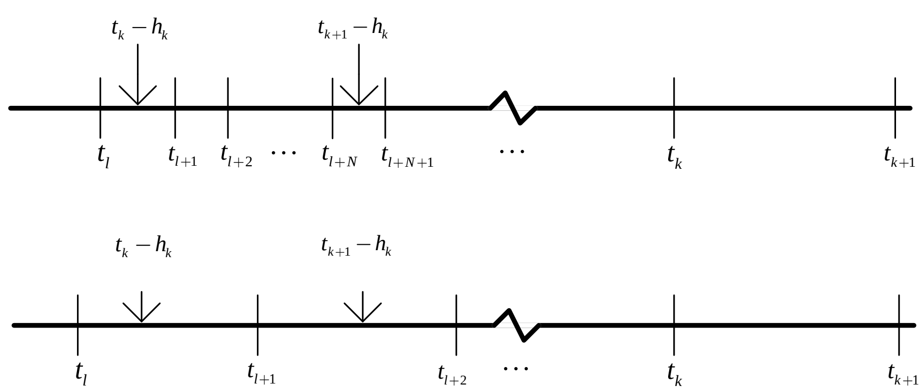

Figure 4.2: Sampling scenario.

not fixed, the number of samples may change by the passage of time. However, for the sake of simplicity, we assume the sampling rate does not change drastically and there is only one sample in the interval of $t \in [t_k - h_k, t_{k+1} - h_k]$. It is noted that, this approach implies that we need to save the previous controller states in advanced up to a certain time depending on maximum delay h_m. Before simplifying I_1, it is noted that

$$I_1 = \int_{t_k - h_k}^{t_{k+1} - h_k} e^{(t_{k+1} - h_k - s) A_K(\rho(t_k))} A_{Kh}(\rho(t_k)) x_K(s) ds. \tag{4.66}$$

In order to approximately find I_1 with a good precision, we use the method proposed in [71] as following. Expanding the integral interval and using the trapezoidal approximation, we have

$$\begin{aligned} I_1 \quad \approx \quad & \int_{t_k - h_k}^{t_{l+1}} e^{(t_{k+1} - h_k - s) A_K(\rho(t_k))} A_{Kh}(\rho(t_k)) \frac{x_K(t_k - h_k) + x_K(t_{l+1})}{2} \, ds \\ + \quad & \int_{t_{l+1}}^{t_{k+1} - h_k} e^{(t_{k+1} - h_k - s) A_K(\rho(t_k))} A_{Kh}(\rho(t_k)) \frac{x_K(t_{l+1}) + x_K(t_{k+1} - h_k)}{2} \, ds \end{aligned}$$

which leads in

$$\begin{aligned} I_1 \quad \approx \quad & \left(e^{(t_{k+1} - t_k) A_K(\rho_k)} - e^{(t_{k+1} - h_k - t_{l+1}) A_K(\rho_k)} \right) A_K^{-1}(\rho_k) A_{Kh}(\rho_k) \ \frac{x_K(t_k - h_k) + x_K(t_{l+1})}{2} \\ + \quad & \left(e^{(t_{k+1} - h_k - t_{l+1}) A_K(\rho_k)} - I \right) A_K^{-1}(\rho_k) A_{Kh}(\rho_k) \ \frac{x_K(t_{l+1}) + x_K(t_{k+1} - h_k)}{2}. \end{aligned} \tag{4.67}$$

In (4.67), $x_K(t_k - h_k)$ and $x_K(t_{k+1} - h_k)$ are not known and should be estimated from the pair $\{x_K(t_l), x_K(t_{l+1})\}$ and $\{x_K(t_{l+1}), x_K(t_{l+2})\}$ respectively (see Figure 4.2). Getting average , we have

$$\begin{aligned} x_K(t_k - h_k) \quad & \approx \quad \frac{x_K(t_l) + x_K(t_{l+1})}{2}, \text{ and} \\ x_K(t_{k+1} - h_k) \quad & \approx \quad \frac{x_K(t_{l+1}) + x_K(t_{l+2})}{2}, \end{aligned}$$

or more accurately, using the linear interpolation

$$x_K(t_k - h_k) \quad \approx \quad \underbrace{\frac{t_{l+1} - (t_k - h_k)}{t_{l+1} - t_l}}_{c_1} x_K(t_l) + \underbrace{\frac{(t_k - h_k) - t_l}{t_{l+1} - t_l}}_{c_2} x_K(t_{l+1}), \text{ and}$$

$$x_K(t_{k+1}-h_k) \approx \underbrace{\frac{t_{l+2}-(t_{k+1}-h_k)}{t_{l+2}-t_{l+1}}}_{c_3} x_K(t_{l+1}) + \underbrace{\frac{(t_{k+1}-h_k)-t_{l+1}}{t_{l+2}-t_{l+1}}}_{c_4} x_K(t_{l+2}). \tag{4.68}$$

Therefore, we may find an equivalent discrete-time state space representation that produces the values of the controller (4.45) approximately described by

$$\begin{aligned} x_d(k+1) &= A_d(\rho(k))x_d(k) + \sum_{i=l}^{l+2} A_i x_d(i) + B_d(\rho(k))y(k) \\ u_d(k) &= C_d(\rho(k))x_d(k) + D_d(\rho(k))y(k), \end{aligned} \tag{4.69}$$

where

$$\begin{aligned} A_d &= e^{(t_{k+1}-t_k)A_K(\rho(t_k))} + \left(e^{(t_{k+1}-t_k)A_K(\rho(t_k))} - I\right) A_K^{-1}(\rho(t_k))A_{K\tau}(\rho(t_k)), \\ B_d &= \left(e^{(t_{k+1}-t_k)A_K(\rho(t_k))} - I\right) A_K^{-1}(\rho(t_k))B_K(\rho(t_k)), \\ C_d &= C_K(\rho(t_k)), \text{ and} \\ D_d &= D_K(\rho(t_k)), \end{aligned} \tag{4.70}$$

and also

$$\begin{aligned} A_l &= \frac{c_1}{2}\left(e^{(t_{k+1}-t_k)A_K(\rho(t_k))} - e^{(t_{k+1}-h_k-t_{l+1})A_K(\rho(t_k))}\right) A_K^{-1}(\rho(t_k))A_{Kh}(\rho(t_k)), \\ A_{l+1} &= \left(\frac{1+c_2}{2}e^{(t_{k+1}-t_k)A_K(\rho(t_k))} - \frac{c_2-c_3}{2}e^{(t_{k+1}-h_k-t_{l+1})A_K(\rho(t_k))} - \frac{1+c_3}{2}I\right) A_K^{-1}(\rho(t_k))A_{Kh}(\rho(t_k)), \text{ and} \\ A_{l+2} &= \frac{c_4}{2}\left(e^{(t_{k+1}-h_k-t_{l+1})A_K(\rho(t_k))} - I\right) A_K^{-1}(\rho(t_k))A_{Kh}(\rho(t_k)). \end{aligned} \tag{4.71}$$

As one can see, at the last stage of the proposed algorithm, the discrete-time controller model is achieved applying an approximation. This is unavoidable since the parameters are measured only at discrete instants. However, the advantages of the proposed sampled-data control is that the effect of sampling and holding devices is considered completely as mentioned earlier by employing input delay method. Also the sampling rate in this method is quite flexible and may vary according to the rate of variation.

4.3.6 Summery of the sampled-data controller design

The computation required in determination of the sampled-data control problem are twofold. First in an offline primary stage, LMI (4.59) of Theorem 4.7 is solved. Consequently, the basis function associated with controller matrices namely $X, Y, \hat{A}, \hat{A}_h, \hat{A}_\tau, \hat{B}, \hat{C}$, and D_K are determined for the second stage, that is the real time operation of the controller, as following: At each sampling instant, the scheduling parameter is measured and all aforementioned LMI variables are updated accordingly. Employing Step 1 and 2 in Theorem 4.7, the continuous-time controller matrices, i.e., $A_K(\rho(t_k)), B_K(\rho(t_k)), C_K(\rho(t_k))$ and $A_{K\tau}(\rho(t_k))$ are determined. For implementation purpose, the digital equivalence controller (4.69) is to be found. This is carried out by means of (4.70) and (4.71). While the controller is working in the closed-loop system, the most recent states of the controller during the previous h_m seconds are recorded to be used in approaching calculation. The designed controller is the one shown in Figure 4.1.

4.3.7 State Feedback Control Problem

Similar to the section 4.2.6, here we consider the state-feedback controller structure as an special case individually. It is assumed that in the plant model (4.43), $C_2 = I$ or $y(t) = x(t)$. The controller output at each sampling instant is updated based on the value of the state vector and corresponding LPV parameter vector, that is,

$$u_d(t_k) = K(\rho(t_k))x(t_k). \tag{4.72}$$

Using a zero-order hold, the controller output becomes a piecewise-constant signal and is fed to the continuous-time plant, i.e.,

$$u(t) = u_d(t_k) \quad t_k \le t < t_{k+1}. \tag{4.73}$$

Therefore, the state-feedback controller is characterized as $u(t) = K(\rho(t_k))x(t - \tau_k(t))$. According to the assumption (4.5), we would rather formulate (4.73) as

$$u(t) = K(\rho(t))x(t - \tau_k(t)). \tag{4.74}$$

Augmenting the plant (4.43) and state feedback controller (4.74) yields in the closed loop system (4.47) with $\bar{x} = x$, and also

$$
\begin{aligned}
&\bar{A} = A, \quad \bar{A}_h = A_h, \quad \bar{A}_\tau = B_2 K, \quad \bar{B} = B_1 \quad \text{and} \\
&\bar{C} = C_1, \quad \bar{C}_h = C_{1h}, \quad \bar{C}_\tau = D_{12} K.
\end{aligned} \tag{4.75}
$$

Similar to the output-control case, we aim to design the state feedback controller matrix K, such that the closed loop system (4.47) with matrices (4.75) is stable and its energy-to-energy gain satisfy (4.8). The discussions of sections 4.3.1 and 4.3.2 are still valid for this closed loop system, however, simpler LMI condition is obtained in comparison with Theorem 4.7, as we will state in the following theorem.

Lemma 4.3. Consider the time-delay LPV system (4.43). There exists a sampled-data controller of the form (4.73) such that the corresponding hybrid closed-loop system is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}$ for all $0 < h(t) \leq h_m$ and $\tau(t) \leq \tau_m$, $\dot{\tau} = 1$ if there exist parameter dependent matrices $\tilde{P}(\rho), \tilde{Q}_h(\rho), \tilde{R}_h(\rho), \tilde{R}_\tau(\rho), U(\rho) : \mathbb{R}^s \to \mathbb{S}_{++}^{n \times n}$, and $\tilde{K}(\rho) : \mathbb{R}^s \to \mathbb{R}^{n_u \times n}$, a positive scalar γ and real scalars λ_2, λ_3 and λ_4 such that the LMI condition

$$
\begin{bmatrix}
-2U & \tilde{P} - \lambda_2 U + AU & -\lambda_3 U + A_h U & -\lambda_4 U + B_2 \tilde{K} & B_1 & 0 & h_m \tilde{R}_h & \tau_m \tilde{R}_\tau \\
\star & \tilde{\Sigma}_{2,2} & \tilde{R}_h + \lambda_3 U A^T + \lambda_2 A_h U & \tilde{R}_\tau + \lambda_4 U A^T + \lambda_2 B_2 \tilde{K} & \lambda_2 B_1 & U C^T & 0 & 0 \\
\star & \star & -(1-\dot{h})\tilde{Q}_h - \tilde{R}_h + \lambda_3 (U A_h^T + A_h U) & \lambda_4 U A_h^T + \lambda_3 B_2 \tilde{K} & \lambda_3 B_1 & U C_h^T & 0 & 0 \\
\star & \star & \star & -\tilde{R}_\tau + \lambda_4 (\tilde{K}^T B_2^T + B_2 \tilde{K}) & \lambda_4 B_1 & \tilde{K}^T D_2^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & D_1^T & 0 & 0 \\
\star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -\tilde{R}_h & 0 \\
\star & \star & \star & \star & \star & \star & 0 & -\tilde{R}_\tau
\end{bmatrix} < 0, \tag{4.76}
$$

with $\tilde{\Sigma}_{2,2} = \tilde{Q}_h - \tilde{R}_h - \tilde{R}_\tau + \lambda_2(UA^T + AU)$ is feasible for any admissible trajectory $\rho(t) \in \mathcal{E}_{\mathcal{P}}^v$. Then, the corresponding sampled-data controller gain in (4.72) is obtained from

$$K(\rho(t))|_{t=t_k} = \tilde{K}(\rho(t))U^{-1}(\rho(t))|_{t=t_k}. \tag{4.77}$$

Proof: Starting from the LMI (4.56), we substitute the system matrices from (4.75). Next, we impose a constraint on the slack variables as $V_1 = V$, $V_i = \lambda_i V$ $(i = 2, 3, 4)$, where λ_i's are real scalars. We note that the above choices are made to ensure that the resulting matrix inequality problem is linear. Next, applying the congruent transformation $T = \text{diag}(U, U, U, U, I, I, U)$ with $U = V^{-1}$ and considering $U^T PU = \tilde{P}$, $U^T Q_h^T U = \tilde{Q}_h$, $U^T R_h U = \tilde{R}_h$, $U^T R_\tau U = \tilde{R}_\tau$ and $KU = \tilde{K}$, the LMI condition (4.76) is obtained and consequently (4.77) gives the controller gain. This completes the proof. Interested readers are referred to [80] for more details on state-feedback design.

4.4 Simulation Results

In this section, we demonstrate the validity of the proposed sampled-data control design methods for LPV systems without delay and with internal delay, using several numerical examples.

4.4.1 Example of Dynamic Output-Feedback Control of an LPV System without Delay

We consider the following linear time-varying system

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 2\sin(0.2t) & 1.1 + \sin(0.2t) \\ -2.2 + \sin(0.2t) & -3.3 + 0.1\sin(0.2t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} w(t) + \begin{bmatrix} 2\sin(0.2t) \\ 0.1 + \sin(0.2t) \end{bmatrix} u(t) \\
z(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).
\end{aligned} \tag{4.78}$$

A sampled-data controller is sought to stabilize the plant and attenuate the effect of disturbance $w(t)$ on the state $x_2(t)$ by measuring the state $x_1(t)$, while maintaining a reasonable control action. We are also interested in determining the maximum sampling period such that the system remains stable. In the all simulations, it is assumed that the system is affected by a pulse disturbance $w(t) = 1$ for $t \in [0, 5]$ and zero elsewhere. In the model shown above, the *sine* term is assumed to be an LPV parameter, whose functional representation is not known a priori but can be measured in real time. Defining $\rho(t) = \sin(0.2t)$, we obtain an LPV state-space representation with the parameter space $\rho \in [-1, 1]$ and $v = 0.2$. Due to the slow variation of the scheduling parameter, it can be treated as if it remains unchanged between two consecutive samples provided that the sampling rate is chosen properly. As mentioned earlier, one easy solution to the sampled-data control problem is to design a continuous-time controller for the continuous-time LPV plant and then discretize it. Since the parameter in this example does not change significantly, the rectangular approximation of the continuous-time design appears suitable for discrete representation. Figure 4.3 shows the response of a continuous-time controller designed for the system (4.78). For this design, optimal γ is obtained to be 0.26. However, if we discretize the obtained controller using the rectangular approximation, even for a very small sampling period $h = 0.001$ the output becomes unstable. Next, we employ the proposed sampled-data design for this system as described in Theorem 4.3. Since this procedure takes into account the inter-sample behavior of the plant, we expect that the sampled-data controller can handle even relatively large sampling rates. As discussed previously, to solve the feasibility problem in Theorem 4.3, we have to decide on the structure of LMI variables X, Y and $\hat{A}$, $\hat{A}_\tau$, $\hat{B}$, $\hat{C}$, $\hat{C}_\tau$ and D_{11}, as well as the scalars λ_2 and λ_3. Selecting the structure of those matrix functions is part of the design and is an ad hoc procedure. For this example, we first use a parameter-independent structure for the aforementioned LMI variables and solve the LMI problem in Theorem 4.3. Next, we repeat the process by employing a quadratic structure similar to (4.34). Table 1 summarizes the energy-to-energy gain of the closed-loop system for various sampling rates and two choices of parameter-dependency for the LMI variables. The results highlight that selecting a quadratic structure can handle sampling rates larger than those in the parameter-independent structure. It is also noted that the value of γ corresponding

Table 4.1: Optimal values of γ for various sampling rates with respect to the chosen structure for LMI variables.

τ_m	λ_2	λ_3	γ for parameter-independent structure	λ_2	λ_3	γ for quadratic structure
0.1	2.8	1.7	0.32	1.4	1.5	0.31
0.2	3.9	-0.2	0.54	1.8	0.6	0.35
0.3	2.8	-0.3	1.62	1.2	0.4	0.67
0.4	-	-	Infeasible	0.8	1	1.36
0.5	-	-	Infeasible	0.6	0	3.44

to a given bound on sampling rate, i.e., τ_m, is sensitive to the scalars λ_2 and λ_2. These two scalars can be optimized by performing a 2-dimensional search. In what follows, we utilize the parameter-dependent LMI variables to achieve an improved performance. We begin the sampled-data control design for the system for a constant sampling rate $\tau_m = 0.1$ sec. Figure 4.4 shows that the closed-loop system is stable and has an acceptable performance in terms of disturbance rejection. Next, we examine the sampled-data control design for a large constant sampling period $\tau_m = 0.5$ sec. Figure 4.5 indicates that although the sampling period is quite large, the system is still stable and a satisfactory response is obtained. Finally, we consider a variable sampling rate, in which the sampling rate varies corresponding to the pattern

$$t_{k+1} = t_k + 0.2(1 + 0.5\sin(0.2t_k)). \tag{4.79}$$

Starting from $t_0 = 0$, the pattern above is associated with a sampling that depends on the parameter variable ranging from 0.1 to 0.3 sec. Figure 4.6 shows an acceptable disturbance attenuation that the sampled-data LPV controller can provide.

4.4.2 Example of State-Feedback Control of an LPV System without Delay

We consider again the linear time varying system (4.78) with

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t).$$

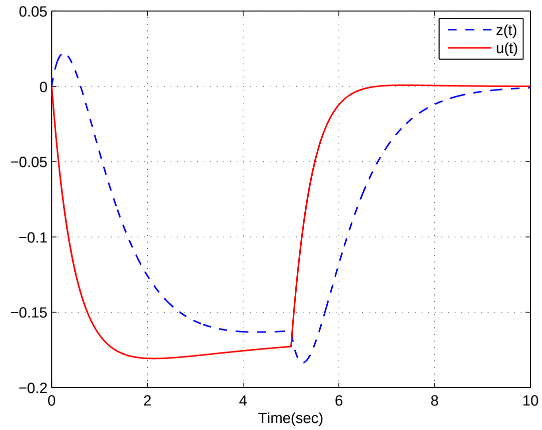

Figure 4.3: Continuous-time controller response.

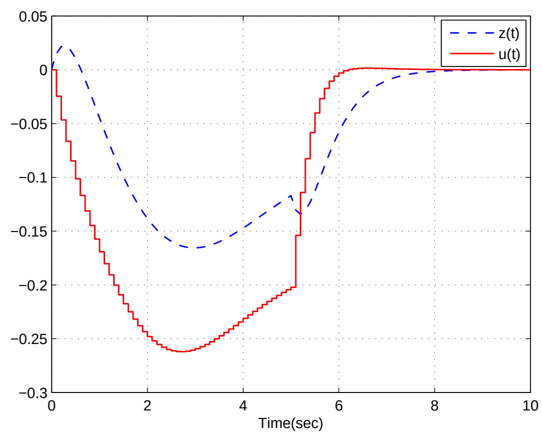

Figure 4.4: Sampled-data controller for $\tau_m = 0.1$ sec.

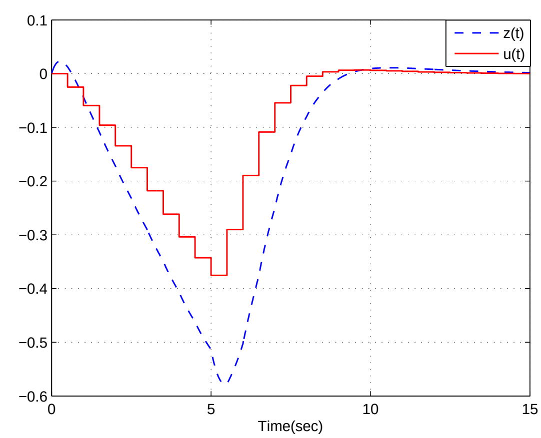

Figure 4.5: Sampled-data controller for $\tau_m = 0.5$ sec.

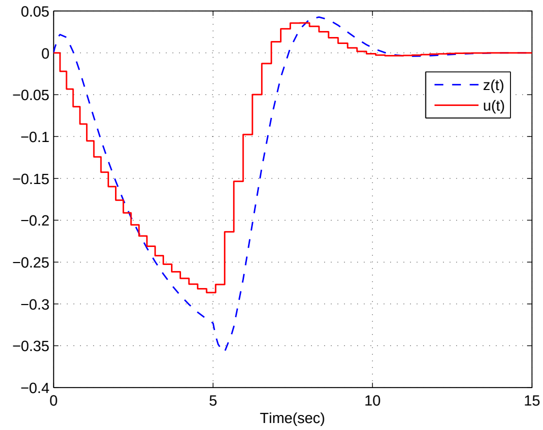

Figure 4.6: Sampled-data controller for variable sampling rate.

Similar to the previous example, a sampled-data controller is designed to stabilize the plant and attenuate the effect of disturbance $w(t)$ on the state $x_2(t)$ while maintaining a reasonable control action. We are also interested in determining the maximum sampling period such that the system remains stable. To solve the LMI problem (4.42), we have to decide on the function variables such as $\tilde{K}$, as well as the scalars λ_2 and λ_3. To solve the LMI problem corresponding to the sampled-data control design problem, we consider two structures for $\tilde{K}$: (i) with parameter-independent gain matrix, i.e., $\tilde{K}(\rho) = \tilde{K}_0$, and (ii) with second-order polynomials, i.e., $\tilde{K}(\rho) = \tilde{K}_0 + \rho\tilde{K}_1 + \frac{\rho^2}{2}\tilde{K}_2$. Table 4.2 shows the optimized $\mathcal{H}_\infty$-norm of the closed-loop system with respect to different maximum sampling rates and two choices of parameter-dependency for the matrix function $\tilde{K}$. We note that tuning the two scalar parameters λ_2 and λ_3 is done independently for each case shown in Table 4.2. In what follows, for the simplicity of implementation, we utilize a parameter-independent state-feedback gain.

Next, we illustrate time-domain simulations to show the effectiveness of the sampled-data control design method to guarantee stability and $\mathcal{H}_\infty$ performance of the closed-loop hybrid systems. To this purpose, we examine two designs corresponding to different sampling rates. Shown in Figure 4.7 is the response of the system (4.78) to the disturbance w with a constant sampling rate $\tau_m = 0.1$. The dashed line corresponds to the state x_2 of the continuous-time system and the solid line indicates the staircase control action. For this example, our search results in $\lambda_2 = 1.3$ and $\lambda_3 = 0.2$. Finally, we examine the case of variable sampling rate, that changes corresponding to the periodic pattern $\{0.3, 0.4, 0.5, 0.3, 0.4, 0.5, \ldots\}^{sec}$. This implies the sampling time instants $\{0, 0.3, 0.7, 1.2, 1.5, 1.9, 2.4, 2.7, \ldots\}^{sec}$. For this example the two scalars are tuned as $\lambda_2 = 1.6$ and $\lambda_3 = 1$. Figure 4.8 shows the simulation results demonstrating an acceptable disturbance attenuation obtained by the controller with a reasonable control action.

Table 4.2: $\mathcal{H}_\infty$-norm for different sampling rates and basis functions.

τ_m	parameter-independent function	2^{nd}-order polynomial
0.1	0.328	0.325
0.2	0.379	0.375
0.3	0.513	0.495
0.4	1.2	0.905
0.5	Infeasible	1.93
0.6	Infeasible	8.04

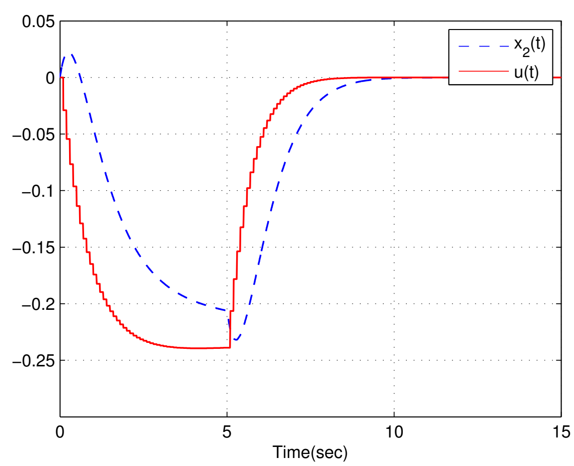

Figure 4.7: Output response for $\tau_m = 0.1$.

4.4.3 Example of Dynamic Output-Feedback Control of an LPV System with Delay

As an illustrative example, we design a sampled-data controller to control the chattering of a milling machine, whose simplified mechanical model is depicted in Figure 4.9. The

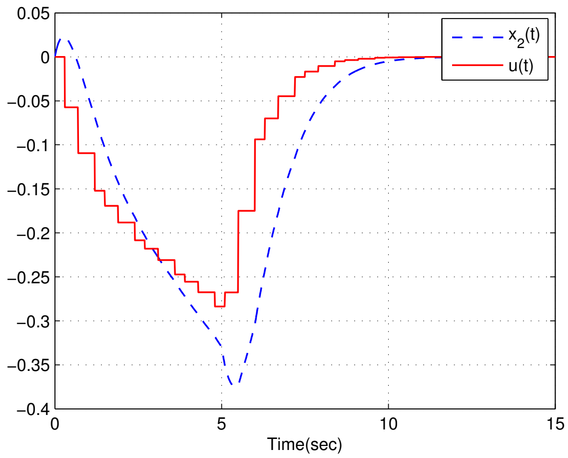

Figure 4.8: Output response for the defined variable sampling rate.

dynamic model of this system can be formulated as an LPV system containing a parameter-dependent time delay [39, 56, 54]. The system consists of a spindle of mass m_2 and a two-blade cutter of mass m_1. Also two springs with stiffness k_1 and k_2 and a damping with the coefficient c are lumped in the model. The rotation of the cutter causes the removal of workpiece material from the surface resulting in a force acting on the cutter denoted by f in the figure. If no control force is applied to the spindle, the machine exhibits chattering. To reduce chattering during the milling process, a force u is to be applied to the spindle dictated by a controller. To this aim, we first derive the dynamic equation associated with the model in Figure 4.9. Introducing the displacement and velocity of the cutter and spindle as state variables, the dynamic model of the system is described by

$$\begin{aligned} &m_1\ddot{x}_1 + k_1(x_1 - x_2) = f\sin(\phi+\beta) + w \quad \text{and} \\ &m_2\ddot{x}_2 + c\dot{x}_2 + k_1(x_2 - x_1) + k_2x_2 = u, \end{aligned} \tag{4.80}$$

where w represents the external disturbance input. Modeling the contact at the point, where the blade touches the surface with a spring of stiffness k, the displacement of this spring equals the difference between the tip position of the consecutive blades that touch this point. Assuming that the angular velocity of the cutter $\omega(t)$ remains constant in a revolution, the time interval between two consecutive blades is equal to $\frac{\pi}{\omega}$. As a result, the corresponding reaction force of the surface is

$$f = -k[x_1(t) - x_1(t - \frac{\pi}{\omega})]\sin(\phi).$$

Therefore, we can rewrite the system equations in (4.80) as

$$\begin{aligned}
\ddot{x}_1 &= \frac{1}{m_1}[-k_1 - k\sin(\phi)\sin(\phi+\beta)]x_1 + \frac{k_1}{m_1}x_1(t - \frac{\pi}{\omega}) + \frac{k_1}{m_1}x_2 + \frac{k_1}{m_1}w(t) \\
\ddot{x}_2 &= \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 - \frac{c}{m_2}\dot{x}_2 + \frac{k_1}{m_2}u.
\end{aligned} \tag{4.81}$$

Table 4.3 summarizes the data corresponding to this example. The obtained model relies on two measurable parameters ϕ and ω. First, we note that

$$\begin{aligned}
\sin(\phi)\sin(\phi+\beta) &= 0.5[\cos(\beta) - \cos(2\phi+\beta)] \\
&= 0.1710 - 0.5\cos(2\phi+\beta).
\end{aligned}$$

Next, we define the scheduling parameter vector as $\rho(t) = [\rho_1(t), \rho_2(t)]^T$ with $\rho_1(t) = \cos(2\phi(t)+\beta)$ and $\rho_2(t) = \omega(t)$. The rotation speed ω is assumed to vary between 200 rpm (20.94 rad/sec) and 2000 rpm (209.4 rad/sec), and maximum variation rate is assumed to be 500 rpm/sec (52.35 rad/sec^2). The parameter space associated with the LPV parameters is as follows:

$$\rho_1(t) \in [-1 \quad 1], \qquad |\dot{\rho}_1| = |-2\omega\sin(2\phi(t)+\beta)| \leq 418.9 \text{ (rad/sec) and}$$

Table 4.3: The milling system parameters.

Parameter	Value	Unit
m_1	1	Kg
m_2	2	Kg
k_1	10	N/m
k_2	20	N/m
k	3	N/m
c	0.5	N/m.s
β	70	degree

$$\rho_2(t) \in [20.94 \quad 209.4] \text{ (rad/sec)}, \quad |\dot{\rho}_2| = 52.35 \text{ (rad/sec}^2).$$

For the parameter-dependent time delay $h(t) = \pi/\omega(t)$, we have

$$\begin{aligned} &0.015 < h(t) < 0.15 \text{ and} \\ &|\dot{h}(t)| = |-\frac{\pi}{\omega^2} \times \dot{\omega}| \leq 0.75. \end{aligned} \tag{4.82}$$

Considering the state vector to be $x = [x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2]^T$, the state-space LPV representation of the system is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.34 + \rho_1(t) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.34 - \rho_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x\left(t - \frac{\pi}{\rho_2(t)}\right) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t), \end{aligned} \tag{4.83}$$

where $z(t)$ is a fictitious system output reflecting the control design objectives. That implies that an $\mathcal{H}_\infty$ controller is sought to reduce the displacement of the elements, as well as penalizing large control actions. The measurement vector $y(t)$ implies that among all states of the system, only two of them are available, namely the position of the spindle and cutter. For effective control of the system, the choice of sampling frequency is crucial and is a trade-off between the quality of the closed-loop system response and the implementation cost. Since the angular velocity could vary during the milling process, the sampling period is not fixed and changes accordingly. Since two blades touch the surface twice per revolution, it is quite reasonable that sampling frequency be at least twice in a rotation. In this example, we consider two samples per revolution, i.e.,

$$t_{k+1} = t_k + \frac{1}{2}\frac{2\pi}{\omega(t_k)}, \tag{4.84}$$

with $t_0 = 0$. Considering the bounds on the time delay in (4.82), we have $h_m = 0.15$ sec and $\tau_m = 0.15$ sec to be used in the LMI (4.59) corresponding to the sampled-data control design. In addition, we have to decide on the structure of the function variables involved in this LMI problem. The structure of the matrix functions is part of the design and is an ad hoc procedure (see first Remark of section 4.2.4). Here, we assume a constant function variables (parameter-independent) to reduce the computational cost. It is noted that the optimal value of γ is quite sensitive to the value of the scalars λ_2, λ_3 and λ_4. These scalars can be optimized by performing a 3-dimensional search. For the milling example, our search resulted in $\lambda_2 = 10$, $\lambda_3 = 1$ and $\lambda_4 = 1$. Also the obtained $\mathcal{H}_\infty$ performance level was calculated to be $\gamma = 0.56$. Shown in Figure 4.10 is the simulation results of the proposed control scheme for the milling machine example, indicating the displacement of the cutter x_1 and that of the spindle x_2 for a predefined test condition. It is assumed that the system is perturbed by a rectangular disturbance $w(t)$ of magnitude one over the time interval $t \in [0, 4]$ and zero elsewhere. The blade rotational speed profile is shown in Figure 4.11. It is apparent that the proposed controller attenuates the disturbance successfully under the variable rotational speed. The control effort required for this study is shown in Figure 4.12, in which the different length of sampling periods is a result of using (4.84). Finally, for comparison we examine the open-loop response of the milling machine, while no control command exists. It is assumed that the system is perturbed by the same disturbance signal as in the previous simulation. Figure 4.13 shows the displacements of the masses while lasting too long to vanish compared to the closed-loop system. This justifies the employment of a controller to diminish the fluctuations as well as settling time.

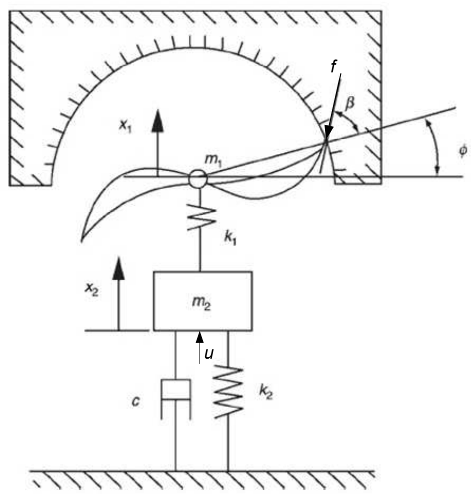

Figure 4.9: A simplified schematic of milling process.

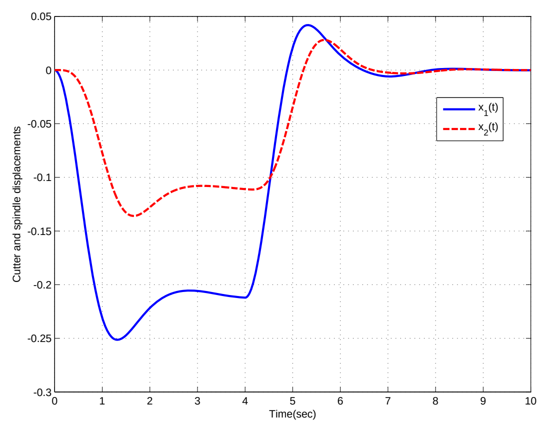

Figure 4.10: Displacement of cutter and spindle.

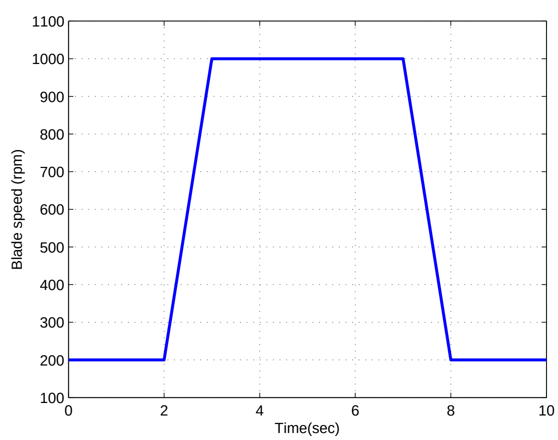

Figure 4.11: Blade rotation speed profile (rpm).

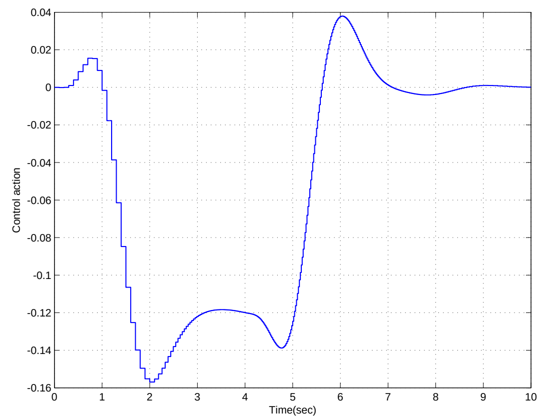

Figure 4.12: Control effort.

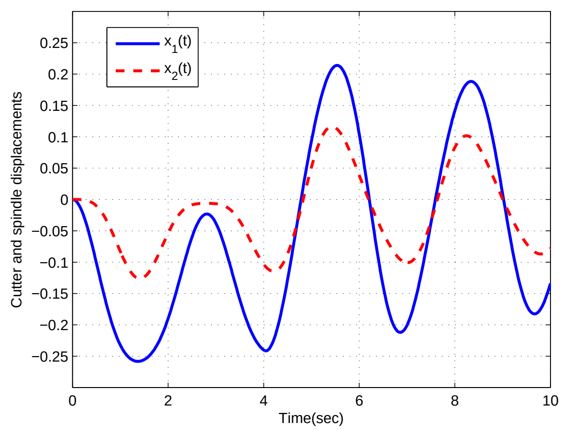

Figure 4.13: Open loop response of the milling machine.

4.4.4 Example of State-Feedback Control of an LPV System with Delay

We consider again the sampled-data control of the milling machine in Example 4.4.3, using state-feedback approach. To this aim in (4.83) we assume

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t).$$

Here, the external disturbance, blade speed profile and sampling scenario are identical to the previous example. For a parameter-independent controller, our search results in $\lambda_2 = 12$,

$\lambda_3 = 0.3$, $\lambda_4 = 1$ and $\gamma = 0.52$. Shown in Figure 4.14 is the simulation results of the proposed control scheme for the milling machine example, indicating the displacement of the cutter x_1 and that of the spindle x_2 for the predefined test condition. It is apparent that the proposed controller attenuates the disturbance successfully under the variable rotational speed. The control effort required for this study is shown in Figure 4.15, in which the different length of sampling periods is a result of using $t_{k+1} = t_k + \frac{1}{2}\frac{2\pi}{\omega(t_k)}$, that is, two samples per revolution of the cutter.

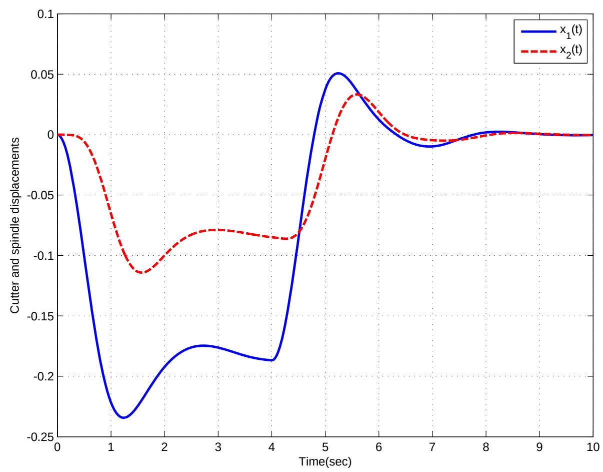

Figure 4.14: Displacement of cutter and spindle.

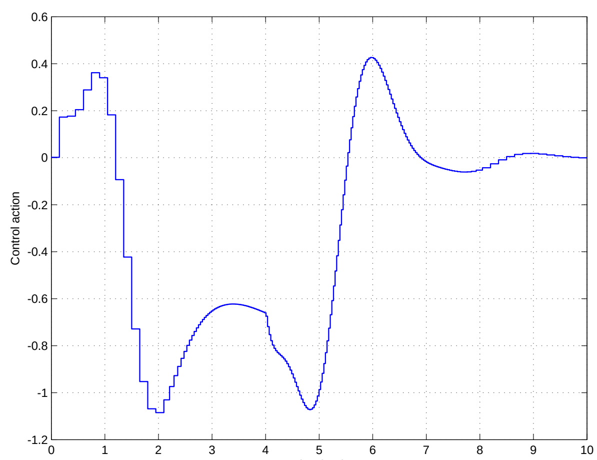

Figure 4.15: Control effort.

4.5 Chapter Conclusions

In this chapter we addressed the problem of sampled-data control for continuous-time LPV systems in two cases, namely LPV systems without delay and with internal delay. Sampled-data control design of a continuous-time system leads to a hybrid closed-loop system, including both continuous and discrete time domains. For analysis and synthesis purposes, the hybrid closed-loop system must be represented in a unified domain, either continuous-time or discrete-time. In this chapter we utilized the input delay approach to map the hybrid closed-loop system into a continuous-time state-delayed LPV system. For the case of LPV system with internal delay, input-delay approach yielded a closed-loop system with two types of delay. To ensure the asymptotic stability and $\mathcal{H}_\infty$ performance of the resulting closed-loop continuous-time state-delayed system, a Lyapunov-Krasovskii functional was introduced with an appropriate structure to take into account different types of delay including the intrinsic system delay and the delay imposed by input delay technique. The use of the proposed Lyapunov-Krasovskii functional led to a delay-dependent matrix inequality condition. Next we employed the slack variables to relax the resulting inequality condition in terms of an LMI problem. In this study, we considered two structures for the controller, i.e., dynamic output-feedback and state-feedback structures. Finally, by means of multiple numerical examples, we examined the validity of the proposed methods.

Chapter5

Conclusions, Contributions and Future Work

5.1 Summary and Assessment of the Dissertation

The main goal of this dissertation was to develop and implement advanced filter and control design methods for dynamical systems. Our major concern in this study was to digitally carry out the filtering and control algorithms without compromising the stability and performance, where we acquired system output data only at discrete time instants. The aforementioned problems became particularly challenging, since we considered a family of time-varying systems whose dynamic change according to an external parameter, namely linear parameter varying (LPV) systems. An additional complexity that we considered in our discussion was that an LPV system might have contained internal delay in its model. For example, in the problem of chattering control in a milling machine, the model of the machine has a delay that depends on the cutter rotary speed. This example and numerous other applications in the literature, endorse the significance of our study to develop advanced methods in filtering and control of LPV systems. It is stressed that the obtained results in this dissertation can be readily used for linear time invariant (LTI) systems, for which more simple synthesis conditions can be exploited.

In the first part of this dissertation, we focused on the filtering problem for continuous-time LPV systems. In chapter 2, we addressed the filtering problem for state-delayed LPV systems, where the state space model of the system contains a delayed version of the state vector and the delay could be parameter dependent. There, we assumed our plant was stable, however, the obtained result could be readily extended for unstable plants where an observer would be designed for state estimation. In addition, the results of this section could be employed for fault detection and isolation in LPV systems. With an appropriate choice of Lyapunov-Kravoskii functional, we derived synthesis conditions that were dependent on the maximum size of delay and its rate. Next, we utilized the slack variables to relax the synthesis conditions in terms of linear matrix inequality (LMI) optimization problems. It was shown that our synthesis criteria have resulted in a less conservative design approach compared to the existing works in the literature.

In chapter 3, we presented a sampled-data filter design method for stable continuous-time LPV systems. We noticed that the sampled-data filter design for a continuous-time system would lead to a hybrid closed-loop system that would be difficult to analyze mathematically. In other words, we have to represent the hybrid system in a unified domain, either continuous or discrete, to do the design in the corresponding time domain. The proposed design method consisted of few intermediate steps to map the hybrid sampled-data closed-loop system into a discrete-time LPV system, using lifting technique, and then to design a discrete-time LPV filter in the corresponding time domain. By means of a numerical example, we showed that this method could handle large and even variable sampling rates. In this chapter, we also modified an indirect method for discretization of a continuous-time LPV model by means of trapezoidal approximation to handle a non-uniform sampling frequency scenario.

In the second part of this dissertation, we investigated the sampled-data control problem for continuous-time LPV systems. First, an LPV system without delay was considered. There, we employed the input-delay approach for sampled-data control design of continuous-time LPV systems by mapping the hybrid closed-loop system to a continuous-time state-delayed LPV system. Then, to ensure the asymptotic stability and $\mathcal{H}_\infty$ performance of the resulting closed-loop hybrid system, we utilized slack variables to relax the resulting inequality condition in terms of an LMI problem. Next, we extended the obtained results in the previous sections for an LPV system with state delay. Applying the input-delay method, we ended up with a system containing two delays in its state-space model, namely the internal delay of the model and the one imposed by mapping the discrete signals into continuous-time domain. To complete the discussion, we proposed synthesis conditions for two structures of the controller, that is, the output-feedback controller and state-feedback controller. Finally, by means of multiple numerical examples, we demonstrated that the proposed method could handle the sampled-data control problem even with varying sampling periods.

5.2 Future Research Directions

Here we list some future directions than can be studied in order to extend and improve the obtained results of this dissertation.

- The filter design problem was considered in chapters 2 and 3. As mentioned earlier, the observer design problem and also fault detection and isolation of dynamical systems are special cases of the filtering problem. One can specialize the presented results for these problems in the sampled-data framework.
- Throughout this dissertation, we assumed the scheduling parameters are piecewise constant. However, in practice, for many applications they may change continuously and fast enough, such that their variations between samples could not be neglected. Special care should be taken to derive new results for such systems
- We derived conditions for synthesis of state-feedback and output-feedback controllers using a delay-dependent Lyapunov-Krasovskii functional. The choice of Lyapunov-Krasovskii functional was rather standard. This work can be extended further by investigating the use of other forms of Lyapunov-Krasovskii functionals.
- The proposed control and filter synthesis conditions satisfy a prescribed energy-to-energy gain (or equivalently $\mathcal{H}_\infty$-norm) of the closed loop system. In a straightforward way, one can extend the obtained results for similar performance objectives such as energy-to-peak gain, peak-to-peak gain, $\mathcal{H}_2$-norm, as well as, multi objective problems.
- In this dissertation, we considered two different structures for the controller, namely state-feedback and output-feedback. For the former case, the availability of all states is a must, otherwise, the latter one has to be chosen. However, the synthesis conditions based on state-feedback design are significantly simpler and computationally more efficient. Therefore as a future study, we can consider the combination of a sampled-data observer in conjunction with a sampled-data state-feedback controller when the full-state measurement is not possible. In addition, one can consider a third structure for the controller, namely static output feedback (possibly parameter dependent), where the control action is obtained from measured output through a static gain.

Appendix A

Evaluation of Operators

In this section, we summarize the procedure to compute $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$, as well as the matrix-valued representation for the operator compositions (3.14) and (3.15). The interested reader is referred to [25], where a complete discussion is presented for LTI systems. In order to compute $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$, we define

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \exp\left\{ \tau_k \begin{bmatrix} -A^T(\rho(t_k)) & -C_1^T(\rho(t_k))C_1(\rho(t_k)) \\ \gamma^{-2}B_1(\rho(t_k))B_1^T(\rho(t_k)) & A(\rho(t_k)) \end{bmatrix} \right\},$$

for $\gamma > 0$. It is shown by [25] that $\|\underline{D}_{11}\|_{\mathcal{L}_2[0,\tau_k)}$ is equal to the largest value of γ, for which the matrix S_{11} has a zero eigenvalue. The procedure for evaluating the operator compositions at each sampling instant is performed by taking the following steps

Step 1: Define the square matrix U as

$$U = \begin{bmatrix} A(\rho(t_k)) & 0_{n\times n_z} \\ 0_{n_z\times n} & 0_{n_z\times n_z} \end{bmatrix}$$

and

$$\begin{aligned} E &= \begin{bmatrix} -A^T(\rho(t_k)) & -C_1^T(\rho(t_k))C_1(\rho(t_k)) \\ \frac{1}{\gamma^2}B_1(\rho(t_k))B_1^T(\rho(t_k)) & A(\rho(t_k)) \end{bmatrix} \\ X &= \begin{bmatrix} C_1(\rho(t_k)) & D_{12}(\rho(t_k)) \end{bmatrix}^T \begin{bmatrix} 0 & C_1(\rho(t_k)) \end{bmatrix} \\ Y &= \begin{bmatrix} C_1(\rho(t_k)) & 0 \end{bmatrix}^T \begin{bmatrix} C_1(\rho(t_k)) & D_{12}(\rho(t_k)) \end{bmatrix}. \end{aligned}$$

Using the aforementioned definitions, we introduce

$$\begin{bmatrix} P & M & L \\ 0 & Q & N \\ 0 & 0 & R \end{bmatrix} = \exp\left\{ \tau_k \begin{bmatrix} -U^T & X & 0 \\ 0 & E & Y \\ 0 & 0 & U \end{bmatrix} \right\}.$$

Next, we partition Q and R in the above equation as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & 0 \\ 0 & I \end{bmatrix}.$$

Consequently, the matrix A_d, that appears in the first subequation in (3.14), is determined to be

$$A_d(\rho(k)) = R_{11}. \tag{1}$$

Step 2: In this step the system matrices A_{dd} and B_{2d} in (3.13) are obtained. Having

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} (Q_{11}^{-1})^T & 0 \end{bmatrix} M^T R$$

and utilizing (1), we determine

$$\begin{aligned} A_{dd}(\rho(k)) &= A_d(\rho(k)) + F_1 \\ B_{2d}(\rho(k)) &= F_2. \end{aligned}$$

Step 3: By means of a matrix factorization (e.g., using Cholesky factorization), one can find B_{1d} in (3.13) satisfying

$$B_{1d}(\rho(k))B_{1d}^T(\rho(k)) = \gamma^2 Q_{12} Q_{11}^{-1}.$$

Step 4: Finally, C_{1d} and D_{12d} in (3.13) are found. To this end, we first define the matrix V as

$$V = [C_1(\rho(t_k)) \quad D_{12}(\rho(t_k))]. \tag{2}$$

Defining

$$\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} = \exp\left\{ \tau_k \begin{bmatrix} -U & V^T V \\ 0 & U \end{bmatrix} \right\}$$

and

$$J = R^T M \begin{bmatrix} Q_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} N - R^T L + P_{22}^T P_{12},$$

the two matrices C_{1d} and D_{12d} are found satisfying

$$\begin{bmatrix} C_{1d}(\rho(k)) & D_{12d}(\rho(k)) \end{bmatrix}^T \begin{bmatrix} C_{1d}(\rho(k)) & D_{12d}(\rho(k)) \end{bmatrix} = J$$

through a matrix factorization. It is emphasized that since in this study our focus is on the matrices depending on a piecewise-constant parameter, we need to repeat the aforementioned steps at each sampling instant. This is done to update the system matrices in (3.14) and (3.15).

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