AN INVESTIGATION OF SENSITIVITY ANALYSIS AS AN ANALYTICAL TOOL

A Thesis

Presented to

the Faculty of the Department of Industrial Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

John Adams Langston, Jr.

January 1969

ACKNOWLEDGMENT

The author wishes to thank his advisor, Dr. B. T. Rhodes, Jr. for his direction and assistance during the months of preparation of this thesis. The author also wishes to thank his wife, Rosemary, for her understanding and encouragement.

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ABSTRACT

The purpose of this study was to examine sensitivity analysis as an analytical tool. The method of procedure was to examine the definition, use and techniques of sensitivity analysis. In addition the application of sensitivity analysis was illustrated on models tractable to partial derivative, simulation and linear programming solution methods. As opposed to an analysis of specific problems, the study was a procedure oriented approach. Sensitivity analysis is useful in identifying variables which require precise estimates, and in observing the best and worst possible outcomes of decisions under uncertainty. Three steps were identified which constitute an analysis: (1) determine which variables to examine; (2) establish the range of each variable examined; and (3) determine the measure of sensitivity. The problems of size of the model and interaction of variables are limiting factors in an analysis. Experimental design and linear programming range techniques were illustrated as procedures for resolving such limitations.

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CHAPTER I

INTRODUCTION

Analysis of a system by Operations Research techniques involves several well defined steps including (1) constructing a model of the system; (2) estimating the operating conditions included in the model; and (3) deriving results from that model. If the actual operating conditions of a system are different than the estimated, a significant difference in results may occur.* One can observe the effect variations in operating conditions have upon the behavior of a system by using sensitivity analysis. For the user of sensitivity analysis, a reference containing the concepts and applications would be helpful. It was the intent of this study to contribute a reference of this nature.

The purpose of this study was to examine sensitivity analysis as an analytical tool. Specifically the following points were examined: (1) the use of sensitivity analysis; (2) the procedures that constitute an analysis; and (3) the techniques and limitations of applying sensitivity analysis.

The study was limited to the problem of examining

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*In this study significant was used in a non-statistical sense. When a statistical sense applied, "statistically significant" was used.

the behavior of a given model. Model formulation, validation, and such factors as stability, and economics are only briefly mentioned in this study. Some limitations on the number of models examined was necessary. As a result the application of sensitivity analysis was illustrated on three models grouped by solution method. The user of sensitivity analysis should find that in most cases the concepts illustrated herein either apply or can be extended to a particular problem.

The general approach of this thesis was to present the concepts of sensitivity analysis, and to illustrate the application of these concepts on a representative group of models. The concepts section contains the definition, uses, and techniques of applied sensitivity analysis along with the limitations to such applications. As opposed to an analysis of specific problems, the study was a procedure oriented approach. The organization of the remainder of the thesis is as follows: (1) review of related work; (2) concepts; (3) applications; and (4) conclusions.

CHAPTER II

REVIEW OF RELATED WORK

In reviewing previous related work it was found that Demski provided a sound basis for this study. He stressed that the theoretical limitations have not been documented, rather work has been mostly concerned with either technique or use.[1] Tomovic attempted to establish sensitivity analysis as an autonomous scientific and engineering field. Specifically his work was in the area of dynamic systems, which he defined as physical systems whose behavior is described by linear or nonlinear differential equations.[2] Boot reported on sensitivity analysis for the purpose of illustrating the concepts.[3] These three references were the only previous work found which were related to examining sensitivity analysis as an analytical tool. However, none of the three provide a comprehensive examination of that concept. Other references found were problem oriented. For example, Bonini

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[1]J. S. Demski, "Some Considerations in Sensitizing an Optimization Model," Industrial Engineering, vol. XIX (September 1968), pp. 463-467.

[2]Rajko Tomovic, Sensitivity Analysis of Dynamic Systems, New York: McGraw-Hill, 1963.

[3]J. C. Boot, Mathemetical Reasoning in Economics and Management Science, Englewood Cliffs, New Jersey: Prentice-Hall, 1967, pp. 126-137.

developed a simulation model of a firm, and used an experimental design and classical analysis of variance techniques to test the sensitivity to selected variables.[4] Forrester reported on his simulation model of industrial systems in which the effect of variations in selected operating conditions was determined.[5] In addition sensitivity analysis was found to be widely used in the linear programming and Lagrangian multiplier sense.

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[4]C. P. Bonini, Simulation of Information and Decision Making in the Firm, Englewood Cliffs, New Jersey: Prentice-Hall, 1963.

[5]J. W. Forrester, Industrial Dynamics, New York: The M.I.T. Press and John Wiley & Sons, 1961.

CHAPTER III

CONCEPTS OF SENSITIVITY ANALYSIS

Definition. Various definitions of sensitivity analysis were found in the literature. Boot states that the basic problem of sensitivity analysis is "how do the results of a model change when the data or parameters or assumptions of the model change?"[6] Essentially the definition can be stated as observing the effect on system behavior of a variation in operating conditions. For example, if y is determined from estimated values for x_1 and x_2 in the model $y = f(x_1, x_2)$, a change in x_1 causes some change in the resulting y. One approach is to find out what change in y results from a specified change in x_1. An alternative approach is to determine by how much x_1 can vary before y changes a significant amount. The question of determining what degree of change is considered significant is relative to the problem context. One index of significance is per cent change.

Uses. Sensitivity studies provide practical information in several cases. Some of the more obvious are as follows: (1) identifying variables which significantly effect

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[6] Boot, op. cit., p. 128.

the behavior of a system; (2) identifying interaction of variables; and (3) observing the distribution of possible outcomes due to uncertainty in estimating variables. An illustration is the case of a model with one hundred data elements. Actual cost data may be infeasible to collect for all elements. One can use estimated values for costs, and observe the sensitivity of the results to variations in each variable over a specified range. If the results are judged insensitive to certain variations, it may not be necessary to collect more precise values for the corresponding variables.

Interaction occurs if the effect of one variable on the result is dependent upon the level of another variable. For example, if y changes 5 units for a 10 unit change in x_1 when x_2 is 50, and y changes 1 unit for a 10 unit change in x_1 when x_2 is 25, then interaction exists between x_1 and x_2. By testing the effect of changing variables simultaneously one can identify interaction.

If one is reasonably certain of the estimates included in a model, and that model is a realistic abstraction of the system under study, then the results will be reasonably valid. This may not be the case in a practical situation. In attempting to forecast demand, for example, only some degree of certainty can be expected. By establishing a range of probable values for each uncertain variable it is possible to determine the best and worst possible outcomes of a decision. Sensitivity analysis is also useful in determining what variables to include in the model.

Techniques and limitations. A sensitivity analysis can be performed by changing each variable in the model, and observing the resulting effect. However, the problems of size and interaction place limitations on such a simplified approach. In the case of the one hundred data element model, for example, the problem of size is a limiting factor. Also, unless the assumption can be made that interaction is insignificant, the problem of testing a large number of variables simultaneously becomes a limiting factor. Even in the case of a model with four variables, a simultaneous test procedure would require sixteen runs for all combinations of changes with one alternative value for each variable.

In order to reduce the problem of size to a workable level, certain techniques can be used. One possibility is to test the variables that are most likely to have a significant effect on the result, and hold other variables constant. Statistical methods are also useful in performing an analysis; for example, experimental design techniques.[7]

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[7]O. C. Davies, The Design and Analysis of Industrial Experiments, New York: Hafner Publishing Co., 1960.

In any case the first step in an analysis is determining which variables to examine.

Step two is to establish the range of each variable that is to be examined. In arriving at estimates for the variables, the likely procedure is to decide what possible values each variable can assume. Then one assigns the corresponding probability of occurrence of each value. The objective is to observe system behavior for variations over the established possible range.

The third step is that of measuring sensitivity, i.e. what change is significant. A standard decision rule cannot be established, as each model must be evaluated in each problem context. Statistical tests for statistical significance are an alternative means of interjecting objectivity in determining sensitivity. In most cases the decision rule is a subjective one.

The solution method is an important factor in the application of an analysis. For example, a linear programming method provides for sensitivity information in addition to computing the results. For a simulation model an analysis involves changing variable values and rerunning the model. Economic factors play a major role in the method of solution because of the cost of performing the computations. One may find that the cost of computer time offsets any economic benefits realized from the sensitivity information. The problem of stability, which is not examined in this study, is essentially the problem of whether sensitivity identified for a time period will in fact remain constant over additional time periods.

CHAPTER IV

APPLICATION OF SENSITIVITY ANALYSIS

In general a sensitivity analysis is performed by changing the operating conditions, and recomputing the results. The complexity of an analysis is determined by the number of variables and by interaction. In addition the type of model is an important factor in applying sensitivity analysis. For example the problem of determining the change in results is greatly facilitated, if the model is tractable to a solution method such as partial derivatives. In any case, an analysis is based on three decision rules: (1) selecting which variables to test for variations; (2) determining the range of each variable tested; and (3) determining the significance of effect upon the results. Applications of sensitivity analysis are illustrated in the sequel, using the concepts established in this study. Three types of models are illustrated corresponding to the following solution methods: (1) partial derivatives; (2) simulation; and (3) linear programming.

Model 1--Partial derivatives method.

The economic lot size equation was selected for the purpose of illustrating solutions by partial derivatives. For the purpose of this study, the following model was used:

$$K = \frac{bn}{x} + \frac{cx}{2}$$

where:

K is total cost per year in dollars

b is cost per order in dollars

n is sales demand per year in pounds

x is lot size in pounds

c is average inventory cost per pound in dollars

The objective for the problem was to minimize K. The minimum cost equation was obtained as follows:

$$\frac{dK}{dx} = 0 = \frac{-bn}{x^2} + \frac{c}{2}$$

$$x^2 = \frac{2bn}{c}$$

$$x^* = \sqrt{\frac{2bn}{c}}$$

Using estimates of n = 12,000 pounds, b = \$40, and c = \$6/pound resulted in an optimum lot size of 400 pounds. The corresponding minimum cost was computed as follows:

$$K = \frac{(\$40)(12{,}000\text{ Lbs.})}{(400\text{ Lbs.})} + \frac{(\$6)}{(\text{Lb.})}\frac{(400\text{ Lbs.})}{2} = \$2400$$

If the model is a valid abstraction of the system under analysis, and the variable estimates are reasonably accurate, then the results are reasonably valid. However, if uncertainty prevails in predicting n (the actual demand), one is concerned with the effect upon K and x* if actual demand differs from the estimated. The effect can be observed by changing demand for each possible value over the specified range, and recomputing cost. An analysis was performed for a range of possible values from a minimum of 6,000 pounds to a maximum of 24,000 pounds. The results are shown in Table I. For example if n is actually 14,400 lbs. (20% increase), then x* is 438 lbs. (10% increase). However, the relevant question, that of effect on K, reveals that an x of 400 lbs. results in K = $2,640 while an x* of 438 lbs. results in K* = $2,629, a difference of less than .05% in K. From Table I one observes that the change in K was less than 5% for variations in n from -40% to +50%. In this case K was relatively insensitive to variations in n, assuming a corresponding decision rule.

The existence of interaction is apparent from observing the model, i.e. the effect of changes in n becomes more significant as b becomes larger. For the purpose of illustration, x was held constant at 400 pounds. In the analysis each of the variables b, c and n were tested at two possible values. The results are shown in Table II. Eight computations were required to account for all combinations of values. The estimated values were repeated as follows: (1) n = 12,000 pounds; (2) c = $6; and (3) b = $40; with corresponding K = $2,400. Each variable was assigned an alternative higher value. For example, if n is 14,400 pounds, the resulting

TABLE I

ANALYSIS OF ECONONIC LOT SIZE MODEL
FOR
UNCERTAIN DEMAND

n (Lbs)	x* (Lbs)	%Δn	K ($)	K* ($)	%ΔK
6,000	282	-50	1800	1697	-6.1
7,200	309	-40	1920	1859	-3.3
8.400	334	-30	2040	2008	-1.6
9,600	357	-20	2160	2146	-0.7
10,800	379	-10	2280	2277	-0.1
12,000	400			2400	
13,200	419	10	2520	2517	0.1
14,400	438	20	2640	2629	0.4
15,600	455	30	2760	2736	0.9
16,800	473	40	2880	2840	1.4
18,000	489	50	3000	2939	2.1
24,000	565	100	3600	3394	6.1

NOTE:
n is Demand (12,000 Lbs. is estimated demand)
x* is Optimun lot size for corresponding n
%Δn is Percent change between actual and estimated demand
K is Cost for actual demand with anticipated optimum lotsize
K* is Cost for actual demand with actual optimum lot size
%ΔK is Percent difference in K and K*

K is $2,640. However, changing n and b simultaneously resulted in the following:

$$K = \frac{bn}{x} + \frac{cx}{2} = \frac{(50)(14,400)}{400} + \frac{(6)(400)}{2} = \$3,000$$

As shown in Table II the increase in n resulted in a ΔK of $240 for b = $40 and a difference in ΔK of $300 for b = $50. Thus the observation that interaction existed is verified. The number of computations can be reduced if interaction does not occur between all variables. In this model c does not interact with either variables b or n. As a result the differences in ΔK were the same at both values of c for corresponding changes in n and b. For example, with b = $40 and c = $6 an increase in n resulted in a ΔK of $240, and with c = $8 an increase in n resulted in a difference in ΔK of $240.

An alternative approach to the treatment of uncertainty in demand, is to assign probabilities to the possible outcomes. As a result n is treated as a random variable. The resulting expected cost is computed by using the expected value for demand:

$$E[K] = \frac{b\ E[n]}{x} + \frac{cx}{2}$$

In addition the variance of cost is computed as follows:

$$V[K] = \frac{b^2\ V[n]}{x^2}$$

The procedure for a sensitivity analysis is to test the

TABLE II

ANALYSIS OF ECONOMIC LOT SIZE MODEL
FOR
CHANGES IN THREE VARIABLES

n (Lbs)	c ($)	b ($)	K ($)	ΔK	ΔK^1
12,000	6	40	2400		
14,400	6	40	2640	240	240
12,000	6	50	2700	300	60
14,400	6	50	3000	600	300
12,000	8	40	2800	400	
14,400	8	40	3040	640	240
12,000	8	50	3100	700	60
14,400	8	50	3400	1000	300

NOTE:
n is Demand
c is Inventory cost
b is Order cost
K is Total cost
ΔK is Difference in corresponding K and the value of K for n = 12,000 Lbs.
ΔK^1 is Difference in K and preceeding K

effect of changes in $E[n]$ and $V[n]$ rather than changes in point estimates as previously illustrated.

One can simplify the problem of finding the change in result by the use of partial derivatives:

$$\frac{\partial K}{\partial n} = \frac{b}{x} = \frac{40}{400} = .1$$

$$\Delta K = .1\ (\Delta n)$$

The resulting equation for ΔK indicates that for each pound increase in demand, cost increases by 10 cents. Extending the approach to b and c results in the following:

$$\frac{\partial K}{\partial c} = \frac{x}{2} = \frac{400}{2} = 200$$

$$\Delta K = 200\ (\Delta c)$$

$$\frac{\partial K}{\partial b} = \frac{n}{x} = \frac{2400}{400} = 6$$

$$\Delta K = 6\ (\Delta b)$$

If demand is actually 14,400 pounds or a Δn of 2,400 pounds, $\Delta K = .1(2400) = 240$. To illustrate the effect of interaction, since $\frac{\partial K}{\partial n} = \frac{b}{x}$, for b = \$60 each pound change in demand results in a 15 cent change in cost.

The economic lot size model was a case where the equations and interaction were linear. As a result the problem of testing the range of each variable was simplified by treating the variables as discrete, and using relatively large intervals. In addition, size and interaction were not significant factors in the analysis.

Model 2--Simulation method.

If the problem cannot be structured as a mathematical model, an alternative method is a simulation model. The following problem was used to illustrate this case. Passengers arrive at a bus stop in a poission fashion with a mean arrival rate, λ = 4 per hour. Buses arrive normally with a mean interval μ = 15 minutes and a standard deviation σ = 3 minutes. The number of seats available on each bus is a poission distribution with a mean m = 1.5. The objective was to minimize the average waiting time of passengers at the bus stop. Average waiting time for the parameter estimates given was 18.2 minutes. A FORTRAN program was used to perform the simulation computations. By changing the parameter estimates and rerunning the simulation model, any change in average waiting time was observed. Random variates were selected from the probability distribution specified for each variable, in order to determine the occurrence of each event. Four variables were included in the model. Therefore, to test the sensitivity for each variable at two levels of values involves 16 computer runs. If it is not feasible to test all variables, a decision is required to determine which variables are held constant. For the purpose of this analysis, λ was held constant. The range of possible values was determined as follows: (1) μ, 14-15; (2) σ, 2-3; and

TABLE III

ANALYSIS OF SIMULATION MODEL
FOR CHANGES IN THREE VARIABLES

Variables			Average Waiting Time	Reduction Avg. Wait Time
μ	σ	m		
15	3	1.5	18.2	
14	3	1.5	17.1	1.1
15	2	1.5	14.8	3.4
14	2	1.5	14.5	3.7
15	3	2.0	15.5	2.7
14	3	2.0	15.1	3.1
15	2	2.0	13.6	4.6
14	2	2.0	11.4	6.8

NOTE:
μ is Mean interval between buses
σ is Standard deviation of interval
m is Number of seats available

(3) m, 1.5-2.0. Even though information was not available as to whether linearity existed over the range of values, the interval was small enough to run a discrete case. In addition, it was decided to check for interaction; therefore each combination of levels was run. Results of the analysis are shown in Table III. Interaction is illustrated between μ and σ by the difference in effect of changes in μ on the average waiting time for the two levels of σ. For example the average waiting time decreases by 1.1 minutes for a change in μ with σ = 3 minutes, and by 0.3 minutes for a change in μ with σ = 2 minutes. The results in Table III show that as μ decreases 1 minute for σ constant, the average waiting time decreases 1.1 minutes. The average waiting time was relatively sensitive to changes in each variable for this case.

Results of a classical analysis of variance are shown in Table IV. Applying an F-test, the effect of a change in either σ or m for the range given was statistically significant at the 75% level. The interaction term for all three variables was used as the error term. Next all interaction terms were pooled as an error term. The results in Table IV show at least a 95% confidence level for each of the three variables.

Experimental designs are a possible means of reducing

TABLE IV

ANALYSIS OF SIMULATION MODEL
USING
ANALYSIS OF VARIANCE

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F Statistic	Level of Significance
μ	1	16.00	2.00	2.4	50%
σ	1	134.56	16.82	19.8	75%
m	1	81.00	10.12	11.9	75%
$\mu\sigma$	1	2.00	0.25		
μm	1	1.44	0.18		
σm	1	0.16	0.02		
$\mu\sigma m$ (Error)	1	6.76	0.85		
(Pooling of all interaction terms)					
μ	1		2.00	6.2	95%
σ	1		16.82	51.8	99%
m	1		10.12	31.1	99%
Error	4		1.30		

the problem of analyzing a large number of variables. For example in this case, the mean passenger arrival rate was held constant. In addition only two levels of each variable were examined in order to keep the number of computer runs at a minimum. It was possible that μ and σ could vary over a wider range than the two levels in Table III. More specifically the range for μ is 13-15, and the range for σ is 2-4 with λ and m unchanged from the previous problem. A random balance design was used whereby λ was fixed and values of μ, σ and m were selected randomly.[8] The procedure is intended to screen a large number of samples in an effort to identify those contributing significantly to the results. Table V shows the results of this procedure. The results indicate that a change in each variable is significant. The greatest change in results of -6.5 minutes occurs when all three variables are changed simultaneously. Identifying which of the three variables has the most significant effect is not readily apparent from Table V. However, since the result decreased -4.6 minutes for a change in m with the other variables fixed, m was the most likely choice.

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[8]F. E. Satterthwaite, "Random Balance Experimentation," Techometrics, vol. I, (1959), p. 111.

TABLE V

ANALYSIS OF SIMULATION MODEL
USING A RANDOM BALANCE EXPERIMENTAL
DESIGN

Variables			Average Waiting Time	Change Avg.Wait Time
μ	σ	m		
15	2	1.5	14.8	-3.4
15	2	2.0	13.6	-4.6
13	4	1.5	15.8	-2.4
15	3	2.0	15.5	-2.7
15	4	1.5	21.3	3.1
13	3	2.0	12.9	-5.3
14	2	1.5	14.5	-3.7
13	4	2.0	11.7	-6.5

NOTE:
μ is Mean interval between buses
σ is Standard deviation of interval
m is Mean number of seats available

Model 3--Linear programming method.

Once an optimal solution has been reached, one can obtain information about that particular optimum. IBM 360 system's mathematical programming package provides for determining the effect of varying the constraints, objective function coefficients and matrix coefficients upon this optimal solution.[9] The question answered is how far can a given constraint value move in either direction while holding all other constraint values constant, before the optimal basis changes? This procedure is a one at a time analysis. As a result no test for interaction is included. The same procedure is used for other elements in the model. However, by using parametric programming a change in variables can be defined as input. The function is to retain optimality and feasibility as the problem continues to change. In this manner variables can be changed simultaneously. In addition as in Model 1 and Model 2, one can change the specified variables and recompute the solution. However, an advantage of range and parametric procedures is that the initial optimum is retained.

A transportation problem shown in Table VI was chosen for the purpose of this study. The problem involved the

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[9] International Business Machines Corporation. Mathematical Programming System/360, Reference 360 A-CO-14X. IBM, 1967.

distribution of inventory from five warehouses to twenty-eight customers. Given the constraints on demand and warehouse capacity, the objective was to minimize the total distribution cost. More than a hundred cost variables were included in the model. The procedure of changing variables in order to observe their effect was illustrated in Model 1 and Model 2. One limitation in Model 3 was the number of variables, since it was not feasible to test a change in all cost variables. A possible solution to this limitation was the random balance technique illustrated in Model 2.

For the purpose of this study an illustration of the range feature in mathematical programming systems was chosen. The use was that of determining which distribution costs required more precise data. It was not feasible in the case of Model 3 to collect actual data on all distribution costs. As a result estimates were used as input to obtain the optimum solution. Shown in Table VII are the results of the analysis. For example in Table VI, the estimated cost of distribution from W1 to C1 was .185 per unit. It was found that the maximum level of C1 before the results changed was .890 per unit. In most cases a wide range of cost was allowable. The cost of distributing from W2 to C15 was rather critical, indicating a need for more precise data. One can only conclude that costs such as W2 to C15 are sensitive over the

TABLE VI

TRANSPORTATION PROBLEM
FOR
LINEAR PROGRAMMING
SOLUTION METHOD

	Distribution Costs Per Unit					
	Warehouse					
Customer	W1	W2	W3	W4	W5	
C1	.185	.950	1.090	1.130	1.410	880
C2	1.430	1.480	1.160	.660	.820	270
C3	.720	.410	.810	.750	1.700	1350
C4	.980	.545	1.030	1.800	.695	900
C5	.200	.295	.080	.230	1.060	2490
C6	1.010	.840	.840	.750	.950	200
C7	1.960	1.150	.470	1.700	.470	120
C8	.275	.870	.400	.600	.855	10
C9	1.430	1.270	1.070	1.490	.375	60
C10	1.655	1.270	1.070	1.750	.600	130
C11	1.290	.655	.060	.202	.720	140
C12	1.430	1.270	.890	1.490	1.375	70
C13	.890	1.090	1.290	.125	1.025	30
C14	1.740	1.590	1.550	2.040	1.180	30
C15	1.000	.800	.675	1.360	.980	10
C16	.275	.870	.200	.600	.855	120
C17	.920	1.290	.920	1.040	.420	170
C18	1.290	.655	.220	.760	.720	50
C19	.545	1.030	.880	.620	.370	70
C20	1.550	1.220	.100	.235	.310	450
C21	.760	1.030	.110	.410	.300	330
C22	.545	1.030	.880	.620	.545	30
C23	1.650	1.350	1.160	1.850	.750	20
C24	.275	.870	.200	.600	.855	50
C25	1.910	1.760	1.650	2.220	1.270	210
C26	1.800	1.630	1.560	2.100	1.200	20
C27	1.550	1.240	1.090	1.740	.470	20
C28	.050	.700	.300	.100	1.275	2600
	1100	1800	500	4980	2450	Equality Constraints (Units)

TABLE VII

ANALYSIS OF TRANSPORTATION PROBLEM
USING LINEAR PROGRAMMING
RANGE TECHNIQUE

	Upper Limits of Cost in Optimum Solution				
	W1	W2	W3	W4	W5
C1	.890				
C2				.695	.860
C3		.760			
C4		.600			.725
C5			.088	.250	
C6				.790	
C7					.620
C8	.500				
C9					1.290
C10					1.380
C11				.210	
C12					1.400
C13				.260	
C14					1.670
C15		.830			
C16	.300				
C17					1.130
C18			.410		
C19					.755
C20					.395
C21					.420
C22					.755
C23					1.430
C24	.300				
C25					1.910
C26					1.740
C27					1.220
C28	.070			.116	

range analyzed. In addition an analysis of interaction was excluded.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions.

Sensitivity analysis is used to observe changes in the results of a model caused by changes in the variables of that model. An analysis is useful in identifying variable estimates which require more precise data, and in observing the best and worst possible outcomes of a decision under uncertainty. In addition interaction of variables is observed from using sensitivity analysis. The procedure of performing an analysis may be a case of observing all possible changes, provided the model is simple enough. In any case three rules apply to an analysis: (1) determine which variables to examine; (2) establish the range of values to examine; and (3) determine the measure of sensitivity. The problems of size and interaction are limitations for the case of a complex model. At the present time techniques to resolve these problems are limited. If it is not possible to subjectively choose which variables to examine, experimental design techniques are an alternative. In most cases the measure of sensitivity is in terms of change in the results, with significance of change determined subjectively. However, features of linear programming methods generate sensitivity information as to the allowable change in variables before a change in results occurs. In addition, the range and parametric procedures of linear programming methods facilitate the use of sensitivity analysis. For simulation models comparable algorithms are not available. This study was somewhat general in approach, but then the topic of sensitivity analysis is general in nature. The concepts presented herein hopefully will contribute to a more effective use of sensitivity analysis.

Recommendations.

It is possible to devote entire studies to specific problems that were noted in this work. For example, the problem of size is a major limitation in applications. Statistical techniques appear to be a fruitful area of research related to that problem. Another possible area of study is in development of mathematical algorithms; for example the ranging feature of linear programming, for use in other solution methods. In addition a more extensive examination of the effect of interaction on the validity of an analysis appears justified. Finally, the work of this thesis on the definition, uses and techniques of sensitivity analysis requires additional investigation.

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