ON THE NATURE OF SOLUTIONS OF THE LINEAR HOMOGENEOUS FOURTH ORDER DIFFERENTIAL EQUATION

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

John Basil Scott

December 1970

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ABSTRACT

This work is a study of the classical linear homogeneous differential equation

$$\text{(L)} \qquad y^{(4)} = p(t)y'' + q(t)y' + r(t)y.$$

The following properties of solutions of (L) are considered:

(a) boundedness

(b) asymptotic behavior

(c) behavior for large t values

(d) behavior of solutions possessing multiple zeros

(e) disconjugacy

(f) distribution of zeros.

A sufficient condition for disconjugacy of (L) is given, and conditions are stated which guarantee the existence of three linearly independent uniformly bounded solutions whose first three derivatives tend to zero.

TABLE OF CONTENTS

INTRODUCTION

This dissertation is a study of the linear homogeneous fourth order differential equation

$$\text{(L)} \qquad y^{(4)} = p(t)y'' + q(t)y' + r(t)y$$

where $p(t)$, $q(t)$, and $r(t)$ are assumed to be continuous functions defined on the infinite half axis $[a,\infty)$ for some real constant a. A variety of results for solutions of (L) are established using the techniques of Barrett [1], Lazer [5], Leighton [6], and Peterson [7].

Among the properties considered are distribution of zeros, asymptotic behavior, boundedness, and infinite limits of solutions of (L). Also the topics of oscillation and nonoscillation are discussed in relation to solutions of (L).

We consider a <u>solution</u> of (L) to be <u>nonoscillatory</u> provided its set of zeros is bounded above, and we say equation (L) is <u>nonoscillatory</u> provided all its solutions are nonoscillatory. A solution of (L) is called <u>oscillatory</u> if its set of zeros is not bounded above.

Also we note that most of the previous work on the fourth order linear homogeneous equations has dealt with the self-adjoint type of differential equation. Leighton [6] studied the equation

$$(a(t)y'')'' + c(t)y = 0,$$

and, recently, Peterson [7] studied a more general equation of a similar type. In this paper, however, the equation (L) is not self-adjoint and many standard techniques employed by Leighton, Barrett, and Peterson simply are not applicable. For this reason much of the work of the first two chapters is based on the methods of Lazer [5] used in his study of the non self-adjoint third order equation

$$y''' + p(t)y' + q(t)y = 0.$$

Finally, we define the symbol $y(t) \varepsilon C^K[a,\infty)$ where $K = 0,1,2,\ldots$ It is equivalent to the statement: "$y(t)$ and its first K derivatives are continuous on the interval $[a,\infty)$."

CHAPTER I

PRELIMINARY RESULTS

In this chapter we shall consider results for equation (L) which are independent of the signs of the coefficient functions $p(t)$, $q(t)$, and $r(t)$. Various identities involving solutions of (L) are established, and a relationship between (L) and its adjoint (L^*) is obtained. The lemmas and theorems of this chapter are preliminary to those considered in the following chapters.

We begin by stating four lemmas which are easily proved using methods of calculus. These lemmas are independent of equation (L).

<u>Lemma 1.1</u> Let $y(t) \in C'[a,\infty)$ and assume $\int_a^\infty y(s)^2 ds < \infty$. Also suppose there exists a number $M > 0$ such that $|y'(t)| < M$ for all t on $[a,\infty)$. Then $\lim_{t\to\infty} y(t) = 0$.

<u>Lemma 1.2</u> Let $y(t) \in C^{K+1}[a,\infty)$. If $y(t)$ and its first $K+1$ derivatives are bounded on $[a,\infty)$ and $\lim_{t\to\infty} y(t) = R < \infty$, then

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = \ldots = \lim_{t\to\infty} y^{(K)}(t) = 0.$$

<u>Lemma 1.3</u> Let $y(t) \in C^2[a,\infty)$. Assume there exists $M_0 > 0$, $M_2 > 0$ such that $|y(t)| < M_0$ and $|y''(t)| < M_2$ for all $t \in [a,\infty)$. Then there exists M_1 such that $|y'(t)| < M_1$ for all $t \in [a,\infty)$.

Lemma 1.4 Let $y(t) \in C'[a,\infty)$ and assume $y'(t)$ vanishes for arbitrarily large values of t. If $y'(\bar{t}) = 0$ implies $|y(\bar{t})| < M$ for all $\bar{t}$ where $y'(\bar{t}) = 0$, then $y(t)$ is bounded for all $t \in [a,\infty)$.

We now establish a basic identity which every solution of (L) must satisfy.

Lemma 1.5 Assume $p(t), r(t) \in C'[a,\infty)$ and let $y(t) \not\equiv 0$ be any solution of (L). Define $H[y(t)]$ by

$$(1.1) \qquad H[y(t)] = r(t)y(t)^2 + p(t)y'(t)^2 + y''(t)^2 - 2y'(t)y'''(t).$$

If t_1 is a point such that $a \leq t_1 < \infty$, then

$$(1.2) \qquad H[y(t)] = H[y(t_1)] + \int_{t_1}^{t} [(p'(s) - 2q(s))y'(s)^2 + r'(s)y(s)^2]ds$$

Proof. Differentiating both sides of (1.1) with respect to t yields

$$\frac{dH[y(t)]}{dt} = r'y^2 + 2ryy' + p'(y')^2 + 2py'y'' + 2y''y''' - 2y''y''' - 2y'y^{(4)}.$$

Substituting $y^{(4)} = py'' + qy' + ry$, we see

$$\frac{dH[y(t)]}{dt} = r'y^2 + 2ryy' + p'(y')^2 + 2py'y'' - 2y'[py'' + qy' + ry];$$

therefore,

$$\frac{dH[y(t)]}{dt} = r'y^2 + (p' - 2q)(y')^2.$$

Integrating both sides from t_1 to t yields the desired result.

The following lemma yields an identity which will be useful in studying solutions of (L) having double zeros.

Lemma 1.6 Assume $p(t) \varepsilon C^2[a,\infty)$ and $q(t) \varepsilon C'[a,\infty)$. Let $y(t) \not\equiv 0$ be any solution of (L) and define $K[y(t)]$ by

$$(1.3) \quad K[y(t)] = y(t)y'''(t) - y'(t)y''(t) - p(t)y(t)y'(t) + \tfrac{1}{2}(p'(t) - q(t))y(t)^2.$$

If t_1 is a point such that $a \leq t_1 < \infty$, then

$$(1.4) \quad K[y(t)] = K[y(t_1)] + \int_{t_1}^{t} [\tfrac{1}{2}(p''(s) + 2r(s) - q'(s))y(s)^2 - p(s)y'(s)^2 - y''(s)^2]ds.$$

Proof. Differentiating $K[y(t)]$ in (1.3) yields

$$\frac{dK[y(t)]}{dt} = yy^{(4)} + y'y''' - y'y''' - (y'')^2 - p'yy' - p(yy''+(y')^2) + \tfrac{1}{2}(p'' - q')y^2 + (p' - q)yy'.$$

Substituting $y^{(4)} = py'' + qy' + ry$, we have

$$\frac{dK[y(t)]}{dt} = \tfrac{1}{2}(p'' - q' + 2r)y^2 - (y'')^2 - p(y')^2.$$

The result follows by integrating both sides from t_1 to t.

We now establish an identity which will be valuable in relating the behavior of solutions of (L) to the behavior of solutions of its

its adjoint (L^*).

Lemma 1.7 Given the two fourth order equations:

$$(L_1) \quad y^{(4)} = p_1(t)y'' + q_1(t)y' + r_1(t)y,$$

$$(L_2) \quad y^{(4)} = p_2(t)y'' + q_2(t)y' + r_2(t)y.$$

If $u(t)$ and $v(t)$ are solutions of (L_1) and (L_2) respectively and if $F(t)$ and $M(t)$ are defined

$$(1.5) \quad F(t) = v(t)u'''(t) - u(t)v'''(t) - v'(t)u''(t) + u'(t)v''(t),$$

$$(1.6) \quad M(t) = F(t) - p_1u'v + p_2v'u + (q_2 - p_2')uv,$$

then

$$(1.7) \quad M'(t) = (p_2 - p_1)u'v' + (q_1 + q_2 - p_1' - p_2')u'v + (r_1 - r_2 + q_2' - p_2'')uv.$$

Proof. Differentiating (1.6) yields:

$$M'(t) = F'(t) - p_1(u'v' + u''v) + p_2(v''u + v'u') - p_1'u'v + p_2'v'u + (q_2 - p_2')u'v + (q_2 - p_2')uv' + (q_2' - p_2'')uv.$$

But $F'(t) = vu^{(4)} - uv^{(4)}$ and

$$vu^{(4)} - uv^{(4)} = p_1u''v - p_2v''u + q_1u'v - q_2v'u + (r_1 - r_2)uv.$$

Substituting this latter quantity for $F'(t)$ yields:

$$M'(t) = p_1u''v - p_2v''u + q_1u'v - q_2v'u + (r_1 - r_2)uv - p_1u'v'$$
$$- p_1u''v + p_2v''u + p_2v'u' - p_1'u'v + p_2'v'u + (q_2-p_2')u'v$$
$$+ q_2uv' - p_2'uv' + (q_2' - p_2'')uv.$$ Canceling, we have

$$M'(t) = (p_2 - p_1)u'v' + (q_1 + q_2 - p_2' - p_1')u'v$$
$$+ (r_1 - r_2 + q_2' - p_2'')uv.$$

The identity follows.

If we define the linear differential operator $L(y)$ by

$$L(y) = y^{(4)} - p(t)y'' - q(t)y' - r(t)y, \tag{1.8}$$

then our original differential equation may be written $L(y) = 0$. Associated with the operator $L(y)$ is an adjoint operator $L^*(z)$ which may be written

$$L^*(z) = z^{(4)} - p(t)z'' + (q(t) - 2p'(t))z' + (q'(t) - r(t) - p''(t))z. \tag{1.9}$$

Therefore, associated with our original differential equation $L(y) = 0$ is an adjoint differential equation

$$L^*(z) = 0. \tag{1.10}$$

It is a simple procedure to prove that the operators $L(y)$ and $L^*(z)$ satisfy Lagrange's identity

$$zL(y) - yL^*(z) = [y;z]' \tag{1.11}$$

where $[y;z]'$ indicates the derivative of a bilinear form in y and z.

We now state a theorem relating solutions of $L(y) = 0$ and $L^*(z) = 0$, the original equation and its adjoint equation.

Theorem 1.8 Let $u(t)$, $v(t)$ and $w(t)$ be three linearly independent solutions of (L). Define the function $z(t)$ by the determinant

$$z(t) = \begin{vmatrix} u(t) & v(t) & w(t) \\ u'(t) & v'(t) & w'(t) \\ u''(t) & v''(t) & w''(t) \end{vmatrix} . \tag{1.12}$$

Then $z(t)$ is a solution of $L^*(z) = 0$.

Proof. Using the assumption that $u(t)$, $v(t)$, and $w(t)$ are solutions of (L) (e.g. $u^{(4)}(t) = p(t)u'' + q(t)u' + r(t)u$) the following matrix identities are easily established:

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix} = p(t) \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = p(t)\cdot z(t)$$

$$\begin{vmatrix} u & v & w \\ u'' & v'' & w'' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix} = q(t) \begin{vmatrix} u & v & w \\ u'' & v'' & w'' \\ u' & v' & w' \end{vmatrix} = -q(t)\cdot z(t)$$

$$\begin{vmatrix} u' & v' & w' \\ u'' & v'' & w'' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix} = r(t) \begin{vmatrix} u' & v' & w' \\ u'' & v'' & w'' \\ u & v & w \end{vmatrix} = r(t)\cdot z(t)$$

Now we merely differentiate z(t) using the normal rules for determinant differentiation and we find:

$$z'(t) = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u''' & v''' & w''' \end{vmatrix}$$

$$z''(t) = \begin{vmatrix} u & v & w \\ u'' & v'' & w'' \\ u''' & v''' & w''' \end{vmatrix} + \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix}$$

$$z''(t) = \begin{vmatrix} u & v & w \\ u'' & v'' & w'' \\ u''' & v''' & w''' \end{vmatrix} + p(t)\cdot z(t)$$

$$z'''(t) = \begin{vmatrix} u' & v' & w' \\ u'' & v'' & w'' \\ u''' & v''' & w''' \end{vmatrix} + \begin{vmatrix} u & v & w \\ u'' & v'' & w'' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix} + (pz)'$$

$$z'''(t) = \begin{vmatrix} u' & v' & w' \\ u'' & v'' & w'' \\ u''' & v''' & w''' \end{vmatrix} - q(t)\cdot z(t) + p(t)\cdot z'(t) + p'(t)\cdot z(t).$$

Finally, calculating $z^{(4)}(t)$ yields

$$z^{(4)}(t) = \begin{vmatrix} u' & v' & w' \\ u'' & v'' & w'' \\ u^{(4)} & v^{(4)} & w^{(4)} \end{vmatrix} + [pz' + p'z - qz]'.$$

Thus $z^{(4)}(t) = rz + [pz' + p'z - qz]'$ and it follows that

$$z^{(4)}(t) = pz'' + (2p' - q)z' + (r + p'' - q')z.$$

Therefore $z(t)$ satisfies (1.10).

Corollary 1.9 Let $u(t)$, $v(t)$ and $w(t)$ be three linearly independent solutions of $L^*(z) = 0$. Define the function $z(t)$ by the determinant (1.12). Then $z(t)$ is a solution of (L).

The proof of Corollary 1.9 is similar to that of Theorem 1.8 and is therefore omitted. We note that the assumption of linear independence on the three solutions is a necessary but not sufficient condition that $z(t)$ be not identically zero.

We complete this chapter by stating a lemma which holds for certain real valued functions independent of our differential equation (L).

Lemma 1.10 Let $u(t), v(t) \in C'(a,b)$ and assume $v(t)$ is of constant sign in this interval. If $u(t)$ has two distinct zeros in (a,b), then the function $D(t) = v(t)u'(t) - u(t)v'(t)$ cannot be of constant sign in the interval bounded by these zeros.

Proof. Suppose $t = \alpha$ and $t = \beta$ are consecutive zeros of $u(t)$ where $a < \alpha < \beta < b$. Assume, without loss of generality, that $D(t) > 0$ in (α,β). Then we see

$$0 < \int_{\alpha}^{\beta} \frac{D(s)}{v(s)^2}\, ds = \frac{u(\beta)}{v(\beta)} - \frac{u(\alpha)}{v(\alpha)} = 0$$

and this contradiction proves the lemma.

For most theoretical uses it is more convenient to state the above lemma in the following equivalent form.

<u>Lemma 1.11</u> Let $u(t)$, $v(t) \varepsilon C'(a,b)$ and assume $v(t)$ is of constant sign in this interval. If $t = \alpha$ and $t = \beta$, $a < \alpha < \beta < b$, are consecutive zeros of $u(t)$, then there exists a constant K such that the function $u(t) - Kv(t)$ has a double zero in (α,β).

By a double zero we mean, as usual, that both the function and its derivative vanish at the point in question. The equivalence of Lemma 1.10 and Lemma 1.11 follows from the remark that $D(t_0) = 0$ is equivalent to the existence of two constants c_1 and c_2, $c_1^2 + c_2^2 > 0$, such that $c_1u(t_0) - c_2v(t_0) = 0$ and $c_1u'(t_0) - c_2v'(t_0) = 0$. Since $v(t) \neq 0$ in $[\alpha,\beta]$, c_1 cannot be zero and may therefore be taken as unity.

CHAPTER II

CONSTRUCTION OF THREE LINEARLY INDEPENDENT BOUNDED SOLUTIONS WHOSE FIRST THREE DERIVATIVES TEND TO ZERO

We shall now be concerned with the asymptotic behavior of solutions of equation (L). Throughout this chapter certain sign assumptions will be required on the coefficient functions $p(t)$, $q(t)$, and $r(t)$. In most of the following lemmas and theorems we shall assume $r(t) \geq 0$ and, occasionally, $p(t) \geq 0$ for all t on the infinite half axis $[a,\infty)$. Furthermore we require that the functions $p(t)$, $q(t)$, and $r(t)$ are continuous on $[a,\infty)$, and certain results will require that various combinations of their derivatives be continuous.

Recall that the set of solutions of a homogeneous linear differential equation of order four forms a finite dimensional vector space of dimension four. One of the major results of this chapter will be the establishment of conditions on $p(t)$, $q(t)$, and $r(t)$ which imply that the subspace of bounded solutions is of dimension three.

Finally it should be noted that motivation for these results is given by the equation

$$y^{(4)} = py'' + qy' + ry \tag{2.1}$$

where p, q, and r are real constants. This constant coefficient equation is completely solvable by methods of elementary differential equations.

We begin by proving a fundamental lemma relating to the "boundedness" of solutions of (L) over certain intervals. In this lemma, as in many others that follow, the need exists for the following conditions:

$$
\begin{aligned}
&p(t), r(t) \in C'[a,\infty) \\
&r'(t) \leq 0,\ p'(t) - 2q(t) \leq 0
\end{aligned}
\tag{2.2}
$$

for all $t \in [a,\infty)$ and not both of the latter inequalities are to be identically zero on any subinterval of $[a,\infty)$.

<u>Lemma 2.1</u>

(a) Let conditions (2.2) hold and suppose $r(t) \geq M > 0$ for all t on $[a,\infty)$. If t_1 and t_2 are points such that $a \leq t_1 < t_2$ and $y(t)$ is a solution of (L) satisfying $y'(t_1) = y'(t_2) = 0$, then

$$\max_{t_1 \leq t \leq t_2} [y(t)]^2 \leq \frac{H[y(t_1)]}{M} .$$

(b) Let conditions (2.2) hold and suppose $r(t) \geq 0$ and $p(t) \geq 0$ for all t on $[a,\infty)$. If t_1 and t_2 satisfy $a \leq t_1 < t_2$ and $y'''(t_1) = y'''(t_2) = 0$ where $y(t)$ is a solution of (L), then

$$\max_{t_1 \leq t \leq t_2} [y''(t)]^2 \leq H[y(t_1)] .$$

In both (a) and (b) $H[y(t)]$ is given by equations (1.1) and (1.2).

Proof.

(a) Let $\overline{X}$ be such that $t_1 \leq \overline{X} \leq t_2$ and assume $\overline{X}$ is the point where the maximum value of $y(t)$ occurs in $[t_1, t_2]$. Since $y'(t_1) = y'(t_2) = 0$ we have $y'(\overline{X}) = 0$. Calculating $H[y(\overline{X})]$ from equations (1.1) and (1.2) yields:

$$H[y(\overline{X})] = r(\overline{X})y(\overline{X})^2 + y''(\overline{X})^2$$

$$= H[y(t_1)] + \int_{t_1}^{\overline{X}} [(p' - 2q)\cdot(y')^2 + r'y^2]ds.$$

Thus conditions (2.2) imply $r(\overline{X})y(\overline{X})^2 + y''(\overline{X})^2 \leq H[y(t_1)]$ and, therefore, $r(\overline{X})y(\overline{X})^2 \leq H[y(t_1)]$. Also $r(\overline{X}) \geq M > 0$ implies $\frac{1}{M} \geq \frac{1}{r(\overline{X})} > 0$.

Finally, $y(\overline{X})^2 \leq \dfrac{H[y(t_1)]}{r(\overline{X})} \leq \dfrac{H[y(t_1)]}{M}$ and (a) is proven.

To prove (b), assume $t_1 \leq \overline{X} \leq t_2$ where $\overline{X}$ is the point where the maximum of $y''(t)$ occurs. Hence $y'''(t_1) = y'''(t_2) = 0$ implies $y'''(\overline{X}) = 0$. Calculating $H[y(\overline{X})]$ yields

$$r(\overline{X})y(\overline{X})^2 + p(\overline{X})y'(\overline{X})^2 + y''(\overline{X})^2 = H[y(t_1)] + \int_{t_1}^{\overline{X}} (p' - 2q)\cdot(y')^2 + r'y^2 ds.$$

Hence $r(t) \geq 0$, $p(t) \geq 0$ and conditions (2.2) imply $y''(\overline{X})^2 \leq H[y(t_1)]$ and (b) is proven.

We remark here that a slight modification of the above proof yields the following lemma which is a variation of Lemma 2.1.

Lemma 2.2 Let conditions (2.2) hold and suppose $r(t) \geq M > 0$ on $[a,\infty)$. If t_1 and t_2 are points such that $a \leq t_1 < t_2$ and $y(t)$ is a solution of (L) satisfying $y'(t_2) = 0$, then

$$\max_{t_1 \leq t \leq t_2} [y(t)]^2 \leq \max \left\langle y(t_1)^2, \frac{H[y(t_1)]}{M} \right\rangle . \tag{2.3}$$

An immediate consequence of Lemma 2.1 is now considered.

Lemma 2.3

(a) Let conditions (2.2) hold and suppose $r(t) \geq M > 0$ on $[a,\infty)$. If $y(t)$ is a solution of (L) such that the set of zeros of $y'(t)$ is not bounded above, then $y(t)$ is bounded on $[a,\infty)$.

(b) Let conditions (2.2) hold and suppose $p(t) \geq 0$, $r(t) \geq 0$ on $[a,\infty)$. If $y(t)$ is a solution of (L) such that the set of zeros of $y'''(t)$ is not bounded above, then $y''(t)$ is bounded on $[a,\infty)$.

Proof. We prove only part (a) and (b) follows similarly. Let t_1 be the first zero of $y'(t)$ to the right of a. Since $y(t) \in C[a,t_1]$ there exists $K_1 > 0$ such that $|y(t)| \leq K_1$ for all $t \in [a,t_1]$. Let $\bar{t} > t_1$ be arbitrary. Since $y'(t)$ vanishes for arbitrarily large t values, there exists a point t_2 such that $t_1 < \bar{t} < t_2$ and $y'(t_1) = y'(t_2) = 0$.

But Lemma 2.1 (a) implies that there exists a constant $K_2 > 0$, depending only on t_1 and M, such that $|y(\bar{t})| \leq K_2$. Since $\bar{t} > t_1$ was arbitrary, we see $|y(t)| \leq K_2$ for all t on (t_1,∞). Defining $K = \max\{K_1,K_2\}$ we conclude $|y(t)| \leq K$ for all t on $[a,\infty)$.

We now record a result which follows directly from Lemma 1.5 and conditions (2.2). Its proof is obvious from equation (1.2).

Lemma 2.4 Suppose conditions (2.2) hold and $y(t)$ is a solution of (L) which is not identically zero. Then $H[y(t)]$, as defined by equation (1.2), is a strictly decreasing function on $[a,\infty)$.

We continue by considering the behavior of solutions of (L) possessing a triple zero at some point of $[a,\infty)$. By triple zero we mean that $y(t)$, $y'(t)$, and $y''(t)$ all vanish at the point in question. Recall from the uniqueness theorem that if a solution of (L) has a quadruple zero at $t = c$, i.e. $y^{(i)}(c) = 0$, $i = 0,1,2,3$, then $y(t) \equiv 0$ on $[a,\infty)$. Hence if we assume $y(t) \not\equiv 0$ and $y(t)$ has a triple zero at $t = c$, then $y'''(c) \neq 0$.

Lemma 2.5 Suppose conditions (2.2) hold and that $r(t) \geq 0$ on $[a,\infty)$. Let $y(t) \not\equiv 0$ be a solution of (L) with a triple zero at some point $c \in [a,\infty)$. Then

$$
\begin{array}{rll}
\text{(i)} & H[y(t)] < 0 & \text{on } (c,\infty), \\
\text{(ii)} & y'(t) \neq 0 & \text{on } (c,\infty), \\
\text{(iii)} & y(t)\cdot y'(t) > 0 & \text{on } (c,\infty).
\end{array}
$$

Proof. Since $y(c) = y'(c) = y''(c) = 0$, we have from equation (1.1) $H[y(c)] = 0$. By Lemma 2.4, $H[y(t)]$ is strictly decreasing and hence $H[y(t)] < 0$ for $t > c$.

To prove (ii), suppose $y'(\bar{c}) = 0$ for some $\bar{c} > c$. Then, equation (1.1) implies $H[y(\bar{c})] = r(\bar{c})y(\bar{c})^2 + y''(\bar{c})^2 \geq 0$ and this contradicts (i).

The proof of (iii) follows from (ii) and the fact that $y(c) = y'(c) = 0$.

We are now ready to proceed with one of the most important results of this chapter. Conditions are given which are sufficient to guarantee that the subspace of bounded solutions of (L) is of dimension three.

<u>Theorem 2.6</u> Suppose conditions (2.2) hold and that $r(t) \geq M > 0$ on $[a,\infty)$. Then there exist three linearly independent uniformly bounded solutions of (L).

<u>Proof</u>. Let $z_i^{(j)}(a) = \delta_{ij}$, $i,j = 0,1,2,3$, be a canonical basis of solutions of (L) at the point $t = a$, where δ_{ij} is Kronecker's δ function. For each integer $n > a$ there exist numbers b_{on}, b_{3n}, c_{1n}, c_{3n}, d_{2n}, and d_{3n} such that

$$b_{on}^2 + b_{3n}^2 = c_{1n}^2 + c_{3n}^2 = d_{2n}^2 + d_{3n}^2 = 1 \tag{2.4}$$

and

$$\begin{aligned} b_{on}z_o'(n) + b_{3n}z_3'(n) &= 0 \\ c_{1n}z_1'(n) + c_{3n}z_3'(n) &= 0 \\ d_{2n}z_2'(n) + d_{3n}z_3'(n) &= 0 \;. \end{aligned} \tag{2.5}$$

Let $u_n(t)$, $v_n(t)$ and $w_n(t)$ be solutions of (L) defined by

$$
\begin{aligned}
u_n(t) &= b_{on} z_o(t) + b_{3n} z_3(t) \\
v_n(t) &= c_{1n} z_1(t) + c_{3n} z_3(t) \qquad (2.6) \\
w_n(t) &= d_{2n} z_2(t) + d_{3n} z_3(t) .
\end{aligned}
$$

Evaluating equations (2.6) at $t = a$, yields

$$u_n'(a) = u_n''(a) = v_n(a) = v_n''(a) = w_n(a) = w_n'(a) = 0$$

$$
\begin{array}{lll}
u_n(a) = b_{on} & v_n'(a) = c_{1n} & w_n''(a) = d_{2n} \\
u_n'''(a) = b_{3n} & v_n'''(a) = c_{3n} & w_n'''(a) = d_{3n} .
\end{array}
$$

Using the identity (1.1) and the above values, we see

$$
\begin{aligned}
H[u_n(a)] &= r(a) b_{on}^2, \\
H[v_n(a)] &= p(a) c_{1n} - 2c_{1n} c_{3n}, \\
H[w_n(a)] &= d_{2n}^2 .
\end{aligned}
$$

Also, equations (2.5) and (2.6) imply $u_n'(n) = v_n'(n) = w_n'(n) = 0$. Hence using the inequality (2.3) of Lemma 2.2, we find that for $t \in [a,n]$

$$
\begin{aligned}
u_n(t)^2 &\le \max \left\langle b_{on}^2, \frac{r(a)}{M} b_{on}^2 \right\rangle \\
v_n(t)^2 &\le \max \left\langle 0, \frac{p(a) c_{1n}^2 - 2c_{1n} c_{3n}}{M} \right\rangle \\
w_n(t)^2 &\le \max \left\langle 0, d_{2n}^2/M \right\rangle .
\end{aligned}
$$

Equations (2.4) imply there exists a number $A > 0$, independent of n, such that

$$(2.7)\qquad \left.\begin{aligned} u_n(t)^2 &\le A \\ v_n(t)^2 &\le A \\ w_n(t)^2 &\le A \end{aligned}\right\rangle \text{ on } [a,n].$$

Again, using equations (2.4) and the compactness of the unit circle, there exists a sequence of integers $\{n_j\}$ such that

$$\lim_{n_j\to\infty} b_{on_j} = b_o \qquad \lim_{n_j\to\infty} c_{1n_j} = c_1 \qquad \lim_{n_j\to\infty} d_{2n_j} = d_2$$

$$\lim_{n_j\to\infty} b_{3n_j} = b_3 \qquad \lim_{n_j\to\infty} c_{3n_j} = c_3 \qquad \lim_{n_j\to\infty} d_{3n_j} = d_3$$

where b_o, b_3, c_1, c_3, d_2, and d_3 are numbers satisfying

$$(2.8)\qquad b_o^2 + b_3^2 = c_1^2 + c_3^2 = d_2^2 + d_3^2 = 1.$$

We now define three solutions of (L), $u(t)$, $v(t)$, and $w(t)$, by

$$(2.9)\qquad \begin{aligned} u(t) &= b_o z_o(t) + b_3 z_3(t) \\ v(t) &= c_1 z_1(t) + c_3 z_3(t) \\ w(t) &= d_2 z_2(t) + d_3 z_3(t). \end{aligned}$$

By the "unit circle" relations (2.8), we have $u(t) \not\equiv 0$, $v(t) \not\equiv 0$, $w(t) \not\equiv 0$. Also, the sequences $\{u_{n_j}(t)\}$, $\{v_{n_j}(t)\}$, and $\{w_{n_j}(t)\}$ converge pointwise on $[a,\infty)$ to $u(t)$, $v(t)$, and $w(t)$ respectively.

Now let $\bar{t}$ be arbitrary in $[a,\infty)$. Then by (2.7), we see

$$\left.\begin{array}{l} u_{n_j}(\bar{t})^2 \leq A \\ v_{n_j}(\bar{t})^2 \leq A \\ w_{n_j}(\bar{t})^2 \leq A \end{array}\right\rangle \text{ whenever } n_j > \bar{t} \ .$$

Thus $u(t)^2$, $v(t)^2$, and $w(t)^2$ are each less than or equal to A for all $t \in [a,\infty)$. The boundedness is proven.

Before proving the linear independence of the three solutions $u(t)$, $v(t)$ and $w(t)$, we prove the following useful lemma:

<u>Lemma 2.7</u> Suppose conditions (2.2) hold and that $r(t) \geq M > 0$ on $[a,\infty)$. Let $y(t)$ represent any one of the solutions $u(t)$, $v(t)$ or $w(t)$ constructed above. Then $H[y(t)] > 0$ for all $t \in [a,\infty)$.

<u>Proof</u>. Suppose, for example $y(t) = u(t)$. Recall $u(t) = \lim_{n_j \to \infty} u_{n_j}(t)$ where $u_n(t) = b_{on}z_o(t) + b_{3n}z_3(t)$. Also recall $u_n'(n) = 0$. Hence we see that $H[u_n(n)] = r(n)u_n(n)^2 + u_n''(n)^2 \geq 0$. But, by Lemma 2.4, $H[u_n(t)]$ is strictly decreasing. Thus $H[u_n(t)] > 0$ for $t \in [a,n)$. Let $\bar{t}$ be an arbitrary point in $[a,\infty)$. Then $H[u_{n_j}(\bar{t})] > 0$ for all $n_j > \bar{t}$. We conclude, therefore, $H[u(\bar{t})] = \lim_{n_j \to \infty} H[u_{n_j}(\bar{t})] \geq 0$. Since $H[u(t)]$ is decreasing it follows that $H[u(t)] > 0$ on $[a,\infty)$.

We now proceed with proving the linear independence of the three solutions: $u(t)$, $v(t)$ and $w(t)$. Suppose $u(t)$, $v(t)$, and $w(t)$ are linearly dependent. Then there exist constants k_1, k_2, and k_3, not all zero, such that

$$k_1u(t) + k_2v(t) + k_3w(t) \equiv 0. \tag{2.10}$$

In particular, at $t = a$ we have $k_1u(a) + k_2v(a) + k_3w(a) = 0$. But equations (2.9) imply $v(a) = w(a) = 0$. Thus, we have $k_1 \cdot b_o = 0$, and two cases are possible:

(i) Suppose $b_o = 0$. This implies $b_3 \neq 0$ and $u(t) = b_3z_3(t)$.

(ii) Suppose $k_1 = 0$. This implies $k_2^2 + k_3^2 \neq 0$. Thus, from (2.10), $k_2v'(t) + k_3w'(t) \equiv 0$ and at $t = a$ we find $k_2 \cdot c_1 = 0$. If $c_1 = 0$, then $c_3 \neq 0$ and $v(t) = c_3z_3(t)$. If $k_2 = 0$, then $k_3 \neq 0$ and $k_3w(t) \equiv 0$. This contradicts $w(t) \not\equiv 0$. Hence in all cases, we find either

$$\text{(2.11)} \qquad \begin{array}{ll} & \text{(i)} \quad u(t) = b_3z_3(t), \quad b_3 \neq 0, \\ \text{or} & \text{(ii)} \quad v(t) = c_3z_3(t), \quad c_3 \neq 0. \end{array}$$

Now, from the identity (1.1), we see $H[b_3z_3(t)] = b_3^2H[z_3(t)]$ and $H[c_3z_3(t)] = c_3^2H[z_3(t)]$. Moreover, $z_3(a) = z_3'(a) = z_3''(a) = 0$, and $z_3'''(a) = 1$. Hence by Lemma 2.5, $H[z_3(t)] < 0$ on (a,∞). But $H[u(t)] = H[b_3z_3(t)] = b_3^2H[z_3(t)] < 0$ for all $t > a$. Similarly $H[v(t)] < 0$ for all $t > a$. In either case, this is a contradiction to Lemma 2.7. Hence the three solutions are independent.

<u>Lemma 2.8</u> Suppose $r(t) \geq M > 0$, $p'(t) - 2\,q(t) \leq d < 0$ and $r'(t) \leq 0$ on $[a,\infty)$. Let $y(t)$ denote any one of the three solutions of Theorem 2.6. Then $\int_a^\infty y'(s)^2 ds < \infty$.

Proof. Let $t > a$ be arbitrary. By Lemma 2.7, $H[y(t)] > 0$ for all $t \geq a$. Hence the identity (1.2) implies

$$-\int_a^t [(p' - 2q)(y')^2 + r'y^2]ds < H[y(a)].$$

But $r'(t) \leq 0$ implies

$$-\int_a^t (p'(s) - 2q(s))y'(s)^2 ds < H[y(a)].$$

Also, $p' - 2q \leq d < 0$ implies $0 < -d \leq (p' - 2q)(-1)$. Thus we see

$$-d\int_a^t y'(s)^2 ds \leq -\int_a^t (p' - 2q)(y')^2 ds < H[y(a)].$$

Dividing both sides of the latter inequality by $(-d)$, yields

$$\int_a^t y'(s)^2 ds < \frac{H[y(a)]}{(-d)} \ .$$

We now consider the final theorem of this chapter. One should note that its hypothesis includes no explicit assumption as to the sign of $q(t)$ on $[a,\infty)$. Scrutiny of the explicit assumptions shows, however, that they are vacuous unless $q(t) \geq 0$ on $[a,\infty)$.

The motivation for this theorem is evident if one considers the constant coefficient equation (2.1) where p, q, and r are positive constants.

Theorem 2.9 Assume $p(t), r(t) \in C'[a,\infty)$. Let $p(t) \geq 0$, $r(t) \geq M > 0$, $r'(t) \leq 0$, and $p'(t) - 2q(t) \leq d < 0$ on $[a,\infty)$. Also assume $p(t)$ and $q(t)$ are bounded on $[a,\infty)$. Then if $y(t)$ represents any one of the three linearly independent uniformly bounded solutions u(t), v(t) or w(t) we have $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'''(t) = 0$. Moreover, if $\lim_{t\to\infty} y(t)$ exists, then $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y^{(4)}(t) = 0$.

Proof. Let y(t) represent any one of the three solutions u(t), v(t), or w(t). We consider two cases:

(i) the set of zeros of $y'''(t)$ is not bounded above,

(ii) the set of zeros of $y'''(t)$ is bounded above.

In case (i), Lemma 2.3 part (b) implies $y''(t)$ is bounded on $[a,\infty)$. Moreover Lemma 2.8 implies $\int_a^\infty y'(s)^2 ds < \infty$. Hence by Lemma 1.1 we have $\lim_{t\to\infty} y'(t) = 0$. Moreover, since $y(t)$, $y'(t)$, $y''(t)$, $p(t)$, $q(t)$, and $r(t)$ are bounded on $[a,\infty)$, we have $y^{(4)} = py'' + qy' + ry$ is also bounded on $[a,\infty)$. Therefore Lemma 1.3 implies $y'''(t)$ is bounded on $[a,\infty)$. Summarizing, we have found $\lim_{t\to\infty} y'(t) = 0$ and $y''(t)$, $y'''(t)$, and $y^{(4)}(t)$ are bounded. Thus Lemma 1.2 implies $\lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'''(t) = 0$. Now, using equation (L), we conclude

$$\lim_{t\to\infty} y^{(4)}(t) = \lim_{t\to\infty} r(t)\cdot\lim_{t\to\infty} y(t) . \tag{2.12}$$

But, by hypothesis $\lim_{t\to\infty} r(t) = K > 0$. We claim $\lim_{t\to\infty} y(t) = 0$. If not, y(t) bounded implies $\lim_{t\to\infty} y(t) = C$ where either $C > 0$ or $C < 0$. Hence (2.12) implies $\lim_{t\to\infty} y^{(4)}(t) = K\cdot C$ where either $K\cdot C > 0$ or $K\cdot C < 0$. If

$K \cdot C > 0$, there exists a point t_1 such that $y^{(4)}(t) > 0$ for all $t > t_1$. If $K \cdot C < 0$ there exists a point t_2 such that $y^{(4)}(t) < 0$ for all $t > t_2$. In either case we contradict the assumption that $y'''(t)$ has arbitrarily large zeros. Hence $\lim_{t \to \infty} y^{(4)}(t) = \lim_{t \to \infty} y(t) = 0$.

We proceed with case (ii) and suppose the set of zeros of $y'''(t)$ is bounded above. Hence there exists a point $\bar{t} > a$ such that $y'''(t) \neq 0$ for all $t > \bar{t}$. Without loss of generality assume $y'''(t) < 0$ for all $t > \bar{t}$. This implies $y''(t)$ is eventually of one sign. We claim $y''(t)$ is eventually positive. If not, we have $y''(t) \leq 0$, $y'''(t) < 0$ for large t and $\lim_{t \to \infty} y'(t) = -\infty$ which contradicts $\int_a^t y'(s)^2 ds < \infty$. Hence there exists a point $\bar{t}_1 \geq \bar{t}$ such that $y''(t) > 0$, $y'''(t) < 0$ for all $t > \bar{t}_1 \geq \bar{t}$.

Now, we see that $y'(t)$ is eventually of one sign. We claim $y'(t)$ is eventually negative. If not, we would have $y'(t) \geq 0$, $y''(t) > 0$ for large t and $\lim_{t \to \infty} y(t) = +\infty$ which contradicts $y(t)$ is bounded. In summary, we have shown that there exists a point $\bar{t}_2 \geq \bar{t}_1 \geq \bar{t}$ such that

$$\left.\begin{aligned} y'(t) &< 0 \\ y''(t) &> 0 \\ y'''(t) &< 0 \end{aligned}\right\rangle \text{ for all } t > \bar{t}_2. \tag{2.13}$$

The inequalities (2.13) imply $\lim_{t \to \infty} y''(t) = 0$ and $y''(t)$ is bounded. The boundedness of $y''(t)$, $\int_a^\infty y'(s)^2 ds < \infty$, and Lemma 1.1

imply $\lim_{t\to\infty} y'(t) = 0$. As in the proof of (i), we have $y^{(4)}(t)$ is bounded, $\lim_{t\to\infty} y'''(t) = 0$, and $\lim_{t\to\infty} y^{(4)}(t) = \lim_{t\to\infty} r(t)\cdot\lim_{t\to\infty} y(t)$.

Again we have $\lim_{t\to\infty} r(t) = K > 0$ and we assume $\lim_{t\to\infty} y(t) = C$ where either $C > 0$ or $C < 0$. If $C > 0$, there exists on integer N_1 such that for all $t > N_1$, $y^{(4)}(t) > \frac{K\cdot C}{2} > 0$. This implies $y'''(t)$ is eventually positive contradicting (2.13). If $C < 0$, there exists an integer N_2 such that for all $t > N_2$, $y^{(4)}(t) < \frac{K\cdot C}{2} < 0$. This implies $y'''(t)$ tends to $-\infty$, contradicting that the $\lim_{t\to\infty} y'''(t) = 0$. Our proof is now complete.

In Theorem 2.9 if we relax the assumption that $p(t)$ and $q(t)$ are bounded, we obtain a slightly weaker result. We record this result in the following corollary.

Corollary 2.10 Assume $p(t), r(t) \in C'[a,\infty)$. Let $p(t) \geq 0$, $r(t) \geq M > 0$, $r'(t) \leq 0$, and $p'(t) - 2q(t) \leq d < 0$ on $[a,\infty)$. If $y(t)$ denotes any one of the three linearly independent bounded solutions $u(t)$, $v(t)$ or $w(t)$, and if $\lim_{t\to\infty} y(t)$ exists, then

(i) $\lim_{t\to\infty} |y(t)| = C < \infty$

(ii) $\lim_{t\to\infty} y'(t) = 0$

(iii) $y''(t)$ is bounded on $[a,\infty)$.

CHAPTER III

STUDY OF ASYMPTOTIC BEHAVIOR OF SOLUTIONS POSSESSING MULTIPLE ZEROS, AND THE CONSTRUCTION OF A NONOSCILLATORY BASIS OF SOLUTIONS FOR (L)

In this chapter we shall continue our study of the asymptotic behavior of solutions of equation (L). Most of the results of this section will be dependent on the sign of the coefficient function $q(t)$, unlike the lemmas and theorems of the majority of the preceding chapter. More specifically we shall be concerned with the behavior of solutions possessing double or triple zeros at some point of the infinite half axis $[a,\infty)$. The asymptotic nature on both sides of such points will be considered, and infinite limits of such solutions will be discussed.

At this point we recall conditions (2.2) of Chapter II:

$$\begin{aligned} &p(t), r(t) \in C'[a,\infty) \\ &r'(t) \leq 0, \ p'(t) - 2q(t) \leq 0 \end{aligned} \tag{2.2}$$

for all $t \in [a,\infty)$ and not both of the inequalities are to vanish identically on any subinterval of $[a,\infty)$.

These conditions will again be extensively used, and, in addition, the following similar conditions will be important.

$$\begin{aligned} &p(t), r(t) \in C'[a,\infty) \\ &r'(t) \geq 0, \ p'(t) - 2q(t) \geq 0 \end{aligned} \tag{3.1}$$

for all $t \in [a,\infty)$ and not both of the inequalities are to vanish identically on any subinterval of $[a,\infty)$.

We begin our discussion with two well known lemmas.

Lemma 3.1 Assume $p(t) > 0$, $q(t) > 0$, and $r(t) > 0$ for all $t \in [a,\infty)$. Let $y(t)$ be a solution of (L) such that for some point $c \in [a,\infty)$ $y^{(i)}(c) \geq 0$ for $i = 0,1,2,3$ with strict inequality holding for at least one value of i. Then $y^{(i)}(t) > 0$ for all $t > c$ and for $i = 0,1,2,3$; moreover $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = +\infty$. If $y^{(i)}(c) \leq 0$ for $i = 0,1,2,3$ with strict inequality holding for at least one value of i, then $y^{(i)}(t) < 0$ for all $t > c$ and for $i = 0,1,2,3$; moreover, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = -\infty$.

Proof. Assume $y^{(i)}(c) \geq 0$ for all i and let $y'''(c) > 0$. By continuity there exists $\delta > 0$ such that

$$(3.2) \qquad y(t) > 0,\ y'(t) > 0,\ y''(t) > 0, \text{ and } y'''(t) > 0$$

on the interval $(c,c+\delta)$.

On the other hand suppose strict inequality holds on $y^{(i)}(c)$ for either $i = 0$ or 1 or 2. Then $y^{(4)}(c) = p(c)y''(c) + q(c)y'(c) + r(c)y(c)$ and the sign assumptions on p, q, and r imply $y^{(4)}(c) > 0$ and again there exists a $\delta > 0$ such that the inequalities (3.2) hold on $(c,c+\delta)$.

We claim inequalities (3.2) hold for all $t > c$, for, if not, then there exists a first point $\bar{c} > c$ where they fail (i.e. $y(\bar{c})\cdot y'(\bar{c})\cdot y''(\bar{c})\cdot y'''(\bar{c}) = 0$). Consider the following identity

$$(3.3)\qquad (yy'y''y''')' = y''y'''(y')^2 + yy'''(y'')^2 + yy'(y''')^2 + yy'y''[py'' + qy' + ry].$$

By (3.2) we see $(yy'y''y''')' > 0$ for $t \varepsilon (c,\bar{c})$. Hence it follows that $\int_c^{\bar{c}} (yy'y''y''')' > 0$. However

$$\int_c^{\bar{c}} (yy'y''y''')' = -[y(c)y'(c)y''(c)y'''(c)] \leq 0.$$

This contradiction proves that inequalities (3.2) hold for all $t > c$. The infinite limits are an immediate consequence of the inequalities. The proof in the case $y^{(i)}(c) \leq 0$ is similar and is omitted.

Lemma 3.2 Assume $p(t) > 0$, $q(t) < 0$, and $r(t) > 0$ for all $t \varepsilon [a,\infty)$. Let $y(t)$ be a solution of (L) such that for some point $c \varepsilon [a,\infty)$ $(-1)^i \cdot y^{(i)}(c) \geq 0$ for $i = 0,1,2,3$ with strict inequality holding for at least one value of i. Then $(-1)^i \cdot y^{(i)}(t) > 0$ for all $t \varepsilon [a,c)$ and for all i.

If $(-1)^i \cdot y^{(i)}(c) \leq 0$ for $i = 0,1,2,3$ with strict inequality holding for at least one value of i, then $(-1)^i \cdot y^{(i)}(t) < 0$ for all $t \varepsilon [a,c)$ and for all i.

Proof. Assume $(-1)^i \cdot y^{(i)}(c) \geq 0$ for all i and suppose strict inequality when $i = 3$, i.e. $y'''(c) < 0$. By continuity there exists a $\delta > 0$ such that

$$(3.4)\qquad y(t) > 0,\ y'(t) < 0,\ y''(t) > 0, \text{ and } y'''(t) < 0$$

on the interval $(c-\delta,c)$.

On the other hand suppose strict inequality holds for $y^{(i)}(c)$ for either $i = 0$, 1 or 2. Then

$y^{(4)}(c) = p(c)y''(c) + q(c)y'(c) + r(c)y(c) > 0$ and again there exists a $\delta > 0$ such that the inequalities (3.4) hold on $(c-\delta, c)$.

We claim the inequalities (3.4) hold for all $t \in [a,c)$, for, if not, there exists a first point to the left of c, say $\bar{c}$, where $a \leq \bar{c} < c$, such that $y(\bar{c})\cdot y'(\bar{c})\cdot y''(\bar{c})\cdot y'''(\bar{c}) = 0$. Then by the identity (3.3) of the previous lemma we see $(yy'y''y''')' < 0$ for $t \in (\bar{c},c)$. Hence $\int_{\bar{c}}^{c}(yy'y''y''')' < 0$. However

$\int_{\bar{c}}^{c} (yy'y''y''')' = y(c)y'(c)y''(c)y'''(c) \geq 0$. This contradiction proves the first part of the lemma. The proof in the case $(-1)^i y^{(i)}(c) \leq 0$ is similar and is omitted.

Two lemmas are now considered which are valuable in studying solutions of (L) which possess double zeros.

<u>Lemma 3.3</u> Assume $p(t) > 0$, $q(t) > 0$ and $r(t) > 0$ on $[a,\infty)$ and let conditions (3.1) hold. If $y(t) \not\equiv 0$ is a solution of (L) such that $y(c) = y'(c) = 0$ for some point $c \in (a,\infty)$, then in at least one of the two intervals $[a,c)$, (c,∞) all of the functions $y(t)$, $y'(t)$, $y''(t)$, and $y'''(t)$ are different from zero. Moreover if $y''(c)\cdot y'''(c) < 0$, then $y(t)\cdot y'(t)\cdot y''(t)\cdot y'''(t) \neq 0$ for $t \in [a,c)$. If $y''(c)\cdot y'''(c) \geq 0$ then $y(t)\cdot y'(t)\cdot y''(t)\cdot y'''(t) \neq 0$ for $t \in (c,\infty)$.

<u>Proof.</u> We construct the following chart exhibiting all possible sign

combinations on $y''(t)$ and $y'''(t)$ at $t = c$. Note that $y(t) \not\equiv 0$ fills in the middle blank. Moreover Lemma 3.1 fills in all the remaining blanks except for the two cases

$$\text{(i)}\quad \begin{matrix} y''(c) > 0 \\ y'''(c) < 0 \end{matrix} \qquad\qquad \text{(ii)}\quad \begin{matrix} y''(c) < 0 \\ y'''(c) > 0 \end{matrix}$$

	$y'''(c) > 0$	$y'''(c) = 0$	$y'''(c) < 0$
$y''(c) > 0$	$y > 0, y' > 0$ $y'' > 0, y''' > 0$ for all $t > c$	$y > 0, y' > 0$ $y'' > 0, y''' > 0$ for all $t > c$	$y > 0, y' < 0$ $y'' > 0, y''' < 0$ $t \in [a,c)$
$y''(c) = 0$	$y > 0, y' > 0$ $y'' > 0, y''' > 0$ for all $t > c$	$y(t) \equiv 0$	$y < 0, y' < 0$ $y'' < 0, y''' < 0$ for all $t > c$
$y''(c) < 0$	$y < 0, y' > 0$ $y'' < 0, y''' > 0$ $t \in [a,c)$	$y < 0, y' < 0$ $y'' < 0, y''' < 0$ for all $t > c$	$y < 0, y' < 0$ $y'' < 0, y''' < 0$ for all $t > c$

We consider case (i) $\quad y''(c) > 0$

$$y'''(c) < 0.$$

The proof of case (ii) is omitted since we may regard it as the solution in case (i) multiplied by (-1). Since $y(c) = y'(c) = 0$, $y''(c) > 0$, $y'''(c) < 0$ there exists a $\delta > 0$ such that the inequalities

$$y(t) > 0,\ y'(t) < 0,\ y''(t) > 0, \text{ and } y'''(t) < 0 \tag{3.5}$$

hold in the interval $(c-\delta, c)$. We claim that $y'''(t) < 0$ for all $t \in [a,c)$ and hence that the inequalities (3.5) hold for all $t \in [a,c)$.

Suppose there exists $\bar{c}$, $a \leq \bar{c} < c$ such that $y'''(\bar{c}) = 0$. Then $y'''(t) < 0$ for $t \in (\bar{c}, c]$. Thus $y''(t)$ is decreasing on $(\bar{c}, c]$ and $y''(\bar{c}) > y''(c) > 0$. Recall $H[y(t)]$ as defined by (1.1) and (1.2). Conditions (3.1) imply that $H[y(t)]$ is strictly increasing. But $H[y(c)] = y''(c)^2$ and since $y''(\bar{c}) > y''(c)$ we see

$$(3.6) \qquad y''(\bar{c})^2 + r(\bar{c})y(\bar{c})^2 + p(\bar{c})y'(\bar{c})^2 > y''(c)^2.$$

The left side of (3.6) is $H[y(\bar{c})]$, and the right side is $H[y(c)]$, Therefore $H[y(\bar{c})] > H[y(c)]$ and $H[y(t)]$ has decreased. This contradiction proves the lemma.

Lemma 3.4 Assume $p(t) > 0$, $q(t) < 0$, and $r(t) > 0$ on $[a,\infty)$, and let conditions (2.2) hold. If $y(t) \not\equiv 0$ is a solution of (L) such that $y(c) = y'(c) = 0$ for some point $c \in (a,\infty)$, then in at least one of the two intervals $[a,c)$, (c,∞) all of the functions $y(t)$, $y'(t)$, $y''(t)$, and $y'''(t)$ are different from zero. Moreover, if $y''(c) \cdot y'''(c) > 0$, then $y(t) \cdot y'(t) \cdot y''(t) \cdot y'''(t) \neq 0$ for $t \in (c,\infty)$. If $y''(c) \cdot y'''(c) \leq 0$, then $y(t) \cdot y'(t) \cdot y''(t) \cdot y'''(t) \neq 0$ for $t \in [a,c)$.

Proof. We again construct a chart exhibiting all possible sign combinations on $y''(t)$ and $y'''(t)$ at $t = c$. As before $y(t) \not\equiv 0$ fills in the middle blank, and Lemma 3.2 fills in all remaining blanks except for the two cases:

(i) $y''(c) > 0$	(ii) $y''(c) < 0$
$y'''(c) > 0$	$y'''(c) < 0$

	$y'''(c) > 0$	$y'''(c) = 0$	$y'''(c) < 0$
$y''(c) > 0$	$y > 0, y' > 0$ $y'' > 0, y''' > 0$ for all $t > c$	$y > 0, y' < 0$ $y'' > 0, y''' < 0$ $t \varepsilon [a,c)$	$y > 0, y' < 0$ $y'' > 0, y''' < 0$ $t \varepsilon [a,c)$
$y''(c) = 0$	$y < 0, y' > 0$ $y'' < 0, y''' > 0$ $t \varepsilon [a,c)$	$y(t) \equiv 0$	$y > 0, y' < 0$ $y'' > 0, y''' < 0$ $t \varepsilon [a,c)$
$y''(c) < 0$	$y < 0, y' > 0$ $y'' < 0, y''' > 0$ $t \varepsilon [a,c)$	$y < 0, y' > 0$ $y'' < 0, y''' > 0$ $t \varepsilon [a,c)$	$y < 0, y' < 0$ $y'' < 0, y''' < 0$ for all $t > c$

As before we need prove only (i). In this case there exists a $\delta > 0$ such that the inequalities

(3.7) $$y(t) > 0,\ y'(t) > 0,\ y''(t) > 0, \text{ and } y'''(t) > 0$$

hold for $t \varepsilon (c,c+\delta)$. We claim $y'''(t) > 0$ for all $t > c$ which will imply that inequalities (3.7) hold for all $t > c$. If there exists a first point $\overline{c} > c$ such that $y'''(\overline{c}) = 0$, then $y''(t)$ is increasing in $[c,\overline{c})$ and $0 < y''(c) < y''(\overline{c})$. Thus we see

(3.8) $$y''(\overline{c})^2 + r(\overline{c})y(\overline{c})^2 + p(\overline{c})y'(\overline{c})^2 > y''(c)^2.$$

But the left side of (3.8) is $H[y(\overline{c})]$, and the right side is $H[y(c)]$. Therefore $H[y(\overline{c})] > H[y(c)]$ and $H[y(t)]$ has increased. However conditions (2.2) imply $H[y(t)]$ is decreasing. This contradiction proves the lemma.

We now state two corollaries to the preceding lemmas. The proofs are immediate and, therefore, are omitted.

Corollary 3.5 If either of the following two hypotheses holds:

a) $p(t) > 0$, $q(t) > 0$, $r(t) > 0$ on $[a,\infty)$ and conditions (3.1),

b) $p(t) > 0$, $q(t) < 0$, $r(t) > 0$ on $[a,\infty)$ and conditions (2.2),

then every solution $y(t) \not\equiv 0$ of (L) has at most two double zeros. Moreover if a solution $y(t)$ possesses two double zeros say at $t=b$ and $t=c$, $a \leq b < c$, then the only zeros of $y(t)$ occur in the interval $[b,c]$, and $y(t)$ is nonoscillatory.

Corollary 3.6 Let either of the two hypotheses of Corollary 3.5 hold, and let $y(t) \not\equiv 0$ be a solution of (L) such that for $a \leq b < c$, $y(b) = y(c) = y'(c) = 0$. Then $y(t)$ is nonoscillatory. Indeed, $\lim_{t\to\infty} |y(t)| = +\infty$.

The following two lemmas shed light on solutions of (L) having triple zeros at a point of $[a,\infty)$.

Lemma 3.7 Assume $p(t) \geq 0$ and $r(t) \geq 0$ on $[a,\infty)$ and let conditions (2.2) hold. If $y(t)$ is a solution of (L) such that $y(c) = y'(c) = y''(c) = 0$, $y'''(c) > 0$ for some point $c \in [a,\infty)$, then

(3.9) $$y(t) > 0,\ y'(c) > 0,\ y''(t) > 0, \text{ and } y'''(t) > 0$$

for all $t > c$ and $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = +\infty$.

If $y(c) = y'(c) = y''(c) = 0$, $y'''(c) < 0$, then $y^{(i)}(t) < 0$ for

$i = 0,1,2,3$ and for all $t > c$, and $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = -\infty$.

Proof. By continuity there exists a $\delta > 0$ such that the inequalities (3.9) hold in $(c, c+\delta)$. Suppose there exists a point such that one of the inequalities (3.9) fails. Let $\bar{c} > c$ be the first such point. Then by Rolle's theorem we must have $y'''(\bar{c}) = 0$. Hence $H[y(\bar{c})] = r(\bar{c})y(\bar{c})^2 + p(\bar{c})y'(c)^2 + y''(\bar{c})^2 \geq 0$ and $H[y(c)] = 0$. Therefore $H[y(\bar{c})] - H[y(c)] \geq 0$. But $H[y(\bar{c})] - H[y(c)] = \int_c^{\bar{c}} (p' - 2q)(y')^2 + r'y^2 < 0$. This contradiction yields the result.

Lemma 3.8 Assume $p(t) \geq 0$ and $r(t) \geq 0$ on $[a,\infty)$ and let conditions (3.1) hold. If $y(t)$ is a solution of (L) such that $0 = y(c) = y'(c) = y''(c)$, $y'''(c) < 0$ for some point $c \in [a,\infty)$, then $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$, and $y'''(t) < 0$ for all $t \in [a,c)$. If $0 = y(c) = y'(c) = y''(c)$, $y'''(c) > 0$, then $y(t) < 0$, $y'(t) > 0$, $y''(t) < 0$, and $y'''(t) > 0$ for all $t \in [a,c)$.

Proof. Apply the identity for $H[y(t)]$ to the left of $t=c$; the details are so similar to those of Lemma 3.7 that they are omitted.

We are now ready to state a theorem describing the nature of solutions possessing a triple zero at some point of $[a,\infty)$.

Theorem 3.9 Suppose that either of the two hypotheses of Corollary 3.5 holds. If $y(t)$ is a solution of (L) such that

$y(c) = y'(c) = y''(c) = 0$ and $y'''(c) > 0$, then

$$\left.\begin{array}{r} y(t) > 0 \\ y'(t) > 0 \\ y''(t) > 0 \\ y'''(t) > 0 \end{array}\right\rangle \text{for all } t > c$$

and

$$\left.\begin{array}{r} y(t) < 0 \\ y'(t) > 0 \\ y''(t) < 0 \\ y'''(t) > 0 \end{array}\right\rangle \text{for all } t \in [a,c).$$

Moreover $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = +\infty$.

If we assume instead that $y(c) = y'(c) = y''(c) = 0$ and $y'''(c) < 0$, then

$$\left.\begin{array}{r} y(t) < 0 \\ y'(t) < 0 \\ y''(t) < 0 \\ y'''(t) < 0 \end{array}\right\rangle \text{for all } t > c$$

and

$$\left.\begin{array}{r} y(t) > 0 \\ y'(t) < 0 \\ y''(t) > 0 \\ y'''(t) < 0 \end{array}\right\rangle \text{for all } t \in [a,c).$$

In this case $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = -\infty$.

Proof. In hypothesis (a) of Corollary 3.5 apply Lemma 3.1 to the right of $t=c$ and Lemma 3.8 to the left of $t=c$.

In hypothesis (b) of Corollary 3.5 apply Lemma 3.2 to the left of $t=c$ and Lemma 3.7 to the right of $t=c$.

We now proceed with a theorem which guarantees the existence of a solution which asymptotically approaches a finite constant.

Theorem 3.10 If either of the hypotheses of Corollary 3.5 holds, then (L) possesses a solution $w(t)$ with the following properties:

(i) $w(t)\cdot w'(t)\cdot w''(t) \neq 0$ on $[a,\infty)$

(ii) $\operatorname{sgn} w(t) = \operatorname{sgn} w''(t) \neq \operatorname{sgn} w'(t)$ on $[a,\infty)$

(iii) $\lim_{t\to\infty} w'(t) = \lim_{t\to\infty} w''(t) = 0$

(iv) $\lim_{t\to\infty} w(t) = K$ (finite)

Proof. Let $z_i(t)$, $i = 0,1,2,3$ be the solutions of (L) defined by the initial conditions $z_i^{(j)}(a) = \delta_{ij}$ where $j = 0,1,2$, and 3. For each positive integer $n > a$ let c_{on}, c_{1n}, c_{2n}, and c_{3n} be real numbers such that

$$\sum_{i=0}^{3} c_{in}\, z_i(n) = 0$$

$$\sum_{i=0}^{3} c_{in}\, z_i'(n) = 0 \qquad (3.10)$$

$$\sum_{i=0}^{3} c_{in}\, z_i''(n) = 0$$

This is possible since the number of equations is less than the number of unknowns. Also we may assume

$$\sum_{i=0}^{3} c_{in}^2 \neq 0 \quad \text{for all } n.$$

Indeed, we may assume further that

$$\sum_{i=0}^{3} c_{in}^2 = 1 \quad \text{for all } n. \tag{3.11}$$

Moreover since $\{z_i(t) \mid i=0,1,2,3\}$ forms a linearly independent set of solutions of (L), we must have

$$\sum_{i=0}^{3} c_{in} z_i'''(n) \neq 0 \quad \text{for all } n.$$

Assume without loss of generality that

$$\sum_{i=0}^{3} c_{in} z_i'''(n) < 0 \quad \text{for all } n. \tag{3.12}$$

Consider the solution $w_n(t)$ of (L) defined by:

$$w_n(t) = c_{on} z_o(t) + \dots + c_{3n} z_3(t). \tag{3.13}$$

Since $w_n(n) = w_n'(n) = w_n''(n) = 0$, and $w_n'''(n) < 0$, Theorem 3.9 implies

$$\begin{array}{c} w_n(t) > 0,\ w_n'(t) < 0 \\ w_n''(t) > 0,\ w_n'''(t) < 0 \end{array} \quad \text{for } t \in [a,n). \tag{3.14}$$

By the compactness of the 4-sphere and equation (3.11), there exists a subsequence of integers $\{n_j\}$ such that $\lim_{n_j \to \infty} c_{in_j} = c_i$ for $i = 0,1,2,3$

and

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1. \tag{3.15}$$

Consider the solution $w(t)$ of (L) defined by

$$w(t) = c_0 z_0(t) + \dots + c_3 z(t). \tag{3.16}$$

Clearly (3.15) implies $w(t) \not\equiv 0$. Moreover the functions $w_{n_j}^{(i)}(t)$, $i = 0,1,2,3$, converge uniformly to $w^{(i)}(t)$ for $i = 0,1,2,3$, respectively, on any finite subinterval of $[a,\infty)$.

Now let $\bar{t} \in [a,\infty)$ be an arbitrary point. By inequalities (3.14), we see that $w_{n_j}(\bar{t}) > 0$, $w'_{n_j}(\bar{t}) < 0$, $w''_{n_j}(\bar{t}) > 0$, and $w'''_{n_j}(\bar{t}) < 0$ for all $n_j > \bar{t}$. Hence

$$w(\bar{t}) = \lim_{n_j \to \infty} w_{n_j}(\bar{t}) \geq 0$$

$$w'(\bar{t}) = \lim_{n_j \to \infty} w'_{n_j}(\bar{t}) \leq 0$$

$$w''(\bar{t}) = \lim_{n_j \to \infty} w''_{n_j}(\bar{t}) \geq 0$$

$$w'''(\bar{t}) = \lim_{n_j \to \infty} w'''_{n_j}(\bar{t}) \leq 0.$$

Since $\bar{t}$ was arbitrary in $[a,\infty)$ we conclude

$$\left.\begin{array}{lr} (3.17) & w(t) \geq 0 \\ (3.18) & w'(t) \leq 0 \\ (3.19) & w''(t) \geq 0 \\ (3.20) & w'''(t) \leq 0 \end{array}\right\} \text{ for all } t \in [a,\infty).$$

Now we rule out equality in (3.17)--(3.20).

In (3.17): Suppose there exists a point $\bar{t}$ such that $w(\bar{t}) = 0$. But $w'(t) \leq 0$ for all t implies $w(t) \equiv 0$ for all $t \geq \bar{t}$. This contradicts (3.15).

In (3.18): Suppose there exists a point $\bar{t}$ such that $w'(\bar{t}) = 0$. But $w''(t) \geq 0$ for all t implies $w'(t) \equiv 0$ for all $t > \bar{t}$. Hence $w(t) \equiv K > 0$ for $t \geq \bar{t}$. Then we have $w(t) \equiv K > 0$, $w'(t) = w''(t) = w'''(t) = w^{(4)}(t) \equiv 0$ for all $t \geq \bar{t}$. From (L), we see $w^{(4)}(t) = r(t)\cdot K > 0$ for all $t \geq \bar{t}$. This contradiction shows $w'(t) < 0$ on $[a,\infty)$.

In (3.19): Suppose there exists a point $\bar{t}$ such that $w''(\bar{t}) = 0$. Since $w'''(t) \leq 0$ we have $w''(t) \equiv 0$ for all $t \geq \bar{t}$. Thus $w'(t) \equiv K_1 < 0$ and $w(t) \equiv K_1 t + K_2$ for all $t \geq \bar{t}$. But for large t values $w(t) \equiv K_1 t + K_2$ is eventually negative contradicting (3.17).

In (3.20): Here we prove only that $w'''(t) \not\equiv 0$. Suppose $w'''(t) \equiv 0$. Then we conclude $w''(t) \equiv K_1 > 0$ and $w'(t) \equiv K_1 t + K_2 < 0$. But for large t values $K_1 t + K_2$ is eventually positive. This contradiction proves that $w'''(t) \not\equiv 0$.

In summary then, we have the following inequalities:

$$
\begin{aligned}
w(t) &> 0 \\
w'(t) &< 0 \\
w''(t) &> 0 \\
w'''(t) &\le 0
\end{aligned}
\tag{3.21}
$$

for all $t \in [a,\infty)$, where $w'''(t) \not\equiv 0$.

Thus properties (i) and (ii) are proven. Properties (iii) and (iv) follow trivially from inequalities (3.21).

Before proceeding, we note that it is always possible to construct a solution of (L) with three zeros. These zeros may be either multiple or distinct. For example, suppose we desire a solution of (L), say $u(t)$, such that $u(b) = u(c) = u(d) = 0$ where $a \le b < c < d$. Let $\{z_i(t)\}$, $i = 0,1,2,3$ be a canonical basis at $t = a$. Since the number of unknowns exceeds the number of equations there exist constants c_0, c_1, c_2, and c_3 with $c_0^2 + c_1^2 + c_2^2 + c_3^2 \neq 0$ such that

$$
\begin{aligned}
c_0 z_0(b) + c_1 z_1(b) + \ldots + c_3 z_3(b) &= 0 \\
c_0 z_0(c) + c_1 z_1(c) + \ldots + c_3 z_3(c) &= 0 \\
c_0 z_0(d) + c_1 z_1(d) + \ldots + c_3 z_3(d) &= 0.
\end{aligned}
$$

Now define a solution $u(t)$ of (L) by $u(t) = c_0 z_0(t) + \ldots + c_3 z_3(t)$. Clearly $u(t) \not\equiv 0$ and $u(b) = u(c) = u(d) = 0$.

Similarly, if we desire a solution $z(t)$ of (L) such that

$z(b) = z'(b) = 0$, $z(c) = 0$, we consider

$$c_0z_0(b) + \dots + c_3z_3(b) = 0$$
$$c_0z_0'(b) + \dots + c_3z_3'(b) = 0$$
$$c_0z_0(c) + \dots + c_3z_3(c) = 0$$

and define $z(t)$ by

$$z(t) = c_0z_0(t) + \dots + c_3z_3(t)$$

where c_0, c_1, c_2, c_3 is a solution of this system of equations.

We now proceed with the construction of a nonoscillatory basis of solutions for equation (L).

Theorem 3.11 If either of the two hypotheses of Corollary 3.5 holds, then there exists a basis of solutions for (L) consisting of four linearly independent nonoscillatory solutions. Three of these solutions tend to infinity and one decreases asymptotically to a finite constant as t tends to infinity.

Proof. Consider the following four solutions:

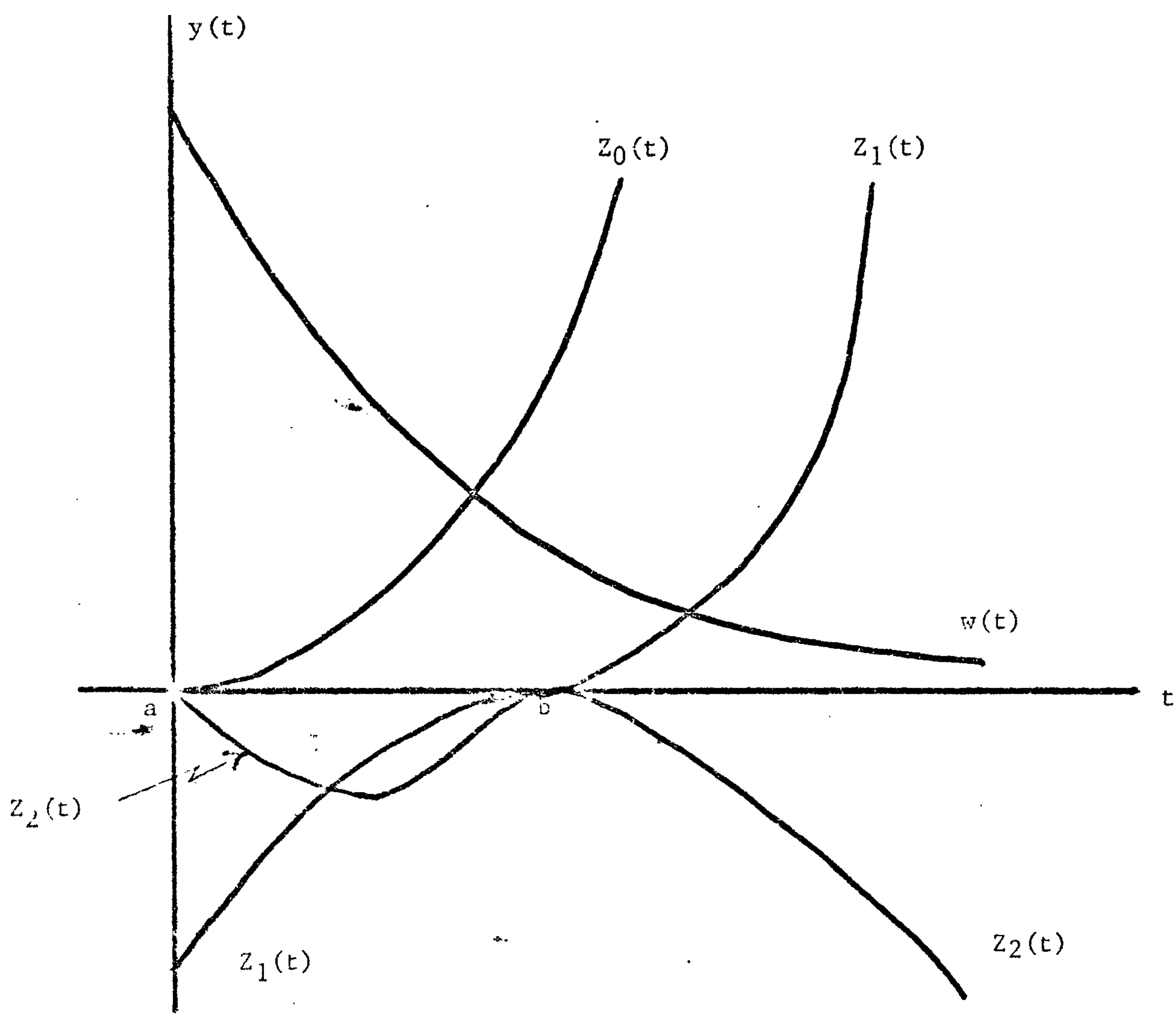
(i) $w(t)$: the solution constructed in Theorem 3.10. Here we assume $w(t) > 0$, $w'(t) < 0$, $w''(t) > 0$, $w'''(t) \leq 0$ and $\lim_{t\to\infty} w(t) = K \geq 0$ (finite).

(ii) $z_0(t)$: the solution defined by $z_0(a) = z_0'(a) = z_0''(a) = 0$, $z_0'''(a) = 1$.

(iii) $z_1(t)$: the solution defined by $z_1(b) = z_1'(b) = z_1''(b) = 0$, $z_1'''(b) = 1$ where b is an arbitrary point such that $a < b$.

(iv) $z_2(t)$: the solution defined by $z_2(a) = z_2(b) = z_2'(b) = 0$ where b is defined in (iii). This solution exists by the comments preceding this theorem.

We note that Theorem 3.9 completely describes the behavior of $z_0(t)$ and $z_1(t)$. Moreover Corollary 3.6 describes the behavior of $z_2(t)$ as $t \to \infty$.



We now prove that these four solutions are linearly independent. Suppose they are dependent. Then there exist constants c_0, c_1, c_2, c_3 such that $c_0^2 + c_1^2 + c_2^2 + c_3^2 > 0$ and

$$c_0 z_0(t) + c_1 z_1(t) + c_2 z_2(t) + c_3 w(t) \equiv 0 \tag{3.22}$$

$$c_0 z_0'(t) + c_1 z_1'(t) + c_2 z_2'(t) + c_3 w'(t) \equiv 0 \tag{3.23}$$

$$c_0 z_0''(t) + c_1 z_1''(t) + c_2 z_2''(t) + c_3 w''(t) \equiv 0 \; . \tag{3.24}$$

Evaluating equations (3.22) and (3.23) at t=b yields

$$\begin{aligned} c_0 z_0(b) + c_3 w(b) &= 0 \\ c_0 z_0'(b) + c_3 w'(b) &= 0 \; . \end{aligned} \tag{3.25}$$

But equations (3.25) have a nontrivial solution (c_1, c_3) if and only if $w(b)z_0'(b) - w'(b)z_0(b) = 0$. However $w(b) > 0$, $z_0'(b) > 0$, $w'(b) < 0$, amd $z_0(b) > 0$ imply $w(b)z_0'(b) - w'(b)z_0(b) > 0$. Hence we conclude $c_0 = c_3 = 0$. Now we evaluate equation (3.24) at t=b and using $c_0 = c_3 = 0$, we find $c_2 z_2''(b) = 0$. Suppose $z_2''(b) = 0$. Then $z_2(t)$ has a triple zero at t=b and Theorem 3.9 implies $z_2(a) \neq 0$ which is a contradiction. Hence $c_2 = 0 = c_0 = c_3$.

Finally we evaluate equation (3.22) at t=a and use $c_2 = c_0 = c_3 = 0$. We find $c_1 z_1(a) = 0$. But $c_2 = c_0 = c_3 = 0$ implies $c_1 \neq 0$, and hence $z_1(a) = 0$. This is a contradiction of Theorem 3.9 which implies $z_1(a) < 0$.

Thus the four solutions are independent and the theorem is proven.

We note that this theorem does not necessarily imply that the equation itself is nonoscillatory. The question must first be answered as to whether every linear combination of these four non-oscillatory solutions is again a nonoscillatory solution. In the next chapter we shall give an added condition which will suffice to guarantee the nonoscillation of equation (L).

CHAPTER IV

STUDY OF CONJUGATE POINTS, DISCONJUGACY, AND DISTRIBUTION OF ZEROS OF SOLUTIONS OF (L)

In order to continue our discussion of equation (L), we must introduce the concepts of conjugate points and distribution of zeros of solutions of (L).

Recall from the previous chapter that given three points, not necessarily distinct, there exists a nontrivial solution of (L) having zeros at these points. The next two definitions refer to situations where the total number of zeros of a solution is critical.

Definition 4.1 If no nontrivial solution of (L) has more than three zeros on $[a,\infty)$, counting multiplicities, then (L) is said to be disconjugate on $[a,\infty)$. (Hence disconjugacy is an extreme case of non-oscillation.)

Definition 4.2 For $n \geq 1$ and $t \in [a,\infty)$, the number $\eta_n(t)$ is the infimum of the set of numbers $b > t$ such that there is a nontrivial solution y of (L) for which $y(t) = 0$ and y has at least $n + 3$ zeros, counting multiplicities, on $[t,b]$. If $\eta_n(t)$ exists, it is called the n^{th} conjugate point of t. By $\eta_n(t) = \infty$ we mean there is no nontrivial solution of (L) with a zero at t and having $n + 3$ zeros on $[t,\infty)$.

We note that disconjugacy is equivalent to $\eta_1(t) = \infty$ for all

$t \in [a,\infty)$. The functions $\eta_K^*(t)$, $t \in [a,\infty)$, $K = 1,2,\ldots$ are similarly defined for the adjoint equation $L^*(z) = 0$.

Definition 4.3 A nontrivial solution y of (L) is said to have an $i_0 - i_1 - \ldots - i_\alpha$ ($\alpha = 1,2,3$; $i_K = 1,2,3$) distribution of zeros on $[t,b] \subset [a,\infty)$ provided there are numbers $t_0, t_1, \ldots, t_\alpha$ such that $t \leq t_0 < \ldots < t_\alpha \leq b$ and y has a zero at each t_K of order at least i_K.

Definition 4.4 For $t \in [a,\infty)$, the number $r_{i_0 i_1 \ldots i_\alpha}(t)$ is the infimum of the set of numbers $b > t$ such that there is a nontrivial solution y of (L) having an $i_0 - i_1 - \ldots - i_\alpha$ distribution of zeros on $[t,\infty)$. The number $r^*_{i_0 i_1 \ldots i_\alpha}(t)$ is defined similarly for the adjoint equation $L^*(z) = 0$.

Consider the following example:

Example 4.1 The equation $y^{(4)} + 10y'' + 9y = 0$ has the following four linearly independent solutions:

$$y_1(t) = \sin^3 t \qquad y_3(t) = \cos^2 t \sin t$$
$$y_2(t) = \cos^3 t \qquad y_4(t) = \sin^2 t \cos t.$$

(i) $y_1(t)$ has a 3-3 distribution of zeros on $[0,\pi]$,

(ii) the solutions $y_3(t)$ and $y_4(t)$ have respectively a 1-2-1 and a 2-1-2 distribution of zeros on $[0,\pi]$,

(iii) $\eta_1(0) = r_{33}(0) = r_{31}(0) = r_{13}(0) = r_{22}(0) = \pi$,

(iv) $\eta_1(o) = r_{121}(o) = r_{212}(o) = r_{1111}(o) = r_{112}(o) = r_{211}(o)$.

Example 4.2 We refer to the various results of Chapter III.

(i) Suppose $p(t)$, $q(t)$ and $r(t)$ are positive for all $t \in [a,\infty)$. By Lemma 3.1 $r_{31}(t) = \infty$ for all $t \in [a,\infty)$.

(ii) Assume $p(t) > 0$, $q(t) < 0$, and $r(t) > 0$ for all $t \in [a,\infty)$. By Lemma 3.2 $r_{13}(t) = \infty$ for all $t \in [a,\infty)$.

(iii) Suppose that either hypothesis of Corollary 3.5 holds. Then by Lemmas 3.3 and 3.4 we have $r_{121}(t) = \infty$ for all $t \in [a,\infty)$.

(iv) Assume that either hypothesis of Corollary 3.5 holds. Then by Theorem 3.9 we see $r_{13}(t) = r_{31}(t) = \infty$.

The following series of lemmas and theorems were first developed by Leighton and Nehari [6] for the differential equation

$$(4.1) \qquad (ry'')'' + py = 0$$

where $p(t)$ is assumed negative on $[a,\infty)$.

These results depend not on equation (4.1) but, instead, on the fact that

$$(4.2) \qquad r_{121}(t) = r_{13}(t) = r_{31}(t) = \infty$$

if $p(t) < 0$ for all $t \in [a,\infty)$.

Returning to equation (L), we consider the following two hypotheses:

$$(4.3)\quad \begin{aligned} &\text{(a)}\quad p(t) > 0,\ q(t) > 0,\ r(t) > 0 \text{ on } [a,\infty) \text{ and conditions (3.1).} \\ &\text{(b)}\quad p(t) > 0,\ q(t) < 0,\ r(t) > 0 \text{ on } [a,\infty) \text{ and conditions (2.2).} \end{aligned}$$

As pointed out in Example 4.2, either hypothesis yields the equalities in (4.2). Therefore _in the following series of lemmas and theorems we assume that either hypothesis_ (a) _or_ (b) _of_ (4.3) _holds_.

Lemma 4.1 If two nontrivial solutions of (L) have three zeros in common (of any type), then they are constant multiples of each other.

Proof. Let $u(t)$ and $v(t)$ be two such solutions of (L). Suppose first the zeros are distinct. Let $a \leq b < c < d$ be such that $u(b) = u(c) = u(d) = v(b) = v(c) = v(d) = 0$. Since $r_{121}(t) = \infty$ we have $u'(c) \neq 0$ and $v'(c) \neq 0$. Define a solution of (L) by $w(t) = v'(c)u(t) - u'(c)v(t)$. Clearly $w(b) = w(c) = w'(c) = w(d) = 0$ and $r_{121}(t) = \infty$ implies $w(t) \equiv 0$ and $u(t)$ and $v(t)$ are dependent.

Suppose $b = c$. Then we have $u(b) = u'(b) = 0$, $v(b) = v'(b) = 0$ and $u(d) = v(d) = 0$. Since $r_{31}(t) = \infty$ we must conclude $u''(b) \neq 0$, $v''(b) \neq 0$. Define a solution of (L) by $w(t) = v''(b)u(t) - u''(b)v(t)$. Then $w(b) = w'(b) = w''(b) = 0 = w(d)$. But $r_{31}(t) = \infty$. Hence $w(t) \equiv 0$ and $u(t)$, $v(t)$ are again dependent.

Finally, if $b = c = d$ then $u(t)$ and $v(t)$ would have a common "triple zero" and would thus be dependent.

Lemma 4.2 Suppose $u(t)$ and $v(t)$ are two linearly independent solutions of (L) such that for $a \leq b < c$ we have $u(b) = v(b) = u(c) = v(c) = 0$. Then the zeros of $u(t)$ and $v(t)$ separate each other in (b,c).

Proof. The hypothesis is not empty. If $u(t)$ is one such solution take $v(t)$ such that $v(b) = v(c) = v(d) = 0$ where d is any point such that $u(d) \neq 0$. By Lemma 4.1 no zero of $u(t)$ in (b,c) coincides with a zero of $v(t)$. Suppose the lemma is false. Then there would exist two consecutive zeros $t = \alpha,\beta$ $(b < \alpha < \beta < c)$ of one of the solutions, say $u(t)$, such that $v(t) \neq 0$ for all $t \in [\alpha,\beta]$. But Lemma 1.11 implies there exists a constant K such that $w(t) = u(t) - Kv(t)$ has a double zero in (α,β). However, $w(b) = w(c) = 0$ where $b < \alpha < \beta < c$ and $w(\gamma) = w'(\gamma) = 0$ for some γ in (α,β). This contradicts $r_{121}(t) = \infty$.

We now state an immediate corollary.

Corollary 4.3 If two nontrivial solutions $u(t)$ and $v(t)$ of (L) have two zeros $t = b,c$ in common where $a \leq b < c$, then the number of zeros of $u(t)$ in (b,c) differs from the number of zeros of $v(t)$ in (b,c) by at most one.

We note in Corollary 4.3 that if the solutions are linearly dependent they have exactly the same zeros. If they are linearly independent we apply Lemma 4.2 realizing that the first and last zeros in (b,c) may both belong to the same solution. Hence we content ourselves with the conclusion of the corollary.

We are now in a position to discuss the concept of the first conjugate point introduced above. Consider the class of solutions $y(t)$ of (L) which vanish at $t = a$ and have at least four zeros in $[a,\infty)$, assuming this class is not empty. We denote by a_1, a_2, a_3, a_4 $(a = a_1)$

the first four zeros of $y(t)$ in $[a,\infty)$, and we desire an extremal solution which minimizes the value of a_4 in this class. We may restrict ourselves to the class of solutions which have a double zero at $t = a$. Indeed, if $y(t)$ is the solution from the class which is extremal and if $a_1 < a_2$, we may compare the number of its zeros in (a,a_4) with that of the solution $v(t)$ for which $v(a) = v'(a) = v(a_4) = 0$. By Corollary 4.3 the extra zero which $y(t)$ may have in (a,a_4) is made up for by the double zero of $v(t)$ at $t = a$ (zeros are counted by multiplicities), and $v(t)$ therefore has at least as many zeros in $[a,a_4]$ as $y(t)$.

We again denote by a_1, a_2, a_3, a_4 ($a = a_1 = a_2$) the first four zeros of $v(t)$ in $[a,\infty)$, and we introduce a solution $w(t)$ of (L) which has a double zero at $t = a$ and is positive for $t > a$. We could take the solution determined by $w(a) = w'(a) = w'''(a) = 0$, $w''(a) = 1$. We search for an extremal solution in two categories of solutions:

(i) solutions for which $a_3 \neq a_4$,

(ii) solutions for which $a_3 = a_4$.

In (i) we see $w(t) \neq 0$ on $[a_3,a_4]$ and $v(t)$ has simple zeros at $t = a_3,a_4$. Therefore Lemma 1.11 implies there exists a constant K such that the solution $u(t) = v(t) - Kw(t)$ has a double zero in the interval (a_3,a_4), say at the point $t = \alpha$, $a_3 < \alpha < a_4$. Clearly $u(t)$ also has a double zero at $t = a$. Thus for the solution $u(t)$ we have $a = a_1 = a_2$ and $a_3 = a_4 = \alpha$, which shows that $u(t)$ belongs to solution category (ii); furthermore, this shows that the extremal solution we desire belongs to category (ii). Indeed, we claim that $u(t)$ is the essentially unique extremal solution. If not, there would exist another solution, say $z(t)$, such that $z(a) = z'(a) = z(\beta) = z'(\beta) = 0$ where $a < \beta < \alpha$. Since both $u(t)$ and $z(t)$ have double zeros to the right of $t = a$, previous lemmas imply $u''(a)\cdot u'''(a) < 0$ and $z''(z)\cdot z'''(a) < 0$. We assume without loss of generality that $u''(a)\cdot z''(a) > 0$, and it follows that $u'''(a)\cdot z'''(a) > 0$. As a result of Corollary 3.5, $u(t)$ and $z(t)$ are nonzero on (α,∞) and (β,∞) respectively. Moreover, $u''(a)\cdot z''(a) > 0$ implies that $u(\beta)\cdot z(\alpha) > 0$.

Finally, it is easily verified that $z(t) = c_1u(t) + c_2w(t)$ for appropriate constants c_1 and c_2. Solving for c_1 by calculating the third derivative at $t = a$ we find $c_1 = \dfrac{z'''(a)}{u'''(a)} > 0$. We assume therefore that $c_1 = 1$. Now consider c_2. At $t = \beta$ we have $0 = u(\beta) + c_2w(\beta)$ and thus $c_2 = -u(\beta)/w(\beta)$. At $t = \alpha$ we find $z(\alpha) = [-u(\beta)/w(\beta)]\cdot w(\alpha)$ implying $z(\alpha)\cdot u(\beta) < 0$ contradicting our result above.

We have proven the following theorem.

<u>Theorem 4.4</u> If there exists a solution $y(t)$ of (L) which vanishes at $t = a$ and has at least four zeros in $[a,\infty)$, there then exists a point $t = \alpha$ and an essentially unique solution $u(t)$ with the following properties:

(a) $u(t)$ has double zeros at $t = a$ and $t = \alpha$.

(b) $u(t)$ has exactly four zeros in $[a,\alpha]$.

(c) any other solution $y(t)$ such that $y(a) = 0$ has fewer than four zeros in $[a,\alpha]$.

Properties (a) and (c) show that $\eta_1(a) = \alpha$, where $\eta_1(a)$ is the first conjugate point of $t = a$ defined above.

We are now in a position to obtain a sufficient condition for disconjugacy of equation (L). This will be accomplished using an identity from Lemma 1.6.

Theorem 4.5 Let $q(t) \in C'[a,\infty)$ and $p(t) \in C^2[a,\infty)$. If $p''(t) + 2r(t) \leq q'(t)$ and $p(t) \geq 0$ on $[a,\infty)$, then every solution of (L) has at most one double zero in $[a,\infty)$. (NOTE: Neither hypothesis (a) nor (b) of (4.3) is needed here.)

Proof Let $y(t)$ be a solution of (L) such that for $a \leq b < c$ we have $y(b) = y'(b) = y(c) = y'(c) = 0$. By (1.3) and (1.4) of Lemma 1.6 we have $K[y(t)] = yy''' - y'y'' - pyy' + \frac{1}{2}(p' - q)y^2$ and

$$K[y(t)] = K[y(t_1)] + \int_{t_1}^{t} \tfrac{1}{2}(p'' + 2r - q')y^2 - p(y')^2 - (y'')^2 ds.$$

Hence $K[y(b)] = K[y(c)] = 0$. Also

$$I = \int_b^c [\tfrac{1}{2}(p'' + 2r - q')y^2 - p(y')^2 - (y'')^2]ds < 0 \text{ by hypothesis.}$$

However, $I = K[y(c)] - K[y(b)] = 0$. This contradiction proves the theorem.

We now state an immediate corollary.

Corollary 4.6 Assume that one of the hypotheses (a) or (b) of (4.3) holds. Let $p(t) \in C^2[a,\infty)$ and $q(t) \in C'[a,\infty)$. If $p''(t) + 2r(t) \leq q'(t)$

on $[a,\infty)$, then equation (L) is disconjugate and hence nonoscillatory.

Proof. We need merely show $\eta_1(t) = \infty$ for $t \in [a,\infty)$. Using the proof of Theorem 4.4, $\eta_1(t)$ will occur at a point $\bar{t}$, $\bar{t} > t$ such that there exists a solution $y(t)$ having double zeros at t and $\bar{t}$. But by Theorem 4.5 there exists no solution having two double zeros and hence $\eta_1(t) = \infty$. This proves the claim.

Note that this result yields much more than (L) being nonoscillatory. Nonoscillation is equivalent to saying the set of zeros of every solution of (L) is bounded above, but the conditions of Corollary 4.6 yield that every solution of (L) has at most three zeros on $[a,\infty)$.

This concludes our section of lemmas and theorems based on conditions (4.3). We shall return to these conditions a little later in this chapter. We now consider a result which is valid for equation $L^*(z) = 0$ as defined by (1.9). One should recall the meaning of $r_{13}(t)$, $r_{31}(t)$, $r^*_{13}(t)$, and $r^*_{31}(t)$ as given by Definition 4.4. Note that no sign assumptions are placed on $p(t)$, $q(t)$, or $r(t)$.

Theorem 4.7 Consider equation (L) and its adjoint equation (L^*)

(L) $$y^{(4)} = py'' + qy' + ry$$

(L^*) $$y^{(4)} = py'' + (2p' - q)y' + (p'' + r - q')y.$$

The following hold:

(a) $r_{13}(t) < \infty$ if and only if $r^*_{31}(t) < \infty$ and $r_{13}(t) = r^*_{31}(t)$ for all $t \in [a,\infty)$.

(b) $r_{31}(t) < \infty$ if and only if $r^*_{13}(t) < \infty$ and $r_{31}(t) = r^*_{13}(t)$ for all $t \in [a,\infty)$.

Proof It suffices to prove the following four implications:

(i) $r^*_{13}(t) < \infty$ implies $r_{31}(t) \leq r^*_{13}(t)$,

(ii) $r^*_{31}(t) < \infty$ implies $r_{13}(t) \leq r^*_{31}(t)$,

(iii) $r_{13}(t) < \infty$ implies $r^*_{31}(t) \leq r_{13}(t)$,

(iv) $r_{31}(t) < \infty$ implies $r^*_{13}(t) \leq r_{31}(t)$

for $t \in [a,\infty)$.

We shall prove only (i) and (iii) since (ii) and (iv) follow similarly.

Consider (i). Suppose $r^*_{13}(t) = \beta < \infty$ and the conclusion false. In other words we assume $\beta = r^*_{13}(t) < r_{31}(t)$. We appeal to identities (1.5), (1.6), and (1.7) of Lemma 1.7 in which we assume (L_1) is (L) and (L_2) is (L^*). We have $p_1(t) \equiv p_2(t) \equiv p(t)$, $q_1(t) \equiv q(t)$, $r_1(t) \equiv r(t)$, $q_2(t) \equiv 2p'(t) - q(t)$, $r_2(t) \equiv p''(t) + r(t) - q'(t)$.

Calculating the coefficients on the right side of (1.7):

$p_2 - p_1 = p(t) - p(t) \equiv 0$, $q_1 + q_2 - p_1' - p_2' = q + 2p' - q - 2p' \equiv 0$, $r_1 - r_2 + q_2' - p_2'' = r - p'' - r + q' + 2p'' - q' - p'' \equiv 0$. Hence, for the case where (L_1) is (L) and (L_2) is (L^*), we see identity (1.7) becomes

$$M'(t) \equiv 0\,, \tag{4.4}$$

where

$$\begin{cases} M(t) = F(t) - p_1u'v + p_2v'u + (q_2 - p_2')uv\,, \\ F(t) = v(t)u'''(t) - u(t)v'''(t) - v'(t)u''(t) + u'(t)v''(t). \end{cases} \tag{4.5}$$

Since $\beta = r_{13}^*(t) < r_{31}(t)$ there exists a solution $v(t)$ of (L^*) such that $v(\beta) = v'(\beta) = v''(\beta) = 0$, $v'''(\beta) = 1$ and there exists a point $\alpha \in [t,\beta)$ such that $v(\alpha) = 0$. Also any solution $u(t)$ of (L) such that $u(\alpha) = u'(\alpha) = u''(\alpha) = 0$ satisfies $u(t) \neq 0$ for $\alpha < t \leq \beta$.

Now integrating (4.4) above from α to β yields

$$0 = \int_\alpha^\beta M'(t)dt = M(\beta) - M(\alpha). \tag{4.6}$$

But from (4.5) we see $M(\beta) = -u(\beta) \neq 0$, $M(\alpha) = v(\alpha)\cdot u'''(\beta) = 0$. This contradicts (4.6).

We now prove (iii). Suppose $r_{13}(t) = \beta < \infty$ and $\beta = r_{13}(t) < r_{31}^*(t)$. This implies there exists a solution $u(t)$ of (L) and a point $\alpha \in [t,\beta)$

such that $u(\alpha) = 0$, $u(\beta) = u'(\beta) = u''(\beta) = 0$, and $u'''(\beta) = 1$. Also any solution $v(t)$ of (L^*) such that $v(\alpha) = v'(\alpha) = v''(\alpha) = 0$, satisfies $v(t) \neq 0$ for $\alpha < t \leq \beta$. Now (4.6) still holds and $M(\beta) - M(\alpha) = 0$. But from (4.5) we see $M(\beta) = v(\beta) \neq 0$, $M(\alpha) = -u(\alpha)\cdot v'''(\alpha) = 0$, and this contradicts (4.6).

Cases (ii) and (iv) follow similarly.

<u>Corollary 4.8</u> Suppose that <u>either</u> of the two following hypotheses holds:

(a) $p(t)$, $q(t)$, $r(t)$ positive on $[a,\infty)$,

(b) $p(t) \geq 0$, $r(t) \geq 0$ on $[a,\infty)$ together with conditions (2.2).

Then $r^*_{13}(t) = \infty$.

<u>Proof</u> In (a) Lemma 3.1 implies $r_{31}(t) = \infty$.

In (b) Lemma 3.7 implies $r_{31}(t) = \infty$.

In either case $r_{31}(t) = \infty$ implies $r^*_{13}(t) = \infty$ by Theorem 4.7.

<u>Corollary 4.9</u> Suppose that <u>either</u> of the following two hypotheses holds:

(a) $p(t) > 0$, $q(t) < 0$, $r(t) > 0$ on $[a,\infty)$,

(b) $p(t) \geq 0$, $r(t) \geq 0$ on $[a,\infty)$ together with conditions (3.1).

Then $r^*_{31}(t) = \infty$.

<u>Proof</u> In (a) Lemma 3.2 implies $r_{13}(t) = \infty$.

In (b) Lemma 3.8 implies $r_{13}(t) = \infty$.

In either case $r_{13}(t) = \infty$ implies $r^*_{31}(t) = \infty$ by Theorem 4.7.

We now state a corollary summarizing the results of Corollaries 4.8 and 4.9.

Corollary 4.10 If either hypothesis (a) or (b) of conditions (4.3) holds, then $r^*_{13}(t) = r^*_{31}(t) = \infty$.

We now prove a theorem yielding sufficient conditions to guarantee that $r_{22}(t) = r^*_{22}(t)$.

Theorem 4.11 Suppose that either hypothesis (a) or (b) of conditions (4.3) holds. Then $r_{22}(t) = r^*_{22}(t)$.

Proof. It suffices to prove that if there exists a solution $v(t) \not\equiv 0$ of (L^*) having double zeros at $t = \alpha,\beta$, then there exists a solution $z(t)$ of (L) having double zeros at $t = \alpha,\beta$ and conversely. We prove this in one direction only since the proof of the converse is so similar.

Suppose there exists a solution $v(t) \not\equiv 0$ of (L^*) such that $v(\alpha) = v'(\alpha) = v(\beta) = v'(\beta) = 0$ where $a \leq \alpha < \beta$. Since $r^*_{13}(t) = r^*_{31}(t) = \infty$ we must have $v''(\alpha) \neq 0$ and $v''(\beta) \neq 0$.

We define two more solutions $u(t)$ and $w(t)$ of (L^*) by:

$$u^{(i)}(\alpha) = 0 \text{ for } i = 0,1,2$$

$$w^{(i)}(\beta) = 0 \text{ for } i = 0,1,2$$

and since $r^*_{13}(t) = r^*_{31}(t) = \infty$, we see $u(\beta) \neq 0$ and $w(\alpha) \neq 0$. We claim u, v, and w are independent. For, if not, then there exist c_1, c_2, and c_3, $c_1^2 + c_2^2 + c_3^2 > 0$, such that

$$c_1u(t) + c_2v(t) + c_3w(t) \equiv 0 \tag{4.7}$$

Evaluating (4.7) at $t = \alpha$ yields $c_3w(\alpha) = 0$ which implies $c_3 = 0$. Evaluating (4.7) at $t = \beta$ yields $c_1u(\beta) = 0$ which implies $c_1 = 0$. Hence we see that $c_2v(t) \equiv 0$ and $c_2 \neq 0$ which implies $v(t) \equiv 0$. This contradiction shows the linear independence of u, v, and w.

Now recall Corollary 1.9. Using this corollary let us define the following solution z(t) of (L):

$$z(t) = \begin{vmatrix} u(t) & v(t) & w(t) \\ u'(t) & v'(t) & w'(t) \\ u''(t) & v''(t) & w''(t) \end{vmatrix} .$$

We note that

$$z(\alpha) = \begin{vmatrix} 0 & 0 & w(\alpha) \\ 0 & 0 & w'(\alpha) \\ 0 & v''(\alpha) & w''(\alpha) \end{vmatrix} = 0,$$

$$z'(\alpha) = \begin{vmatrix} 0 & 0 & w(\alpha) \\ 0 & 0 & w'(\alpha) \\ u'''(\alpha) & v'''(\alpha) & w'''(\alpha) \end{vmatrix} = 0,$$

$$z''(\alpha) = \begin{vmatrix} 0 & 0 & w(\alpha) \\ 0 & v''(\alpha) & w''(\alpha) \\ u'''(\alpha) & v'''(\alpha) & w'''(\alpha) \end{vmatrix} = -u'''(\alpha)\cdot v''(\alpha)\cdot w(\alpha).$$

Since $u(t) \not\equiv 0$, $u'''(\alpha) \neq 0$. Also $v''(\alpha) \neq 0$ from above, as well as $w(\alpha) \neq 0$. Hence $z''(\alpha) \neq 0$ and we see $z(t) \not\equiv 0$. Similarly it is shown that $z(\beta) = z'(\beta) = 0$, $z''(\beta) \neq 0$. Hence $z(t) \not\equiv 0$ is a solution of (L) having double zeros at $t = \alpha, \beta$ and we are through. The converse follows in a similar manner.

Theorem 4.12

(a) If for equation (L) we have $r_{13}(t) = r_{31}(t) = r_{22}(t) = \infty$ for $t \in [a,\infty)$, then $r_{1111}(t) = r_{211}(t) = r_{121}(t) = r_{112}(t) = \infty$ for $t \in [a,\infty)$.

(b) If for equation (L^*) we have $r^*_{13}(t) = r^*_{31}(t) = r^*_{22}(t) = \infty$ for $t \in [a,\infty)$, then $r^*_{1111}(t) = r^*_{211}(t) = r^*_{112}(t) = r^*_{121}(t) = \infty$ for $t \in [a,\infty)$.

Proof The same proof holds for (L) and (L^*) and we prove it only for (L).

(i) Suppose $r_{211}(\bar{t}) = \gamma < \infty$ for some $\bar{t}$. Then there exists a solution $y(t)$ of (L) and points α, β such that $\bar{t} \leq \alpha < \beta < \gamma$ and $y(\alpha) = y'(\alpha) = y(\beta) = y(\gamma) = 0$. Consider another solution $z(t)$ of (L) such that $z(\alpha) = z'(\alpha) = z''(\alpha) = 0$, $z'''(\alpha) \neq 0$. Since $r_{31}(\alpha) = \infty$, we see $z(t) \neq 0$ for $t \in [\beta,\gamma]$. Hence by Lemma 1.11 there exists a constant K such that the solution $w(t) = y(t) - Kz(t)$ has a double zero at some point $c \in (\beta,\gamma)$. But then $w(\alpha) = w'(\alpha) = w(c) = w'(c) = 0$ contradicting $r_{22}(\alpha) = \infty$.

(ii) Suppose $r_{112}(\overline{t}) = \gamma < \infty$ for some $\overline{t}$. Then there exists a solution $y(t)$ of (L) and points α,β such that $\overline{t} \leq \alpha < \beta < \gamma$ and $y(\alpha) = y(\beta) = y(\gamma) = y'(\gamma) = 0$. Consider another solution $z(t)$ of (L) such that $z(\gamma) = z'(\gamma) = z''(\gamma) = 0$, $z'''(\gamma) \neq 0$. Since $r_{13}(t) = \infty$, for all t we see $z(t) \neq 0$ for $t \in [\alpha,\beta]$. Hence by Lemma 1.11 there exists a constant K such that the solution $w(t) = y(t) - Kz(t)$ has a double zero at some point $c \in (\alpha,\beta)$. But then $w(c) = w'(c) = w(\gamma) = w'(\gamma) = 0$ contradicting $r_{22}(c) = \infty$.

(iii) Suppose $r_{121}(\overline{t}) = \gamma < \infty$ for some $\overline{t}$. Then there exists a solution $y(t)$ of (L) and points α,β such that $\overline{t} \leq \alpha < \beta < \gamma$ and $y(\alpha) = y(\beta) = y'(\beta) = y(\gamma) = 0$. Clearly $r_{22}(\alpha) = \infty$ implies $y'(\alpha) \neq 0$. Consider another solution $z(t)$ of (L) defined by $z(\alpha) = z'(\alpha) = z(\beta) = 0$. By part (i) $r_{211}(\alpha) = \infty$ and we must have $z(t) \neq 0$ for $t \in (\beta,\gamma]$. An easy check shows that Lemma 1.11 still applies since $z(t)$ has a <u>single zero</u> at $t = \beta$, and $y(t)$ has a double zero at this point. Hence there exists a constant K such that the solution $w(t) = y(t) - Kz(t)$ has a double zero at a point $c \in (\beta,\gamma)$. But then $w(\alpha) = w(\beta) = w(c) = w'(c) = 0$ contradicting $r_{112}(\alpha) = \infty$ established in part (ii).

(iv) Suppose $r_{1111}(\overline{t}) = \gamma < \infty$ for some $\overline{t}$. Then there exists a solution $y(t)$ of (L) and points α, β, η such that $\overline{t} \leq \alpha < \beta < \eta < \gamma$ and $y(\alpha) = y(\beta) = y(\eta) = y(\gamma) = 0$. Consider another solution $z(t)$ of (L) defined by $z(\alpha) = z'(\alpha) = z(\beta) = 0$. Since $r_{211}(\alpha) = \infty$ by part (i) we have $z(t) \neq 0$ for $t \in [\eta,\gamma]$. Hence Lemma 1.11 implies there exists a constant K such that the solution $w(t) = y(t) - Kz(t)$ has a double zero at some point $c \in (\eta,\gamma)$. But then $w(\alpha) = w(\beta) = w(c) = w'(c) = 0$ contradicting $r_{112}(\alpha) = \infty$. The proof is now complete.

We now state an immediate corollary.

Corollary 4.13 If $t \in [a,\infty)$ and $\eta_1(t) < \infty$, then $\eta_1(t) = \min \{r_{13}(t), r_{31}(t), r_{22}(t)\}$. Moreover, if $\eta_1^*(t) < \infty$, then $\eta_1^*(t) = \min \{r_{13}^*(t), r_{31}^*(t), r_{22}^*(t)\}$.

We now state an important theorem relating $\eta_1(t)$ and $\eta_1^*(t)$.

Theorem 4.14 Suppose that either hypothesis (a) or (b) of (4.3) holds. Then $\eta_1(t) = \eta_1^*(t)$.

Proof By Theorem 4.11 $r_{22}(t) = r_{22}^*(t)$, and by Theorem 4.7 we have $r_{13}^*(t) = r_{31}(t)$ and $r_{31}^*(t) = r_{13}(t)$. The result follows immediately from Corollary 4.13.

The final theorem of this chapter is an extension of Corollary 4.6.

Theorem 4.15 Assume that one of the hypotheses (a) or (b) of (4.3) holds. Let $p(t) \in C^2[a,\infty)$ and $q(t) \in C'[a,\infty)$. If $p''(t) + 2r(t) \leq q'(t)$ on $[a,\infty)$, then both equation (L) and equation (L^*) are disconjugate and hence nonoscillatory.

Proof: By Corollary 4.6 $\eta_1(t) = \infty$, and Theorem 4.14 implies $\eta_1(t) = \eta_1^*(t)$. Hence $\eta_1^*(t) = \infty$ and the result follows.

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