THE MIKUSIŃSKI OPERATIONAL CALCULUS

A Thesis
Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

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by
Don A. Edwards
May, 1964

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ABSTRACT

The Mikusinski Operational Calculus is derived from a commutative ring of continuous functions a, b, ... in which the addition and multiplication (convolution) operations are

$$a + b = \{a(t) + b(t)\}$$

$$ab = \left\{\int_0^t a(t-\tau)b(\tau)d\tau\right\}.$$

By the Theorem of Titchmarsh, this ring has no divisors of zero—if ab = 0, then a = 0 or b = 0. Thus the ring may be extended to a field of convolution quotients a/b, b ≠ 0. Here a/b is of an equivalence class such that

(1) $\frac{a}{b} = \frac{a'}{b'}$ if and only if $ab' = ba'$.

The operations on this equivalence class are defined by

(2) $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

(3) $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

where the right hand member is equivalent in the sense of (1) when a/b is replaced by another member of the class a'/b'. Thus we have the one-to-many correspondence

$$a = ak/k \quad \text{for all } k \neq 0.$$

Originally, there was no multiplicative element, but by the introduction of the Dirac Delta Operator $\delta(t) = \{1\}/\{1\}$, the convolution of δ with any function f gives

$$\delta f = \int_0^t \delta(\tau)f(t-\tau)d\tau = f.$$

The convolution quotient a/b may represent either continuous or discontinuous functions, but it is not necessarily either. The reciprocal of the unit function $\ell = \{1\}$, a continuous function, is the differential operator $s = 1/\{1\}$ which is not a function at all.

By considering polynomials in s as rational operators, a system of operators similar to the Laplace Transformations is obtained. Consideration of convergence is not necessary. By means of these operators, there is a convenient method not only to evaluate in terms of $\{f(t)\}$ the polynomials in s, but by the same methods, to solve ordinary differential equations with constant coefficients. These methods may be extended to the finding of general solutions, and to the solving of boundary condition problems of two or more points.

Discontinuous functions (of class K) are those functions defined in the interval $0 \leq t < \infty$ such that

i) $\{f(t)\}$ has at most a finite number of discontinuities in every finite interval, and

ii) the Riemannian integral $\int_0^t |f(\tau)|\, d\tau$ has a finite value for all $t \geq 0$.

Therefore if f is of class K, it may be represented as the quotient of two continuous functions ($f = a/\ell$) such that

$$\ell\{f(t)\} = \{1\}\{f(t)\} = \left\{\int_0^t f(\tau)\, d\tau\right\} = a.$$

Thus operations on operators defined previously still hold although the solution of differential equations involving discontinuous functions or those with discontinuous derivatives require that other methods be developed.

A discussion of the jump function (Heaviside's function) gives the last operator, h^{λ}, for the expression of jump and translated functions. Lastly, a method is given for representing a function a with jumps β_ν at the points $t = t_\nu$ as the sum of two functions, a continuous part b, and a jump part such that:

$$a = b + \frac{1}{s}\sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}.$$

We then find that

$$sa = a' + a(0) + \sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}.$$

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CHAPTER 1

OPERATIONAL CALCULUS

General Definition. The Mikusinski Operational Calculus may be defined as a commutative ring of continuous (of class $\mathcal{C}_2$) functions, f(t), in the interval $t \geq 0$, which satisfy the following definitions:

$$(1.1) \qquad a + b = \{a(t)\} + \{b(t)\} = \{a(t) + b(t)\}$$

$$(1.2) \qquad ab = \{a(t)\}\{b(t)\} = \int_0^t a(t-\tau)b(\tau)d\tau.$$

This calculus differentiates between a function, the value of a function at a point, and a constant function. A function f of a variable t, the entire function, will be denoted by $\{f(t)\}$. The symbol f(t) will denote the value of the function f at the point t. The symbol $\{c\}$ will imply a constant function which equals the value c for all values of t. Here the brackets become important since in ordinary arithmetic

$$(2)(3) = 6$$

while in operational calculus

$$\{2\}\{3\} = \int_0^t (2)(3)d\tau = \{6t\}.$$

Commutative Ring. A commutative ring is defined by means of the following six postulates:

(i) $a + b = b + a$ (commutativity of addition)

(ii) $(a+b) + c = a + (b+c)$ (associativity of addition)

(iii) For any pair of elements a,b there is a third element x satisfying the equation $a + x = b$.

(iv) $ab = ba$ (commutativity of multiplication)

(v) $(ab)c = a(bc)$ (associativity of multiplication)

(vi) $a(b+c) = ab + ac$ (distributivity of multiplication with respect to addition).

With the definitions (1.1) and (1.2), these postulates may be easily verified. Postulates (i) and (ii) are obvious from (1.1). Postulate (iii) says that for every pair of elements, a,b there exists a unique element x which is the difference of a and b and is written b-a. This postulate is sometimes called the feasibility of subtraction.

Verification of Postulate (iv):

With the change of variables $\sigma = t-\tau$,

$$\int_0^t a(t-\tau)b(\tau)d\tau = \int_t^0 a(\sigma)b(t-\sigma)(-d\sigma)$$

$$= \int_0^t a(\sigma)b(t-\sigma)d\sigma$$

which is the same as the original convolution.

Verification of Postulate (v):

If

$$f(t) = \int_0^t a(x)b(t-x)dx; \quad g(t) = \int_0^t b(u)c(t-u)du$$

then (v) states $fc = ag$ and

$$fc(t) = \int_0^t f(y)c(t-y)dy = \int_0^t\int_0^y a(x)b(y-x)c(t-y)dxdy$$

is the double integral over the triangle $0 \leq x \leq y \leq t$,

$$ag(t) = \int_0^t a(t-v)g(v)dv = \int_0^t\int_0^v a(t-v)b(u)c(v-u)dudv$$

is over the triangle $0 \leq u \leq v \leq t$.

With the substitution $x = t-v$, $y-x = u$, $(y = t+u-v$, $t-y = v-u)$ whose Jacobian $\partial(x,y)/\partial(u,v) = 1$, the first integral is converted into the second.

Verification of Postulate (vi):

Given:

$$\int_0^t a(t-\tau)\left[b(\tau) + c(\tau)\right] d\tau$$

$$= \int_0^t \left[a(t-\tau)b(\tau) + a(t-\tau)c(\tau)\right] d\tau$$

$$= \int_0^t a(t-\tau)b(\tau)d\tau + \int_0^t a(t-\tau)c(\tau)d\tau.$$

Thus we have verified the postulates of a commutative ring for the operational calculus. With them we have for example:

$$\{\cos^2 t\}\{t\} + \{t\}\{\sin^2 t\}$$

$$= \{t\}\{\cos^2 t\} + \{t\}\{\sin^2 t\} \qquad \text{P(iv)}$$

$$= \{t\}\{\cos^2 t + \sin^2 t\} \qquad \text{P(vi)}$$

$$= \{t\}\{1\} \qquad \text{trigonometry}$$

$$= \left\{\int_0^t \tau\, d\tau\right\} \qquad \text{definition}$$

$$= \{\tfrac{1}{2}t^2\}.$$

<u>The</u> <u>Operator</u> ℓ. For simplicity, let us denote the unit function $\{1\}$ by ℓ. Then by definition (1.1) and iteration, it may be seen:

$$\ell^2 = \{1\}\{1\} = \left\{\int_0^t d\tau\right\} = \{t/1!\}$$

$$\ell^3 = \{1\}\{t\} = \left\{\int_0^t \tau\, d\tau\right\} = \{t^2/2!\}$$

$$\ell^4 = \{1\}\{t^2/2!\} = \{\tfrac{1}{2}\int_0^t \tau^2 d\tau\} = \{t^3/3!\}$$

$$\ell^5 = \{1\}\{t^3/3!\} = \{\tfrac{1}{6}\int_0^t \tau^3 d\tau\} = \{t^4/4!\}$$

etc. such that we obtain the general formula:

$$\ell^n = \left\{ \frac{t^{n-1}}{(n-1)!} \right\}. \tag{1.3}$$

Cauchy's Formula for Integration. Using this notation, let us consider the product of ℓ^n with $\{f(t)\}$. We thus have Cauchy's formula for reducing an n-tuple integral into a simple integral:

$$\underbrace{\int_0^t dt \ldots \int_0^t}_{n} f(t)dt = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)d\tau. \tag{1.4}$$

Quotient Field. We may say a ring has no divisors of zero (where a-a = 0 for any a) if $ab = \{0\}$ implies that either $a = \{0\}$ or $b = \{0\}$.

If a commutative ring has no divisors of zero, it may be extended to a quotient field, i.e., we have a set of fractions (b/a), $a \neq 0$, with the following equivalence relation:

$$(b/a) = (d/c) \text{ if and only if } ad = bc. \tag{1.5}$$

Let us also define

$$a = a/1 = (ak)/k \text{ for any } k \neq 0. \tag{1.6}$$

The latter gives a one-to-many correspondence of the elements of a ring to the elements of a quotient field, namely an equivalence class in the sense of (1.5). In this field, we define addition and multiplication by the equations:

$$\frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac} \tag{1.7}$$

$$\frac{b}{a} \cdot \frac{d}{c} = \frac{bd}{ac}. \tag{1.8}$$

Here a/b denotes the class of all ordered pairs equivalent to a/b. Thus if

$$\frac{a}{b} = \frac{a'}{b'} \quad (ab' = ba'), \quad \frac{c}{d} = \frac{c'}{d'} \quad (cd' = dc')$$

it may be shown that

$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a}{b} + \frac{c}{d}, \quad \frac{b'}{a'} \cdot \frac{d'}{c'} = \frac{b}{a} \cdot \frac{d}{c}.$$

As long as there are no divisors of zero, the elements of the quotient field with the above definitions obviously fulfill Postulates (i)-(vi) for a commutative ring. In fact any set of arbitrary elements which form a commutative ring without any divisors of zero <u>may</u> be extended into a quotient field although only those fields whose elements are constructed as above are quotient fields.

The commutative ring definition by Mikusinski with the operations (1.1) and (1.2) above has no divisors of zero. This theorem was first proved by E. Titchmarsh in 1924. We shall give a more restricted proof for one portion based on the Laplace Transformations and shall then sketch a simplified general proof due to C. Ryll-Nardzewski which was first presented in 1952.

CHAPTER II

THEOREM OF TITCHMARSH

<u>Theorem of Titchmarsh</u>. In the previous chapter, we have considered the transitivity, associativity, and distributivity of the convolution product with respect to addition. A much more important property is given by the following

Theorem. If two functions f and g of class $\mathcal{C}$ are not identically equal to 0, then neither is their convolution identically equal to 0.

This theorem was first presented and proved by E. Titchmarsh in 1924. A rigorous proof based exclusively on the methods of functions of real variables was given by C. Ryll-Nardzewski in 1952.[1] However, in this chapter, we shall not give this general proof. We shall instead present a more restricted proof based on a different method.

First we need to consider several theorems and some important properties of the Laplace Transformation.

<u>Product of Transformations</u>. If F(s) and G(s) are the transformations of two functions f(t) and g(t), which are sectionally continuous (a finite number of discontinuities) in each finite interval $0 \leqslant t \leqslant T$ and are of the order $e^{\alpha t}$ as t approaches ∞, then the transformation of the convolution $f(t)*g(t)$ exists when $s > \alpha$ and is given by F(s)G(s). Then

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[1] Mikusinski, <u>Operational Calculus</u>, pp. 15-22.

the inverse thansformation of the product F(s)G(s) is given by the equation

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = f(t)\star g(t).^{2}$$

<u>Convergence of Improper Integrals</u>. Let us now consider two theorems, written as one. For the proofs, see Brand, <u>Advanced Calculus</u>.[3]

Theorem. If $F(s) = \mathcal{L}\left\{f(t)\right\}$ converges for $s = \alpha$, then the improper integral $\int_0^\infty e^{-st} f(t)dt$ will converge uniformly when $s \geqslant \alpha$, and will define a continuous function there.

<u>Theorem on Moments</u>. In the later discussion we shall make use of the following theorem on moments.

Theorem. If f(t) is continuous in the interval $0 \leqslant t \leqslant a$, and if

$$\int_0^a t^n f(t)dt = 0, \quad n = 0, 1, 2, \ldots$$

then $f(t) \equiv 0$ in the interval.[4]

Proof. The proof is based on the fact that any continuous function f(t) in the interval $a \leqslant t \leqslant b$ can be uniformly approximated by a polynomial P(t) to any desired degree of accuracy, i.e.; by the Approximation Theorem of Weierstrass[5]

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[2]Churchill, <u>Operational Mathematics</u>, p. 37.

[3]Brand, <u>Advanced Calculus</u>, pp. 427-430.

[4]Ford, <u>Differential Equations</u>, p, 104.

[5]Brand, <u>Op. Cit.</u>, p. 529.

$$\left|f(t) - P(t)\right| < \varepsilon, \quad a \geq t \geq 0.$$

Since P(t) is a polynomial in powers of t, we have

$$\int_0^a f(t)P(t)dt = 0$$

from the equation of the hypothesis. Hence

$$\int_0^a f(t)f(t)dt = \int_0^a f(t)\left[f(t) - P(t)\right]dt$$

and thus

$$\left|\int_0^a f(t)\left[f(t) - P(t)\right]dt\right| \leq \int_0^a \left|f(t)\right|\left|f(t) - P(t)\right|dt$$

$$\leq \varepsilon \int_0^a \left|f(t)\right| dt;$$

hence

$$\int_0^a f^2(t)\, dt = 0;$$

Since f(t) is continuous, f(t) = 0 in the interval $0 \leq t \leq a$.

<u>Titchmarsh's</u> <u>Theorem</u> <u>When</u> <u>f = g</u>. When f = g and f is a continuous function of exponential order, the convolution $f*f = 0$ implies f = 0.

Proof: Let

$$F(s) = \mathcal{L}\left\{f(t)\right\}$$

$$F(s)G(s) = \mathcal{L}\left\{f(t)*g(t)\right\}.$$

Then when f = g

$$\mathcal{L}\left\{f(t)*f(t)\right\} = \left\{F(s)\right\}^2.$$

But by the hypothesis

$$\left\{f(t)*f(t)\right\} = \left\{0\right\}$$

and f(t) is of exponential order α so that

$$\left|f(t)\right| < Me^{\alpha t}.$$

Since

$$\mathcal{L}\left\{f(t)*f(t)\right\} = \left\{F(s)\right\}^2, \quad s > \alpha.$$

and therefore

$$\{F(s)\}^2 = \{0\}, \quad s > \alpha$$

hence

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \{0\},$$

and consequently, by Lerch's Theorem*

$$f(t) = \{0\}.$$

Thus if the convolution of equal functions is zero, then the function itself is zero under the conditions above.

General Proof. C. Ryll-Nardzewski has shown that the above case may be easily generalized for any arbitrary functions f and g. By hypothesis, we are given that the convolution fg = 0, that is

$$\int_0^t f(t-\tau)g(\tau)d\tau = 0, \quad 0 \leq t < \infty.$$

We also have

$$t\int_0^t f(t-\tau)g(\tau)d\tau = 0$$

and by adding and subtracting

$$\int_0^t f(t-\tau)\tau g(\tau)d\tau$$

and then combining the terms, we obtain

$$\int_0^t (t-\tau)f(t-\tau)g(\tau)d\tau + \int_0^t f(t-\tau)\tau g(\tau)d\tau = 0.$$

Introducing the notation

$$f_1(t) = tf(t); \; g_1(t) = tg(t), \; 0 \leq t < \infty,$$

we may write the last equality as

$$\int_0^t f_1(t-\tau)g(\tau)d\tau + \int_0^t g(t-\tau)g_1(\tau)d\tau = 0$$

or in convolution notation

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*Ford, Op. Cit., pp. 103-105.

$$f_1 g + fg_1 = 0.$$

Now multiplying by fg_1 we also have

$$fg_1(f_1 g + fg_1) = 0.$$

Since the convolution is distributive with respect to addition, commutative and associative,

$$fg \cdot f_1 g_1 + (fg_1)^2 = 0.$$

Since $fg = 0$ by hypothesis, $(fg_1)^2 = 0$ or $fg_1 = ftg = 0$; that is

$$\int_0^t f(t-\tau)\tau g(\tau)d\tau = 0, \quad 0 \leq t < \infty.$$

By repeating this process, we have

$$\int_0^t f(t-\tau)\tau^2 g(\tau)d\tau = 0,$$

and in general after n repetitions,

$$\int_0^t f(t-\tau)\tau^n g(\tau)d\tau = 0, \quad 0 \leq t < \infty$$

for every natural n. Now by the Theorem on Moments proved previously

$$f(t-\tau)g(\tau) = 0.$$

If $g(\tau) = 0$, there is nothing to prove. If $g(\tau_0) \neq 0$, then $f(t-\tau_0)g(\tau_0) = 0$ which implies $f(t-\tau_0) = 0$ when $t \geq \tau_0$ or $t-\tau_0 \geq 0$. Then putting $\tau = t-\tau_0$ we have $f(\tau) = 0$ when $\tau \geq 0$.

We have thus proved that if a convolution fg is identically equal to zero, then at least one of the functions f or g is identically equal to zero. This is merely another way of stating Titchmarsh's Theorem as given at the start of the chapter.

CHAPTER III

OPERATORS

<u>Ordered Pairs</u>. We have seen that in the operational calculus, just as in algebra, fractions of the type

$$a/b$$

can be introduced. This is not to be considered as in ordinary division, but as an inverse operation to convolution.

The construction is done by means of ordered pairs (a,b) of continuous functions where it is to be assumed that $b \neq \{0\}$; this ordered pair will correspond to the solution of a = bc. We now make the following definition:

> (a,b) and (c,d) are equivalent if and only if ad = bc; a/b is the class of all ordered pairs equivalent to (a,b). Each class is called an equivalence class or convolution quotient and each ordered pair corresponding to a/b is a representative of a/b.[6]

The relation defined above is a true equivalence relation--it is reflexive since (a,b) is equivalent to (a,b); symmetric because if (a,b) is equivalent to (c,d) then (c,d) is equivalent to (a,b); and transitive since if (a,b) is equivalent to (c,d) which is equivalent to (e,f), then (a,b) is equivalent to (e,f).

We also have that for any non-zero integer, y,

$$(a,b) = (ay,by).$$

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[6] Erdelyi, <u>Operational Calculus</u>, p. 21.

The fractions of form (ay,y) are isomorphic to the integers, a, such that we have the many-to-one correspondence

$$(ay,y) \leftrightarrow a.$$

Also, the unicity of c in the equation a = bc is insured by the Theorem of Titchmarsh proved in the previous chapter.

For example, let us verify:

$$\{t^3 - 6t\}/\{t - 1\} = \{6t + 6\}$$

$$\begin{aligned}\{t^3 - 6t\} &= 6\{t - 1\}\{t + 1\} \\ &= 6\{t\}\{t\} - 6\{1\}\{1\} \\ &= 6\int_0^t (t-\tau)\tau d\tau - 6\int_0^t d\tau \\ &= 6\left[t\frac{\tau^2}{2} - \frac{\tau^3}{3} - \tau\right]_0^t \\ &= \{t^3 - 6t\}.\end{aligned}$$

<u>Inverse to Convolution</u>. It may occur that for given functions a and b $\neq \{0\}$, that there exists no functions c satisfying a = bc. Take for example a = b = $\{1\}$. But there exists no c = $\{c(t)\}$ such that

$$1 = \int_0^t c(\tau)d\tau$$

for any $t \geq 0$, which is not true even if t = 0.

<u>Operators</u>. The non-performability of the inverse operation leads to a new concept, one of operators. The fraction $\{1\}/\{1\}$ represents an operator. No longer is it a function. In this paper however, we shall admit all fractions a/b as operators whether or not a $\{c(t)\}$ exists. In other words, all

functions, c, defined above are operators, but not all operators are functions.

For operators, we adopt the same definitions as for a quotient field, (1.5) through (1.8). Due to the analogy between operators and fractions in arithmetic, operations on operators are performed identically as those on ordinary fractions.

Numerical Operators. We shall now consider operators of the form

$$[a] = \{a\}/\{1\} \tag{3.1}$$

where $\{a\}$ is any arbitrary constant function with a value of a everywhere. Then the formulas:

$$[a] + [b] = [a + b] \tag{3.2}$$

$$[a][b] = [ab] \tag{3.3}$$

may be easily verified.

For (3.2):

$$[a] + [b] = \frac{\{a\}}{\ell} + \frac{\{b\}}{\ell} = \frac{\{a\} + \{b\}}{\ell}$$

$$= \frac{\{a + b\}}{\ell} = [a + b];$$

and for (3.3):

$$[a][b] = \frac{\{a\}}{\ell}\frac{\{b\}}{\ell} = \frac{\{abt\}}{\ell^2}$$

$$= \frac{\ell\{ab\}}{\ell^2} = \frac{\{ab\}}{\ell} = [ab].$$

It may be seen by these formulas that the braces are now superfluous. Operators of type $[a]$ are termed numerical operators in contrast to constant function operators of type $\{a\}$.

<u>Products</u> <u>and</u> <u>Sums</u>. We also have that for any numerical operator a (henceforth to be called simply numbers due to the operational analogy) and an arbitrary constant function $\{b\}$:

$$a\{b\} = \{ab\}$$

or in general

$$a\{f(t)\} = \{a \cdot f(t)\}. \tag{3.4}$$

Compare, for example, the formulas

$$(2)(3) = 6; \quad 2\{3\} = \{6\}; \quad \{2\}\{3\} = \{6t\}.$$

There exists no formula with respect to addition however, analogous to (3.4). The sum of the number a and the function $\{f(t)\}$ can be written only in the form $a + \{f(t)\}$. In consideration of commutativity and associativity, the following is very important:

$$(a + \{f(t)\})(b + \{g(t)\}) =$$
$$ab + \left\{ bf(t) + ag(t) + \int_0^t f(t-\tau)g(\tau)d\tau \right\}.$$

For example, verify:

$$\begin{aligned}
&(1 + \{t\})(1 - \{\sin t\}) = 1 \\
&= 1 + \left\{t - \sin t - \int_0^t (t-\tau)\sin\tau \, d\tau\right\} \\
&= 1 + \left\{t - \sin t - t\int_0^t \sin\tau \, d\tau + \int_0^t \tau\sin\tau \, d\tau\right\} \\
&= 1 + \left\{t - \sin t + t\cos t - t - t\cos t + \int_0^t \cos\tau \, d\tau\right\} \\
&= 1 + \{- \sin t + \sin t\} \\
&= 1 + \{0\} = 1 + 0 = 1.
\end{aligned}$$

<u>Numbers</u> <u>0</u> <u>and</u> <u>1</u>. Using formulas (3.1) through (3.4), it is easily seen by substitution that if c is an arbitrary operator, we may make the following definitions:

$$1c = c;\ 0c = 0;\ c+0 = c, \tag{3.5}$$

and

$$\{0\} = 0. \tag{3.6}$$

<u>Differential</u> <u>Operator</u> <u>s</u>. Operators may be divided by one another. For example, if $g = a/b$ and $h = c/d$, then

$$\frac{g}{h} = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

The fraction $1/h$ is called the inverse of operator h. It may be seen that if h is a function, then $1/h$ cannot be a function. A particularly important operator is the inverse of the integral operator $\ell = \{1\}$ which is denoted by

$$s = 1/\ell$$

such that we have

$$\ell s = s\ell = 1.$$

Now consider the following important

Theorem. If a function $a = \{a(t)\}$ has a derivative $a' = \{a'(t)\}$, continuous for $0 \leq t < \infty$, we have

$$sa = a' + a(0)$$

where $a(0)$ is a number.

Proof.

$$\{a(t)\} = \left\{\int_0^t a'(\tau)d\tau\right\} + \{a(0)\}$$
$$= \ell\{a'(t)\} + \ell a(0)$$

and multiplying both sides by s

$$s\{a(t)\} = s\{\{a'(t)\} + s\{a(0)$$

(3.7) $$sa = a' + a(0).$$

If the function $\{a(t)\}$ equals zero at $t = 0$, then

$$sa = a'$$

such that multiplication of the function by s gives the derivative of the function. Therefore s is called the differential operator. The restriction that $a(0) = 0$ is important since the value of the function at $t = 0$ must otherwise be added.

<u>Powers of s</u>. If a function $a = \{a(t)\}$ has a second derivative $a'' = \{a''(t)\}$, continuous in the interval $0 \leq t < \infty$, then by multiplying (3.7) by s, we obtain

$$s^2 a = sa' + sa(0)$$

and applying (3.7) to a'

$$s^2 a = a'' + a'(0) + sa(0).$$

By induction then, we may obtain the general theorem:

Theorem. If a function $a = \{a(t)\}$ has an n-th derivative of $a^{(n)} = \{a^{(n)}(t)\}$, continuous on the interval $0 \leq t < \infty$, then

(3.8) $$s^n a = a^{(n)} + a^{(n-1)}(0) + sa^{(n-2)}(0) + \dots + s^{n-1}a(0).$$

Equation (3.8) may be written for convenience as

$$a^{(n)} = s^n a - s^{n-1}a(0) - \dots - sa^{(n-2)}(0) - a^{(n-1)}(0).$$

<u>Polynomials in s</u>. A polynomial operator in s

$$a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s^1 + a_0$$

where $a_n, \ldots, a_0$ are arbitrary numbers is very important in this discussion. Operations on these polynomials are performed as in ordinary algebra. We thus have: If two polynomials in s are equal, then their coefficients are respectively equal; i.e.:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0$$

implies the equality $a_n = b_n$; $(n = 0, 1, \ldots, n)$.

Exponential Functions. Applying equation (3.7) to $\{e^{at}\}$ we have that:

$$s\{e^{at}\} = 1 + a\{e^{at}\}$$

or

$$\{e^{at}\} = 1/(s-a). \tag{3.9}$$

Then with the definition of convolution,

$$\frac{1}{(s-a)^2} = \{e^{at}\}^2 = \left\{\int_0^t e^{a(t-\tau)} e^{a\tau} d\tau\right\} = \left\{e^{at}\int_0^t d\tau\right\} = \left\{\frac{t}{1!} e^{at}\right\}$$

and

$$\frac{1}{(s-a)^3} = \{e^{at}\}\{te^{at}\} = \left\{\int_0^t e^{a(t-\tau)}\tau e^{a\tau} d\tau\right\}$$

$$= \left\{e^{at}\int_0^t \tau\, d\tau\right\} = \left\{\frac{t^2}{2!} e^{at}\right\}$$

or in general,

$$\frac{1}{(s-a)^n} = \left\{\frac{t^{n-1}}{(n-1)!} e^{at}\right\} \qquad (n = 1, 2, \ldots). \tag{3.10}$$

Trigonometric Functions. Using Euler's equations:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}; \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

we obtain

$$\left\{e^{at}\sin bt\right\} = \frac{1}{2i}\left\{e^{(a+ib)t} - e^{(a-ib)t}\right\}$$

and

$$\left\{e^{at}\cos bt\right\} = \tfrac{1}{2}\left\{e^{(a+ib)t} + e^{(a-ib)t}\right\}$$

Now using (3.9), we may write

$$(3.11)\quad \left\{\frac{1}{b}\, e^{at}\sin bt\right\} = \frac{1}{2ib}\left(\frac{1}{s-a-ib} - \frac{1}{s-a+ib}\right) = \frac{1}{(s-a)^2 + b^2}\,.$$

and

$$(3.12)\quad \left\{e^{at}\cos bt\right\} = \frac{1}{2}\left\{\frac{1}{s-a-ib} + \frac{1}{s-a+ib}\right\} = \frac{s-a}{(s-a)^2 + b^2}\,.$$

Powers of the operators of the above two formulas may be calculated by repeated convolution but become very complex. Of more interest is the case where a = 0 such that we obtain:

$$(3.13)\qquad \left\{\sin bt\right\} = b/(s^2 + b^2)$$

$$(3.14)\qquad \left\{\cos bt\right\} = s/(s^2 + b^2).$$

<u>Rational</u> <u>Operators</u>. By the expression rational operator, we mean a fraction of the form

$$(3.15)\qquad \frac{\gamma_m s^m + \ldots + \gamma_1 s + \gamma_0}{\delta_n s^n + \ldots + \delta_1 s + \delta_0} \qquad (m < n)$$

where $\gamma_m, \ldots, \gamma_0; \delta_n, \ldots, \delta_0$ are complex and $\delta_n \neq 0$.

From algebra we know that if $m < n$ and if γ_i and δ_i are real, the above expression may be resolved by the methods of partial fractions and undetermined coefficients into simple fractions of the types:

$$\frac{1}{(s-a)^p};\quad \frac{b}{[(s-a)^2 + b^2]^p};\quad \frac{s}{[(s-a)^2 + b^2]^p}$$

where p = 1, 2, 3, ... The first type is given by (3.10).

To obtain the second type, convolute the function (3.11) with itself; thus

$$(3.16) \qquad \frac{1}{[(s-a)^2 + b^2]^2} = \left\{ \frac{e^{at}}{2b^2}\left(\frac{1}{b} \sin bt - t \cos bt\right)\right\}.$$

Apply s to this formula to obtain the third type using (3.7) and noting that the function vanishes when t = 0; thus

$$(3.17) \qquad \frac{s}{[(s-a)^2 + b^2]^2} = \frac{d}{dt}\left\{\frac{e^{at}}{2b^2}\left(\frac{1}{b} \sin bt - t \cos bt\right)\right\}$$
$$= \frac{e^{at}}{2b^2}\left\{\frac{a+b^2t}{b} \sin bt - at \cos bt\right\}$$

so that we now have a specific example of the third type of the above fractions.

<u>Table of Operators</u>. The use of the above series of equations to resolve rational fractions or to solve differential equations is very similar to the methods of the Laplace Transformations as may be seen in the following table (p. 20) of functions $\{f(t)\}$ and their respective operators, F(s). In all of the cases, the operators F(s) equivalent to the functions $\{f(t)\}$ are also the Laplace Transformations of the functions, i.e.:

$$\mathcal{L}\{f(t)\} = F(s).$$

The equations of convolution theory, however, have been developed without consideration of convergence being necessary. It must be remembered however, that these formulas, although they are general, have been given for f(0) = 0. Otherwise initial conditions must be considered, as discussed in Chapter IV.

TABLE OF OPERATORS

	Function	Operator
1.	$\{e^{at}\}$	$\frac{1}{s-a}$
2.	$\{te^{at}\}$	$\frac{1}{(s-a)^2}$
3.	$\left\{\frac{t^n e^{at}}{n!}\right\}$	$\frac{1}{(s-a)^{n+1}}$
4.	$\{\sin bt\}$	$\frac{b}{(s^2+b^2)}$
5.	$\{\cos bt\}$	$\frac{s}{(s^2+b^2)}$
6.	$\{e^{at} \sin bt\}$	$\frac{b}{[(s-a)^2 + b^2]}$
7.	$\{e^{at} \cos bt\}$	$\frac{s-a}{[(s-a)^2 + b^2]}$
8.	$\{f^{(n)}(t)\}$	$s^n f - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
9.	$\frac{e^{at}}{2b^2}\left\{\frac{1}{b} \sin bt - t \cos bt\right\}$	$\frac{1}{[(s-a)^2 + b^2]^2}$
10.	$\frac{e^{at}}{2b^2}\left\{\frac{a+b^2t}{b} \sin bt - at \cos bt\right\}$	$\frac{s}{[(s-a)^2 + b^2]^2}$

If in the rational operator being considered, the exponent of the numerator equals or exceeds that of the denominator, i.e., $m \geq n$, then the expression (3.13) will be given as the sum of a polynomial in s and a fraction of degree such that $m < n$. Then the methods of partial fractions with undetermined coefficients are acceptable. These methods will be used in the following four examples.

Examples.

Example 1, to evaluate

$$\frac{5s+3}{(s-1)(s \quad + 2s + 5)} = \frac{A}{s-1} + \frac{B(s+1)+C}{(s+1)^2+4}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}$$

$$= \left\{ e^{t} - e^{-t}\cos 2t + \tfrac{3}{2}e^{-t}\sin 2t \right\}.$$

Example 2, to evaluate

$$\frac{2s^6+6s^4+3s^2+5}{s^8+2s^6-2s^2-1} = \frac{2s^6+6s^4+3s^2+5}{(s-1)(s+1)(s^2+1)^3}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{(s^2+1)} + \frac{Es+F}{(s^2+1)^2} + \frac{Gs+H}{(s^2+1)^3}$$

$$= \frac{1}{s-1} - \frac{1}{s+1} - \frac{3}{(s^2+1)^3}$$

$$= \left\{ e^{t} - e^{-t} - \tfrac{3}{8}(3-t^2)\sin t + \tfrac{9}{8}t\cos t \right\}.$$

Example 3, to evaluate

$$\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{s^2+b^2} - \frac{1}{s^2+a^2}\right)$$

$$= \left\{ \frac{1}{a^2-b^2}\left(\cos bt - \cos at\right) \right\}.$$

Example 4, to evaluate

$$\frac{s^3}{s-1} = s^2 + s + 1 + \frac{1}{s-1}$$
$$= s^2 + s + 1 + \{e^t\}.$$

Thus we have seen the ease with which this method will evaluate rational polynomial operators in s, and the similarity of the convolution equations with those of the Laplace Transformations. In the next chapter, we shall discuss the ways in which the convolution operators may be extended to give a general method by which to solve ordinary differential equations with constant coefficients with initial conditions imposed at $t = 0$. In Chapter V, we shall impose conditions at $t_0 \neq 0$.

CHAPTER IV

ORDINARY DIFFERENTIAL EQUATIONS

General Method. Operational calculus now provides a convenient method of solving ordinary linear differential equations with constant coefficients. The previous discussion of polynomials in s and the development of the equations given in the table in the previous chapter prove sufficient for the reduction of homogeneous and non-homogeneous equations to ordinary algebraic ones.

Consider the equation

$$(4.1) \qquad f(x) = a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x$$

where the coefficients a_i are constants and $f(x)$ is an arbitrary function continuous for all $t \geq 0$. We wish a solution of $x(t)$ such that

$$x(0) = \gamma_0;\ x'(0) = \gamma_1;\ \dots\ ;\ x^{(n-1)}(0) = \gamma_{n-1}.$$

In view of equation (3.7), equation (4.1) may be written in the form

$$a_n s^n x + a_{n-1} s^{n-1} x + \dots + a_0 = b_{n-1} s^{n-1} + \dots + b_0 + f(x)$$

where

$$b_k = a_{k+1}\gamma_0 + a_{k+2}\gamma_1 + \dots + a_n \gamma_{n-k-1},\ (k = 0, 1, \dots, n-1).$$

So we now find that

$$x = \frac{b_{n-1} s^{n-1} + \dots + b_0 + f(x)}{a_n s^n + \dots + a_0}$$

Similar methods may be used for a system of n equations in n unknowns:

$$x_1' + a_{11}x_1 + \dots + a_{1n}x_n = f_1(x)$$

$$x_n' + a_{n1}x_1 + \dots + a_{nn}x_n = f_n(x).$$

Assuming that $x_1(0) = \gamma_1, \dots, x_n(0) = \gamma_n$, we may now change the form of this system using #8 of the table, to

$$\begin{array}{ccccc} (a_{11} + s)x_1 + \dots + a_{1n}x_n & = & \gamma_1 + f_1(x) \\ \vdots & & \vdots \\ a_{n1}x_1 + \dots + (a_{nn} + s)x_n & = & \gamma_n + f_n(x) \end{array}$$

and then use matrices, determinants, or some other classical method to solve this system of algebraic equations for all x_i.

Examples. We now give the following three examples.

Example 1, to find $x(t)$ given

$$x'' - x' - 6x = 2, \quad x(0) = 1, \quad x'(0) = 0.$$

$$\{x''(t)\} - \{x'(t)\} - 6\{x(t)\} = \{2\}$$

$$s^2x(t) - sx'(0) - x(0) - sx(t)$$

$$+ x'(0) - 6x(t) = 2/s$$

$$s^2x - sx - 6x = s - 1 + (2/s)$$

$$x(s^2 - s - 6) = (s^2 - s + 2)/s$$

$$x = (s^2 - s + 2)/s(s-3)(s+2)$$

$$= \frac{-1}{3s} + \frac{8}{15}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}$$

$$= \left\{-\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}\right\}.$$

Example 2, to find $x(t)$ given

$$x^{(8)} - 2x^{(6)} - 2x'' - x = 0, \quad x(0) = x''(0) = x^{(4)}(0)$$

$$= x^{(6)}(0) = 0, \; x'(0) = x^{(3)}(0) = 2, \; x^{(5)}(0) = -1, \; x^{(7)}(0) = 11.$$

$$s^8x + 2s^6x - 2s^2x - x = 2s^6 + 6s^4 + 3s^2 + 5$$

$$x = \frac{2s^6 + 6s^4 + 3s^2 + 5}{s^8 + 2s^6 - 2s^2 - 1}$$

which by example 2 in chapter three

$$x = \left\{e^t - e^{-t} - \frac{3}{8}(3 - t)^2 \sin t + \frac{9}{8} t \cos t\right\}.$$

Example 3, to find x(t)

Given $x' - ax - by = be^{at}$

$y' + bx - ay = 0$

where $x(0) = 0$, and $y(0) = 1$.

$$sx - ax - by = b/(s-a)$$

$$sy + bx - ay = 1$$

$$x = \frac{2b}{(s-a)^2 + b^2} = \left\{2e^{at} \sin bt\right\}$$

$$y = \frac{(s-a)^2 - b^2}{(s-a)[(s-a)^2 + b^2]} = \frac{s(s-a)}{(s-a)^2 + b^2} - \frac{1}{s-a} = \left\{2e^{at} \cos bt - e^{at}\right\} = \left\{e^{at}(2 \cos bt - 1)\right\}.$$

We have now shown the methods employed for the solution of both single equations and a simple system of equations.

CHAPTER V

GENERAL SOLUTIONS AND BOUNDARY PROBLEMS

<u>General Solutions</u>. So far we have considered differential equations where we were given initial conditions for $t = 0$, a situation for which the operational calculus is well suited. However, it may also be used in other problems such as finding a general solution or in boundary condition problems involving two points, etc.

First we shall consider the problem of the general solution. We have seen that

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_0 = f$$

may be written as

$$x = \frac{b_{n-1} s^{n-1} + \dots + b_0}{a_n s^n + \dots + a_0} + \frac{1}{a_n s^n + \dots + a_0} f. \tag{5.1}$$

where the values of b_n are dependent on conditions at $t = 0$. If these conditions are not given beforehand, the b's are arbitrary constants and equation (5.1) provides the general solution. It may most easily be reached by decomposition into fractions of the types below where it may be seen that it would not be necessary to find the coefficients A, B, C, D, ... in

$$\frac{A}{(s-a)^p}\,; \qquad \frac{Bs + C}{[(s-a)^2 + b^2]^p}$$

which arise in the decomposition. In this way the calculations are made considerably easier.

The general methods will be shown in the following two examples.

Examples.

Example 1, to find the general solution of the differential equation

$$x^{(4)} - 2x''' + 2x'' - 2x' + x = f.$$

Since the initial conditions are arbitrary, an application of the operator transformations gives

$$s^4 x - 2s^3 x + 2s^2 x - 2sx + x = W + f$$

where $W = b_3 s^3 + b_2 s^2 + b_1 s + b_0$. Therefore we have

$$x = \frac{W}{(s-1)^2(s^2+1)} + \frac{f}{(s-1)^2(s^2+1)};$$

the first part of which may be written

$$\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs + D}{(s^2+1)}$$

considering A, B, C, and D without regard to their connection with W. The coefficients of the second part are uniquely determined to be

$$\frac{1}{(s-1)^2(s^2+1)} = -\frac{1}{2(s-1)} + \frac{1}{2(s-1)^2} + \frac{s}{2(s^2+1)}.$$

So we now have the general solution

$$x(t) = Ae^t + Bte^t + C\sin t + D\cos t$$
$$+ \tfrac{1}{2}\int_0^t f(t-\tau)(-e^\tau + \tau e^\tau + \cos\tau)\,d\tau.$$

Example 2, to find the general solution of the system of equations

$$x' = x - 2y$$
$$y' = x - y + f.$$

We shall now introduce the initial (arbitrary) conditions

$$x(0) = a$$

$$y(0) = b,$$

such that by the transformations, we have

$$sx = x - 2y + a$$

$$sy = x - y + b + f$$

or

$$x = \frac{(a-2b) + as}{s^2+1} - \frac{sf}{s^2+1}$$

$$y = \frac{(a-b) + bs}{s^2+1} + \frac{s-1}{s^2+1} f.$$

Then the general solution is

$$x(t) = (a-2b)\,\sin t + a\,\cos t - 2\int_0^t f(t-\tau)(\sin\tau)d\tau$$

$$y(t) = (a-b)\,\sin t + b\,\cos t + \int_0^t f(t-\tau)(-\sin\tau + \cos\tau)d\tau.$$

Boundary Problems. In a two point boundary problem we are given the values of the required functions, and possibly those of their derivatives, at the two ends of a fixed interval. For instance, let us require a function satisfying the equations

$$x^{(4)} - 2x''' + 2x'' - 2x' + x = \cos 2t \tag{5.2}$$

and

$$x(0) = 1/25,\ x(\pi) = 1/25,\ x'(0) = 2/15,\ x'(\pi) = 2/25.$$

We find a general solution

$$x(t) = Ae^t + Bte^t + C\sin t + D\cos t + \frac{4}{75}\sin 2t + \frac{1}{25}\cos 2t.$$

To determine the coefficients A, B, C, and D, we first need to find $x'(t)$:

$$x'(t) = Ae^{t} + Bte^{t} + Be^{t} + C\cos t$$
$$- D \sin t + \frac{8}{75} \cos 2t - \frac{2}{25} \sin 2t$$

so we may then substitute the values of zero and π to obtain a system of four equations in A, B, C, and D to be solved by any convenient method.

$$x(0) = A + D + 1/25 = 1/25$$
$$x(\pi) = Ae^{\pi} + B\pi e^{\pi} - D + 1/25 = 1/25$$
$$x'(0) = A + B + C + 8/75 = 2/15$$
$$x'(\pi) = Ae^{\pi} + B(\pi e^{\pi} + e^{\pi}) - C + 8/75 = 2/25.$$

Solving this system, we obtain

$$A = B = D = 0, \quad C = 2/75$$

and substituting into (5.2)

$$x(t) = \frac{2}{75} \sin t + \frac{4}{75} \sin 2t + \frac{3}{75} \cos 2t.$$

Similarly, we might have been given values of $x(t)$ for four different values of t. The solution would have been quite similar--differing only in the values assigned for t in obtaining the four equations for determining A, ... , D.

Initial Conditions at $t_0 \neq 0$. The methods just given may of course be used in the solving of problems of this sort but we must take into consideration the necessary translation of coordinates as if the given conditions were for $t_0 = 0$.

Given to find the solution of

$$x^{(4)} - 2x''' + 2x'' - 2x' + x = f(t)$$

such that at $t_o \neq 0$

$$x(t_o) = x'(t_o) = x''(t_o) = x'''(t_o) = 0.$$

First find the solution of the equation

$$x^{(4)} - 2x''' + 2x'' - 2x' + x = \{f(t+t_o)\},$$

such that

$$x(0) = x'(0) = x''(0) = x'''(0) = 0.$$

We then have

$$s^4x - 2s^3x + 2s^2x - 2sx + x = \{f(t + t_o)\},$$

then using partial fractions

$$x = \left(-\frac{1}{2(s-1)} + \frac{1}{2(s-1)} + \frac{2}{2(s\ +1)}\right)\{f(t + t_o)\}$$

and now using the transformations and the definition of multiplication

$$x = \left\{\tfrac{1}{2}\int_0^t f(t + t_o - \tau)(-e^\tau + \tau e^\tau + \cos\tau)d\tau\right\}.$$

Now to obtain the actual desired solution, replace t by $t-t_o$ everywhere such that

$$x(t) = \tfrac{1}{2}\int_0^{t-t_o} f(t-\tau)(-e^\tau + \tau e^\tau + \cos\tau)d\tau.$$

Thus far we have considered only functions of class $\mathcal{C}$, that is functions continuous everywhere on the interval $t \geq 0$. In the next chapter, we shall consider functions which have a finite number of discontinuities in any finite interval.

CHAPTER VI

DISCONTINUOUS FUNCTIONS

<u>Functions of Class K</u>. Thus far we have considered only those functions which were continuous for all $t \geqslant 0$. Now we shall introduce a discussion of some aspects of selected discontinuous functions on the field of operational calculus.

The function $\{f(t)\}$ defined in the interval $0 \leqslant t < \infty$, belongs to class K if and only if:

I) it has at most a finite number of discontinuities in every finite interval,

II) the Riemannian integral $\int_0^t |f(\tau)|\,d\tau$ has a finite value for all $t \geqslant 0$.

One example of a discontinuous function is the square wave function $f(t) = h(t) - 2h(t-a) + 2h(t-2a) - \ldots$ which has amplitude one and period 2a as shown in Figure 1 below. This function fulfills both requirements of the above definition of a discontinuous function. A second example is the function

$$\int_0^t \frac{1}{\sqrt{\tau}}\,d\tau = 2\sqrt{t}$$

shown in Figure 2.

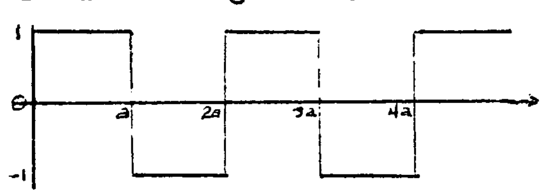

Figure 1

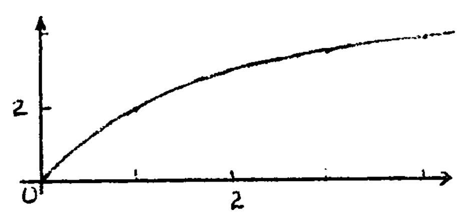

Figure 2

Therefore with the above definition, if $f = \{f(t)\}$ is a function of class K,

$$\ell f = \{1\}\{f(t)\} = \left\{\int_0^t f(\tau)d\tau\right\},$$

where the integral always represents a continuous function such that if we denote the integral by a, we have $\ell f = a$ or

$$f = a/\ell .$$

Thus all functions of class K may now be represented as operators since they may be expressed as the quotient of two continuous functions. Since we may now consider functions of class K as operators, all operations which have been previously defined for operators are applicable to discontinuous functions.

<u>Equality of Functions of Class K</u>. Two functions f and g of class K are defined as equal when the continuous functions given by the integrals

$$a = \left\{\int_0^t f(\tau)d\tau\right\}, \qquad b = \left\{\int_0^t g(\tau)d\tau\right\},$$

are equal, i.e., $\ell f = a = b = \ell g$.

Now $a' = f$ and $b' = g$ at all points where f and g are both continuous; hence when $a = b$, $f = g$ at all points t at which <u>both</u> f and g are continuous and <u>only</u> at those points.

<u>Sums and Products</u>. If f and g are functions of class K, then the sum of f and g is given by the equation

$$f + g = \frac{a}{\ell} + \frac{b}{\ell} = \frac{1}{\ell}\left[\left\{\int_0^t f(\tau)d\tau\right\} + \left\{\int_0^t g(\tau)d\tau\right\}\right]$$

which becomes

$$= \frac{1}{\ell}\left\{\int_0^t \left[f(\tau) + g(\tau)\right] d\tau\right\},$$

and thus

$$\{f(t)\} + \{g(t)\} = \{f(t) + g(t)\}.$$

Similarly we have that

$$\{f(t)\} - \{g(t)\} = \{f(t) - g(t)\}$$

and

$$\alpha\{f(t)\} = \{\alpha f(t)\}.$$

It is obvious then that a sum or a difference of any two functions of class K, as well as the product of a number and a function of class K remains a function of class K. It may also be shown that the convolution of any two functions of class K is still a function of class K.[7]

Using this as a basis, it is easy to deduce the qualities of associativity, commutativity, and distributivity of convolution for functions of class K. The methods are the same as those used before.

Euler's Gamma Function. In the first chapter, we developed the formula

$$\ell^n = \left\{\frac{t^{n-1}}{(n-1)!}\right\} \tag{6.1}$$

for all natural values of n. In this calculus, it is also necessary to consider the non-integer powers of ℓ. We shall now consider Euler's Gamma Function:

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[7]Mikusinski, Operational Calculus, p. 109.

$$\Gamma(\lambda) = \int_0^t t^{\lambda-1} e^{-t}\, dt,$$

which is equal to $(\lambda-1)!$ for all positive integers λ. Here it will be sufficient to consider the integral $\Gamma(\lambda)$ for $\lambda \geq 0$. We will utilize the following necessary properties in the discussion:

(I) $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$,

(II) as a deduction from (I)

$\Gamma(n) = (n-1)!$, n an integer,

(III) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

(IV) $B(\lambda,\mu) = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\,dt$

$= \dfrac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$ where $B(\lambda,\mu)$ is Euler's Beta Function.

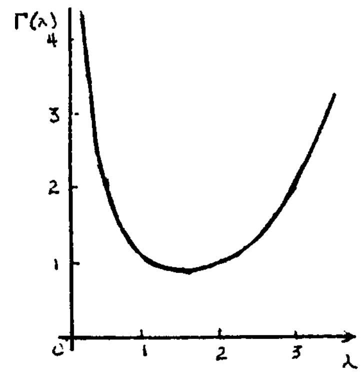


Figure 3

From formula (I) we may evaluate $\Gamma(\lambda)$ for every λ if we know the values assumed in any interval of length one. Property (III) implies the important formula:

$$\int_0^\infty e^{-\sigma^2}\, d\sigma = \sqrt{\pi}/2. \tag{6.2}$$

We have that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t}\, dt$$

and by means of the substitution $t = \sigma^2$

$$= 2\int_0^\infty e^{-\sigma^2}\, d\sigma$$

and formula (6.2) follows.

<u>Non-integer Powers of</u> ℓ <u>and</u> $s-\alpha$. Since formula (6.1) may be written as:

$$\ell^n = \left\{ \frac{t^{n-1}}{\Gamma(n)} \right\},$$

We may generalize it to

(6.3) $$\ell^\lambda = \left\{ \frac{t^{\lambda-1}}{\Gamma(\lambda)} \right\}$$

for all positive λ. With this definition, the property of powers is retained:

(6.4) $$\ell^\eta \cdot \ell^\mu = \ell^{\eta+\mu} \qquad (\eta, \mu > 0).$$

We shall now define, and prove, the formula for a more general operator

(6.5) $$(s-\alpha)^{-\lambda} = \left\{ \frac{t^{\lambda-1}}{\Gamma(\lambda)} e^{\alpha t} \right\}$$

where λ is a positive number and α is arbitrary. If $\alpha = 0$, (6.5) is reduced to (6.3) and if λ is natural, it is then identical to the formula developed in chapter three for operators. With the definition of convolution, we may obtain

$$\begin{aligned}
(s-\alpha)^{-\lambda}(s-\alpha)^{-\mu} &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^t (t-\tau)^{\lambda-1} e^{\alpha(t-\tau)} \tau^{\mu-1} e^{\alpha\tau} \, d\tau \\
&= \frac{e^{\alpha t}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^t (t-\tau)^{\lambda-1} \tau^{\mu-1} \, d\tau \\
&= \frac{e^{\alpha t}\, t^{\lambda+\mu-1}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^1 (1-\sigma)^{\lambda-1} \sigma^{\mu-1} \, d\sigma
\end{aligned}$$

due to making the change of variable

$$\tau = t\sigma, \qquad d\tau = t\, d\sigma,$$

Now the last integral is

$$B(\lambda,\mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$$

so that when $\lambda, \mu > 0$,

(6.6) $$(s-\alpha)^{-\lambda}(s-\alpha)^{-\mu} = \frac{e^{\alpha t}\, t^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)} = (s-\alpha)^{-\lambda-\mu}$$

as in (6.5). When $\alpha = 0$, (6.6) reduces to (6.4).

We may extend this definition to all real λ by writing

(6.7) $$(s-\alpha)^0 = 1 \text{ and } (s-\alpha)^{\lambda} = 1/(s-\alpha)^{-\lambda} \quad (\lambda > 0).$$

If $\alpha = 0$, then (6.7) becomes $\ell^0 = 1$ and $\ell^{-\lambda} = 1/\ell^{\lambda}$ $(\lambda > 0)$. If $\lambda \geq 1$, then the operator $(s-\alpha)^{-\lambda}$ is a continuous function; if $0 \leq \lambda < 1$, then the function is discontinuous at $t = 0$; if $\lambda \leq 0$, it is not a function.

<u>Error Function erf t</u>. From formula (6.5) we have in particular that

$$\frac{1}{\sqrt{s+\alpha}} = \left\{\frac{1}{\sqrt{\pi t}}\, e^{-\alpha t}\right\},$$

such that we obtain by the substitution $\alpha\tau = \sigma^2$; $\alpha\, d\tau = 2\sigma\, d\sigma$

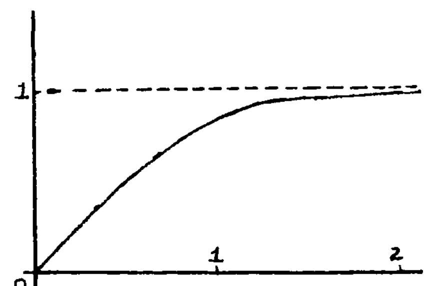
$$\frac{1}{s\sqrt{s+\alpha}} = \left\{\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}}\, e^{-\alpha\tau}\, d\tau\right\} = \left\{\frac{2}{\alpha\sqrt{\pi}} \int_0^{\sqrt{\alpha t}} e^{-\tau^2}\, d\tau\right\} \quad (\alpha > 0).$$

Now introducing the notation

$$\text{erf } t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2}\, d\tau$$

we write

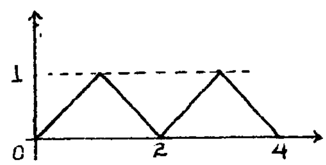
$$\frac{1}{s\sqrt{s+\alpha}} = \left\{\frac{1}{\alpha} \text{ erf } \sqrt{\alpha t}\right\}.$$



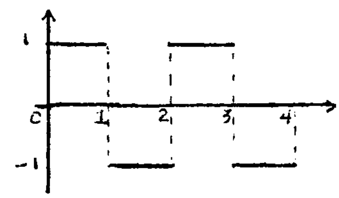
erf t
Figure 4

It may easily be seen that $\{\text{erf } t\}$ is a continuous function increasing from 0 to 1 in the interval $0 \leq t < \infty$. This function occurs in the theory of probability and is called the error function, hence the "erf".

<u>Derivatives of Class K</u>. A function $\{a\}$ will have a derivative of class K if it has a derivative in the interval $0 \leqslant t < \infty$ except for a finite number of points in every finite interval. For example, consider the function shown in the graph in Figure 5. The function is continuous everywhere; it is differentiable for all t except for t when t is an integer, but the derivative is discontinuous at those points as shown in Figure 6.



F(t)
Figure 5



F'(t)
Figure 6

We have proved in chapter three that if a function a has a derivative of class $\mathcal{C}$, then a is also continuous, and that

$$sa = a' + a(0).$$

With the preceding definition of equality, we now may prove the more general

Theorem. If the continuous function a has a derivative a' of class K, then

$$sa = a' + a(0). \tag{6.8}$$

Proof: If a' is of class K, then we have

$$\{a(t)\} = \left\{\int_0^t a'(\tau)d\tau\right\} + \{a(0)\} \tag{6.9}$$

since a' has only a finite number of discontinuities in the interval $0 \leq \tau \leq t$ and the integral, if improper, has a finite value. We may write the equation as

$$a = \ell a' + \{a(0)\}.$$

Multiply by $s = 1/\ell$; then because

$$s\ell a' = a'$$

at all points where a' is continuous, the definition of equality enables us to write

$$sa = a' + \frac{\{a(0)\}}{\ell} = a' + a(0)$$

where $a(0)$ is a number (cf. 3.1).

We notice that if a were discontinuous, equation (6.9), upon which the proof rests, would be false because when a is of class K, the right member is necessarily continuous and could not equal the discontinuous function a on the left.

For example, the discontinuous function

$$a = \begin{Bmatrix} 0 & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < \infty \end{Bmatrix},$$

which has a derivative of class K, $a' = \{0\}$ except when $t = 1$ where a' is undefined. Then we have that $a' = 0$ with our definition of equality and thus equation (6.9) becomes

$$a = \ell\{0\} + \{0\} = \{0\}$$

which is false.

<u>Differential Equations with Discontinuous Right Side</u>. Let us now consider a differential equation of the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_0 x = f$$

where f is any arbitrary discontinuous function. A function x is the solution of the equation if

1) it has n-1 continuous derivatives, and

2) it has the n-th derivative $x^{(n)}$ at all points where f is continuous, and

3) the equation is satisfied at all points where f is a continuous function.

If we give initial or boundary conditions at points at which f is continuous, then the entire theory is applicable.

Examples. Let us now consider two different examples, the first in which the input function has no Laplace transform; the second having an input function with a discontinuous derivative.

Example 1, given:

$$x' - x = f = \left\{(2t - 1)\, e^{t^2}\right\}, \qquad x(0) = 2.$$

We thus have

$$sx - 2 - x = \left\{(2t - 1)\, e^{t^2}\right\}$$

$$(s - 1)x = 2 + \left\{(2t - 1)\, e^{t^2}\right\}$$

$$x = \frac{2}{s-1} + \frac{1}{s-1}\left\{(2t - 1)\, e^{t^2}\right\}$$

$$= \left\{2e^t\right\} + \left\{e^t\right\}\left\{(2t - 1)\, e^{t^2}\right\}.$$

Considering the second part of the right side

$$\left\{e^t\right\}\left\{(2t - 1)\, e^{t^2}\right\} = \left\{\int_0^t e^{(t-\tau)}(2\tau-1)e^{\tau^2}\, d\tau\right\}$$

$$= \left\{e^t \int_0^t e^{\tau^2-\tau}\,(2\tau-1)\, d\tau\right\}$$

$$\{e^t\}\{(2t - 1)\ e^{t^2}\} = \{e^t[e^{\tau^2-\tau}]_0^t\}$$

$$= \{e^t[e^{t^2-t} - 1]\}$$

$$= \{e^{t^2} - e^t\}$$

so that we now have

$$x = \{2e^t\} + \{e^{t^2-t}\}$$

$$x = \{e^{t^2} + e^t\}.$$

For the second example, let us consider an integro-differential equation of a square wave input function:

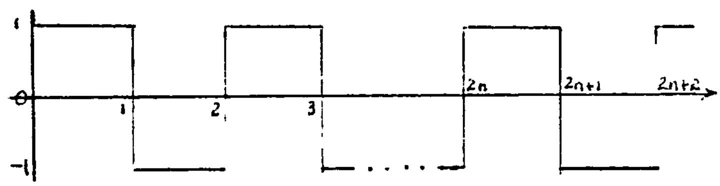


Figure 7

$$x' + 4x + 3\int_0^t x\ dt = f(t), \quad x(0) = 0.$$

Now applying (6.8)

$$sx + 4x + 3\{x = f$$

$$\left(s + 4 + \frac{3}{s}\right)x = f$$

$$x(s+1)(s+3)/s = f$$

$$x = \frac{sf}{(s+1)(s+3)}.$$

Now using the operators developed in chapter three:

$$x = [3/2(s+3) - 1/2(s+1)]\,f$$

$$x = \{\tfrac{3}{2}e^{-3t} - \tfrac{1}{2}e^{-t}\}\{f(t)\},$$

where $f = (-1)^n$ in $n\pi < t < (n+1)\pi$.

When t lies in the interval $n < t < n+1$, x has the value

$$x(t) = \frac{3}{2}\sum_{j=0}^{n-1}(-1)^j \int_0^{j+1} e^{-3(t-\mu)}\,du - \frac{1}{2}\sum_{j=0}^{n-1}(-1)^j \int_0^{j+1} e^{-(t-\mu)}\,du$$

$$+ \frac{3}{2}(-1)^n \int_n^t e^{-3(t-\mu)}\,du - \frac{1}{2}(-1)^n \int_n^t e^{-(t-\mu)}\,du$$

$$= \frac{3}{2} e^{-3t}\sum_{j=0}^{n-1}(-1)^j \frac{(e^3-1)e^{3j}}{3} - \frac{1}{2} e^{-t}\sum_{j=0}^{n-1}(-1)^j \frac{(e-1)e^j}{1}$$

$$+ \tfrac{1}{2}(-1)^n e^{-3t}(e^{3t} - e^{3n}) - \tfrac{1}{2}(-1)^n e^{-t}(e^t - e^n)$$

$$= e^{-3t}(e^3-1)\sum_{j=0}^{n-1}(-1)^j e^{3j} - \tfrac{1}{2}e^{-t}(e-1)\sum_{j=0}^{n-1}(-1)^j e^j$$

$$+ \left(\tfrac{1}{2}(-1)^n\left\{1-e^{-3(t-n)}\right\} - \tfrac{1}{2}(-1)^n\left\{1-e^{-(t-n)}\right\}\right).$$

Now evaluating the above summations

$$\sum_{j=0}^{n-1}(-1)^j e^{3j} = \frac{1-(-1)^n e^{3n}}{1+e^3}$$

$$\sum_{j=0}^{n-1}(-1)^j e^j = \frac{1-(-1)^n e^n}{1+e}$$

Hence when t lies in the interval $n < t < n+1$, x(t) equals

$$x_n = \tfrac{1}{2}e^{-3t}\frac{e^3-1}{e^3+1}\left[1-(-1)^n e^{3n}\right] - \tfrac{1}{2}e^{-t}\frac{e-1}{e+1}\left[1-(-1)^n e^n\right] + \tfrac{1}{2}(-1)^n(e^{-\tau}-e^{-3\tau})$$

where $\tau = t-n$. We see that

$$\text{Tr. } x = \frac{1}{2}\,\frac{e^3-1}{e^3+1}\,e^{-3t} - \frac{1}{2}\,\frac{e-1}{e+1}\,e^{-t}$$

$$\xi(\tau) = \frac{(-1)^n}{2}\left[-\frac{e^3-1}{e^3+1}e^{-3\tau} - \frac{e-1}{e+1}e^{-\tau} + e^{-\tau} - e^{-3\tau}\right]$$

$$= \frac{(-1)^n}{2}\left[\left(1+\frac{e-1}{e+1}\right)e^{-\tau} - \left(\frac{e^3-1}{e^3+1}+1\right)e^{-3\tau}\right]$$

$$= (-1)^n\left[\frac{e}{e+1}e^{-\tau} - \frac{e^3}{e^3+1}e^{-3\tau}\right], \quad \tau = t-n.$$

Thus when we consider x(t) in the form

$$x(t) = \mathrm{Tr}\ x(t) + \xi_n(\tau)$$

where Tr x is the transient state and $\xi_n(\tau)$ is the steady state of the input, then we have

$$\mathrm{Tr}\ x(t) = \tfrac{1}{2}\left(\tanh \tfrac{3}{2}\right)e^{-3t} - \tfrac{1}{2}\left(\tanh \tfrac{1}{2}\right)e^{-t}$$

and

$$\xi_n(\tau) = (-1)^n\left(\frac{e}{e+1}\, e^{-\tau} - \frac{e^3}{e^3+1}\, e^{-3\tau} \right)$$

where $\tau = t-n$. $\xi_n(\tau)$ is a wave of period two and is continuous over the interval $0 \leq t < \infty$. This may be verified by showing

$$\xi_0(0) = \xi_1(1)$$

$$(-1)^0\left(\frac{e\cdot e^0}{e+1} - \frac{e^3 e^0}{e^3+1} \right) = (-1)^1\left(\frac{e\cdot e^{-1}}{(e+1)} - \frac{e^3 e^{-3}}{(e^3+1)} \right)$$

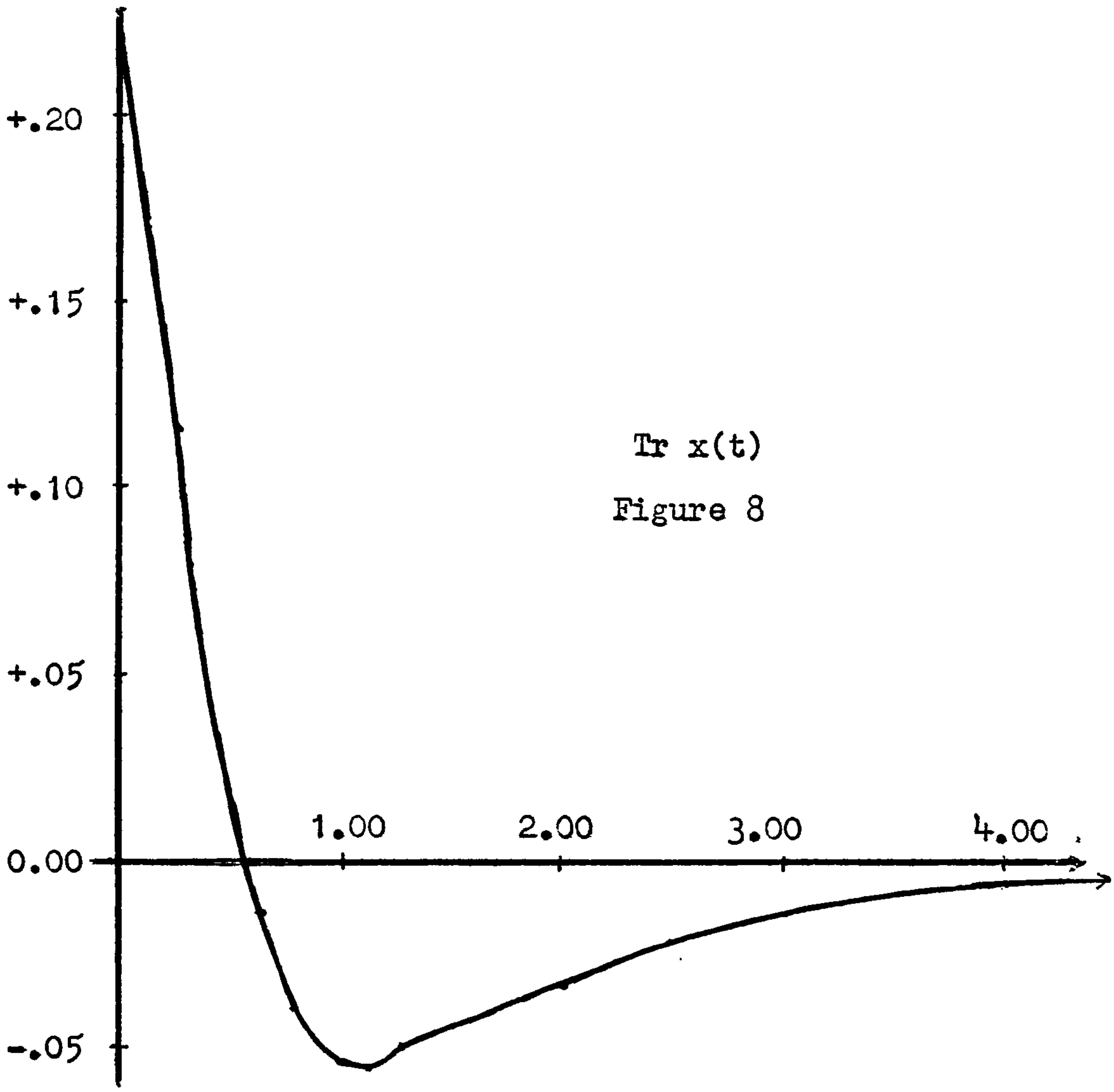
$$\frac{e^4 + e - e^4 - e^3}{(e+1)(e^3+1)} = \frac{(-1)(e^3 + 1 - e - 1)}{(e+1)(e^3+1)}$$

$$e - e^3 = e - e^3.$$

The graphs on the following two pages show the transient state (Figure 8) and the steady state (Figure 9). The minimum point of Tr x(t) is at t = 1.11+ which may be verified by means of the derivative

$$\mathrm{Tr}'\ x = \frac{-1}{4}\left[\frac{9z^2 \cosh^2 \frac{1}{2}z - \cosh^2 \frac{3}{2}z^3}{(\cosh^2 \frac{3}{2}z^3)(\cosh^2 \frac{1}{2}z)}\right]e^{-t}$$

where $z = e^{-t}$, and substituting t = 1.11+, the numerator of the fraction in braces is equal to zero. The value for t is much closer to 1.11 than to 1.12.



Tr x(t)

Figure 8

Table of Values

t	Tr x	t	Tr x
0.00	+.2215	1.10	-.0548
.10	.1904	1.11	-.0553
.25	.1186	1.25	-.0485
.50	.0145	1.50	-.0474
.55	.0000	2.00	-.0320
.60	-.0122	2.50	-.0201
.80	-.0431	3.00	-.0124
1.00	-.0541	4.00	-.0092

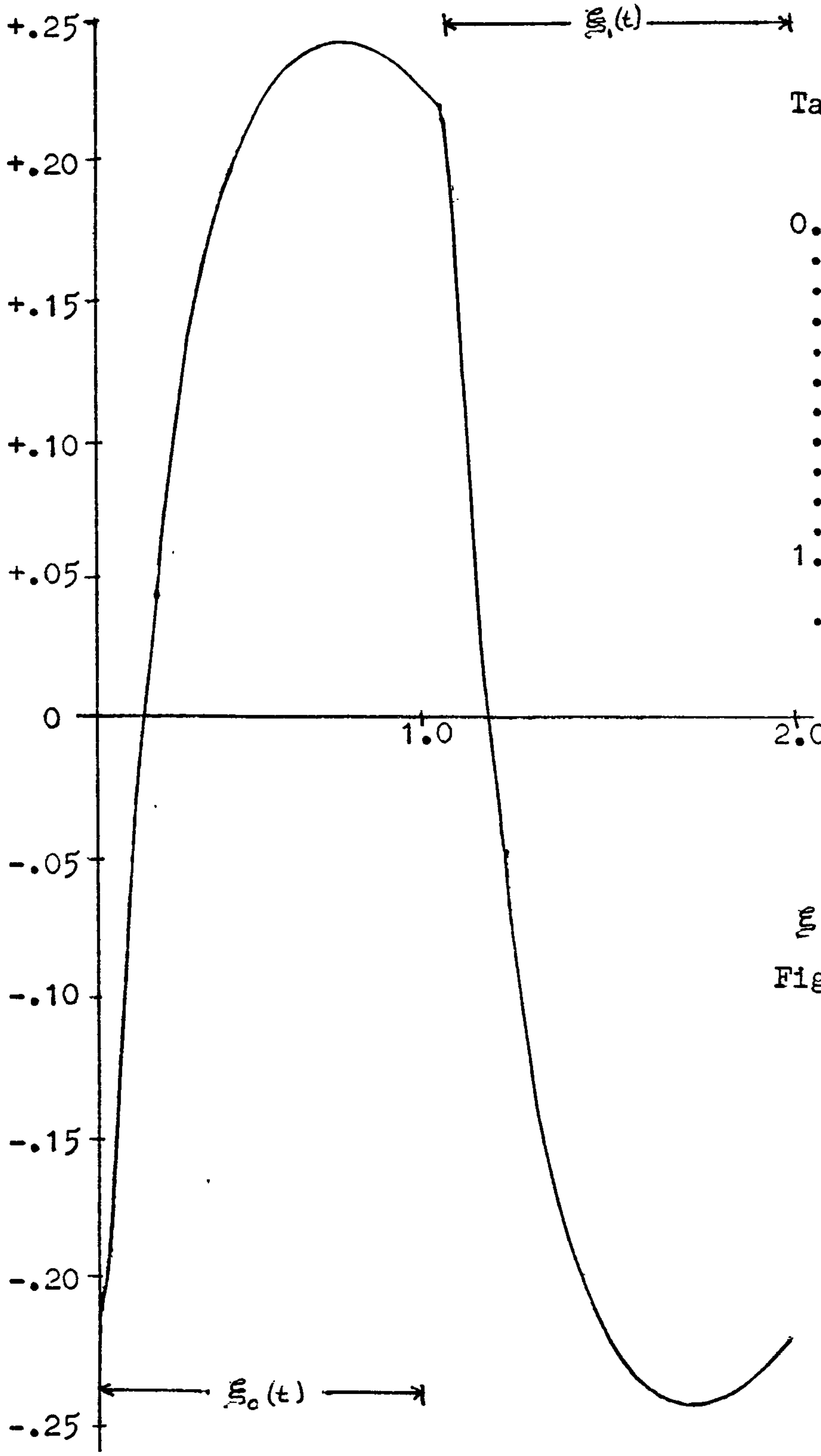


Table of Values

t	$\xi_n(\tau)$
0.00	-.2215
.05	-.1245
.10	-.0442
.20	+.0743
.30	.1542
.40	.2031
.50	.2309
.60	.2418
.70	.2461
.80	.2420
.90	.2332
1.00	.2215
.125	.0096

$\xi_n(\tau)$

Figure 9

Jump Function and Translation Operator. Let us denote by $\{H_\lambda(t)\}$ a function which is equal to zero in the interval $0 \leq t < \lambda$ and is equal to one for all $t \geq \lambda$. The function has only the one jump at $t = \lambda$ and is continuous thereafter. This function is called the Jump Function or Heaviside's Function. Actually, it is the operator

$$(6.10) \qquad h^\lambda = s\{H_\lambda(t)\}, \qquad (\lambda > 0)$$

associated with it (the translation operator) which is so important to this calculus. This is shown by the following

Theorem. If $f(t)$ is an arbitrary function of class K, then

$$h^\lambda\{f(t)\} = \begin{Bmatrix} 0 & \text{for } 0 \leqslant t < \lambda \\ f(t-\lambda) & \text{for } 0 \leqslant \lambda < t \end{Bmatrix}.$$

Proof. $h^\lambda\{f(t)\} = s[H_\lambda(t) * f(t)]$

$$= s\int_0^t f(\tau)H_\lambda(t-\tau)d\tau$$

where

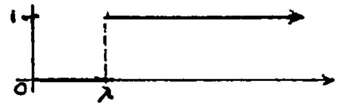
$$H_\lambda(t-\tau) = \begin{Bmatrix} 0 & \text{for } t-\tau < \lambda \\ 1 & \text{for } t-\tau \geq \lambda \end{Bmatrix}$$

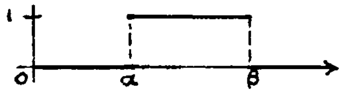
such that

$$h^\lambda\{f(t)\} = s\begin{Bmatrix} 0 & \text{for } 0 \leqslant t \leqslant \lambda \\ \int_0^{t-\lambda} f(\tau)d\tau & \text{for } 0 \leqslant \lambda < t \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & \text{for } 0 \leqslant t \leqslant \lambda \\ f(t-\tau) & \text{for } 0 \leqslant \lambda < t \end{Bmatrix}.$$

Thus we may write:

$\frac{h^\lambda}{s} = H_\lambda(t) \longrightarrow$ 

$\frac{h^\alpha}{s} - \frac{h^\beta}{s} \longrightarrow$ 

As a direct result of this theorem, the multiplication of any given function $\{f(t)\}$ by h^λ yields a translation of the graph a distance λ in the positive direction as seen in Figure 10.

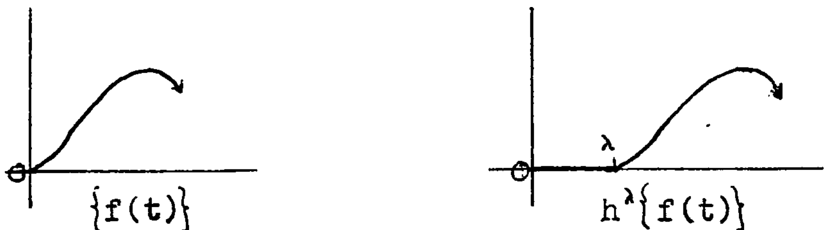


Figure 10

The theorem implies that

$$h^\lambda h^\mu = h^{\lambda+\mu}, \tag{6.11}$$

and

$$h^\lambda h^\mu f = h^{\lambda+\mu} f \quad (\lambda, \mu > 0). \tag{6.12}$$

Therefore we may write the translation operator as a power. Instead of h^1, we shall write simply h. Thus we also have that

$$h^0 = 1 \text{ and } h^{-\lambda} = 1/h^\lambda \quad (\lambda > 0).$$

We should now consider the more general

Theorem. If a function a has jumps $\beta_1, \ldots, \beta_n$ at the points $t_1, \ldots t_n$, is elsewhere continuous and has a derivative of class K, then

$$\text{(6.13)} \qquad sa = a' + a(0) + \sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}$$

where $a(0)$ is the value of the function at the point $t = 0$.

Proof. Let b be the continuous part of the function a, then we have

$$a = b + \frac{1}{s}\sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}.$$

Since b is continuous, from (6.8)

$$sb = b' + b(0).$$

Then

$$sa = sb + \sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}$$

$$= b' + b(0) + \sum_{\nu=1}^{n} \beta_\nu h^{t_\nu}$$

and since $b(0) = a(0)$ and $b' = a'$ by the definition of equality, we thus obtain equation (6.13).

<u>Examples</u>. Example 1, the operator $(h^\alpha - h^\beta)/s$ is equal to $\{H_\alpha(t) - H_\beta(t)\}$ as shown in Figure 11.

Example 2, the operator of the function in Figure 12 is obtained by first considering the slope graph to get $f'(t)$ and then integrating to obtain $f(t)$. The process may be seen as: in considering the slope graph (Figure 13) we obtain

$$f'(t) = 1 - 2h + h^2/s$$

and integrating

$$f(t) = (1 - 2h + h^2)/s^2.$$

Example 3, the function in Figure 14 is derived similarly. $f'(t)$ is shown in Figure 15. The functions are as follows:

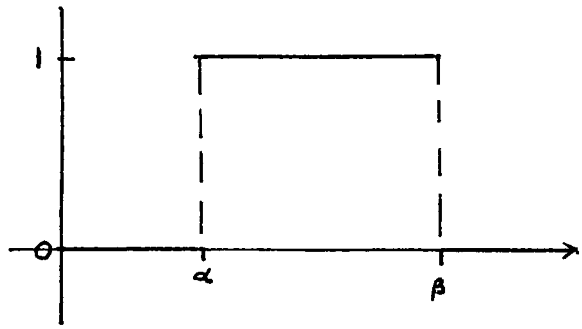


Figure 11

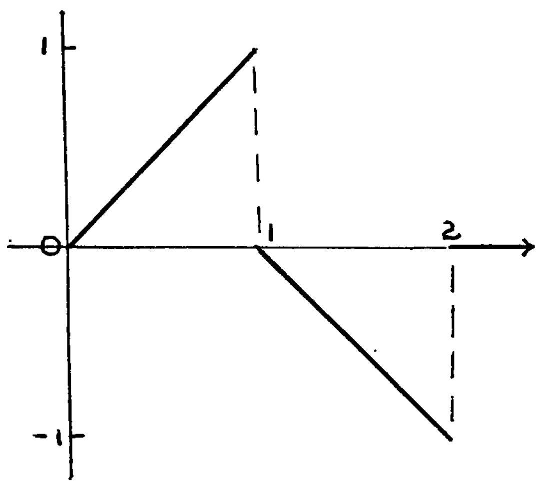


Figure 12

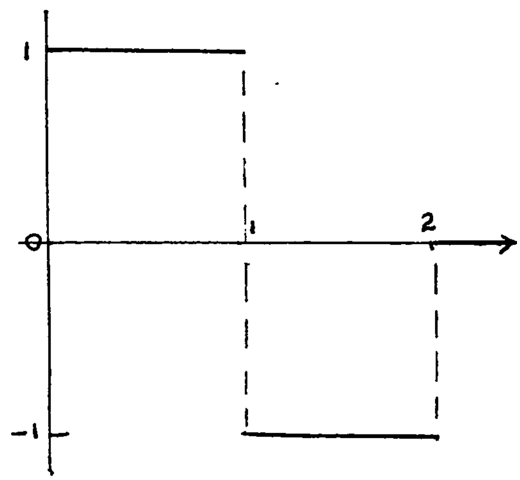


Figure 13

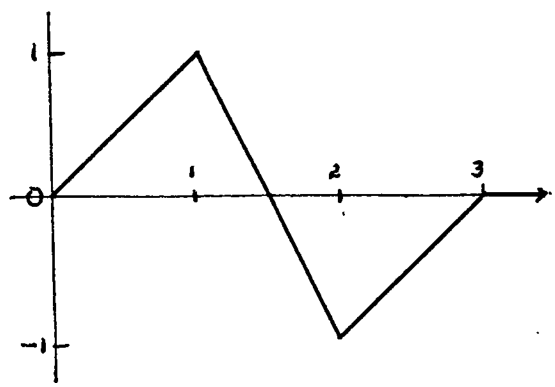


Figure 14

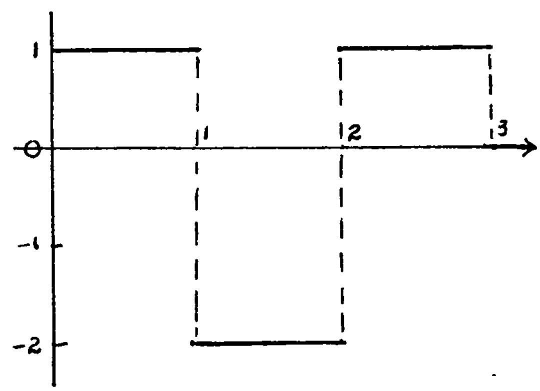


Figure 15

$$f'(t) = 1 - 3h + 3h^2 - h^3/s$$

and

$$f(t) = (1 - 3h + 3h^2 - h^3)/s^2.$$

Example 4, an example of the theorem expressed in (6.13) may be seen in Figures 16-18 showing how a function with multiple jumps may be given by a continuous part and a jump part. Figure 16 shows the original part; Figure 17, the continuous part of the function, and Figure 18, the jump-forming part of the function.

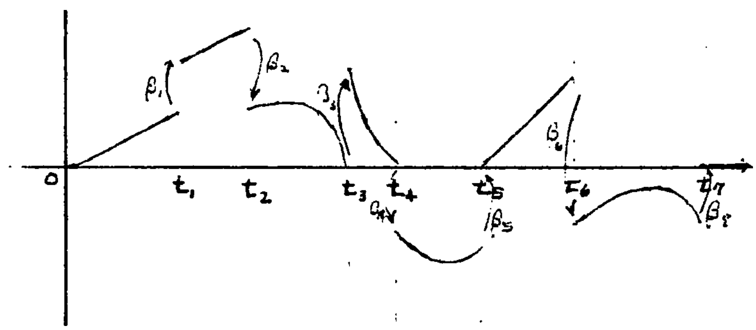


Figure 16

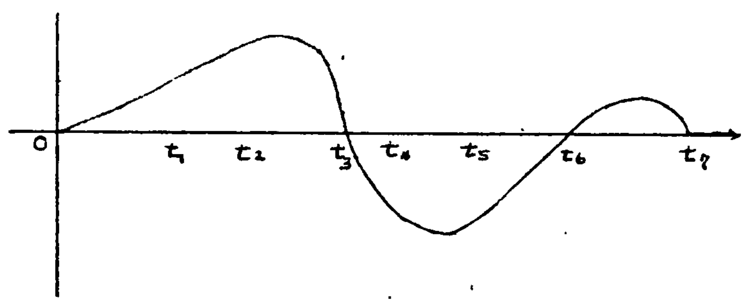


Figure 17

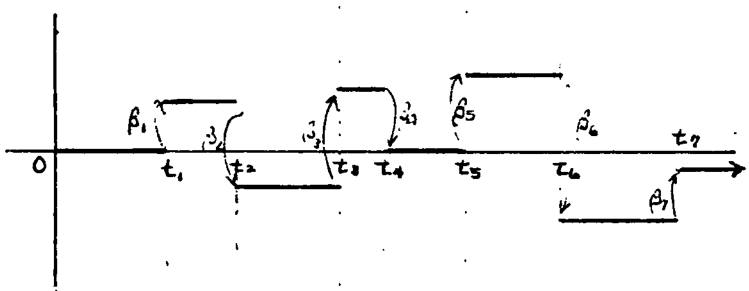
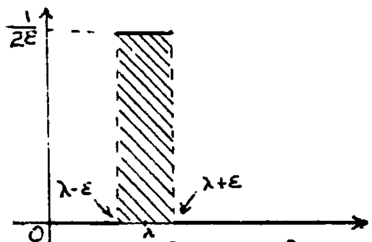


Figure 18

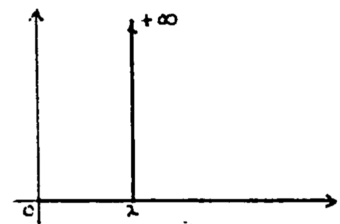
<u>Dirac</u> <u>Delta</u> <u>Operator</u>. This operator is most easily seen

by taking the limit of the function shown in Figure 19 as ε approaches 0. This gives the function shown in Figure 20 which is equal to zero everywhere except at the point $t = \lambda$ at which value it tends to $+\infty$. This operator is called the Dirac Delta Operator. It is not a true jump function as it is sometimes considered to be.



$$\frac{1}{s}\cdot\frac{h^{\lambda-\varepsilon} - h^{\lambda+\varepsilon}}{2\varepsilon}$$

Figure 19



h^{λ}

Figure 20.

However, in this work, we need only consider the operator in the form $\delta(t)$ such that

$$\delta(t) = f/f = \{1\}/\ell$$

for any arbitrary continuous function f. This allows us to have an operator (it is not a function) such that for any $\{f\}$:

$$\{\delta\}\{f\} = \left\{\int_0^t \delta(\tau) f(t-\tau)\,d\tau\right\} = \{f\}.$$

The Dirac Operator is therefore the multiplicative operator for operational calculus as the number 1 is for algebra. Thus we may now complete the extention of the commutative ring into a field of convolution quotients.[8]

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[8]Brand, Differential and Difference Equations.

CHAPTER VII

SUMMARY

Definition. The Mikusinski Operational Calculus is derived from a commutative ring of continuous functions, a, b, c, ... in which the operations of addition and multiplication (convolution) are

$$a + b = \{a(t) + b(t)\}$$

$$ab = \left\{\int_0^t a(t-\tau)b(\tau)d\tau\right\}.$$

From Titchmarsh's Theorem, this ring has no divisors of zero, i.e.: if $ab = 0$, then either $a = 0$, or $b = 0$. Therefore the ring may be extended into a field of convolution quotients a/b where $b \neq 0$. Here a/b represents an equivalence class of quotients such that

(7.1) $$\frac{a}{b} = \frac{a'}{b'} \text{ if and only if } ab' = ba'.$$

The operations on these equivalence classes are defined by the relationships

(7.2) $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

(7.3) $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

where the right hand members are equivalent in the sense of (7.1) when a/b is replaced by any other member of the equivalence class, a'/b'. Thus we have that

$$a = ak/k \quad \text{for all } k \neq 0.$$

In the original ring, there was no multiplicative identity element for convolution; that is there was no continuous function x such that the convolution

$$xa = \left\{ \int_0^t x(t-\tau)a(\tau)d\tau \right\} = \{a\}$$

holds for all functions a, for if $a = \{1\}$, the equation is $x\{1\} = \{1\}$ which does not hold when $t = 0$: the left hand member is zero and the right hand member is one.

But in the quotient field, the unit for convolution is f/f where f is any continuous function not equal to zero, and from (7.3)

$$\frac{a}{b} \cdot \frac{f}{f} = \frac{af}{bf} = \frac{a}{b}$$

since convolution is associative and distributive. Thus the multiplicative element is

$$1 = \frac{\{1\}}{\{1\}} = \frac{\{1\}}{\ell}.$$

It may be identified with the Dirac symbol $\delta(t)$ for from the "sifting property" of the Dirac symbol, the convolution of $\delta\{f\}$ gives

(7.4) $$\delta\{f(t)\} = \left\{ \int_0^t \delta(\tau)f(t-\tau)d\tau \right\} = \{f(t)\}.$$

<u>Operators</u>. The convolution quotients a/b may represent continuous functions or discontinuous functions, but they are not necessarily either. In all cases, however, they are operators. An operator may be a function such as $\ell = \{1\}$, but its reciprocal is the differential operator $s = 1/\ell$ which is

not a function. The reciprocal of any function cannot be a function, either continuous or discontinuous.

By considering polynomials in s in terms of a rational operator, a system of operators identical with those of the Laplace Transformations were obtained. Consideration of the convergence was not necessary.

Differential Equations. By means of these operators, there is a convenient method not only to evaluate, in terms of functions of t, the rational polynomials in s, but by the same methods, to solve ordinary differential equations with constant coefficients. These same methods were also extended to the solutions of problems involving the finding of general solutions, or in boundary condition problems involving two or more points.

Discontinuous Functions. Discontinuous functions (of class K) were said to be those functions $\{f(t)\}$ defined in the interval $0 \leq t < \infty$, such that

i) $\{f(t)\}$ has at most a finite number of discontinuities in every finite interval, and

ii) the Riemannian integral $\int_0^t |f(\tau)|\,d\tau$ has a finite value for all $t \geq 0$.

Therefore, if $\{f(t)\}$ were of class K, it could be represented as $f = a/\ell$ where a is a continuous function:

$$\ell\{f(t)\} = \{1\}\{f(t)\} = \left\{\int_0^t f(\tau)\,d\tau\right\} = a.$$

Thus every function of class K may be considered as an operator since it may be expressed as the quotient of two continuous functions. Then although previously defined operations on operators held, the methods for solving differential equations did not, and new methods were developed.

With the consideration of the jump function (Heaviside's function) the last operator, the translation operator h^{λ}, was developed for the expression of jump and translated functions. Lastly, a theorem was given whereby a function with jumps β_{ν} at $t = t_{\nu}$ could be expressed as the sum of two functions, a continuous part b, and a jump part:

$$a = b + \frac{1}{s}\sum_{\nu=1}^{n} \beta_{\nu} h^{t_{\nu}}.$$

We then found that

$$sa = a' + a(0) + \sum_{\nu=1}^{n} \beta_{\nu} h^{t_{\nu}}.$$

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