Bubbles

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Vanitas vanitatum et omnia vanitas: bubbles are emptiness, non-liquid, a tiny cloud shielding a mathematical singularity. Born from chance, a violent and brief life ending in the union with the (nearly) infinite. But a wealth of phenomena spring forth from this nothingness: underwater noise, sonoluminescence, boiling, and many others. Some recent results on a "blinking bubble" micropump and vapor bubbles in sound fields are outlined. The last section describes Leonardo da Vinci's observation of the non-rectlinear ascent of buoyant bubbles and justifies the name *Leonardo's paradox* recently attributed to this phenomenon. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1695308]

I. INTRODUCTION

Vanitas vanitatum, et omnia vanitas or, as the King James Bible has it, "Vanity of vanities, and all is vanity" (Eccles. 1:2), continuing: "What profit hath a man of all his labour which he taketh under the sun? ... There is no remembrance of former things ... and, behold, all is vanity and vexation of spirit."

The connection between this quotation and the subject of this paper is provided by the classical Greek proverb Πομφολυξ ο ανθροπως: *Homo bulla* in Latin, or "Man is a bubble," was popular in antiquity, and resurfaced in the culture of the Renaissance through Erasmus of Rotterdam's best-seller *Adagia* (1500).[1] With his customary wit, Lucian of Samosata (ca. 117–180 AD) explains the meaning of the dictum in this way: "I've thought of a simile to describe the human life as a whole ... . You know the bubbles that rise to the surface below a waterfall—those little pockets of air that combine to produce foam? ... Well, that's what human beings are like. They're more or less inflated pockets of air ... but sooner or later they're bound to go pop."[2] Many writers embroidered on and enlarged the scope of the idea. For example, Arthur Golding (ca. 1536–1605) says: "When man seemeth to bee at his best, he is altogither nothing else but a bubble blowen togither of vanitie."[3] Several other examples are given in Ref. 4.

It is through this association with the ultimate fleetingness of human life that bubbles came to play an important iconographic role in the Western figurative tradition. Figure 1 reproduces a painting by the Italian artist Dosso Dossi (1486–1542) titled *Allegory of Fortune* (ca. 1535), in which the female figure, representing Fortune, unstably sits on a large bubble. The allusion to the changing fortunes of man is reinforced by the goddess's billowing cloak (an allusion to the tempestuous changes that Fortune brings about), the bundle of lottery tickets in the man's hand, and the semi-transparent band around the bubble (perhaps difficult to discern in the reproduction) carrying the signs of the zodiac.[5]

As far as I know, this early instance remained for a while an isolated occurrence of the use of this symbol, which really burst onto the scene only half a century later and in a different cultural context—the Netherlands.[6] In 1574 Cornelis Ketel (1548–1616) painted the portrait of a gentleman and, on its back, surmounted by the Greek proverb mentioned at the beginning, a boy blowing bubbles out of a mussel shell (Fig. 2). This image is a moral exhortation: the bubbles are a metaphor for the frailty of human life and the boy also is, as he will soon age and die. The sitter, though young and (presumably) affluent, should meditate on the finiteness of his life and "while living, learn to die" (*vivus nunc age disce mori*). As attested by Pieter Bruegel (ca. 1520–1569) in his painting *Children's games* (Kunsthistorisches Museum, Vienna), the blowing of soap bubbles in this manner had become a popular pastime for children, who delighted in the varycolored reflections of the mussel shells.[8]

A more striking example is the 1594 print *Allegory of Transience* (Fig. 3) by Hendrik Goltzius (1558–1617). Here, the mussel shell and bubbles are accompanied by other reminders of the fleetingness of human life: the flowers which will soon wither, the grass which will dry into hay, the smoke quickly dissipated by the wind,[9] the skull, a reminder of the boy's own skeleton temporarily hidden under a beautiful flesh cover.[10]

Another early (1603) example is the *Vanitas* by De Gheyn (1565–1629) shown in Fig. 4. Here the two most obvious symbols—a skull surmounted by a bubble (or a glass globe?)—are surrounded by a veritable catalogue of symbols of transience. The arch bears the inscription *Humana vana* ("human things are useless"). The two figures in the upper corners are the Greek philosophers Democritus (ca. 460–370 BC) and Heraclitus (?–460 BC), the former always depicted laughing and the latter crying—both for the same reason, namely the folly of humankind. They point to the bubble, which is also a stand-in for the world.[12] Fleeting smoke issues from the urn on the right. The beauty of the tulip will soon be gone.[13] Money is useless in the other world.[14,15]

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FIG. 1. (Color) Dosso Dossi (1486–1542), *Allegory of Fortune* (about 1530, oil on canvas, 179.1×217.2 cm^2), The J. Paul Getty Museum, Los Angeles.

The images reproduced in Figs. 2–4 are early examples of the so-called *Vanitas* pictorial genre, which encountered a tremendous fortune in Dutch art in the 17th century and beyond. The viewer is invited to contemplate his/her own mortality through the use of a series of symbols of transience which, in addition to those encountered before, often include musical instruments (the fading of sound), time pieces (the passage of time), glass (fragile, like human pleasures), candlesticks with a stump of a candle, dice and cards (games of chance), and others. "One of the most notable characteristics of 17th-century Dutch culture was its relentless addiction to taking everyday things and occurrences and either searching out their inherent deeper meanings or, conversely, using them as vehicles to be loaded with ready-made ideas[16] … Dominating everything is the tendency to moralization, as a rule resulting in the encouragement of virtue and in reminders of transitoriness and death" (Ref. 18, p. 27).

Figure 5 shows a later (ca. 1650), more complex example by David Bailly (1584–1657). In addition to many of the symbols of transience mentioned before, the miniature portrait held by the black figure and the painter's palette allude to the immortality of Art, as opposed to the transience of life, a theme also stressed in another well-known painting by the same artist.[19] An ambiguous master/servant role reversal has also been seen in this work.[20]

There is such a large number of paintings containing references to the *Homo bulla* theme that many more examples could be easily given. As a final one, in Fig. 6 we see a female figure, personifying the world, who turns away from the earthly pursuits signified by the objects strewn at her feet: military prowess, wealth, games, music, knowledge. The figure's eyes look up to heaven and show that she has chosen to devote herself to God. As always, the putto blowing bubbles recalls the brevity of human life and also hints at the childishness of earthly concerns.[21]

FIG. 2. Verso of a portrait by Cornelis Ketel (1548–1616); courtesy of the Rijksmuseum, Amsterdam.

Very many occurrences of the same bubble theme can also be found in emblems—images accompanied by a short, usually moralizing text—which flourished in the 17th and 18th centuries. An example, from the collection *Emblemata*

FIG. 3. *Allegory of Transience* (1594), by Hendrik Goltzius (1558–1617); courtesy of the Boijmans Museum, Rotterdam, The Netherlands.

by Hadrianus Iunius (1511–1575) published in 1565 by the famous Antwerp publisher Plantin, is shown in Fig. 7. The image shows children busily blowing and chasing bubbles. The Latin motto at the top says "it's foolish to desire to embrace everything" (*Cuncta complecti velle, stultum*), and the Italian motto below, from Petrarch (1304–1374), "And everything I embrace and nothing I clasp" (*Et tutto abbraccio et nulla stringo*), followed by a quatrain elaborating the same theme ("... he who handles diverse studies or hunts for dubious honors, seems to me sillier than children"). This example is typical of many others which can be found in analogous collections by Quarles, Alciati, Heinsius, Hohberg, to name but a few.

It is now time to turn to fluid mechanics.

II. THE RADIAL MOTION OF A LIQUID SHELL

Consider a centrally symmetric shell of incompressible liquid extending between $r=R(t)$ and $r=S(t)>R(t)$. Assume that the region $0\leq r<R(t)$, the bubble, is occupied by a mixture of gas and vapor, and that, at $r=S$, the pressure is specified to be P_∞, which may be time dependent. This quantity may be used, for example, to represent the ambient pressure, or the pressure of a sound field with a wavelength much larger than the bubble size.

Let N be the number of space dimensions, which can be left unspecified for the moment. It is easy to derive the following equation of motion for the bubble interface:

$$\frac{1}{N-2}\left[1-\left(\frac{R}{S}\right)^{N-2}\right][R\ddot{R}+(N-1)\dot{R}^2] \\ -\frac{1}{2}\left[1-\left(\frac{R}{S}\right)^{2N-2}\right]\dot{R}^2=\frac{p_B-P_\infty}{\rho}, \tag{1}$$

where dots denote time derivatives and p_B, the pressure on the liquid side of the bubble surface, will equal the pressure of the gas–vapor mixture in the bubble, corrected for the effects of surface tension σ and viscosity μ:

$$p_B=p_{GV}-\frac{(N-1)\sigma}{R}-2(N-1)\mu\frac{\dot{R}}{R}. \tag{2}$$

In the following, the last two terms will be often neglected for simplicity.

By setting $N=2+\epsilon$ and taking the limit $\epsilon\to 0$, one finds the equation for a cylindrical bubble while, for $N=1$, one has the motion of a liquid slug of length $S-R$. When $N>2$, it makes physical sense to take the limit $S\to\infty$ to find

$$R\ddot{R}+\frac{N}{2}\dot{R}^2=\frac{p_B-P_\infty}{\rho}. \tag{3}$$

In particular, with $N=3$, this is just the well-known Rayleigh–Plesset equation describing the radial motion of a spherical bubble in an unbounded liquid.

There are situations in which the bubble internal pressure p_{GV} may approximately be regarded as constant. Consider for example a bubble containing mostly vapor—so that $p_{GV}\simeq p_V$—collapsing due to an increase of the ambient pressure. If the liquid is sufficiently cold that vapor can condense with negligible latent heat effects, its pressure p_V may be

FIG. 4. (Color) Jacques De Gheyn the Elder (1565–1629), *Vanitas* (1603, oil on wood). The Metropolitan Museum of Art, Charles B. Curtis, Marquand, Victor Wilbour Memorial, and Alfred N. Punnett Endowment Funds, 1974. (1974.1) Photograph © 1984 The Metropolitan Museum of Art.

assumed to remain equal to the saturation pressure at the liquid temperature. If, conversely, the bubble expands upon a decrease of the ambient pressure, the contribution of the gas to p_{GV} quickly decreases and becomes negligible compared with the vapor pressure. A time-dependent ambient pressure P_∞ may be approximated as a sequence of steps. In these cases, with the neglect of surface tension and viscosity, $p_B - P_\infty$ may be assumed to be (at least piecewise) constant and (3) integrated to find

$$\dot{R}^2 = \frac{2}{N}\,\frac{p_B - P_\infty}{\rho} + \left(\frac{R_i}{R}\right)^N \left[\dot{R}_i^2 - \frac{2}{N}\,\frac{p_B - P_\infty}{\rho}\right], \tag{4}$$

where the subscript i denotes initial values. Of particular interest is the case in which $p_B < P_\infty$, so that the bubble implodes, or collapses. When the collapse has proceeded far enough, with $R_i = 0$ the previous relation gives

$$\dot{R} \simeq -\sqrt{\frac{2}{N}\,\frac{P_\infty - p_B}{\rho}\left(\frac{R_i}{R}\right)^N}, \tag{5}$$

which exhibits one aspect of the mathematical singularity alluded to in the abstract of this paper. In practice, as the collapse proceeds, the singularity is removed by several effects, the most important of which are the increase of the gas pressure, the failure of the vapor to condense at a sufficient rate to keep up with the decrease of the bubble volume, and the possible fragmentation of the bubble. For an adiabatic compression, the gas pressure p_G is proportional to the $R^{-N\gamma}$, where γ is the ratio of the gas specific heats, and

FIG. 5. (Color) *Vanitas* (ca. 1650), by the Dutch painter David Bailly (1584–1657). Gift of Louis V. Keeler, Class of 1911, and Mrs. Keeler. Courtesy of the Herbert F. Johnson Museum of Art, Cornell University.

FIG. 6. (Color) *Allegory of Transience* by Jan van den Hoecke (1611–ca. 1651), and Ambrosius Francken II (died 1632, who added the still life); formerly attributed to Pieter van Mol. Courtesy of the Kunsthandel/Art Gallery Hoogsteder & Hoogsteder, The Hague, The Netherlands.

FIG. 7. An example of the many emblems in which the bubble theme recurs. This is No. 16 from *Emblemata* by Hadrianus Iunius (1511–1575). The Latin motto at the top says ''It's foolish to desire to embrace everything'' (*Cuncta complecti velle, stultum*). Reproduced from Ref. 22, courtesy of Georg Olms Verlag AG.

therefore its rise with decreasing R is abrupt. Before the inward bubble wall motion stops and reverses under the action of this increasing pressure, therefore, the bubble contents have undergone a very significant compressional heating, which is responsible for major effects such as sonoluminescence and sonochemistry.[23–25] The shock wave radiated into the liquid as the bubble wall slows down and, eventually, reverses its motion, strongly contributes to cavitation noise. (Of course, the appearance of a shock wave in the liquid has absolutely no implication on the presence or absence of a shock in the gas.) Both the order of the singularity and the abruptness of the rise of p_G increase with N, which implies increasingly dramatic phenomena as the dimensionality of the space increases.

III. BLINKING BUBBLE PUMP

The device shown in Fig. 8 is a first, rather crude implementation of a surprisingly powerful ''blinking bubble'' micropump. The thin line traversing horizontally the transparent rectangle is a 200-μm-wide groove formed in a plastic plate bonded to an equal one by baking in an oven. Short metal needles attached to either end of the groove permit one to connect the pump to a hydraulic line. The two vertical wires, spaced by 760 μm, are embedded in the plastic in such a way that they are exposed to the saturated sodium chloride–water solution filling the channel. With this arrangement, a current pulse through the wires heats up the liquid and generates a bubble, which condenses and disappears when the current stops. A sequence depicting this process is shown in Fig. 9, in which the time interval between successive images is 100 μs. In this example the current is applied for 200 μs, the rms voltage is 92 V, and the resistance 500 Ω, so that each current pulse carries an energy of approximately 3.4 mJ. The current-carrying wires appear as vertical dark bars. It is apparent here that, since the bubble is nucleated to the left of the place where it ultimately condenses, the liquid contained in the channel between these two positions is pushed to the right every time the bubble ''blinks.''

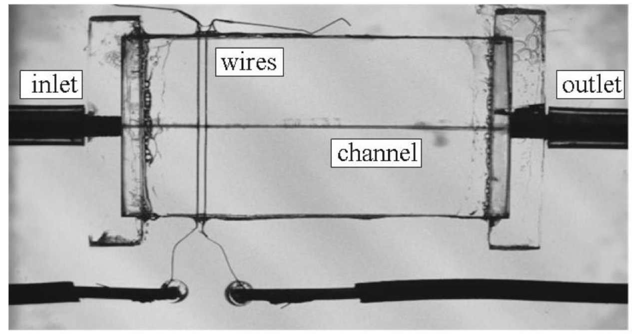


FIG. 8. A crude implementation of the ''blinking bubble'' pump. The groove has a diameter of 200 μm; the plastic plates (dimensions $26\times 12.5\ \mathrm{mm}^2$, thickness 1.59 mm), are bonded together by baking in an oven. The inlet and outlets are 1.19-i.d. metal needles. The two vertical wires (platinum, 100 μm diameter, spacing 760 μm) are baked in the plastic so as to be exposed to the liquid in the channel. By using an electrically conducting liquid (a saturated sodium chloride–water solution), a short pulse of current periodically applied to the wires heats up the liquid and generates a bubble, which condenses and disappears when the current stops.

It is somewhat surprising that this periodic process results in a dc effect. The key to the phenomenon is the asymmetric position of the bubble with respect to the midpoint of the channel. A simple explanation is the following. The bubble divides the liquid occupying the channel into two columns, a shorter one on the left and a longer one on the right. Let us apply to this latter longer column the one-dimensional version $(N=1)$ of Eq. (1):

$$(S-R)\ddot{R}=\frac{p_B-P_o}{\rho}. \tag{6}$$

Since the channel empties into the outlet needle, which has a diameter more than five times as large, we can take S to be the end of the channel and the corresponding pressure P_∞ to be the pressure in the outlet needle P_o. Furthermore, $\dot{R}$ is just the velocity U_ℓ of the longer liquid column. Let us now

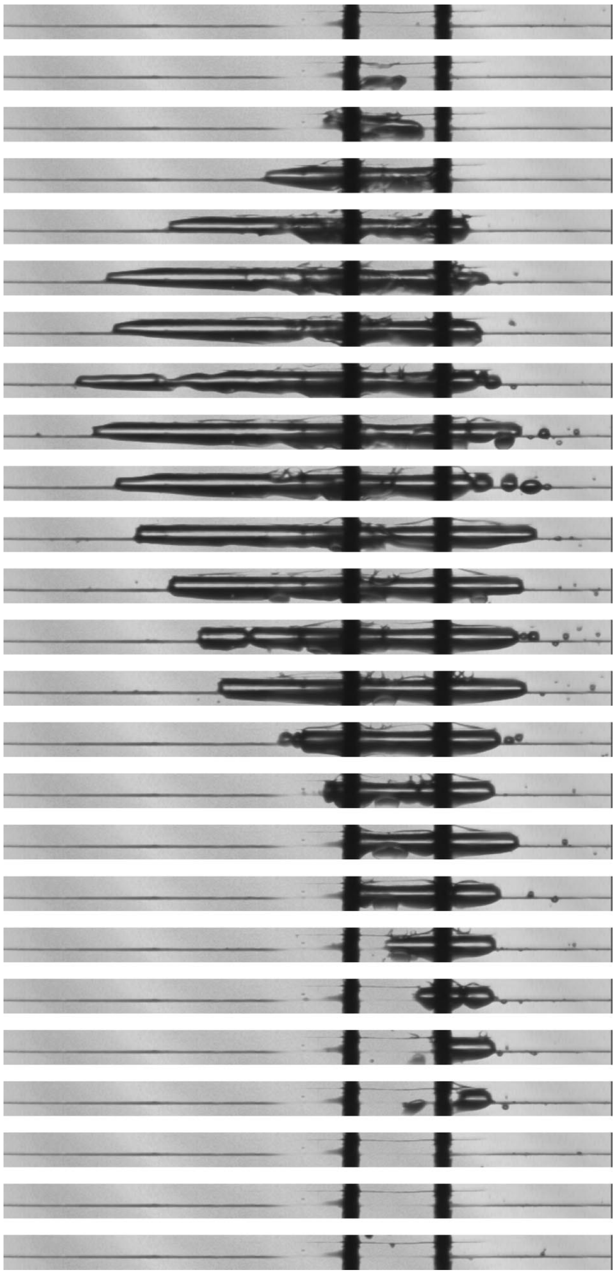


FIG. 9. An example of the bubble evolution in the pump of Fig. 8. The current pulse (92 V rms, 200 kHz ac) is applied for 200 μs with a total energy expenditure of about 3.4 mJ. The two vertical dark bars are the current-carrying wires. Note that the bubble appears near the left wire, while it condenses near the right one. Thus, an amount of liquid approximately equal to the volume between the wires is pushed to the right. The time between successive frames is 100 μs.

assume periodic operating conditions with no net flow in the device. Upon averaging the previous equation over a period we then find

$$\rho\overline{U_\ell^2}=\overline{p_B}-\overline{P_o}, \tag{7}$$

where the overline denotes the time average. A similar argument applied to the shorter liquid column gives

$$\rho\overline{U_s^2}=\overline{p_B}-\overline{P_i}, \tag{8}$$

where P_i is the pressure in the inlet needle and, in view of the low vapor density which causes the pressure in the bubble to be spatially uniform, we may use the same $\overline{p_B}$ in both Eqs. (7) and (8). Upon subtracting (7) from (8), we then find

$$\overline{P_o}-\overline{P_i}=\rho(\overline{U_s^2}-\overline{U_\ell^2}). \tag{9}$$

Since the shorter column has a smaller inertia, its average velocity is larger and therefore the difference in the right-hand side is positive. From this argument we conclude that, in order to maintain the hypothesized periodic conditions with no net flow, the pressure at the pump outlet must exceed that at the inlet. When this pressure excess is insufficient, therefore, a flow will be established in the direction of the longer column.

A numerical simulation of this effect, obtained by solving the Navier–Stokes equations, is described in Ref. 26 and some additional theoretical considerations along the previous line in Refs. 27–29. The measured pressure difference across the pump versus flow rate for a few cases is shown in Fig. 10. It is remarkable that, extrapolated to zero flow rate, the data for the 200 μm channel give a pressure difference of about 25 kPa. It may also be noted that, of the two sets of data for the 127-μm-diam channel (circles and triangles), the bigger pressure difference is obtained when the difference between the lengths of the two liquid columns is greater, i.e., when the asymmetry in the location of the heater is more pronounced.

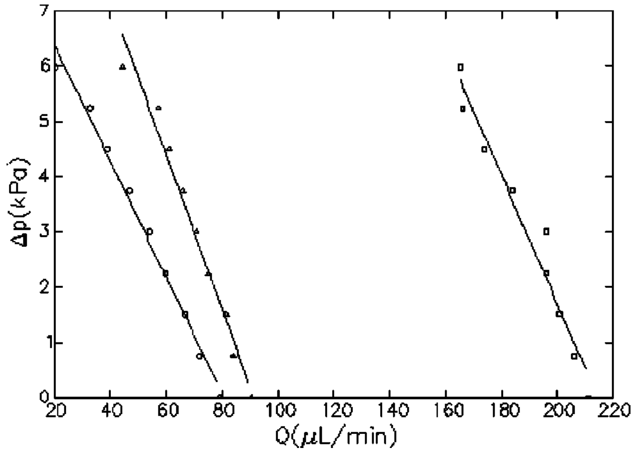
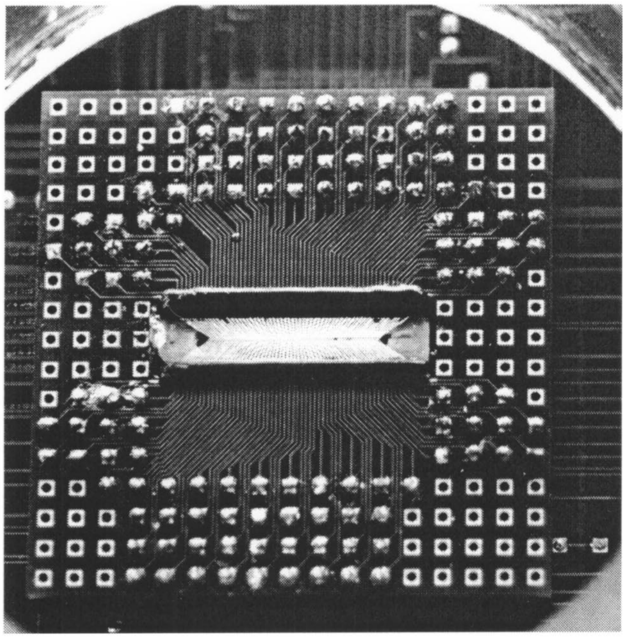
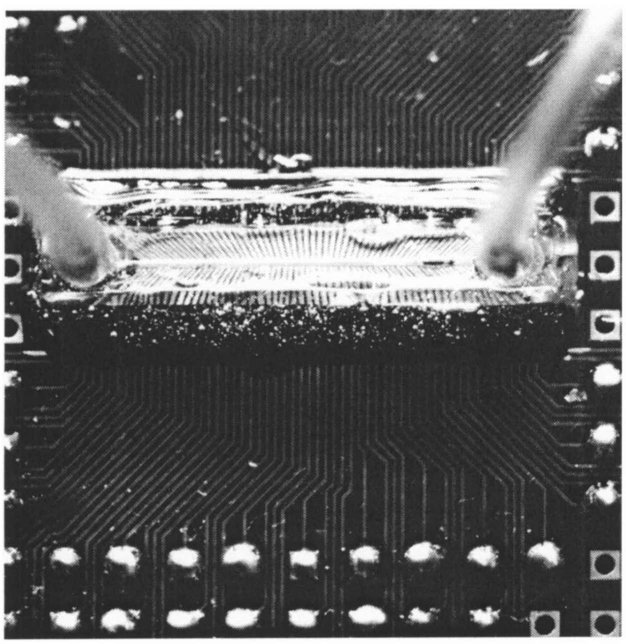


FIG. 10. Pressure difference (kPa) vs flow rate (μl/min) measured on three different pumps similar to the one shown in Fig. 8. (○) Channel diameter 127 μm, lengths of short and long liquid columns 6.60 and 19.8 mm; (△) channel diameter 127 μm, lengths of short and long liquid columns 5.30 and 21.1 mm; (□) channel diameter 203 μm, lengths of short and long liquid columns 6.60 and 19.8 mm. In all cases the bubble generation frequency was 200 Hz.

A more sophisticated implementation of this principle is shown in Fig. 11. The elongated rectangle in the center of the upper image is a silicon chip on which a row of heaters has been vacuum-deposited. The lower image is the complete pump, with a transparent plastic cover in which the channel has been etched, and the inlet and outlet tubing which, in this case, is perpendicular to the channel. The row of heaters permits a study of the dependence of the pumping efficiency on the heater position along the channel.



(a)



(b)

FIG. 11. (a) The elongated rectangle in the center is a silicon chip with a row of vacuum-deposited heaters, each one with its own separate electrical connections. (b) The complete pump; the channel is etched in the transparent plastic cover placed over the heaters and the inlet and outlet tubes are visible at the edges of the image. The use of many independent heaters permits the study of the effect of the heater location along the channel.

IV. BUBBLE OSCILLATIONS IN A SOUND FIELD

The action of an acoustic field on a bubble can be specified by prescribing a time-dependent ambient pressure in the relations of Sec. II, $P_\infty = p_\infty - p_A \sin \omega t$, where p_∞ is the static pressure, p_A the amplitude of the acoustic wave, and $\omega/2\pi$ the sound frequency.

Consider small-amplitude oscillations and set $R = R_e[1 + X(t)]$, where R_e is the equilibrium radius. Upon linearization, (1) gives then

$$\ddot{X} + \frac{N-2}{1-(R_e/S)^{N-2}} \frac{1}{\rho R_e^2}\left[-\frac{\partial p_B}{\partial \dot{X}}\dot{X} - \frac{\partial p_B}{\partial X}X\right] = \frac{N-2}{1-(R_e/S)^{N-2}} \frac{p_A}{\rho R_e^2}\sin \omega t, \tag{10}$$

where we have set

$$p_B = p_\infty + \frac{\partial p_B}{\partial \dot{X}}\dot{X} + \frac{\partial p_B}{\partial X}X; \tag{11}$$

the second term allows for the presence of dissipative effects due, e.g., to heat exchange between the bubble and the liquid. Equation (10) has the standard form of a linear oscillator with damping parameter b and natural frequency $\omega_0/2\pi$ given by

$$\begin{aligned} 2b &= \frac{N-2}{1-(R_e/S)^{N-2}} \frac{1}{\rho R_e^2}\left(-\frac{\partial p_B}{\partial \dot{X}}\right), \\ \omega_0^2 &= \frac{N-2}{1-(R_e/S)^{N-2}} \frac{1}{\rho R_e^2}\left(-\frac{\partial p_B}{\partial X}\right). \end{aligned} \tag{12}$$

In the particular case in which vapor and surface tension effects are negligible and the bubble contents undergo a polytropic process with polytropic index κ, with $1 \leqslant \kappa \leqslant \gamma$, linearization of (2) gives

$$\frac{\partial p_B}{\partial X} = -p_\infty N\kappa, \quad \frac{\partial p_B}{\partial \dot{X}} = -2(N-1)\mu, \tag{13}$$

so that, from (12),

$$\omega_0^2 = \frac{N(N-2)}{1-(R_e/S)^{N-2}} \frac{\kappa p_\infty}{\rho R_e^2}, \tag{14}$$

which, for $N=3$, is

$$\omega_0^2 = \frac{3}{1-R_e/S} \frac{\kappa p_\infty}{\rho R_e^2}, \tag{15}$$

for $N=2$,

$$\omega_0^2 = \frac{2}{\log(S/R_e)} \frac{\kappa p_\infty}{\rho R_e^2}, \tag{16}$$

and, for $N=1$,

$$\omega_0^2 = \frac{\kappa p_\infty}{\rho R_e(S-R_e)}. \tag{17}$$

While, for $S \gg R_e$, (17) and (16) tend to zero, (15) tends to a well-defined limit which, for an air–water system, approximately reduces to the well-known approximate relation $R_e(\omega_0/2\pi) \simeq 3$ kHz$\times$mm. When the liquid film has a thickness much smaller than the bubble radius, (14) reduces to (17), as it should, except for a factor N which accounts for the stronger dependence of the pressure on the volume in a space of higher dimensionality. From (12) and (13) the viscous component of the damping parameter is found to be

$$2b_v = \frac{2(N-1)(N-2)}{1-(R_e/S)^{N-2}} \frac{\mu}{\rho R_e^2}. \tag{18}$$

It is only for very small bubbles that this is the dominant dissipative mechanism; thermal losses are usually more important.

Let us now consider the opposite limit case of a pure vapor bubble. Assume that, during a time of the order of ω^{-1}, the bubble is compressed so that its volume decreases by $\Delta\mathcal{V}$. This volume decrease will tend to cause the condensation of an amount of vapor $\rho_V \Delta\mathcal{V}$ and the consequent release of an amount $L\rho_V\Delta\mathcal{V}$ of latent heat at the bubble wall, with L the latent heat per unit mass. This latent heat will diffuse in a liquid shell of thickness $\sqrt{D/i\omega}$, where D is the liquid thermal diffusivity and the imaginary unit has been introduced to account for the proper phase relationship. The corresponding temperature rise ΔT can be estimated from

$$L\rho_V\Delta\mathcal{V} \simeq \rho c_L \mathcal{A} \sqrt{\frac{D}{i\omega}} \Delta T, \tag{19}$$

where $\mathcal{A}$ is the bubble surface area and c_L is the liquid specific heat. From the Clausius–Clapeyron relation we then have a pressure increase of the order of

$$\Delta p \simeq (1+i) \frac{(L\rho_V)^2}{T\rho c_L} \sqrt{\frac{\omega}{2D}} \frac{\Delta\mathcal{V}}{\mathcal{A}}. \tag{20}$$

But $\Delta\mathcal{V}/\mathcal{A} = N\Delta R = NR_eX$ so that we may write

$$\Delta p \simeq \frac{(L\rho_V)^2}{T\rho c_L} \sqrt{\frac{\omega}{2D}} NR_e \left(X + \frac{\dot{X}}{\omega} \right). \tag{21}$$

By using this result in (12) with, for simplicity of writing, $N=3$ and $S \gg R_e$, we have the following estimate for the thermal damping of vapor bubble oscillations:

$$2b_{\text{th}} \simeq \frac{(L\rho_V)^2}{T\rho^2 c_L R_e \sqrt{2D\omega}}, \tag{22}$$

while the effective natural frequency is of the order of

$$\omega_0^2 = \frac{(L\rho_V)^2}{T\rho^2 c_L R_e} \sqrt{\frac{\omega}{D}}. \tag{23}$$

True resonance occurs for $\omega = \omega_0$, which results in a radius dependence $\omega_0 \propto R_e^{-2/3}$ rather than the inverse proportionality $\omega_0 \propto R_e^{-1}$ found for a gas bubble in (12). From (22) and (23) one can calculate the Q factor of the resonance, which is found to be of order 1: the phase change process which provides the stiffness of vapor bubbles is therefore also responsible for a strong damping of their motion.

Figure 12 shows the linear resonance frequency of vapor bubbles in water caculated on the basis of more precise arguments[30] for temperatures, in ascending order, of 50, 70, 100, and 110 °C. The lower branch of the curves (dashed) is a stability limit. The dotted line is Eq. (23) with $\omega = \omega_0$ multiplied by a constant of order 1 so as to fit the $T = 100\,°\mathrm{C}$ exact result. The approximation (23) is seen to be fairly accurate in spite of the simplicity of the argument.

It is very difficult in practice to observe the resonance just described because a vapor bubble in a sound field quickly grows even in a slightly subcooled liquid.[30] This phenomenon, termed rectified diffusion of heat, is due to a net accumulation of vapor over each cycle caused by a combination of nonlinear effects: (i) due to the geometry-induced stretching for $N > 1$, the thermal boundary layer is thinner during expansion, and therefore the heat flux greater; (ii) for the same change in radius, the surface area is greater upon expansion than contraction; (iii) the saturation pressure–temperature relation is strongly concave upward.[31] Figure 13 shows three examples of this effect in the course of simulated nonlinear forced vapor bubble oscillations at liquid temperatures of 95, 100, and 110 °C.[30] Traces of the underlying resonance structure are evident in the small-scale oscillations superimposed on the large-scale ones. This is a

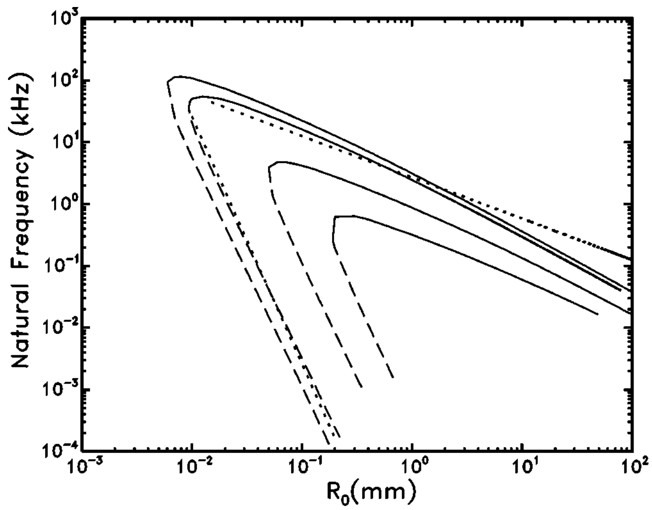


FIG. 12. Resonance frequency of vapor bubbles in water according to linear theory (Ref. 30) for liquid temperatures of 50, 70, 100, and 110 °C, in ascending order. The lower branch of the curves (dashed) is a stability limit. The dotted line is Eq. (23) with $\omega = \omega_0$ fitted to the $T = 100\,°\mathrm{C}$ exact result by multiplying by a numerical constant of order 1. In all cases the ambient pressure has the saturation value at the appropriate temperature.

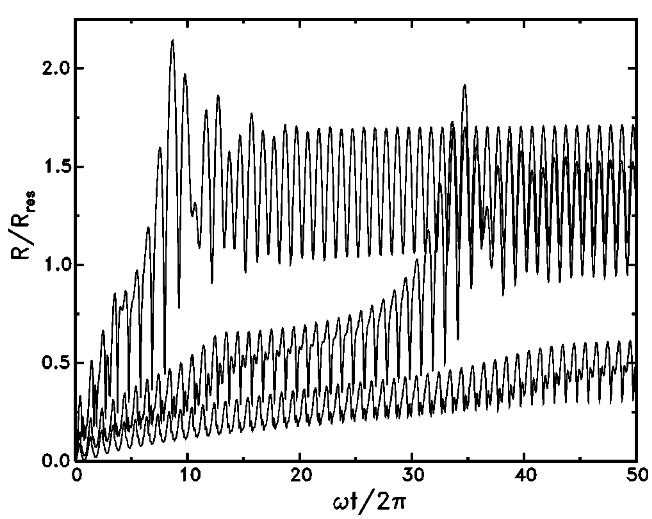


FIG. 13. Growth of a vapor bubble by rectified diffusion in a 1 kHz sound field with $p_A = 40$ kPa, $p_\infty = 101.3$ kPa, and liquid temperatures of 95, 100, and 110 °C in ascending order; $R_{\text{res}} = 2.71$ mm is the linear resonance radius at 100 °C.

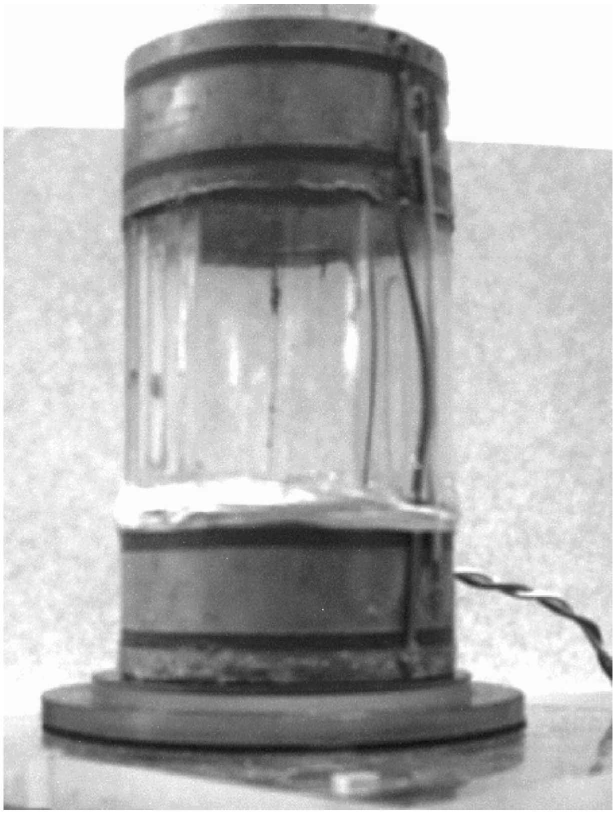


FIG. 14. The resonant acoustic cell (height 150 mm) used for demonstrating the action of Bjerknes forces on vapor bubbles. The two horizontal metal rings are piezoelectric transducers which set up an acoustic standing wave. The wire visible near the cell axis is a heater on which bubbles grow.

FIG. 15. Two frames separated by 1 ms showing the downward motion of a vapor bubble along the heater wire (diameter 200 μm) in the acoustic cell of Fig. 14. The sound frequency is about 17 kHz, the amplitude 20–30 kPa, and the liquid temperature close to 90 °C. The resonant radius in these conditions is about 20 μm. The bubble is nearly ten times as large and, therefore, it is driven below its resonant frequency. Accordingly, the Bjerknes force is toward the pressure antinode located near the center of the frames.

nonlinear phenomenon which occurs when the resonance frequency corresponding to the cycle-averaged radius is close to a rational multiple of the driving frequency.

The same time variation of the bubble contents described for vapor bubbles also occurs in the case of gas bubbles, and for the same reasons. However, due to the fact that the diffusion coefficient of dissolved gases in ordinary liquids is much smaller than the coefficient of thermal diffusion (typically by two orders of magnitude), the process of rectified diffusion of gases is very much slower and the approximation of a constant gas content often justified, at least over a limited number of acoustic cycles.

A bubble of volume $\mathcal{V}$ subjected to a pressure gradient $\boldsymbol{\nabla} p$ experiences an effective buoyancy force $-\mathcal{V}\boldsymbol{\nabla} p$. In a sound field, this force is often referred to as Bjerknes force and, if the oscillation amplitude is small, it has an average value $F_B = -\langle \Delta\mathcal{V} \boldsymbol{\nabla} p \rangle$ where $\Delta\mathcal{V}$ is the amplitude of the volume oscillations. In this case, if the phase of the bubble oscillations with respect to the sound is denoted by ϕ, one finds $F_B \propto \cos\phi$ and, therefore, F_B changes sign as the bubble goes through resonance. A simple argument shows that the force is toward the pressure antinode when the bubble is driven below its resonance frequency and toward the pressure node above resonance. This phenomenon is well known in the case of gas bubbles (see, e.g., Refs. 32 and 33), but has not been studied much in the case of vapor bubbles.[34] A simple demonstration of the effect can be given using the resonant cell shown in Fig. 14, consisting of a glass cylinder between two ring piezoelectric transducers. When the cell is filled with water close to its boiling point, a pulse of current applied to the vertical wire heater visible near the axis causes a local superheat of the liquid which generates bubbles. Figure 15 demonstrates the downaward motion of one such bubble under the action of the Bjerknes force. This phenomenon could be useful for the control of vapor bubbles in microgravity. For example, in the absence of buoyancy, vapor bubbles tend to linger in the vicinity of heating surfaces thus promoting an early transition to the undesirable film boiling regime. An acoustic field might be employed to prevent this phenomenon by pushing the bubbles away.

V. LEONARDO'S PARADOX

It is well known that, in a still liquid, gas bubbles rise along a rectilinear path only when they are sufficiently small (see, e.g., Refs. 35 and 36): larger bubbles follow either a zig-zag or spiral trajectory. In a recent investigation of the added mass of an expanding bubble,[37] an air bubble was released at the bottom of a pressurized water column, the pressure of which was then brought back to atmospheric by a fast-opening valve. An example of the bubble radius versus time measured in this device is shown in Fig. 16; here the initial and final radii are 0.435 and 0.695 mm, respectively.

Figure 17, in which the numbers are keyed to the photos of the preceding figure, shows the projection of the bubble trajectory onto a horizontal plane. Due to a very slight misalignment of the optics (note the fine scale), the initial straight ascent between points 1 and 2 ($t = 0$ and 0.05 s)

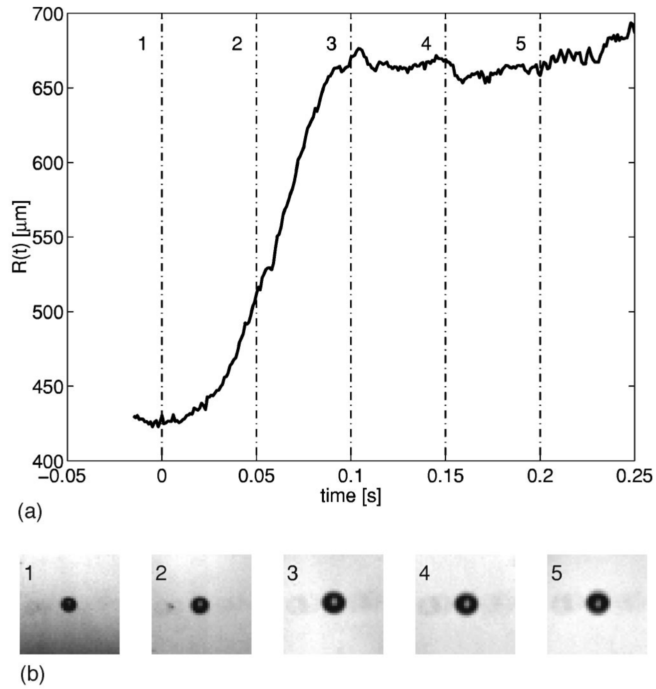


FIG. 16. The radius vs time of an air bubble rising in a water column undergoing a de-pressurization from 405.2 kPa (4 atm) to 101.3 kPa (1 atm). The thick line is the measured radius, while the thin line is the radius predicted from the measured pressure by assuming an isothermal expansion. The photos show the bubble at intervals of 50 ms starting from $t=0$, the beginning of the pressure release.

appears to be along an inclined, rather than vertical, path. The more interesting part of the figure is the very rapid transition to a spiralling motion as the bubble expands and grows to its final size.

I have proposed to refer to the failure of bubbles to follow a straight path as *Leonardo's paradox*, as Leonardo da Vinci's annotations are possibly the first scientific references

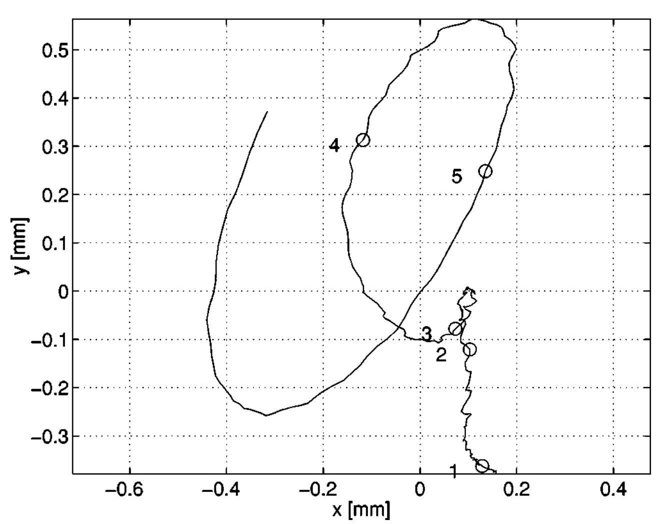


FIG. 17. Horizontal projection of the trajectory of the rising bubble of Fig. 16; the numbers are keyed to the photos of Fig. 16. Note the rapid onset of a spiral motion between 2 and 3. Due to a very slight misalignment of the optics, the initial straight rise between points 1 and 2 ($t=0$ and 0.05 s) appears to be along an inclined path.

to the phenomenon. There are already many ''paradoxes'' in the normal lexicon of fluid mechanics—D'Alembert's, Stokes's, Whitehead's, and others (see, e.g., Ref. 38). In all these cases, the paradoxes' namesakes and their fellow scientists were puzzled by the mismatch between a theoretical deduction based on the available knowledge and observation. Leonardo's paradox has a similar origin and, furthermore, it has the distinction of having remained a puzzle longer than any of the other ones: it is only now that its cause begins to be understood in spite of many earlier attempts (see, e.g., Refs. 39–41), starting with Leonardo's own. According to the current explanation,[42–45] the transition to a nonrectilinear path is caused by an instability of the axisymmetric wake which becomes two-threaded when the bubble is large enough to deform into a strongly oblate spheroid. The two threads have opposite circulation and thus provide a lift force, much like the trailing vortices of an airplane. This view replaces earlier hypotheses which relied on the periodic discharge of vorticity in analogy with the flow past bluff bodies.

Leonardo's attention to the phenomenon was motivated by his lifelong interest in mechanics, a discipline to which he devoted a great deal of thought and a large number of notes in his extant manuscripts. At his time, the view prevailing in the major Italian universities, such as Padua and Bologna, was essentially that of Aristotle,[46] according to whom velocity was proportional to force, so that the permanence of motion required the constant action of a force.[48] To explain phenomena like the motion of a stone thrown by the hand, this theory postulated an active role of the medium through which the body moved: the agent which originally set the body in motion, also imparted movement to the surrounding air, and it was this moving air which dragged the body along.

The obvious difficulties which this theory encountered (e.g., a wheel spinning by inertia) had prompted the formulation of other hypotheses, notably by John Philoponus (5th century A.D.), who postulated that the hand throwing the stone imparts to it a ''quality''—which, after Galilei, we call momentum—which carries its motion.[50,51] It is ironic that this correct explanation was dismissed for a millennium by the majority of the most influential thinkers.[52,53] Nevertheless, it was not forgotten, and it returned to flourish in the Middle Ages, especially at the Universiy of Paris, and in particular thanks to Jean Buridan (ca. 1300–ca. 1358)[54] and Albert of Saxony (ca. 1316–1390), with whose works Leonardo was familiar as Duhem has demonstrated (Ref. 49, Vol. 1, pp. 19–33). Albert so defines *impetus*, one of the words used by Buridan: ''It is a certain quality which is, by its very nature, apt to cause motion in the same direction in which the motor made its projection'' (Ref. 49, Vol. 2, p. 196).[55,56]

Leonardo sometimes uses the word *impeto* and sometimes the word *forza* (force). *Impetus* he defines as ''a power transmitted from the mover to the movable thing'' [57,58] (Cod. Atl. 219 v. *a*, Ref. 59, Vol. 1, p. 529) and recognizes its role by saying ''Impetus is frequently the cause why movement prolongs the desire of the thing moved'' (Cod. Atl. 123 r. *a*, Ref. 59, Vol. 1, p. 526).[60,61]

Whether one looked at the rising bubble from the point

FIG. 18. (Color) Fol. 25 *r* of Leonardo's manuscript known as *Codex Leicester*. The small sketch in the upper right-hand corner, enlarged in Fig. 19, shows the spiralling motion of a rising bubble. (Reproduced from Ref. 63 with the kind permission of the Armand Hammer Foundation.)

FIG. 19. Detail of Fig. 18 showing Leonardo's sketch of the spiralling motion of a rising bubble. (Reproduced from Ref. 63 with the kind permission of the Armand Hammer Foundation.)

of view of Aristotelian mechanics or the impetus theory, the deviation from a straight path posed a puzzle because ''Every natural and continuous movement desires to preserve its course on the line of its inception, that is, however its locality varies, it proceeds according to its beginning'' (ms. I, 68[20] r, Ref. 59, Vol. 1, p. 76).[62]

Leonardo's attempt at an explanation can be found in the *Codex Leicester* (formerly Hammer; see Ref. 63); Figure 18 shows fol. 25 *r* of this manuscript. The sketch in the upper right corner, enlarged in Fig. 19, shows the spiralling motion of a bubble and is accompanied by the following text:

> ''The air which is submerged together with the water ... returns to the air, penetrating the water in sinuous movement... . And this occurs because the light thing cannot remain under the heavy ...; and because the water that stands there perpendicular is more powerful than the other in its descent, this water is always driven away by the part of the water that forms its coverings, and so moves continually sideways where it is less heavy and in consequence offers less resistance ... . And because this has to make its movement by the shortest way it never spreads itself out from its path except to the extent to which it avoids that water which covers it above'' (Ref. 59, Vol. 1, p. 112).[64]

A similar explanation is given in a passage in ms. F 37 *r*, accompanied by a sketch reproduced, among others, in Fig. 14 of Ref. 37:

> ''Whether the air escapes from beneath the water by its nature or through its being pressed and driven by the water. The reply is that since a heavy substance cannot be supported by a light one this heavy substance will proceed to fall and seek what may support it, because every natural action seeks to be at rest; consequently that water which surrounds this air above, on the sides and below finds itself all spread against the air enclosed by it, and all that which is above *d e n m*, [the reference is to the sketch reproduced in Ref. 37] pushes this air downwards, and would keep it below itself if it were not that the laterals *a b e f* and *a b c d* which surround this air and rest upon its sides came to be a more preponderant weight than the water which is above it; consequently this air escapes by the angles *n m* either on one side or on the other, and goes winding as it rises'' (Ref. 59, Vol. 1, p. 557).

The ascending bubble was not the only phenomenon of non-rectilinear propagation to attract Leonardo's prodigious observational powers. In ms. F 52 *r*, he writes

> ''If every movable thing pursues its movement along the line of its commencement, what is that causes the movement of the arrow or thunderbolt to swerve and bend in so many directions whilst still in the air? What has been said may spring from two causes, one of which ... is as in the third [section] of the fifth [book] concerning water, where it is shown how sometimes the air issuing out of the beds of

> swamps in the form of bubbles comes to the surface of the water with sinuous curving movement'' (Ref. 59, Vol. 1, pp. 558, 559).

In spite of the similarity with the bubble case, he thought that, unlike water, motion in air involved compression of the medium (see Ref. 60) and therefore the detailed mechanics would be different. For example, concerning lightning, he writes:

> ''The movement of the thunderbolt which originates in the cloud is curved, because it bends from thickness to thinness, this thickness being occasioned by the fury of the aforesaid movement. For this thunderbolt not being able to extend in the direction in which it commenced, bends into the course that is freest and proceeds by this until it has created a second obstacle, and so following this rule it continues on to the end'' (C.A. 121 r. b, Ref. 59, Vol. 1, p. 396).

Not satisfied with simply collecting and describing instrances of this general phenomenon, Leonardo tried to formulate a general law:

> ''Every impetuous movement bends towards the less resistance as it flies from the greater'' (Cod. Atl. 315 r *b*, Ref. 59, Vol. 1, p. 532).

This article started with soap bubbles, and it is therefore fitting to end it with an annotation by Leonardo on this topic. The small bottle-shaped sketch under the spiraling bubble in Fig. 18 depicts a soap bubble formed at the end of a straw. The accompanying text says: ''Water attracts other water to itself when it touches it: this is proved from the bubble formed by a reed with water and soap, because the hole, through which the air enters there into the body and enlarges it, immediately closes when the bubble is separated from the reed, running one of the sides of its lip against its opposite side, and joins itself with it and makes it firm'' (Cod. Leic. 25 r, Ref. 59, Vol. 2, pp. 114–115). And ''It may be shown with a bubble of water how this water is of such uniform fineness that it clothes an almost spherical body formed out of air somewhat thiker than the other; and reason shows us this because as it breaks it makes a certain amount of noise'' (Cod. Leic. 23 v, Ref. 59, Vol. 2, p. 111).

VI. CONCLUSIONS

As witnessed by the many thousands of papers devoted to it, the subject of bubbles is vast. In this article, and in the talk on which it is based, I have only included a few recent results, mostly unpublished, obtained with my collaborators to all of whom I express my sincere gratitude.

I have also taken the opportunity for a detour away from science and into another context in which bubbles play a prominent role. The number of occurrences of the *Homo bulla* theme that one encounters visiting the world's museums is quite remarkable, and not limited to the West. For example, a beautiful *Still Life with a Boy Blowing Soap Bubbles* painted in 1635/36 by Gerrit Dou (1613–1675), one of Rembrandt's most gifted pupils, can be admired in the National Museum of Western Art in Tokyo. Such paintings continued to be produced until the late 19th century, long after their original moral message had been forgotten. One of the latest examples—*Bubbles*, depicting a beautiful dreamy little boy (the painter's own grandson) painted in 1886 by J. E. Millais (1829–1896)—was sold by its first purchaser to the proprietor of Pear's soap, who used it in advertising for many years with great scandal of the British academic community.

As for Leonardo, one of my colleagues observed that he does not need us for his reputation. I could not agree more—it is fluid mechanics which can benefit from the association!

ACKNOWLEDGMENTS

I am very grateful to the Division of Fluid Dynamics of the American Physical Society for the Otto Laporte Award. A warm vote of thanks goes to the students and post-doctoral associates who were instrumental in obtaining the results described in this paper: Dr. Hasan N. Oğuz, Dr. He Yuan, Dr. Claus Dieter Ohl, Dr. Yue Hao, Dr. Shu Takagi, Xu Geng, Zhizhong Yin, Arjan Tijink, Xiao Hua. The intellectual stimulation and friendship of Professor Leen van Wijngaarden and Professor Detlef Lohse has meant more to me than they realize. Thanks are also due to NSF and NASA for their support of the work described in this paper. *In quibus si quid minus bene dixero, benigne correctioni melius dicentium me subjicio. Pro bene dictis autem non mihi soli, sed magistris meis reverendis de nobili Instituto Technologico Californiensi, qui me talia docuerunt, peto dari gratias et exhibitionem honoris et reverentae*. (Adapted from Albert of Saxony, cited in Ref. 49, Vol. 1, p. 4.) To $\mathcal{L}$, *primum mobile*.

[1] M. M. Phillips and R. A. B. Mynors, *Erasmus's Adagia* (University of Toronto Press, Toronto, 1982).

[2] Lucian of Samosata, ''Charon or the Inspectors,'' in *Works* (Harvard University Press, Cambridge MA, first printing 1915; reprinted 1991), Vol. 2, p. 434.

[3] A. Golding, *The Psalms of David and Others*, XXXIX 6, II 155 (Arion, San Francisco, CA, 1977).

[4] A. Prosperetti, ''Bubble mechanics: Luminescence, noise, and two-phase flow,'' in *Theoretical and Applied Mechanics*, edited by S. R. Bodner, J. Singer, A. Solan and Z. Hashin (Elsevier, Amsterdam, 1992), pp. 355–369.

[5] P. Humfrey, ''Allegory of Fortune,'' in *Dosso Dossi*, *Court Painter in Renaissance Ferrara*, Exhibition Catalogue, edited by A. Bayer (The Metropolitan Museum of Art, New York, 1998), pp. 215–218.

[6] Around 1521 Joos van Cleve (1485–1540) painted a *Saint Jerome in His Study* (Louvre) in which the saint is represented pointing to a skull and a scroll on the wall carries the words *Homo bulla*; the picture however does not contain a pictorial representation of bubbles. An *Allegory* (Germanisches Nationalmuseum, Nuremberg), attributed to Georg Pencz with a date of 1541 (Ref. 7, p. 300), includes a boy blowing bubbles, but seems to present uncertainties of date and/or attribution.

[7] E. de Jongh and G. Luijten, *Mirror of Everyday Life: Genreprints in the Netherlands 1550–1700* (Rijksmuseum, Amsterdam, 1997).

[8] The popularity of the pastime was no doubt also related to the recent major advances in the art of soap making, which had replaced the basically abrasive concoctions of antiquity and the Middle Ages with something more similar to what we know today.

[9] ''For my days are consumed like smoke,'' Psalms 102:3.

[10] The quatrain under the image, by the poet F. Estius, leaves no ambiguity: ''The fresh silvery flower, fragrant with the breath of spring,/ withers instantly and its beauty wanes;/ Likewise the life of man, already ebbing in the newborn babe,/ disappears like a bubble or fleeting smoke'' (Ref. 11). (*Flos novus et verna fragrans argenteus aura/ Marcescit subito, perit,*

ali, perit illa venustas/ Sic et vita hominum iam nunc nascentibus, eheu / Instar abit bullae vanitas clapsa vaporis.)

[11] W. L. Strauss, *Hendrik Goltzius 1558–1617. The Complete Engravings and Woodcuts* (Abaris, New York, 1977).

[12] A transparent globe to signify the world is a standard symbol in Byzantine, Medieval, and Renaissance paintings. The band with the signs of the zodiac over the bubble in Dosso's painting (Fig. 1) also refers to the world. Dosso's picture thus embodies the statement "The world is a bubble," which was actually put into words by Francis Bacon (1561–1626) in his *The World* (1629), and by others (Ref. 4).

[13] At the time this picture was painted, the tulip had recently (1593) been introduced to Holland from Turkey via Vienna and the country was about to precipitate in a veritable tulip obsession. Investing and trading in tulips reached absurd proportions and the price of rare specimens soared to incredible heights. Many fortunes were made and lost until everything came to a screeching halt with the bursting of the "tulip bubble" in 1636. An amusing account of the events is given in the classic *Extraordinary Popular Delusions and the Madness of Crowds* by C. Mackay, written in 1841 and reprinted many times, most recently (1995) by Wordsworth Editions (Ware, UK). Another savory tidbit in the same book—which fits in the general theme of this paper—is the chapter on "The South Seas Bubble" which roiled the British financial markets in the 1720s.

[14] I. Bergström, "De Gheyn as a vanitas painter," Oud Holland **85**, 143 (1970).

[15] I. Bergström, "Homo bulla," in *Les Vanités dans la Peinture au XVII^e Siecle*, edited by A. Tapié (Michel, Caen, France, 1990), pp. 49–54.

[16] In his *Houwelijk*, the Dutch poet J. Cats (1577–1660) writes: "Attend to the child that blows bubbles/ And see how much he is amazed/That so much blown up froth and slobber/ Endures but so brief a phase" (cited in Ref. 17, pp. 512–513).

[17] S. Schama, *The Embarrassement of Riches: An Interpretation of Dutch Culture in the Golden Age* (Knopf, New York, 1987).

[18] E. de Jongh, *Still-Life in the Age of Rembrandt* (Auckland City Art Gallery, Auckland, New Zealand, 1982); reprinted in E. de Jongh, *Questions of Meaning. Theme and Motif in Dutch Seventeen-Century Painting*, translated and edited by M. Hoyle (Primavera Pers, Leiden, 2000).

[19] S. Alpers, *The Art of Describing: Dutch Art in the Seventeenth Century* (University of Chicago Press, Chicago, 1983).

[20] P. Erickson, "Representations of race in renaissance art," Upstart Crow **18**, 2 (1998).

[21] Witness the contemporary French expression "Faire des bulles."

[22] H. I. Medicus, *Emblemata et Aenigmata* (Olms, Hildesheim, 1987).

[23] K. S. Suslick, "Sonochemistry," in *Kirk-Othmer Encyclopedia of Chemical Technology* (Wiley, New York, 1998), Vol. 26, pp. 517–541.

[24] K. S. Suslick and G. Price, "Applications of ultrasound to materials chemistry," Annu. Rev. Mater. Sci. **29**, 295 (1999).

[25] M. P. Brenner, S. Hilgenfeldt, and D. Lohse, "Single-bubble sonoluminescence," Rev. Mod. Phys. **74**, 425 (2002).

[26] E. Ory, H. Yuan, A. Prosperetti, S. Popinet, and S. Zaleski, "Growth and collapse of a vapor bubble in a narrow tube," Phys. Fluids **12**, 1268 (2000).

[27] H. Yuan and A. Prosperetti, "The pumping effect of growing and collapsing bubbles in a tube," J. Micromech. Microeng. **9**, 402 (1999).

[28] A. Prosperetti, H. Yuan, and Z. Yin, "A bubble-based micropump," in *Proceedings of the 2001 ASME Fluids Engineering Division Summer Meeting* (CD-ROM) (ASME, New York, 2001).

[29] A. Prosperetti, H. Yuan, and Z. Yin, "A bubble-based micropump," in *Proceedings of the ASME FEDSM 2001* (CD-ROM) (ASME, New York, 2001).

[30] Y. Hao and A. Prosperetti, "The dynamics of vapor bubbles in acoustic pressure fields," Phys. Fluids **11**, 2008 (1999).

[31] N. A. Gumerov, "Dynamics of vapor bubbles with non-equilibrium phase transitions in isotropic acoustic fields," Phys. Fluids **12**, 71 (2000).

[32] L. A. Crum, "Bjerknes forces on bubbles in a stationary sound field," J. Acoust. Soc. Am. **57**, 1363 (1975).

[33] A. A. Doinikov, "Acoustic radiation force on a bubble: Viscous and thermal effects," J. Acoust. Soc. Am. **103**, 143 (1998).

[34] Y. Hao and A. Prosperetti, "Rectified heat transfer into translating and pulsating vapor bubbles," J. Acoust. Soc. Am. **112**, 1787 (2002).

[35] R. A. Hartunian and W. R. Sears, "On the stability of small gas bubbles moving uniformly in various liquids," J. Fluid Mech. **3**, 27 (1957).

[36] K. Ellingsen and F. Risso, "On the rise of an ellipsoidal bubble in water: Oscillatory paths and liquid-induced velocity," J. Fluid Mech. **440**, 235 (2001).

[37] C. D. Ohl, A. Tijink, and A. Prosperetti, "The added mass of an expanding bubble," J. Fluid Mech. **482**, 271 (2003).

[38] G. Birkhoff, *Hydrodynamics: A Study in Logic, Fact, and Similitude*, 2nd ed. (Princeton University Press, Princeton, 1960).

[39] P. G. Saffman, "On the rise of small air bubbles in water," J. Fluid Mech. **1**, 249 (1956).

[40] J. Mercier, A. Lyrio, and R. Forslund, "Three-dimensional study of the nonrectilinear trajectory of air bubbles rising in water," ASME J. Appl. Mech. **40**, 650 (1973).

[41] D. I. Meiron, "On the stability of gas bubbles rising in an inviscid fluid," J. Fluid Mech. **198**, 101 (1989).

[42] J. Magnaudet and I. Eames, "The motion of high-Reynolds-number bubbles in inhomogeneous flows," Annu. Rev. Fluid Mech. **32**, 659 (2000).

[43] A. W. G. de Vries, A. Biesheuvel, and L. van Wijngarden, "Notes on the path and wake of a gas bubble rising in pure water," Int. J. Multiphase Flow **28**, 1823 (2002).

[44] G. Mougin and J. Magnaudet, "Path instability of a rising bubble," Phys. Rev. Lett. **88**, 014502 (2002).

[45] G. Mougin and J. Magnaudet, "The generalized Kirchhoff equations and their application to the interaction between a rigid body and an arbitrary time-dependent viscous flow," Int. J. Multiphase Flow **28**, 1837 (2002).

[46] "... for they consider Aristotle not subject to human limitations, but rather as having some divine power, and they hold it a sin if one differs even in a nail's breadth from him," was the complaint of G. B. Benedetti (1530–1590; Ref. 47 p. 154).

[47] S. Drake and I. E. Drabkin, *Mechanics in Sixteenth-Century Italy* (University of Wisconsin Press, Madison, WI, 1969).

[48] "If a certain force or power moves a certain body with a certain velocity, a double force or power is needed to move the body with double the velocity" cited in Ref. 49, Vol. 3, p. 58. A significant implication of this notion was the necessary existence of a *primum mobile*, or first mover, which could maintain the motion of the celestial spheres constituting the universe, an entity which the medieval theology had identified with God.

[49] P. M. M. Duhem, *Études sur Léonard de Vinci* (Hermann, Paris, 1906–1913; reprinted by Librairie F. de Nobele, Paris, 1955), Vols. 1–3.

[50] It is necessary "to assume that some incorporeal motive force is imparted by the projector to the projectile, and that the air set in motion contributes either nothing at all, or else very little to this motion of the projectile" [Ref. 51, p. 223].

[51] M. R. Cohen and I. E. Drabkin, *A Source Book in Greek Science* (Harvard University Press, Cambridge, 1948).

[52] The most illustrious example is Thomas Aquinas (1225–1274) who writes: "However, it ought not to be thought that the force of the violent motor impresses in the stone which is moved by violence [as in throwing, as opposed to the "natural" movement of falling] some force (*virtus*) by means of which it is moved.... For, if so, violent motion would arise from an intrinsic source, which is contrary to the nature of violent motion. It would also follow that a stone would be altered by being violently moved in local motion, which is contrary to sense" (Ref. 53, p. 517; Ref. 49, Vol. 2, p. 192).

[53] M. Clagett, *The Science of Mechanics in the Middle Ages* (University of Wisconsin Press, Madison, WI, 1959).

[54] Buridan recognized that the *impetus* was proportional to the density and volume of the body, and was an increasing function of velocity (Ref. 49, Vol. 3, p. 49). Perhaps worried about the possible theological implications of his dismissal of the necessity of a *primum mobile*, he says: "In the creation of the world, God set into motion each [celestial] sphere with the velocity which His will assigned to it, and then stopped moving it; in the following time, these movements have always persisted by virtue of the *impetus* imparted to those spheres. This is why it is said that God rested the seventh day" (reported in Ref. 49, Vol. 3, p. 53; see also p. 42).

[55] For a history of the impetus theory see Ref. 53, pp. 505–525 and Ref. 56.

[56] A. Franklin, *The Principle of Inertia in the Middle Ages* (Colorado Associated University Press, Boulder, CO, 1976).

[57] "L'impeto è una virtú trasmutata dal motore al mobile" (Ref. 58, p. 350).

[58] A. M. Brizio, *Scritti Scelti di Leonardo da Vinci* (UTET, Torino, Italy, 1973).

[59] E. MacCurdy, *The Notebooks of Leonardo da Vinci* (Reynal & Hitchcock, New York, 1938), Vols. 1 and 2.

[60] The use of the adverb "frequently" betrays some hesitation in Leonardo's adherence to this concept, which is confirmed by several other passages in which, echoing the old idea of the "antiperistasis" already refuted by Aristotle, he attributes an active effect of the medium through which the

body moves. For example: ''Impetus at every stage of time becomes less by degrees, and the prolongation of its essence is caused by the air or the water, which closes up behind the movable thing, filling up the vacuum which the movable thing that penetrates it leaves of itself. And this air is more powerful to strike and compress the movable thing by direct percussion, than is the air which is so placed as to resist the penetration of this movable thing by becoming compressed; and it is this compression of the air which diminishes the fury of the aforesaid impetus in the movable thing'' (Cod. Atl. 168 v. *b*, Ref. 59, Vol. 1, p. 526). The same idea is expressed in Cod. Atl. 168 v. *b* (Ref. 59, Vol. 1, p. 526) and 219 v. *a* (Ref. 59, Vol. 1, p. 529). It has been argued that Leonardo's frequent vacillation between the traditional and the new makes him the very embodiment of ''the irreconcilable conflict from which the scientific revolution came to be born'' (Ref. 61). Note also in the passage quoted the erroneous idea that the air ahead of the body is compressed. In this respect Leonardo states: ''The impetus of the movable thing within the water is different from the impetus of the movable thing within the air, and these differences result from the varieties of the aforesaid liquids, because air is condensable to infinity and water is not'' (Cod. Atl. 168 v. *b*, Ref. 59, Vol. 1, p. 527).

[61] M. Kemp, ''The crisis of received wisdom in Leonardo's late thought,'' in *Leonardo e l'Eta' della Ragione*, edited by E. Bellone and P. Rossi (Scientia, Milano, 1982), pp. 27–39.

[62] ''Ogni moto naturale e continuo desidera conservare suo corso per la linia del suo principio, cioè in qualunque loco esso si varia domanda principio.''

[63] J. Roberts, *The Codex Hammer of Leonardo da Vinci* (Giunti Barbera, Florence, 1981).

[64] ''L'aria che si sommerse insieme coll'acqua che sopra l'altr'acqua percosse, ritorna alla aria penetrando l'acqua in moto fressuoso, variando il suo corpo in moltissime forme. E questo accade, perche' il lieve non po' stare sotto il grieve, anzi al continuo e' premuto dalla parte del liquido che sopra se li posa . . . .''