A TWO-STATE ANALYSIS OF THE ONE-DIMENSIONAL RELATIVISTIC PARTICLE

A Thesis

Presented to

the Faculty of the Department of Physics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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John Festus Pierce

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ABSTRACT

The quantum description of a one-dimensional relativistic particle can be formulated in terms of a Feynman two-state analysis. The formalism presents the main physical features of the relativistic particle in a concise, simple form. A Hamiltonian is developed in analogy with the ammonia molecule in an electric field. Using this Hamiltonian the conditions under which a particle loses its positive definite energy quality can be determined. Zitterbewegung, the Klein paradox, and the symmetry between particles of negative energy and positive energy anti-particles can be developed as a consequence of this condition. A second order propagation equation for the state vector is formulated which may be interpreted in two ways: (1) the state space is flat and the state vectors satisfy a Feynman Gell-Mann propagation equation; (2) the state vectors satisfy a Klein Gordon equation, but the state space is structured or curved. The structure of the manifold, given by a Weyl geometry, is due to the presence of an electromagnetic field.

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1. INTRODUCTION

The mathematical formalism associated with a relativistic particle in one dimension suggests that a two-state analysis may be used to describe such a physical system. Accordingly, the two-state techniques utilized by Feynman[1] to describe with great clarity the quantum concepts of the ammonia molecule, and the spinning electron in a magnetic field, are herein applied to the problem of the relativistic particle in motion. The technique gives rise to a representation for the Hamiltonian of the relativistic particle which may be interpreted by comparison to the more familiar two-state systems. Using this Hamiltonian, many of the basic physical features of the relativistic particle may be developed in a natural way with only a minimum of mathematical formalism. First, the conditions under which a particle loses its positive definite energy quality are determined by adiabatic perturbation theory. These conditions are then used to provide a way of conceptually constructing physical systems whose general state is of definite or indefinite energy quality: those systems initially possessing a positive definite energy quality and whose interactions are such as to be within the demands of the adiabatic condition, maintain their positive definite energy nature; those systems

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[1] R. P. Feynman, Lectures on Physics, Volume III, (Addison-Wesley Publishing Co., Reading, 1965), pp. 8.11-9.9.

whose interactions are such as to violate the adiabatic condition lose their definite energy nature, and an appreciable negative energy state component develops.

The free or weakly interacting particle, a system for which the adiabatic condition is fulfilled, is investigated, and the notion of describing the components of the state vector as "large" and "small" is developed. This "large and small" characteristic is then used to account for the disappearance of a degree of freedom in the state vector as the non-relativistic limit is considered.

Systems for which the adiabatic condition is violated, and which are characterized by a non-negligable probability of being in a negative energy state, are next examined. Sudden perturbation theory is used to formulate a description of such systems. Examination of the probability current of a particle undergoing such interaction leads to a model of Zitterbewegung in terms of transitions from the positive energy state to the negative energy state. This model for Zitterbewegung is then related to its geometric manifestation as a "deviation from the classical trajectory"[2] and the frequency of jitter is determined.

Also an investigation of the Klein paradox provides a direct insight into the physical consequences of the violation of the adiabatic condition, and the subsequent transitions to the negative

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[2]Albert Messiah, Quantum Mechanics, Volume II, (North-Holland Publishing Co., Amsterdam, 1962), p. 951.

energy states. These consequences are shown to manifest themselves in the problem of the localization of a particle.

A method for identifying negative energy particles and positive energy "anti-particles" is also developed. By seeking a positive energy state which can be put into one-to-one correspondence with the negative energy eigenstates, the "charge conjugate" or "anti-particle" state is developed. The analysis is then reformulated in terms of the particle/anti-particle states to show how the degree of freedom originally associated with the energy, now manifests itself as a degree of freedom in charge.

From the representation for the relativistic Hamiltonian derived above, a set of field equations can be formulated. The field equations may be interpreted in two ways. If the commutator of differential operators in the equations is assumed to be zero, the state vectors are solutions of a Feynman Gell-Mann propagation equation. However, an alternative interpretation is available in which the state vectors are required to satisfy a Klein Gordon equation. In this case, the commutator of differential operators is non-zero, and determines a constraint relationship. This constraint is interpreted from a differential geometry point of view as defining a curvature of a two-dimensional state manifold. The components of the curvature tensor, and thereby the structure of the state manifold, is found to be proportional to the strength of the applied electromagnetic field. The contracted curvature tensor for the two-state manifold is anti-symmetric, indicating that the geometry of the manifold is not the usual Riemannian variety encountered in gravitational theory. Rather, the space fits the form suggested by Weyl for describing in a differential geometry format, the motion of a charged particle in an electro-magnetic field.

Finally, the two-state analysis can be related to the four-dimensional Dirac theory by means of a projection operator. This is interpreted physically as corresponding to a projection along a line of fixed spin--the two-state theory is a theory in which spin is a constant of the motion. This interpretation is emphasized by the fact that for those interactions amenable to treatment by the two-state formalism, there is no spin flip.

The idea that one can go from studying the ammonia molecule directly to an introductory theory of the relativistic particle is of value from a pedantical viewpoint. The limitations of the two-state theory illustrate the necessity for a more complete theory of larger dimension, and give some insight into why the more complete theory is developed the way it is (with charge conjugation, operators for internal degrees of freedom, Foldy-Wouthuysen representations, etc.). Also, one is introduced to the fertile idea of describing the spin-field interactions in terms of a curvature of state space. Such an approach could provide the mathematical formalism necessary for describing, in a unified way, particles possessing various degrees of spin.

In Section 2 the two-state formalism is reviewed, the isomorphism to the relativistic particle in motion is made, and the physical features associated with the particle in motion are investigated. In Section 3 the structure of the state manifold is developed. In Section 4 the correspondence to the Dirac theory is made, and possible relationships to other particle theories are suggested.

2-a. TWO-STATE ANALYSIS

Consider a physical system which admits a description in terms of a two-dimensional vector space. By such a statement it is meant that an arbitrary state of the system, $|\psi\rangle$, can be adequately described by a linear combination of two time independent base states, $|1\rangle$ and $|2\rangle$:

$$|\psi(t)\rangle = |1\rangle C_1(t) + |2\rangle C_2(t) \,. \tag{2.1}$$

Furthermore, assume that the time development of the arbitrary state is governed by the Schroedinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \mathbb{H} |\psi(t)\rangle. \tag{2.2}$$

The time development of the physical system can be completely described, without explicit knowledge of the base states, if the matrix elements, H_{ij}, of the Hamiltonian of the system with respect to those base states are known. It is emphasized here that the physical interpretation of the time development of the quantum system is with respect to the chosen set of base states. This notion is of primary importance and will form the central part of the arguments which follow. In such a case, the basic equation (2.2) can be expressed in terms of the probability amplitudes, C_1 and C_2:

$$i\hbar \frac{d}{dt} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} . \tag{2.3}$$

The eigenvalues of $\mathbb{H}$ are easily evaluated from the formula

$$E_{\substack{\mathrm{I}\\ \mathrm{II}}} = \frac{(H_{11}+H_{22})}{2} \pm \left[\frac{(H_{11}+H_{22})^2}{4} + H_{12}H_{21}\right]^{1/2}. \qquad (2.4)$$

The eigenvectors of $\mathbb{H}$, or the states of definite energy, can be expressed in terms of the original base states, $|1\rangle$ and $|2\rangle$:

$$|\psi_{\mathrm{I}}(t)\rangle = |1\rangle a_1 e^{-i/\hbar E_{\mathrm{I}} t} + |2\rangle a_2 e^{-i/\hbar E_{\mathrm{I}} t} \equiv |\mathrm{I}\rangle e^{-i/\hbar E_{\mathrm{I}} t} \qquad (2.5)$$

$$|\psi_{\mathrm{II}}(t)\rangle = |1\rangle a_1 e^{-i/\hbar E_{\mathrm{II}} t} + |2\rangle a_2 e^{-i/\hbar E_{\mathrm{II}} t} \equiv |\mathrm{II}\rangle e^{-i/\hbar E_{\mathrm{II}} t}, \qquad (2.6)$$

where

$$a_1/a_2 = \frac{H_{12}}{E-H_{11}} = \frac{E-H_{22}}{H_{21}}, \qquad (2.7)$$

and a_1 is determined by the requirement of normalization.

As a first example of the two-state technique, and in order to set the stage for the relativistic analysis, consider the treatment of the ammonia molecule, as given by Feynman.[3] The structure of the NH_3 molecule is simplified to that of a tetrahedron with the nitrogen atom on either side of the plane defined by the three hydrogen atoms. The base states are chosen to be the geometrical arrangements $|1\rangle$ and $|2\rangle$, corresponding to the nitrogen atom "up", or on top of the plane of hydrogen atoms, and to the nitrogen atom "down", or below the hydrogen atom

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[3]R. P. Feynman, Loc. Cit.

plane, respectively. The general state of the NH_3 system may then be represented in terms of $|1\rangle$ and $|2\rangle$ by the linear combination (2.1). The time evolution of the state of the NH_3 system is governed by the basic equation (2.3), where the matrix elements of the Hamiltonian are with respect to the nitrogen "up" and the nitrogen "down" base states.

The explicit form for the Hamiltonian matrix may be gained from physical argument. A first suggestion is that the two states are degenerate with common energy E_0. If the Hamiltonian matrix with respect to the base states $|1\rangle$ and $|2\rangle$ were diagonal, then equation (2.2) would imply that if the NH_3 molecule were in a definite state, $|1\rangle$ or $|2\rangle$, initially, it would remain there for ever. Observation indicates, however, that the NH_3 molecule does not remain in a pure "up" or "down" state. This suggests off-diagonal or "mixing terms" in the Hamiltonian matrix with respect to $|1\rangle$ and $|2\rangle$. These off-diagonal terms represent an inherent property of the NH_3 molecule and would exist in the absence of any external forces on the NH_3 molecule system. The Hamiltonian with respect to the base states $|1\rangle$ and $|2\rangle$ would then be

$$H_{ij} = \begin{bmatrix} E_0 & -A \\ -A & E_0 \end{bmatrix} . \tag{2.8}$$

The eigenstates of the Hamiltonian (2.8) can be expressed in terms of the "up-down" base states, $|1\rangle$ and $|2\rangle$, by equations (2.5) and (2.6). With respect to these eigenstates, $|I\rangle$ and $|II\rangle$, the Hamiltonian matrix has the diagonal form

$$H_{IJ} = \begin{bmatrix} E_0+A & 0 \\ 0 & E_0-A \end{bmatrix}, \tag{2.9}$$

where $E_I = E_0 + A$ and $E_{II} = E_0 - A$ are the energy eigenvalues.

Next, consider the system of the NH_3 molecule in an external static electric field of magnitude $\mathcal{E}$ chosen perpendicular to the plane of hydrogen atoms for convenience. If one assigns a dipole moment $\vec{\mu}$ to the NH_3 molecule, directed along the altitude of the tetrahedron, then with respect to the base states, $|1\rangle$ and $|2\rangle$, it may be argued that the effect of the dipole moment interaction with the electric field will be to further split the energy levels of the unperturbed Hamiltonian. Specifically, the new Hamiltonian matrix, with respect to the $|1\rangle$, $|2\rangle$ basis set, becomes

$$H_{ij} = \begin{bmatrix} E_0+\mu\mathcal{E} & -A \\ -A & E_0-\mu\mathcal{E} \end{bmatrix}. \tag{2.10}$$

With respect to the base states $|I\rangle$ and $|II\rangle$, the states of definite energy in the absence of the field, the Hamiltonian would be represented by the matrix

$$H_{IJ} = \begin{bmatrix} E_0+A & -\mu\mathcal{E} \\ -\mu\mathcal{E} & E_0-A \end{bmatrix}. \tag{2.11}$$

The energy eigenvalues for the Hamiltonian of the NH_3 molecule-in-the-field system can be obtained from (2.4) and are

$$\begin{aligned} E_I &= E_0+\left(A^2+\mu^2\mathcal{E}^2\right)^{1/2} \\ E_{II} &= E_0-\left(A^2+\mu^2\mathcal{E}^2\right)^{1/2}. \end{aligned} \tag{2.12}$$

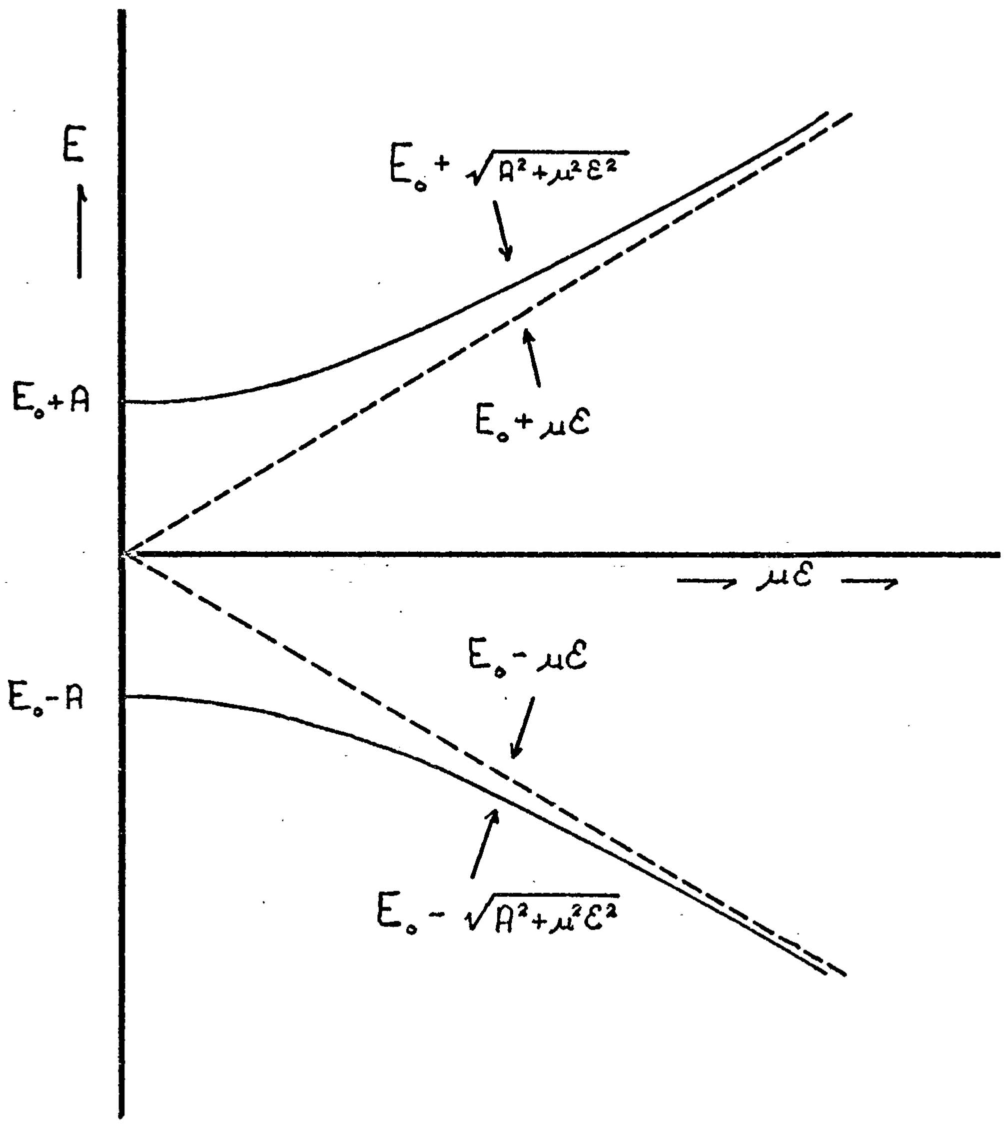
A plot of the energy eigenvalues of the NH_3 system as a function of the electric field dipole interaction energy, $\mu\mathcal{E}$, is given in Figure I.

The two-state technique as applied above to the NH_3 molecule also characterizes the problem of a spin one-half particle in a magnetic field. The correspondence between the two systems is established by describing the base states of the system and by writing the matrix elements for the Hamiltonian.

For the spin one-half particle, two base states are defined from the angular momentum projection along a chosen Z-axis. In such a case $|1\rangle$ is chosen to correspond to the state of a particle whose Z-component of spin is $+\hbar/2$, and $|2\rangle$ is chosen to correspond to the state of a particle whose Z-component of spin is $-\hbar/2$.

The interaction of the particle with the magnetic field is accounted for by assigning to the particle a magnetic moment as

FIGURE I. Two-State Energy Eigenspectrum for the NH_3 Molecule as a Function of the Dipole-Field Interaction Energy



E
$E_0 + \sqrt{A^2+\mu^2\mathcal{E}^2}$
$E_0 + A$
$E_0 + \mu\mathcal{E}$
$\mu\mathcal{E}$
$E_0 - \mu\mathcal{E}$
$E_0 - A$
$E_0 - \sqrt{A^2+\mu^2\mathcal{E}^2}$

a consequence of its intrinsic spin property. The interaction is then given by the classical expression

$$\mathcal{E} = \vec{\mu} \cdot \vec{B} . \tag{2.13}$$

For the case of $\vec{B}$ along the chosen Z-axis, in analogy to the classic dipole, the $\vec{B}$ field is assumed to cause no "flipping" from state $|1\rangle$ to $|2\rangle$, or vice-versa, so that $|1\rangle$ and $|2\rangle$ are stationary states, or states of definite energy. The Hamiltonian with respect to $|1\rangle$ and $|2\rangle$ will then be diagonal and of the form

$$H_{ij} = \begin{bmatrix} \mu B & 0 \\ 0 & -\mu B \end{bmatrix} . \tag{2.14}$$

For the case of $\vec{B}$ arbitrarily directed, with components in all three directions of the chosen reference frame XYZ, the above development permits one to assume that the base states of definite energy, $|I\rangle$ and $|II\rangle$, refer to the measurement of the component of spin along a Z'-axis. This Z'-axis is chosen to be the axis of the $\vec{B}$ field. Then with respect to the states $|I\rangle$ and $|II\rangle$, the Hamiltonian for this system is as above

$$H_{IJ} = \begin{bmatrix} \mu B & 0 \\ 0 & -\mu B \end{bmatrix} . \tag{2.15}$$

If B_x, B_y, B_z, are the components of the $\vec{B}$ field with respect to the laboratory frame XYZ, then the eigen energies are

$$E_I = \mu\left(B_x^2+B_y^2+B_z^2\right)^{1/2}$$
$$E_{II} = \mu\left(B_x^2+B_y^2+B_z^2\right)^{1/2}, \qquad (2.16)$$

For the arbitrarily oriented magnetic field a linear representation for the Hamiltonian operator is

$$H_{ij} = \begin{bmatrix} \mu B_z & \mu(B_x+iB_y)e^{i\delta} \\ \mu(B_x-iB_y)e^{-i\delta} & -\mu B_z \end{bmatrix}. \qquad (2.17)$$

The phase factor $e^{i\delta}$ is arbitrary and following convention, is set equal to -1 .

In spherical coordinates (2.18) may be written as

$$H_{ij} = \begin{bmatrix} \mu B\cos\theta & -\mu B\sin\theta e^{-i\phi} \\ -\mu B\sin\theta e^{i\phi} & -\mu B\cos\theta \end{bmatrix}. \qquad (2.18)$$

2-b. CONNECTION TO THE RELATIVISTIC PARTICLE

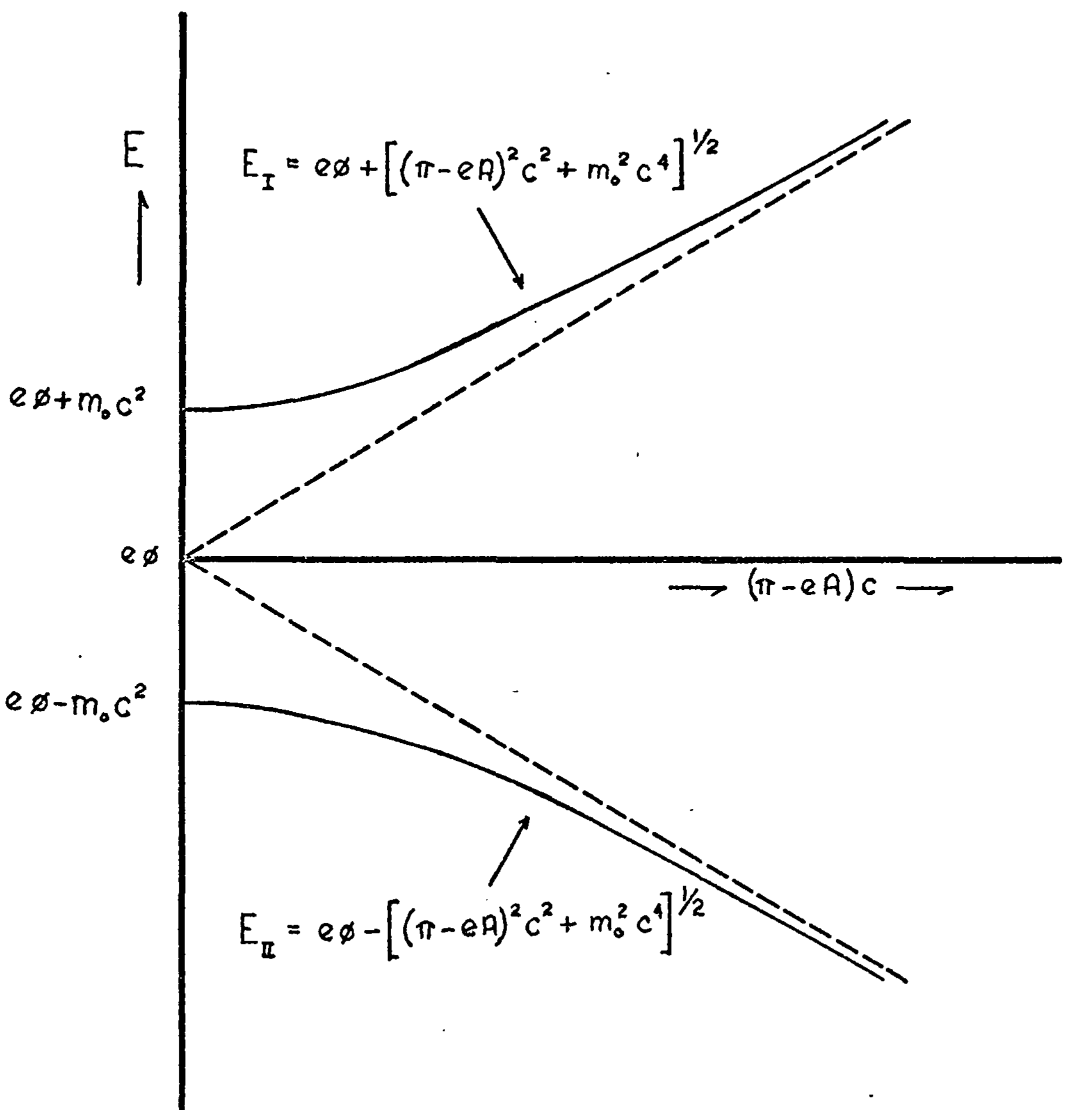
The connection between the one-dimensional relativistic particle and the two-state technique is most easily made at this point by considering the Einstein expression for the energy of the one-dimensional relativistic particle of mass and charge in an external field, which is

$$E = e\phi \pm \left[(\pi - eA)^2 c^2 + m_0^2 c^4\right]^{1/2}. \qquad (2.19)$$

If the particle momentum is assumed to be an independent variable, a plot of the total energy E versus $(\pi - eA)c$ exhibits the same bi-valued behavior as does the energy level plot of the NH_3 two-state system. (Compare Figure I and II.) Thus if one makes the correspondence to the NH_3 molecule in the static $\mathcal{E}$ field, as shown in Table I, the rest mass term, $m_0 c^2$, can be associated with the "internal" transition amplitude of the NH_3 molecule, and the particle momentum term, $(\pi - eA)c$, can be associated with the external interaction energy, $\mu\mathcal{E}$. The correspondence leads to the following representation for the Hamiltonian for the relativistic particle in motion, with respect to the base states $|I\rangle$ and $|II\rangle$:

$$H_{IJ} = \begin{bmatrix} e\phi + m_0 c^2 & (\pi - eA)c \\ (\pi - eA)c & e\phi - mc^2 \end{bmatrix} \qquad (2.20)$$

FIGURE II. The Energy Eigenspectrum for the One-Dimensional Relativistic Particle as a Function of the Particle Momentum


E
$E_I = e\phi + \left[(\pi - eA)^2 c^2 + m_o^2 c^4\right]^{1/2}$
$e\phi + m_o c^2$
$e\phi$
$(\pi - eA)c$
$e\phi - m_o c^2$
$E_{II} = e\phi - \left[(\pi - eA)^2 c^2 + m_o^2 c^4\right]^{1/2}$

The base states $|I\rangle$ and $|II\rangle$ are, respectively, the positive and negative definite energy eigenstates when the particle is at rest with respect to the observer. It will be this representation for the Hamiltonian of the relativistic particle which will be used herein for further study.

TABLE I

Table of Isomorphism between the
Three Systems Treated by the Two-State Technique

NH_3 in Static $\mathcal{E}$ Field	Spin One-Half Particle in $\vec{B}$ Field	Relativistic Particle in Electromagnetic Field

A. With respect to the base states $|1\rangle$ and $|2\rangle$:

$$H_{ij} = \begin{bmatrix} E_o + \mu\mathcal{E} & A \\ A & E_o - \mu\mathcal{E} \end{bmatrix} \qquad H_{ij} = \begin{bmatrix} \mu B_z & \mu(B_x - iB_y)e^{i\delta} \\ \mu(B_x + iB_y)e^{-i\delta} & -\mu B_z \end{bmatrix} \qquad H_{ij} = \begin{bmatrix} e\phi + (\pi - eA)c & m_oc^2 \\ m_oc^2 & e\phi - (\pi - eA)c \end{bmatrix}$$

B. With respect to the base states $|I\rangle$ and $|II\rangle$:

$$H_{IJ} = \begin{bmatrix} E_o + A & \mu\mathcal{E} \\ \mu\mathcal{E} & E_o + A \end{bmatrix} \qquad H_{IJ} = \begin{bmatrix} \mu B & 0 \\ 0 & \mu B \end{bmatrix} \qquad H_{IJ} \begin{bmatrix} e\phi + m_oc^2 & (\pi - eA)c \\ (\pi - eA)c & e\phi - m_oc^2 \end{bmatrix}$$

Table of Parametric Isomorphisms

$$E_o \longleftrightarrow 0 \longleftrightarrow e\phi$$
$$\mu\mathcal{E} \longleftrightarrow \mu B_z \longleftrightarrow (\pi - eA)c$$
$$A \longleftrightarrow \mu(B_x - iB_y)e^{i\delta} \longleftrightarrow m_oc^2$$

2-c. THE FREE PARTICLE FEATURES

The basic features of the relativistic particle in motion are now developed using the Hamiltonian representation (2.20). A key point in understanding these features is that they depend on the fact that the general state of the system has two degrees of freedom in energy. The system may possess a "positive definite energy", by which is meant the system resides in a state which is a positive energy eigenstate of the Hamiltonian, or the state of the system may be a linear combination of both positive and negative energy eigenstates, in which case it is said to be in a state of "indefinite energy". Thus to understand the features of the relativistic particle, it is first necessary to determine how a particle can be known to be in a state of definite energy, and secondly, under what conditions does the particle maintain this characteristic.

If the particle is at rest with respect to the observer, the question of whether the particle possesses a definite or indefinite energy is readily answered. For this case, the base states $|1\rangle$ and $|2\rangle$ are the positive and negative energy eigenstates of the Hamiltonian. Thus if the general state of the particle at rest with respect to the observer can be completely specified in terms of either the base state $|1\rangle$, or the base state $|2\rangle$, but not a combination of both, then the particle possesses a definite energy. Further since the states $|1\rangle$ and $|2\rangle$ are stationary states for the particle at rest, the particle will maintain its definite or indefinite energy quality if it is not set into motion.

If the particle is set into motion it may or may not maintain its energy quality, depending on the conditions under which it was set into motion. The question is now asked, under what conditions can a particle, which is initially at rest with respect to the observer and possessing a positive definite energy, be put into motion and still maintain its definite energy quality. Adiabatic perturbation theory provides the necessary conditions for the maintainence of the definite energy.[4] The general two-state equation for the time-dependent Hamiltonian is

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \mathbb{H}(t) |\Psi(t)\rangle \quad . \tag{2.21}$$

It is assumed from the development that, since the particle is accelerated slowly, at each instant in time, t , the Hamiltonian $\mathbb{H}(t)$ possesses a complete set of energy eigenstates, denoted

$$|\phi_K(p,t)\rangle \, , \; K = 1,2 \, . \tag{2.22}$$

These eigenstates are functions of the momentum of the particle at the time t. They satisfy the eigen-value equation (for $t = \theta$)

[4] D. Bohm, Quantum Theory, (Prentice-Hall Inc., New York, 1951), p. 496 ff.

$$\mathbb{H}(\theta)\,|\phi_K(p,\theta)\rangle = |\phi_K(p,\theta)\rangle E_K(\theta). \tag{2.23}$$

A complete set of stationary states may be built up at each moment of time from these "instantaneous" eigenstates

$$|\Psi_K(pt)\rangle = |\phi_K(p,t)\rangle e^{-\frac{i}{\hbar}\int_0^t E_K(\theta)d\theta}, \quad K=1,2. \tag{2.24}$$

The general state may be expressed in terms of the instantaneous stationary states as

$$|\Psi(t)\rangle = \sum |\phi_K(p,t)\rangle e^{-\frac{i}{\hbar}\int_0^t E_K(\theta)d\theta}\, C_K(t)\,. \tag{2.25}$$

Thus if the system is initially prepared at $t=0$, such that

$$C_1(0) = 1.0 \qquad\qquad C_2(0) = 0.0 \tag{2.26}$$

and if E_1 , E_2 , and $\langle\phi_2|\frac{\partial H}{\partial t}|\phi_1\rangle$ are slowly varying functions of time, then to the first approximation, $C_2(t)$ is (see Appendix A-1)

$$C_2(t) \cong \frac{\hbar}{i(E_1-E_2)^2}\langle\phi_2(t)|\frac{\partial H}{\partial t}|\phi_1(t)\rangle\left(e^{-\frac{i}{\hbar}(E_1-E_2)t}-1\right). \tag{2.27}$$

The probability of transition thus becomes

$$|C_2(t)|^2 \leq \frac{\hbar^2}{(E_1-E_2)^4}\left|\langle\phi_2(t)|\frac{\partial H}{\partial t}|\phi_1(t)\rangle\right|^2. \tag{2.28}$$

If the change in $H(t)$ during a time $\tau \sim \frac{\hbar}{E_2 - E_1}$, is small in comparison to the energy difference between the states, or more explicitly,

$$\frac{\left(\frac{\partial H}{\partial t}\right)_{21} \tau}{E_1 - E_2} \ll 1 \tag{2.29}$$

the probability of a transition to the negative energy state can be considered negligible.

For the case of the particle initially at rest,

$$|\phi_1(p)\rangle = |\mathrm{I}\rangle, \qquad |\phi_2(p)\rangle = |\mathrm{II}\rangle,$$
$$\left(\frac{\partial H}{\partial t}\right)_{21} = \frac{\partial pc}{\partial t}, \tag{2.30}$$

and the adiabatic condition is

$$\frac{\frac{\partial pc}{\partial t}}{E_1 - E_2} \tau \ll 1, \qquad \tau = \frac{\hbar}{E_1 - E_2}. \tag{2.31}$$

The adiabatic condition is seen to be related to the rate of change of momentum. If, during a period of time τ corresponding to the Bohr period associated with the positive and negative energy levels for a given value of momentum, the change in momentum is small compared to the difference in energy between these energy levels, then the particle maintains itself in a positive definite energy eigenstate of the Hamiltonian at each instant in time, even though these eigenstates are constantly changing in time.

Thus we can determine in an operational sense if a particle possessing an arbitrary momentum is in a state of definite energy. If the particle is brought to rest with respect to the observer, or conversely, in such a manner so as not to violate the adiabatic condition during the transition, the complete specification of the state of the particle in terms of either the state $|I\rangle$ or $|II\rangle$, but not both, is a sufficient indication that the particle is one of definite energy.

An investigation of the order of magnitude of the change in momentum required to violate the adiabatic condition indicates that ordinary accelerations are quite well within the demands of the condition.[5] Particles undergoing ordinary accelerations thus will not exhibit those features dependent on the existence of a non-negligable negative energy amplitude. One would expect that systems satisfying the adiabatic condition could be accurately described in terms of a one-component state vector, the second degree of freedom being suppressed. A graphic example of the disappearance of a degree of freedom is present in the development of the non-relativistic limit of the two-state equations.[6]

From the above development it has been learned that a particle initially at rest with respect to the observer and in a positive

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[5]For example, the free electron initially at rest must experience a force of .5 newtons before its energy state becomes indefinite.

[6]Davydov, Quantum Mechanics, (Addison-Wesley Publishing Co., Reading, 1965), p. 223 ff.

definite energy state can be set into motion and still reside in a positive definite energy state, provided the adiabatic condition is satisfied. Since the particle's state $|+(p)\rangle$ is an eigenstate of the Hamiltonian, it is governed by the equation

$$\mathbb{H}|+(p)\rangle = |+(p)\rangle E_+(p) \tag{2.32}$$

With respect to the observer's states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ it is represented by the following expanded matrix equations:

$$|+(p)\rangle = |\mathrm{I}\rangle C_{\mathrm{I}} + |\mathrm{II}\rangle C_{\mathrm{II}} \tag{2.33}$$

$$\begin{aligned} E_+(p) C_{\mathrm{I}} &= m_0 c^2 C_{\mathrm{I}} + pc C_{\mathrm{II}} \\ E_+(p) C_{\mathrm{II}} &= pc C_{\mathrm{I}} - m_0 c^2 C_{\mathrm{II}} \end{aligned} \tag{2.34}$$

Consider the non-relativistic limit of these equations. For (2.34-b) can be approximated as

$$C_{\mathrm{II}} \cong \frac{pc}{2m_0c^2} C_{\mathrm{I}} \tag{2.35}$$

So, in the non-relativistic limit

$$\frac{C_{\mathrm{II}}}{C_{\mathrm{I}}} = \frac{pc}{2m_0c^2} \ll 1 \tag{2.36}$$

For this reason C_{II} is termed the "small component" of the state vector. Introducing the equation (2.35) into (2.34-a), the equation governing the evolution of the state is determined:

$$E(p) C_{\mathrm{I}} = m_0 c^2 C_{\mathrm{I}} + \frac{p^2}{2m_0} C_{\mathrm{I}} \tag{2.37}$$

This is the Schrodinger time independent equation with the usually suppresses rest mass factor. Thus for particles whose motions are non-relativistic and whose interactions satisfy the adiabatic condition, the two-state equations specialize to the non-relativistic Schrodinger equation and a subsidary condition. The second degree of freedom dissolves, and the system is for all practical purposes specified by a one-component wave function.

Physical systems whose state is a linear combination of positive and negative energy states are now examined. Such a system is the free particle of initially positive definite energy which is perturbed into motion in such a way as to violate the adiabatic condition. Sudden approximation theory should then provide the means of developing the basic characteristics of this system.[7]

Consider a particle initially at rest with respect to the observer and possessing a positive definite energy, which is given a momentum in a "sudden manner" at $t=0$. The time dependent Hamiltonian may then be written as

$$\mathbb{H}(t) = \mathbb{H}_o + \mathbb{V}(t) \qquad \mathbb{V}(t) = \begin{cases} 0 & t<0 \\ V & t \geq 0 \end{cases} \tag{2.38}$$

$\mathbb{H}_o$ is the Hamiltonian for a particle at rest with respect to the observer. Its energy eigenstates are the observer's base states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$. Since for $t<0$ the particle is in a positive definite energy state, it may be represented as

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[7]Bohm, Op. Cit., p. 408 ff.

$$|\Psi(t)\rangle = |\mathrm{I}\rangle\, e^{-\frac{i}{\hbar}E_{\mathrm{I}}t}, \qquad t<0\,. \tag{2.39}$$

The Hamiltonian for $t>0$, $\mathbb{H}(t)$, has associated with it a complete set of energy eigenstates, $|+(p)\rangle$ and $|-(p)\rangle$, for p a constant momentum. Thus the state vector of the system may be expanded in terms of this set as

$$|\Psi(t)\rangle = \sum_{K=+}^{-} |K(p)\rangle\, C_K(t)\, e^{-\frac{i}{\hbar}E_K t} \tag{2.40}$$

Sudden approximation theory, with the above assumption of a prepared initial state then leads to the following equation for the state of the system in terms of the base states $|+(p)\rangle$ and $|-(p)\rangle$:

$$|\Psi(t)\rangle = \left(|+(p)\rangle\, e^{-\frac{i}{\hbar}E_+t} - |-(p)\rangle \frac{pc}{E+mc^2}\, e^{-\frac{i}{\hbar}E_-t} \right) \sqrt{\frac{E+mc^2}{2E}} \tag{2.41}$$

In terms of the observer's base states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$,

$$\begin{aligned} |\Psi(t)\rangle = \Big\{ &|\mathrm{I}\rangle \left(e^{-\frac{i}{\hbar}E_+t} + \left(\frac{pc}{E+mc^2}\right) e^{-\frac{i}{\hbar}E_-t} \right) + \\ &|\mathrm{II}\rangle \frac{pc}{E+mc^2} \left(e^{-\frac{i}{\hbar}E_+t} - e^{-\frac{i}{\hbar}E_-t} \right) \Big\} \frac{E+mc^2}{2E} \end{aligned} \tag{2.42}$$

If one now constructs the first order correction to the probability current associated with the one-dimensional particle (see Appendix B) the following current is obtained:

$$J_z = \frac{p}{m} - \frac{p}{m}\cos\frac{2mc^2}{\hbar}t \quad . \tag{2.43}$$

Interpreting this probability current from a Schroedinger viewpoint, the first term, p/m , is what is expected classically as the "average motion" of the particle. However, the sudden perturbation causes a deviation from this average motion, a deviation which is oscillatory in time, and which does not correspond to any classical motion. Schroedinger termed this oscillatory deviation from the classical trajectory "Zitterbewegung"[8] and explained its nature in the following way. A quantum particle at low velocities has associated with its wave packet a mean position which maps out the uniform motion trajectory associated with the particle when viewed classically. However, as the motion becomes relativistic, this mean trajectory deviates from the classical uniform rectilinear motion. This new mean trajectory is a superposition of the classical motion and an oscillatory piece whose frequency of oscillation is, to the first approximation, $\frac{2m_0c^2}{\hbar}$. This behavior is caused by an interaction between the positive and negative energy states associated with the particle.

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[8]Bjorken and Drell, Relativistic Quantum Mechanics, (Mc Graw-Hill Book Co., New York, 1964), p. 38.

A model of this interaction between the states may be gained by exploiting the isomorphism between the NH_3 molecule in an electric field, and the relativistic particle in motion. For the case of the NH_3 molecule, the flip-flop is induced by the interaction of the electric field and the dipole moment of the molecule. For the relativistic particle case, the flip-flop is induced by the sudden existence of a particle momentum. The Zitterbewegung disappears (has zero amplitude) when the particle is in a positive energy state.

Thus the phenomena of Zitterbewegung may be characterized as a coordinate manifestation of the existence in state space of a time-dependent probability amplitude which is a consequence of two facts: the particle is in motion with respect to the observer, and the particle is in an indefinite energy state.

A graphic illustration of the significance of the non-negligible transitions to the negative energy state is provided by the paradox of the localization of the electron, as originally proposed by Klein.[9] The paradox concerns the attempt to localize a relativistic particle to within a distance d . Klein attempted this localization by means of a potential barrier which rises appreciably within the distance d of localization. However, if this distance of localization becomes comparable to the Compton wavelength of the particle, $\frac{\hbar}{m_0 c}$, while the potential barrier changes by an amount

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[9] O. Klein, Zeitschrift fur Physik, 53:157, 1929.

$E + m_0c^2$ within this range for d (E is the energy of the impinging particles), unordinary results are achieved. Specifically, the exponential decay in the potential barrier wall changes to an oscillatory behavior, and a reflected current is produced which is greater than the incident current. These results are detrimental to a theory allowing for only positive energy solutions.

However, by the two-state analysis, this unordinary result is precisely what should happen. For $d \sim \frac{\hbar}{m_0c^2}$, the associated Bohr period is

$$\tau = d/c \sim \frac{\hbar}{m_0c^2} . \qquad (2.44)$$

During this period of time, the change in the Hamiltonian is

$$\tau \frac{\partial H}{\partial t} \sim E_1 - E_2 = E + m_0c^2 . \qquad (2.45)$$

The gradient in energy is too severe to satisfy the adiabatic condition. Consequently transitions to the negative energy state become non-negligible, "anti-particles" are produced, and these manifest themselves as an addition to the original current.

Now the identification of negative energy particles with the above mentioned positive energy "anti-particles" can easily be developed. The motivation for this identification lies in the seemingly paradoxical behavior which would be attributed to a negative energy particle by virtue of its relativistic features, in comparison to the "normal behavior" associated with positive definite energy non-relativistic particles. A way of reformulating the two-state development is sought which would allow the degree of freedom now associated with energy, to manifest itself in some other way. This desire is expressed quantitatively by asking the question, does there exist a positive energy state which can be put into one-to-one correspondence with the negative energy eigenstate, and if so, what is its equation of motion? One way of answering this question is provided by the following development utilizing the representation of the two-state problem in terms of the base states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$.

Let $|-(p)\rangle$ represent the negative energy eigenstate for the Hamiltonian, for the given value of momentum. Appendix A-2 provides the first order expansion of the $|-\rangle$ in terms of the observer's base states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$:

$$|-(p)\rangle = |\mathrm{I}\rangle C'_{\mathrm{I}} + |\mathrm{II}\rangle C'_{\mathrm{II}}$$

$$|-(p)\rangle = \left(\frac{-pc}{E+mc^2}|\mathrm{I}\rangle + |\mathrm{II}\rangle\right)\sqrt{\frac{E+mc^2}{2E}} \quad . \tag{246}$$

Consider the state $|*\rangle$ constructed form the negative energy eigenstate in the following operational manner:

$$|*\rangle = |\mathrm{I}\rangle \gamma_{\mathrm{I}} + |\mathrm{II}\rangle \gamma_{\mathrm{I}}$$

$$\begin{pmatrix}\gamma_{\mathrm{I}}\\ \gamma_{\mathrm{II}}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}C'_{\mathrm{I}}\\ C'_{\mathrm{I}}\end{pmatrix}^{*} \tag{247}$$

$$|*\rangle \equiv \mathbb{K}_1|-\rangle \quad .$$

Now $|*\rangle$ is a positive definite energy state, since as $p \to 0$, $|*\rangle \to |I\rangle$. In Appendix C it is shown that the equation of motion for $|*\rangle$ is

$$\begin{bmatrix} -e\phi+m_oc^2 & (\pi+eA)c \\ (\pi+eA)c & -e\phi-m_oc^2 \end{bmatrix} \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix}, \tag{248}$$

which involves the original Hamiltonian with the substitution $e \to -e$. For this reason the operation from which $|*\rangle$ was gained from the negative energy eigenstate is termed the "charge conjugation operation". $|*\rangle$ is termed the state "conjugate" to the negative energy eigenstate $|-\rangle$, and it is the "anti-particle" state associated with the positive energy eigenstate $|+\rangle$. Thus formally one can equally well consider combinations of particle and anti-particle states, or positive and negative energy states. The degree of freedom which had originally been associated with energy now manifests itself as a degree of freedom in charge.

The above charge conjugation operation is not unique. A second possibility for the charge conjugate state is $|*\rangle = \mathbb{K}_2|-\rangle$, and its expansion in terms of the base states $|I\rangle$ and $|II\rangle$ is

$$\begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_I' \\ C_{II}' \end{pmatrix}^* . \tag{249}$$

Its Hamiltonian with respect to $|I\rangle$ and $|II\rangle$ is

$$H = \begin{bmatrix} -e\phi+m_oc^2 & -(\pi+eA)c \\ -(\pi+eA)c & -e\phi-m_oc^2 \end{bmatrix} . \tag{250}$$

The interesting point about this Hamiltonian is the -1 factor in the off-diagonal elements. This second charge conjugation operation causes a shift in the arbitrary off-diagonal phase factor, $e^{i\delta}$, which had originally been set equal to +1. In Section 4 it will be seen that this second charge conjugation operation corresponds to that charge conjugation operation associated with the Dirac theory in the original representation. The change in sign of the off-diagonal term is a manifestation of the flip in spin involved in the charge conjugation operation in Dirac's theory. On the two-state level, however, the change in the phase effects in no way the measurable quantities like currents, and energy expectation values. Thus one can at best say that the change in the value of the phase factor $e^{i\delta}$ represents an "internal degree of freedom" which for the systems treated by the two-state analysis, remains "hidden". This point will again be discussed in Section 4.

Once an insight into the physics associated with the relativistic quantum particle has been gained, the above development may be recast into a more powerful mathematical form. A representation for the Lorentz group in the state space may be developed, representations for the various dynamic variable operators can be derived, the concept of "even" and "odd" operators can be introduced, and a form of the Foldy-Wouthuysen problem can be considered. These developments, however, will be defered until the results of the correspondence to Dirac's theory are available from Section 4.

At present, it is enough to realize that the two-state analysis has provided a simple means of illustrating the conditions under which an object will maintain or change its energy status, has characterized the phenomena of Zitterbewegung, and has provided a general insight into the features of the relativistic particle of definite and indefinite energy.

3. CURVATURE OF THE STATE MANIFOLD

The relativistic two-state Hamiltonian (2.20) may be combined with the De Broglie operator correspondence rule,

$$\Pi_\mu = hk_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}, \tag{3.1}$$

to yield a set of field equations for the amplitudes Ψ_1 and Ψ_2, where Ψ_1 and Ψ_2 are taken as an equivalent notation for C_1 and C_2. By direct substitution of (3.1) into the two-state equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{bmatrix} e\phi + m_0c^2 & (\pi - eA)c \\ (\pi - eA)c & e\phi - m_0c^2\end{bmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \tag{3.2}$$

it may be shown (see Appendix D) that each amplitude satisfies the expression

$$\begin{aligned}\Box^2\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = {} & \left[\left(\frac{m_0c}{\hbar}\right)^2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} - \frac{2ie}{\hbar}A_\mu\frac{\partial}{\partial x^\mu}\right]\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \\ & + \frac{1}{c}\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right)\begin{pmatrix}\Psi_2\\ \Psi_1\end{pmatrix} - \frac{ie}{\hbar c}\left(\frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial t}\right)\begin{pmatrix}\Psi_2\\ \Psi_1\end{pmatrix}.\end{aligned} \tag{3.3}$$

If the partial derivative commutator $\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right)\begin{pmatrix}\Psi_2\\ \Psi_1\end{pmatrix}$ is taken to be zero, equation (3.3) reduces to a form comparable to the Feynman, Gell-Mann propagation equation,[10] as will be shown in the next section. However, if the commutator is not set identically equal to zero, but rather used to determine a constraint

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[10] R. Feynman and M. Gell-Mann, Physical Review, 109:193, January 1958.

relationship, the above equation provides an interesting interpretation of the interaction between the particle and the electromagnetic field in terms of a geometric structure of a state space set into correspondence with $\psi_1(xt)$ and $\psi_2(xt)$.

Specifically, it is noted from equation (3.3) that the state amplitudes ψ_1 and ψ_2 satisfy the Klein Gordon equation,

$$\Box^2 \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left[\left(\frac{m_0 c}{\hbar}\right)^2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} - \frac{2ie}{\hbar} A_\mu \frac{\partial}{\partial x^\nu}\right] \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad (3.4)$$

subject to the constraint that

$$\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left(-\frac{ie}{\hbar}\right)\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial t}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \qquad (3.5)$$

Geometrically speaking, the non-vanishing of such a commutator may be interpreted as suggesting that a two-dimensional surface, when viewed as imbedded in a higher dimensional space, has certain characteristics, generally termed a "structure", which can be described in terms of its curvature, torsion, and other qualities. This may be seen in the following way.

Assume that the two-dimensional surface related to the state amplitudes, presently unspecified, is in actuality imbedded in a higher dimensional space. The coordinate variables x and t are assumed to be intrinsic to this surface, that is, they identify points uniquely on this surface, but not in the imbedding space outside the surface. Since the surface is imbedded in the space, a field of vectors defined on the surface may be characterized as having components which lie in the surface (intrinsic components), and components which are orthogonal to the surface (extrinsic components). Although the extrinsic components of such a vector field are not immediately known to the observer confined to the surface, the change in this "normal component" of the vector field as the observer moves from one point to another on the surface may possess a component which lies in the surface. This tangent component of the derivative of the "normal" (unobserved) vector field manifests itself to the surface observer by a non-commutation of differential displacements. This the observer can relate to the apparent shape or "structure" of the surface in the imbedding space.

The constraint relation (3.5) suggests explicitly how this "structure" will manifest itself, in that the commutator is related to a physical entity, the electromagnetic force on a charge e. Thus the observation of an electromagnetic force on the one-dimensional particle can be interpreted as requiring the assignment of a structure to this two-dimensional surface related to the state amplitudes, which we will call the "state manifold". This interpretation is quite analogous to the general relativistic situation where the existence of a gravitational force is interpreted by assigning a structure to the three-dimensional coordinate hypersurface at an instant in time.

From an analytical viewpoint, the commutator in partial derivatives has the significance of defining the components of a curvature tensor associated with the surface. Non-zero components of the curvature tensor implies that the state manifold may not be flat; it may or may not have curvature.

From the general relations governing the geometric character of a curved manifold (Appendix E), the curvature tensor and the contracted curvature tensor of the state manifold may be determined. Assume an origin and a coordinate system on the state manifold. Consider $\vec{r}(xt)$ as a position vector to any point on the state manifold, and regard Ψ_1 as $\Psi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial \vec{r}}{\partial x} \triangleq \vec{a}_1$, and Ψ_2, as $\Psi_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial \vec{r}}{\partial t} \triangleq \vec{a}_2$, that is, covariant base vectors in the space tangent to the manifold at the point $\vec{r}(xt)$.

The components of the curvature tensor follow immediately from (3.5) by the relations

$$\left(\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial \xi_2}\frac{\partial}{\partial \xi_1}\right)\Psi_1 = \Psi_\alpha L^\alpha_{\ 112} \tag{3.5}$$

$$\left(\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial \xi_2}\frac{\partial}{\partial \xi_1}\right)\Psi_2 = \Psi_\alpha L^\alpha_{\ 212} \tag{3.6}$$

They are:

$$\begin{aligned} L^1_{\ 112} &= -\frac{ie}{\hbar}\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right) = -L^1_{\ 121} \\ L^2_{\ 212} &= -\frac{ie}{\hbar}\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial x}\right) = -L^2_{\ 221} \end{aligned} \tag{3.7}$$

All other components are zero.

In Einstein's general relativity theory, the components of the Riemann tensor are said to account for the acceleration induced by the gravitational field.[11] On the state manifold considered above, the components of the curvature tensor account for the accelerations induced by the electromagnetic field. The components of the curvature tensor (3.7) are directly proportional to the $\vec{E}$ field strength, which classically is the acceleration producing field on a charged particle. Such a result suggests that a study of the structure of the state manifold may yield a better understanding of the electromagnetic field.

A significant deviation from the gravitational theory is exhibited by the fact that the contracted curvature tensor for the manifold is antisymmetric,

$$L_{\mu\nu} = L^{\tau}_{\mu\nu\tau} = \begin{bmatrix} 0 & +\frac{ie}{\hbar}\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial t}\right) \\ -\frac{ie}{\hbar}\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A}{\partial t}\right) & 0 \end{bmatrix} \tag{3.8}$$

Several points become clear from the above results. First, the motion of the particle is not that associated with a "flat space", since $L_{\mu\nu} \neq 0$ for some components. Also, the anti-symmetry of the contracted curvature tensor indicates that the geometry of the state manifold in non-Riemannian. Of particular interest is the fact that the form of the contracted curvature tensor is exactly that proposed by Weyl for describing in a differential geometry format, the motion of a charged particle in an electromagnetic field, as will now be developed.

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[11]Adler, Introduction to General Relativity, (Mc Graw-Hill Book Co., New York, 1964), p. 186.

Geometrically, the extension of the Riemann geometry initiated by Weyl is the following.[12] The basic relations for the general connection and the curvature tensor are as given in Appendix E. If the connection is separated into symmetirc and anti-symmetric parts

$$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk} \tag{3.9}$$

the curvature tensor may also be separated as follows:

$$L^i_{jkl} = B^i_{jkl} + \Omega^i_{jkl} \tag{3.10}$$

where

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^h_{jl}\Gamma^i_{hk} - \Gamma^h_{jk}\Gamma^i_{hl} \tag{3.11}$$

$$\Omega^i_{jkl} = \Omega^i_{jl|k} - \Omega^i_{jk|l} + \Omega^i_{hl}\Omega^h_{jk} - 2\Omega^i_{jh}\Omega^h_{kl} \tag{3.12}$$

the slash indicating a covariant derivative. For the present study, consider the connection to be symmetric:

$$\Omega^i_{jk} = 0 \tag{3.13}$$

Examination of the contracted curvature tensor reveals that it may be separated into symmetric and anti-symmetric parts:

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[12] L. P. Eisenhart, Non-Riemannian Geometry, (Volume VIII of the American Mathematical Society Colloquium Publications; American Mathematical Society, New York, 1927), pp. 8-10.

$$B^i_{jki} \equiv B_{jk} = b_{jk} + \beta_{jk} \tag{3.14}$$

where

$$b_{jk} = \tfrac{1}{2}\left(\frac{\partial \Gamma^h_{hj}}{\partial x^k} + \frac{\partial \Gamma^h_{hk}}{\partial x^j}\right) - \frac{\partial \Gamma^h_{jk}}{\partial x^h} + \Gamma^h_{ji}\Gamma^i_{hk} - \Gamma^h_{jk}\Gamma^i_{hi} \tag{3.15}$$

$$\beta_{jk} = \tfrac{1}{2}\left(\frac{\partial \Gamma^h_{hj}}{\partial x^k} - \frac{\partial \Gamma^h_{hk}}{\partial x^j}\right). \tag{3.16}$$

If the symmetric connection is expressed in terms of the ordinary Riemann connection, the Christoffel connection, as

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + a^i_{jk}, \tag{3.17}$$

then from the properties of the Christoffel connection it may be shown[13] that

$$\beta_{jk} = \tfrac{1}{2}\left(\frac{\partial a^i_{ij}}{\partial x^k} - \frac{\partial a^i_{ik}}{\partial x^j}\right) \equiv \tfrac{1}{2}\left(\frac{\partial a_j}{\partial x^k} - \frac{\partial a_k}{\partial x^j}\right). \tag{3.18}$$

Likewise if R_{ij} is the ordinary contracted Riemann tensor, the Ricci tensor, the symmetric part of the contracted curvature tensor is

$$b_{jk} = R_{jk} + \tfrac{1}{2}\left(a_{j|k} + a_{k|j}\right) - a^i_{jk|i} + a^h_{ji}a^i_{hk} - a^h_{jk}a_h \tag{3.19}$$

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[13] Eisenhart, Loc. Cit.

The important thing to note here is that the deviation of the contracted curvature tensor from the ordinary Ricci tensor, both in the additional symmetric parts and the entirely additional anti-symmetric components, is dependent upon the extension of the symmetric connection beyond the usual Christoffel part. Thus, the fact that (3.8) is anti-symmetric indicates that the geometry of the state manifold is not strictly Riemannian, and the connection associated with the manifold, though symmetric, is not strictly the Christoffel connection.

The general analytical features of Weyl's theory are now summarized.[14] The general relativistic connection of gravity (physics) to geometry involves the assignment of a manifold in a space somehow to the physical system studied, and the characterizing of the qualities of the manifold in terms of a metric, g_{ij}, and a connection derivable from the metric, the Christoffel connection. This derivation involves the requirement of the conservation of the length or "norm" of a vector as it is "parallel displaced" from one point to another on the surface. Weyl relaxed this requirement and allowed the norm of the vector to vary as it was displaced infinitesimally from a given point. This variation was chosen to be proportional to the norm of the vector at the given point, and

[14] Adler, Op. Cit., pp. 401-10.

the differential vector of the displacement:

$$dl = (\phi_\beta dx^\beta) l \tag{3.20}$$

where ϕ_β is to be termed the "gauge vector". By so doing, the connection for the manifold was generalized beyond the Christoffel connection:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \delta^i_j \phi_k + \delta^i_k \phi_j - g_{jk} \phi^i . \tag{3.21}$$

If one constructs the curvature tensor from this symmetric connection, and contracts it as shown, one finds the following:

$$B^i_{ikl} = n \left(\frac{\partial \phi_l}{\partial x^k} - \frac{\partial \phi_k}{\partial x^l} \right) \tag{3.22}$$

where n is the number of dimensions.

The analytical structure is now given a physical interpretation. The basic idea is that "forces" acting on a physical system can be interpreted by assigning a "structure" to the geometry of a manifold which somehow has been put into correspondence with the physical system. For Weyl, the structure of the manifold, and thereby the physics of the system, is determined when, not only the metric associated with the manifold, but also the gauge vector is determined. Further, equations involving which are identical in form to the equations describing the electromagnetic field can be derived. Thus by analogy, Weyl suggests that the geometric manifestation of an interaction of a physical system with an electromagnetic field is a "shortening of norms of vectors", or a "change in scale" at different points on the manifold.

We now connect what has been summarized here, and the state manifold characterized previously. It is noticed that the contracted curvature tensor (3.8) is anti-symmetric. Thus if one relates (3.22) with the anti-symmetric part of the contracted curvature tensor given by (3.14) and (3.16), and sets the symmetric part of (3.14) equal to zero, one finds:

$$B_{jk} = B^{i}_{jki} = \tfrac{1}{2} B^{i}_{ikj} = \left(\frac{\partial \phi_j}{\partial x^k} - \frac{\partial \phi_k}{\partial x^j}\right) \tag{3.23}$$

where $n = 2$, since the state manifold is two-dimensional.

The comparison of (3.23) and (3.8) immediately indicates that the geometry of the state manifold can be characterized as a Weyl geometry, and that the gauge vector ϕ_k is indeed exactly that of the electromagnetic potential. Thus we may characterize the "physics" of the state manifold as follows. Its curvature tensor has no symmetric part, indicating that the manifold may be thought of as "gravitationally flat". However, there are other effects, manifesting themselves as "changes in scale" as one moves from point to point on the manifold. These changes in scale may be identified with the existence of an electromagnetic field interacting with the physical system.

Thus, permitting the differential cummutator to determine a constraint on the state manifold has allowed the Weyl theory "to enter by the back door", in the sense that such a constraint determines a structure for the manifold which is directly attributable to the electromagnetic field in a way proposed by Weyl.

Once again the basic simplicity and clarity inherent in the two-state analysis of the one-dimensional relativistic particle has allowed the development and understanding of an intriguing and fruitful area of physics in terms of two-dimensional geometry. Such an insight is usually impeded by the presence of a more complex form, more degrees of freedom, and more dimensions.

4. THE RELATION OF TWO-STATE TO THE DIRAC FORMALISM

The correspondence of the preceeding analysis to the Dirac formalism is quite direct. If the Dirac equation is formulated for the special case of motion in one spatial dimension, the Hamiltonian display, with respect to those base states for which the operators α and β are displayed as

$$\hat{\alpha}_i = \begin{bmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{4.1}$$

is

$$H_D = \begin{bmatrix} e\phi + m_o c^2 & 0 & (\pi - eA)_z c & 0 \\ 0 & e\phi + m_o c^2 & 0 & -(\pi - eA)_z c \\ (\pi - eA)_z c & 0 & e\phi - m_o c^2 & 0 \\ 0 & -(\pi - eA)_z c & 0 & e\phi - m_o c^2 \end{bmatrix} \tag{4.2}$$

If the Hamiltonian is reduced by excluding the second and fourth rows and columns, as indicated by the arrows, the result is the Hamiltonian (2.20):

$$H = \begin{bmatrix} e\phi + m_o c^2 & (\pi - eA)_z c \\ (\pi - eA)_z c & e\phi - m_o c^2 \end{bmatrix} \tag{4.3}$$

The mathematical basis for this reduction is provided by the use of the projection operation. The two-state forms may be considered as a "projection" of the Dirac forms. In addition, associated with a given set of projection operators is a "symmetry property", a property in terms of which the system may be characterized, and for which there exists a distinct set of values. A set of projection operators allows the "classification" of a system in terms of its associated symmetry. Thus the problem that will now be considered is to determine a representation for the projection operator which gives rise to the above two-state form for the Hamiltonian, and subsequently to interpret physically its associated symmetry.

Consider the operator ρ_z represented as

$$\rho_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{4.4}$$

It has the property that $\rho_z^2 = \mathbb{I}$, therefore it is a reflexive operator. Since it is reflexive, it may be shown by theorem that two projection operators may be constructed from the operator:

$$\mathbb{P}_+ = \tfrac{1}{2}(\mathbb{I} + \rho_z) \tag{4.5}$$

$$\mathbb{P}_- = \tfrac{1}{2}(\mathbb{I} - \rho_z) \tag{4.6}$$

If one operates on (4.2) with the projection operators (4.5) and (4.6), the results are

$$H = \begin{bmatrix} e\phi + m_0c^2 & 0 & (\pi - eA)_z c & 0 \\ 0 & 0 & 0 & 0 \\ (\pi - eA)_z c & 0 & e\phi - m_0c^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.7}$$

and

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & e\phi + m_0c^2 & 0 & -(\pi - eA)_z c \\ 0 & 0 & 0 & 0 \\ 0 & -(\pi - eA)_z c & 0 & e\phi - m_0c^2 \end{bmatrix} \tag{4.8}$$

The Hamiltonian (4.3) can be directly identified with (4.7), and the Hamiltonian (4.8) may be identified with (2.50) in Section 2.

The physical interpretation of the symmetry property associated with the projection operator set $\mathbb{P}_+$ and $\mathbb{P}_-$ is best seen by examining the energy eigenfunctions for the free particle with momentum in the Z-direction, in the 4-dimensional representation. These are conveniently given by Messiah,[15] classified in terms of their energy and spin character, and here displayed in Table II. Again, projection of the wave functions by $\mathbb{P}_+$ gives rise to the two, two-state eigenfunctions of the Hamiltonian, represented with respect to the observer's base states $|I\rangle$ and $|II\rangle$. The non-zero reduced eigenfunctions are

$$|\psi_1\rangle \sim \begin{bmatrix} C_I \\ C_{II} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{pc}{E+m_0c^2} \end{bmatrix} \qquad |\psi_3\rangle \sim \begin{bmatrix} C_I' \\ C_{II}' \end{bmatrix} = \begin{bmatrix} \frac{-pc}{E+m_0c^2} \\ 1 \end{bmatrix} \tag{4.9}$$

From the spin-energy classification of the four-dimensional picture, $|\psi_1\rangle$ describes an object of positive energy, whereas $|\psi_3\rangle$ describes a negative energy object. Moreover, the spin associated with $|\psi_1\rangle$ and $|\psi_3\rangle$ is, in both cases, $\frac{1}{2}$, indicating that a degree of freedom has been removed in the two-state analysis. If the 4-dimensional wave functions are projected by the operator $\mathbb{P}_-$, the result is a prepared spin state $-\frac{1}{2}$. The results in

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[15] Messiah, Op. Cit., p. 924.

TABLE II

The Energy Eigenfunctions for the
Spin $\frac{1}{2}$ Free Particle of Momentum (0,0,p)

Energy	$+E_p$		$-E_p$	
Spin	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
	1	0	$-\frac{pc}{E_p+m_0c^2}$	0
ψ	0	1	0	$\frac{pc}{E_p+m_0c^2}$
	$\frac{pc}{E_p+m_0c^2}$	0	1	0
	0	$\frac{-pc}{E_p+m_0c^2}$	0	1
	ψ_1	ψ_2	ψ_3	ψ_4

either case are isomorphic. Thus the above investigation indicates that all features of the two-state analysis may be carried over to the full Dirac formalism, as characteristics attributable to particles in one-dimensional motion, and in a prepared spin state.

That spin is a constant of the motion can also be seen directly from the relation between the two-state and Dirac theory. Using the Dirac theory, Mendlowitz[16] has shown that for a particle with its spin and velocity vectors parallel to the electric field, the spin configuration is a constant of the motion. The restriction to one spatial dimension, inherent in the structure of the two-state theory, makes this spin-velocity-field configuration the only one admissible to two-state description.

One further point is mentioned in this context. The projection of the Dirac Hamiltonian along the $-\frac{1}{2}$ line of spin gives rise to the Hamiltonian (4.8), which is identical to the original two-state Hamiltonian, except for the -1 factors on the off-diagonal. In Section 2 it was mentioned that such an off-diagonal factor was due to the specification of the phase factor $e^{i\delta}$ in the Hamiltonian. Thus there is a relationship between the specification of the phase factor and the spin state of the system. In particular, under the second charge conjugation operation, the change in the phase factor would indicate a flip in spin. This is in keeping with the Dirac

[16] Mendlowitz, American Journal of Physics, 26:19, 1958.

theory, for which charge conjugation involves spin flip. The first conjugation operation is the one associated with the Dirac theory in the Majorana representation.[17] The two representations are identical for the case of one-dimensional motion.

Also, as a consequence of the direct link between the two-state analysis and the Dirac formalism, one would expect correspondences in two-state theory for the operators and transformations included in the full Dirac theory. This is indeed the case. Specifically, the diagonalization of the free or weakly coupled particle Hamiltonian, developed by Foldy and Wouthuysen,[18] has a counterpart in two-state. It is identical to the Foldy and Wouthuysen transformation matrix, projected by the operator $\mathbb{P}_+$. The conditions for such diagonalization, specified by Foldy and Wouthuysen, are equivalent to the adiabatic condition of Section 2. The two-state analogues of the various "dynamic variable" operators in the original and in the Foldy-Wouthuysen representation may also be easily developed. Finally, the representation of the Lorentz group in the two-dimensional state space is identical to the representation in Dirac theory, again projected by $\mathbb{P}_+$.

The developments of Section 3 may also be generalized. If the space-time commutator in the propagation equation (3.3) is set to zero, the resulting equation is

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[17] Davydov, Op. Cit., p. 261.

[18] Foldy and Wouthuysen, Physical Review, 78:29, April 1950.

$$\Box^2 \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left| \left(\frac{mc}{\hbar}\right)^2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} - \frac{2ie}{\hbar} A_\mu \frac{\partial}{\partial x^\mu} \right| \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} - \frac{ie}{\hbar c}\left(\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad (4.10)$$

If one defines the 4-dimensional field intensity tensor

$$F_{\mu\nu} = \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) \qquad (4.11)$$

where

$$A_\mu = \left(\vec{A}, \frac{i\phi}{c}\right) \qquad x^\mu = (\vec{x}, ict) \qquad (4.12)$$

then the interaction term in equation (4.10) may be written as

$$-\frac{e}{\hbar} F_{34} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad (4.13)$$

Now if one defines a set of 4-dimensional matrices

$$\gamma_4 = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix} \qquad \gamma_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \qquad (4.14)$$

and a set of operators

$$\sigma_{\mu\nu} = \tfrac{1}{2} i \left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right) \qquad (4.15)$$

then

$$\sigma_{34} = \tfrac{1}{2}\, i(\gamma_3\gamma_4 - \gamma_4\gamma_3) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (4.16)$$

Projection by the operator $\mathbb{P}_+$ gives rise to a reduced operator

$$\sigma_{34}^{(r)} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad (4.17)$$

Thus (4.13) may be written as

$$\frac{e}{\hbar} F_{34}\, \sigma_{34}^{(r)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad (4.18)$$

and consequently, the 4-dimensional generalization of (4.10) is

$$\Box^2 \psi = \left[\left(\frac{m_0 c}{\hbar}\right)^2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} - \frac{2ie}{\hbar} A_\mu \frac{\partial}{\partial x^\mu} + \frac{e}{\hbar}\,\sigma_{\mu\nu} F_{\mu\nu} \right] \psi \qquad (4.19)$$

This result corresponds to the second order propagation equation suggested by Feynman and Gell-Mann for describing beta decay.[19]

Of greater interest, however, is the generalization of the alternative procedure of Section 3, in which the spin-field interaction is described in terms of a structured state manifold. The 4-dimensional constraints define a curvature tensor associated with a 4-dimensional state manifold, and the Weyl characteristics of

[19] Feynman and Gell-Mann, Op. Cit., p. 193.

the two-dimensional surface associated with the spin-electric field interaction would be carried over directly. Further, the spin-magnetic field interaction would have a description in terms of the curvature of the 4-dimensional state manifold. Indeed, by carrying out the same development in 4-dimensions that was done in 2-dimensions, it is found that the constraint relationships which follow require that the contracted curvature tensor for the 4-dimensional state manifold be proportional to the field intensity tensor, $F_{\mu\nu}$. This result is quite similar to the conclusions drawn by Flint,[20] and Haskey,[21] in a modification of Kaluza's theory, in which the 4-dimensional trajectory of a charged particle moving in an electromagnetic field is related to the null geodesics in a 5-dimensional "structured" space. They found that in order to make this correspondence, it was necessary to assume that the contracted curvature tensor of the 5-dimensional space was proportional to the field intensity tensor $F_{\mu\nu}$. Further, they suggested that the Klein Gordon equation would be the wave equation associated with the null geodesics in the 5-dimensional space, analogous to the D'alembertian wave equation's association with the null geodesics in four dimensional space-time.

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[20] H. Flint, Proceedings of the Royal Society, London, Series A, 131:170, 1931.

[21] H. Haskey, Philosophical Magazine, 27:221, 1939.

Two points of difference between Flint's work and what is described in this paper are worth noting. Whereas Flint postulated the structure of the 5-dimensional space so as to be able to make the correspondence described above, the structure of the state manifold was not postulated, but rather was displayed explicitly in the formalism. Secondly, whereas the role played by the state vector in Flint's theory is vague, the state vector in the preceding work defines the state manifold, as seen in Section 3. The exact relationship between this extension of the two-state theory and the five dimensional theories describing a charged particle's motion requires further study.

A consequence of this reasoning is the possible correspondence between the description of particles of fixed spin, and those of spin zero. The parallels in the description of the two types of systems are well known.[22,23] Though the descriptions are parallel, attempts to unify the two types of systems under one theory have not been completely successful. The developments in this paper suggest another approach to the problem. Spin zero particles are governed by a Klein Gordon equation. In Section 3, it was found that fixed spin particles could satisfy a Klein Gordon equation, subject to a constraint, which could be interpreted from a differential geometric point of view. Thus one can regard spin zero

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[22] Feshbach and Villars, Reviews of Modern Physics, 30:25, 1958.

[23] K. Case, Physical Review, 95:1323, 1954.

particles as described in terms of a state manifold which is geometrically "flat", and particles of spin other than zero as described in terms of a state manifold which is "structured", rather than flat. The degree of structure is dependent upon the spin quality of the particle and the field present, as seen in Sections 3 and 4.

In conclusion, the investigation of the one-dimensional particle in terms of a two-state analysis has lead to an elegant introduction into the basic features of Relativistic Quantum Mechanics. Further, the direct nature of the generalization of the two-state description to Dirac theory has provided a good insight into the relation between the physics of the relativistic quantum particle and the mathematical formalism used to describe it. Finally, the simplicity of the two-state formalism allows connections to other modes of description, such as differential geometry, to be seen quite readily. By so doing, the two-state description provides a clear and versatile model, something which is necessary for gaining insight into phenomena, and germinating original thought.

APPENDIX A

THE PERTURBATION ANALYSIS OF THE TIME DEPENDENT TWO-STATE HAMILTONIAN

Part 1: The adiabatic perturbation analysis in terms of "instantaneous eigenstates".

Let

$$|\Psi(t)\rangle = |\phi_K\,(p,t)\rangle\, e^{-\frac{i}{\hbar}\int_0^t E_K(\theta)\,d\theta}\, C_K(t) \quad . \qquad (2.24)$$

Placing this into the Schroedinger equation (2.21) and using (2.23):

$$\frac{i\hbar^2}{\hbar}\sum_K\left(|\phi_K\rangle\frac{\partial C_K}{\partial t}+\frac{\partial}{\partial t}|\phi_K\rangle\, C_K\right)e^{-\frac{i}{\hbar}\int_0^t E_K d\theta} + \sum_K|\phi_K\rangle C_K E_K e^{-\frac{i}{\hbar}\int_0^t E_K d\theta}$$
$$= \sum_K|\phi_K\rangle C_K E_K e^{-\frac{i}{\hbar}\int_0^t E_K d\theta} \quad .$$

Multiplying by

$$e^{+\frac{i}{\hbar}\int_0^t E_m d\theta}\,\langle\phi_m|$$

and assuming

$$\langle\phi_m|\phi_n\rangle = \delta_{mn} \; ,$$

then

$$\frac{\partial C_m}{\partial t} + \sum_K\langle\phi_m|\frac{\partial}{\partial t}|\phi_K\rangle\, e^{-\frac{i}{\hbar}\int_0^t (E_K-E_m)\,d\theta}\, C_K = 0 \quad . \qquad (A.1)$$

It is now shown that the $K=m$ term in the summation may be transformed away. First it is shown that $\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle$

is imaginary:

$$\frac{\partial}{\partial t}\langle\phi_m|\phi_m\rangle = \left(\frac{\partial}{\partial t}\langle\phi_m|\right)|\phi_m\rangle + \langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle$$
$$= \left(\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle\right)^* + \langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle = 0 \ .$$

Thus

$$\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle = -\left(\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle\right)^*,$$

or $\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle$ is imaginary. Since it is imaginary, represent the product as:

$$\langle\phi_m|\frac{\partial}{\partial t}|\phi_m\rangle = \frac{i}{\hbar}\beta_m,$$

with β_m a real function.

Now consider the transformation

$$|\phi_K'\rangle = |\phi_K\rangle\, e^{+\frac{i}{\hbar}\int_0^t \beta_K(\theta)\,d\theta}$$
$$E_K' = E_K + \beta \ .$$

Substituting this into (A.1) one obtains

$$\begin{aligned}\frac{\partial C_m}{\partial t} &+ \langle\phi_m'|\frac{\partial}{\partial t}|\phi_m'\rangle - \frac{i}{\hbar}\beta_m\langle\phi_m'|\phi_m'\rangle \\ &+ \sum_{K\neq m}\langle\phi_m'|\frac{\partial}{\partial t}|\phi_K'\rangle\, e^{-\frac{i}{\hbar}\int_0^t (E_K'-E_m')\,d\theta} \\ &+ \sum_{K\neq m} -\frac{i}{\hbar}\beta_m\,\langle\phi_m'|\phi_K'\rangle\, e^{-\frac{i}{\hbar}\int_0^t (E_K'-E_m')\,d\theta} = 0 \ .\end{aligned} \tag{A.2}$$

Now since

$$\langle\phi_m|\phi_m\rangle = \langle\phi_m'|\phi_m'\rangle$$

and since $\langle \phi_m' | \frac{\partial}{\partial t} | \phi_m' \rangle$ is imaginary by an argument similiar to the one above, it may be shown that

$$\langle \phi_m' | \frac{\partial}{\partial t} | \phi_m' \rangle = \frac{i}{\hbar} \beta_m .$$

Thus since $\langle \phi_m' | \phi_K' \rangle = \delta_{mK}$, (A.2) reduces to

$$\frac{\partial C_m}{\partial t} + \sum_{K \neq m} \langle \phi_m' | \frac{\partial}{\partial t} | \phi_K' \rangle e^{-\frac{i}{\hbar} \int_0^t (E_K' - E_m') d\theta} .$$

We now relate $\langle \phi_m' | \frac{\partial}{\partial t} | \phi_K' \rangle$ to the Hamiltonian. Multiply (2.23) by the phase factor

$$e^{\frac{i}{\hbar} \int_0^t \beta_K \, d\theta}$$

then

$$\mathbb{H}(t) \, | \phi_K' \rangle = | \phi_K' \rangle \, E_K(t) .$$

Differentiating with respect to time and multiplying by $\langle \phi_m' |$, taking into account the Hermitian property of the Hamiltonian results in

$$\langle \phi_m' | \frac{\partial \mathbb{H}}{\partial t} | \phi_K' \rangle = (E_K - E_m) \langle \phi_m' | \frac{\partial}{\partial t} | \phi_K' \rangle ,$$

or

$$\langle \phi_m' | \frac{\partial}{\partial t} | \phi_K' \rangle = \frac{\langle \phi_m' | \frac{\partial \mathbb{H}}{\partial t} | \phi_K' \rangle}{E_K - E_m} .$$

Thus

$$\frac{\partial C_m}{\partial t} = \sum_{K \neq m} \frac{C_K \langle \phi_m' | \frac{\partial H}{\partial t} | \phi_K' \rangle e^{-\frac{i}{\hbar}\int_0^t (E_K' - E_m') d\theta}}{E_K - E_m} = 0 .$$

Now if the initial conditions are chosen such that

$$C_1(0) = 1.0, \quad C_2(0) = 0.0,$$

then to the first approximation:

$$\frac{\partial C_2}{\partial t} + \frac{\langle \phi_2' | \frac{\partial H}{\partial t} | \phi_1' \rangle}{E_1 - E_2} e^{-\frac{i}{\hbar}\int_0^t (E_1' - E_2') d\theta} = 0 .$$

If E_1' and E_2' are slowly varying functions of time, and treating β_m as small,

$$\frac{\partial C_2}{\partial t} \cong - \frac{\langle \phi_2' | \frac{\partial H}{\partial t} | \phi_1' \rangle e^{-\frac{i}{\hbar}(E_1 - E_2)t}}{E_1 - E_2} .$$

Finally if $\langle \phi_2' | \frac{\partial H}{\partial t} | \phi_1' \rangle$ is slowly varying over the period of time involved,

$$C_2 \cong \left(\frac{\hbar}{i\,(E_1 - E_2)^2} \right) \left\langle \phi_2' \left| \frac{\partial H}{\partial t} \right| \phi_1' \right\rangle \left(e^{-\frac{i}{\hbar}(E_1 - E_2)t} - 1 \right). \qquad (2.27)$$

The condition of "smallness" imposed here is the adiabatic condition; (2.27) is valid if

$$\frac{\hbar}{E_1 - E_2} \left\langle \phi_2' \left| \frac{\partial H}{\partial t} \right| \phi_1' \right\rangle \ll E_1 - E_2 \quad . \qquad (2.29)$$

Part 2: The sudden approximation analysis of the free particle given a momentum "suddenly".

Problem: Consider a particle initially at rest with respect to the observer and possessing a positive definite energy. At $t=0$ let the particle be given a momentum p in a "sudden" manner. Find the state vector describing the system.

Write

$$\mathbb{H}(t) = \mathbb{H}_0 + \mathbb{V}(t),$$

where

$$\mathbb{V}(t) = \begin{cases} 0 & t<0 \\ V & t \geq 0 \end{cases}$$

and $\mathbb{H}_0$ is the Hamiltonian for a particle at rest with repsect to the observer. For $t<0$, the system is in a positive definite energy state, thus its state may be represented as

$$|\Psi(t)\rangle = |I\rangle e^{-\frac{iE_I}{\hbar}t}.$$

Now for $t>0$, $\mathbb{H}(t)$ is the Hamiltonian associated with a particle possessing constant momentum p with respect to the observer. It has a complete set of energy eigenvectors, $|+(p)\rangle$ and $|-(p)\rangle$. It is assumed that this set is complete for all time. Thus the general state of the system may be represented in terms of these energy eigenvectors as:

$$|\Psi(t)\rangle = \sum_{K=+}^{-} |K\rangle e^{-\frac{i}{\hbar}E_K t} C_K \, .$$

If continuity of the state vector is demanded at ,

$$|\Psi(0)\rangle = |\mathrm{I}\rangle = |+(p)\rangle C_+ + |-(p)\rangle C_- \, .$$

Using the information in Appendix A, part 3, the expansion factors can be determined:

$$C_+ = \langle +|\mathrm{I}\rangle \, , \qquad C_- = \langle -|\mathrm{I}\rangle \, ,$$

or

$$|\Psi(t)\rangle = \left\{ |+\rangle e^{-\frac{iE_+t}{\hbar}} - |-\rangle \frac{pc}{E+mc^2} e^{-\frac{iE_-t}{\hbar}} \right\} \sqrt{\frac{E+mc^2}{2E}} \, .$$

In terms of the observer's base states $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$, Appendix A, part 3, implies that the state vector is represented as:

$$|\Psi(t)\rangle = \frac{E+mc^2}{2E} \left\{ |\mathrm{I}\rangle \left(e^{-\frac{iE_+t}{\hbar}} + \left(\frac{pc}{E+mc^2}\right)^2 e^{-\frac{iE_-t}{\hbar}} \right) + |\mathrm{II}\rangle \frac{pc}{E+mc^2} \left(e^{-\frac{iE_+t}{\hbar}} - e^{-\frac{iE_-t}{\hbar}} \right) \right\} .$$

Part 3: The expansion of the energy eigenstates $|+(p)\rangle$ and $|-(p)\rangle$ in terms of the observer's base states $|I\rangle$ and $|II\rangle$.

$|+(p)\rangle$ is the positive energy eigenstate associated with the momentum p . With respect to the observer's base states $|I\rangle$ and $|II\rangle$, the eigenstate is expanded as

$$|+(p)\rangle = |I\rangle C_I(p) + |II\rangle C_{II}(p),$$

where C_I and C_{II} satisfy

$$\begin{bmatrix} mc^2 & pc \\ pc & -mc^2 \end{bmatrix}\begin{pmatrix} C_I \\ C_{II} \end{pmatrix} = E_+(p)\begin{pmatrix} C_I \\ C_{II} \end{pmatrix},$$

for $E_+(p) = \sqrt{(pc)^2+(mc^2)^2} \triangleq +E(p)$.

This implies

$$C_{II} = \frac{pc}{E+mc^2} C_I \quad ,$$

C_I arbitrary.

Thus $|+(p)\rangle$ may be represented as

$$|+(p)\rangle = \sqrt{\frac{E+mc^2}{2E}}\left(|I\rangle + \frac{pc}{E+mc^2}|II\rangle\right).$$

Likewise $|-(p)\rangle$ is the negative energy eigenstate for the momentum p . With respect to $|I\rangle$ and $|II\rangle$, it is expanded as

$$|-(p)\rangle = |I\rangle C_I'(p) + |II\rangle C_{II}'(p) \quad ,$$

where C_I' and C_{II}' satisfy

$$\begin{bmatrix} mc^2 & pc \\ pc & -mc^2 \end{bmatrix} \begin{pmatrix} C_I' \\ C_{II}' \end{pmatrix} = E_-(p) \begin{pmatrix} C_I' \\ C_{II}' \end{pmatrix},$$

for $E_-(p) = -\sqrt{(pc)^2 + (mc^2)^2} \triangleq -E(p)$.

This implies

$$C_I' = \frac{-pc}{E+mc^2} C_{II}' ,$$

C_{II}' arbitrary.

Thus $|-(p)\rangle$ may be represented as

$$|-(p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \left(|I\rangle \frac{-pc}{E+mc^2} + |II\rangle \right).$$

Conversely, the observer's states $|I\rangle$ and $|II\rangle$ can be expressed in terms of $|+(p)\rangle$ and $|-(p)\rangle$ by regarding the observer's frame as moving with velocity $-\frac{p}{m}$ with respect to the particle's frame. The substitution $p \rightarrow -p$ in the above development then yields:

$$|I\rangle = \sqrt{\frac{E+mc^2}{2E}} \left(|+(p)\rangle - |-(p)\rangle \frac{pc}{E+mc^2} \right)$$

$$|II\rangle = \sqrt{\frac{E+mc^2}{2E}} \left(|+(p)\rangle \frac{pc}{E+mc^2} + |-(p)\rangle \right) .$$

Finally, the transformation between the eigenstates $|+(p)\rangle$ and $|-(p)\rangle$, and $|I\rangle$, and $|II\rangle$ may be written as

$$|\mathcal{K}\rangle = SS\,|K\rangle,$$

where

$$|\mathcal{K}\rangle = \{\,|+(p)\rangle,\ |-(p)\rangle\,\}$$

$$|J\rangle = \{\,|I\rangle,\ |II\rangle\,\}\ .$$

In terms of a matrix representation

$$\langle J\,|\,\mathcal{K}\rangle = \langle J|SS|L\rangle\langle L|K\rangle\ .$$

From the above knowledge of the expansion factors of $|+\rangle$ and $|-\rangle$ in terms of $|I\rangle$ and $|II\rangle$, the matrix display is

$$S_{JK}(p) = \langle J\,|SS|\,K\rangle = \sqrt{\frac{E+mc^2}{2E}}\begin{bmatrix} 1 & \frac{-pc}{E+mc^2} \\ \frac{pc}{E+mc^2} & 1 \end{bmatrix}$$

Since $|+(p)\rangle$ and $|-(p)\rangle$ are the energy eigenstates for the momentum p, and $|I\rangle$ and $|II\rangle$ are the eigenstates for the momentum zero with respect to the observer, SS is the transformation on the state vectors induced by a Lorentz transformation of coordinates. Thus S_{JK} is the representation of the Lorentz group in the state vector space for the observer.

APPENDIX B

THE TWO-STATE PROBABILITY CURRENT

Part 1: The development of a two-state probability current.

Given

$$\begin{bmatrix} mc^2 & pc \\ pc & -mc^2 \end{bmatrix}\begin{pmatrix} C_I \\ C_{II} \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} C_I \\ C_{II} \end{pmatrix}$$

make the operator correspondence

$$p \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial z}$$

and the corresponding transformation on the expansion factors:

$$C_i \rightarrow C_i e^{\frac{i}{\hbar c}pz} \equiv a_i$$

Then

$$\begin{bmatrix} mc^2 & c\frac{\hbar}{i}\frac{\partial}{\partial z} \\ c\frac{\hbar}{i}\frac{\partial}{\partial z} & -mc^2 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{B.4}$$

Taking the Hermitian conjugate of both sides of this equation:

$$-i\hbar\frac{\partial}{\partial t}[a_1^*, a_2^*] = [a_1^*, a_2^*]\begin{bmatrix} mc^2 & -c\frac{\hbar}{i}\frac{\partial}{\partial z} \\ -c\frac{\hbar}{i}\frac{\partial}{\partial z} & -mc^2 \end{bmatrix} \tag{B.5}$$

Post multiplying (B.5) by $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and promultiplying (B.4) by $[a_1^*, a_2^*]$, and subtracting the former from the latter leads to the following equation:

$$\frac{\partial}{\partial t}(a_1^* a_1 + a_2^* a_2) = -c\frac{\partial}{\partial z}(a_1^* a_2 + a_2^* a_1)$$

If this is interpreted as a one-dimensional continuity equation, then a probability current may be defined as

$$J_z = c(a_1^* a_2 + a_2^* a_1) = [a_1^*, a_2^*]\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

If the current is interpreted in terms of the classical analogue

$$J_z = \rho V_z$$

then the matrix

$$\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

may be regarded as representing the velocity operator.

Part 2: The two-state current for the free particle set into motion "suddenly".

Consider a free particle initially at rest with respect to the observer and possessing a positive definite energy, and which is given a constant momentum p at $t = 0$ in a "sudden" manner so as to violate the adiabatic condition. Its state vector may be represented in terms of the observer's base states $|I\rangle$ and $|II\rangle$ as shown in Appendix A-2:

$$|\Psi(t)\rangle = \frac{E+mc^2}{2E}\left\{|I\rangle\left(e^{-\frac{iE_+t}{\hbar}} + \left(\frac{pc}{E+mc^2}\right)^2 e^{-\frac{iE_-t}{\hbar}}\right) + |II\rangle\frac{pc}{E+mc^2}\left(e^{-\frac{iE_+t}{\hbar}} - e^{-\frac{iE_-t}{\hbar}}\right)\right\}$$

Thus the probability current associated with the particle by the observer is

$$J_z = c\left[\frac{pc(E+mc^2)}{2E^2}\left(1-\frac{1}{2}\cos\frac{E_+-E_-}{\hbar}t\right) - \frac{(pc)^3}{2E^2(E+mc^2)}\left(1-\frac{1}{2}\cos\frac{E_+-E_-}{\hbar}t\right)\right]$$

For convenience of interpretation, consider the case where $pc \ll m_0c^2$, so that the approximation $E_+ - E_- \approx 2m_0c^2$ can be made. Then up to first order terms in pc, the probability current with respect to the observer is

$$J_z = c\left[\frac{pc}{mc^2}\left(1-\frac{1}{2}\cos\frac{2mc^2}{\hbar}t\right)\right] = \frac{p}{m} - \frac{p}{2m}\cos\frac{2mc^2}{\hbar}t$$

where again, $E_+ = E$, and $E_- = -E$.

APPENDIX C

THE DEVELOPMENT OF THE CHARGE CONJUGATE STATE AND ITS EQUATION OF MOTION

Let

$$\left|-(p)\right\rangle = \left(\left|\mathrm{I}\right\rangle - \left|\mathrm{II}\right\rangle \frac{pc}{E+mc^2} \right) \sqrt{\frac{E+mc^2}{2E}}$$

Then the expansion factors for $\left|-(p)\right\rangle$ satisfy the equation

$$\begin{bmatrix} e\phi+mc^2 & (\pi-eA)c \\ (\pi-eA)c & e\phi-mc^2 \end{bmatrix} \begin{pmatrix} C_{\mathrm{I}}' \\ C_{\mathrm{II}}' \end{pmatrix} = \begin{pmatrix} C_{\mathrm{I}}' \\ C_{\mathrm{II}}' \end{pmatrix} E_{-}(p) = -\begin{pmatrix} C_{\mathrm{I}}' \\ C_{\mathrm{II}}' \end{pmatrix} E(p)$$

consider the operator correspondence

$$\pi \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial z}$$

and the induced transformation

$$C_i' \longrightarrow C_i' e^{\frac{i}{\hbar}\pi z} \equiv a_i'$$

Then the equation (C.2) becomes

$$\begin{bmatrix} e\phi+mc^2 & \left(\frac{\hbar}{i}\frac{\partial}{\partial z}-eA\right)c \\ \left(\frac{\hbar}{i}\frac{\partial}{\partial z}-eA\right)c & e\phi-mc^2 \end{bmatrix} \begin{pmatrix} a_{\mathrm{I}}' \\ a_{\mathrm{II}}' \end{pmatrix} = -\begin{pmatrix} a_{\mathrm{I}}' \\ a_{\mathrm{II}}' \end{pmatrix} E(p) .$$

The complex conjugate of the last equation is:

$$\begin{bmatrix} -(e\phi+mc^2) & (\frac{\hbar}{i}\frac{\partial}{\partial z}+eA)c \\ (\frac{\hbar}{i}\frac{\partial}{\partial z}+eA)c & -e\phi+mc^2 \end{bmatrix} \begin{pmatrix} a_I'^{*} \\ a_{II}'^{*} \end{pmatrix} = + \begin{pmatrix} a_I'^{*} \\ a_{II}'^{*} \end{pmatrix} E(p) \qquad (C.6)$$

Now define the state $|*\rangle$ constructed from the negative energy eigenstate in the following manner:

$$|*\rangle = |K|-\rangle$$

$$|*\rangle = |I\rangle \gamma_I + |II\rangle \gamma_{II}$$

$$\begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_I'^{*} \\ C_{II}'^{*} \end{pmatrix}$$

Then equation (C.6) may be written as

$$\begin{bmatrix} -(e\phi+mc^2) & (\frac{\hbar}{i}\frac{\partial}{\partial z}+eA)c \\ (\frac{\hbar}{i}\frac{\partial}{\partial z}+eA)c & -e\phi+mc^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \gamma_I e^{-\frac{i}{\hbar}\pi z} \\ \gamma_{II} e^{-\frac{i}{\hbar}\pi z} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \gamma_I e^{-\frac{i}{\hbar}\pi z} \\ \gamma_{II} e^{-\frac{i}{\hbar}\pi z} \end{pmatrix} E(p)$$

Multiplying by the inverse matrix, and inverting the operator correspondence,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -(e\phi+mc^2) & (\pi+eA)c \\ (\pi+eA)c & -e\phi+mc^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} = \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} E(p)$$

Thus the equation of motion for $|*\rangle$ with respect to $|I\rangle$ and $|II\rangle$ is

$$\begin{bmatrix} -e\phi+mc^2 & (\pi+eA)c \\ (\pi+eA)c & -e\phi-mc^2 \end{bmatrix} \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} = \begin{pmatrix} \gamma_I \\ \gamma_{II} \end{pmatrix} E(p)$$

Since $E(p) = E_+(p)$ in the notation, it is concluded that $|*\rangle$ is a positive energy eigenstate of its Hamiltonian, and that its Hamiltonian is related to the free particle Hamiltonian by the substitution $e \to -e$. Thus $|*\rangle$ is interpreted as representing a particle of positive energy and charge $-e$.

APPENDIX D

THE DEVELOPMENT OF THE KLEIN GORDON EQUATION

Re-write the state vector for the relativistic particle with respect to the states $|I\rangle$ and $|II\rangle$, the states of definite energy when the particle is at rest, as

$$|\Psi\rangle = |I\rangle \Psi_1(t) + |II\rangle \Psi_2(t) \ .$$

Then the equation of motion for the state amplitudes becomes, using Hamiltonian (2.20) and the operator correspondence:

$$i\hbar \frac{d}{dt}\begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix} = \begin{vmatrix} e\phi + mc^2 & \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA\right)c \\ \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA\right)c & e\phi - mc^2 \end{vmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

or

$$i\hbar \frac{\partial \Psi_1}{\partial t} = (e\phi + mc^2)\Psi_1 + \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA\right)c\,\Psi_2 \qquad (D.3)$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA\right)c\,\Psi_1 + (e\phi - mc^2)\Psi_2 \qquad (D.4)$$

Taking symmetric and anti-symmetric combinations of Ψ_1 and Ψ_2 :

$$\begin{aligned} \mathbb{S} &= \Psi_1 + \Psi_2 \\ \mathbb{K} &= \Psi_1 - \Psi_2 \end{aligned} \qquad (D.5)$$

the following equations result, through the addition and subtraction of (D.3) and (D.4)

$$mc^2\mathbb{K} = i\hbar\frac{\partial \mathbb{S}}{\partial t} - e\phi\mathbb{S} - \frac{\hbar c}{i}\frac{\partial \mathbb{S}}{\partial x} + eAc\,\mathbb{S}$$

$$mc^2\mathbb{S} = i\hbar\frac{\partial \mathbb{K}}{\partial t} - e\phi\mathbb{K} - \frac{\hbar c}{i}\frac{\partial \mathbb{S}}{\partial x} + eAc\,\mathbb{K}$$

From the last equations, the following second order equations result:

$$\Box^2 \mathbb{K} = \left(\frac{mc}{\hbar}\right)^2 \mathbb{K} + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu \mathbb{K} - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu}\mathbb{K} - \frac{2ie}{\hbar} A_\mu \frac{\partial \mathbb{K}}{\partial x^\mu}$$
$$+ \frac{ie}{\hbar c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\mathbb{K} - \frac{1}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right]\mathbb{K} \qquad (D.7)$$

$$\Box^2 \mathbb{S} = \left(\frac{mc}{\hbar}\right)^2 \mathbb{S} + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu \mathbb{S} - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu}\mathbb{S} - \frac{2ie}{\hbar} A_\mu \frac{\partial \mathbb{S}}{\partial x^\mu}$$
$$- \frac{ie}{\hbar c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\mathbb{S} + \frac{1}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right]\mathbb{S} \qquad (D.8)$$

where

$$A_\mu = \left(A, \frac{i\phi}{c}\right) \quad x^\mu = (x, ict).$$

Adding and subtracting (D.7) and (D.8), and using (D.5) yields

$$\Box^2 \psi_1 = \left(\frac{mc}{\hbar}\right)^2 \psi_1 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu \psi_1 - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu}\psi_1 - \frac{2ie}{\hbar} A_\mu \frac{\partial \psi_1}{\partial x^\mu}$$
$$- \frac{ie}{\hbar c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\psi_2 + \frac{1}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right]\psi_2$$

$$\Box^2 \psi_2 = \left(\frac{mc}{\hbar}\right)^2 \psi_2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu \psi_2 - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu}\psi_2 - \frac{2ie}{\hbar} A_\mu \frac{\partial \psi_2}{\partial x^\mu}$$
$$- \frac{ie}{\hbar c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\psi_1 + \frac{1}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right]\psi_1$$

or more compactly,

$$\Box^2 \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left[\left(\frac{mc}{\hbar}\right)^2 + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} - \frac{2ie}{\hbar} A_\mu \frac{\partial}{\partial x^\mu}\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$
$$\left\{+ \frac{1}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right] - \frac{ie}{\hbar c}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\right\}\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}$$

Now the Klein Gordon equation, in terms of the function $f(x)$, is

$$\Box^2 f(x) = \left(\frac{mc}{\hbar}\right)^2 f(x) + \left(\frac{e}{\hbar}\right)^2 A_\mu A_\mu f(x) - \frac{ie}{\hbar}\frac{\partial A_\mu}{\partial x^\mu} f(x) - \frac{2ie}{\hbar} A_\mu \frac{\partial f(x)}{\partial x^\mu}$$

Thus the amplitudes Ψ_1 and Ψ_2 can satisfy a Klein-Gordon equation, if they are subjected to the constraints

$$\left[\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right]\Psi_1 = \frac{ie}{\hbar}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\Psi_1$$

$$\left|\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial t}\right|\Psi_2 = \frac{ie}{\hbar}\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}\right]\Psi_2 \quad .$$

APPENDIX E

BASIC DIFFERENTIAL GEOMETRY RELATIONS

Let the position of any point P on a manifold be specified by a vector $\vec{r}(x^{\mu})$ relative to some origin, where x^{μ}; $\mu = 1, \cdots, n$ are a set of parameters whose number is such as to allow the unique specification of the point on the manifold. The number n is the dimension of the manifold.

Assuming $\vec{r}(x^{\mu}) \in C^{1}$ at P, the set of vector functions $\vec{a}_{\nu} = \frac{\partial \vec{r}}{\partial x^{\nu}}$ $\nu = 1, \cdots, n$, span the n-dimensional space tangent to the manifold at P. Since the vecotr functions are linearly independent, they form a basis in the tangent space.

The coefficients of connection at the point P on the manifold is given by the following relation:

$$\frac{\partial \vec{a}_{\nu}}{\partial x^{\tau}} = \vec{a}_{\rho} L^{\rho}_{\nu\tau} .$$

It is noted that the definition of the coefficients of connection involve only components of the differentiation lying in the tangent space at the point P. The curvature tensor $L^{\alpha}_{\beta\gamma\delta}$ is defined by the following equation:

$$\frac{\partial^2 \vec{a}_{\nu}}{\partial x^{\sigma} \partial x^{\tau}} - \frac{\partial^2 \vec{a}_{\nu}}{\partial x^{\tau} \partial x^{\sigma}} = \vec{a}_{\mu} L^{\mu}_{\nu\sigma\tau} .$$

From (E.1) it follows that

$$L^{\mu}_{\nu\sigma\tau} = \frac{\partial L^{\mu}_{\nu\tau}}{\partial x^{\sigma}} - \frac{\partial L^{\mu}_{\nu\sigma}}{\partial x^{\tau}} + L^{\rho}_{\nu\tau} L^{\mu}_{\rho\sigma} - L^{\rho}_{\nu\sigma} L^{\mu}_{\rho\tau} .$$

The contracted curvature tensor is defined by

$$L^{\mu}_{\nu\sigma\mu} = L_{\nu\sigma} .$$

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