OSCILLATORY BEHAVIOR OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATION IN A B*-ALGEBRA

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CHAPTER I

INTRODUCTION

This dissertation is concerned with the oscillatory behavior of solutions of the second order linear differential equation

$$\text{(E)} \qquad [R(x)Y']' + Q(x)Y = 0$$

on $[a,\infty)$, where R and Q are continuous symmetric matrix-valued functions or, more generally, are operators which assume their values in a B*-algebra, and where R is positive definite.

Since matrices form particular B*-algebras, we concentrate on them in certain instances in order to contribute to the research in this area. However, the principle objective of this dissertation is to extend the research which established oscillation criteria for the case where R and Q are $n\times n$ matrices to the generalized B*-algebra case.

Several authors have obtained results for the case where R and Q are $n\times n$ matrices. Among the most notable are F. V. Atkinson [2], G. J. Etgen [9], H. C. Howard [14], C. A. Swanson [24], and E. C. Tomastik [27]. Recently C. M. Williams [28] has extended the results of Hille [15] on the nonoscillation properties of equation (E) by allowing R and Q to take their values on a B*-algebra.

In Chapter II the properties of a B*-algebra are delineated, and a characterization of B*-algebras is given. Also, the set $\mathcal{G}$ consisting of positive functionals which operate on a B*-algebra is introduced and characterized.

The existence and uniqueness of solutions of equation (E) are discussed in Chapter III. In addition, basic properties of solutions are delineated.

Chapter IV establishes oscillation criteria for equation (E) using the characteristics of the set G and allowing R and Q to take their values in a B*-algebra.

The trigonometric differential system and the polar coordinate transformation are treated in Chapter V. Sufficient conditions for oscillation are established for the trigonometric differential system and then are extended to equation (E) by means of the polar coordinate transformation.

In Chapter VI we extend the results of K. Kreith [17] which formulate comparison criteria for both oscillation and nonoscillation.

The dissertation is concluded in Chapter VII with some remarks concerning extensions to nonlinear differential equations and to nonlinear differential inequalities.

CHAPTER II

B*-ALGEBRAS

2.1 Characteristics of B*-Algebras.

The purpose of this section is to define a B*-algebra and to present some of the basic properties concerning B*-algebras which will be required throughout this paper.

A Banach space is a normed linear space over a scalar field, which is complete in the metric determined by its norm, $||\cdot||$.

A Banach algebra is a Banach space with an associative multiplication defined and such that the inequality $||AB|| \leq ||A|| \cdot ||B||$ holds for all elements A,B in the space. A Banach algebra is called unital if there is an element I such that $IA = AI = A$ for each element A in the algebra and $||I|| = 1$. An element A in a unital Banach algebra is called nonsingular, or regular, if there is an element A^{-1} in the algebra such that $AA^{-1} = A^{-1}A = I$. If an element does not meet this requirement, it is called singular.

If $\mathcal{B}$ is a unital Banach algebra over the complex scalar field $\mathcal{C}$, *i.e.*, a complex unital Banach algebra, and $B \in \mathcal{B}$, then the spectrum of B is defined:

$$\sigma(B) = \{\lambda \in \mathcal{C} \;/\; \lambda I - B \quad \text{is singular}\}.$$

A complex unital Banach algebra $\mathcal{B}$ is called a B*-algebra provided it has an involutory operation $(\cdot)^*$ with the following properties:

(*i*) If $A \in \mathcal{B}$, there is a unique $A^* \in \mathcal{B}$ such that $(A^*)^* = A$.

(*ii*) If $A, B \in \mathcal{B}$, then $(A+B)^* = A^*+B^*$.

(*iii*) If $A \varepsilon \mathcal{B}$ and $\alpha \varepsilon \mathcal{C}$, then $(\alpha A)^* = \bar{\alpha}A^*$, where $\bar{\alpha}$ is the complex conjugate of α.

(*iv*) If $A, B \varepsilon \mathcal{B}$, then $(AB)^* = B^*A^*$.

(*v*) For all $A \varepsilon \mathcal{B}$, $||A^*A|| = ||A||^2$.

An element $A \varepsilon \mathcal{B}$ is called <u>symmetric</u>, or <u>hermitian</u>, if $A^* = A$. Furthermore the following properties are assumed to hold:

(*i*) Each symmetric element has a real spectrum.

(*ii*) The set of symmetric elements with nonnegative real spectra is closed under addition, multiplication by positive scalars, and passage to a limit.

(*iii*) Each element of the form A^*A has a nonnegative real spectrum.

A symmetric element A having a positive (nonnegative), (negative) spectrum will be called <u>positive</u> (<u>nonnegative</u>), (<u>negative</u>) <u>definite</u> and will be denoted $A > 0$, $(A \geq 0)$, $(A < 0)$. From (*ii*) above, sums and positive multiples of nonnegative definite elements are nonnegative definite.

The preceding definition is the one employed in [15, pg. 110] and in [28, pg. 3].

An example of a B*-algebra is $\mathcal{M}_n$, the algebra of n×n matrices with complex entries. For this B*-algebra the operation $(\cdot)^*$ is defined so that A^* is the conjugate transpose of the matrix A. Another B*-algebra, and one that is most important from the standpoint of this paper, is the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Rickart [23, pg. 244] has shown that every B*-algebra is isometrically*-isomorphic to an algebra of bounded linear operators on a complex Hilbert space.

Assuming $\mathcal{B}$ is a B*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and $B \varepsilon \mathcal{B}$, the spectrum of B consists of:

(*i*) $\{\lambda \varepsilon \mathcal{C} \;/\; \lambda I - B$ is 1-1 but not onto$\}$.

(*ii*) $\{\lambda \varepsilon \mathcal{C} \;/\; \lambda I - B$ is not 1-1$\}$.

The elements in (*ii*) are called <u>eigenvalues</u> of B, and any nonzero element $\alpha \varepsilon \mathcal{H}$ such that $(\lambda I - B)\,\alpha = 0$ is called an <u>eigenvector</u> of B.

For an element $B \varepsilon \mathcal{B}$, the number $\gamma(B) = \sup \{|\lambda| \;/\; \lambda \varepsilon \sigma(B)\}$ is called the <u>spectral radius</u> of B. Clearly, the spectrum of B lies in the disc centered at 0 with radius $\gamma(B)$.

The following theorems describe some of the properties of the spectrum of an element A in a B*-algebra $\mathcal{B}$. Their proofs are in Hille [15] so they will not be repeated here.

<u>Theorem 2.1.1</u>. If A is a positive definite element of a B*-algebra, then A^{-1} is positive definite.

<u>Theorem 2.1.2</u>. The unit element I is positive definite.

<u>Theorem 2.1.3</u>. If A is a symmetric element and $\sigma(A) \subset [a,b]$ where $a > 0$, then

$$aI \leq A \leq bI \quad \text{and} \quad b^{-1}I \leq A^{-1} \leq a^{-1}I.$$

Theorem 2.1.4. If A is positive definite then for any positive integer n, A^n is positive definite. In addition, if $\sigma(A) \subset [a,b]$ where $a > 0$, then $a^2 I \leq A^2 \leq b^2 I$.

It has been pointed out by Hille [15] and Taylor [25] that nonsingular elements (other than the identity) exist in a B*-algebra $\mathcal{B}$, and that the inverse operation is continuous.

Theorem 2.1.5. If $A \in \mathcal{B}$ is nonsingular, and $B \in \mathcal{B}$ is such that $||A - B|| < \frac{1}{||A^{-1}||}$, then B is nonsingular. Also,

$$||A^{-1} - B^{-1}|| \leq ||A^{-1}||^2 \; ||A - B|| \; / \; (1 - ||A||^{-1} \; ||A - B||).$$

The following theorems are well known (see Hille [15], Taylor [25], or Williams [28]), and are used extensively throughout this paper. The first theorem provides the existence of "square roots" in a B*-algebra $\mathcal{B}$.

Theorem 2.1.6. If $A \in \mathcal{B}$, and $A > 0$, $(A \geq 0)$, then there is an element $M \in \mathcal{B}$ such that $M > 0$, $(M \geq 0)$ and $A = M^2$.

Theorem 2.1.7. If $A \in \mathcal{B}$ and $A \geq 0$, $(A \leq 0)$, then for any $C \in \mathcal{B}$ $C^*AC \geq 0$, $(C^*AC \leq 0)$. If C is nonsingular and $A > 0$, $(A < 0)$, then $C^*AC > 0$, $(C^*AC < 0)$.

Theorem 2.1.8. If $A \in \mathcal{B}$, $A \geq 0$, and $C \in \mathcal{B}$ such that $C^*AC = 0$, then $AC = 0$.

Theorem 2.1.9. If $A, C \in \mathcal{B}$ such that $0 < A \leq C$, then $0 < C^{-1} \leq A^{-1}$.

Integrals and derivatives of B*-algebra-valued functions will be required throughout this paper. The definition of an integral will be that of the ordinary Riemann-type integral. In particular, if $B(x)$ is a function defined on the compact interval $[a,b]$ which takes its values in a B*-algebra $\mathcal{B}$, and $\{a = x_0, x_1, \ldots, x_n = b\}$ denotes a partition of this interval, then $B(x)$ is called integrable on $[a,b]$ provided $\sum_{i=1}^{n} B(x_i)(x_i - x_{i-1})$ has a limit in $\mathcal{B}$ as the norm of the partition approaches zero. This limit is called the integral of $B(x)$ on $[a,b]$ and is denoted $\int_a^b B(x)dx$. It can be verified that $B(x)$ is integrable on $[a,b]$ whenever $B(x)$ is continuous or piecewise continuous on $[a,b]$.

By using the definition and basic properties of a B*-algebra $\mathcal{B}$, and the definition of the integral, we obtain the following useful results.

Theorem 2.1.10. If $B(x)$ is a continuous $\mathcal{B}$-valued function on $[a,b]$, then $\left\|\int_a^b B(x)dx\right\| \leq \int_a^b \|B(x)\|dx$.

Theorem 2.1.11. If $B(x)$ is a continuous $\mathcal{B}$-valued function on $[a,b]$ and if $B(x) > 0$ $(B(x) \geq 0)$ on $[a,b]$, then $\int_a^b B(x)dx > 0$, $(\int_a^b B(x)dx \geq 0)$. Likewise $B(x) < 0$ $(B(x) \leq 0)$ on $[a,b]$ implies

$\int_a^b B(x)dx < 0$ ($\int_a^b B(x)dx \leq 0$).

If B(x) is a $\mathcal{B}$-valued function on the interval (a,b), it is called <u>differentiable</u> at $x_0 \varepsilon$ (a,b) provided $\lim_{x \to x_0} \frac{B(x) - B(x_0)}{x - x_0}$ exists in $\mathcal{B}$. The limit is denoted $B'(x_0)$ and is called the <u>derivative</u> of B at x_0. In addition, if B is a continuous $\mathcal{B}$-valued function on [a,b], and $A(x) = \int_a^x B(t)dt$, then for each x ε (a,b) the derivative A'(x) exists and equals B(x).

2.2 Positive Functionals.

<u>This section introduces the concept of functionals operating on a</u> B*-algebra.

Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted by $\langle , \rangle$. Assume that $\mathcal{B}^*$ is the space of bounded linear operators on $\mathcal{H}$. Denote by $\mathcal{S}$, the subspace of $\mathcal{B}^*$ consisting of the symmetric operators. Define the set $\mathcal{G} = \{g:\mathcal{B}^* \to C \;/\; g$ is linear and $g(A^*A) \geq 0$ for each $A \;\varepsilon\; \mathcal{B}^*\}$. Then each $g \;\varepsilon\; \mathcal{G}$ is a <u>positive functional</u> as defined by Rickart [23, pg. 212]. It is clear that $g \;\varepsilon\; \mathcal{G}$ is positive if and only if $g(A) \geq 0$ for all nonnegative definite elements A in $\mathcal{S}$. According to Rickart [23, pg. 215], every positive functional is bounded, *i.e.*, continuous.

It is easy to show that the set $\mathcal{G}$ is not empty. For example, for $\alpha \;\varepsilon\; \mathcal{H}$, define g_α as follows: $g_\alpha(A) = \langle A\alpha,\alpha \rangle$ where $A \;\varepsilon\; \mathcal{B}^*$. Certainly g_α is linear and $g_\alpha(A^*A) = \langle A^*A\alpha,\alpha \rangle = \langle A\alpha,A\alpha \rangle \geq 0$, so g_α is positive.

The theorem which follows proves that G is closed under multiplication by nonnegative numbers and summation.

<u>Theorem 2.2.1</u>. If $g_1, g_2, \ldots, g_n$ are elements of G and $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then $\sum_{i=1}^{n} a_i g_i \ \varepsilon\ G$.

<u>Proof</u>. It is clear that $\sum_{i=1}^{n} a_i g_i$ is a linear functional on $\mathcal{B}^*$. Choose any $A \ \varepsilon\ \mathcal{B}^*$. Then $\left(\sum_{i=1}^{n} a_i g_i\right)(A^*A) = \sum_{i=1}^{n} a_i g_i(A^*A) \geq 0$ since $a_i \geq 0$ and $g_i(A^*A) \geq 0$ for $i = 1, 2, \ldots, n$. Thus $\sum_{i=1}^{n} a_i g_i \ \varepsilon\ G$.

It follows from Theorem 2.2.1 that G is a <u>convex</u> set, that is, $\{af + (1 - a)g \ / \ 0 \leq a \leq 1\} \subset G$ whenever $f, g \ \varepsilon\ G$.

As another example of a positive functional, let $\mathcal{B}^* = M_n$, the set of n×n matrices, and consider the functional <u>trace (•)</u> defined by trace $(A) = \sum_{i=1}^{n} a_{ii}$, where $A = (a_{ij}) \ \varepsilon\ M_n$. It is well known that

trace $(A) = \Sigma\lambda_i$ where λ_i, $1 \leq i \leq n$, are the eigenvalues of A. Clearly trace (•) is linear, and since A^*A is nonnegative definite, trace $(A^*A) \geq 0$. Therefore, trace (•) is a positive functional. Also, trace $(\cdot) = \sum_{i=1}^{n} g_{e_i}(\cdot)$, where e_i is the n component vector with "1" as its i^{th} component and "0's" elsewhere.

The following theorem is easily established using the linearity and continuity of $g \in G$.

Theorem 2.2.2. If A is a continuous B^*-valued function on [a,b], then $g[\int_a^b A(x)dx] = \int_a^b g[A(x)]dx$. In addition, if $F(x) = \int_a^x A(t)dt$, then F is a differentiable B^*-valued function on [a,b], g(F) is a differentiable complex-valued function on [a,b], and $\{g[F(x)]\}' = g[F'(x)]$.

If F is a differentiable B^*-valued function, then the derivative of g(F) will be denoted g'(F).

Lemma 2.2.3 is a Cauchy-Schwartz inequality for positive functionals. It is taken from Rickart [23, pg. 213].

Lemma 2.2.3. If $g \in G$, then $|g(A^*B)|^2 \leq g(A^*A)\, g(B^*B)$ for all $A,B \in B^*$.

This lemma is used to establish several basic properties of positive functionals.

Theorem 2.2.4. Let $g \in G$. Then $g = 0$, the zero functional, if and only if $g(I) = 0$.

Proof. Clearly, if $g = 0$, then $g(I) = 0$. On the other hand, suppose $g \in G$ and, $g(I) = 0$. If $A \in B^*$, then by Lemma 2.2.3,

$$|g(A)|^2 = |g(IA)|^2 \leq g(I)\ g(A^*A).$$

Since $g(I) = 0$, $|g(A)| = 0$ for all $A \in \mathcal{B}^*$. Thus g is the zero functional.

Theorem 2.2.5. If $g \in \mathcal{G}$, $g \neq 0$, and $A \in \mathcal{B}^*$ is positive definite, then $g(A) > 0$.

Proof. Let $g \in \mathcal{G}$. Choose $A \in \mathcal{B}^*$ such that $A > 0$. There is, by Theorem 2.1.6, a $B \in \mathcal{B}^*$ such that $B > 0$ and $A = B^2$. Therefore, by Lemma 2.2.3, $0 < |g(I)|^2 = |g(BB^{-1})|^2 \leq g(BB)\ g(B^{-1}B^{-1}) = g(A)g(A^{-1})$. Hence $g(A) > 0$.

Finally we arrive at a theorem which associates a positive number with each nonzero element of $\mathcal{G}$.

Theorem 2.2.6. If $g \in \mathcal{G}$ and $g \neq 0$, then there exists a positive number ρ_g such that $g(A^*A) \geq \rho_g |g(A)|^2$ for all $A \in \mathcal{B}^*$.

Proof. Let $A \in \mathcal{B}^*$. Choose any $g \in \mathcal{G}$ such that $g \neq 0$. By Lemma 2.2.3 $|g(A)|^2 = |g(IA)|^2 \leq g(A^*A)g(I)$. This reduces to $g(A^*A) \geq \frac{1}{g(I)} |g(A)|^2$.

CHAPTER III

SECOND ORDER DIFFERENTIAL EQUATIONS IN A B*-ALGEBRA

Throughout the remainder of this paper we shall assume that $\mathcal{H}$ is a complex Hilbert space with inner product denoted $\langle , \rangle$, and that $\mathcal{B}^*$ is the B*-algebra of bounded linear operators on $\mathcal{H}$. We shall also be concerned with the special case where $\mathcal{H}$ is the linear space of ordered n-tuples and $\mathcal{B}^* = \mathcal{M}_n$ is the collection of n×n matrices. In this special case we shall assume, without loss of generality, that $\mathcal{H}$ is the real vector space of ordered n-tuples of real numbers and $\mathcal{B}^*$ is the set of n×n matrices ~~with real entries. For convenience this will be referred to~~ as the "finite dimensional" case.

In this chapter we will consider existence, uniqueness, and other basic properties of the solutions of the linear differential equation

$$\text{(E)} \qquad [R(x)Y']' + Q(x)Y = 0,$$

where R and Q are continuous, symmetric, $\mathcal{B}^*$-valued functions on the interval $[a,\infty)$, and R is positive definite.

A <u>solution</u> of (E) is a $\mathcal{B}^*$-valued function Y on $[a,\infty)$ such that each of Y and RY' is continuously differentiable and $[R(x)Y'(x)]' + Q(x)Y(x) \equiv 0$ on $[a,\infty)$, *i.e.*, Y satisfies (E).

Our first theorem establishes the existence and uniqueness of solutions of equation (E). Basically its proof is that of Picard and Lindelöf, which utilizes the classical methods of successive approximations, and can be found in Hille [15, chapter 6].

Theorem 3.1. If Y_0 and Z_0 are elements of $\mathcal{B}*$ and $b \varepsilon [a,\infty)$, then there is a unique solution Y of equation (E) such that $Y(b) = Y_0$ and $R(b)Y'(b) = Z_0$.

The next theorem introduces a concept which is made use of extensively throughout this paper.

Theorem 3.2. If Y_1 and Y_2 are solutions of equation (E) on $[a,\infty)$, then

$$Y_1^*(x)R(x)Y_2'(x) - Y_1^{*\prime}(x)R(x)Y_2(x) \equiv K, \text{ a constant}$$

on $[a,\infty)$.

Proof. Let $M(x) = Y_1^*(x)R(x)Y_2'(x) - Y_1^{*\prime}(x)R(x)Y_2(x)$. Then, by taking the derivative, we have

$$Y_1^{*\prime}RY_2' + Y_1^*(RY_2')' - (RY_1')^{*\prime}Y_2 - Y_1^{*\prime}RY_2'$$

$$= -Y_1^*QY_2 + Y_1^*QY_2 = 0.$$

This implies $M(x) \equiv K$, a constant.

Two solutions, Y_1 and Y_2, of (E) are said to be mutually conjoined if $Y_1^*(x)R(x)Y_2'(x) - Y_1^{*\prime}(x)R(x)Y_2(x) \equiv 0$ on $[a,\infty)$. In particular, a solution Y of equation (E) is called self conjoined whenever $Y^*(x)R(x)Y'(x) \equiv Y^{*\prime}(x)R(x)Y(x)$ on $[a,\infty)$.

The concept of conjoined solutions was used by M. Morse [19, chapter 3] in his research in the calculus of variations. Conjoined solutions of (E), also referred to as <u>prepared</u> solutions, can be shown to exist merely by requiring a solution Y to satisfy

$$Y^*(b)R(b)Y'(b) = Y^{*\prime}(b)R(b)Y(b)$$

at some point b. Then since M as defined in Theorem 3.2 is constant, $M = 0$ and Y is a conjoined solution.

A solution Y of equation (E) is called <u>nontrivial</u> if there exists at least one point $c \varepsilon [a,\infty)$ such that $Y(c)$ is nonsingular.

<u>Theorem 3.3</u>. If Y is a nontrivial solution of equation (E), then

$$Y^*(x)Y(x) + [R(x)Y'(x)]^* R(x)Y'(x) > 0$$

on $[a,\infty)$.

<u>Proof</u>. Let Y be a nontrivial solution of (E) and assume $Y(c)$ is nonsingular, where $c \varepsilon [a,\infty)$.

Suppose there is a point $b \varepsilon [a,\infty)$ and a vector γ, $\gamma \neq 0$, such that

$$\langle [Y^*(b)Y(b) + (R(b)Y'(b))^*R(b)Y'(b)]\gamma, \gamma\rangle = 0.$$

Then it follows that

$$Y(b)\gamma = R(b)Y'(b)\gamma = 0.$$

Let $y(x) = Y(x)\gamma$. The vector y is a solution of the operator-vector equation

$$[R(x)y']' + Q(x)y = 0. \tag{3.1}$$

Since the basic existence and uniqueness theorem (Theorem 3.1) also holds for (3.1), we can conclude that $y(x) \equiv 0$ on $[a,\infty)$. But $y(c) = Y(c)\gamma = 0$ contradicts the fact that $Y(c)$ is nonsingular.

Corollary 3.4. Let Y be a solution of (E). If there is a point $c \in [a,\infty)$ such that $Y(c)$ is nonsingular and

$$Y^*(c)R(c)Y'(c) = [R(c)Y'(c)]^*Y(c),$$

then Y is nontrivial and conjoined.

A slightly stronger result holds in the finite dimensional case, see [2, chapter 10] or [19, chapter 3].

Corollary 3.5. Consider equation (E) where R and Q are $n \times n$ matrices. A solution is nontrivial and conjoined if and only if there is a point b such that

$$Y^*(b)Y(b) + [R(b)Y'(b)]^*R(b)Y'(b) > 0$$

and

$$Y^*(b)R(b)Y'(b) = [R(b)Y'(b)]^*Y(b).$$

Hereafter, the term "solution" will be interpreted to mean nontrivial and conjoined solution.

A solution Y of (E) is called oscillatory if for each $b \in [a,\infty)$, there is a $c \geq b$ such that $Y(c)$ is singular. A solution Y of (E) is nonoscillatory if it is not oscillatory.

The differential equation (E) is said to be oscillatory provided it has an oscillatory solution; otherwise it is called nonoscillatory.

In the finite dimensional case, *i.e.*, R and Q are $n \times n$ matrices, we have the following result attributed to Morse [19, Theorem 5.1]. This theorem gives us important information about the oscillation properties of equation (E). It is known as the Morse Separation Theorem.

Theorem 3.6. If Y is a solution of equation (E) which is nonsingular on $[b,c]$, and if U is any other solution of equation (E), then U has at most n singularities on $[b,c]$, multiple singularities being counted according to their multiplicities. (Note: A singularity of U is a zero of $\det U$.)

It follows from this theorem that if (E) has an oscillatory solution, then all solutions of (E) oscillate. On the other hand, if (E) has a nonoscillatory solution then no solution of (E) oscillates.

For many theorems in the following chapters we shall restrict equation (E) by setting $R(x) \equiv I$, the identity element. With this restriction, equation (E) becomes

$$Y'' + Q(x)Y = 0, \tag{e}$$

where $Q(x)$ is a continuous symmetric function on $[a,\infty)$ taking its values in $\mathcal{B}^*$.

CHAPTER IV

OSCILLATION CRITERIA

FOR LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

4.1 Introduction.

In Sections 4.2 and 4.3 of this chapter, oscillation criteria for the differential equations (E) and (e) are investigated. The previous results of Swanson [24], Etgen [10], Hayden and Howard [13], and Allegretto and Erb [1] are shown to be special cases of the theorems in these sections.

In Section 4.4, the research of P. Hartman [11], W. J. Coles [4], and J. W. Macki and J. S. W. Wong [18] on weighted averages and averaging pairs is extended.

In Section 4.5 oscillation criteria for equation (E) are established and our results are compared with those of E. C. Tomastik [27].

Throughout this chapter we shall make extensive use of the properties of the set $\mathcal{G}$ of positive functionals on $\mathcal{B}^*$ introduced in Chapter II.

4.2 Oscillation Criteria for Equation (e).

In this section we establish oscillation criteria for

(e) $$Y'' + Q(x)Y = 0,$$

where Q is a continuous symmetric $\mathcal{B}^*$-valued function on $[a,\infty)$. We then demonstrate that many known oscillation criteria are special cases of our results.

Theorem 4.2.1. If there exists a $g \varepsilon G$ such that $\lim_{x\to\infty} g[\int_a^x Q(t)dt] = +\infty$, then equation (e) is oscillatory.

Proof. Assume the theorem is false. Then there is a nontrivial conjoined solution Y of (e) which is nonsingular on $[b,\infty)$ for some $b \geq a$. Define the operator S on $[b,\infty)$ by $S(x) = -Y'(x)Y^{-1}(x)$. Using the fact that Y is conjoined

$$S^*(x) = -[Y^*(x)]^{-1}[Y^*(x)]' = -[Y^*(x)]^{-1}Y^*(x)Y'(x)Y^{-1}(x) = S(x).$$

Hence, S(x) is symmetric.

By differentiating S(x) we find that

$$S'(x) = Q(x) + S^2(x),$$

and integrating this equation yields

$$S(x) = S(b) + \int_b^x Q(t)dt + \int_b^x S^2(t)dt.$$

Now, utilizing the functional $g \varepsilon G$ mentioned in the hypothesis, we have

$$g[S(x)] = g[S(b)] + g[\int_b^x Q(t)dt] + g[\int_b^x S^2(t)dt].$$

Since $\lim_{x\to\infty} g[\int_a^x Q(t)dt] = +\infty$, there is a $c \geq b$ such that

$$(4.2.1) \qquad g[S(x)] > g[\int_b^x S^2(t)dt] \qquad \text{for } x \varepsilon [c,\infty).$$

Let $W(x) = \int_b^x S^2(t)dt$. Then $W'(x) = S^2(x)$. From (4.2.1) we have $g[S(x)] > 0$ for $x \varepsilon [c,\infty)$. This fact, together with the properties of g introduced in Chapter II, gives

$$g[W(x)] = g[\int_b^x S^2(t)dt] = \int_b^x g[S^2(t)]dt \geq \int_b^x \rho_g\{g[S(t)]\}^2 dt > 0$$

on $[c,\infty)$. Also, we have

$$g'[W(x)] = g[W'(x)] = g[S^2(x)] \geq \rho_g\{g[S(x)]\}^2.$$

By using (4.2.1),

$$g'[W(x)] > \rho_g\{g[\int_b^x S^2(t)dt]\}^2 = \rho_g\{g[W(x)]\}^2.$$

Thus $\dfrac{g'[W(x)]}{\{g[W(x)]\}^2} > \rho_g$.

Integration yields the inequality

$$\frac{1}{g[W(c)]} > \frac{1}{g[W(c)]} - \frac{1}{g[W(x)]} > \rho_g(x - c).$$

Clearly this inequality cannot hold on $[c,\infty)$, and we have a contradiction.

The following corollary investigates a particular form of equation (E).

Corollary 4.2.2. If in equation (E) $R(x) = K$, where K is a nonsingular constant $\mathcal{B}^*$-valued function on $[a,\infty)$ such that $K^{-1}Q(x)$ is symmetric and continuous on $[a,\infty)$, and if there is a $g \in \mathcal{G}$ such that $\lim_{x\to\infty} g[\int_a^x K^{-1}Q(t)dt] = +\infty$, then (e) is oscillatory.

Proof. Equation (E) reduces to

$$KY'' + Q(x)Y = 0.$$

Since K is nonsingular on $[a,\infty)$, we can write this equation as

$$Y'' + K^{-1}Q(x)Y = 0.$$

Because $K^{-1}Q(x)$ is a symmetric, continuous $\mathcal{B}^*$-valued function, the hypothesis of Theorem 4.2.1 is satisfied.

The following theorems contain a variety of well known oscillation criteria. It will be demonstrated that they are special cases of Theorem 4.2.1.

Theorem A. If Q in equation (e) is a continuous, symmetric, n×n matrix which has a diagonal element $q_{ii}(x)$ such that $\lim_{x\to\infty} \int_a^x q_{ii}(t)dt = +\infty$, then (e) is oscillatory.

Let e_i denote the constant vector with "1" as its i^{th} component and "0's" elsewhere. Define g_{e_i} by $g_{e_i}(A) = \langle Ae_i, e_i \rangle = a_{ii}$ for any n×n matrix $A=(a_{ij})$. As indicated in Chapter II, $g_{e_i} \varepsilon \mathcal{G}$.

Now,

$$g_{e_i}[\int_a^x Q(t)dt] = \int_a^x g_{e_i}[Q(t)]dt = \int_a^x q_{ii}(t)dt$$

so by Theorem 4.2.1, equation (e) is oscillatory.

This result can also be found in the work of C. A. Swanson [24, Theorem 1].

The next theorem has been established by T. L. Hayden and H. C. Howard [13, Theorem 2].

Theorem B. Let Q in equation (e) be a continuous, symmetric, $\mathcal{B}^*$-valued function. If $\lambda(x)$ denotes the minimum eigenvalue of $\int_a^x Q(t)dt$ and $\lim_{x\to\infty} \lambda(x) = +\infty$, then equation (e) is oscillatory.

Suppose $\lim_{x\to\infty} \lambda(x) = +\infty$. Choose any constant vector α such that $||\alpha|| = 1$. Then

$$\lambda(x) \leq \langle[\int_a^x Q(t)dt]\alpha,\alpha\rangle \qquad \text{on } [a,\infty).$$

As indicated in Section 2.2, the vector α determines a functional $g_\alpha \in \mathcal{G}$, so we have

$$g_\alpha[\int_a^x Q(t)dt] \geq \lambda(x).$$

Hence, the hypothesis of Theorem 4.2.1 is satisfied.

In order to discuss the following theorem, extracted from the work of Allegretto and Erb [1], it is necessary to introduce some additional notation. Let k and n be positive integers with $k \leq n$, and let $O_{k,n}$ denote the set of strictly increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$. If F is an $n\times n$ matrix and $\gamma = \{m_1, m_2, \ldots, m_k\}$ is an element of $O_{k,n}$, then $F(\gamma,\gamma)$ denotes the $k\times k$ submatrix of F which is obtained by deleting all rows and columns from F except for the rows and columns $m_1, m_2, \ldots, m_k$. For any matrix A, let ΣA denote the sum of all entries of A.

Theorem C. Let Q in equation (e) be a continuous, symmetric, n×n matrix. If there exists a $\gamma \in O_{k,n}$ such that $\lim_{x\to\infty} \int_a^x \Sigma Q(\gamma,\gamma)(t)dt = +\infty$, then equation (e) is oscillatory.

By definition $Q(\gamma,\gamma)$ is the k×k submatrix of Q obtained by removing all rows and columns except for the rows and columns $m_1, m_2, \ldots, m_k$. Let $e(\gamma)$ denote the constant n-component vector whose $m_1, m_2, \ldots, m_k$ entries are ones, while all other entries are zero. Then

$$\langle Q(t)e(\gamma),\ e(\gamma)\rangle = \Sigma Q(\gamma\gamma)(t),$$

so

$$\langle [\int_a^x Q(t)dt]e(\gamma), e(\gamma)\rangle = \int_a^x \Sigma Q(\gamma,\gamma)(t)dt.$$

The vector $e(\gamma)$ determines the functional $g_{e(\gamma)} \in G$, so we have

$$g_{e(\gamma)}[\int_a^x Q(t)dt] = \int_a^x \Sigma Q(\gamma,\gamma)(t)dt.$$

Therefore, by Theorem 4.2.1, (e) is oscillatory.

The next theorem, proven for n×n matrices, is also a special case of Theorem 4.2.1 since the functional "trace" is an element of G.

Theorem D. (Etgen [10, Theorem 2]). If $\lim_{x\to\infty}$ trace $[\int_a^x Q(t)dt] = +\infty$, then equation (e) is oscillatory.

4.3 Other Oscillation Theorems for Equation (e).

This section is a further extension of the research of Allegretto and Erb [1], Etgen [10], and Hayden and Howard [13]. The results of these authors are demonstrated to be special cases of the main theorems of this section.

<u>Theorem 4.3.1</u>. Let Q be a continuous symmetric $\mathcal{B}^*$-valued function on $[a,\infty)$. If there exists a positive differentiable scalar function q such that $\lim_{x\to\infty} \int_a^x \frac{1}{q(t)}\,dt = +\infty$, and if

$$J(x) = \int_a^x \{q(t)Q(t) - [(q'(t))^2/4q(t)]I\}dt + \frac{q'(t)}{2}\, I$$

has the property that there exists a $g \in \mathcal{G}$ such that $\lim_{x\to\infty} g[J(x)] = +\infty$, then equation (e) is oscillatory.

<u>Proof</u>. Assume the theorem is false. Then equation (e) has a nonsingular conjoined solution Y on the interval $[b,\infty)$ for some $b \geq a$. On this interval we define S by

$$\frac{1}{q(x)}\, S(x) = -Y'(x)Y^{-1}(x).$$

Then $S(x) = -q(x)Y'(x)Y^{-1}(x)$ and

$$S'(x) = -q'(x)Y'(x)Y^{-1}(x) + q(x)Q(x) + q(x)Y'(x)Y^{-1}(x)Y'(x)Y^{-1}(x)$$

which can be written

$$S'(x) = \frac{1}{q(x)}\,[S(x) + \frac{q'(x)}{2}\, I]^2 + q(x)Q(x) - \frac{[q'(x)]^2}{4q(x)}\, I.$$

Integration yields

$$S(x) + \frac{q'(x)}{2}\, I = S(b) + \int_b^x \frac{1}{q(t)}\,[S(t) + \frac{q'(t)}{2}\, I]^2 dt$$

$$+ \int_b^x \{q(t)Q(t) - \frac{[q'(t)]^2}{4q(t)}\, I\}dt + \frac{q'(x)}{2}\, I.$$

Hence, utilizing the $g \in G$ mentioned in the hypothesis, we have

$$g[S(x) + \frac{q'(x)}{2} I] = g[S(b)] + g\{\int_b^x \frac{1}{q(t)} [S(t) + \frac{q'(t)}{2} I]^2 dt\} + g[J(x)].$$

Since $\lim_{x\to\infty} g[J(x)] = +\infty$, there is a number $c \geq b$ such that

(4.3.1) $$g[S(x) + \frac{q'(x)}{2} I] > g\{\int_b^x \frac{1}{q(t)} [S(t) + \frac{q'(t)}{2} I]^2 dt\}$$

on $[c,\infty)$.

Let $W(x) = \int_b^x \frac{1}{q(t)} [S(t) + \frac{q'(t)}{2} I]^2 dt.$

Then $W'(x) = \frac{1}{q(x)} [S(x) + \frac{q'(x)}{2} I]^2$. By making use of the properties of g, we have

$$g'[W(x)] = g[W'(x)]$$

$$= \frac{1}{q(x)} g[S(x) + \frac{q'(x)}{2} I]^2 \geq \frac{\rho_g}{q(x)} \{g[S(x) + \frac{q'(x)}{2} I]\}^2.$$

Then from (4.3.1),

$$g'[W(x)] > \frac{\rho_g}{q(x)} \{g[\int_b^x \frac{1}{q(t)} (S(t) + \frac{q'(t)}{2} I)^2 dt]\}^2$$

on $[c,\infty)$. This implies

(4.3.2) $$g'[W(x)] > \frac{\rho_g}{q(x)} \{g[W(x)]\}^2 \quad \text{on } [c,\infty).$$

Now (4.3.1) also implies $g[S(x) + \frac{q'(x)}{2} I] > 0$ on $[c,\infty)$ and from this fact plus the properties of g

$$g[W(x)] = \int_b^x \frac{1}{q(t)} g[S(t) + \frac{q'(t)}{2} I]^2 dt$$

$$\geq \int_b^x \frac{\rho_g}{q(t)} \{g[S(t) + \frac{q'(t)}{2} I]\}^2 dt > 0$$

on $[c,\infty)$. Therefore (4.3.2) can be written

$$\frac{g'[W(x)]}{\{g[W(x)]\}^2} > \frac{\rho_g}{q(x)} \qquad \text{on } [c,\infty).$$

By integrating this inequality we get

$$\frac{1}{g[W(c)]} > \frac{1}{g[W(c)]} - \frac{1}{g[W(x)]} > \rho_g \int_c^x \frac{1}{q(t)}\, dt.$$

Since $\lim_{x\to\infty} \int_c^x q^{-1}(t)\,dt = +\infty$, this inequality cannot hold on $[c,\infty)$, and we have a contradiction.

Referring to the notation introduced for $n\times n$ matrices prior to Theorem C, we can use Theorem 4.3.1 to prove the following corollary.

Corollary 4.3.2. Let Q be a continuous symmetric $n\times n$ matrix and let f be a positive differentiable scalar function on $[a,\infty)$ such that $\lim_{x\to\infty} \int_a^x \frac{1}{f(x)}\, dx = +\infty$. If there exists a $\gamma \in O_{k,n}$ such that

$$\lim_{x\to\infty} \int_a^x f(t)\{\Sigma Q(\gamma,\gamma)(t) - [\frac{f'(t)}{2f(t)}]^2 \Sigma I(\gamma,\gamma)\}dt + \frac{f'(x)}{2}\, \Sigma I(\gamma,\gamma) = +\infty,$$

then equation (e) is oscillatory.

Proof. By definition $Q(\gamma,\gamma)$ is the $k\times k$ submatrix of Q obtained by deleting all rows and columns of Q except for the rows and columns $m_1, m_2, \ldots, m_k$ where $\{m_i\}_{i=1}^{k}$ is an increasing subsequence of $\{j\}_{j=1}^{n}$. Let $e(\gamma)$ be the constant n-component vector whose $m_1, m_2, \ldots, m_k$ entries are ones, while all other entries are zeros. Then

$$\langle[\int_a^x f(t)\{Q(t) - [\frac{f'(t)}{2f(t)}]^2 I\}dt + \frac{f'(x)}{2} I]e(\gamma),\ e(\gamma)\rangle$$

$$= \int_a^x f(t)\{\Sigma Q(\gamma,\gamma)(t) - [\frac{f'(t)}{2f(t)}]^2 \Sigma I(\gamma,\gamma)\}dt + \frac{f'(x)}{2}\ \Sigma I(\gamma,\gamma).$$

The vector $e(\gamma)$ determines a functional $g_{e(\gamma)} \varepsilon\ G$, so we have $\lim_{x\to\infty} g_{e(\gamma)}\{\int_a^x f(t)Q(t) - \frac{[f'(t)]^2}{4f(t)}\ I\ dt + \frac{f'(t)}{2}\ I\} = +\infty$ which satisfies the hypothesis of Theorem 4.3.1.

By using the same terminology we can show that the next theorem, found in the work of Allegretto and Erb [1, Corollary 2], is a special case of Corollary 4.3.2.

Theorem E. If there exists a $\gamma\ \varepsilon\ O_{k,m}$ such that $\lim_{x\to\infty} \int_a^x t[\Sigma Q(\gamma,\gamma)(t) - \frac{\Sigma I(\gamma,\gamma)}{4(t)^2}]dt = +\infty$, then equation (e) is oscillatory.

Let $f(t) = t$. Then f is a positive differentiable scalar function on $[a,\infty)$ and $\lim_{x\to\infty} \int_a^x t^{-1}dt = +\infty$. In addition

$$\int_a^x t[\Sigma Q(\gamma,\gamma)(t) - \frac{\Sigma I(\gamma,\gamma)}{4t^2}]dt + \frac{1}{2}\ \Sigma I(\gamma,\gamma)$$

$$= \int_a^x f(t)\{\Sigma Q(\gamma,\gamma)(t) - [\frac{f'(t)}{2f(t)}]^2 \Sigma I(\gamma,\gamma)\}dt + \frac{f'(x)}{2}\ \Sigma I(\gamma,\gamma).$$

Since the first term on the left side of this equation has limit $+\infty$ as $x \to \infty$, and the second term is constant, the hypothesis of Corollary 4.3.2 is satisfied.

If the reasoning which led to Theorem D, Section 4.2, is repeated, we have the next result.

Theorem F. (Etgen [10, Theorem 3]). Let Q be a continuous, symmetric, n×n matrix and define r and J as in Theorem 4.3.1. If $\lim_{x\to\infty}$ trace $[J(x)] = +\infty$, then equation (e) is oscillatory.

The very same logic used to demonstrate that Theorem B, Section 4.2, was a special case of Theorem 4.2.1 can be used to show that the following result of Hayden and Howard [13, Theorem 3] is a special case of Theorem 4.3.1.

Theorem G. Let Q be a continuous, symmetric function taking its values in $\mathcal{B}^*$. If there is a positive differentiable scalar function q such that $\lim_{x\to\infty} \int_a^x \frac{1}{q(t)}\, dt = +\infty$ and if J, defined in Theorem 4.3.1, has the property that the minimum eigenvalue of J(x) has limit $+\infty$ as $x \to \infty$, then equation (e) is oscillatory.

4.4 Weighted Averages and Averaging Pairs.

Additional information about the oscillation of equation (e) can be obtained by considering the concepts of weighted averages of $\int_a^x Q(t)dt$ and of averaging pairs as introduced by Coles [4] and Macki and Wong [18], respectively. These concepts lead to oscillation criteria which differ extensively from those developed in sections 4.2 and 4.3.

A weighted average of $\int_a^x Q(t)dt$ is defined by W. J. Coles [4] as follows: Let f be a nonnegative continuous scalar function on $[a,\infty)$ such that $\int_a^x f(t)dt \not\equiv 0$, and there is a number $b > a$ with the property that

$$(4.4.1) \qquad A(x) = \frac{\int_a^x f(t)[\int_a^t Q(s)ds]dt}{\int_a^x f(t)dt}$$

exists on $[b,\infty)$.

Coles' results were obtained for the scalar equation

$$(4.4.2) \qquad y'' + q(x)y = 0.$$

Our next theorem extends his work to equation (e).

Theorem 4.4.1. Let Q be a continuous, symmetric function on $[a,\infty)$ which takes its values in $\mathcal{B}^*$. If there exists a $g \in \mathcal{G}$ and a nonnegative continuous scalar function f on $[a,\infty)$ satisfying

$$\lim_{x\to\infty} \int_b^x \{f(t)[\int_a^t f(s)ds]^k \,/\, \int_a^t f^2(s)ds\}dt = +\infty$$

for some k, $0 \leq k < 1$, and for $b \geq a$, and

$$\lim_{x\to\infty} g[A(x)] = +\infty,$$

where A is given by (4.4.1), then (e) is oscillatory.

Proof. Assume (e) is nonoscillatory. Then there is a nontrivial conjoined solution Y of (e) such that Y is nonsingular on $[b,\infty)$ for some $b \geq a$. We can define $S(x) = -Y'(x)Y^{-1}(x)$ on $[b,\infty)$. Since Y is conjoined, S is symmetric. Differentiation of S(x) and subsequent integration of the derivative yields

$$S(x) = S(b) + \int_b^x Q(t)dt + \int_b^x S^2(t)dt.$$

Multiplication by f(x) followed by integration results in

$$\int_b^x f(t)S(t)dt = \int_b^x f(t)S(b)dt + \int_b^x f(t)\int_b^t Q(s)dsdt$$

$$+ \int_b^x f(t)\int_b^t S^2(s)dsdt.$$

Let $g \varepsilon G$ be the positive functional given in the hypothesis. Then

$$g[\int_b^x f(t)S(t)dt] = \left\{\int_b^x f(t)dt\right\}\{g[S(b) + \frac{\int_b^x f(t)\int_b^t Q(s)dsdt}{\int_b^x f(t)dt}]\}$$

$$+ g\ [\int_b^x f(t)\int_b^t S^2(s)dsdt].$$

By hypothesis, the first term on the right side of this equation increases to $+\infty$, so there exists a number $c > b$ such that for $x \geq c$,

$$g[\int_b^x f(t)S(t)dt] > g[\int_b^x f(t)\int_b^t S^2(s)dsdt]. \tag{4.4.3}$$

Let $W(x) = \int_b^x f(t)\int_b^t S^2(s)dsdt]$. Then $W'(x) = f(x)\int_b^x S^2(t)dt$. By using the properties of g,

$$g'[W(x)] = g[W'(x)] = f(x)\int_b^x g[S^2(t)]dt \geq \rho_g f(x)\int_b^x \{g[S(t)]\}^2 dt.$$

Also

$$g'[W(x)] \geq \frac{\rho_g f(x)}{\int_b^x f^2(t)dt}\left[\int_b^x f^2(t)dt\right]\left[\int_b^x \{g[S(t)]\}^2 dt\right].$$

By using the Cauchy-Schwartz inequality,

$$g'[W(x)] \geq \frac{\rho_g f(x)}{\int_b^x f^2(t)dt}\{\int_b^x f(t)g[S(t)]dt\}^2,$$

so
$$g'[W(x)] \geq \frac{\rho_g f(x)}{\int_b^x f^2(t)dt} \{g[\int_b^x f(t)S(t)dt]\}^2.$$

This implies, by (4.4.3), that

$$g'[W(x)] > \frac{\rho_g f(x)}{\int_b^x f^2(t)dt} \{g[\int_b^x f(t)\int_b^t S^2(s)dsdt\}^2,$$

or

$$g'[W(x)] > \frac{\rho_g f(x)}{\int_b^x f^2(t)dt} \{g[W(x)]\}^2. \tag{4.4.4}$$

From the definition of W(x), we derive

$$g[W(x)] \geq g[\int_b^x f(t)\int_b^d S^2(s)dsdt]$$

for $d \geq c$ because $\int_b^t S^2(s) \geq \int_b^d S^2(s)ds$ on $[d,\infty)$.

Since $\int_b^d S^2(s)ds$ is a constant,

$$g[W(x)] \geq [\int_b^x f(t)dt][g\{\int_b^d S^2(s)ds\}].$$

Let $g[\int_b^d S^2(s)ds] = m$, a positive constant. This implies

$$\{g[W(x)]\}^k \geq m^k[\int_b^x f(t)dt]^k$$

on $[d,\infty)$, for $0 \leq k < 1$. Now combine this with (4.4.4) to obtain

$$\{g[W(x)\}^k g'[W(x)] > \rho_g m^k \frac{f(x)[\int_b^x f(t)dt]^k}{\int_b^x f^2(t)dt} \{g[W(x)]\}^2 \tag{4.4.5}$$

on $[d,\infty)$.

It can be deduced from (4.4.3) that $\int_b^x f(t)g[S(t)]dt > 0$ on $[c,\infty)$.

This implies that $f(x) \not\equiv 0$ and $g[S(x)] \not\equiv 0$ on $[b,c]$. Hence

$\int_b^x\{g[S(t)]\}^2dt > 0$ on $[c,\infty)$ and $\int_b^x f(t)\int_b^t\{g[S(s)]\}^2dsdt > 0$ on $[c,\infty)$.

Therefore

$$g[W(x)] = \int_a^x f(t)\int_b^t g[S^2(s)]dsdt \geq \rho_g\int_b^x f(t)\int_b^t\{g[S(s)]\}^2dsdt > 0$$

on $[c,\infty)$ so we can rewrite (4.4.5) as

$$\left\{\frac{1}{g[W(x)]}\right\}^{2-k} g'[W(x)] > \rho_g m^k \frac{f(x)[\int_b^x f(t)dt]^k}{\int_b^x f^2(t)dt}$$

on $[d,\infty)$.

By integrating, we get

$$\frac{1}{1-k}\left\{\frac{1}{g[W(c)]}\right\}^{1-k} > \frac{1}{1-k}\left\{\frac{1}{g[W(c)]}\right\}^{1-k} - \left\{\frac{1}{g[W(x)]}\right\}^{1-k}$$

$$> \rho_g m^k \int_c^x f(t)\frac{[\int_b^t f(s)ds]^k}{\int_b^t f^2(s)ds}\,dt.$$

Since $\lim_{x\to\infty}\int_c^x f(t)\dfrac{[\int_b^t f(s)ds]^k}{\int_b^t f^2(s)ds}\,dt = +\infty$, this inequality cannot hold on

$[d,\infty)$, and we have a contradiction.

The following theorem uses the concept of averaging pairs to develop oscillation criteria for equation (e).

A pair of scalar functions (σ,α) is called an <u>averaging pair</u> provided

(*i*) σ is continuous on $[a,\infty)$, $\sigma \geq 0$, $\alpha > 0$, and α is differentiable on $[a,\infty)$.

(*ii*) $\lim_{x\to\infty} \int_a^x \sigma(t)(\int_a^t \alpha(s)\sigma^2(s)ds)^{-1}dt = +\infty.$

This is the definition used by Macki and Wong [18] in their study of equation (4.4.2). Our theorem extends their results to equation (e).

<u>Theorem 4.4.2.</u> Let Q be a continuous symmetric function which takes its values in $\mathcal{B}^*$. If there exists an averaging pair (σ,α) and a $g \in G$ such that

$$\lim_{T\to\infty} g \underset{T}{\Pi}(K_\alpha + \frac{\alpha'}{2} I) = +\infty$$

where
$$K_\alpha(s) = \int_a^s [\alpha(u)Q(u) - \frac{1}{4}\frac{(\alpha'(u))^2}{\alpha(u)} I]du$$

and
$$\underset{T}{\Pi}(K) = (\int_a^T \sigma(s)ds)^{-1}\int_a^T \sigma(s)K(s)ds,$$

then equation (e) is oscillatory.

<u>Proof.</u> Assume the theorem is false. Then there is a nontrivial conjoined solution Y of (e) which is nonsingular on $[b,\infty)$ for some $b \geq a$. We can define $S(x) = -\alpha(x)Y'(x)Y^{-1}(x)$ on $[b,\infty)$. Since Y is conjoined S is symmetric.

By differentiating S we have

$$S'(x) = \frac{1}{\alpha(x)} [S(x) + \frac{\alpha'(x)}{2} I]^2 + [\alpha(x)Q(x) - \frac{(\alpha'(x))^2}{4\alpha(x)}].$$

Put $V(x) = S(x) + \frac{\alpha'(x)}{2} I$ and integrate to get

$$S(x) = \int_b^x \frac{V^2(t)}{\alpha(t)} dt + K_\alpha(x) + C,$$

where $C \in \mathcal{B}^*$ is a constant of integration. Then adding $\frac{\alpha'(x)}{2} I$ to both sides yields

$$V(x) = S(x) + \frac{\alpha'(x)}{2} I = \int_b^x \frac{V^2(t)}{\alpha(t)} dt + K_\alpha(x) + \frac{\alpha'(x)}{2} I + C.$$

This implies

$$\underset{T}{\Pi}(V) = \underset{T}{\Pi}(\int_b^x \frac{V^2(t)}{\alpha(t)} dt) + \underset{T}{\Pi}(K_\alpha + \frac{\alpha'(x)}{2} I) + C.$$

Now, using the g described in the hypothesis,

$$g\left[\underset{T}{\Pi}(V)\right] = g\left[\underset{T}{\Pi}(\int_b^x \frac{V^2(t)}{\alpha(t)} dt)\right] + g\left[\underset{T}{\Pi}(K_\alpha + \frac{\alpha'(x)}{2} I)\right] + g(C).$$

Since $\lim_{x\to\infty} g\left[\underset{T}{\Pi}(K_\alpha + \frac{\alpha'(x)}{2} I)\right] = +\infty$, we have $g\left[\underset{T}{\Pi}(V)\right] > g\left[\underset{T}{\Pi}(\int_b^x \frac{V^2(t)}{\alpha(t)} dt)\right]$ on $[c,\infty)$ for some $c \geq b$. This implies

$$(4.4.6) \qquad g[\int_b^x \sigma(t)V(t)dt] > g[\int_b^x \sigma(t)\int_b^t \frac{V^2(t)}{\alpha(s)} dsdt].$$

Then, using the properties of g and squaring both sides, we get

$$\left\{\int_b^x \sigma(t)\sqrt{\alpha(t)}\; g\left[\frac{V(t)}{\sqrt{\alpha(t)}}\right]dt\right\}^2 > \left\{g\; \int_b^x \sigma(t)\int_b^t \frac{V^2(s)}{\alpha(s)} dsdt\right\}^2.$$

Consider the left side of this inequality. From the Cauchy-Schwartz inequality, we have

$$\frac{1}{\rho_g} \int_b^x \sigma^2(t)\alpha(t)dt \int_b^x g\left[\frac{V^2(t)}{\alpha(t)}\right]dt \geq \int_b^x \sigma^2(t)\alpha(t)dt \int_b^x g^2\left[\frac{V(t)}{\sqrt{\alpha(t)}}\right]dt$$

$$\geq \left\{\int_b^x \alpha(t)\sqrt{\alpha(t)}\; g\left[\frac{V(t)}{\sqrt{\alpha(t)}}\right] dt\right\}^2.$$

Therefore,

$$(4.4.7)\qquad \frac{1}{\rho_g}\left(\int_b^x \sigma^2(t)\alpha(t)dt\right)\left(\int_b^x g[\frac{V^2(t)}{\alpha(t)}]dt\right) > \left\{g[\int_b^x \sigma(t)\int_b^t \frac{V^2(s)}{\alpha(\bar{s})}\, dsdt]\right\}^2.$$

Let $W(x) = \int_b^x \sigma(t)\int_b^t \frac{V^2(s)}{\alpha(s)} dsdt$. Then $W'(x) = \sigma(x)\int_b^x \frac{V^2(t)}{\alpha(t)}\, dt$, so

$$(4.4.8)\qquad g'[W(x)] = g[W'(x)] = \sigma(x)\int_b^x g[\frac{V^2(t)}{\alpha(t)}]dt.$$

In addition (4.4.7) becomes

$$(4.4.9)\qquad \frac{1}{\rho_g}\left(\int_b^x \sigma^2(x)\alpha(t)dt\right)\left(\int_b^x g[\frac{V^2(t)}{\alpha(t)}]dt\right) > \{g[W(x)]\}^2.$$

It can be deduced from (4.4.6) that $\int_b^x \sigma(t)g[V(t)]dt > 0$ on $[c,\infty)$. This implies that $\sigma(t) \not\equiv 0$ and $g[V(t)] \not\equiv 0$ on $[b,c]$. Hence $\int_b^x \left\{g\left[\frac{V(t)}{\sqrt{\alpha(t)}}\right]\right\}^2 dt>0$ on $[c,\infty)$, and $\int_b^x \sigma(t) \int_b^t \left\{g\left[\frac{V(s)}{\sqrt{\alpha(s)}}\right]\right\}^2 dsdt > 0$ on $[c,\infty)$.

Therefore,

$$g[W(x)] = \int_a^x \sigma(t) \int_b^t g[\frac{V^2(s)}{\alpha(s)}]dsdt \geq \rho_g \int_b^x \sigma(t) \int_b^t \left\{g\left[\frac{V(s)}{\sqrt{\alpha(s)}}\right]\right\}^2 dsdt > 0$$

on $[c,\infty)$.

This fact, plus (4.4.8) and (4.4.9), allows us to write

$$\frac{g'[W(x)]}{\{g[W(x)]\}^2} > \frac{\sigma(x)\int_b^x g[\frac{V^2(t)}{\alpha(t)}]dt}{\frac{1}{\rho_g}\left(\int_b^x \sigma^2(t)\alpha(t)dt\right)\left(\int_b^x g[\frac{V^2(t)}{\alpha(t)}]dt\right)}.$$

By integrating we get

$$\frac{1}{g[W(c)]} > \frac{1}{g[W(c)]} - \frac{1}{g[W(x)]} > \rho_g \int_c^x \sigma(t)\left(\int_b^t \sigma^2(s)\alpha(s)ds\right)^{-1}dt.$$

Since (σ,α) is an averaging pair, $\lim_{x\to\infty} \int_c^x \sigma(t)(\int_b^t \sigma^2(s)\alpha(s)ds)^{-1}dt = +\infty$, so this inequality cannot hold on $[c,\infty)$. Therefore, we have a contradiction.

4.5 Oscillation Criteria for Equation (E).

In this section, oscillation criteria are developed for the differential equation

$$\text{(E)} \qquad [R(x)Y']' + Q(x)Y = 0,$$

where R and Q are continuous symmetric $\mathcal{B}^*$-valued functions on $[a,\infty)$ and R is positive definite. Results by Howard [14] and Etgen [10] are shown to be special cases of our criteria.

<u>Theorem 4.5.1</u>. If there exists a positive differentiable function q and a $g \in \mathcal{G}$ such that the function P defined by

$$P(x) = \int_a^x \{q(t)Q(t) - R(t)\frac{[q'(t)]^2}{4q(t)}\}dt + \frac{q'(x)}{2}R(x)$$

has the property that $\lim_{x\to\infty} g[P(x)] = +\infty$, and if $[q(x)R(x)]^{-1} \geq k(x)I$, where k is a positive scalar function and $\lim_{x\to\infty} \int_a^x k(t)dt = +\infty$, then equation (E) is oscillatory.

<u>Proof</u>. Suppose the theorem is false. Then there exists a nontrivial conjoined solution Y of (E) such that Y(x) is nonsingular on $[b,\infty)$ for some $b \geq a$. Define S(x) by

$$q^{-1}(x)S(x) = -R(x)Y'(x)Y^{-1}(x) \qquad \text{on } [b,\infty).$$

Then S(x) is symmetric because Y(x) is conjoined and R(x) is symmetric.

By taking the derivative

$$S'(x) = \frac{1}{q(x)}\,[S(x)R^{-1}(x)S(x) + q'(x)S(x)] + q(x)Q(x)$$

which factors into

$$S'(x) = (S + \tfrac{1}{2}\,Rq')(q^{-1}R^{-1})(S + \tfrac{1}{2}\,Rq') + qQ - \frac{R(q')^2}{4q}\,.$$

If we integrate and add $\frac{q'R}{2}$ to both sides, we get

$$S(x) + \frac{q'(x)R(x)}{2} = S(b) + \int_b^x\{q(t)Q(t) - \frac{R(t)[q'(t)]^2}{4q(t)}\}dt + \frac{q'(x)R(x)}{2}$$

$$+ \int_b^x(S + \tfrac{1}{2}\,Rq')(q^{-1}R^{-1})(S + \tfrac{1}{2}\,Rq')dt.$$

Let $\hat{S}(x) = S(x) + \frac{q'(x)R(x)}{2}$. Then

$$\hat{S}(x) = S(b) + \int_b^x\{q(t)Q(t) - \frac{R(t)[q'(t)]^2}{4q(t)}\}dt + \frac{q'(x)R(x)}{2}$$

$$+ \int_b^x \hat{S}(t)[q(t)R(t)]^{-1}\,\hat{S}(t)dt.$$

By hypothesis

$$\hat{S}(x) \geq S(b) + \int_b^x\{q(t)Q(t) - \frac{R(t)[q'(t)]^2}{4q(t)}\}dt + \frac{q'(x)R(x)}{2} +$$

$$+ \int_b^x k(t)[\hat{S}(t)]^2dt.$$

Also by hypothesis, there is a $g \in G$ and a $c \geq b$ such that

(4.5.1) $$g[\hat{S}(x)] > g\{\int_b^x k(t)[\hat{S}(t)]^2dt\} \quad \text{on } [c,\infty).$$

Let $W(x) = \int_b^x k(t)[\hat{S}(x)]^2dt$. Then $W'(x) = k(x)[\hat{S}(x)]^2$. By using the properties of g,

$$g'[W(x)] = g[W'(x)] = k(x)g\{[\hat{S}(x)]^2\} \geq \rho_g k(x)\{g[\hat{S}(x)]\}^2.$$

Substituting (4.5.1) into this equation, we get

(4.5.2) $g'[W(x)] > \rho_g k(x)\{g[\int_b^x k(t)[\hat{S}(t)]^2 dt\}^2 = \rho_g k(x)\{g[W(x)]\}^2.$

Now (4.5.1) implies $g[\hat{S}(x)] > 0$ on $[c,\infty)$, so

$$g[W(x)] = \int_b^x k(t) g\{[\hat{S}(x)]^2\} dt \geq \rho_g \int_b^x k(t)\{g[\hat{S}(t)\}^2 dt > 0.$$

This means we can write (4.5.2) as

$$\frac{g'[W(x)]}{\{g[W(x)]\}^2} > \rho_g k(x) \quad \text{on } [c,\infty).$$

We can integrate to get

$$\frac{1}{g[W(c)]} > \frac{1}{g[W(c)]} - \frac{1}{g[W(x)]} > \rho_g \int_c^x k(t)\,dt$$

on $[c,\infty)$.

Since $\lim_{x\to\infty} \int_c^x k(t)\,dt = +\infty$, this inequality cannot hold on $[c,\infty)$.

The resulting corollary has been proven by Howard [14, Theorem 3] for the case where R and Q are $n\times n$ matrices. We can use our theorem to extend his results to the $\mathcal{B}^*$-valued case.

Corollary 4.5.2. Let R and Q be continuous symmetric $\mathcal{B}^*$-valued functions on $[a,\infty)$, and let R be positive definite. Let P and q be defined as in Theorem 4.5.1. If $\lim_{x\to\infty} \lambda(x) = +\infty$, where $\lambda(x)$ is the minimum eigenvalue of $P(x)$, then (E) is oscillatory.

Proof. Choose any constant unit vector α. Then $\lambda(x) \leq \langle P(x)\alpha,\alpha\rangle$. This inner product determines a $g_\alpha \in \mathcal{G}$ as was demonstrated in Chapter II. Therefore $g_\alpha[P(x)] \geq \lambda(x)$, so the hypothesis of Theorem 4.5.1 is satisfied.

Theorem H. (Etgen [10, Theorem 3]). Let R and Q be continuous symmetric $n \times n$ matrices on $[a,\infty)$, and let R be positive definite. Define P and q as in Theorem 4.5.1. If $\lim\limits_{x\to\infty}$ trace $[P(x)] = +\infty$, then (E) is oscillatory.

Since trace is an example of a $g \in \mathcal{G}$ as was demonstrated in Chapter II, this theorem is a special case of Theorem 4.5.1.

Equation (e) is a special case of equation (E) by having $R(x) \equiv I$, the identity element. The next theorem extends this thinking by letting $R(x) = r(x)I$, where $r(x)$ is a continuous positive scalar function.

Theorem 4.5.3. In equation (E), let Q be a continuous symmetric $\mathcal{B}^*$-valued function on $[a,\infty)$ and let $R(x) = r(x)I$, where r is a continuous positive scalar function on $[a,\infty)$. If there is a $g \in \mathcal{G}$ such that $\lim\limits_{x\to\infty} g[\int_a^x Q(t)dt] = +\infty$ and $\lim\limits_{x\to\infty} \int_a^x \frac{1}{r(t)}\, dt = +\infty$, then (E) is oscillatory.

The proof is an obvious modification of the proof of Theorem 4.2.1.

The following theorem extends the work of Tomastik [27]. In particular, we allow the coefficients in equation (E) to take their values in $\mathcal{B}^*$ rather than being limited to n×n matrices.

Theorem 4.5.4. Let R and Q in equation (E) be continuous positive definite symmetric $\mathcal{B}^*$-valued functions on $[a,\infty)$. If there exists a $g \in G$ such that $\lim_{x\to\infty} g[\int_a^x R^{-1}(t)dt] = +\infty$ and $\lim_{x\to\infty}$ minimum eigenvalue $[\int_a^x Q(t)dt] = +\infty$, then equation (E) is oscillatory.

Proof. Assume (E) is nonoscillatory. Then there is a nontrivial conjoined solution Y of (E) such that Y(x) is nonsingular on $[b,\infty)$ for some $b \geq a$.

Since Y(x) is nonsingular on $[b,\infty)$, we can define S by

$$S(x) = -R(x)Y'(x)Y^{-1}(x).$$

The conjoined property of Y and the symmetry of R imply that S is symmetric.

Taking the derivative of S(x) and integrating S'(x) yields

$$S(x) = S(b) + \int_b^x Q(t)dt + \int_b^x S(t)R^{-1}(t)S(t)dt.$$

Now $\int_b^x S(t)R^{-1}(t)S(t)dt$ is positive definite, and, by the hypothesis, $S(b) + \int_b^x Q(t)dt$ is positive definite on $[c,\infty)$ for some point $c \geq b$. Hence $S(x) > 0$ on $[c,\infty)$. This implies that S^{-1} exists and that $S^{-1}(x) > 0$ on $[c,\infty)$.

Let g be the positive functional described in the hypothesis. Then $g[S^{-1}(x)] > 0$ on $[c,\infty)$.

We now take the derivative of $S^{-1}(x)$ and integrate $[S^{-1}(x)]'$. This gives

$$S^{-1}(x) = S^{-1}(c) - \int_c^x R^{-1}(t)dt - \int_c^x S^{-1}(t)Q(t)S^{-1}(t)dt.$$

Therefore,

$$g[S^{-1}(x)] = g[S^{-1}(c)] - g[\int_c^x R^{-1}(t)dt] - g[\int_c^x S^{-1}(t)Q(t)S^{-1}(t)dt].$$

Since $g[S^{-1}(c)]$ is a constant, $g[\int_c^x S^{-1}(t)Q(t)S^{-1}(t)dt] > 0$, and $\lim_{x\to\infty} g[\int_c^x R^{-1}(t)dt] = +\infty$, we have that $\lim_{x\to\infty} g[S^{-1}(x)] = -\infty$ which contradicts $g[S^{-1}(x)] > 0$.

Corollary 4.5.5. Let R and Q in equation (E) be continuous positive definite symmetric n×n matrices. If $\lim_{x\to\infty} \int_a^x r_{ii}(t)dt = +\infty$, where r_{ii} is the i^{th} diagonal element of R^{-1}, and $\lim_{x\to\infty}$ minimum eigenvalue $[\int_a^x Q(t)dt] = +\infty$, then equation (E) is oscillatory.

Proof. In order to satisfy the hypothesis of Theorem 4.5.4, we must only show that there exists a $g \in G$ such that $\lim_{x\to\infty} g[\int_a^x R^{-1}(t)dt] = +\infty$.

As in Theorem A, Section 2, we can define a $g_{e_i} \in G$ such that $g_{e_i}[\int_a^x R^{-1}(t)dt] = \int_a^x r_{ii}(t)dt$. Hence $\lim_{x\to\infty} g_{e_i}[\int_a^x R^{-1}(t)dt] = +\infty$.

CHAPTER V

TRIGONOMETRIC MATRICES AND THE POLAR COORDINATE TRANSFORMATION

Let Q be a continuous symmetric $n\times n$ matrix on $[a,\infty)$, and let $\{S,C\}$ be a solution pair of the second order system

$$(5.1) \qquad Y' = Q(x)Z\ , \quad Z' = -Q(x)Y.$$

This differential system is referred to as a trigonometric differential system because the solution pairs have many of the properties of the sine and cosine functions. See, for example, Etgen [6].

In particular it is easy to verify that if $\{S,C\}$ is a solution of (5.1), then the matrices $S^*C - C^*S$ and $S^*S + C^*C$ are constant on $[a,\infty)$. We can, in fact, show that the following identities hold on $[a,\infty)$.

$$(5.2) \qquad \begin{aligned} S^*C &\equiv C^*S, \\ SC^* &\equiv CS^*, \\ S^*S + C^*C &\equiv I, \\ SS^* + CC^* &\equiv I. \end{aligned}$$

This can be done if we impose the initial condition

$$Y(b) = Y_0\ , \quad Z(b) = Z_0$$

such that $Y_0^*Z_0 = Z_0^*Y_0$. This implies $S^*C \equiv C^*S$ on $[a,\infty)$, and the pair $\{S,C\}$ is conjoined. Also, the pair is nontrivial if and only if the constant matrix $S^*S + C^*C$ is positive definite. In fact, assuming that $\{S,C\}$ is nontrivial, we can without loss of generality assume that

$$S^*S + C^*C \equiv I$$

on $[a,\infty)$.

These identities can be used to show that

$$SC^* \equiv CS^*,$$

$$SS^* + CC^* \equiv I.$$

In addition, if A and B are n×n matrices such that $A^*B = B^*A$, $AB^* = BA^*$, $A^*A + B^*B = I$, and $AA^* + BB^* = I$, then the matrix functions

$$U = SA - CB,$$

$$V = CA + SB,$$

are solutions of (5.1) which also satisfy the identities corresponding to (5.2). These can be compared with the trigonometric addition formulas. If Q is positive definite on $[a,\infty)$, then $\{S,C\}$ has oscillatory properties similar to those of $\{\sin\int_a^x q(t)dt, \cos\int_a^x q(t)dt\}$ where q is a positive continuous scalar function on $[a,\infty)$ [7]. Finally, if Q is nonsingular on $[a,\infty)$, then (5.1) can be written

$$[Q^{-I}(x)Y']' + Q(x)Y = 0,$$

a special case of equation (E).

The solution pair $\{S,C\}$ of (5.1) was introduced by J. H. Barrett [3] in order to study the oscillation properties of the second order differential equation

$$(5.3) \qquad [P(x)Y']' + F(x)Y = 0,$$

where P and F are continuous n×n matrices on $[a,\infty)$. He accomplished this by performing a generalized polar coordinate (or Prüfer) transformation

$$Y(x) = S^*(x)M(x)\ , \quad P(x)Y'(x) = C^*(x)M(x),$$

where M satisfies the matrix differential equation

$$M'(x) = [S(x)P^{-1}(x)C^*(x) - C(x)F(x)S^*(x)]M(x),$$

and Q is the matrix

$$Q(x) = C(x)P^{-1}(x)C^*(x) + S(x)F(x)S^*(x).$$

Clearly, Q is symmetric.

Theorems which are handled with greater ease using the trigonometric differential system have their results transformed back to equation (5.3).

The following lemma and theorems establish a condition which guarantees the oscillation of the trigonometric differential system (5.1) in the case where $Q > 0$ on $[a,\infty)$.

If we choose a nontrivial conjoined solution pair $\{S,C\}$ of equation (5.1), then the matrix defined by

(5.4) $$\Theta(x) = [C(x) - iS(x)]^{-1}[C(x) + iS(x)]$$

exists on $[a,\infty)$. The following lemma describing the properties of Θ is well known. (See [2, Chapter 10], [7], or [9].)

Lemma A. The matrix Θ defined by (5.4) has the following properties on $[a,\infty)$.

(*i*) Θ is unitary, *i.e.*, $\Theta\Theta^* = \Theta^*\Theta = I$.

(*ii*) Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ denote the eigenvalues of Θ. Then $|\gamma_j| = 1$ for $j = 1, 2, \ldots, n$, and $\gamma_j(c) = 1$, $c \geq a$, for at least one j, $1 \leq j \leq n$ if and only if $S(c)$ is singular, *i.e.*, $\det S(c) = 0$. Moreover, the multiplicity of the zero of $\det S(c)$ equals the number of eigenvalues of Θ having the value $+1$.

(*iii*) The eigenvalues of Θ move monotonically and positively on the unit circle as x increases on $[a,\infty)$.

(*iv*) Let $\omega_j(x) = \arg[\gamma_j(x)]$, $j = 1, 2, \ldots, n$, and assume that $0 \leq \omega_j(a) \leq 2\pi$, $j = 1, 2, \ldots, n$, and that every ω_j is continued as a continuous function on $[a,\infty)$. Then the functions $\omega_j(x)$ are increasing functions on $[a,\infty)$.

(*v*) If for some point $b \geq a$, all of the eigenvalues of $\Theta(b)$ are in the upper half-plane, then the symmetric matrix $C(b)S^{-1}(b)$ is positive definite. Similarly, if for some point $c \geq a$, all of the eigenvalues of $\Theta(c)$ are in the lower half-plane, then $C(c)S^{-1}(c)$ is negative definite.

(*vi*) Both of the matrices $C^*C - S^*S$ and $2S^*C$ are real and symmetric. In addition, they have the same eigenvectors.

<u>Theorem 5.1</u>. Let Θ be the continuous, unitary $n \times n$ matrix defined by (5.4) on $[a,\infty)$. If the differential system (5.1) is nonoscillatory, then there is a number $c \geq a$ and a nontrivial conjoined solution pair $\{U,V\}$ of (5.1) such that $V(x)U^{-1}(x)$ is negative definite on $[c,\infty)$.

<u>Proof</u>. Let $\{S,C\}$ be any nontrivial conjoined solution pair of (5.1) which satisfies the identities (5.2). Since (5.1) is nonoscillatory, there is a number $b \geq a$ such that S is nonsingular on $[b,\infty)$. Define the matrix T by

$$T(x) = C(x)S^{-1}(x) \qquad \text{on } [b,\infty).$$

Then using the identities (5.2), we can show that T is symmetric on $[b,\infty)$.

By taking the derivative of $T(x)$, we have

$$T'(x) = -Q(x) - T(x)Q(x)T(x).$$

Since Q is positive definite on $[b,\infty)$, $T'(x)$ is negative definite which means $T(x)$ decreases on $[b,\infty)$ in the sense that each of its eigenvalues is a decreasing function. Note that if $C(x)S^{-1}(x)$ is negative definite at some point $c \geq b$, then it is negative definite on $[c,\infty)$. Because it is not necessarily the case that $C(x)S^{-1}(x)$ is eventually negative definite, we will proceed to construct a solution pair such that this condition does hold.

Consider the matrix Θ mentioned in the hypothesis. Since S is nonsingular on $[b,\infty)$ we can conclude from *(ii)* of Lemma A that no eigenvalue, $\gamma_j(x)$, of $\Theta(x)$ passes through the point $+1$ as x increases on $[b,\infty)$. It then follows that the increasing functions

$$\omega_j(x) = \arg[\gamma_j(x)], \quad j = 1, 2, \ldots, n,$$

are bounded above on $[b,\infty)$. Hence

$$\omega_j(\infty) = \lim_{x\to\infty} \omega_j(x), \quad j = 1, 2, \ldots, n,$$

exists, and we can define

$$\alpha_j \equiv \omega_j(\infty) \pmod{2\pi}, \quad j = 1, 2, \ldots, n.$$

From this we can conclude that $\lim_{x\to\infty} \Theta(x) = \Theta(\infty)$ exists.

Let $G = C^*C - S^*S$ and $H = 2S^*C$. Then by *(vi)* of Lemma A, the matrices G and H are real, symmetric, and have the same eigenvectors. Furthermore $GH = HG$, $G^2 + H^2 = 0$, and since $C^* + iS^* = (C - iS)^{-1}$, we can write (5.4) as

$$\Theta(x) = [C^*(x) + iS^*(x)]\,[C(x) + iS(x)].$$

This expands to

$$\Theta(x) = C^*(x)C(x) - S^*(x)S(x) + i\,2S^*(x)C(x)$$

which can be written

$$\Theta(x) = G(x) + iH(x).$$

Since $\lim_{x\to\infty} \Theta(x) = \Theta(\infty)$ exists, both $\lim_{x\to\infty} G(x) = G(\infty)$ and $\lim_{x\to\infty} H(x) = H(\infty)$ exist and the identities $G(\infty)H(\infty) = H(\infty)G(\infty)$ and $G^2(\infty) + H^2(\infty) = I$ hold because they hold for all $x \geq b$. Hence, $G(\infty)$ and $H(\infty)$ are real, symmetric, and have the same eigenvectors, so they can be diagonalized by the same real, orthogonal, matrix J, *i.e.*, $J^* = J^{-1}$. This means that

$$J^*\Theta(\infty)J = J^*G(\infty)J + iJ^*H(\infty)J = D$$

where D is of the form

$$D = \begin{pmatrix} e^{i\alpha_1} & & & \\ & e^{i\alpha_2} & & \\ & & \ddots & \\ & & & e^{i\alpha_n} \end{pmatrix}$$

For any n×n matrix M we shall denote the transpose of M as M^T. If M is an n×n matrix with real entries, then $M^T = M^*$. Hence $J^T = J^*$, so $J^T\Theta(\infty)J = D$.

We can choose a matrix K such that

$$(JK)^T\Theta(\infty)(JK) = K^TJ^T\Theta(\infty)JK = K^TDK = N,$$

and N has all its eigenvalues on the lower half-plane. This is accomplished by setting

$$K = \begin{pmatrix} e^{i\lambda_1} & & & \\ & e^{i\lambda_2} & & \\ & & \ddots & \\ & & & e^{i\lambda_n} \end{pmatrix}$$

where $\pi < 2\lambda_j + \alpha_j < 2\pi, \quad j = 1, 2, \ldots, n.$

Hence we have

$$N = K^T DK = \begin{pmatrix} e^{i(2\lambda_1 + \alpha_1)} & & & \\ & e^{i(2\lambda_2 + \alpha_2)} & & \\ & & \ddots & \\ & & & e^{i(2\lambda_n + \alpha_n)} \end{pmatrix},$$

so N has all its eigenvalues in the lower half-plane.

We can rewrite K in the form

$$K = \begin{pmatrix} \cos(\lambda_1) & & & \\ & \cos(\lambda_2) & & \\ & & \ddots & \\ & & & \cos(\lambda_n) \end{pmatrix} + i \begin{pmatrix} \sin(\lambda_1) & & & \\ & \sin(\lambda_2) & & \\ & & \ddots & \\ & & & \sin(\lambda_n) \end{pmatrix}$$

so

$$JK = J\begin{pmatrix} \cos(\lambda_1) & & & \\ & \cos(\lambda_2) & & \\ & & \ddots & \\ & & & \cos(\lambda_n) \end{pmatrix} + iJ\begin{pmatrix} \sin(\lambda_1) & & & \\ & \sin(\lambda_2) & & \\ & & \ddots & \\ & & & \sin(\lambda_n) \end{pmatrix}$$

If we let

$$A = J\begin{pmatrix} \cos(\lambda_1) & & & \\ & \cos(\lambda_2) & & \\ & & \ddots & \\ & & & \cos(\lambda_n) \end{pmatrix}$$

and

$$B = -J\begin{pmatrix} \sin(\lambda_1) & & & \\ & \sin(\lambda_2) & & \\ & & \ddots & \\ & & & \sin(\lambda_n) \end{pmatrix}$$

then the matrices A and B satisfy the identities

$$\begin{aligned} A^*B &= B^*A, \\ AB^* &= BA^*, \\ A^*A + B^*B &= I, \\ AA^* + BB^* &= I, \end{aligned} \tag{5.5}$$

and

$$N = (JK)^T\Theta(\infty)(JK) = (A - IB)^T\Theta(\infty)\ (A - iB). \tag{5.6}$$

We now let

$$U(x) = S(x)A - C(x)B$$

and

$$V(x) = C(x)A + S(x)B.$$

Clearly $\{U,V\}$ is a solution pair of (5.1), and by using (5.2) and (5.5) the following identities hold on $[b,\infty)$.

$$(5.7)\qquad \begin{aligned} U^*V &= V^*U \\ UV^* &= VU^* \\ U^*U + V^*V &= I \\ UU^* + VV^* &= I. \end{aligned}$$

Denote the matrix Ψ on $[b,\infty)$ by

$$(5.8)\qquad \Psi = [V - iU]^{-1}[V + iU].$$

Then

$$\begin{aligned} \Psi &= [(C - iS)(A + iB)]^{-1}[(C + iS)(A - iB)] \\ &= (A + iB)^{-1}\,\Theta(A - iB). \end{aligned}$$

Since $\lim\limits_{x\to\infty} \Theta(x) = \Theta(\infty)$ exists, $\lim\limits_{x\to\infty} \Psi(\infty)$ exists and

$$(5.9)\qquad \Psi(\infty) = (A + iB)^{-1}\,\Theta(\infty)(A - iB).$$

By using the fact that $A^* = A^T$ and $B^* = B^T$ along with the identities (5.5), we can show that

$$(A - iB)^T = (A + iB)^{-1}.$$

This implies by (5.6) and (5.9) that

$$N = \Psi(\infty).$$

Hence, $\Psi(\infty)$ has all its eigenvalues in the lower half-plane. This implies there exists a number $c \geq b$ such that $\Psi(x)$ has all its eigenvalues in the lower half-plane on $[c,\infty)$. Therefore, by (v) of Lemma A, we can conclude that $V(x)U^{-1}(x)$ is negative definite on $[c,\infty)$.

<u>Theorem 5.2</u>. If Q is a continuous positive definite symmetric $n\times n$ matrix on $[a,\infty)$, and if there exists a positive functional $g \in G$ such that $\lim\limits_{x\to\infty} g[\int_a^x Q(t)dt] = +\infty$, then equation (5.1) is oscillatory.

Proof. By the technique demonstrated in Theorem 5.1, we can construct a nontrivial conjoined solution pair $\{S,C\}$ such that $C(x)S^{-1}(x)$ is negative definite on $[c,\infty)$ for some $c \geq a$.

Let $W(x) = C(x)S^{-1}(x) < 0$ on $[c,\infty)$. Then $W(x)$ is symmetric, and $W^{-1}(x)$ exists and is negative definite on $[c,\infty)$. By differentiating $W^{-1}(x)$ and integrating the derivative, we have

$$W^{-1}(x) = W^{-1}(c) + \int_c^x Q(t)dt + \int_c^x W(t)Q(t)W(t)dt.$$

Hence, utilizing the positive functional g mentioned in the hypothesis, we get

$$g[W^{-1}(x)] = g[W^{-1}(c)] + g[\int_c^x Q(t)dt] + g[\int_c^x W^{-1}(t)Q(t)W^{-1}(t)dt].$$

Since $\lim_{x\to\infty} g[\int_c^x Q(t)dt] = +\infty$, there is a point $d \geq c$ such that

$$g[W^{-1}(x)] > g[\int_b^x W^{-1}(t)Q(t)W^{-1}(t)dt] > 0 \quad \text{on } [d,\infty).$$

This contradicts $W^{-1}(x) < 0$ on $[b,\infty)$.

As a corollary to Theorem 5.1 we get an extension of a result of W. T. Reid [21, Theorem 5.4].

Corollary 5.3. Suppose there is a nontrivial conjoined solution Y of the differential equation (5.3) such that

$$Q(x) = M^{*-1}(x)[Z^*(x)P^{-1}(x)Z(x) + Y^*(x)F(x)Y(x)]M^{-1}(x),$$

where $Z(x) = P(x)Y'(x)$ and $M^*(x)M(x) = Y^*(x)Y(x) + Z^*(x)Z(x)$, is positive definite on $[b,\infty)$ for some $b \geq a$. If there is a $g \in G$ such that

$\lim_{x\to\infty} g[\int_b^x Q(t)dt] = +\infty$, then equation (5.3) is oscillatory.

Proof. Consider the solution Y called out in the hypothesis. We can perform the polar coordinate transformation by setting $Y(x) = S^*(x)M(x)$ and $Z(x) = C^*(x)M(x)$, where M satisfies the differential equation

$$M'(x) = [S(x)P^{-1}(x)C^*(x) - C(x)F(x)S^*(x)]M(x)$$

and

$$Q(x) = C(x)P^{-1}(x)C^*(x) + S(x)F(x)S^*(x).$$

It can be shown that $M^*(x)M(x) = Y^*(x)Y(x) + Z^*(x)Z(x) > 0$ because Y is nontrivial. Thus M(x) is nonsingular on $[b,\infty)$. In addition,

$$M^*(x)Q(x)M(x) = Z^*(x)P^{-1}Z(x) + Y^*(x)F(x)Y(x),$$

so

$$Q(x) = M^{*-1}(x)[Z^*(x)P^{-1}(x)Z(x) + Y^*(x)F(x)Y(x)]M^{-1}(x).$$

Since Q(x) is positive definite on $[b,\infty)$ and there is a $g \in \mathcal{G}$ such that $\lim_{x\to\infty} g[\int_b^x Q(t)dt] = +\infty$, we have, by Theorem 5.2, that the trigonometric differential equation (5.1) is oscillatory. Because we are dealing with an n×n matrix differential equation, Theorem 3.6 implies all the solutions of equation (5.1) are oscillatory, in particular S. Therefore $Y = S^*M$ is oscillatory, so equation (5.3) is oscillatory.

CHAPTER VI

COMPARISON CRITERIA

In this chapter we shall extend the work of K. Kreith [17] by generalizing his comparison lemma [17,Lemma 2]. This is done by making use of the set of positive functionals $\mathcal{G}$ defined in Chapter II to relax his hypotheses. In addition, we shall introduce comparison criteria for oscillation and nonoscillation by using both Kreith's work and our extensions.

Consider the differential equation

$$[P(x)V']' + G(x)V = 0 \tag{6.1}$$

and compare it with equation

$$[R(x)Y']' + Q(x)Y = 0 \tag{E}$$

where P, G, R, and Q are continuous symmetric functions on $[a,\infty)$ which take their values in $\mathcal{B}^*$. In addition, we assume that P and R are positive definite.

Theorem 6.1. Suppose equations (6.1) and (E) are $\mathcal{B}^*$-valued differential equations, and V is a non-identically zero solution of (6.1) satisfying

(*i*) $V^*(x)[Q(x) - G(x)]V(x) \geq 0$,

(*ii*) $V^{*\prime}(x)[P(x) - R(x)]V'(x) \geq 0$ on $[b,c]$, and

(*iii*) $V(b) = V(c) = 0$.

If Y is a conjoined solution of equation (E), then Y(x) is singular for some point x on $[b,c]$.

Kreith proved this theorem for the finite dimensional case [17, Lemma 2], and his proof extends without modification to the $\mathcal{B}^*$-valued case.

The following theorem is an extension of Theorem 6.1.

Theorem 6.2. Suppose equations (6.1) and (E) are $\mathcal{B}^*$-valued differential equations, $g \in \mathcal{G}$, and V is a non-identically zero solution of (6.1) satisfying

(*i*) $g[V^*(Q - G)V] \geq 0$ on $[b,c]$

(*ii*) $g[V^{*\prime}(P - R)V'] \geq 0$ on $[b,c]$

(*iii*) $g[V^*(b)V(b)] = g[V^*(c)V(c)] = 0$

(*iv*) For any point d, if $g[V^*(d)V(d)] = 0$, then $g[V^{*\prime}(d)R(d)V'(d)] > 0$.

If Y is a conjoined solution of (E), then $Y(x)$ is singular at some point $x \in [b,c]$.

Proof. Assume Y is a conjoined solution of (E) which is nonsingular on $[b,c]$. Then Y^{-1} exists on $[b,c]$. Let V be a non-identically zero solution of (6.1) satisfying (*i*) through (*iv*) of the hypothesis.

Now

$$(V^*PV' - V^*RY'Y^{-1}V)' = V^*(PV')' - V^*(RY')'Y^{-1}V + V^{*\prime}(P - R)V' + (V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V).$$

By letting $S = RY'Y^{-1}$ and integrating, we obtain

$$V^*PV' - V^*SV\Big|_b^c = \int_b^c V^*(Q - G)V\,dt + \int_b^c V^{*\prime}(P - R)V'\,dt$$
$$+ \int_b^c (V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)\,dt.$$

Therefore, using the positive functional g mentioned in the hypothesis, we get

$$(6.2)\quad g[V^*(c)P(c)V'(c)] - g[V^*(c)S(c)V(c)] - g[V^*(b)P(b)V'(b)$$
$$+ g[V^*(b)S(b)V(b)]$$
$$= g[\int_b^c V^*(Q - G)V\,dt] + g[\int_b^c V^{*\prime}(P - R)V'\,dt]$$
$$+ g[\int_b^c (V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)\,dt].$$

By Lemma 2.2.3

$$|g[V^*(x)P(x)V'(x)]|^2 \leq g[V^*(x)V(x)]\ g[(P(x)V'(x))^*\ P(x)V'(x)]$$

and

$$|g[V^*(x)S(x)V(x)]|^2 \leq g[V^*(x)V(x)]\ g[(S(x)V(x))^*S(x)V(x)].$$

By (*iii*) of the hypothesis, $g\ [V^*(x)V(x)] = 0$ when $x = b$ or $x = c$. Hence, the left side of equation (6.2) is zero and we have

$$0 = \int_b^c g[V^*(Q - G)V]\,dt + \int_b^c g[V^{*\prime}(P - R)V']\,dt$$
$$+ \int_b^c g[(V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)]\,dt.$$

Each term on the right side of this equation is nonnegative, so it suffices to show that at least one term is positive. Concentrating on the third term, we have

$$g[(V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)] = g[V^{*\prime}RV'] + g[(Y'Y^{-1}V)^*R(Y'Y^{-1}V)]$$
$$- g[V^{*\prime}R(Y'Y^{-1}V)] - g[(Y'Y^{-1}V)^*RV'].$$

By evaluating this at $x = b$, and using hypothesis (*iii*) and (*iv*), we get

$$g[(V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)(b)] = g[V^{*\prime}(b)R(b)V(b)] > 0.$$

Since $g[(V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)]$ is continuous, there is an interval $[b, b']$ on which it is positive, and we have

$$\int_b^c g[V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)\,dt$$
$$\geq \int_b^{b'} g[(V' - Y'Y^{-1}V)^*R(V' - Y'Y^{-1}V)]dt > 0.$$

The following result was obtained by Kreith [17, Theorem 1]. We offer an alternate proof.

<u>Corollary 6.3</u>. Let J be a nonzero n×n matrix with "ones" and "zeros" on the main diagonal and "zeros" elsewhere. Assume that the scalar equation

$$(6.3) \qquad [r(x)y']' + q(x)y = 0,$$

where r and q are continuous functions on $[a,\infty)$ with $r > 0$, is oscillatory. If $J[Q(x) - q(x)I]J \geq 0$ and $J[r(x)I - R(x)]J \geq 0$ on $[a,\infty)$, then equation (E) is oscillatory.

<u>Proof</u>. Let v be a nontrivial solution of (6.3), and let $V(x) = v(x)I$. In equation (6.1), let $P(x) = r(x)I$ and $G(x) = q(x)I$. Thus V is a solution of equation (6.1).

Since v is oscillatory, given any number $b \geq a$, there are numbers c and d, $b \leq c < d$ such that $V(c) = V(d) = 0$. Define e to be the vector whose entries are the diagonal elements of the matrix J. Let g_e be the positive functional defined by $g_e(\cdot) = \langle(\cdot)e,e\rangle$. Then

$$g_e[V^*(c)V(c)] = g_e[V^*(d)V(d)] = 0,$$

$$g_e[V^*(Q - qI)V] = \sum_D v^2J[Q - qI]J \geq 0,$$

and

$$g_e[V^{*\prime}[rI - R]V'] = \sum_D (v')^2J[rI - R]\, J \geq 0,$$

where $\sum_D A$ is the sum of the elements of the matrix A. It is easy to see that for any point z, $g_e[V^*(z)V(z)] = 0$ if and only if $v(z) = 0$. Since v is a nontrivial solution of (6.3) we can conclude that if $v(z) = 0$, then $v'(z) \neq 0$ so $[v'(z)]^2 > 0$. In addition,

$$V^{*\prime}(z)R(z)V'(z)] = [v'(z)]^2R(z)$$

which implies

$$g_e[V^{*\prime}(z)R(z)V'(z)] = [v'(z)]^2g_e[R(z)] > 0$$

since R is positive definite.

Therefore the hypotheses of Theorem 6.2 are satisfied and we can conclude that (E) is oscillatory.

The next corollary compares equation (E) with a scalar equation to obtain an oscillation criterion when R and Q are $\mathcal{B}^*$-valued functions.

Corollary 6.4. Suppose the scalar equation

(6.4) $$[p(x)y']' + f(x)y = 0,$$

where p and f are continuous functions on $[a,\infty)$ and $p > 0$, is oscillatory. If there is a positive functional $g \in \mathcal{G}$ such that (*i*) $g[Q - fI] \geq 0$,

(*ii*) $g[pI - R] \geq 0$

on $[a,\infty)$, then equation (E) is oscillatory.

Proof. Let Y be a nontrivial conjoined solution of equation (E). Let v be a nontrivial solution of the scalar equation (6.4). Since (6.4) is oscillatory, for each number $b \geq a$, there exist numbers c and d, $b \leq c < d$, such that $v(d) = v(c) = 0$. Let V be the $\mathcal{B}^*$-valued function on $[a,\infty)$ defined by $V(x) = v(x)I$. It is easy to verify that the hypotheses of Theorem 6.2 are satisfied.

It is clear that Theorem 6.1 can also be used for nonoscillation criteria. We first consider the finite dimensional case.

Theorem 6.5. Suppose the scalar equation (6.4) is nonoscillatory. If there exists a positive functional $g \in \mathcal{G}$ such that

(*i*) $g[fI - Q] \geq 0$

(*ii*) $g[R - pI] \geq 0$

on $[a,\infty)$, then (E) is nonoscillatory.

Proof. Let y be a nontrivial solution of the scalar equation (6.4). Then there is a number $b \geq a$ such that $y(x) \neq 0$ on $[b,\infty)$. Define the matrix Y by $Y(x) = y(x)I$. Then Y is nonsingular on $[b,\infty)$.

Assume the theorem is false, that is, assume (E) is oscillatory. Let U be a nontrivial conjoined solution of (E) such that $U(b) = 0$.

There is a point $c > b$ such that $U(c)$ is singular. Let γ be a nonzero constant vector such that $U(c)\gamma = 0$, and let z be the vector defined by $z(x) = U(x)\gamma$. Then $z(b) = z(c) = 0$. Also, since U is nontrivial, $z'(b) \neq 0$ and $z'(c) \neq 0$.

Let $V_1, V_2, \ldots, V_n$ denote the $n \times n$ matrices such that V_i has a "one" in the i,i position and "zeros" elsewhere, $1 \leq i \leq n$. Clearly

$$I = \sum_{i=1}^{n} V_i.$$

Therefore

$$0 < g(I) = \sum_{i=1}^{n} g(V_i).$$

Since the V_i's are nonnegative definite, it follows that $g(V_i) > 0$ for at least one i, $1 \leq i \leq n$.

Fix an integer i, $1 \leq i \leq n$, such that $g(V_i) > 0$, and let V be the matrix whose i^{th} column is the vector z and whose remaining columns are all "zero." Suppose d is a number such that $z(d) = 0$, then $V(d)$ is the zero matrix, and

$$g[V^*(d)V(d)] = 0.$$

Also, $z'(d) \neq 0$ and

$$\begin{aligned} g[V^{*\prime}(d)\ p(d)I\ V'(d)] &= p(d)g[V^{*\prime}(d)V'(d)] \\ &= p(d) \sum_{j=1}^{n} [z_j'(d)]^2\ g[V_i] > 0, \end{aligned}$$

where $z_j'(d)$, $j = 1, 2, \ldots, n$, are the components of $z'(d)$.

It now follows that the hypotheses of Theorem 6.2 are satisfied. Thus $Y = yI$ is singular on $[b,c]$, that is, y has a zero on $[b,c]$ and we have a contradiction.

The difficulty of extending the preceeding theorem to the $\mathcal{B}^*$-valued case lies in the construction of a solution V such that V is zero at two points. If we use the more restrictive definition of oscillation as presented by Hille [15, pg. 486], we are able to obtain an extension. According to Hille, a solution Y of (E) is <u>oscillatory</u> on $[a,\infty)$ provided it has an algebraic singularity on every interval $[b,\infty)$ where $b \geq a$. The $\mathcal{B}^*$-valued function Y on $[a,\infty)$ is said to have an <u>algebraic singularity</u> at a point $x \in [a,\infty)$ if $Y(x)$ is not 1-1.

<u>Lemma 6.6</u>. If A is an algebraically singular element of $\mathcal{B}^*$, then there is a nonzero element, K, of $\mathcal{B}^*$ such that $AK = 0$.

<u>Proof</u>. Let A be an algebraically singular element of $\mathcal{B}^*$. Then there is a nonzero element, γ, of $\mathcal{H}$ such that $A\gamma = 0$, the zero element of $\mathcal{H}$. Define the nonzero element K in $\mathcal{B}^*$ so that K maps $\mathcal{H}$ into the space generated by γ, that is, into multiples of γ. Then AK is the zero element of $\mathcal{B}^*$ since for any $\alpha \in \mathcal{H}$, $AK\alpha = A(m\gamma) = mA\gamma = 0$ for some scalar m.

<u>Theorem 6.7</u>. Suppose equations (6.1) and (E) are $\mathcal{B}^*$-valued differential equations, and for every non-identically zero solution V of equation

(6.1), the following conditions hold on $[a,\infty)$:

(*i*) $V^*(x)[Q(x) - G(x)]V(x) \geq 0$

(*ii*) $V^{*\prime}(x)[P(x) - R(x)]V'(x) \geq 0$.

If equation (E) is nonoscillatory on $[a,\infty)$, then equation (6.1) is nonoscillatory on $[a,\infty)$.

Proof. Since (E) is nonoscillatory, we are able to choose a solution Y such that Y is nonsingular on $[b,\infty)$ for some $b > a$.

Assume equation (6.1) is oscillatory on $[a,\infty)$. By requiring the initial condition that $W(b) = 0$, there is a nontrivial conjoined solution W of equation (6.1) such that W satisfies the initial condition, and $W(c)$ is singular for some point $c > b$. Then by Lemma 6.6 there exists a nonzero constant element $K \in \mathcal{B}^*$ such that $W(c)K = 0$. Denote $V(x) = W(x)K$. Then $V(x)$ is a non-identically zero solution of equation (6.1) such that $V(b) = V(c) = 0$. This, plus (*i*) and (*ii*) of the hypothesis, implies, by Theorem 6.1, that Y is singular at some point on $[b,c]$. This contradicts the fact that Y is nonsingular on $[b,\infty)$.

CHAPTER VII

NONLINEAR DIFFERENTIAL EQUATIONS

Let K be a subset of real Euclidean n^2 space and let Φ_K be the collection of functions to which ϕ belongs only in case ϕ is a real-valued function on K. If Y is an n×n matrix and $\phi \in \Phi_K$, then denote

$$\phi(Y) = \phi(y_{11}, y_{12}, \ldots, y_{1n}, y_{21}, \ldots, y_{nn}).$$

Let ϕ_{ij}, σ_{ij}, τ_{ij}, δ_{ij}, $(i,j = 1,2,\ldots,n)$, be members of Φ_K and consider the following system of n^2 differential equations.

$$\sum_{h=1}^{n} [r_{ih}(x, \phi_{ih}(Y), \sigma_{ih}(Y'))y'_{hj}]'$$

$$+ \sum_{h=1}^{n} q_{ih}(x, \tau_{ih}(Y), \delta_{ih}(Y'))y_{hj} = 0$$

where $i,j = 1,2,\ldots,n$, q_{ih} and r_{ih} are continuous, real-valued functions, and satisfy conditions which will insure the existence of solutions when appropriate initial conditions are specified. In addition, assume that $r_{ih} \equiv r_{hi}$ and $q_{ih} \equiv q_{hi}$ for all i,h.

This system was introduced by Etgen [6] and represented in the form

$$[R(x,Y,Y')Y']' + Q(x,Y,Y')Y = 0, \tag{7.1}$$

where R and Q are n×n symmetric matrices for all pairs of n×n continuous matrices (Y,Y').

Nonlinear matrix differential equations of this form have also been studied by Howard [14], Tomastik [26], and Kartsatos [16]. These authors apply to the nonlinear differential equation (7.1) the same basic techniques and ideas that had been developed for the linear differential equation (E) in order to characterize its oscillation properties.

Swanson [24] has considered nonlinear matrix differential inequalities of the form

$$Y^*(L(Y)) \leq 0 \tag{7.2}$$

where $\quad L(Y) = [R(x,Y,Y')Y']' + Q(x,Y,Y')Y.$

Let R and Q be $\mathcal{B}^*$-valued functions defined on $[a,\infty) \times \mathcal{B}^* \times \mathcal{B}^*$ such that each of $R(x,A,B)$ and $Q(x,A,B)$ is symmetric for all $x \in [a,\infty)$ and $A,B \in \mathcal{B}^*$, and R is positive definite. Consider the differential equation

$$[R(x,Y,Y')Y']' + Q(x,Y,Y')Y = 0, \tag{7.3}$$

as well as the differential inequality

$$Y^*L(Y)Y \leq 0, \tag{7.4}$$

where $L(Y) = [R(x,Y,Y')Y']' + Q(x,Y,Y')Y$.

It is easy to verify the theorems and definitions in Chapter III extend to the nonlinear case. Also, the methods that were developed for the linear equation can be extended to (7.3) and (7.4).

The following theorems are nonlinear versions of some of the main theorems proven in previous chapters. Since their proofs are analogous, these theorems are stated without proof.

Theorem 7.1. If $P(x,Y,Y') = I$ and if there is a $g \varepsilon \mathcal{G}$ such that

$$\lim_{x\to\infty} g[\int_a^x Q(t,Y,Y')dt] = +\infty$$

for every nonsingular differentiable $Y \varepsilon \mathcal{B}^*$, then equation (7.3) is oscillatory.

The proof is analogous to the proof of Theorem 4.2.1. Similar results for the case of n×n matrices were established by Etgen [10, Theorem 2] and Howard [14, Theorem 2].

The following theorem is similar to Theorem 4.5.4, and its proof is analogous.

Theorem 7.2. If there is a $g \varepsilon \mathcal{G}$ such that $\lim_{x\to\infty} g[\int_a^x R^{-1}(t,Y,Y')dt] = +\infty$, $\lim_{x\to\infty}$ minimum eigenvalue $[\int_a^x Q(t,Y,Y')dt] = +\infty$, and $R(x,Y,Y')$ and $Q(x,y,Y')$ are positive definite for every nonsingular differentiable $Y \varepsilon \mathcal{B}^*$, then equation (7.3) is oscillatory.

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