CONCERNING ORDERED SPACES

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Kenneth E. Oberhoff

January, 1968

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ABSTRACT

Ordered spaces are an abstraction of the real line. This paper shows in Chapter 1 that all ordered spaces are hereditarily normal. In Chapter 2, necessary and sufficient conditions are given for a separable ordered space to be completely separable, and hence metrizable. In semi-metrizable ordered spaces the following are shown to be equivalent:

(a) The space is completely separable.

(b) The space is separable.

(c) The space is hereditarily separable.

(d) The space has the Lindelöf property.

(e) If M is an uncountable subset of the space, then some point of M is a limit point of M.

(f) There does not exist an uncountable collection of mutually exclusive open sets in the space.

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CHAPTER I

INTRODUCTION

<u>Undefined Terms</u>. The term "point" and the relation "precedes" are undefined.

<u>Axiom 1</u>. If p is a point, then p does not precede p.

<u>Axiom 2</u>. If p and q are two distinct points, then p precedes q or q precedes p.

<u>Axiom 3</u>. If p precedes q and q precedes r, then p precedes r.

<u>Definition</u>. The statement that (p,q) is a <u>segment</u> means that p is a point, q is a point, p precedes q, there is a point r such that p precedes r and r precedes q, and (p,q) is the set to which the point x belongs if and only if p precedes x and x precedes q. The statement that (p,q) is empty means that p precedes q and there is no point r such that p precedes r and r precedes q.

<u>Definition</u>. The statement that [p,q) is an <u>end</u> segment means that p is a point, q is a point, p precedes q, there is no point r such that r precedes p, and [p,q) is the set to which the point x belongs if and only if x precedes q.

Similarly, the statement that (p,q] is an <u>end segment</u> means that p is a point, q is a point, p precedes q, there is no point r such that q precedes r, and (p,q] is the set to which the point x belongs if and only if p precedes x.

<u>Definition.</u> The statement that the point p is a <u>right end point</u> means that if x is a point distinct from p, then x precedes p.

Similarly, the statement that the point p is a <u>left end point</u> means that if x is a point distinct from p, then p precedes x.

The statement that the point p is an <u>end point</u> means that p is a left end point or p is a right end point.

<u>Definition.</u> If M is a set, then the statement that the point p is a <u>limit point</u> of M means that if X is a segment or end segment containing p, then X contains a point of M distinct from p.

<u>Remarks.</u> (1) If S is a set containing more than one point and if there is a meaning for the word "precedes" so that Axioms 1,2, and 3 are satisfied, then the collection of segments and end segments form a basis for a topology on S. This topology is called the <u>order topology</u>.

(2) When we say S is an <u>ordered space</u>, we mean S is a set, there is a meaning for the word "precedes" so that Axioms 1,2, and 3 are satisfied, and the topology on S is the order topology.

Definition. If each of H and K is a set, then the statement that H and K are mutually exclusive means that neither contains a point of the other.

Definition. If each of H and K is a set, then the statement that H and K are mutually separated means that H and K are mutually exclusive and neither contains a limit point of the other.

Notation. The statement that H is the union of the sets M and N is written $H = M \cup N$. If G is a collection of sets, G^* denotes the union of all sets belonging to G.

Theorem. If S is an ordered space and H and K are mutually separated subsets of S, then there exist mutually exclusive open sets H' and K' which contain H and K respectively.

Proof. Suppose S is an ordered space and H and K are mutually separated subsets of S.

For each point h in H which is not an end point of S, there exists a segment (h',h") containing h which has the following properties:

(1) No point of K belongs to (h',h").

(2) The point h" (similarly h') belongs to H, or else there is no point of H between h and h" (h').

(3) The point h" (similarly h') is not in K, or else there is no point between h and h" (h').

To show this, let h be a point of H which is not an end point of S. Since h is not a limit point of K, there is a segment (m,n) which contains h and no point of K. If there is no point between h and n, let h" be n. If there is a point p between h and n, then:

(a) if there is no point of H between h and n, let h" be p.

(b) if there is a point q of H between h and n, let h" be q.

The point h' is determined similarly.

For each h in H which is not an end point of S, let M(h) denote the collection of segments which contain h and satisfy properties 1,2, and 3.

If h belongs to H and h is an end point of S, then there exists an end segment [h,h") or (h',h] which contains h and no point of K and satisfies properties 2 and 3. For each h in H where h is an end point of S, let M(h) be the collection of end segments which contain h and no point of K and satisfy properties 2 and 3.

Let M be the collection of segments and end segments formed by choosing one and only one element from each M(h). Then M* is an open set which contains H and no point of K.

We now show that if k is a point of K which is not an end point of S, then there is a segment (k',k") which contains k and no point of M^*. Since k is not a limit point of H, there is a segment (p,q) which contains k and no point of H. If no element in M contains p, let k' be p. If some element in M contains p, then there is only one. For suppose there are two, (m',m'') and (h',h''), respectively containing the points m and h of H. Then h is not m so h precedes m or m precedes h. Suppose h precedes m. Then h'' belongs to H since m is between h and h'' and since p belongs to (h',h''), p precedes h'' so that h'' belongs to (p,q) which is impossible. Hence there is only one segment in M which contains p, say (h',h'') containing the point h of H. If h'' is k, let k' be h. If h'' is not k, then h'' precedes k. Let k' be h''.

The point k'' is determined similarly.

If there is a point z common to M^* and (k',k''), then z belongs to a segment in M which contains either k' or k'' which is impossible. Therefore k is not a limit point of M^*.

If k is a point of K which is an end point of S, then there is an end segment $[k,k'')$ or $(k',k]$ which contains k and no point of M^*. The proof is similar to that for a non-end point.

Hence no point of K is a point of M^* or a limit point of M^*. Therefore K is a subset of the complement of the closure of M^*.

Let H' be M^* and K' the complement of the closure of M^*. Then H' and K' are mutually exclusive open sets containing H and K respectively, and the theorem is proved.

Remark. A space S is said to be hereditarily normal if and only if for every two mutually separated sets H and K there exist mutually exclusive open sets H' and K' containing H and K respectively. If S is an ordered space, then S is hereditarily normal.

CHAPTER II

SEPARABLE ORDERED SPACES

Definition. The statement that the set M is separable means that there exists a countable subset K of M such that each point of M is a point of K or a limit point of K.

Definition. The statement that the set M is first countable means that if x is a point of M, then there exists a countable collection G of open sets containing x such that if R is any open set containing x, then some member of G is a subset of R.

Theorem 1. If S is a separable ordered space, then S is first countable.

Proof. Suppose S is a separable ordered space. Then there exists a countable subset K of S such that each point of S is a point of K or a limit point of K.

Let x be a point of S which is not an end point of S.

Case 1. Suppose there exist points m and n in S such that x belongs to (m,n) and (m,x) is empty and (x,n) is empty. Then the segment (m,n) is a subset of every open set which contains x.

Case 2. Suppose there exists no point m in S such that (m,x) is empty, but there exists a point n in S such that (x,n) is empty. Then let L be the collection of all segments (k,n) where k belongs to K and k precedes x. The collection L is countable since K is countable.

Let (p,q) be a segment which contains x. Then (p,x) is not empty so there exists a point k in K such that k belongs to (p,x). Thus x belongs to (k,n) which is a subset of (p,q) and (k,n) belongs to L.

Case 3. Suppose there exists no point n in S such that (x,n) is empty, but there exists a point m in S such that (m,x) is empty. Then let L be the collection of segments (m,k) where k belongs to K and x precedes k. The collection L is countable since K is countable.

Let (p,q) be a segment which contains x. Then (x,q) is not empty so there exists a point k in K such that k belongs to (x,q). Thus x belongs to (m,k) which is a subset of (p,q) and (m,k) belongs to L.

Case 4. Suppose there exists no point m in S such that (m,x) is empty and there exists no point n in S such that (x,n) is empty. Then let L be the collection of all segments (k,j) where k and j belong to K and k precedes x and x precedes j. The collection L is countable since K is countable.

Let (p,q) be a segment which contains x. Then (p,x) is not empty so there exists a point k in K such that k belongs to (p,x). Also (x,q) is not empty so there exists a point j in K such that j belongs to (x,q). Thus x belongs to (k,j) which is a subset of (p,q) and (k,j) belongs to L.

From cases 1,2,3, and 4, we have that if x is not an end point of S, then there exists a countable collection G of open sets containing x such that if R is any open set containing x, then some member of G is a subset of R.

If x is an end point of S, then by arguments similar to that for a non-end point, there exists a countable collection G of open sets containing x such that if R is any open set containing x, then some member of G is a subset of R. Thus the theorem is proved.

Definition. The statement that the set M is completely separable (second countable) means that there exists a countable collection G of open sets such that if x is a point of M and R is an open set containing x, then some member of G contains x and is a subset of R.

Theorem 2. If S is a separable ordered space, then S is completely separable if and only if the set of points in S which have immediate predecessors is countable.

Proof. Suppose S is a separable ordered space. Let M be the set of all points in S which have immediate predecessors.

Suppose M is not countable. Let G be a countable collection of segments. Then there exists a point y in M such that y is not the right end point of any segment in G. If not, then G would not be countable.

Let L be the set of all points of S which precede y. The set L is open and there is a point y' in L such that (y',y) is empty. There is no segment in G which contains y' and is a subset of L. Thus S is not completely separable.

Therefore if S is completely separable, then M is countable.

Now suppose M is countable. Let M' be the set of all points y' such that there exists a point y in M where (y',y) is empty. The set M' is countable since M is countable. Let K be a countable subset of S such that each point of S is a point of K or a limit point of K. Let $L = K \cup M \cup M'$.

Let T be the collection of segments (p,q) where p is a point of L, q is a point of L, p precedes q, and (p,q) is not empty. The collection T is countable since L is countable.

If S has end points, then by Theorem 1 there exist countable collections H and H' of open sets about these points which satisfy the first countable property. Let $T' = T \cup H \cup H'$.

If x is an end point and R is an open set containing x, then there is an open set V in H or H' which contains x and is a subset of R.

Now let x be a non-end point of S and let (m,n) be a segment containing x.

Case 1. If (m,x) is empty and (x,n) is empty, then m and n are in L and hence (m,n) belongs to T'.

Case 2. If (m,x) is not empty and (x,n) is empty, then n belongs to L and there is a point k in K such that k belongs to (m,x). Thus x belongs to (k,n) which is a subset of (m,n) and (k,n) belongs to T'.

Case 3. If (m,x) is empty and (x,n) is not empty, then m belongs to L and there is a point k in K such that k belongs to (x,n). Thus x belongs to (m,k) which is a subset of (m,n) and (m,k) belongs to T'.

Case 4. If (m,x) is not empty and (x,n) is not empty, then there is a point k in K such that k belongs to (m,x) and there is a point j in K such that j belongs to (x,n). Thus x belongs to (k,j) which is a subset of (m,n) and (k,j) belongs to T'.

From cases 1,2,3, and 4 we have that if x is a point of S and R is an open set containing x, then some member of the countable collection T' of open sets contains x and is a subset of R. Therefore if M is countable, then S is completely separable. Thus the theorem is proved.

Remark. A regular T_1 space whose topology has a countable base is metrizable, [1, Theorem 16, p. 125] . Therefore if S is a separable ordered space, then S is metrizable if and only if the set of all points in S which have immediate predecessors is countable.

Theorem 3. If S is a separable ordered space and M is a subset of S, then M is separable.

Proof. Suppose S is a separable ordered space and M is a subset of S. Let K be a countable subset of S such that every point of S is a point of K or a limit point of K.

If M is countable, there is nothing to prove.

Suppose M is not countable. Let L be the collection of all segments (k,j) where k and j are points of K and there is a point m in M such that m belongs to (k,j). Let L' be the subset of M formed by choosing a point of M from each segment in L.

Let K' be the set of all points of K which are also in M.

Let R be the set of all points m in M such that there exists a point k in K where m precedes k and (m,k) contains no point of M. Similarly let R' be the set of all points m in M such that there exists a point k in K where k precedes m and (k,m) contains no point of M.

Let E be the set of points in M which are end points of S.

Let $T = L' \cup K' \cup R \cup R' \cup E$. The set T is a countable subset of M.

Suppose x is a point of M which is not in T. Let (p,q) be a segment which contains x.

Case 1. Suppose (p,x) is empty. Then (x,q) is not empty since x is not a point of K. Therefore there is a point k in K such that k belongs to (x,q). If (x,k) contains no point of M, then x belongs to T. Thus there is a point m of M such that m belongs to (x,k). Also (x,m) is not empty so there is a point j in K such that j belongs to (x,m). Hence m belongs to (j,k) which is a subset of (p,q) and there is a point of T in (j,k). Therefore, in this case, there is a point of T in (p,q).

Case 2. Suppose (x,q) is empty. Then (p,x) is not empty so there is a point k in K such that k belongs to (p,x). Also (k,x) contains a point m of M since x is not in T and (m,x) is not empty. Thus there is a point j in K such that j belongs to (m,x). Thus m belongs to (k,j) which is a subset of (p,q) and (k,j) contains a point of T. Therefore, in this case, there is a point of T in (p,q).

Case 3. Suppose (p,x) is not empty and (x,q) is not empty. Then there exists points k and j in K such that k belongs to (p,x) and j belongs to (x,q). Thus x belongs to (k,j) which is a subset of (p,q) and (k,j) contains a point of T. Therefore, in this case, there is a point of T in (p,q).

From cases 1,2, and 3 we have that if (p,q) is a segment containing the point x, then (p,q) contains a point of T. Therefore x is a limit point of T, and the theorem is proved.

<u>Theorem 4</u>. If S is a separable ordered space and M is an uncountable subset of S, then some point of M is a limit point of M.

Proof. Suppose S is a separable ordered space and M is an uncountable subset of S. From Theorem 3, there is a countable subset T of M such that each point of M is a point of T or a limit point of T. Since T is countable there is a point x of M which is not in T. The point x is a limit point of T and is therefore a limit point of M. Thus the theorem is proved.

<u>Definition</u>. The set S is said to be <u>semi-metrizable</u> if and only if there exists a real-valued function f defined on $S \times S$ such that if p and q are points of S, then

(1) $f(p,q)$ is zero if and only if p is q,

(2) $f(p,q) = f(q,p)$,

(3) $f(p,q)$ is non-negative, and

(4) the point t is a limit point of the set M if and only if for every positive number r, there exists a point x of M distinct from t such that $f(t,x)$ is less than r.

If p and q are points of S, then $f(p,q)$ is called the distance from p to q.

Definition. If S is a semi-metrizable ordered space and x is a point of S, then the statement that D is a disk of radius r about x means that D is the set to which the point y belongs if and only if the distance from x to y is less than r.

Theorem 5. If S is a separable semi-metrizable ordered space, then S is completely separable.

Proof. Suppose S is a separable semi-metrizable ordered space. Let M be the set of all points of S which have immediate predecessors. Suppose M is uncountable.

For each y in M, let $L(y)$ be the set of all points x in S such that x is y or y precedes x. The set $L(y)$ is an open set.

Consider the collection R of ordered pairs (y,n) where y is a point of M and n is the smallest natural number such that the disk of radius $1/n$ about y is a subset of $L(y)$. There exists a natural number m such that uncountably many points of M are first terms of ordered pairs in R which have m as the second term. Let M' be the set of all points y in M such that (y,m) belongs to R. No point of M' is a limit point of M', since if y and z are two distinct points of M', the distance from y to z is greater than or equal to $1/m$. Thus S contains an uncountable subset M' with the property that no point of M' is a limit point of M'. This is a contradiction to Theorem 4.

Hence the set M is countable and from Theorem 2 we have that S is completely separable. Thus the theorem is proved.

Definition. The statement that the set S has the Lindelöf property means that whenever G is a collection of open sets such that S is a subset of G^*, then there exists a countable subcollection F of G such that S is a subset of F^*.

Theorem 6. If S is a semi-metrizable ordered space which has the Lindelöf property, then S is separable.

Proof. Suppose S is a semi-metrizable ordered space which has the Lindelöf property.

For each point x in S and for each natural number n, there is an open set containing x which is a subset of the disk of radius $1/n$ about x. The interior of the disk is the largest open subset of the disk. Thus for each natural number n, let $G(n)$ be the collection of interiors of disks of radius $1/n$ about the points of S.

Since S has the Lindelöf property and S is a subset of $G(n)^*$, then there is a countable subcollection $F(n)$ of $G(n)$ such that S is a subset of $F(n)^*$. This is true for each natural number.

Let $K(n)$ be the collection of points x in S such that the interior of the disk of radius $1/n$ about x belongs to $F(n)$. The set $K(n)$ is countable. Let K denote the collection of sets A where A is $K(n)$ for some n. Then K^* is countable.

Suppose x is a point of S which is not in K^*. Let r be a positive number. There is a natural number n such that $1/n$ is less than r. Since x is a point of S and S is a subset of $F(n)^*$, there is an open set in $F(n)$ which contains x, and this open set is a subset of the disk of radius $1/n$ about a point y which belongs to K^*. Thus the distance from x to y is less than r. Therefore x is a limit point of K^*.

The set K^* is countable and every point of S is a point of K^* or a limit point of K^*. Therefore S is separable, and the theorem is proved.

<u>Theorem 7</u>. If S is a semi-metrizable ordered space which has the Lindelöf property, then S is completely separable.

Proof. Suppose S is a semi-metrizable ordered space which has the Lindelöf property. Then by Theorem 6, S is separable, and by Theorem 5, S is completely separable. Thus the theorem is proved.

Theorem 8. If S is an ordered space, then the following are equivalent:

(a) There does not exist an uncountable collection of mutually exclusive open sets.

(b) If M is an uncountable set, then some point of M is a limit point of M.

Proof. Suppose S is an ordered space.

If property (a) is not true, then there exists an uncountable collection G of mutually exclusive open sets. Choose one and only one point from each set in G, and denote this set by M. The set M is uncountable and no point of M is a limit point of M. Therefore if property (b) is true, then property (a) is true.

Suppose property (a) is true and suppose property (b) is not true. Then there exists an uncountable set M with the property that no point of M is a limit point of M.

For each x in M, let B(x) be the collection of all segments (a,x) such that (a,x) contains no point of M. If x and y are two distinct points of M, and if (a,x) belongs to B(x) and (b,y) belongs to B(y), then (a,x) and (b,y) are disjoint. Since property (a) is true, only countably many of the sets B(x) are non-empty. If x belongs to M and if B(x) is empty, then x has an immediate predecessor. Therefore, let M' be the set of all points in M which have immediate predecessors. The set M' is uncountable.

For each x in M', let C(x) be the collection of segments (x',b) where (x',x) is empty, no point of M' different from x belongs to (x',b), and b is not in M' unless (x,b) is empty. Each set in C(x) contains the point x. Suppose x and y are two distinct points of M', and suppose (x',b) belongs to C(x) and (y',d) belongs to C(y). Then (x',b) and (y',d) are disjoint, since x is not an element of (y',d) and y is not an element of (x',b).

Choose one and only one set from each C(x), and denote this collection by H. The collection H is an uncountable collection of mutually exclusive open sets. This contradicts property (a). Therefore if property (a) is true, then property (b) is true. Thus the theorem is proved.

<u>Theorem 9</u>. If S is a semi-metrizable ordered space with the property that each uncountable subset of S contains one of its limit points, then S is completely separable.

Proof. Suppose S is a semi-metrizable ordered space with the property that each uncountable subset of S contains one of its limit points.

Suppose S is not completely separable. Then S is not separable and S does not have the Lindelöf property.

For each natural number n, let G(n) be the collection of disks of radius 1/n about the points of S. The space S is a subset of $G(n)^*$. If, for each n, there is a countable subcollection of G(n) which covers S, then S is separable by techniques similar to those of the proof of Theorem 6. So there exists a natural number m such that S is a subset of $G(m)^*$ and no countable subcollection of G(m) covers S. Let R be a well-ordering of the collection G(m).

Form a new uncountable ordering R' as follows: Let the first member A of R be the first member of R'. The set A does not cover S, so there exists a point y in S which is not in A. There exists a disk in G(m) which contains y as its center point. Let B be the first member in R which has the property that its center point is not in A. Let B be the second member of R'. If Q is an initial segment in R', then if Q is countable, Q^* does not cover S so there exists a point x in S which is not in Q^*. Let C be the first member in R which has the property that its center point is not in Q^*. Then C is the next set in R'.

Thus there exists an uncountable subcollection F of G(m) having the property that if x and y are distinct center points of sets in F, then the distance from x to y is greater than or equal to 1/m. Therefore there exists an uncountable set M having the property that no point of M is a limit point of M. This is a contradiction to our assumption. Therefore S is completely separable, and the theorem is proved.

Remark. Theorem 10 is stated as a summary of the theorems of Chapter 2.

Theorem 10. If S is a semi-metrizable ordered space, then the following are equivalent:

(a) The space S is completely separable.

(b) The space S is separable .

(c) The space S is hereditarily separable.

(d) The space S has the Lindelöf property.

(e) If M is an uncountable subset of S, then some point of M is a limit point of M.

(f) There does not exist an uncountable collection of mutually exclusive open sets.

CHAPTER III

EXAMPLES

<u>Example 1</u>. Let S be the set of ordered pairs of real numbers (p,q) such that p and q are greater than or equal to zero and less than or equal to one. Order S as follows: (p,q) precedes (x,y) if and only if p is less than x or if p = x, then q is less than y. With this order, S is an ordered space.

Properties:

(a) S is compact.

(b) S is connected.

(c) S is first countable.

(d) S is not separable.

(e) S is not semi-metrizable. (Theorem 6).

<u>Example 2</u>. Let S be the set of all ordered pairs of rational numbers (p,q) such that p and q are less than one and greater than zero, with the same order as in example one.

Properties:

(a) S is metrizable.

(b) S is not locally compact.

(c) S is not limit compact.

(d) S is not connected or locally connected.

(e) S is not separable.

(f) S does not have the Lindelöf property.

Example 3. Let S be the set of all ordered pairs (p,0) and (q,1) where p and q are real numbers and p and q are less than or equal to one and greater than or equal to zero. Order S as in example one.

Properties.

(a) S is separable.

(b) S is compact.

(c) S is first countable. (Theorem 1).

(d) S is not connected or locally connected.

(e) S is not completely separable. (Theorem 2).

(f) S is not semi-metrizable. (Theorem 5).

Example 4. Let M be an uncountable set and let R be a well ordering of M. Let A be the set of all points y of M such that there are uncountable many points of M which precede y in R. Let x be the first element in A.

Let S be the subset of M consisting of x and all points of M which precede x in R. The set S is an ordered space with respect to the ordering in R.

Properties:

(a) S is compact.

(b) S is not first countable.

(c) S is not separable.

(d) S is not semi-metrizable. (Theorem 6).

<u>Example 5</u>. Let S be the subset of example 4 of all points of M which precede x.

Properties:

(a) S is first countable.

(b) S is limit compact.

(c) S is locally separable and locally compact.

(d) S does not have the Lindelöf property.

(e) S is not separable.

(f) S is not semi-metrizable.

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