EMPIRICAL INTERPRETATION OF ECONOMIC GROWTH MODELS OF THE UNITED STATED

A Thesis

Presented to

The Faculty of the Department of Economics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

To Sang Kong

January, 1966

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ABSTRACT

This is a study of economic growth theories. This study is not a traditional approach using general treatises, but an approach utitizing a combination of mathematical economics and econometrics.

In recent years the study of economic growth has become common among economists. Before the Great Depression little attention was paid to the many factors affecting the growth of national income. The Great Depression encouraged economists to give closer scrutiny to this subject. During the Second World War the analysis of national income was largely aimed at controlling the war economy in the United States. The national income concept has been considered very useful in understanding and explaining what takes place in the economy. Thus the study of economic growth has become more popular. New courses in this subject have been introduced in colleges and universities. Also, new books, institutes, and conferences in this field are continually increasing in number. The topic of economic growth is extremely broad. It may be divided into two categories:

1) the growth involved in the shift of an economy from the stage of "underdeveloped" to the stage of "developed;"
2) the growth of the already "developed" economy.

This study is confined to the second category, particularly to the growth of the national income of the United States.

Due to the scarcity of data, a complete and more sophisticated analysis is a matter of difficulty. Collection of reliable statistics is tedious. Often the desirable raw material is hard to obtain. Even

in the United States, income statistics on the state level do not exist.

Moreover, in this study of national income accomplishment is unlikely without tedious calculation. The deeper one goes into the study, the more calculation becomes necessary. Also if a better result is to be expected, a more complicated analysis has to be undertaken.

In the preparation of this study, an attempt has been made to give an empirical interpretation of economic growth models for the United States, and to bring out the consistencies or inconsistencies between reality and theories. Chapter I is an introduction to the study. The subject of Chapter II is the discussion of some major economic growth theories, which constitute the basis of this study. The method of estimation of parameters and the structure of models are explained in Chapter III. Chapter IV covers statistical results, the empirical interpretation of various arguments in economic growth. Finally, a summary and a conclusion are presented in Chapter V. The author wishes to thank Dr. Z. A. Eltezam for his guidance, patience, understanding, and encouragement in the supervision of this study. He also appreciates the valuable suggestions and time-consuming efforts of Dr. Henry C. Chen.

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CHAPTER I

INTRODUCTION

I THE PURPOSE OF THE STUDY

Economic problems can be studied in two distinct ways. One is the traditional approach, by using general treatises dealing with the variables that affect the economy. The other one is the mathematical or econometric approach, by the use of mathematics and statistics. The second approach has received wide acceptance during the last few decades. It was originated at the end of the 1920's, and became institutionalized in the succeeding years.[1] It is considered as a scientific development of economics, and is still developing and growing in importance.

The approach by using mathematics and statistics has become a tool of analysis widely used in economic growth. Still there are many arguments about this approach. Economists have different opinions on it.[2] We all realize that mathematics has been successfully applied in natural sciences, but can it achieve the same success in economics? Economists have been trying hard to fit mathematics into economic theories, intending to discover some laws governing the growth or development of an

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[1] Institutions and journals for this purpose have been established in the past decades, such as: The Econometric Society, The Cowles Commission for Research in Economics, _Econometrica_, _Review of Economic Statistics_, _et cetera_.

[2] Professor R. G. D. Allen has said; "Whether mathematical techniques can be, or should be, used in economics is a much-discussed question." See R. G. D. Allen, _Mathematical Economics_ (London: Macmillan & Company, Ltd., 1959), Introduction, p. xv.

economy. Such a scientific development in economics, however, is still considered in an experimental stage. More effort is needed. This study may be said to be one of the experiments in this respect.

There is one thing to be noted that every economic phenomenon is characterized by change. Consumption is subject to the change of con-

Morris Copeland.[3] The flow-of-funds accounts divide the economy into various sectors or different economic decision-making groups, showing for each sector the main sources of funds with which it makes payments and the principal uses of funds in connection with such payments. For instance, in the United States, in the Federal Reserve flow-of-funds accounts, as shown in the Federal Reserve Bulletin, there are eleven sectors, e.g., consumer and nonprofit, corporate business, farm business, federal government, et cetera. Thus the flow-of-funds accounts show how money and credit perform, and also the pattern of financial assets and debts after such transactions. On the other hand, the national income and product accounts provide a measure of the nation's current productive efforts. There are two main streams in these accounts. They are the stream of consumption, and the stream of investment. Consumption consists of personal consumption and government consumption, while investment is the aggregate of private investment (including foreign investment) and government investment.

According to the national income and product accounts, aggregate income equals the sum of aggregate consumption and aggregate investment. This concept has become the basis of macro-economics, and is widely used in the analysis of economic growth. Nowadays, in most of the national income models, the national income and product accounts

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[3]Morris Copeland, Study of Moneyflows in the United States (New York: National Bureau of Economic Research, 1952). This is the first fully developed publication in the flow-of-funds accounts.

are used, such as in the Keynesian model[4]

$$Y = C + I$$

where Y designates the aggregate income, C, the aggregate consumption, and I, the aggregate investment. In this study the national income and product accounts rather than the flow-of-funds accounts are considered.

The period for this study is from the year 1929 to the year 1963. During this thirty-five year period, there were a great depression at the beginning and a great war in the middle.. The great depression, of course, was an economic phenomenon, but the great war was certainly not. Moreover, during the war, a substantial increase in government expenditures did not represent the normal behavior of the economy. For this reason, the war-years are excluded from the data used in this study. Also, gross national product is used rather than net national product, since the value of capital depreciation of the private sector and the government sector is not easily traceable. Because the period for this study starts in the year 1929, all data are converted into 1929 dollars to avoid inconsistency.

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[4] John M. Keynes, The General Theory of Employment, Interest and Money (New York: Harcourt, Brace and Company, 1936), p. 63.

A REVIEW OF ECONOMIC GROWTH THEORIES

In this chapter an attempt is made to review some of the major theories in economic growth. Economic growth is actually a complex resultant of many factors. Economists agree that there is a very close relationship among aggregate income, aggregate consumption and aggregate investment, and that their interactions play a chief role in the economic growth, but there is little agreement among them as to the nature of this relationship. Let us examine some of the most widely recognized relationships, which exist among these economic aggregates.

I THE CLASSICAL THEORY

Although basically this study is related to modern growth theories, a brief review of the classical theory is helpful in understanding the position of modern theorists. Here the classical theory refers to the traditional or orthodox principles of economics handed down and generally accepted by Western economists from somewhere around the time of David Ricardo (1772-1823) to 1930. According to this theory, output is a function of labor. By the assumption of Say's Law,[1] supply creates its

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[1]This is named after the French economist, J. B. Say, 1767-1832. His theory is usually summarized as "supply creates its own demand," which is best expressed in the following quotation from his writings: "The total supply of products and the total demand for them must of necessity be equal, for the total demand is nothing but the whole mass of commodities which have been produced; a general congestion would consequently be an absurdity." From J. B. Say, Traite (1st ed., 1803), Vol. II,p. 175, as quoted in C. Gide and C. Rist, A History of Economic Doctrines (2nd English ed.; New York: D. C. Heath & Co., 1948), p. 131.

own demand, and there will always be a sufficient rate of spending to maintain full employment. Thus, income is spent automatically at a rate keeping all resources employed (including labor supply). Income is either spent currently on consumer goods or saved for the future spending on producer goods; i.e., all income is spent, partly on consumption and partly on investment. The equality of saving and investment is attributed to the rate of interest. An increase in interest rate will increase saving, and a decrease in interest rate will decrease saving. On the other hand, the lower rate of interest will increase the incentive to invest, leading to the elimination of the excess of saving over investment. By this principle, since saving is spent on investment sooner or later, the volume of consumption does not seem to be important. Classical economists, however, did not realize that a fall in consumption, instead of leading to an increase in investment, may lead to a fall in total demand and therefore in employment.

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II THE KEYNESIAN THEORY OF ECONOMIC GROWTH

The Keynesian theory is a turning point from the classical theory. According to Say's Law, if more resources are employed in one industry or in one firm, they are assumed to be drawn away from other industries or other firms, because supply cannot be increased without the increase of demand as both of them are equated to one another. Thus the classical theory primarily relates, but not entirely, to the use of a given quantity of resources by individual firms and individual industries within the economic system as a whole. On the contrary, the Keynesian theory relates to economic aggregates, such as the aggregates of employment, national income, consumption, saving, and investment. Keynes recognizes income as a function of laber supply (or employment), but also presumes that aggregate income Y is derived from aggregate consumption C and aggregate investment I, as we have indicated in the previous chapter, such that

$$Y = C + I \tag{2-1}$$

In other words, aggregate consumption and aggregate investment determine the amount of aggregate income.

Keynes accepts the classical proposition of equality of saving and investment but attributes the equality to changes in the level of income rather than to the rate of interest. He first sets consumption as a function of income.[2] Let us see how this consumption function is arrived at. It is assumed that income is either consumed or saved or both. According to Keynes, saving S is a function of income rather than a function of interest as in the classical theory, i.e.,

$$S = sY \tag{2-2}$$

where s is a constant, and less than one. s is called the marginal propensity to save. According to the proposition of equality of saving and investment, and from equation (2-2), then (2-1) becomes

$$Y = C + sY$$

yielding the consumption function,

$$C = (1-s)Y$$

or

$$C = cY \tag{2-3}$$

where c is equal to (1-s) and is called the marginal propensity to

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[2]Keynes, *op. cit.*, p. 27.

consume.

By substituting (2-3) into (2-1), there results

$$Y = cY + I$$

After transposing, it gives

$$(1-c)Y = I$$

or

$$Y = [1/(1-c)]I$$

$$= kI$$

where k is a positive constant and greater than one as (1-c) is less than one; k is called the multiplier.[3] Since 1-c=s, the multiplier can be described as the reciprocal of the marginal propensity to save. The multiplier implies that an increase in investment will create k times the original increment in new income. The greater the marginal propensity to consume, the greater the multiplier will be. In the Keynesian theory, price level, rate of interest, quantity of money or total assets, distribution of income, and other such factors are of little or no importance to the consumption function.

There is an argument that consumption may not behave as a constant proportion with the level of national income. As one moves along the distribution from lower income to higher income, average consumption will rise, but less than income in proportion, and the higher the income the less the rise in consumption from a further increase in income. Although the marginal propensity to consume is still positive and less than one, it declines as income rises. It will not stay with the same

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[3]Actually this multiplier concept was originated by R. F. Kahn. See R. F. Kahn, "The Relation of Home Investment to Unemployment," Economic Journal, June, 1931. Also ibid., pp. 113-115.

constant.

However, for J. S. Duesenberry[4] and M. Friedman,[5] the long-run relationship between consumption and income appears to be constant so that the average propensity to consume is constant and equal to the marginal propensity to consume. If the consumption function is

$$C = a + cY$$

where a is a constant term, then the average propensity to consume will be

$$\begin{aligned} APC &= C/Y \\ &= (a + cY)/Y \\ &= a/Y + c \end{aligned}$$

but c is the marginal propensity to consume in the case where

$$C = cY$$

i.e., $$\begin{aligned} MPC &= C/Y \\ &= cY/Y = c \end{aligned}$$

Thus in the first case

$$APC = a/Y + MPC$$

which shows that the average propensity to consume is not a constant and is greater than the marginal propensity to consume, but declines as income increases. Mathematically, if income Y increases substantially, a/Y will approach zero; thus, in the long-run APC is approaching a limit of the MPC as income Y increases.

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[4] J. S. Duesenberry, Income, Saving and the Theory of Consumer Behavior (Cambridge: Harvard University Press, 1949), pp. 32-37.

[5] M. Friedman, A Theory of the Consumption Function (Princeton: Princeton University Press, 1957), pp. 7-14.

Perhaps there is such a situation that a long-run consumption function merely relates to the MPC, while a short-run function involves the APC, which is greater than the MPC. How can the two be related? Arthur Smithies has tried to make a reconciliation.[6] He has argued that the consumption function is basically nonproportional to the fluctuations of income drifting slowly upward over time as income grows slowly, and that its upward drift will just happen to offset the tendency for the average propensity to consume to decline with the growing of income. His argument for the upward drift in consumption is as follow:

a) Population has been moving from rural to urban residence where people usually spend more out of a given income.

b) The older age bracket is becoming bigger because of the successful advancement of medical science, and these older people consume without earning.

c) The introduction of new consumer commodities is increasing, stimulating people to spend additional money on consumption.

Therefore, Smithies has suggested that the consumption function is also a function of time t, such that

$$C = a + bY + ct$$

where a, b and c are some constants, and t, a positive integer, designates the time period, such as

$$t = 0, 1, 2, \dots .$$

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[6]Arthur Smithies, "Forecasting Postwar Demand:I," Econometrica, Vol. 13, January, 1945, pp. 1-14.

Next on the investment side, Keynes' assumption is that investment is autonomous, that is, investment is determined outside the model. In mathematical interpretation, the assumption is

$$Y = C + I$$

$$C = cY$$

$$I = I_o$$

where $I=I_o$ means that I_o is given, and its volume is not dependent upon either the volume of consumption or the level of income inside the model. According to Keynes, the volume of investment depends on the marginal efficiency of capital and the rate of interest.[7] For the classical theory investment is a function of interest alone, which the Keynesian theory does not totally accept. The Keynesian theory incorporates the marginal efficiency of capital into investment.[8] The marginal efficiency of capital is a rate of discount which will make the present value of all the prospective returns from an investment just equal to the cost of the investment. Take a simple example, assuming that the cost of a building is \$20,000. The building will yield \$1,200 per year in rental and has depreciation of \$200 per year, giving a net return of \$1,000 per year. Then the marginal efficiency of capital is 5% (i.e., \$1,000/\$20,000=0.05). If the rate of interest of 4%, this building is worth \$25,000 (i.e., \$1,000/0.04=\$25,000); then it will be preferable

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[7]Keynes, op. cit., pp. 27-28.

[8]The concept of marginal efficiency of capital is actually not originated by Keynes. Professor Irvin Fisher, at an earlier date, has provided a similar phrase to Keynes' marginal efficiency of capital, "rate of return over cost;" ibid., pp. 140-141.

to invest in this building rather than to lend out \$20,000 at 4% yielding \$800 a year. In other words, if a man expects to yield \$1,000 at the end of the year, he has to invest \$952 only if the marginal efficiency of capital is 5% (\$1,000/(1+0.05)=\$952); but if he lends out his money at 4%, he has to lend out \$961 in order to get \$1,000 at the end of the year (\$1,000/(1+0.04)=\$961). This \$952 is synonymous to Keynes' supply price.[9] The general formulation of the supply price may be expressed as:

$$\text{Supply Price} = \frac{A_1}{(1+r)} + \frac{A_2}{(1+r)^2} + \frac{A_3}{(1+r)^3} + \dots + \frac{A_n}{(1+r)^n}$$

where A's are the annual returns, and r is the marginal efficiency of capital. By using this formula we can also calculate the principal on money lent out by substituting the rate of interest into r. According to this investment theory, therefore, when the marginal efficiency of capital is above the going rate of interest, investment will be considered as profitable and will tend to expend, and when it is below the rate of interest, investment will be discouraged. Hence, the volume of investment is determined by the relation between marginal efficiency of capital and the rate of interest.

According to this principle, however, the judgement of the value of marginal efficiency of capital may not be accurate for the later years or periods due to the fluctuation of economic phenomenon, Furthermore, the behavior of investment does not seem to be so simple. Increases of investment may bring a higher level of employment, which

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[9] Ibid., p. 135.

will give a higher level of aggregate income. Then consumption will be increased as a result of increased income. Increases in consumption mean increases of consumer demands which will cause additional investment. This "feed-back" reaction cannot be neglected. Moreover, for aggregate investment, the availability of funds are more important than the market rate of interest, and, on the other hand, the availability of funds may influence the market rate of interest.

In generalizing the Keynesian model, the multiplier is a relation between output and investment, and thus the effect of a change in investment is examined by means of the multiplier. This is one-sided only because it ignores the reciprocal relations between investment and output. As we have seen, investment does influence output, but output also affects investment. Investment which arises due to a change in output is called induced investment. In the Keynesian model, induced investment is neglected. So Keynes' model is clearly defective as a description of economic reality.

III THE ACCELERATION PRINCIPLE

We have seen that the multiplier is concerned only with original investment as a stimulus to consumption and then to income. It is not the way the "real world" seems to be, because the multiplier does not involve the question whether additional consumption will induce further investment or not. Output can reproduce the course of autonomous investment suitably "multiplied up," but otherwise it tends steadily to its equilibrium level. This is because the multiplier uses only one relation, the consumption function; it gives no consideration to the side of investment. Induced investment does occur with additional consumption. As mentioned in the preceding section, additional consumption will enlarge consumer demand which will induce additional investment. the effect of added consumption upon the demand for investment is called the acceleration principle.

The acceleration principle has long been used in the theory of investment, which is now recognized to be of crucial importance in almost all macro-economic models. It was first formally presented by J. M. Clark in 1917.[10] It is held that the demand for investment is derived from the demand for consumption. An increase in consumption will tend to induce an increase in investment; this relation can be expressed as

$$I(t) = i[C(t) - C(t-1)] \tag{2-4}$$

where i is called the accelerator, t is referred to the time period.

In this consumption-investment relation, investment I will be zero when the volume of consumption does not change between two periods, that is, when consumption is constant. If consumption changes by a positive or negative amount, investment or disinvestment will occur at a rate which is small or large depending on whether the change in consumption is small or large. Since consumption is a function of income, then investment is also a function of income. Mathematically, if we substitute (2-3) into (2-4), then we have

$$I(t) = i[cY(t) - cY(t-1)]$$

or

$$I(t) = ci[Y(t) - Y(t-1)] \tag{2-5}$$

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[10] John M. Clark, "Business Acceleration and the Law of Demand: A Technical Factor in Economic Cycle," The Journal of Political Economy, Vol. 25, March, 1917, reprinted in the AEA Readings in Business Cycle Theory (Philadelphia: Blakiston Co., 1944), pp. 235-260.

Thus we obtain investment expressed in terms of income. Since c and i are both positive constants, the product of c and i will also be a positive constant, but, because c is less than one, the product of c and i will be less than the value of i. Therefore, by the above substitution the acceleration coefficient is smaller when we define investment in terms of income. However, the power of the acceleration upon investment does not shrink. For the purpose of illustration, let us take a simple example. Assume c=0.8, i=2, Y(t)=100, and Y(t-1)=80, from (2-5)

$$I(t) = (2)(0.8)(100-80)$$
$$= (1.6)(20) = 32$$

From (2-3)

$$C(t) = cY(t)$$
$$= (0.8)(100) = 80$$

and

$$C(t-1) = cY(t-1)$$
$$= (0.8)(80) = 64$$

Then from (2-4), we also get

$$I(t) = (2)(80-64)$$
$$= (2)(16) = 32$$

In both ways we obtain the same amount of investment.

The acceleration principle then overcomes the deficiency of the multiplier and accomplishes the "feed-back" of investment. It shows us that investment can be induced rather than being autonomous only. Subsequently, income is not merely a function of the level of employment, as indicated by the classical economists and Keynes, but is also influenced by investment. But what will happen when the multiplier and the acceleration principles are acting together? Professor Paul A.

Samuelson has given a clear analysis, which will be discussed later.

IV DYNAMIC ECONOMIC ANALYSIS

In economic analysis involving different time periods, two methods may be used. One is called the continuous analysis by using differential calculus, which we shall not use in this study. The other one is called the period analysis. The period analysis is usually referred to as time lags analysis. Suppose that there are two variables, X and Y, in an analysis; if their relation involves the same time period, it is said that there is no time lag, e.g.,

$$X(t) = aY(t)$$

where a is a constant and t designates the time period. In case these two variables X and Y are not related in the same time period, such as

$$X(t) = aY(t-1)$$

then there is a time lag, and X of the present time is in terms of Y of one period ago. Now economists call those economic relations, not involving difference in time, static. Similar equations with time lags are called difference equations. The above equation with one time lag is a first-order difference equation; since there is no constant term in it (or the constant term is zero), it is a homogeneous equation. If an equation has two time lags, such as

$$X(t) = aY(t-1) + bY(t-2)$$

it is called a second-order difference equation. Similarly, an equation with three time lags is called a third-order difference equation, and so on.

Now let us look back at Keynes' model, where current consumption is a fixed proportion of the current income. This is a static relation, being without any lag. Many economists do not think that this relation can describe the dynamic situation of the "real world." D. H. Robertson first introduces a time significance known as the "period analysis."[11] he suggests a lag between the receipt of income and its expenditures, i.e., the total consumption in this year is a function of income earned last year, or

$$C(t) = f[Y(t-1)]$$

Logically his suggestion that consumption follows income is not without truth, because cash cannot be spent before it is received. But under his assumption, a zero cash balance at the beginning and at the end of each period must be presumed, and book credit facilities are not available - which is not necessarily true. The lag should probably be represented as a complex one. However, Robertson's simple lag relationship between income and consumption seems to be more reasonable than Keynes' consumption function without a lag.

Some other economists, like Duesenberry, also advocate the introduction of a lag to the consumption function. Duesenberry argues that the reason why consumption falls less than income in a depression is that consumers adjust their consumption not only to current income but to their previous income, particularly previous peak income.[12] The previous peak level of income has a persisting influence in maintaining

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[11] D. H. Robertson, "Some Notes on Mr. Keynes' General Theory of Employment," Quarterly Journal of Economics, November, 1936, p. 168ff.

[12] Duesenberry, op. cit., pp. 76-89.

consumption expenditures during a period of cyclical decline. His concept implies a lag, but a lag without regular length.

P. A. Samuelson[13] and J. R. Hicks[14] both agree with Robertson's suggestion: consumption as a proportion of income of the previous period

$$C(t) = cY(t-1) \qquad (2\text{-}6)$$

For investment, a lag should exist so that current investment is a function of the change in income occurring in the period before the last one, such that

$$I(t) = i[Y(t-1) - Y(t-2)] \qquad (2\text{-}7)$$

Thus this concept in the "period analysis" is more specified than Duesenberry's.

V SAMUELSON'S INTERACTIONS BETWEEN THE MULTIPLIER AND THE ACCELERATOR

Professor Paul A. Samuelson on the basis of a suggestion by Professor Alvin Hansen puts the multiplier principle and the acceleration principle together, contributing a famous analysis in aggregate income.[15] His basic assumption is also

$$Y(t) = C(t) + I(t) \qquad (2\text{-}8)$$

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[13] P. A. Samuelson, "Interactions Between the Multiplier Analysis and the Principle of Acceleration," Review of Economic Statistics, May, 1939, pp. 75-78, reprinted in the AEA Readings in Business Cycle Theory (Philadelphia: The Blakiston Co., 1944), pp. 261-69.

[14] J. R. Hicks, A Contribution to the Theory of the Trade Cycle (Oxford: The Clarendon Press, 1950), pp. 21-23.

[15] Samuelson, loc. cit.

having the meaning that national income Y of a certain period t is the sum of estimated consumer demand C and estimated investment demand I of the same period t. In his analysis there is a time lag, as mentioned in the previous section, between income and consumption

$$C(t) = cY(t-1) \tag{2-6}$$

where c is the marginal and average propensity to consume. This consumption function with a lag is the major difference with the Keynesian model. With regard to estimated investment, it is basically a function of the change in consumption, exactly the same as in the acceleration principle, being the same relation as expressed by (2-4)

$$I(t) = i[C(t) - C(t-1)] \tag{2-4}$$

Mathematically the investment function may also be expressed in terms of income Y by substitution, yielding a similar expression to (2-7)

$$I(t) = ci[Y(t-1) - Y(t-2)] \tag{2-9}$$

with the same meaning, i.e., present investment is a function of the change in income of the period before the last one. According to (2-9), investment depends upon the change in income. If there is no change in income (income is constant), then investment will be zero, that is, no investment is induced. Perhaps, in addition to induced investment, there is some autonomous investment, which is independent of income, such as the government demand for armaments, then (2-4) becomes

$$I(t) = i[C(t) - C(t-1)] + A \tag{2-10}$$

and (2-9) becomes

$$I(t) = ci[Y(t-1) - Y(t-2)] + A \tag{2-11}$$

where A is the autonomous investment.[16] From (2-8), (2-6) and (2-10), therefore

$$Y(t) = cY(t-1) + ci[Y(t-1) - Y(t-2)] + A$$

or $$Y(t) = (c + ci)Y(t-1) - ciY(t-2) + A \qquad (2\text{-}12)$$

This is a non-homogeneous second-order difference equation. It tells us that the current income depends on the income of the two previous periods, plus the current autonomous investment. At equilibrium, that means income is stable,

$$Y(E) = Y(t) = Y(t-1) = Y(t-2)$$

where Y(E) designates the income at equilibrium. Then (2-12) becomes

$$Y(E) = c(1+i)Y(E) - ciY(E) + A$$

or $$Y(E) = A/(1-c)$$

This is the multiplier formulation, in which income is equal to autonomous investment (which does not depend upon income) times the multiplier. It should be noted that the accelerator, i, drops out of the expression for equilibrium income. This is because induced investment occurs as a result of the acceleration principle only when income is changing. Since at equilibrium income is considered to be stable, the role for the acceleration principle in equilibrium does not exist.

This analysis seems quite reasonable at first sight, but a careful examination discloses the difficulties with this model. Let us look at equation (2-12) again

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[16] This modification is also made by P. A. Samuelson. With this modification, the model is sometimes called Samuelson's "second interaction model" to differentiate his original model which does not involve any autonomous investment. See P. A. Samuelson, "A Synthesis of the Principle of Acceleration and the Multiplier," The Journal of Political Economy, Vol. 47, December, 1939, pp. 786-797.

$$Y(t) = (c+ci)Y(t-1) - ciY(t-2) + A$$

It is apparent that if $Y(t-1)$ is greater than $Y(t-2)$, then income is growing. Once $Y(t-2)$ is greater than $Y(t-1)$ so that $ciY(t-2)$ is greater than $(c+ci)Y(t-1)$, then income will be falling to a negative value. Assume $c=0.5$, $i=2$, $Y(t-1)=20$, $Y(t-2)=35$, and $A=2$, then

$$Y(t) = 1.5(20) - 35 + 2 = -3$$

$$Y(t+1) = 1.5(-3) -20 + 2 = -22.5$$

$$Y(t+2) = 1.5(-22.5) + 3 + 2 = -28.75$$

Unless, at the first period, A is great enough to cover the decrease of income, keeping the incomes of the succeeding periods with a positive value, the level of income will fall to a negative value. Suppose $A=20$, and other assumptions remain unchanged, then

$$Y(t) = 1.5(20)- 35 + 20 = 15$$

$$Y(t+1) = 1.5(15) - 20 + 20 = 22.5$$

But if A is not great enough, say $A=6$, then

$$Y(t) = 1.5(20) - 35 + 6 = 1$$

$$Y(t+1) = 1.5(1) - 20 + 6 = -12.5$$

and income goes on the negative side again.

Therefore, initial conditions are very important to this model in predicting the growth of national income (product) for the future. This model cannot represent a general growth model, but may be a particular model for the situation where national income is in progress. When using this model one should avoid using depression periods as the initial condition in predicting the growth of an economy. We shall present more evidence about the difficulties of this model in the later chapter.

VI HARROD-DOMAR GROWTH THEORY

According to Keynes' theory, if today's productive capacity in not adequately used, that is, if today's investment is not big enough to meet the productive capacity, tomorrow's investment will also be discouraged.[17] If investment declines tomorrow there will be an increase of the surplus of idle capital, making the problem more difficult. If, however, total demand tomorrow is sufficiently greater than today's demand, then today's productive capacity can be fully employed, and there will be room for new investment again tomorrow, creating productive capacity that may in turn find full outlet if only demand would continue to grow day-after-tomorrow. Now the problem of growth is on the demand side. R. F. Harrod, recognizes this "growth problem," and tries to provide a theory which can explain how steady growth occurs in an economy, and also how, if this growth is interrupted - if this growth once diverges from its equilibrium path - the aggregate income may either explode into too rapid growth, producing inflation, or stop growing, producing depression.[18]

Harrod's analysis also incorporates both the multiplier concept and the acceleration principle. His concept is represented in terms of saving and investment, an alternative to stating national income in terms of consumption and investment. If consumption is a function of national income, then saving is also a function of national income.

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[17] Keynes, op. cit., pp. 141-146.

[18] R. F. Harrod, Towards a Dynamic Economics (London: The Macmillan & Co., Ltd., 1956), Lecture Three, pp. 63-100.

He considers both present consumption and present saving to be fixed by present national income, and at equilibrium saving equal to investment. The relations are:

$$C(t) = cY(t)$$

$$S(t) = sY(t)$$

$$S(t) = I(t)$$

Also desired investment is proportional to the change in income between the present period and the period immediately past, i.e.,

$$I(t) = g[Y(t) - Y(t-1)]$$

Then investment will be a constant proportion, g, of the difference of Y(t)-Y(t-1). By substitutions, the equation may be written as:

$$sY(t) = g[Y(t) - Y(t-1)]$$

or $$(s/g)Y(t) = Y(t) - Y(t-1)$$

where s/g will be a constant, and is the rate of growth which will just keep saving and investment equal. This rate of growth is known as the "warranted rate of growth," and designated by G_w,

$$G_w = s/g = [Y(t) - Y(t-1)]Y(t)$$

Since marginal propensity to consume plus marginal propensity to save is unity, i.e., c+s=1, then

$$[Y(t) - Y(t-1)]Y(t) = (1-c)/g$$

$$1- Y(t-1)/Y(t) = (1-c)/g$$

$$Y(t-1)/Y(t) = (g+c-1)/g$$

or $$Y(t) = Y(t-1)[g/(g+c-1)] \qquad (2\text{-}13)$$

Professor Evsey D. Domar has independently produced an analysis very similar to Harrod's. He gets the same relation but with a different interpretation.[19] His argument is that g is the reciprocal of the average investment productivity, i.e., 1/g is the ratio of the additional output (income) from investment to the amount of investment:

$$1/g = [Y(t) - Y(t-1)]/I(t)$$

Thus investment is not taken as dependent on Y(t)-Y(t-1) as in Harrod's principle; it is Y(t)-Y(t-1) which is dependent on investment through the productivity of investment. In mathematics, however, we may get the same formulation for both Harrod's and Domar's principles.

In the Harrod-Domar model, there is no lag either in the multiplier or in the accelerator; the multiplier and the accelerator are found to act together to produce a steady and progressive growth in income over time. Many economists criticize this model because the complete absence of time lags reduces its plausibility. It has been said that it is in "a world without history."[20] Nevertheless, one may introduce a lag either in the consumption function or in the investment function to eliminate the sense of "without history," such as

$$C(t) = cY(t-1)$$

and

$$I(t) = i[Y(t-1) - Y(t-2)]$$

These two functions with a lag have been discussed previously (see (2-6) and (2-7)), but they do not possess any of Harrod's properties. They are, however, analogous to that in Samuelson's model, and here is

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[19]E. D. Domar, "Capital Expansion, Rate of Growth, and Employment," Econometrica, Vol. 14, April, 1946, pp. 137-147.

[20]Joan Robinson, "Mr. Harrod's Dynamics," Economic Journal, March, 1949, p. 69.

the comment of Professor G. Ackley:

> ... we again get (above two functions with time lag), not Harrod's result of a "warranted" rate of steady growth with cumulative instability on either side, but an accelerator model permitting various kinds of fluctuations, but whose equilibrium solution is always a constant income, not a steadily-growing one.[21]

Professor W. J. Baumol, therefore, makes a modification.[22] In addition to the original relation in the investment function, he adds some autonomous investment demand A, and some investment demands which are proportioned to income, such as the community's trade balance. Since the community's trade balance provides a net non-consumption demand for the community's products, Baumol considers it as a form of investment. Then the total investment demand during period t is given by

$$I(t) = g[Y(t) - Y(t-1)] + jY(t) + A \qquad (2\text{-}14)$$

and in order that this be equal to realized investment (saving)

$$S(t) = sY(t)$$

i.e., in order that investment desires be satisfied, then

$$sY(t) = g[Y(t) - Y(t-1)] + jY(t) + A$$

By substituting 1-c into s and by transposing, yields

$$Y(t) = [g/(g+j+c-1)]Y(t-1) - A/(g+j+c-1) \qquad (2\text{-}15)$$

The final result of the modified model as expressed in (2-15) is

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[21] Gardner Ackley, Macroeconomic Theory (New York: The Macmillan Co., 1961), p. 524.

[22] W. J. Baumol, "Yet Another Note on the Harrod-Domar Model," Economic Journal, Vol. 62, June, 1952, pp. 422-27; also, W. J. Baumol, Economic Dynamics: An Introduction (New York: The Macmillan Co., 1960), pp. 44-46.

more or less similar to Harrod's original model (see (2-13)). In (2-15), there is a constant term, but in both (2-13) and (2-15) the current income is a constant proportion of the level of income of the last period.

In summarizing, the Harrod-Domar growth model seems to be somewhat unrealistic. It implies perfect forecasting for all concernec. Producers must always perfectly forecast their sales, and this forecast of sales must include the sale of capital goods in an amount determined by the simultaneous growth of sales. Also consumers must perfectly forecast their incomes. With all the perfect forecasting, the rate of growth can then be kept going without interruption. There is no time lag, no chance for error either in the investor's forecast of production, or the producer's forecast of sales. Moreover, the assumption that present income is constantly proportional to the level of income of the last period cannot be valid all the time. The annual income may not grow with the same percentage of the income of preceding year. Even if this percentage is measured by period, different periods will yield different percentages. For example, according to R. Goldsmith's report, the percentage increased per year of the gross national product of the United States in constant prices from 1839 to 1959 is 3.66, but for the period 1839-1879 it is 4.31, for 1879-1919 it is 3.72 it is 3.72, and for 1919-1959 it is 2.97.[23] It will be seen later, however, in Chapter IV how the Harrod-Domar model works empirically.

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[23]Raymond Goldsmith, "Historical and Comparative Ratio of Production, Productivity and Price," Employment, Growth and Price Levels, hearings before the Joint Economic Committee, 86th Congress, 1st Sess. (Washington, D. C.: U. S. Government Printing Office, 1959), Part 2, p.271.

CHAPTER III

THE MAKING OF MODELS

I THE ESTIMATION OF PARAMETERS

A. The Method of Least Squares

One of the simplest and most widespread methods of estimation used by economists is the least squares method. The least squares method suggests that the sum of the squares of the deviations between the actual values and the estimated values of observations be a minimum. More precisely, the method of least squares possesses the following properties.

1) The sum of the squares of the deviations of the sample observations from the sample values is a minimum.
2) The estimates are such that the estimated line passes through the point of means of the variables.
3) The estimated values for the parameters are the best unbiased linear estimates. "Best" means that the estimates have the smallest variance among all linear unbiased estimates.

For example, in a two-variables case, which is sometimes referred to as the simple regression, assuming that X and Y are two variables, and there is a linear relationship between them such that

$$Y = a + bX \tag{3-1}$$

where X is the independent variable and Y is the dependent variable, and a, b are the unknown parameters indicating the intercept and slope of the function, respectively. Now suppose that the unknown parameters have been estimated as a' and b' by using the least squares method, and

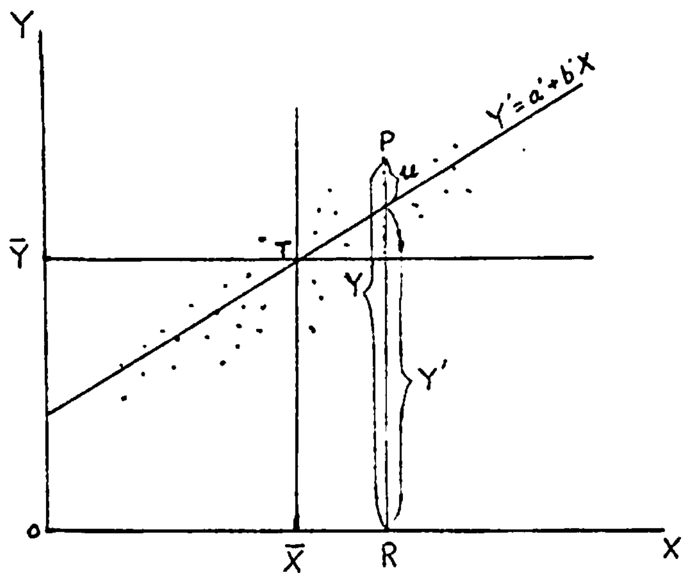

Fig. 3-1

then the estimated line is

$$Y' = a' + b'X \tag{3-2}$$

where Y' is the ordinate on the estimated line for any given value of X. It is obvious that not all the observations or sample points fall on the estimated line, such as shown in Fig. 3-1. If p is any point which is not on the estimated line, then there is a deviation u which is the difference between Y and Y', i.e.,

$$u = Y - Y'$$

The deviation from the estimated line may be positive or negative as the sample point lies above or below the line. Thus, if all these deviations are squared and summed, the resultant quantity must be non-negative and will vary directly with the spread of the points from the line:

$$\sum u^2 = \sum (Y - Y')^2$$

Then from (3-2)

$$\sum u^2 = \sum (Y-a'-b'X)^2$$

Since the property of least squares is that $\sum u^2$ be minimized, a necessary condition is that the partial derivatives of the sum with respect to a' and b' should both be zero. Therefore,

$$\frac{\partial}{\partial a'} \sum u^2 = -2 \sum (Y-a'-b'X) = 0$$

$$\frac{\partial}{\partial b'} \sum u^2 = -2 \sum X(Y-a'-b'X) = 0$$

Divide both sides of each equation by 2, and by transposing we then obtain the standard form of the normal equations:

$$\sum Y = a'n + b' \sum X \tag{3-3}$$

$$\sum XY = a' \sum X + b' \sum X^2 \tag{3-4}$$

The simultaneous solution of these two normal equations yields the values of a' and b' that minimize the sum of the squares of the deviations u's. This is the basic property of the least squares method.

If divide through equation (3-3) by n, then

$$\bar{Y} = a' + b'\bar{X} \tag{3-5}$$

where $\bar{Y}=\sum Y/n$, and $\bar{X}=\sum X/n$. Thus it shows that the estimated line $Y'=a'+b'X$ passes through the point T of means $\bar{X}$ and $\bar{Y}$, as shown in Fig. 3-1. This fulfills the second property of the method of least squares.

By substracting (3-5) from (3-1), there results

$$Y' - \bar{Y} = b'(X - \bar{X})$$

Let $x=X-\bar{X}$, $y=Y-\bar{Y}$, and $y'=Y'-\bar{Y}$, then

$$y' = b'x \tag{3-6}$$

that is an alternative way of writing the equation of the least-squares line. Also

$$
\begin{aligned}
u &= Y - Y' \\
&= (y + \bar{Y}) - (y' + \bar{Y}) \\
&= y - y' \\
&= y - b'x \qquad\qquad \text{(from (3-6))}
\end{aligned}
$$

so that the sum of the squares of the deviations is

$$\sum u^2 = \sum (y - b'x)^2$$

Minimizing it by taking partial derivatives with respect to b' gives

$$-2 \sum x(y-b'x) = 0$$

$$\sum xy - b' \sum x^2 = 0$$

$$b' = \sum xy / \sum x^2 \tag{3-7}$$

and a' can be obtained from (3-5)

$$a' = \bar{Y} - b'\bar{X} \tag{3-8}$$

Next we are going to show that a' and b' are unbiased linear estimates. Let us assume

$$Y = a + bX + u \tag{3-9}$$

and the expectation of u is zero, i.e.,

$$E(u) = 0$$

From (3-7) we have

$$
\begin{aligned}
b' &= \sum xy / \sum x^2 \\
&= \sum x(Y-\bar{Y}) / \sum x^2 \\
&= \sum xY / \sum x^2 - \bar{Y} \sum x / \sum x^2
\end{aligned}
$$

Since the sum of deviations from the mean is zero, i.e.,

$$\sum (X - \bar{X}) = 0$$

then $\sum x = 0$

Hence

$$
\begin{aligned}
b' &= \sum xY / \sum x^2 \\
&= \sum wY
\end{aligned}
$$

where $w = x/\sum x^2$.[1] From (3-9)

$$b' = \sum w(a + bX + u)$$
$$= a \sum w + b \sum wX + \sum wu$$
$$= b + \sum wu$$

Then $E(b') = b + \sum wE(u)$

Since $E(u)=0$, then $E(b')=b$. Thus b' is an unbiased linear estimate of b.

Similarly, a' is an unbiased linear estimate of a. From (3-8)

$$a' = \bar{Y} - b'\bar{X}$$
$$= \sum Y/n - (\sum wY)\bar{X}$$
$$= \sum (Y/n) - \sum \bar{X}wY$$
$$= \sum (1/n - \bar{X}w)Y$$
$$= \sum (1/n - \bar{X}w)(a + bX + u)$$
$$= a - a\bar{X} \sum w + b\bar{X} - b\bar{X} \sum wX + \sum (1/n - \bar{X}w)u$$
$$= a + \sum (1/n - \bar{X}w)u$$

Hence $E(a') = a + \sum (1/n - \bar{X}w)E(u)$

or $E(a') = a$

Finally we are going to see that the estimated a' and b' are the best linear unbiased, that is, they have the smallest variance. Let us define any arbitrary linear estimate of b as

$$b'' = \sum cY$$

where $c = w + d$ (3-10)

---

[1] w is constant and since $\sum x=0$, then:

$$\sum w = \sum (x/\sum x^2) = \sum x/\sum x^2 = 0$$
$$\sum w^2 = \sum (x/\sum x^2)^2 = \sum x^2/(\sum x^2 \sum x^2) = 1/\sum x^2$$
$$\sum wx = \sum (x/\sum x^2)x = \sum x^2/\sum x^2 = 1$$
$$\sum wx = \sum w(X - \bar{X}) = \sum wX - \bar{X}\sum w = \sum wX = 1$$

w being defined as previously and d being arbitrary constant. For b'' to be an unbiased estimate of b, d must fulfill certain conditions. From (3-9)

$$b'' = \sum c(a + bX + u)$$

$$= a\sum c + b\sum cX + \sum cu$$

so $$E(b'') = a\sum c + b\sum cX = b$$

if $\sum c=0$ and $\sum cX=1$. These two conditions, from (3-10) and the properties of w, gives the required conditions for d, i.e.,

$$\sum d = 0$$

and $$\sum dX = \sum \dot{d}X = 0$$

The variance of this arbitrary linear unbiased estimate is then

$$\mathrm{Var}(b'') = E((\sum cu)^2)$$

$$= E(u^2)\sum c^2$$

But $$\sum c^2 = \sum w^2 + \sum d^2 + 2\sum wd$$

$$\sum wd = \sum xd/\sum x^2 = 0$$

therefore,

$$\mathrm{Var}(b'') = \mathrm{Var}(b') + E(u^2)\sum d^2$$

where $\sum d^2$ must be nonnegative and is zero only if each value of d is zero. Thus the least-squares estimate has the smallest variance of all linear unbiased estimates. A similar result may be obtained for Var(a').

So with all these properties, we obtain the estimates of the parameters of a linear relation by using the above two normal equations (3-3) and (3-4), or the equations derived from them, such as (3-7) and (3-8). Moreover, by solving the two normal equations, we may get

$$b' = \frac{n\sum XY - \sum X\sum Y}{n\sum X^2 - (\sum X)^2} \qquad (3\text{-}11)$$

$$a' = \frac{\sum X^2 \sum Y - \sum X \sum XY}{n \sum X^2 - (\sum X)^2} \qquad (3\text{-}12)$$

More detail about the technique of the least squares method may be seen in Appendix A.

B. Problems with the Least Squares Method[2]

We have seen the properties of the least squares method, but there are some consequences arising from it.[3]

1) The sampling variances of these estimates may be unduly large compared with other methods.

2) By the usual least squares method, we are likely to obtain a serious underestimate of the variances.

3) We shall obtain inefficient prediction, that is, predictions with needlessly large sampling variances.

These consequences, however, may be eliminated by making use of autocorrelation, i.e., making use of the autoregressive structure of the disturbances (the measurement errors, i.e., the differences between the estimated values and the actual values of observations). This process is equivalent to a two-steps procedure.

The first step is to transform the original variables according to

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[2]The aim of this study is not to discuss the econometric method, so in this section we simply point out the problems with the least squares method, and the difficulty of this study. More detail about the methods which can overcome the consequences of the least squares method may be obtained in J. Johnston, Econometric Methods (New York: McGraw-Hill Book Co., Inc., 1963), pp. 177-295.

[3]Ibid., p. 179.

the autoregressive structure of the disturbance term, which refers to u such as in the equation

$$Y = bX + u$$

where X and Y are variables, b is the parameter, and u is the measurement error such that

$$u = Y - Y'$$

where Y' is the estimated value of Y. The second step is to apply the usual least squares to the transformed variables. This process can be made by the use of matrix algebra. We may write Y=bX+u in matrix notation such that

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} = \begin{bmatrix} X_{11} & X_{21} & \cdots & X_{k1} \\ X_{12} & X_{22} & \cdots & X_{k2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ X_{1n} & X_{2n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \qquad (3\text{-}13)$$

By applying a transformation matrix to (3-13), gives

$$\begin{bmatrix} -p & 1 & 0 & \cdots & 0 \\ 0 & -p & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix} = \begin{bmatrix} -p & 1 & 0 & \cdots & 0 \\ 0 & -p & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_{11} & X_{21} & \cdots & X_{k1} \\ X_{12} & X_{22} & \cdots & X_{k2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ X_{1n} & X_{2n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \end{bmatrix}$$

$$+ \begin{bmatrix} -p & 1 & 0 & \cdots & 0 \\ 0 & -p & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \qquad (3\text{-}14)$$

where p is the coefficient of autocorrelation, and its absolute value is less than one i.e., $|p| < 1$. Its value can be obtained by applying the usual least squares method from

$$u_t = pu_{t-1} + e_t$$

where e_t is not a random variable but a residual such that

$$e_t = Y - b'X$$

where b' is the estimated parameters. Solving (3-14) we then obtain the value of b, which will not yield the consequences as stated above. In this study, however, we do not concentrate on one or two equations, but have a number of equations with a large sample. Therefore, it is impossible for us to use this two-steps procedure.

The second problem arising from the usual least squares method is that there will be a negative bias if there is any time lag in the variables, such as in the equation

$$Y(t) = a + bY(t-1) + u$$

Especially in small samples, the bias is even more serious. Professor J. Johnston has introduced some ways of lessening the difficulty of the least squares method in the estimation when time lags are involved,[4] but they are too complicated for us to apply in so many equations as in this study. Furthermore, a complete elimination of the bias in lagged variables is still in discussion.

So far we have merely pointed out the problems with the usual least squares in a single equation. In an economic model there is usually more than one equation. When single equations interact together, the

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[4] *Ibid.*, pp. 211-221.

direct application of the usual least squares method to the single equations will not yield unbiased estimates of the parameters. Let us take a simple model such as

$$C = a + bY + u \tag{3-15}$$

$$Y = C + I \tag{3-16}$$

$$I = I_o \tag{3-17}$$

where C = consumption expenditure

Y = income

I = investment, which is autonomous and is determined outside the model

I_o = given value for I

We also assume that there are some data given as in the following table:

Y	C	I_o
5	5	0
10	10	0
15	15	0
15	10	5
20	15	5
25	20	5
$\sum Y=90$	$\sum C=75$	$\sum I_o=15$
$\bar{Y}=15$	$\bar{C}=12.5$	$\bar{I}_o=2.5$

By applying directly the usual least squares method to (3-15) gives

$$b = \frac{n \sum CY - \sum C \sum Y}{n \sum Y^2 - (\sum Y)^2}$$

where $n=6$, $\sum CY=1,300$, $\sum C\sum Y=6,750$, $\sum Y^2=1,600$, and $(\sum Y)^2=8,100$. Thus we obtain $b=0.7$, and $a=\bar{C}-b\bar{Y}=2$. Re-write (3-15)

$$C(t) = 2 + 0.7Y(t) + u_t \tag{3-18}$$

where t designates the time period. We may apply the least squares method to the reduced form of (3-15) by substituting (3-16) into (3-15) such that

$$C = a + b(C + I) + u$$

from which we obtain the reduced form of C

$$C = \frac{a}{1-b} + \frac{b}{1-b} I + \frac{1}{1-b} u$$

By applying the least squares method to the reduced form, gives

$$b/(1-b) = 1, \text{ or}, b=0.5$$

and $\quad a/(1-b)=10$, or, $a=5$

Re-write (3-15)

$$C(t) = 5 + 0.5Y(t) + u_t \tag{3-19}$$

The method employed in the above by using the reduced form of the model is that of indirect least squares; that is, usual least squares method is applied to find estimates of the parameters of the reduced from, and from these, in turn, estimates of the structural parameters are obtained. The indirect least squares method will yield unbiased estimates of a and b for the model as a whole system, such as in (3-19). It is obvious that the result of the direct application of least squares method, as shown in (3-18), yields upward biased estimates, because the slope of (3-18) is greater than that of (3-19).

Although the indirect least squares method yields unbiased estimates of the parameters for the whole model, it is sometimes feasible. There is another method of more general applicability called the two-stage least squares method. The first stage of this method is, from (3-16), to compute

$$Y = h + gI + e$$

where $$g = \frac{n \sum YI - \sum Y \sum I}{n \sum I^2 - (\sum I)^2}$$

and $$h = \bar{Y} - g\bar{I}$$

The estimated Y values are given by

$$Y' = h + gI \tag{3-20}$$

Now the second stage is to substitute these Y' values in (3-15) to give

$$C = a + bY' + (u + be) \tag{3-21}$$

The second stage is then completed by applying least squares directly to (3-21) to obtain estimates of a and b.

Besides the above two methods, there are still some other methods which can eliminate the bias arising from the direct application of the least squares method to the single equations, such as the least-variance-ratio method, the full-information maximum likelihood method, and the three-stage least squares method. All these methods are quite complicated. If we apply any one of them to this study, we have to compute the parameters model by model. Also, in so doing, we will have different values of parameters for the same function in different models. In this study, however, we are also interested in the values of the parameters of each single function, so that we can compare the various theories in the consumption and investment functions empirically. For these reasons, we have to apply directly the usual least squares method to this study.

II BASIC ASSUMPTIONS

The Keynesian theory assumes that aggregate income is equal to the sum of aggregate consumption and aggregate investment, making a definitional equation in macro-economics

$$Y(t) = C(t) + I(t) \tag{3-22}$$

All theories which we have discussed in the previous chapter use this definition. In it we do not see how the government spends its money. In theory, all government expenditures belong either in the category of consumption or investment expenditures. Hence, according to the above definition (3-22), the total government expenditures are merged in the consumption and investment categories. In Keynes' model, however, it actually does not neglect the government sector. It does emphasize the role of government in an economy, for the impact of government action can help to stabilize economic growth. Therefore, some economists introduce the government sector as a separate item among the components of national income, such that

$$Y(t) = C(t) + I(t) + G(t) \tag{3-23}$$

where G is the volume of government expenditures on goods and services. There are also some other reasons for separating the government sector, such as

1) the difficulty in breaking down government expenditures into consumption and investment expenditures,
2) some special interest which centers around government expenditures.

The determination of the volume of government expenditures is not simple. There are many factors which affect government expenditures, although usually government accounts are under the annual budget.

Disgarding those factors arising from politics and wars, economists usually consider the total volume of government expenditures as a function of the level of income with direct proportion:

$$G(t) = gY(t)$$

where g is a positive constant and less than one. In economics, the purpose of a government is not to make money to increase its assets such as by investments. Its major receipts are from taxation. It is clear that receipts arising from taxation depend upon the level of national income. Government receipts and expenditures are supposed to be balanced. Therefore government expenditures depend largely upon the level of national income.

In this study, we use both arguments as our definitional equations i.e., (3-22) and (3-23); also as mentioned in Chapter I, $Y(t)$ designates gross national income, thus $I(t)$ here is gross investment.

III THE CONSUMPTION FUNCTION

As seen in the previous chapter, there are many arguments about the consumption function. Basically, in this study we adopt the income-consumption relation and its extensiors. The followings are the consumption functions which we are going to test:

$$C(t) = cY(t) \tag{3-24}$$

$$C(t) = cY(t) + a \tag{3-25}$$

$$C(t) = cY(t-1) \tag{3-26}$$

$$C(t) = cY(t-1) + a \tag{3-27}$$

$$C(t) = cY(t) + bt + a \tag{3-28}$$

Equations (3-24), (3-25) and (3-28) are called static formulations, while equations (3-26) and (3-27) are dynamic in the sense that there is a time lag in the level of income.

Equation (3-24) is from the Keynesian model, which is critized as being too static; and equation (3-26) involves a lag on the income side, as suggested by Robertson and other economists trying to overcome the deficiency of the static Keynesian model. The difference between (3-24) and (3-25) and that between (3-26) and (3-27) is the same; it is only a matter of whether there is a constant term or not. For those two equations, (3-25) and (3-27), the marginal propensity to consume is less than the average propensity to consume, as discussed in Chapter II. This situation, according to Duesenberry and Friedman, should not be the case in long-run. They suggest that in long-run, the marginal propensity to consume should be the same as average propensity to consume, and so the consumption function should be formulated like (3-24) and (3-26).

The consumption function (3-28) is formulated according to the suggestion of Arthur Smithies. There is no lag at all, but instead, he argues, as seen in Chapter II, that consumption should also be a function of time, not only of income, in order to reconcile the short-run long-run problem.

Besides, there is another approach concerning the consumption function, which we have not discussed before. The underconsumptionists say that aggregate consumption is determined by two things: (1) the level of aggregate income of the previous period, and (2) by the distribution of that income between wages and profits, such that[5]

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[5]Howard J. Sherman, Macrodynamic Economics: Growth, Employment and Prices (New York: Appleton-Century-Crofts, 1964), p. 79 and p. 235

$$C(t) = a + bW(t-1) + cP(t-1) \tag{3-29}$$

where P designates profit income, and W is wage income proportional to net national income, such that[6]

$$W(t) = d + eY(t) \tag{3-30}$$

Also it is assumed that

$$W(t) + P(t) = Y(t) \tag{3-31}$$

where Y is net national income, so that

$$W(t-1) + P(t-1) = Y(t-1)$$

Therefore, the relation of (3-29) implies that consumption is a function of the income of the last period.

Thus, all of these consumption functions as stated above are not without reasons. In the next chapter, we shall see how they work empirically, and which of them is closest to the "real world."

IV THE INVESTMENT FUNCTION

Our investment functions are based upon the acceleration principle. They are:

$$I(t) = i[C(t) - C(t-1)] \tag{3-32}$$

$$I(t) = i[C(t) - C(t-1)] + A \tag{3-33}$$

$$I(t) = i[C(t-1) - C(t-2)] \tag{3-34}$$

$$I(t) = i[Y(t) - Y(t-1)] \tag{3-35}$$

$$I(t) = i[Y(t-1) - Y(t-2)] \tag{3-36}$$

$$I(t) = i[Y(t) - Y(t-1)] + jY(t) \tag{3-37}$$

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[6] *Ibid.*, p. 235.

$$I(t) = i[Y(t) - Y(t-1)] + jY(t) + A \qquad (3\text{-}38)$$

Equation (3-32) is the original formulation of investment function under the acceleration principle. Equation (3-33) is similar to (3-32) except that it has a constant term A, which designates autonomous investment, according to the argument that in addition to induced investment there is also autonomous investment.

The investment function (3-34) involves a time lag on the consumption side. In either (3-32) or (3-33), the induced investment arising from the action of the accelerator is considered to be completed in the same period as that in which the additional consumer goods output occurs which required the investment. Some economists argue that if income rises, people buy more consumer goods; but in order to make more consumer products, more machines are required (subsequently more machines have to be made), and thus it requires some times for the construction and installation of machines and plants. Therefore, it is suggested that the accelerator effect should be lagged.

Equations (3-32), (3-33) and (3-34) express investment as a function of the change in consumption, which is a function of income. But some economists apply the acceleration principle to the change in income directly, such as (3-35), which is suggested by Professor Harrod. As seen in the previous chapter, in the discussion of the acceleration principle, no matter whether we put investment as a function of the changed consumption or as a function of the changed income, mathematically the acceleration effect is the same when we use the same marginal propensity of consume. Empirically, however, it will not, because the change in consumption does not vary in a constant proportion to the change in income over time. That is to say, the acceleration coefficient in (3-35) will not equal the product of the marginal propensity to consume times the acceleration coefficient in (3-34). For example, according to the result of our estimation, we have

$$C(t) = 0.7972Y(t) \tag{3-39}$$

$$I(t) = 2.8452[C(t) - C(t-1)] \tag{3-40}$$

$$I(t) = 2.0266[Y(t) - Y(t-1)] \tag{3-41}$$

If we substitute (3-39) into (3-40), we have

$$\begin{aligned} I(t) &= 0.7972(2.8452)[Y(t) - Y(t-1)] \\ &= 2.2682[Y(t) - Y(t-1)] \end{aligned}$$

where the acceleration coefficient is different from that of (3-41). Therefore, we cannot view (3-32) and (3-35) as the same thing, because they will render different effects. By the same token, (3-36), a function of the change in income occurring in the period before the last one, will have a different effect from (3-34). The time lag in (3-36) is introduced for the same reason as (3-34).

Equations (3-37) and (3-38) are the modification of Harrod-Domar model, as seen in the previous chapter. In (3-37) it is assumed that some investment demand is proportional to income, such as the community's trade balance, designated by $jY(t)$, where j is a constant and may be negative; also, there is still some investment demand based on the acceleration principle as in the Harrod-Domar model. In (3-38) in addition to the assumption of (3-37), there is some autonomous investment, written as A, independent both of the lavel of income and the change in income.

In addition to the investment functions as listed in the above, according to the theory of "overinvestment," there is another one, which is proportional to the change in profit of the previous period, such that[7]

$$I(t) = g + h[P(t-1) - P(t-2)] \tag{3-42}$$

where I is net investment. The basic reason of this proposition is that the aim of investment is to [illegible] profit. Additional profit will attract additional investment, but, on the other hand, a decline in profit will discourage investment.

The combinations of the basic definitions, consumption functions and investment functions constitute the economic models for us to investigate in the next chapter.

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[7] *Ibid.*, p. 89 and p. 237.

CHAPTER IV

EMPIRICAL INTERPRETATION

Major growth theories were discussed in Chapter II, and the hypotheses of this study were established in Chapter III. Upon these hypotheses and using the least squares method, this chapter presents the results of empirical applications.

I THE FUNCTIONS

According to the basic assumption established in the preceding chapter, there are two definitional equations:

$$Y(t) = C(t) + I(t) \tag{4-1}$$

$$Y(t) = C(t) + I(t) + G(t) \tag{4-2}$$

In this study Y is gross national product, and t is time period. In (4-1), C is the sum of personal consumption and government consumption, and I is the sum of private investment and government investment. In (4-2), C refers to personal consumption, I private investment, and G total government expenditures on goods and services. In (4-2), total government expenditures are not broken down according to consumption and investment. The Office of Business Economics of the United States Department of Commerce published the total government expenditures on goods and services with no distinction between their consumption and investment nature. In Table A-IIa of John W. Kendrick's book, _Productivity Trends in the United States_, total government expenditures are broken down into consumption expenditures and investment expenditures. This break-down is recorded so that column (6) presents government investment, and column (10) total government expenditures on goods and services.[1] By taking the ratio of these two columns, i.e., dividing column (6) by column (10), the annual percentage of government investment out of the total government expenditures on goods and services may be computed. According to these percentages, annual government investment may be apportioned. By substracting government investment from total government expenditures, the resulting amount is government consumption expenditures. The data given by Kendrick, however, is up to the year 1957 only; so for the years 1958-1963, the percentage of government investment out of the total government expenditures is based upon the average of the annual percentages of 1929-1957. This average is approximately 29%. Summary data are recorded in Appendix C. The original data are obtained from Survey of Current Business[2] in current dollars, and converted into 1929 dollars according to the "Consumer Price Index" provided by the Statistical Abstract of the United States.[3]

Under each of the above definitional equation, there are five consumption and seven investment equations. By combining these equations, some fifty-five national income models result, each one of which is tested against observation. Below are the results of the estimation of

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[1] John W. Kendrick, Productivity Trends in the United States (Princeton: Princeton University Press, 1961), pp. 293-295.

[2] U. S. Department of Commerce, Office of Business Economics, Survey of Current Business (Washington, D. C.: U. S. Government Printing Office, 1930-1964).

[3] U. S. Department of Commerce, Bureau of the Census, Statistical Abstract of the United States (Washington, D. C.: U. S. Government Printing Office, 1930-1964).

all the parameters (all constant terms in the following equations are in millions of 1929 dollars).

A) Defining aggregate income as

$$Y(t) = C(t) + I(t) \tag{4-1}$$

we have

$$C(t) = 0.79724Y(t) \tag{4-3}$$

$$C(t) = 0.75841Y(t) + 8,632.65 \tag{4-4}$$

$$C(t) = 0.82906Y(t-1) \tag{4-5}$$

$$C(t) = 0.78035Y(t-1) + 10,313.42 \tag{4-6}$$

$$C(t) = 0.72752Y(t) + 283.83t + 10,289.00 \tag{4-7}$$

and

$$I(t) = 2.84519[C(t) - C(t-1)] \tag{4-8}$$

$$I(t) = 0.95075[C(t) - C(t-1)] + 30,969.63 \tag{4-9}$$

$$I(t) = 2.83189[C(t-1) - C(t-2)] \tag{4-10}$$

$$I(t) = 2.02659[Y(t) - Y(t-1)] \tag{4-11}$$

$$I(t) = 2.01716[Y(t-1) - Y(t-2)] \tag{4-12}$$

$$I(t) = 0.11667[Y(t) - Y(t-1)] + 0.19802Y(t) \tag{4-13}$$

$$I(t) = 0.10027[Y(t) - Y(t-1)] + 0.23799Y(t) - 8,702.24 \tag{4-14}$$

The following estimations are also made:

$$C(t) = 19,618.84 + 0.72414W(t-1) + 0.53286P(t-1) \tag{4-15}$$

$$I(t) = 19,348.34 + 1.00638[P(t-1) - P(t-2)] \tag{4-16}$$

$$W(t) = 0.59518Y(t) - 3,578.03 \tag{4-17}$$

where C is total consumption, W wage income, P profit income, I net investment, and where $Y(t)=W(t)+P(t)$. In this case Y refers to net national product rather than gross national product.

Since the data used cover the years 1929 through 1963 (after excluding the four year war period, 1942-1945) there could have been thirty-one years of observations. In this study, however, period analysis is used; and by taking 1930 as the basic year, there are only thirty years of observations. Furthermore, for investment functions with lags, 1931 is taken as the basic year, which leaves only twenty-nine years of observations. In order to match investment functions with corresponding consumption functions, the following consumption functions are used:

$$C(t) = 0.79699Y(t) \tag{4-3)'}$$

$$C(t) = 0.75670Y(t) + 9{,}016.40 \tag{4-4)'}$$

$$C(t) = 0.82986Y(t-1) \tag{4-5)'}$$

$$C(t) = 0.77034Y(t-1) + 12{,}619.16 \tag{4-6)'}$$

where the basic time period t is 1931.

B) With government expenditures as a separate component of aggregate income, such that

$$Y(t) = C(t) + I(t) + G(t) \tag{4-2}$$

we have

$$C(t) = 0.66299Y(t) \tag{4-18}$$

$$C(t) = 0.58363Y(t) + 17{,}585.89 \tag{4-19}$$

$$C(t) = 0.68946Y(t-1) \tag{4-20}$$

$$C(t) = 0.60235Y(t-1) + 18{,}549.27 \tag{4-21}$$

$$C(t) = 0.59769Y(t) - 132.64t + 16{,}880.40 \tag{4-22}$$

and

$$I(t) = 2.19507[C(t) - C(t-1)] \tag{4-23}$$

$$I(t) = 0.74771[C(t) - C(t-1)] + 23{,}324.20 \tag{4-24}$$

$$I(t) = 2.24533[C(t-1) - C(t-2)] \tag{4-25}$$

$$I(t) = 1.51270[Y(t) - Y(t-1)] \tag{4-26}$$

$$I(t) = 1.43317[Y(t-1) - Y(t-2)] \quad (4\text{-}27)$$

$$I(t) = 0.13835[Y(t) - Y(t-1)] + 0.14249Y(t) \quad (4\text{-}28)$$

$$I(t) = 0.12592[Y(t) - Y(t-1)] + 0.17392Y(t) - 6{,}852.99 \quad (4\text{-}29)$$

Also, we have

$$G(t) = 0.18890Y(t) \quad (4\text{-}30)$$

In (4-25) and (4-27), there are twenty-nine observations only. For those consumption function of twenty-nine observations, there are

$$C(t) = 0.66216Y(t) \quad (4\text{-}18)'$$

$$C(t) = 0.58397Y(t) + 17{,}496.55 \quad (4\text{-}19)'$$

$$C(t) = 0.68938Y(t-1) \quad (4\text{-}20)'$$

$$C(t) = 0.59849Y(t-1) + 19{,}550.00 \quad (4\text{-}21)'$$

Also, for government expenditures with 1931 as the basic year, we have

$$G(t) = 0.18952Y(t) \quad (4\text{-}30)'$$

Although estimation based on either twenty-nine or thirty observations gives approximately the same value of parameter, still the principle of logic cannot be ignored.

To find out which of the above functions is better than the others, their mean-squares-error are used. Mean-squares-error is the average of the sum of the squares of the difference between the estimated value and the actual value of the observation, such that

$$\text{M-S-E} = \frac{\sum[C(t)-C'(t)]^2}{n}, \quad \text{or} \quad \text{M-S-E} = \frac{\sum[I(t)-I'(t)]^2}{n}$$

where C'(t) and I'(t) are the actual value of observations,[4] and n is

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[4]Since we have already used C(t) and I(t) as our estimated values, so we use C'(t) and I'(t) to represent the actual values of observations. This notation is contrary to the usual way.

the number of total observations. The smaller the M-S-E, the better the function. In Table A the consumption and the investment functions are listed with their M-S-E, which correspond to the definitional equation (4-1). Table B consists of the functions related to the definitional equation (4-2). In both tables, the functions are listed in ascending order according to the M-S-E, so that the first model has the smallest M-S-E.

Table A and Table B have one thing in common, that is, the best consumption function is

$$C(t) = cY(t) + a$$

and the best investment function is

$$I(t) = i[Y(t) - Y(t-1)] + jY(t) + A$$

This implies that no matter what definitional equation is used, these two functions are likely to be the best choice. However, the M-S-E's of these two functions are smaller in Table A than those in Table B, that is, when a function is related to the definition, $Y(t)=C(t)+I(t)$, the M-S-E is smaller than that when the function is related to $Y(t)=C(t)+I(t)+G(t)$. In other words, the result by the definition $Y(t)=C(t)+I(t)$ is better than that by $Y(t)=C(t)+I(t)+G(t)$. This may be due to the fact that if total government expenditures on goods and services are separated from consumption and investment, the government sector cannot give the same effect as that produced by consumption and investment.

The consumption function next to the best in both Table A and Table B is

$$C(t) = cY(t) + bt + a$$

Table A

Function	Empirical Result	M-S-E	Ref.
$C(t)=cY(t)+a$	$C(t)=0.75841Y(t)+8{,}632.65$	7,960,528.977	(4-4)
$C(t)=cY(t)+bt+a$	$C(t)=0.72752Y(t)+283.83t+10{,}289.00$	13,157,589.379	(4-7)
$C(t)=cY(t)$	$C(t)=0.79724Y(t)$	19,812,774.227	(4-3)
$C(t)=cY(t-1)+a$	$C(t)=0.78035Y(t-1)+10{,}313.42$	56,695,278.351	(4-6)
$C(t)=cY(t-1)$	$C(t)=0.82906Y(t-1)$	72,380,449.597	(4-5)
$I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$I(t)=0.10027[Y(t)-Y(t-1)]+0.23799Y(t)-8{,}702.24$	6,664,672.650	(4-14)
$I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$I(t)=0.11667[Y(t)-Y(t-1)]+0.19802Y(t)$	18,368,179.157	(4-13)
$I(t)=i[C(t)-C(t-1)]+A$	$I(t)=0.95075[C(t)-C(t-1)]+30{,}969.63$	333,332,094.762	(4-9)
$I(t)=i[C(t)-C(t-1)]$	$I(t)=2.84519[C(t)-C(t-1)]$	949,836,592.234	(4-8)
$I(t)=i[C(t-1)-C(t-2)]$	$I(t)=2.83189[C(t-1)-C(t-2)]$	970,723,641.711	(4-10)
$I(t)=i[Y(t)-Y(t-1)]$	$I(t)=2.02659[Y(t)-Y(t-1)]$	1,001,057,709.943	(4-11)
$I(t)=i[Y(t-1)-Y(t-2)]$	$I(t)=2.01716[Y(t-1)-Y(t-2)]$	1,057,385,770.811	(4-12)

Note: this table is under the assumption, $Y(t)=C(t)+I(t)$

Table B

Function	Empirical Result	M-S-E	Ref.
$C(t)=cY(t)+a$	$C(t)=0.58363Y(t)+17{,}585.89$	8,783,301.679	(4-20)
$C(t)=cY(t)+bt+a$	$C(t)=0.59769Y(t)-132.64t+16{,}880.40$	8,793,634.195	(4-23)
$C(t)=cY(t-1)+a$	$C(t)=0.60235Y(t)+18{,}549.27$	38,767,194.847	(4-22)
$C(t)=cY(t)$	$C(t)=0.66299Y(t)$	57,211,900.490	(4-19)
$C(t)=cY(t-1)$	$C(t)=0.68946Y(t-1)$	92,903,474.109	(4-21)
$I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$I(t)=0.12592[Y(t)-Y(t-1)]+0.17392Y(t)-6{,}852.99$	11,382,610.178	(4-30)
$I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$I(t)=0.13835[Y(t)-Y(t-1)]+0.14249Y(t)$	19,187,303.018	(4-29)
$I(t)=i[C(t)-C(t-1)]+A$	$I(t)=0.74771[C(t)-C(t-1)]+23{,}324.20$	194,030,284.900	(4-25)
$I(t)=i[Y(t)-Y(t-1)]$	$I(t)=1.51270[Y(t)-Y(t-1)]$	529,727,544.859	(4-27)
$I(t)=i[Y(t-1)-Y(t-2)]$	$I(t)=1.43317[Y(t-1)-Y(t-2)]$	553,517,125.649	(4-28)
$I(t)=i[C(t)-C(t-1)]$	$I(t)=2.19507[C(t)-C(t-1)]$	590,710,860.222	(4-24)
$I(t)=i[C(t-1)-C(t-2)]$	$I(t)=2.24533[C(t-1)-C(t-2)]$	598,761,513.228	(4-26)

Note: this table is under the assumption, $Y(t)=C(t)+I(t)+G(t)$.

which is suggested by A. Smithies. As discussed in Chapter II, Smithies argues that consumption is also a function of time, t, i.e., the consumption function slowly drifts upward over time. In that case, the coefficient of t is supposed to be in positive value. According to the above results, when this consumption function is related to the definition, $Y(t)=C(t)+I(t)$, the value of the coefficient b is positive, as shown in (4-7); but, when this function is related to $Y(t)=C(t)+I(t)+G(t)$, as shown in (4-22), b is negative. That is to say that consumption (or income, since consumption is a function of income) may not necessarily drift upward over time empirically. There, it can be said that consumption or income may be a function of time, but not necessarily directly proportional to the time period.

The investment function next to the best in both Table A and Table B is

$$I(t) = i[Y(t)-Y(t-1)] + jY(t)$$

which is similar to the best function except the latter one does not have a constant term. In general, it can be seen that, either in Table A or in Table B, a function will yield a better result if there is a constant term, because a constant term can make the estimated points closer to the average point on the estimated line. It should be noted in both Table A and Table B that the M-S-E's of

$$C(t) = cY(t) + a$$

$$C(t) = cY(t-1) + a$$

$$I(t) = i[Y(t) - Y(t-1)] + jY(t) + A$$

$$I(t) = i[C(t) - C(t-1)] + A$$

are smaller than those of

$$C(t) = cY(t)$$

$$C(t) = cY(t-1)$$

$$I(t) = i[Y(t) - Y(t-1)] + jY(t)$$

$$I(t) = i[C(t) - C(t-1)]$$

respectively. That is, a constant term can make the slope of the function flatter so that the estimated points are closer to the average of the sample.

Whether the investment function can be expressed better in terms of consumption or in term of income cannot be found in Table A and Table B. We cannot obtain a clue because in Table A the investment functions are better expressed in terms of consumption, while in Table B the M-S-E's of

$$I(t) = i[Y(t) - Y(t-1)]$$

$$I(t) = i[Y(t-1) - Y(t-2)]$$

are smaller than those of

$$I(t) = i[C(t) - C(t-1)]$$

$$I(t) = i[C(t-1) - C(t-2)]$$

Another thing to note is that those unlagged function can yield better results than those lagged functions. There may be three reasons for this:

1) Because of the exclusion of the war-period, 1942-1945, there is a large measurement error between 1941 and 1946. For example, in the lagged consumption function, the measurement of the consumption of 1946 based on the income of 1941 is incorrect.
2) As indicated in Chapter III, the estimates of the parameters of lagged functions are biased because of the direct application of the least squares method.

3) The introduction of time lag into a function cannot be more realistic. The first two reasons are quite obvious and positive, but the third reason is still under discussion among economists. In this study the third reason cannot be assured because of the deficiencies in the first two reasons.

II THE MODELS

The empirical results of the various consumption and investment functions have been noted above. Upon combinations of these functions with the two definitional equations, (4-1) and (4-2), the models for this study are established. The combinations are tabulated in Table C and Table D. Table C is related to the definition $Y(t)=C(t)+I(t)$, while Table D is related to $Y(t)=C(t)+I(t)+G(t)$. Both tables record the solution of each model with some remarks. The method for solving these models is to substitute all variables into the definitional equation, formulating a single difference equation in terms of income Y. Solving the difference equation, the general solution is obtained for national income over time, $Y(t)$. For the solution of difference equations, one may refer to Appendix B.

In all there are fifty-five models in Table C and Table D. There is a variety of solutions, so that the common rate of growth calculated by arithmetic mean is used as an index.[5] It is not intended that this

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[5]We do not use the geometric mean because we have some difficulty with it. The formula of the geometric mean is

$$R = \left[\frac{Y(t)-Y(t-1)}{Y(t-1)} \cdot \frac{Y(t-1)-Y(t-2)}{Y(t-2)} \cdots\right]^{1/n}$$

where R, Y, t and n have the same meaning as in (4-31). Since we may occasionally experience a negative value for the change of income between two periods, we may have a complex number for R.

is the "standard" rate of growth, but it is used as a guide,having no better alternative. Based upon the data in Appendix C the rate of growth of the national income of the United States is computed for the period 1930-1963 (excluding the war period, 1942-1945) by using arithmetic mean, such that

$$R = \frac{1}{n}\sum\left[\frac{Y(t) - Y(t-1)}{Y(t-1)}\right] \tag{4-31}$$

where R designates the average rate of growth, n is the number of observations, Y represents gross national product, and t refers to time period. The average rate of growth, from (4-31), for the period 1930-1963 (excluding 1942-1945), is 0.041998 or 4.1998 per cent. From this result the relation of (4-31) may be expressed as

$$Y(t) - Y(t-1) = 0.041998\ Y(t-1)$$

which is a difference equation. By transposing, gives

$$\begin{aligned} Y(t) &= (1 + 0.041998)Y(t-1) \\ &= 1.041998Y(t-1) \end{aligned} \tag{4-32}$$

from which a general solution

$$Y(t) = (1.041998)^t Y_o \tag{4-33}$$

is obtained (see Appendix B), where Y_o is the initial value. In this case, Y_o refers to the gross national product of the year 1930. Accordingly, if any model in both Table C and Table D yields a rate around 0.041998, it might be said that this rate is likely to be a moderate one, otherwise the rate is too high or too low.

Most of the models, either in Table C or in Table D, cannot yield a satisfactory result from the point of view of forecasting. According to the result in the previous section the best functions under both definitional equations are

Table C

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
1) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.79724Y(t)$ $I(t)=2.84519[C(t)-C(t-1)]$	$Y(t)=93,453(1.09816)^{t}$	The rate of growth is too high.
2) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[C(t)-C(t-1)]+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.79724Y(t)$ $I(t)=0.95075[C(t)-C(t-1)]$ $+30,969.63$	$Y(t)=-59,287.08(1.36519)^{t}$ $+152,740.08$	The rate is too high, and income is going to fall in negative value.
3) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[C(t-1)-C(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.79699Y(t)$ $I(t)=2.83189(C(t-1)-C(t-2))$	$Y(t)=88,400.09(1.11103)^{t}$ $-2,487.09(10.0066)^{t}$ (Basic Year: 1931)	The rate of the negative term is greater than that of the positive term, so income is going to be negative.
4) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.79724Y(t)$ $I(t)=2.02659[Y(t)-Y(t-1)]$	$Y(t)=93,453(1.11117)^{t}$	The rate of growth is too high.

Table C (Continued)

Formulation of Model	Model with Est'd parameters	Solution	Remarks
5) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t-1)-Y(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.79699Y(t)$ $I(t)=2.01716[Y(t-1)-Y(t-2)]$	$Y(t)=88,984.31(1.12807)^t$ $-3,071.31(8.80818)^t$ (Basic Year: 1931)	Y(t) is going to be negative because the rate of the negative term is greater than that of the positive term.
6) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$	$Y(t)=C(t)+I(t)$ $C(t)=0.79724Y(t)$ $I(t)=0.11667[Y(t)-Y(t-1)]$ $+0.19802Y(t)$	$Y(t)=93,453(1.04240)^t$	The rate of this model seems to be quite moderate.
7) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.79724Y(t)$ $I(t)=0.10027[Y(t)-Y(t-1)]$ $+0.23799Y(t)-8,702.24$	$Y(t)=247,008.24$ $-153,555.24(0.73999)^t$	In this case, 0.73999=1-0.26001, i.e., the rate is in negative value: -26.001%. When t increases indefinitely, $(0.73999)^t$ will approach zero, and Y(t) will remain at 247,008.24.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
8) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.75841Y(t)+8{,}632.65$ $I(t)=2.84519[C(t)-C(t-1)]$	$Y(t)=57{,}720.84(1.12608)^{t}$ $+35{,}732.16$	The rate is too high.
9) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[C(t)-C(t-1)]+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.75841Y(t)+8{,}632.65$ $I(t)=0.95075[C(t)-C(t-1)]$ $+30{,}969.63$	$Y(t)=-70{,}468.21(1.50389)^{t}$ $+163{,}921.21$	Y(t) is going to be negative over time.
10) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[C(t-1)-C(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.75670Y(t)+9{,}016.40$ $I(t)=2.83189[C(t-1)-C(t-2)]$	$Y(t)=28{,}355.97(1.1491)^{t}$ $-2{,}610.59(7.7151)^{t}$ $+54{,}946.44$ (Basic Year: 1931)	The negative term has a greater rate, so Y(t) will fall into negative value.
11) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.75841Y(t)+8{,}632.65$ $I(t)=2.02659[Y(t)-Y(t-1)]$	$Y(t)=57{,}720.84(1.3535)^{t}$ $+35{,}732.16$	The rate of growth is too high.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
12) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t-1)-Y(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.75670Y(t)+9{,}016.40$ $I(t)=2.01716Y[(t-1)-Y(t-2)]$	$Y(t)=59{,}201.97(1.16320)^t$ $-2{,}807.26(7.12754)^t$ $+37{,}058.30$ (Basic Year: 1931	The negative term has a greater rate, so Y(t) will fall into negative value.
13) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$	$Y(t)=C(t)+I(t)$ $C(t)=0.75841Y(t)+8{,}632.65$ $I(t)=0.11667[Y(t)-Y(t-1)]$ $+0.19802Y(t)$	$Y(t)=198{,}095.75$ $-104{,}642.75(1.59624)^t$	Y(t) falls into negative value.
14) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.75841Y(t)+8{,}632.65$ $I(t)=0.10027[Y(t)-Y(t-1)]$ $+0.23799Y(t)-8{,}702.24$	$Y(t)=112{,}768.58(1.03727)^t$ $-19{,}315.58$	The rate seems quite moderate
15) $Y(t)=C(t)+I(t)$ $C(t)-cY(t-1)$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.82906Y(t-1)$ $I(t)=2.84519[C(t)-C(t-1)]$	$Y(t)=120{,}653.22(1.16753)^t$ $-27{,}200.22(2.02035)^t$	The negative term has a greater rate, so Y(t) will be negative.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
16) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[C(t)-C(t-1)]+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.82906Y(t-1)$ $I(t)=0.95075[C(t)-C(t-1)]$ $+30{,}969.63$	$Y(t)=0.88782^{t}(-87{,}715.65$ $\cos tB-105{,}302.44\sin tB)$ $+181{,}168.65$ where $B=88^{\circ}45.0'=1.555$ Rad.	In this case 0.88782 will approach zero so eventually Y(t) will approach the value 181,168.65.
17) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.82906Y(t-1)$ $I(t)=2.02659[Y(t)-Y(t-1)]$	$Y(t)=93{,}453(1.16652)^{t}$	The rate of growth is too high.
18) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t-1)-Y(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.82986Y(t-1)$ $I(t)=2.01716[Y(t-1)-Y(t-2)]$	$Y(t)=297{,}861.96(1.32746)^{t}$ $-211{,}948.96(1.51957)^{t}$ (Basic year: 1931)	The negative term has a greater rate, so Y(t) will fall into negative value.
19) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$	$Y(t)=C(t)+I(t)$ $C(t)=0.82906Y(t-1)$ $I(t)=0.11667[Y(t)-Y(t-1)]$ $+0.19802Y(t)$	$Y(t)=93{,}453(1.03950)^{t}$	The rate seems quite moderate.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
20) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.82906Y(t-1)$ $I(t)=0.10027[Y(t)-Y(t-1)]$ $+0.23799Y(t)-8{,}702.24$	$Y(t)=129{,}792.63$ $-36{,}339.63(1.10132)^t$	Y(t) is going to be negative.
21) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.78035Y(t-1)+10{,}313.42$ $I(t)=2.84519[C(t)-C(t-1)]$	$Y(t)=46{,}953.53$ $+111{,}233.13(1.32524)^t$ $-64{,}733.67(1.67535)^t$	The negative term has a greater rate, so Y(t) will be negative.
22) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.78035Y(t-1)+10{,}313.42$ $I(t)=2.02659[Y(t)-Y(t-1)]$	$Y(t)=46{,}499.47(1.21396)^t$ $+46{,}953.53$	The rate is too high.
23) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t-1)-Y(t-2)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.77034Y(t-1)+12{,}619.16$ $I(t)=2.01716[Y(t-1)-Y(t-2)]$	$Y(t)=54{,}946.44$ $+1.42027^t(30{,}966.56$ $\cos tB-90{,}704.37\sin tB)$ where $B=78^\circ 55.5'=$ 1.3775 Rad. (Basic year: 1931)	The value of Y(t) is oscillatory and explosive, but the rate is too high.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
24) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$	$Y(t)=C(t)+I(t)$ $C(t)=0.78035Y(t-1)+10,313.42$ $I(t)=0.11667[Y(t)-Y(t-1)]$ $+0.19802Y(t)$	$Y(t)=-383,218.40$ $(0.96843)^t$ $+476,671.40$	Since 0.96843 is less than one, and the coefficient is negative, so Y(t) will increase to a maximum 476,671.4 eventually.
25) $Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.78035Y(t-1)+10,315.42$ $I(t)=0.10027[Y(t)-Y(t-1)]$ $+0.23799Y(t)-8,702.24$	$Y(t)=-87,853.55$ $+181,306.55(1.02771)^t$	The rate seems a little too low, but the coefficient is so large that it may set off the low rate.
26) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.72752Y(t)+283.83t$ $+10,289.00$ $I(t)=2.84519[C(t)-C(t-1)]$	$Y(t)=4.5,162.61+1,041.68t$ $-321,709.61(1.15159)^t$	The rate is too high and the coefficient (negative) is too large.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
27) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[C(t)-C(t-1)]+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.72752Y(t)+283.83t$ $+10{,}289.00$ $I(t)=0.95075[C(t)-C(t-1)]$ $+30{,}969.63$	$Y(t)=155{,}055.91+$ $1{,}041.68t$ $-61{,}602.91(1.64997)^t$	The rate is too high, and Y(t) will be negative.
28) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$	$Y(t)=C(t)+I(t)$ $C(t)=0.72752Y(t)+283.83t$ $+10{,}289.00$ $I(t)=2.02659[Y(t)-Y(t-1)]$	$Y(t)=45{,}508.91+1{,}041.68t$ $+47{,}944.09(1.15534)^t$	The rate is too high.
29) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$	$Y(t)=C(t)+I(t)$ $C(t)=0.72752Y(t)+283.83t$ $+10{,}289.00$ $I(t)=0.11667[Y(t)-Y(t-1)]$ $+0.19802Y(t)$	$Y(t)=144{,}152.79$ $+3{,}811.84t$ $-50{,}699.79(2.76421)^t$	The rate is too high, and Y(t) will become a negative value.
30) $Y(t)=C(t)+I(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$	$Y(t)=C(t)+I(t)$ $C(t)=0.72752Y(t)+283.83t$ $+10{,}289.00$ $I(t)=0.10027[Y(t)-Y(t-1)]$ $+0.23799Y(t)-8{,}702.24$	$Y(t)=69{,}944.12+8{,}230.60t$ $+23{,}508.88(1.52423)^t$	The rate is too high.

Table C (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
31) $Y(t)=C(t)+I(t)$ $C(t)=a+bW(t-1)+cP(t-1)$ $I(t)=i[P(t-1)-P(t-2)]+j$ $W(t)=d+eY(t)$ $P(t)=Y(t)-W(t)$	$Y(t)=C(t)+I(t)$ $C(t)=19,618.84+0.72412W(t-1)$ $+0.53286P(t-1)$ $I(t)=19,348.34+1.00638[P(t-1)$ $-P(t-2)]$ $W(t)=0.59518Y(t)-3,578.03$ $P(t)=Y(t)-W(t)$	$Y(t)=0.59270^t(-35,747.46\cos tB$ $-88,21161\sin tB)+112,462.46$ where $B=37^\circ 35.5'=0,6559$ Rad., and $Y(t)$=NNP	When t increases indefinitely, 0.59270^t approaches zero because 0.59270 is less than one. Then Y(t) will approach the value 112,462.46.

Note: (i) In this table all solutions for Y(t) are measured in millions of dollars (1929), where Y(t) designates gross national product except in model (31).
(ii) The basic year (i.e., at time t=0) of all solutions in this table is 1930, except those with a notification.

Table D

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
1) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[C(t)-C(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66298Y(t)$ $I(t)=2.19507[C(t)-C(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=93{,}453(1.11331)^t$	The rate of growth is too high.
2) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[C(t)-C(t-1)]+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66298Y(t)$ $I(t)=0.74771[C(t)-C(t-1)]$ $+23{,}324.19$ $G(t)=0.18890Y(t)$	$Y(t)=157{,}473.21$ $-64{,}020.21(1.42610)^t$	The rate is too high and income is going to fall in negative value.
3) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[C(t-1)-C(t-2)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66216Y(t)$ $I(t)=2.24533[C(t-1)-C(t-2)]$ $G(t)=0.18952Y(t)$	$Y(t)=107{,}550.72(1.12661)^t$ $-3{,}114.73(8.89807)^t$ (Basic Year: 1931)	The rate of the negative term is greater than that of the positive, so Y(t) is going to be negative.

Table D (Continued)

Formulation of Model	Model with Est'd Parameters	Solution	Remarks
4) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66298Y(t)$ $I(t)=1.51270[Y(t)-Y(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=93{,}453(1.10854)^t$	The rate is too high.
5) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t-1)-Y(t-2)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66216Y(t)$ $I(t)=1.43317[Y(t-1)-Y(t-2)]$ $G(t)=0.189525Y(t)$	$Y(t)=107{,}795.38(1.13279)^t$ $-3{,}359.38(8.53044)^t$ (Basic Year: 1931)	$Y(t)$ is going to be negative because the rate of the negative term is greater than that of the positive term.
6) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66298Y(t)$ $I(t)=0.13835[Y(t)-Y(t-1)]$ $+0.14249Y(t)$ $G(t)=0.18890Y(t)$	$Y(t)=93{,}453(1.04240)^t$	This is the same rate as that in model 6 of Table A. This rate seems quite moderate.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
7) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$ $G(t)=g(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.66298Y(t)$ $I(t)=0.12592[Y(t)-Y(t-1)]$ $+0.17392Y(t)-6,852.99$ $G(t)=0.18890Y(t)$	$Y(t)=265,583.46$ $-172,130.46(0.82993)^t$	0.82993 is less than one, so Y(t) will approach the value 126,583.46.
8) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[C(t)-C(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58363Y(t)+17,585.89$ $I(t)=2.19507[C(t)-C(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=77,309.48$ $+16,143.52(1.21590)^t$	The rate is too high.
9) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[C(t)-C(t-1)]+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58363Y(t)+17,585.89$ $I(t)=0.74771[C(t)-C(t-1)]$ $+23,324.19$ $G(t)=0.18890Y(t)$	$Y(t)=179,845.18$ $-86,392.18(2.08886)^t$	Y(t) is going to be negative over time.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
10) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58363Y(t)+17{,}585.89$ $I(t)=1.51270[Y(t)-Y(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=77{,}309.48$ $16{,}143.52(1.17699)^{t}$	The rate is too high.
11) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t-1)-Y(t-2)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58397(t)+17{,}496.55$ $I(t)=1.43317[Y(t-1)-Y(t-2)]$ $G(t)=0.18952Y(t)$	$Y(t)=77{,}246.89$ $+31{,}786.61(1.24495)^{t}$ $-4{,}597.50(5.08246)^{t}$ (Basic Year: 1931)	The negative term has a greater rate, so Y(t) will be a negative value.
12) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58363Y(t)+17{,}585.89$ $I(t)=0.13835[Y(t)-Y(t-1)]$ $+0.14249Y(t)$ $G(t)=0.18890Y(t)$	$Y(t)=206{,}927.78$ $-113{,}474.78(2.9252)^{t}$	Y(t) will fall into negative value.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
13) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.58363Y(t)+17,585.89$ $I(t)=0.12592[Y(t)-Y(t-1)]$ $+0.17392Y(t)-6,852,99$ $G(t)=0.18890Y(t)$	$Y(t)=200,425.04$ $-106,972.04(1.74007)^t$	Y(t) will be negative.
14) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.68946Y(t-1)$ $I(t)=1.51270[Y(t)-Y(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=93,453(1.17338)^t$	The rate is too high.
15) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.68946Y(t-1)$ $I(t)=0.13835[Y(t)-Y(t-1)]$ $+0.14249Y(t)$ $G(t)=0.18890Y(t)$	$Y(t)=93,453(1.03930)^t$	This rate seems quite moderate.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
16) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.689455Y(t-1)$ $I(t)=0.12592[Y(t)-Y(t-1)]$ $+0.17392Y(t)-6,852.99$ $G(t)=0.18890Y(t)$	$Y(t)=131,095.41$ $-37,642.41(1.10225)^t$	Y(t) will be negative.
17) $Y(t)=C(t)+I(t)+C(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+C(t)$ $C(t)=0.60235Y(t-1)+18,549.27$ $1(t)=1.51270[Y(t)-Y(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=88,858.79$ $+45,942.11(1.29753)^t$	The rate is too high.
18) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.60235Y(t-1)+18,549.27$ $I(t)=0.13835[Y(t)-Y(t-1)]$ $+0.14249Y(t)$ $G(t)=0.18890Y(t)$	$Y(t)=279,939.43$ $-186,486.43(0.87500)^t$	0.87500 is less than one, so Y(t) will approach the value 279,939.43.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
19) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.60235Y(t-1)+18,549.27$ $I(t)=0.12592[Y(t)-Y(t-1)]$ $+0.17392Y(t)-6,852.99$ $G(t)=0.18890Y(t)$	$Y(t)=335,810.69$ $-242,357.69(0.93189)^t$	0.93189 is less than one, so Y(t) will approach the value 335,810.69
20) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+bt+a$ $(I)=i[C(t)-C(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.59769Y(t)-132.64t$ $+16,880.40$ $I(t)=2.19507[C(t)-C(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=73,913.67-621.52t$ $+19,539.33(1.19426)^t$	The rate is too high.
21) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[C(t)-C(t-1)]+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.59769Y(t)-132.64t$ $+16,880.40$ $I(t)=0.74771[C(t)-C(t-1)]$ $+23,324.19$ $G(t)=gY(t)$	$Y(t)=186,626.05-621.52t$ $-93,173.05(1.91399)^t$	Y(t) falls into negative value.

Table D (Continued)

Formulation of Model	Model with est'd Parameters	Solution	Remarks
22) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.59769Y(t)-132.64t$ $+16,880.40$ $I(t)=1.51270[Y(t)-Y(t-1)]$ $G(t)=0.18890Y(t)$	$Y(t)=74,693.37-621.52t$ $+18,759.63(1.16425)^t$	The rate is too high.
23) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$ $+jY(t)$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.59769Y(t)-132.64t$ $+16,880.40$ $I(t)=0.13835[Y(t)-Y(t-1)]$ $+0.14249Y(t)$ $G(t)=0.18890Y(t)$	$Y(t)=16,862.05-1,870.22t$ $+76,590.95(2.05176)^t$	The rate is too high.
24) $Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)+bt+a$ $I(t)=i[Y(t)-Y(t-1)]$ $jY(t)+A$ $G(t)=gY(t)$	$Y(t)=C(t)+I(t)+G(t)$ $C(t)=0.59769Y(t)-13264t$ $16,880.40$ $I(t)=0.12592[Y(y)-Y(t-1)]$ $+0.17392Y(t)-6,852.99$ $G(t)=0.18890Y(t)$	$Y(t)=243,212.40-3,358.76t$ $-149,759.40(1.45690)^t$	$Y(t)$ falls into negative value.

Note: (i) In this table all solutions for $Y(t)$ are measured in millions of dollars (1929), where $Y(t)$ designates gross national product, (ii) the basic year (i.e. at time $t=o$) of all solutions in this table is 1930 except those with a notification.

$$C(t) = cY(t) + a$$

$$C(t) = cY(t) + bt + a$$

and $$I(t) = i[Y(t) - Y(t-1)] + jY(t) + A$$

$$I(t) = i[Y(t) - Y(t-1)] + jY(t)$$

from which under each definitional equation four combinations can be made. They are:

$$\begin{cases} C(t) = cY(t) + a \\ I(t) = i[Y(t) - Y(t-1)] + jY(t) + A \end{cases}$$

$$\begin{cases} C(t) = cY(t) + a \\ I(t) = i[Y(t) - Y(t-1)] + jY(t) \end{cases}$$

$$\begin{cases} C(t) = cY(t) + bt + a \\ I(t) = i[Y(t) - Y(t-1)] + jY(t) + A \end{cases}$$

$$\begin{cases} C(t) = cY(t) + bt + a \\ I(t) = i[Y(t) - Y(t-1)] + jY(t) \end{cases}$$

Accordingly, these combinations, together with the definitional equation, should yield a better result. Let us look at Table C, which is under the definition, $Y(t)=C(t)+I(t)$. Except the first combination which can yield a moderate rate as shown in Table C model 14, the other three combinations cannot give a satisfactory result. The second combination as shown in Table C model 13 yields a negative value for $Y(t)$. This does not make sense. The third combination as in Table C model 30 yields a rate 0.52423 or 52.423 per cent, which is too high in the "real world". The last combination as in Table C model 29 gives a negative value for $Y(t)$. These four combinations under the definition, $Y(t)=C(t)+I(t)+G(t)$, can be seen in Table D, model 13, model 12, model 24 and model 23, respectively. In Table D, the solution for $Y(t)$ from the first three combinations is a negative value. The last combination as in Table D model 23 yields a very high rate of growth, i.e., 105.176%. Although these consumption and investment functions appear to be fairly accurate in describing individual economic relations, when a consumption function and an investment function interact together, the result of this interaction may be absurd. That is to say, the combination of a good consumption function and a good investment function may not necessarily make a good model. The above first combination can make a moderate model in Table C, but it cannot do the same job in Table D.

In general, according to our empirical work, there are four cases in the solutions of both Table C and Table D. The first case refers to those solutions with a moderate, i.e., around 0.041998. Only six out of fifty-five models can yield a moderate rate. In the second case the rate is a negative value. For example, the solution of model 7 in Table C is $Y(t)=247{,}008.24-153{,}335.24(0.73999)^t$, where $0.73999=1-0.26001$, i.e., the rate is -26.001 per cent. Since 0.73999 is less than one, then in long-run as t approaches infinity, $(0.73999)^t$ will approach zero, so that Y(t) will approach asymptotically the value 247,008.24. Thus in this case, Y(t) will approach a constant level asymptotically over time, i.e., after Y(t) has attained to that level, Y(t) will remain there forever and stop growing. There are not many but six models yield a negative rate in this study. In the third case the rate is too high, which refers to the rate close to 10 per cent or above, so that the path of Y(t) over time is steeply explosive. Some models yield an extremely high rate, such as that of model 23 in Table D. Totally there are eighteen models in this case. The remaining models belong to the forth case, that is, they all yield a negative value for Y(t) over time. This is due to the negative value of the coefficients which are determined by the initial value of Y(t) and the value of the constant term (if any). In this study all models involving in second-order difference equation give a negative value for Y(t) over time (the reason for this will be discussed later). For example, model 10 in Table C, the rate of the negative term is higher than that of the positive term so that the value of Y(t) over time will be negative. This negative value is determined by the initial values of Y(0) and Y(1) and the value of the constant term. In Fig. 4-1, four curves, y_1, y_2, y_3, and y_4 are drawn to represent the path of Y(t) over time in the four cases, respectively.

According to our empirical result, the Harrod-Domar model as shown in Table C model 4 yields a solution for Y(t) over time with a pretty high rate which is classified in the above fourth case. The solution is

$$Y(t) = 93{,}453(1.11117)^t$$

Here the rate 0.11117 is not the "warranted rate of growth," but it is the annual rate of growth of income. This rate, however, can also keep saving and investment equal. The assumption of the Harrod-Domar model at equilibrium is as follows:

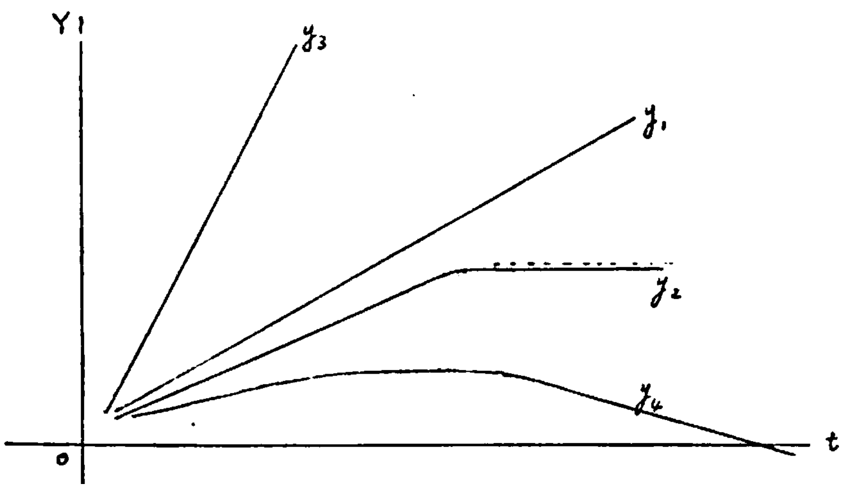

Fig. 4-1

$$C(t) = cY(t)$$

$$S(t) = sY(t)$$

$$c + s = 1$$

$$S(t) = I(t)$$

and $$I(t) = i[Y(t) - Y(t-1)]$$

Therefore,

$$sY(t) = i[Y(t) - Y(t-1)]$$

$$[Y(t) - Y(t-1)]/Y(t) = s/i = (1-c)/i = G_w$$

where G_w is the "warranted rate of growth" which keep saving and investment equal. From empirical results, we have

$$(1-c)/i = (1-0.79724)/2.02659 = 0.10005$$

That is, the "warranted rate of growth" is 0.10005. Since

$$[Y(t) - Y(t-1)]/Y(t) = 0.10005$$

then $$(1-0.10005)Y(t) = Y(t-1)$$

or $$Y(t) = 1.11117Y(t-1)$$

from which the same general solution as shown in Table C model 4 may be obtained. Now judging from the empirical work, indications are that the Harrod-Domar model is not good in forecasting, because it yields a rate too high for the "real world."

The empirical result also seems to be a good reason for the criticism of the unlagged Harrod-Domar model. However, even when a lag is introduced into the consumption function (as discussed in Chapter II, this modified model will not possess any of Harrod's properties), still a better result cannot be obtained than that as shown in Table C model 17:

$$Y(t) = 93{,}453(1.16652)^t$$

where the rate 0.16652 is even higher than that of the original Harrod-Domar model. Furthermore, this modified model with a lagged consumption function does not possess any of Harrod's properties because the rate in the solution cannot keep saving and investment equal. If saving and investment are equated, then (see Table C model 17)

$$S(t) = I(t)$$

or $$sY(t-1) = i[Y(t) - Y(t-1)]$$

By the assumption c+s=1, then

$$(1-c)Y(t-1) = i[Y(t) - Y(t-1)]$$

Hence $$(1.082906)Y(t-1) = 2.02659[Y(t) - Y(t-1)]$$

or $$[Y(t) - Y(t-1)]/Y(t-1) = 0.17094/2.02659 = 0.08435$$

which is the rate equivalent to the "warranted rate of growth" in the Harrod-Domar model, i.e., this rate keeps saving and investment equal. From the above last expression, we have

$$0.08435Y(t-1) = Y(t) - Y(t-1)$$

or $$Y(t) = (1 + 0.08435)Y(t-1) = 1.08435Y(t-1)$$

and the general solution is

$$Y(t) = Y_o(1.08435)^t$$

where Y_o in this case is 93,453. This rate, 0.08435, can also keep saving and investment equal, but it is different from that given in Table C model 17 (i.e., 0.16652). Thus this difference between the two rates implies that the rate given in Table C model 17 does not keep saving and investment equal. Therefore, when a lagged consumption is introduced into the Harrod-Domar model, the modified model will not retain the original properties.

The Samuelson model is also invalid in this empirical work. As shown in Table C model 15, the solution is

$$Y(t) = 120,653.22(1.16753)^t - 27,200.22(2.02035)^t$$

where the rate of the negative term is higher than that of the positive term so that income Y(t) will fall into negative value over time. The modification of this model by adding some autonomous investment in the investment function yields

$$Y(t) = 0.88782^t(-87,715.65\cos tB-105,302.44\sin tB)+181,168.65$$

as shown in Table C model 16, where 0.88782 is less than one so that when t increases indefinitely, 0.88782^t will approach zero and then Y(t) will approach asymptotically the value 181,168.65. In this case, after Y(t) has attained to the value 181,168.65, Y(t) will stop growing. The solutions of these two models, however, do not make any sense from the standpoint of economic growth.

The main reason why the Samuelson model is invalid in this study is that the period starts in the depression time as we have indicated previously in Chapter II. Actually this is a general shortcoming of any model formulated by second-order difference equation. This is the reason why none of the models of second-order difference equation can yield a moderate rate for Y(t), including the model formulated by the underconsumption and the overinvestment theories (the solution of this particular model can be seen in Table C model 31).

It is apparent that a model of second-order difference equation can be applicable only in a certain particular case. The conditions for a model of second-order difference equation are:

1) The necessary condition is that the initial situation of national income Y(t) must be progressive, i.e., income must be higher in the next period.
2) The necessary and sufficient condition is that the initial rate of increase of Y(t) must be great enough (i.e., the value of Y(1)-Y(0) must be great enough).

Take the solution of the Samuelson model again for illustration,

$$Y(t) = A_1(1.16753)^t + A_2(2.02035)^t \tag{4-34}$$

where A_1 and A_2 are constants to be determined by the two initial values $Y(0)$ and $Y(1)$. The necessary condition is that $Y(1)$ must be greater than $Y(0)$. Suppose $Y(1)$ is less than $Y(0)$, such that $Y(0)=2$ and $Y(1)=1$, then at $t=0$

$$Y(0) = A_1 + A_2 = 2$$

and at $t=1$

$$Y(1) = 1.16753A_1 + 2.02035A_2 = 1.$$

Solving for A_1 and A_2 yields

$$A_1 = 3.5654 \qquad \text{and} \qquad A_2 = -1.5654.$$

Thus the solution for (4-34) is

$$Y(t) = 3.5654(1.16753)^t - 1.5654(2.02035)^t$$

where the rate of the negative term is higher than that of the positive term, so that income $Y(t)$ will fall into negative value over time. But, the other way round, when $Y(0)=1$, and $Y(1)=2$, then

$$A_1 = 0.00239 \qquad \text{and} \qquad A_2 = 0.99761$$

and the solution of (4-34) becomes

$$Y(t) = 0.00239(1.16753)^t + 0.99761(2.02035)^t$$

Then $Y(t)$ will be in progressive values, consistent with the sense of growth. For the necessary and sufficient condition, the initial rate of increase of $Y(t)$ must be great enough. Suppose $Y(0)=1$, and $Y(1)=1.1$, then

$$A_1 = 1.07919 \qquad \text{and} \qquad A_2 = -0.07919$$

and (4-34) will become

$$Y(t) = 1.07919(1.16753)^{t} - 0.07919(2.02035)^{t}$$

where the rate of the negative term is higher than that of the positive term so that Y(t) will be in negative value over time. In this case, if the value of Y(1) is only increased by 10% of the value of the preceding period Y(0), the model is not applicable. Therefore, in order to meet the necessary and sufficient condition, the initial rate of increase of Y(t) must be great enough, and its magnitude is determined by the values of the two characteristic roots.

So far we merely use the rate of growth calculated by arithmetic mean as an index to find out which model can yield a moderate rate for Y(t) over time. In Table C, there are four models with a moderate rate, namely, models 6, 14, 19, and 25; and in Table D, there are two models, model 6 and model 15, which have a moderate rate. However, which rate is the best? To answer this question we have to make use of the mean-squares-error again by taking the average of the sum of the squares of the difference between the estimated value and the actual value of the observation of Y, such that

$$\text{M-S-E} = \frac{[Y(t) - Y'(t)]^2}{n}$$

where Y is the estimated value, Y' is the actual value, n is the number of total observations, and t is the time period. Accordingly, the smaller the M-S-E, the better the model. In Table E, the above six models are listed; they appear to reflect better rates as well as their M-S-E. The models are listed in an ascending order according to the value of the M-S-E. The M-S-E of that computed according to the rate of growth by arithmetic mean is also put in Table E. Besides, in Table E, there is a model (C-24) which yields a negative rate (i.e., -0.03157, as 0.96843=1-0.03157); so the aptitude of the M-S-E of a model with a negative rate can be observed.

In Table E, there are more models from Table C than from Table D, that is, the merging of government expenditures into consumption and investment expenditures may yield a better result. The reason for this, as mentioned before, may be due to the fact that the portion of investment arising from government expenditures cannot achieve the function of "feed-back" if the total volume of government expenditures is separated. However, there is one thing to note, that both model C-6 and model D-6 yield the same result. Both of them have the same consumption and investment functions, but in model D-6, the volume of government expenditures is separated. Is this a coincidence because of the rounding up of decimal points, or is there any other reason? To answer this question more investigation and effort are needed, and is beyond the scope of this study.

In Table E all consumption functions are either related to the current income or to that of the previous period. The suggestion of A. Smithies that consumption is also a function of time does not appear in Table E, that is, his consumption function cannot make a good model in this study. With regard to investment functions, in Table E, these are all related to the Harrod-Domar's modified investment function, as suggested by W. Baumol, either with or without the constant

Table E

Model	Solution	M-S-E	Ref.
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$Y(t)=181{,}306.55(1.02771)^t-87{,}853.55$	368,322,920.86	C-25
$Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=93{,}453(1.04240)^t$	476,178,230.92	C-6
$Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$ $G(t)=gY(t)$	$Y(t)=93{,}453(1.04240)^t$	476,178,230.92	D-6
$Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$Y(t)=112{,}768.58(1.03727)^t-19{,}315.58$	494,310,352.74	C-14
Rate of Growth by Arithmetic mean. $Y(t)=(1+R)Y(t-1)$ where $R=\frac{1}{n}\sum\left[\frac{Y(t)-Y(t-1)}{Y(t-1)}\right]$	$Y(t)=93{,}453(1.041998)^t$	536,457,025.87	

Table E (Continued)

Model	Solution	M-S-E	Ref.
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=93{,}453(1.03950)^{t}$	899,614,708.20	C-19
$Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$ $G(t)=gY(t)$	$Y(t)=93{,}453(1.03931)^{t}$	915,532,930.26	D-15
Y9t)=C(t)+I(t) $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=-383{,}218.40(0.96843)^{t}+476{,}671.40$	2,025,732,729.36	C-24

Note: The last column of this Table refers to the model number of Table C and Table D.

term (i.e., autonomous investment). None of the investment functions in Table E is related to the change of consumption. It is also noted that there is no lagged investment function in Table E.

There are four models, in Table E, yielding a better result than that calculated by the method of arithmetic mean, that is, these four models have a smaller M-S-E. It appears then that the common method to find the rate of growth by arithmetic mean is not a very good one. However, if this common method of arithmetic mean is good enough for the prediction of the growth of an economy, econometricians may save a lot of time in studying and formulating economic growth models.

The model with the smallest M-S-E in Table E is model C-25):

$$Y(t) = C(t) + I(t)$$

$$C(t) = cY(t-1) + a$$

$$I(t) = i[Y(t)-Y(t-1)] + jY(t) + A.$$

Included in this model is the best investment function, but the consumption function is not a very good one (see previous section). The solution of this model is

$$Y(t) = 181{,}306.55(1.02771)^t - 87{,}853.55 \qquad (4\text{-}35)$$

The rate in this solution is a little too low, and the constant term is of negative value. Actually, the rate of growth in the United States is higher than 0.02771, and a negative constant term makes Y(t) grow even slower than without that constant term. This model has the smallest M-S-E in Table E because there is a large coefficient (i.e., 181,306.55) so that the deficiency of the low rate can be overcome. Also, this low rate can lessen the deviation of the estimated Y(t) in the depression period. If the initial condition is not cast on the depression period, then this model will not have the smallest M-S-E. Therefore, even though this model has the smallest M-S-E, still it is not an ideal model.

It has been indicated that there are two models which have the same M-S-E. They are in the second place in Table E. These two models have the same consumption and investment functions, and there are no constant terms either in the consumption function or in the investment function. Thus the path of Y(t) over time is strictly proportional to the rate 0.04240 only.

In the third place, there is a model in Table E with the best consumption and the best investment function, as indicated in the previous section. This model yields a rate slightly lower than that of models C-6 and D-6, and its M-S-E is a little larger than that of the latter ones.

The last model in Table E is the one with a negative rate of growth, i.e., -0.03157. When the time period t keeps going indefinitely, the value of Y(t) will approach a maximum level 476,671.40 asymptotically. Also it can be seen that its M-S-E is extremely large. Thus it is obvious that this model is not applicable.

However, the evaluation as shown in Table E is still not satisfactory. In Table E, the M-S-E is calculated according to the general solution which is determined by the initial value of the model. Usually an initial point is quite far away from the average point of the sample, as it may often be arbitrary. A great deviation between the initial value and the average value will not give a good solution for the model, because in a general solution the initial value is the base of prediction. Therefore the M-S-E of the general solution of a good model may not be smaller than those of the others due to a bad initial condition. In this study, the initial condition is not an ideal one because the period studied, as pointed out before, is from the beginning of the Great Depression. Another way of evaluation is possible by avoiding the use of the initial value as a base. In order to avoid the influence of this initial condition, the solution for Y(t) is not performed in the general form in the same way as a difference equation is solved. All variables are substituted into the definitional equation, and then like terms are grouped without transposing. Thus the estimated value of Y(t) can be obtained by substitutions of actual values of variables of each year into the non-general form. Using model C-25 as an example, (4-35) is its general solution as shown above. Alternatively, if all variables are substituted into the definitional equation without transposing, then the estimated values of Y(t) can be obtained by the non-general solution

$$Y(t) = 0.33836Y(t) + 0.68008Y(t-1) + 1{,}611.18 \qquad (4\text{-}36)$$

where the Y(t) on the left hand side is the estimated value, and the one on the right hand side is the actual value. The values of Y(t) on the left hand side can be obtained by substitutions of the annual actual values into the variables Y(t) and Y(t-1) on the right hand side. This process, however, can be said to be a test against economic relation rather than a forecasting. Thus the magnitude of the M-S-E of (4-36) can tell whether or not model C-25 is a good model in describing the relation of economic variables in comparison with other models. Thus according to the non-general solutions, the M-S-E's of these models in Table E are calculated again, as tabulated in Table F. The two Samuelson's models are also put in Table F allowing observation as to whether or not they are good in testing against economic relation even though they are not applicable in forecasting.

In Table F, the model with the smallest M-S-E is not the same one with the smallest M-S-E in Table E. The model with the smallest M-S-E in Table E, i.e., model C-25, now is in fifth place in Table F. The reason for the decline of this model to fifth place is due to the shortcoming discussed above. We have said previously that this is not an ideal model, and now we can get a strong support for this remark.

The smallest M-S-E of the model in Table F is in the third place in Table E, i.e., model C-14. According to the general solution, it does not have the smallest M-S-E in comparison with the general solutions of other models, because the initial condition of this study is not perfect. However, according to the non-general form, it does bave the smallest M-S-E in comparison with the non-general solutions of other models. Thus model C-14 is a model which can describe the economic relation better than any of the others. Therefore, according to the general solution a good model may not yield good results for Y(t) due to a bad initial condition, but according to the non-general solution, it shows that the model is still a good one. Let us

Table F

Model	Solution	M-S-E	Ref.
$Y(t)=C(t)+I(t)$ $C(t)=cY(t)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$Y(t)=1.09666Y(t)-0.10027Y(t-1)-69.58$	1,071.526.17	C-14
$Y(t)=C(t)+I(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=1.11192Y(t)-0.11667Y(t-1)$	1,444,616.42	C-6
$Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$ $G(t)=gY(t)$	$Y(t)=1.3272Y(t)-0.13835Y(t-1)$	2,031,494.66	D-6
$Y(t)=C(t)+I(t)+G(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$ $G(t)=gY(t)$	$Y(t)=0.46974Y(t)+0.55110Y(t-1)$	32,336,354.85	D-15

Table F (Continued)

Model	Solution	M-S-E	Ref.
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)+A$	$Y(t)=0.33826Y(t)+0.68008Y(t-1)+1{,}611.18$	51,195,650.35	C-25
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=0.31468Y(t)+0.71239Y(t-1)$	52,249,847.21	C-19
$T(t)=C(t)+I(t)$ $C(t)=cY(t-1)+a$ $I(t)=i[Y(t)-Y(t-1)]+jY(t)$	$Y(t)=0.31468Y(t)+0.77368Y(t-1)+10{,}313.42$	62,705,407.26	C-24
Rate of growth by arithmetic mean. $Y(t)=(1+R)Y(t-1)$ where $R=\frac{1}{n}\sum\left[\frac{Y(t)-Y(t-1)}{Y(t-1)}\right]$	$Y(t)=1.041998Y(t-1)$	124,437,492.00	
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[C(t)-C(t-1)]+A$	$Y(t)=1.61728Y(t-1)-0.78822Y(t-2)+30{,}969.63$	304,902,615.49	C-16
$Y(t)=C(t)+I(t)$ $C(t)=cY(t-1)$ $I(t)=i[C(t)-C(t-1)]$	$Y(t)=3.18788Y(t-1)-2.35882Y(t-2)$	1,170,181,423.17	C-15

Note: The last column of this Table refers to the model number of Table C and Table D.

look at the model

$$Y(t) = C(t) + I(t)$$

$$C(t) = cY(t) + a$$

$$I(t) = i\left[Y(t)-Y(t-1)\right] + jY(t) + A$$

The consumption function is proportional to the level of the present income; and in addition there is some consumption demand, designated by a, independent of income in the consumption function. The investment function is suggested by W. Baumol. In this investment function there is some investment demand based on the acceleration principle, i.e., $i[Y(t)-Y(t-1)]$, and some investment demand proportioned to income (such as the community's trade balance), written as $jY(t)$, where j is a constant, and some autonomous investment demand independent both of the level of income and of its rate of change (or increase), designated by A. As mentioned before, these consumption and investment functions are the best functions considered by this study. This model should, then, be the best of the fifty-five models, since it consists of the best consumption and investment functions and has the smallest M-S-E in Table F.

It has been indicated that if a function (either consumption function or investment function) is related to the definitional equation, $Y(t)=C(t)+I(t)$, its M-S-E is smaller than that when it is related to the definition, $Y(t)=C(t)+I(t)+G(t)$. It has also been mentioned that four out of the six models in Table E (these six models yield a moderate rate) are related to $Y(t)=C(t)+I(t)$, saying that the definition $Y(t)=C(t)+I(t)$ is better than $Y(t)=C(t)+I(t)+G(t)$. Now, further empirical support is gained for this statement. It has been shown that both model C-6 and model D-6 yield the same rate, but their M-S-E's in Table F (i.e., according to the non-general solution) differ from each other. The M-S-E of model C-6 in Table F is smaller than that of model D-6. Model C-6 still stands in second place, in both Table E and Table F, while model D-6 is in third place in Table F. Both model C-6 and model D-6 consist of the same consumption and investment functions, but model C-6 is related to $Y(t)=C(t)+I(t)$ while model D-6 is related to $Y(t)=C(t)+I(t)+G(t)$. Therefore, a model can describe the relation of economic variables better if it is related to $Y(t)=C(t)+I(t)$.

Model C-24, which yields a negative rate, is still not good according to the non-general solution. As seen in Table F, this model has a large M-S-E, larger than those of the other six models which yield a moderate rate. In Table F, the M-S-E computed according to the rate of growth by arithmetic mean is pretty large too.

Both of Samuelson's models, as shown in Table F, have large M-S-E's, larger than that of any model in Table F. Both of them involve a second-order difference equation. We have already discussed the shortcomings of these two models, as well as any model involving a second-order difference equation. According to the results then in Table F, more evidence has been gained concerning the impracticability of this kind of models.

A summary of interpretations is as follows:

1) The choice of the initial period or year is of the utmost importance in the forecasting of economic growth.
2) Those models involving a second-order difference equation are impracticable in this study.
3) The definition, $Y(t)=C(t)+I(t)$, is better than the definition, $Y(t)=C(t)+I(t)+G(t)$.
4) A function with a constant term is better than one without a constant term.
5) An unlagged function is better than a lagged function.
6) The combination of the best consumption function and the best investment function will make the best model.

CHAPTER V

SUMMARY AND CONCLUSION

I. HYPOTHESES

In this study the hypotheses are primarily generated from the multiplier and the acceleration principles. Criticism and arguments about these two principles have been raised. Economists hold different opinions. R. F. Kahn first originated the idea of the multiplier, which was later fully developed by Keynes. Many economists think that the function of the multiplier is inadequate, since it is merely related to the original investment as stimulus to income. Keynes defines aggregate income as the sum of the aggregates of consumption and investment, and assumes that the volume of investment is autonomous. Then investment is rather stable in the Keynesian model. Will some additional income generated by investment induce some additional investment too? This "feed-back" relation of investment does exist. Therefore, the Keynesian model is one-sided. It only describes the stimulation of investment upon income through the function of the multiplier effect, but it does not explain the stimulation that income has upon investment. Hence, in addition to the multiplier principle, the acceleration principle originated by J. M. Clark is used to reduce the defect of the multiplier.

The principle of acceleration states that when income is increased by the stimulation of investment, there will yield some additional consumption which will in turn stimulate the volume of investment. Thus the circuit of "feed-back" is completed, as shown in Fig. 5-1.

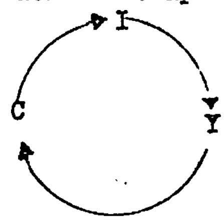

Fig. 5-1

In breaking down the circuit, investment is increased in proportion to the additional consumption. This is the original idea of the acceleration principle. Since consumption is a function of income, as indicated by Keynes, then by substitution, investment is also a function of income, and the circuit of Fig. 5-1 will become

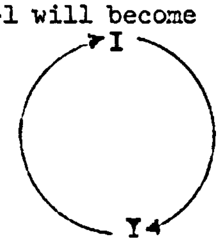

Fig. 5-2

If we are allowed to borrow the terms from mechanical engineering, we may say that the relation in Fig. 5-1 is a "three-stroke cycle" while that in Fig. 5-2 is a "two-stroke cycle" in the aggregate economy. Some economic models are formulated under the concept of "three-stroke cycle" such as the Samuelson model, and some models, like the Harrod-Domar model are based upon the "two-stroke cycle."

Some economists argue about the time lag in economic relations. The Keynesian assumption that current consumption is a function of the level of current income does not involve any time lag, and is said to be "out of history," static, and undynamic. The same time lag argument is also applied in the investment function; that investment should be a function of the change of consumption or income of the preceding period.

There are other arguments. (i) Consumption is a function of both income and time. (ii) In addition to the acceleration relation, investment is also a function of the level of present income. (iii) According to underconsumptionists, consumption is a function of wages and profit, both of which are based on the level of the preceding period, and the sum of which is equal to net national income. (iv) Investment is a function of the change in profit of the preceding period, as assumed in the overinvestment theory. Furthermore, in addition to Keynes' definition of aggregate income, some economists define aggregate income as the sum of the aggregates of consumption, investment and government expenditures, separating the government sector from consumption and investment.

In this study an attempt is made to put all the above arguments into the "empirical world" to see whether they are applicable or not. All these arguments are the hypotheses upon which the models of this study were formulated.

II. METHODOLOGY

How to determine the empirical relationships among economic variables is a tremendous task. This can only be accomplished by estimation based upon the statistical data obtained. The method of estimation used in this study is the simplest one in econometrics. The straightforward least-squares method is used to estimate the parameters of each consumption function and investment function. As shown in Chapter III, our estimated parameters are the best, unbiased estimates in that single equation; but they are still not perfect estimates, as the usual least squares method has some unsatisfactory consequences.

For the estimates of those equations with a time lag, the usual least squares method is again used. It was pointed out in Chapter III that the application of the usual least squares method to a lagged function will yield negative biased estimates, such as in the equation

$$C(t) = cY(t-1)$$

if c is greater than zero.

The consequences of the estimation of parameters for each single equation has been shown. Furthermore, in the model as a whole, when the consumption and the investment functions are combined to express the income function, the estimates of the parameters are no more unbiased. In order to eliminate this shortcoming, the indirect least squares method or some other complicated method (such as the two-stage least squares method and full-information maximum-likelihood method, etc., as indicated in Chapter III) must be used. None of these complicated methods have been applied to this study because there were fifty-five models to estimate and this would have resulted in the computation of parameters model by model. Besides, if the parameters were estimated model by model, different values would have resulted for the parameters of the same function in different models. This would have prevented further comparative analysis. Therefore, in order to overcome these difficulties and permit comparative analysis, the usual least squares method is used to get the same values for the parameters in the same function. In this study there are obvious shortcomings in the method of estimation; however, since all these models are based upon the same method, the comparison of them is still possible. Nevertheless, the aim of this study is not to build up a single model which can truly describe the growth or behavior of an economy, but to give a general interpretation to the various economic growth models.

III. CONCLUSION

In all the consumption and investment functions in this study the following two functions under the assumption, $Y(t)=C(t)+I(t)$, are the best consumption function and the best investment function:

$$C(t) = cY(t) + a$$

$$I(t) = i[Y(t) - Y(t-1)] + jY(t) + A.$$

The consumption function is an unlagged function. It can be said to be a modification of the Keynesian theory, in which the consumption function does not have a constant term. An unlagged consumption function has been criticized as being too static. Is a lagged consumption function more realistic than the unlagged one? The answer may not be positive, especially in the United States where credit facilities are so prevalent. First of all, even though one does not have any income in the last period, one still has to consume in this period (i.e., one does not necessarily have to consume in relation to the income of last period). Secondly, consumption may not be determined by the income earned in the last period. It may be more realistic that people consume according to their present income. Assume that people have a habit or budget to consume 80% of their income, which may not mean the income of last period. Let us take a common example of an individual. Suppose a man earned $100 last week, and he knows that he is going to make $130 this week. Then he will consume $104 ($130 x 80%) instead of $80 ($100 x 80%) this week. He may not obtain his money until the end of the week, but he can consume out of his cash balance and through various credit facilities. The main reason to support a lagged consumption function is that money cannot be spent before it is received. This reasoning is too rigid. It is only in the absence of credit facilities that people cannot consume their present income, having to spend only that income they earned in the preceding period. They could not, then, consume more than their last income. However, people are quite aware that they are entitled to their present income; and if there are credit facilities available, people will consider present consumption and present income together. People would not worry about what they had earned but what they have just earned, because the most realistic thing for human beings is "the present." Therefore, it may be more realistic to work with an unlagged rather than a lagged consumption function in the analysis of economic growth.

The above investment function is not only based upon the acceleration principle but it is also a function of the level of present income. Also, there are some autonomous investment independent of both the level of income and the change of income. As already indicated, this is a modified Harrod-Domar investment function suggested by W. Baumol. An example given by him for that portion of investment arising from the level of income is the community's trade balance, so that the value of j may be negative. According to our results, j is positive. It seems that this portion of investment is not so simple as the community's trade balance. If there is no change of income, according to the acceleration principle only (i.e., $I(t) = i[Y(t) - Y(t-1)]$), then investment will be zero. This situation is out of reality. The volume of investment may decline but will never be zero because people will never stop consuming. Inventories may be drawn down because of the decline of investment. Investment, however, may decline but will not stop. Therefore, investment may not depend upon the change of income only, but may also depend upon the level of income. Thus, the existence of that portion of investment proportional to the level of income is not so simple as a kind of trade balance, and is an important factor in the economy.

The investment functions of all models in this study which yield a moderate rate (as tabulated in Table E in Chapter IV) are related to the "two-stroke cycle," i.e., related to income. None of those models has the investment function related to the "three-stroke cycle," i.e., related to consumption. Perhaps, the "two-stroke cycle" may describe the economic relation more directly. Theoretically as indicated before, both "cycles" can achieve the same effect in the acceleration principle, but empirically they have different effects. Moreover, the "two-stroke cycle" seems to be more realistic than the "three-stroke cycle." It is likely that there is always some extra investment for new products, seeking a new market and more profit. This portion of investment in new products then does not depend upon the consumers' usual spending budget, but depends on how much extra money consumers want to spend out of their income, that is, some investment, as in new products, is determined by income other than consumption. Thus there are two portions of investment: one arising from consumption and the other arising from income. If we designate the former portion as I_c and the latter I_y, then in symbol

$$I = I_c + I_y.$$

New products may be a kind of substitute for old products, so that consumers may shift their consumption to new products. Then the total volume of consumption may not be changed. However, if the new products are not substitutes, then the volume of consumption may be changed by the additional consumption of consumers according to their income. Hence, it is very difficult to tell what portion of investment depends on consumers' usual budgets (i.e., depends on consumption), and what portion of investment depends on income; that is, it is very difficult to identify I_c and I_y. Since I_c is also a function of income, it is better to put the total volume of investment in terms of income. In other words, in the investment function, it is likely that the "two-stroke cycle" is superior to the "three-stroke cycle."

The combination of the above consumption and investment functions under the definition, $Y(t) = C(t)+I(t)$, makes out a model which is the best one in this study. Although, according to the general solution of this model, the mean-squares-error is not the smallest one as shown in Table E in Chapter IV, but this is due to the fact that the initial value is cast upon the depression period of the 1930's. However, this model has the smallest mean-squares-error in Table F, Chapter IV, according to the non-general solution, showing that this model is the nearest one to the "real world" in this study. This model is (model C-14)

$$Y(t) = C(t) + I(t)$$

$$C(t) = cY(t) + a$$

$$I(t) = i[Y(t)-Y(t-1)] + jY(t) + A.$$

The model next to the best in this study is

$$Y(t) = C(t) + I(t)$$

$$C(t) = cY(t)$$

$$I(t) = i[Y(t)-Y(t-1)] + jY(t).$$

The difference between this model and the best model rests on the constant term. In this model there is no constant term, either in the consumption function or in the investment function. As a matter of fact, mathematically, in the long-run a constant term of a function does not give much effect, because the value of a constant term does not change over time. The longer the period the less effective the constant term is in a growing economy. In other words, the constant term must grow relatively more and more insignificant as income grows larger and larger over time. Therefore, in the short-run it is better to use a constant term, while in the long-run the constant term may be ignored.

The consumption and the investment functions of model C-14 (the best model in this study) are subject to the definition, Y(t)=C(t)+I(t), only. They cannot be applicable under the definition, Y(t)=C(t)+I(t)+G(t), in this study, because it is likely that the former definition is more realistic than the latter one.

Those models based upon the simple acceleration principle, which is merely related to either the change of consumption or the change of income, are not applicable in this study. They are likely out of plausibility, because, according to the simple acceleration principle, when there is no change in either consumption or income, investment will be zero. As discussed before, a zero investment does not make any sense in the economic growth. An example of this kind of model is the Harrod-Doma model, which yields a pretty high rate as shown in Table C model 4 in Chapter IV. With regard to those models involving in second-order difference equation, such as the Samuelson model, they cannot be applicable when initial values are cast upon a depression period, or a period with a small rate of increase in national income. They can only be applied in a period starting with a fast growing national income.

Nevertheless, the best choice model in this study (i.e., model C-14) is still not a satisfactory one, because its solution gives a constant rate. Economic phenomenon is characterized by change. It is impossible to have the same constant rate of growth all the time. As pointed out before, the rate of growth of an economy is different for different periods. Therefore, even the solution of model C-14 for Y(t) is pretty close to the "real world," but long-run prediction with this rate may not be correct. In case this model is used for forecasting, parameters must be re-estimated from time to time in order to conciliate the economic change.

Here let us quote what R. G. D. Allen has said to complete our conclusion:

> . . . therefore, the multiplier-accelerator model needs to be modified or supplemented. There are several possible modifications to consider. The period analysis of the model may be too rigid and it may be better to have continuous variation. The linear assumptions may be the reason for the "unrealistic" features of the model and a non-linear accelerator may be the answer.[1]

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[1] R. G. D. Allen, Mathematical Economics (London: Macmillan and Company, Ltd., 1959), p. 219.

APPENDIX A

THE METHOD OF LEAST SQUARES

I SIMPLE LINEAR REGRESSION

Simple regression refers to a relationship between two variables. Assume there is a random variable Y that is related to another variable X by the linear equation

$$Y = a + bX \tag{A-1}$$

where Y is called a dependent variable, and X an independent variable, also a, b are the parameters to be estimated. By using the method of least squares, a regression line (estimated line) is obtained

$$Y' = a' + b'X \tag{A-2}$$

where a', b' = estimates of the two unknown parameters

Y' = ordinate on line for any given value of X.

By the properties of least squares (see Chapter III), gives

$$\sum u^2 = \sum (Y - Y')^2 \tag{A-3}$$

where u=Y-Y'. By substituting (A-1) into (A-3), yields

$$\sum u^2 = \sum (Y - a' - b'X)^2 \tag{A-4}$$

Taking the partial derivatives of (A-4) with respect to a' and b', and setting them equal to zero to minimize the value of $\sum u^2$, we get

$$\frac{\partial}{\partial a'} \sum u^2 = -2 \sum (Y - a' - b'X) = 0 \tag{A-5}$$

$$\frac{\partial}{\partial b'} \sum u^2 = -2 \sum X(Y - a' - b'X) = 0 \tag{A-6}$$

Re-write (A-5) and (A-6)

$$\sum (Y - a' - b'X) = 0$$

$$\sum X(Y - a' - b'X) = 0$$

or, re-written

$$\sum Y = na' + b' \sum X \qquad \text{(A-7)}$$

$$\sum XY = a' \sum X + b' \sum X^2 \qquad \text{(A-8)}$$

Equations (A-7) and (A-8) are the normal equations, from which the values of a' and b' can be obtained:

$$b' = \frac{n \sum XY - \sum X \sum Y}{n \sum X^2 - (\sum X)^2}$$

$$a' = \frac{\sum X^2 \sum Y - \sum X \sum XY}{n \sum X^2 - (\sum X)^2}$$

Alternatively, if we divide through (A-7) by n, then

$$\bar{Y} = a' + b'\bar{X} \qquad \text{(A-9)}$$

Where $\bar{Y}=\sum Y/n$, $\bar{X}=\sum X/n$, means of Y and X, respectively. (A-9) shows that the regression line passes through the point of means. From (A-9), then a' can be obtained in a simple way, such that

$$a' = \bar{Y} - b'\bar{X} \qquad \text{(A-10)}$$

In (A-1), if the constant term a is dropped out (or say, a is equal to zero), then (A-1) becomes

$$Y = bX \qquad \text{(A-11)}$$

In this case, the regression line will be

$$Y' = b'X \qquad \text{(A-12)}$$

Then with the least squares properties

$$\sum u^2 = \sum (Y - Y')^2$$

$$= \sum (Y - b'X)^2$$

By taking the partial derivatives of the last expression with respect to b' and setting it equal to zero for the purpose of minimization, gives

$$\frac{\partial}{\partial b'} \sum u^2 = -2 \sum X(Y - b'X) = 0$$

$$\sum XY = b' \sum X^2$$

or $$b' = (\sum XY)/\sum X^2 \qquad \text{(A-13)}$$

II MULTIPLE LINEAR REGRESSION

When more than two variables are involved, the regression analysis is called multiple regression. As a matter of fact, multiple regression is the extension of simple regression. Assume there are three variables X, Y and Z, such that

$$Y = a + bX + cZ \qquad \text{(A-14)}$$

where a,b and c are the parameters to be estimated. By using the least squares method, gives the multiple regression

$$Y' = a' + b'X + c'Z \qquad \text{(A-15)}$$

where a', b' and c' are the estimates of the three parameters, and Y is the ordinate on line for any given value of X. As done in the two variables case, it gives

$$\sum u^2 = \sum (Y - Y')^2$$
$$= \sum (Y - a' - b'X - c'Z)^2$$

Then take the partial derivatives of the sum with respects to a', b' and c', and set them equal to zero for minimization:

$$\sum (Y - a' - b'X - c'Z) = 0$$
$$\sum X(Y - a' - b'X - c'Z) = 0$$
$$\sum Z(Y - a' - b'X - c'Z) = 0$$

or, re-written

$$\sum Y = na' + b' \sum X + c' \sum Z \qquad \text{(A-16)}$$
$$\sum XY = a' \sum X + b' \sum X^2 + c' \sum XZ \qquad \text{(A-17)}$$
$$\sum ZY = a' \sum Z + b' \sum XZ + c' \sum Z^2 \qquad \text{(A-18)}$$

Solving the normal equations (A-16), (A-17) and (A-18), we can obtain the values of a', b' and c'. Alternatively, since the least squares plane passes through the point of means, we can also obtain a' by

$$a' = \bar{Y} - b'\bar{X} - c'\bar{Z}$$

In the case that the constant term is equal to zero, then

$$Y' = b'X + c'Z$$

Similarly

$$\begin{aligned}\sum u^2 &= \sum (Y - Y')^2 \\ &= \sum (Y - b'X - c'Z)^2\end{aligned}$$

Then take partial derivatives with respects to b' and c', and set both of them to be zero, such that

$$\sum X(Y - b'X - c'Z) = 0$$

$$\sum Z(Y - b'X - c'Z) = 0$$

or

$$\sum XY = b' \sum X^2 + c' \sum XZ$$

$$\sum ZY = b' \sum XZ + c' \sum Z^2$$

By solving the last two simultaneous equations, yields

$$b' = \frac{\sum XY \sum Z^2 - \sum ZY \sum XZ}{\sum X^2 \sum Z^2 - (\sum XZ)^2}$$

$$c' = \frac{\sum X^2 \sum ZY - \sum XY \sum XZ}{\sum X^2 \sum Z^2 - (\sum XZ)^2}$$

If the dependent variable Y is related to three or more variables, still the same principle can be applied. In general if Y is a function of k variables, let the general multiple regression be written as

$$Y' = a' + b_1'X_1 + b_2'X_2 + \dots + b_k'X_k$$

and the normal equations are

$$
\begin{aligned}
\sum Y &= a'n &&+ b_1' \sum X_1 &&+ b_2' \sum X_2 &&+ \cdots + b_k' \sum X_k \\
\sum X_1 Y &= a' \sum X_1 &&+ b_1' \sum X_1^2 &&+ b_2' \sum X_1 X_2 &&+ \cdots + b_k' \sum X_1 X_k \\
\vdots \quad & \quad \vdots && \quad \vdots && \quad \vdots && \qquad\qquad \vdots \\
\sum X_k Y &= a' \sum X_k &&+ b_1' \sum X_k X_1 &&+ b_2' \sum X_k X_2 &&+ \cdots + b_k' \sum X_k^2
\end{aligned}
$$

In this study, however, no dependent variable is a function of more than two variables.

APPENDIX B

SOLUTION OF LINEAR DIFFERENCE EQUATIONS

I FIRST-ORDER DIFFERENCE EQUATIONS

A) $Y(t) = Y(t-1) + c$

In this equation, Y is the variable, c is a constant, and t is a positive integer designating the time period.

When $t=1$, $Y(1) = Y(0) + c$

$t=2$, $Y(2) = Y(1) + c = Y(0) + c + c = Y(0) + 2c$

$t=3$, $Y(3) = Y(2) + c = Y(0) + 2c + c = Y(0) + 3c$

$\vdots \qquad \vdots$

$t=n$, $Y(n) = Y(0) + nc$

Write t for n, then the general solution for Y(t) is

$$Y(t) = Y(0) + tc$$

B) $Y(t) = aY(t-1)$

In this equation, Y and t have the same meaning as before, and a is a coefficient.

When $t=1$, $Y(1) = aY(0)$

$t=2$, $Y(2) = aY(1) = a[aY(0)] = a^2Y(0)$

$t=3$, $Y(3) = aY(2) = a[a^2Y(0)] = a^3Y(0)$

$\vdots \qquad \vdots$

$t=n$, $Y(n) = a^nY(0)$

Write t for n, then the general solution for Y(t) is

$$Y(t) = a^tY(0)$$

C) $Y(t) = aY(t-1) + c$ (B-1)

Since there is a constant term c in this equation, so it is called a non-homogeneous first-order difference equation. In this kind of equation (non-homogeneous difference equation), the general solution consists of two parts: the solution of the homogeneous counterpart, $y(t)$, and the particular solution, $\bar{Y}$, of the complete non-homogeneous equation, such that the general solution is

$$Y(t) = y(t) + \bar{Y} \qquad \text{(B-2)}$$

First, to solve the homogeneous counterpart of (B-1)

$$y(t) = ay(t-1) \qquad \text{(B-3)}$$

let $y(t)=kx^t$, where k is some constant to be found, then

$$y(t-1) = kx^{t-1}$$

From (B-3), it gives

$$kx^t = akx^{t-1}$$

$$kx^{t-1}(x - a) = 0$$

$$x - a = 0$$

$$x = a$$

So $y(t)=kx^t$ is a solution, if $x=a$, i.e., if

$$y(t) = ka^t \qquad \text{(B-4)}$$

which is the solution of the homogeneous counterpart.

Secondly, to find the particular solution $\bar{Y}$, let

$$\bar{Y} = Y(t) = Y(t-1)$$

From (B-1), it gives

$$\bar{Y} = a\bar{Y} + c$$

or, re-written

$$\bar{Y} = c/(1-a) \qquad \text{(B-5)}$$

where $(1-a) \neq 0$.

Now, from (B-2), it yields

$$Y(t) = ka^t + c/(1-a)$$

where k is an arbitrary constant, and can be determined by using the initial value Y(0). When t=0, then

$$Y(0) = k + c/(1-a)$$

Thus,

$$k = Y(0) - c/(1-a)$$

Therefore, the final solution is

$$Y(t) = [Y(0) - c/(1-a)]a^t + c/(1-a)$$

D) $Y(t) = aY(t-1) + bt + c$ (B-6)

First, for the homogeneous counterpart

$$y(t) = ay(t-1)$$

which is the same as (B-3), the solution for y(t) can be obtained as before:

$$y(t) = ka^t$$

where k is a constant.

Secondly, for the particular solution $\bar{Y}(t)$, let

$$\bar{Y}(t) = Y(t) = Y(t-1) = mt + n \qquad \text{(B-7)}$$

where m and n are some constants to be found. Then from (B-6) it gives

$$\bar{Y}(t) = a\bar{Y}(t) + bt + c$$

or, re-written

$$mt + n = a[m(t-1) + n] + bt + c$$

By transposing and grouping the like terms, yields

$$(t - at - a)m + (1-a)n = bt + c \qquad \text{(B-8)}$$

So if $\bar{Y}(t)=mt+n$, m and n must satisfy (B-8) and then (B-7).

Let t=0, then from (B-8) it gives

$$am + (1-a)n = c \qquad \text{(B-9)}$$

and if t=1,

$$m + (1-a)n = b + c \qquad \text{(B-10)}$$

Substract (B-9) from (B-10), then

$$m - am = b$$

or $$m = b/(1-a)$$

From (B-9) it gives

$$a[b/(1-a)] + (1-a)n = c$$

or $$n = c/(1-a) - ab/(1-a)^2$$

Thus, the values of m and n can satisfy both (B-7) and (B-8). From (B-7) it gives

$$\bar{Y}(t) = bt/(1-a) + c/(1-a) - ab/(1-a)^2$$
$$= (bt+c)/(1-a) - ab/(1-a)^2$$

Therefore, finally the general solution is

$$Y(t) = y(t) + \bar{Y}(t)$$
$$= ka^t + (bt+c)/(1-a) - ab/(1-a)^2$$

where the value of k can be found by using the initial value Y(0) by setting t=0, such that

$$k = Y(0) - c/(1-a) + ab/(1-a)^2$$

II SECOND-ORDER DIFFERENCE EQUATIONS

A) Homogeneous Second-order Difference Equation:

$$Y(t) + aY(t-1) + bY(t-2) = 0 \qquad \text{(B-11)}$$

Let $Y(t)=Ax^t$; then from (B-11) it gives

$$Ax^t + aAx^{t-1} + bAx^{t-2} = 0$$

By factorizing, yields

$$Ax^{t-2}(x^2 + ax + b) = 0$$

Thus, if $Y(t)=Ax^t$, x must satisfy the relation

$$x^2 + ax + b = 0$$

which is a quadratic equation. The solution of x can be obtained by completing a square, such that

$$x = \frac{-a \pm (a^2 - 4b)^{\frac{1}{2}}}{2} \tag{B-12}$$

So there are two possible values for x:

$$x_1 = \frac{-a + (a^2 - 4b)^{\frac{1}{2}}}{2}$$

$$x_2 = \frac{-a - (a^2 - 4b)^{\frac{1}{2}}}{2}$$

Thus, there are two solutions for Y(t):

$$Y(t) = A_1x_1^t \qquad \text{and} \qquad Y(t) = A_2x_2^t$$

By adding up these two possible solutions, gives a more general solution:

$$Y(t) = A_1x_1^t + A_2x_2^t \tag{B-13}$$

where A_1 and A_2 are arbitrary constants. This general solution depends on the nature of the characteristic roots x_1 and x_2 of the quadratic equation. There are three possible cases.

1) The roots are real and unequal, i.e., $a^2 > 4b$.

2) The roots are real and equal, i.e., $a^2 = 4b$.

3) The roots are complex, i.e., $a^2 < 4b$.

Case (1): $a^2 > 4b$

From (B-13), the values of A_1 and A_2 can be obtained in terms of the initial values Y(0) and Y(1) by setting t=0 and t=1 respectively, such that

$$Y(0) = A_1 + A_2$$

$$Y(1) = A_1x_1 + A_2x_2$$

By solving these two equations, yields

$$A_1 = \frac{Y(1) - x_2Y(0)}{x_1 - x_2}$$

$$A_2 = \frac{Y(1) - x_1Y(0)}{x_2 - x_1}$$

The general solution thus becomes

$$Y(t) = \left[\frac{Y(1) - x_2Y(0)}{x_1 - x_2}\right]x_1^t + \left[\frac{Y(1) - x_1Y(0)}{x_2 - x_1}\right]x_2^t$$

where x_1 and x_2 are given in the above in terms of a and b.

Case (2): $a^2 = 4b$

In this case the form of the solution is a little difference, i.e.,

$$Y(t) = A_1x^t + A_2tx^t \tag{B-14}$$

where, from (B-12), $x=-a/2$

By introducing the initial values, then from (B-14), gives

$$Y(0) = A_1$$

$$Y(1) = A_1x + A_2x$$

Therefore,

$$A_1 = Y(0), \qquad \text{and} \qquad A_2 = [Y(1) - xY(0)]/x$$

and the final solution is

$$Y(t) = Y(0)x^t + tx^t[Y(1) - xY(0)]/x$$

which can also be written as

$$Y(t) = [x(1-t)Y(0) + tY(1)]x^{t-1}$$

where the value of x as given in the above is (-a/2).

Case (3): $a^2 < 4b$

In this case the roots are complex, involving the imaginary number $\sqrt{-1}$, or i, such that

$$x_1 = c + di$$

$$x_2 = c - di$$

where $c=-a/2$, and $d=(4b-a^2)^{\frac{1}{2}}/2$. The values of (c+di) and (c-di) can be expressed in trigonometric functions such that

$$c + di = (c^2 + d^2)^{\frac{1}{2}}[c/(c^2+d^2)^{\frac{1}{2}} + id/(c^2+d^2)^{\frac{1}{2}}]$$

$$= (c^2 + d^2)^{\frac{1}{2}}(\cos B + i \sin B)$$

$$c - di = (c^2 + d^2)^{\frac{1}{2}}[c/(c^2+d^2)^{\frac{1}{2}} - id/(c^2+d^2)^{\frac{1}{2}}]$$

$$= (c^2 + d^2)^{\frac{1}{2}}(\cos B - i \sin B)$$

where B is some angle such that

$$\cos B = c/(c^2 + d^2)^{\frac{1}{2}}$$

$$\sin B = d/(c^2 + d^2)^{\frac{1}{2}}$$

Therefore,

$$x_1^t = (c + di)^t$$

$$= (c^2 + d^2)^{t/2}(\cos B + i \sin B)^t$$

$$= (c^2 + d^2)^{t/2}(\cos tB + i \sin tB)$$

$$x_2^t = (c - di)^t$$

$$= (c^2 + d^2)^{t/2}(\cos B - i \sin B)^t$$

$$= (c^2 + d^2)^{t/2}(\cos tB - i \sin tB)$$

Then from (B-13) it gives

$$Y(t) = A_1 x_1^t + A_2 x_2^t$$

$$= (c^2 + d^2)^{t/2}[(A_1+A_2)\cos tB + i(A_1-A_2)\sin tB] \qquad \text{(B-15)}$$

Since A_1 and A_2 are arbitrary, so let

$$A_1 + A_2 = g$$

$$(A_1 - A_2)i = h$$

where g and h are real numbers. This implies that (A_1-A_2) is imaginary.

Re-write (B-15):

$$Y(t) = (c^2 + d^2)^{t/2}(g \cos tB + h \sin tB) \qquad \text{(B-16)}$$

when t=0, then

$$Y(0) = g \cos 0^\circ + h \sin 0^\circ = g$$

When t=1, then

$$Y(1) = (c^2 + d^2)^{\frac{1}{2}}(g \cos B + h \sin B)$$

$$= (c^2 + d^2)^{\frac{1}{2}}[Y(0) \cos B + h \sin B]$$

Then solve for h:

$$h = [Y(1)-(c^2+d^2)^{\frac{1}{2}}Y(0)\cos B]/[(c^2+d^2)^{\frac{1}{2}}\sin B]$$

Since the value of c,d,g, h and B are known, therefore, the solution for Y(t) can be obtained from (B-16).

B) Non-homogeneous Second-order Difference Equation:

$$Y(t) + aY(t-1) + bY(t-2) = q \qquad \text{(B-17)}$$

First, with the same procedure as before, find the solution of the homogeneous counterpart which is

$$y(t) + ay(t-1) + by(t-2) = 0 \qquad \text{(B-18)}$$

From (B-18), there are three possible solutions:

$$y(t) = a_1x_1^t + a_2x_2^t$$

$$y(t) = a_1x^t + a_2tx^t$$

$$y(t) = (c^2 + d^2)^{t/2}(g \cos tB + h \sin tB)$$

where the values of c, d, g, h and B are given as before.

Secondly, for the particular solution, as done before, let $\bar{Y}$ be the particular solution so that

$$\bar{Y} = Y(t) = Y(t-1) = Y(t-2)$$

From (B-17) it gives

$$\bar{Y} + a\bar{Y} + b\bar{Y} = q$$

or $$\bar{Y} = q/(1+a+b)$$

where $(1+a+b)\neq 0$, so that the particular solution is acceptable. Since

$$Y(t) = y(t) + \bar{Y}$$

then there are three possible solutions for Y(t):

$$Y(t) = a_1x_1^t + a_2x_2^t + q/(1+a+b)$$

$$Y(t) = a_1x^t + a_2tx^t + q/(1+a+b)$$

$$Y(t) = (c^2+d^2)^{t/2}(g \cos tB + h \sin tB) + q/(1+a+b)$$

Substituting the values of c, d, g, h and B as given in the above, therefore, according to the nature of the characteristic roots x_1 and x_2, the value of Y(t) can be obtained.

APPENDIX C

BASIC DATA
(Money Value in Millions of 1929 Dollars)

Year	Consumer Price Indexes	Gross National Product	Personal Consumption Expenditures	Gross Private Investment
1929	100.000	104,436	78,952	17,002
1930	97.487	93,453	72,797	11,238
1931	88.777	85,913	69,087	6,443
1932	79.732	73,326	61,840	1,357
1933	75.544	74,081	61,411	2,039
1934	78.057	83,240	66,482	4,249
1935	80.067	90,552	70,302	7,773
1936	80.905	102,271	77,394	10,272
1937	83.752	108,391	80,307	14,100
1938	82.245	103,626	78,596	9,447
1939	81.072	112,363	83,356	12,577
1940	81.742	123,092	87,936	17,940
1941	85.930	146,424	95,281	22,339
1946	113.903	184,949	129,153	29,021
1947	130.318	179,782	126,927	31,076
1948	140.369	184,617	127,032	33,181
1949	139.028	185,613	130,303	26,424
1950	140.369	202,750	138,929	36,017
1951	151.591	217,015	138,402	38,729
1952	154.941	223,955	141,844	33,032
1953	156.114	234,050	149,025	31,968
1954	156.784	231,600	151,817	31,785
1955	156.281	154,329	164,409	41,551
1956	158.626	264,257	170,159	44,316
1957	164.154	269,727	173,717	43,294
1958	168.677	263,548	173,822	34,320
1959	170.017	283,915	184,416	42,329
1960	172.697	291,030	190,062	43,286
1961	174.539	297,138	193,279	42,030
1962	176.549	315,039	202,071	47,083
1963	178.727	326,709	209,794	48,342
Column	(1)	(2)	(3)	(4)

BASIC DATA (Continued)

(Money Value in Millions of 1929 Dollars)

Year	Government Expenditures			Net National Product	Wage & Salary	Net Profit
	Total	Government Consumption	Government Investment			
1929	8,482	4,361	4,121	95,819	50,423	45,396
1930	9,418	5,009	4,409	84,692	47,378	37,314
1931	10,383	6,100	4,283	76,715	44,064	32,651
1932	10,131	6,702	3,429	63,777	38,224	25,553
1933	10,631	7,607	3,024	64,602	38,384	26,218
1934	12,509	8,477	4,032	74,129	43,180	30,949
1935	12,477	8,325	4,152	81,515	45,824	35,691
1936	14,605	8,224	6,381	93,007	51,814	41,193
1937	13,984	8,690	5,294	99,143	55,052	44,091
1938	15,583	9,835	5,748	94,163	55,254	38,909
1939	16,430	10,164	6,266	102,695	56,667	46,028
1940	17,216	11,418	5,798	113,124	60,945	52,179
1941	28,804	18,855	9,949	135,902	72,252	63,650
1946	26,775	20,878	5,897	175,561	98,212	77,349
1947	21,779	16,734	5,045	169,784	94,264	75,520
1948	24,604	19,012	5,592	173,795	96,303	77,492
1949	28,886	21,801	7,085	173,188	96,640	76,548
1950	27,804	20,044	7,760	189,169	104,273	84,896
1951	39,884	28,109	11,775	202,522	112,615	89,907
1952	49,079	34,250	14,829	208,461	119,308	89,153
1953	53,057	37,554	15,503	217,059	126,898	90,161
1954	47,998	33,776	14,222	213,225	125,178	88,047
1955	48,369	35,021	13,348	233,863	134,951	98,912
1956	49,782	36,280	13,502	242,563	143,504	99,059
1957	52,716	38,272	14,444	246,918	145,320	101,598
1958	55,406	39,338	16,068	240,662	142,167	98,495
1959	57,170	40,591	16,579	259,822	152,020	107,802
1960	57,682	40,954	16,728	266,135	157,101	109,034
1961	61,849	43,913	17,936	271,653	159,747	111,906
1962	65,885	46,778	19,107	287,434	168,301	119,133
1963	68,573	48,687	19,886	298,265	174,651	123,614
Column	(5)	(6)	(7)	(8)	(9)	(10)

Sources of Data. Except column (1), all data are originally in current dollars from U. S. Department of Commerce, Office of Business Economics, Survey of Current Business (Washington, D. C.: U. S. Government Printing Office, 1930-1964), and converted into 1929 dollars according to the "Consumer Price Index" given in column (1). An attempt has been made to get all revised figures so that a series of Survey of Current Business has been consulted. The following are some special remarks.

Column (1). The data are originally from U. S. Department of Commerce, Bureau of the Census, Statistical Abstract of the United States (Washington, D. C.: U. S. Government Printing Office, 1930-1964), on the base, 1957-1959=100. A series of Statistical Abstract of the United States has been consulted in order to get the revised figues. As shown in this column the data are already converted into 1929 dollars, i.e., 1929=100.

Column (2). This column is the sum of column (3), column (4) and column (5).

Column (5). This column is the sum of column (6) and column(7). For 1929-1957, the apportion of this column into column (6) and column (7) is according to the proportions of column (6) and column (10) of Table A-IIa of J. W. Kendrick's Productivity Trends in the United States (Princeton: Princeton University Press, 1961), pp. 293-295. For 1957-1963, the proportions are based on the average of 1929-1957. The average is approximately 29%.

Column (8). This is the sum of column (9) and column (10).

Column (10). This is obtained by substracting column (9) from (8).

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