HOMOLOGY AND HOMOTOPY OF INFINITE DIMENSIONAL MANIFOLDS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

The University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

Phillip Arthur Martens

January 1969

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ABSTRACT

This paper is concerned with a study of the structure of infinite dimensional manifolds, giving information about the homology and homotopy, and leading to the construction of a codimensional homology functor which distinguishes sets of finite codimension and which satisfies a Poincare duality with respect to the singular cohomology.

Attention is restricted to separable differentiable Hilbert manifolds which are Cauchy and geodesically complete and which support finite dimensional vector valued functions with associated thin singular sets so that these sets can be removed via diffeomorphisms between the manifolds and the complements of the thin subsets. This leads to representations for these manifolds as the inverse limit of finite dimensional manifolds which are the images of the given manifold under a vector valued function, with the structure of the inverse system being determined by a sequence of foliations and an associated sequence of q-parameter groups of diffeomorphisms. There is also a strong homotopy equivalence between the given infinite dimensional manifold and the direct limit of the above mentioned finite dimensional manifolds.

A homology functor $H_{\infty - p}(\cdot, Z)$ is then constructed by using the strong homotopy equivalence and the connecting

homomorphisms of Mayer-Vietoros exact sequences which arise from splittings of the finite dimensional manifolds used in the representations. The ensuing duality is independent of any representation.

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CHAPTER I

INTRODUCTION

The importance of infinite dimensional manifolds in the study of non-linear global analysis has led to a need and interest to determine more information about the homology and homotopy of these manifolds. In particular, there should be natural methods of constructing homology functors that reflect the structure of sets of finite codimension. J. W. Alexander [1] constructed an ∞-p homology in 1935. However, that has the disadvantage that it gives different groups for the same space, essentially depending upon the system of covers used to construct the theory.

K. Geba [12], K. Geba and A. Granas [13], J. Eells [9], and K. K. Mukherjea [17] have constructed and used ∞-p omology functors.

Geba constructed an ∞-p cohomotopy theory giving an Alexander-Pontrjagin type duality between a closed bounded subset of a Banach space and its complement. Granas and Geba gave an ∞-p cohomology functor again giving an Alexander-Pontrjagin type duality relating the same kinds of sets mentioned above.

Eells constructed an ∞-p homology which is canonically isomorphic to the singular cohomology for an open subset of a Hilbert space, assuming that the coefficients are a field or that the homology is finitely generated in each dimension. Mukherjea's theory is a similarly constructed ∞-p cohomology. These last two constructions and their corresponding Poincare dualities are constructed and given for a manifold and not with regard to a set and its complement. They both use a cutting technique which gives a strong homotopy equivalence between the specified manifold and a direct limit of finite dimensional manifolds. But, this surgery is not delicate enough to pick out sets of finite codimension in a fashion that would allow the elimination of the two conditions in Eells above mentioned result and give a complete geometrical determination of finite codimensional cycles.

The main problem is to represent the manifold in such a way as to give a natural procedure to pick out the sets of finite codimension. The obstruction to any such study is the fact that finite dimensional geometry simply doesn't cary over to the infinite dimensional case. The big difference is the fact that the removal of a reasonably thin subset from a well behaved infinite space leaves the space unchanged with respect to homeomorphism type. The trick is to utilize this difference and create a new infinite dimensional geometry; the two principle ingredients being the above mentioned fact and the use of a sequence of foliations.

The present work is concerned with a study of this problem for a class of Hilbert manifolds along with the consequent construction of an ∞-p homology functor $H_{\infty-p}(.,Z)$.

This paper is divided into five main chapters. The first four of these treat the construction of a representation theory for a manifold E of the given class in the form $E \approx \varprojlim_{m \geq p} E_m$, where $\approx$ denotes homeomorphism and E_m is an m dimensional orientable manifold. This is accomplished by a succession of foliations, each one being induced by a global p-parameter group of diffeomorphisms determined by a smooth function without singularities. Chapter II relates a manifold E to $E \setminus K$, for a suitable thin subset K. Chapter III gives the construction of a smooth function with singularities in a thin set K along with controlled associated solution curves.

Chapter IV treats the local p-parameter groups of diffeomorphisms. In Chapter V, these notions are unified to form a sequence of foliations which generates a required representation.

Then the results of Chapter V are used in Chapter VI to pick out sets of finite codimension in the construction of a homology functor, $H_{\infty-p}(.,Z)$, and a canonical duality with the singular cohomology.

CHAPTER II

GEODESICALLY COMPLETE MANIFOLDS AND A STUDY OF DIFFEOMORPHISMS BETWEEN A MANIFOLD E AND THE MANIFOLD $E\setminus K$, FOR A CERTAIN THIN SUBSET K

We establish in this chapter a simple form of the general principle that the removal of a suitable thin subset leaves an infinite dimensional manifold unchanged. This is for a closed locally compact subset of a smooth Hilbert manifold, with the help of a result of Bessaga [3], which gives a diffeomorphism between the separable Hilbert space H and $H\setminus\{o\}$, and then for a sequence of tubes.

<u>Definition 1</u>. The termonology essentially follows that of [9], [16], and [18]. A C^k Banach manifold E without boundary is described as follows: A C^k diffeomorphism of an open subset $U\subset E$, onto an open subset of a fixed Banach space B, $\psi : U \longrightarrow B$, is called a chart. Require two charts, $\psi : U \longrightarrow B$ and $\phi : V \longrightarrow B$, to be related by stipulating that $\psi\phi^{-1}: \phi(U\cap V) \longrightarrow \psi(U\cap V)$ be a C^k isomorphism. A C^k atlas for E is a collection of charts, parwise related as above, which cover E. Take a maximal atlas and denote it by A. The manifold E is then a paracompact Hausdorff space with a maximal chart A denoted simply by E.

<u>Definition 2</u>. This is due to Palais[18]. Fix an

indexed collection of Banach spaces $\{B_i\}_{i\in A}$ and isomorphisms (ϕ_{ij}) where $\phi_{ij}: B_j \longrightarrow B_i$. Then require ϕ_{ii} = identity and $\phi_{ij}\phi_{jk} = \phi_{ik}$. Then construct a Banach space B and canonical isomorphisms $\pi_i: B \longrightarrow B_i \ni \pi_i = \phi_{ij}\pi_j$. Actually $B \subset \prod_i B_i \ni \{b_i\} \in B$ satisfies the condition $b_i = \phi_{ij} b_j$. This is called amalgamation. π_i is an isomorphism since $\amalg_i$, defined by $\amalg_i(b)_j = \phi_{ji}(b)$, is a continuous linear two sided inverse.

Let $\{B'_k, \psi_{k\ell}\}$ be a second such collection with indexing set A'. For $(i,k) \in A \times A'$ specify a bounded linear map $T_{ki}: B_i \longrightarrow B'_k \ni \psi_{k\ell} T_{\ell i}\phi_{ij} = T_{kj}$. Then for the amalgamation B' $\exists$ a unique $T: B \longrightarrow B' \ni \pi_k T = T_{ki}\pi_i$ given by $T\{b_i\} = \{b'_k\}$ for $b'_k = T_{ki} b_i$. T is the amalgamation of T_{ki}.

For E a C^k manifold and $e \in E$, let A_e be the charts at e. Also let B_ϕ = the target of ϕ. Then $d(\psi\phi^{-1})_{\phi(e)}: V_\phi \longrightarrow V_\psi$ is an isomorphism, $d(\phi\phi^{-1})_{\phi(e)}$ = identity, and $d(\phi_3\phi_2^{-1})_{\phi_2(e)} d(\phi_2\phi_1^{-1})_{\phi_1(e)} = d(\phi_3\phi_1^{-1})_{\phi_1(e)}$. Therefore this determines an amalgamation E_e, the tangent space at e.

If F is a second such manifold and $f: E \longrightarrow F$ is C^k and A' the set of charts at f(e), then for $(\phi, \psi) \in A \times A'$, $\exists$ a linear map $d(\psi f \phi^{-1})_{\phi(e)}: B_\phi \longrightarrow B'_\psi$. This then determines an amalgamation map $df_e: E_e \longrightarrow F_{f(e)}$, which is the differential of f at e.

Define the tangent bundle by $T(E) = \bigcup_e E_e$ and define the bundle map $\pi: T(E) \longrightarrow E$ by $\pi(E_e) = e$. For a chart ϕ

with domain U and target B_ϕ define $\Phi : U \times B_\phi \longrightarrow \pi^{-1}(U)$ by requiring $b \longrightarrow \Phi(e,b)$ to be the natural isomorphism of B_ϕ and E_e. For $f : E \longrightarrow f$, define $df : T(E) \longrightarrow T(F)$ by $df | E_e = df_e$.

We consider chiefly real C^K separable Hilbert manifolds so that $B_\phi = H$ = real separable Hilbert space.

Definition 3. (Lang [16 , Ch. IV]) A spray over a Banach manifold E of class C^K is a second order differential equation over E, represented as a vector field η on the tangent bundle T(E) ∋ for the projection $\pi : T(E) \longrightarrow E$ we have $d\pi\,\eta(v) = v$, where $d\pi$ is given by the following:

$$\begin{array}{ccc} TT(E) & \xrightarrow{d\pi} & T(E) \\ \downarrow & & \downarrow \pi \\ T(E) & \xrightarrow{\pi} & E \end{array}$$

The vector field also satisfies additional properties to be described. Furthermore, assume $3 \leq k \leq \infty$.

Of course, to be a second order differential equation η must satisfy the condition that for each integral curve β of η [Ch. IV] we have $(\pi\beta)' = \beta$, where $(\pi\beta)'(a) = d(\pi\beta)_a(1)$, for $a \in R$ = reals. In addition for $v \in T(E)$, let β_v be the unique integral curve β of η with initial condition v, ($\beta_v(o) = v$). If $\Phi \subset T(E)$ is the set of vectors in T(E) ∋ β_v is defined at least on [o,1] , then Φ is open and $v \longrightarrow \beta_v(1)$ is a morphism $\Phi \longrightarrow T(E)$. Define $\exp : \Phi \longrightarrow E$ by the equality $\exp(v) = \pi\,\beta_v(1)$. exp is C^{k-2}.

Now for h to be a spray it must satisfy the following equivalences:

1. $a \in R$ is in the domain of $\beta_v \Longleftrightarrow 1$ is in the domain of β_{av} and $\pi\beta_v(a) = \pi\beta_{av}(1)$
2. $a, b \in R$. ab is in the domain of $\beta_v \Longleftrightarrow a$ is in the domain of β_{bv} and $\beta_{bv}(a) = \beta_v(ab)$.
3. $a \in R$ is in the domain of $\beta_{bv} \Longleftrightarrow ba$ is in the domain of β_v and $\beta_{bv}(a) = b\beta_v(ba)$.
4. $\forall\, a \in R$ and $v \in T(E)$, $h(av) = da(ah(v))$.

Since locally the sprays form a convex set a spray can be constructed if partitions of unity exist. If E is a C^k manifold for $3 \leq k \leq \infty$ and is also modeled on the separable Hilbert space then the results of Bonic and Frampton [4] assure the existence of C^k partitions of unity.

<u>Definition 4</u>.* Call a manifold geodesically complete if the following conditions hold:

1. $\Phi = T(E)$.
2. Every β_v can be extended so that $\beta_v : R \longrightarrow T(E)$. If $\beta_s(1) \in T(E)_{\bar{e}}$, then $T(E)_{\bar{e}} \cap \beta_{s'}(1) = \emptyset$, where $s \neq s'$, for $s, s' \in T(E)_e$; $\bar{e}, e \in E$. Also $\beta_v(R) \cap$ o-section $= \emptyset$, $v \neq$ o.

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*According to a result of Anderson and Schori 2 E H E. It seems possible to use this result and a notion of preferred paths to generate conditions like these topologically.

3. $\forall$ s,v $\in T(E)_e \exists$ a C^k map $g_{s,v}: R \longrightarrow R \ni \pi\beta_{s+av}(1) =$

$\pi\beta_{(a_0+a)v(e)}(g_{s,v}(a))$ where $v(e) \in \pi\beta_v(R)$ and

$$2a_0 \frac{\partial g_{s,v}(a_0)}{\partial a} + g_{s,v}(a_0) = 0.$$

Lemma 1. $\exp_e: T(E)_e \longrightarrow U_e \subset E$ is a diffeomorphism of at least class C^1.

Proof. By condition 2 of Definition 4 $\exp_e$ is defined for all $v \in T(E)_e$ and is also 1 - 1.

By the following inverse function theorem [9] it will suffice to show that $d(\exp)_e$ is a bijection:

Let E and F be Banach spaces, U an open subset of E and $\phi: U \longrightarrow F$ be a C^k map, $k \geq 1$. Then

1. If $x \in U$ is a point $\ni d\phi_x: E \longrightarrow F$ is injective and its image is a direct summand, then $\exists U_x$ in U $\ni$ $\phi | U_x \longrightarrow F$ is a split C^k embedding.

2. If $d\phi_x: E \longrightarrow F$ is surjective and its kernel is a direct summand, then $\exists$ a neighborhood $U_x \ni \phi | U_x \longrightarrow F$ is a split C^k projection.

3. If $d\phi_x$ is a bijection, then ϕ maps a neighborhood U_x of x, C^k diffeomorphically onto a neighborhood of $\phi(x)$.

To see that $d(\exp)_e$ is a bijection, it is only necessary to evaluate along rays of the form x + av. This is done as follows:

$$\frac{d}{da}\pi\beta_{s+av}(1)\Big|_{a_0} = \frac{d}{da}\pi\beta_{(a_0+a)v(e)}(g_{s,v}(a))\Big|_{a_0}$$

$$= d\pi_{B_{V(e)}(2a_0 g_{s,v}(a_0))} d\beta_{V(e)}(2a_0 g_{s,v}(a_0))(2a_0 \frac{\partial g_{s,v}(a_0)}{\partial a} + g_{s,v}(a_0))$$

$$= (2a_0 \frac{\partial g_{s,v}(a_0)}{\partial a} + g_{s,v}(a_0)) d\pi_{B_{V(e)}(2a_0 g_{s,v}(a_0))} \frac{d}{da} \beta_{V(e)}(2a_0 g_{s,v}(a_0))$$

$$= (2a_0 \frac{\partial g_{s,v}(a_0)}{\partial a} + g_{s,v}(a_0)) d\pi_{B_{V(e)}(2a_0 g_{s,v}(a_0))} \eta(\beta_{V(e)}(2a_0 g_{s,v}(a_0))$$

$$= (2a_0 \frac{\partial g_{s,v}(a_0)}{\partial a} + g_{s,v}(a_0)) \beta_{V(e)}(2a_0 g_{s,v}(a_0))$$

$\neq$ o by conditions 2 and 3 of Definition 4 and Definition 3 of the spray. Since this is true for all rays the bijection follows. Q.E.D.

We can now establish a result that shows that the removal of a thin subset leaves a manifold unchanged under suitable circumstances.

Theorem 1. Let E be a geodesically complete, separable C^k Hilbert manifold for $3 \leq k \leq \infty$. Then if $K \subset E$ is closed and locally compact $\exists$ $\tilde{K} \supset K$ and a map $f: E \setminus \tilde{K} \longrightarrow E$ $\ni$ f is a diffeomorphism, where $\tilde{K}$ is also closed.

Note. Since E is a C^k Hilbert manifold it has an induced Riemannian structure, which is a special case of a more general Finsler structure. Hence $\exists$ an induced metric ρ.

Proof. The proof is based upon the result of Bessaga [3] , which will be outlined because of its importance.

Proposition (Bessaga). If H denotes the separable infinite dimensional Hilbert space, then $\exists$ a diffeomorphism $h: H\setminus\{o\} \longrightarrow H$.

The result follows from the following propositions:

Proposition 1. $\exists$ an incomplete norm $w \ni w(x) \leq \|x\|$ where w is of class C^∞ on $H\setminus\{o\}$. Also $\exists$ a point $\tilde{x}$ in the completion of $(H,w) \ni \tilde{x} \notin (H, \|\cdot\|)$ and $\exists$ a function $(o,\infty) \xrightarrow{p} \{x \in H \mid w(x) \leq \frac{1}{2}\}$ which is of class $C^\infty \ni p(a) = o$ for $a \geq 1$, where $\lim_{a\to o} w(p(a) - \tilde{x}) = o$ and where $w(p'(a)) \leq \frac{1}{2}$ for $a > o$.

The proof is built upon the following argument:

Select an orthonormal basis $\{e_m\}$ in H. For any n select an infinitely (C^∞) differentiable monotone decreasing real valued function $\varphi_n \ni \begin{cases} \varphi_n(a) = o, & a \geq \frac{1}{2^{n-1}} \\ \varphi_n(a) = 1, & a \leq \frac{1}{2^n} \end{cases}$

Let $d = \max(2^{n+1}, 2\sup \varphi_n'(a))$ and let $p(a) = \sum_{n=1}^{\infty} \varphi_n(a)e_n$ and let w be defined by $w(x) = \left(\sum_{n=1}^{\infty} \frac{(x,e_n)^2}{d_n^2} + \|x - \sum_{n=1}^{\infty}(x,e_n)e_n\|^2\right)^{1/2}$.

Then a candidate for $\tilde{x}$ is $\tilde{x} = \sum_{n=1}^{\infty} e_n$. $\tilde{x}$ is Cauchy with respect to w, but $x \notin (H, \|\cdot\|)$.

Proposition II. $h_1(x) = p(w(x)) + x$ is a C^∞ isomorphism mapping $H\setminus\{o\} \longrightarrow H \ni h_1(x) = x$ for $w(x) \geq 1$.

The main point to the argument for this proposition is to fix a vector x and use the Banach contraction principle on the function $\phi: [o,\infty) \longrightarrow [o,\infty)$ defined by

$\phi(a) = w(x-p(a))$ for $a > o$ and $\phi(o) = w(x-\tilde{x})$. Proposition $1 \Longrightarrow \phi$ actually maps $[o,\infty) \xrightarrow{\text{onto}} [o,\infty)$. Also $|\phi(a) - \phi(b)| = |w(x-p(a)) - w(x-p(b))| \leq w(p(a)-p(b)) = w(\int_a^b p'(t)dt)$ $\int_a^b w(p'(t))dt \leq |a-b| \sup_{a\leq t\leq b} w(p'(t)) \leq \frac{1}{2}|a-b|$.

Therefore applying the contraction principle to $([o,\infty),\phi)$ one can conclude that $\exists$ a unique solution a for $\phi(a) = a$. This means that $\forall\ x \in H \ \exists$ a unique $a_x \geq o \ \ni$ $w(x-p(a_x)) = a_x$ and since $\tilde{x} \notin H$, then $a_x > o$. Consequently $\exists$ a 1 to 1 mapping $h_1 : H\setminus\{o\} \longrightarrow H$ with $h_1^{-1}(x) = x - p(a_x)$. This is easily checked since with $h_1(x) = x + p(w(x))$, $h_1(h_1^{-1}(x)) = x - p(a_x) + p(w(x-p(a_x))) = x - p(a_x) + p(a_x) = x$.

Proposition III. Let $w(\cdot)$ be a norm in H of class C^∞ on $H\setminus\{o\} \ni w(x) \leq \|x\|$. Then $\exists$ a C^∞ diffeomorphism $h_2 : H \xrightarrow{\text{onto}} H$ $\ni \{x \in H \mid \|x\| \leq 1\} \xrightarrow{\text{onto}} \{x \in H \mid w(x) \leq \frac{1}{2}\}$.

The map h_2 is $\begin{cases} h_2(o) = o \\ h_2(x) = (\lambda(\|x\|)\dfrac{\|x\|}{w(x)} + 1 - \lambda(x))x \end{cases}$

where λ is a monotone increasing real valued function of class $C^\infty \ni \begin{cases} \lambda(a) = o, \ a \leq \frac{1}{2} \\ \lambda(a) = 1, \ a \geq 1 \end{cases}$

The map h is then given by $h(x) = \frac{1}{2}h_2^{-1}(h_1^{-1}(h_2(2x)))$.

The fact that h_1 is a C^∞ diffeomorphism follows from the following: If $\psi(y,a) = a - w(y-p(a))$, then since for $y \in H$, $y - p(a_y) \neq o \Longrightarrow \psi$ is differentiable on a neighborhood of $(y_o, a_{y_o}) \subset H \times (o,\infty)$. Also $D_a\psi \geq 1 - \frac{1}{2} > o$.

Then the following implicit function theorem [7, Ch. 10] can be used to give the result: Let E,F,G be Banach spaces, f a differentiable C^∞ map of an open subset $A \subset E \times F \ni A \xrightarrow{\text{into}} G$. Let $(x_o, y_o) \in A \ni f(x_o, y_o) = o$ and with $D_2 f(x_o, y_o)\colon F \xrightarrow{\text{onto}} G$ a linear homeomorphism. Then $\exists$ an open U_o of x_o in $E \ni \forall$ connected neighborhood U of x_o, with $U \subset U_o$, $\exists$ a unique continuous mapping $\alpha : U \xrightarrow{\text{onto}} F$ $\ni \alpha(x_o) = y_o$ and $(x, \alpha(x)) \in A$ and $f(x, \alpha(x)) = o$ $\forall\, x \in U$. In addition α is C^∞ in U and its derivative is given by $d\alpha_x = -(D_2 f(x, \alpha(x)))^{-1} (D_1 f(x, \alpha(x)))$.

Here we let $E = H$, $F = G = (o, \infty)$. $\|D_2 \Psi(a_o, x_o)\| = \|D_a \Psi(a_o, x_o)\| > o$. This gives the desired conclusion.

Now the generalization to the Hilbert manifold comes by a reduction to the case where we can consider the closed locally compact subset to behave as if it were only a smooth finite dimensional closed submanifold. Then we can construct a tubular neighborhood about this set, which is actually a trivial neighborhood bundle, and give a diffeomorphism by defining the analogue of the function p at each fibre of the neighborhood bundle. Each point of K is pushed to a point in the completion of a new norm.

First assume that K satisfies the condition that if $k \in K$, $\exists$ an arbitrarily small chart about k which contributes only a minimal number, d_k, of coordinates to K in some suitable coordinatization of the chart.

For a set of this type we can choose a cover $(U_{k_i})_{i \in I}$ with the following properties, where disc means open ball:

1. (U_{k_i}) is star finite, with U_{k_i} a disc in a chart.
2. If D_{k_i} = the k_i dimensional disc spanned by d_{k_i} coordinates contributing to K and $\dot{D}_{k_i}$ = the boundary of D_{k_i}, then $(\dot{D}_{k_i})$ is a star finite collection.

For a point $k \in K$, select a disc U_k contained in some chart and contributing the minimal d_k coordinates to K. Then every point in $K \cap U_k$ has at most d_k coordinate contributions to K. Also $\dot{D}_k \cap K$ is compact and closed so we can cover this set by a finite collection of discs (U_{k_i}), where U_{k_i} contributes the minimal d_{k_i} coordinates to K. Now (point set boundary $\bigcup U_{k_i}) \cap K$ is compact. So then for this set select a finite cover $(U_{k_{i_1, i_2}})$ with the property that $U_{k_{i_1, i_2}}$ does not intersect U_k. Continue on inductively $\ni$ at any stage with $(U_{k_{i_1, i_2, \quad, i_n}})$, which is finite then $\overline{U}_{k_{i_1, i_2, \quad, i_n}}$ does not intersect any $\overline{U}_{k_{i_1, i_2, \quad, i_m}}$ for $m \leq n - 2$ and also $\dot{D}_{k_{i_1, i_2, \quad i_n}}$ only intersects a finite collection $(\dot{D}_{k_{i_1, i_2, \quad i_{n-1}}})$.

Now if $U_k \bigcup ((U_{k_{i_1, i_2, \quad i_n}}))_{n \geq 1}$ does not cover K then $K \setminus U_k \bigcup ((U_{k_{i_1, i_2, \quad i_n}}))_{n \geq 1}$ is closed. For if $\exists$ a sequence (k_a) from this set converging to some k' in $U_k \bigcup ((U_{k_{i_1, i_2, \quad i_n}}))_{n \geq 1}$

Then it converges to some $\bar{D}_{k_{i_1,i_2,\ \ i_n}}$ and hence a subsequence $\subset$ a $D_{k_{i_1,\ \ ,i_{n+1}}}$ by the above construction. Therefore $k' \in K \setminus U_k \bigcup ((U_{k_{i_1,i_2,\ \ i_n}}))_{n\geq 1}$.

This shows that $K \bigcap \left\{U_k \bigcup ((U_{k_{i_1,i_2,\ \ i_n}}))_{n\geq 1}\right\}$ is a component of K in view of the fact that $U_k \bigcup ((U_{k_{i_1,i_2,\ \ i_n}}))_{n\geq 1}$ was constructed to contain any limit point of the above intersection, which is then closed and open in K.

Now do the same thing for any point $k \in K$ and then obtain the cover $\bigcup_{k\in K} U_k \bigcup ((U_{k_{i_1,i_2,\ \ i_n}}))_{n\geq 1}$. Then select a denumerable subcover $\bigcup_{\alpha} U_{k^\alpha} \bigcup ((U_{k^\alpha_{i_1,i_2,\ \ i_n}}))_{n\geq 1}$. If a $\bar{D}_{k^\alpha_{i_1,i_2,\ \ i_n}}$ intersects some $\bar{D}_{k^{\alpha'}_{i_1,i_2,\ \ i_m}}$, then by the construction of the collection we need only take that collection which contains maximal components and we have the cover with the desired properties.

Now we generate trivial tubular neighborhoods about closed smooth finite dimensional submanifolds of E. Recall that if $M \subset E$ is a submanifold a tubular neighborhood of M in a vector bundle $\pi: B \longrightarrow M$ is an open neighborhood Z of the zero section of M in B and an isomorphism $g: Z \longrightarrow U$ of Z onto an open set in E containing M $\ni$ the following commutes:

$$\begin{tikzcd} & Z \arrow[dl, "\Gamma"'] \arrow[dr, "g"] & \\ M \arrow[rr, "i"] & & E \end{tikzcd} \qquad \Gamma = \text{section}$$

A vector bundle $B \longrightarrow M$ is compressible if given an open neighborhood Z of the zero section, then $\exists$ an isomorphism $g_1 : B \longrightarrow Z_o$ where Z_o is open in Z and contains the zero section $\ni$ the following commutes:

$$\begin{tikzcd} B \arrow[rr, "g_1"] \arrow[dr, "\pi"'] & & Z_o \arrow[dl, "\pi|Z_o"] \\ & M & \end{tikzcd}$$

Now Lang [16] constructs a tubular neighborhood by considering the exact sequence

$$o \longrightarrow T(M) \longrightarrow T(E) \longrightarrow N(M) \longrightarrow o$$

of vector bundles, for N(M) the normal bundle over M, and then by showing the existence of a set $V_M \subset U_M = \exp(\Phi \cap N(M)) = \exp(N(M))$ so that V_M is a vector bundle and serves as a tubular neighborhood. He also demonstrates the existence of a compression, if we use the fact that for M $\exists$ a C^k partition of unity [4], for a bundle with a structure of a Hilbert bundle. This takes the form of a diffeomorphic retraction for the fibre over $m \in M$ which is expressed as $r_m(v) = \dfrac{(\sum a_{m(i)} h_i(v))}{(1+ \|v\|^2)^{1/2}} v$ for (h_i) a suitable partition of unity and $a_{m(i)}$ denoting a real number so that all vectors in H_m of length $< a_{m(i)}$ are elements contained in the required open neighborhood Z of the zero section and where H_m is the fibre over m. Now since M is finite demensional $N(M)_m$ is diffeomorphic to H and hence N(M) inherits a restricted action of the general infinite separable linear group $GL(\infty)$

from the Hilbert bundle which we can then compress so that we can consider an induced action of $GL(\infty)$ on the tubular neighborhood of M by defining $gv = \exp_m r_m(g\bar{v})$ for $v = \exp_m r_m(\bar{v})$. This is simply a deformation of the action on each fibre in N(M). Therefore, we have a tubular neighborhood which we denote by $\mathcal{T}M$ with an induced action of $GL(\infty)$.

We can now apply Kuiper's result [15] that $GL(\infty)$ is contractable and hence a Hilbert bundle over a space dominated by a CW complex with a $GL(\infty)$ structure is contractable. Therefore, $\mathcal{T}M$ is a trivial tubular neighborhood bundle.

We are now in a position to apply the (Bessaga) result to construct a diffeomorphism $f: E\setminus M \longrightarrow E$ by constructing maps $h_{1,m}$, $h_{2,m}$, h_m, where $h_{1,m}(e) = e + p_m w_m(e)$ for w_m and p_m constructed in the fibres. The map $h_{2,m}$ is then given by
$$\begin{cases} h_{2,m}(o) = o & \text{and} \\ h_{2,m}(e) = \left(\lambda(\|e\|) \dfrac{e}{w_m(e)} + 1 - \lambda(e)\right) \end{cases}$$

$h_m(e) = \frac{1}{2} h_{2,m}^{-1}(h_{1,m}^{-1}(h_{2,m}(2e)))$ for $e \in H \setminus m$. Since this map is the identity outside the corresponding closed tube of radius I the map f is defined to be the identity on the complement of the tubular neighborhood.

Consider now the closed locally compact set to be covered as before by the star finite collection $(U_i)_{i\geq 1} \ni K \subset \bigcup D_{d_i}$ for D_{d_i} a d_i dimensional disc in U_i and $(\dot{D}_{d_i})_{i\geq 1}$

star finite. Define $\tilde{K} = \bigcup \bar{D}_{d_i}$ and $\dot{\tilde{K}} = \bigcup \dot{D}_{d_i}$. Then define the required diffeomorphism by first considering $\dot{\tilde{K}}$. Construct now a collection (f_j) of diffeomorphisms inductively as follows: Let $\dot{D}_{d_i} = \dot{\tilde{K}}_i$ and identify $\dot{D}_{d_i}$ with the M making sure that $\tau \dot{D}_{d_i} \subset \bigcup_{\text{Finite}} U_j$. Then start by setting down the diffeomorphism $f_1 : E \setminus A_1 \longrightarrow E$, where $A_1 = \dot{D}_{d_1}$. Assume that we have given $f_{n-1} : E \setminus A_{n-1} \longrightarrow E$, where

$A_{n-1} = f_{n-2} f_{n-3} \circ \quad \cdot f_1 (\dot{\tilde{K}}_{n-1} \setminus \bigcup_1^{n-2} \dot{\tilde{K}}_i)$. Then since all of the maps are assumed to be diffeomorphisms

$$f_{n-2} f_{n-3} \circ \quad \circ f_1 (\dot{\tilde{K}}_n \setminus \bigcup_1^{n-1} \dot{\tilde{K}}_i) \subset E \setminus A_{n-1}$$ is a closed set in the relative topology. Now the diffeomorphism f_{n-1} maps this set to a closed set. Therefore, define

$$A_n = f_{n-1} f_{n-2} \quad f_1 (\dot{\tilde{K}}_n \setminus \bigcup_1^{n-1} \dot{\tilde{K}}_i).$$

Now since all of the f_j, $j \leq n-1$ are diffeomorphisms, they map the relevent remaining portions of the trivial tubes $\tau \dot{D}_{d_i}$ diffeomorphically so that A_n is a closed set surrounded by a trivial tubular neighborhood. Then define $f_n : E \setminus A_n \longrightarrow E$ by mapping within the fibres as previously described. f_n is the identity outside some closed subtube, so that outside the tubular neighborhood we again define f_n to be the identity.

Then let $f = \prod_1^{\infty} f_i : E \setminus \dot{\tilde{K}} \longrightarrow E$. This is well defined and 1 to 1 because f is a composition of diffeomorphisms which are not the identity only on a star finite collection of tubular neighborhoods. It is also onto. For if $e \in E$ does not intersect any of the tubes then $f(e) = e$. If e intersects some finite collection of tubes then $\exists$ some $e' \in \bigcap_1^n \mathcal{T} \dot{D}_{d_i}$ $\ni e = \prod_1^n f_i(e') = \prod_1^\infty f_i(e')$.

Each $D_{d_i} \setminus \dot{\tilde{K}}$ is closed in $E \setminus \dot{\tilde{K}}$ since $D_{d_i} \setminus \dot{\tilde{K}} =$ $= \bar{D}_{d_i} \cap (E \setminus \dot{\tilde{K}})$. Therefore each $D_{d_i} \setminus \dot{\tilde{K}}$ is a closed finite dimensional submanifold considered to be $\subset E$, as the image of the map f. We can then construct a star finite collection of trivial tubes $\{\mathcal{T} f(D_{d_i} \setminus \dot{\tilde{K}})\}$. Now we can construct a map $\bar{f} : E \setminus f(\tilde{K} \setminus \dot{\tilde{K}}) \longrightarrow E$ in the same way f was constructed. Define $f| = f|E \setminus \dot{\tilde{K}} \setminus \tilde{K}$. Then $\bar{f} f| : E \setminus \tilde{K} \longrightarrow E$ is a diffeomorphism.

To handle the general closed locally compact subset, consider the closed set $K_c = \{ k \in K \mid$ k has a neighborhood chart arbitrarily small with an infinite number of coordinate contributions in K for every coordinatization.$\}$.

This set is closed, for if e is the limit of a sequence $\{k_i \mid k_i \in K_c\}$ then we can take an arbitrarily small chart U_e containing a subsequence of this, $\{k_{i_j}\} \ni \exists$ arbitrarily small neighborhood charts about each k_{i_j} which are contained in U_e with an infinite number of local coordinates contributing to K and hence forcing U_e to also contribute an infinite number of coordinate contributions to K.

Now K_c has the property that for $k \in K_c$ $\exists$ arbitrarily small neighborhood charts V_k contributing only a finite number of coordinate contributions to K_c for some coordinatization of V_k. For if V_k contributes an infinite number of coordinates to K_c and hence to K in every coordinatization, we could then select such a chart V_k containing a set $(k_i) \cup k$, for $k_i, k \in V_k$, constructed $\ni (k_i) \cup k \subset O \subset \overline{O} \subset V_k$ for an open $O, \ni \forall x \in V_k \setminus \{(k_i) \cup k\} \exists$ an open neighborhood $O_x \subset V_k \ni O_x \cap \{(k_i) \cup k\} = \emptyset$. We can assume that O is chosen so that $\overline{O} \cap K$ is compact, and therefore $k_i \longrightarrow k$, and hence the metric length of the coordinates contributing to K in a coordinatization of V_k also must converge to o. Hence every point of $K \cap V_k$ is contained in an arbitrarily small chart with a coordinate structure contributing only a finite number of coordinates to K. Therefore, these points cannot belong to K_c and hence the only contribution for K_c can come from only a finite number of coordinates.

We can apply the previous result to obtain a diffeomorphism $F_1 : E \setminus \widetilde{K}_c \longrightarrow E$, where $K_c \subset \widetilde{K}_c$. $K \setminus \widetilde{K}_c$ is closed in $E \setminus \widetilde{K}_c$ because $K \setminus \widetilde{K}_c = K \cap E \setminus \widetilde{K}_c$. Again since F_1 is a diffeomorphism, $F_1(K \setminus \widetilde{K}_c)$ is closed and satisfies the condition that $\exists$ arbitrarily small charts V_k contributing only a minimal number, d_k, of coordinates to $F_1(K \setminus K_c)$. Therefore $\exists$ a diffeomorphism $F_2 : E \setminus \widetilde{F_1(K \setminus \widetilde{K}_c)} \longrightarrow E$ for some set $\widetilde{F_1(K \setminus \widetilde{K}_c)} \supset F_1(K \setminus \widetilde{K}_c)$. Let $\widetilde{K} = K_c \cup F_1^{-1}\left(\widetilde{F_1(K \setminus \widetilde{K}_c)}\right)$. Then $F_2\, F_1| : E \setminus \widetilde{K} \longrightarrow E$ is a diffeomorphism where $F_1| = F|E \setminus \widetilde{K}$. This is true because F_1 and F_2 are both 1 to 1 and $e \in E$ is the image of the point $F_1^{-1}(F_2^{-1}(e))$, which is an element of $E \setminus \widetilde{K}$ since $F_2^{-1}(e) \in E \setminus \widetilde{F_1(K \setminus \widetilde{K}_c)}$.

Q.E.D.

Theorem 2. Add the additional hypothesis that E is Cauchy complete to those of Theorem 1. Then if $K = \bigcup_{i \geq 1} M_i$, for M_i a closed finite dimensional submanifold, $\exists$ a diffeomorphism $f : E \setminus K \longrightarrow E$.

Proof. Select a sequence of trivial tubes $(\mathcal{T}M_i)$ and the corresponding diffeomorphisms $f_i : E \setminus M_i \longrightarrow E$ so that $\mathcal{T}M_i \subset \left\{ e \mid \rho(M_i, e) < 1/2^{i+1} \right\}$ for the metric ρ of E and so that $\bigcap_{i \geq 1} \mathcal{T}M_i \subset \bigcup_{i \geq 1} M_i$ and so that $\bigcap_{i \in A} \mathcal{T}M_i \subset \bigcup_{i \in A} M_i$ for any infinite set $A \subset Z$.

Then let $f = \prod_{i \geq 1} f_i : E \setminus K \longrightarrow E$. f is 1 to 1 and C^{k-1} since $e \in E$ cannot be an element of more than a finite number, n_e, of tubes. Hence $f(e) = \prod_{1}^{n_e} f_i(e)$. Also this map is onto. For consider the sequence:

$$E \longrightarrow E \setminus M_1 \subset E \longrightarrow E \setminus M_2 \subset E \longrightarrow \cdots \longrightarrow \cdots$$

Then for $F_n = \prod_1^n f_i$ and any given $e \in E$, assume $\exists$ a sequence $(e_n) \ni e = F_i(e_1), \cdots, e_{n-1} = F_n(e_n), \cdots$. Now for $n > m \geqslant N$,

$$\rho(e_n, e_m) \leqslant \sum_0^{n-m-1} \rho(e_{m+i}, e_{m+i+1}) \leqslant \sum_m^{n-1} 1/2^{i+1} \leqslant \sum_m^{\infty} 1/2^{i+1} = 1/2^m$$

Therefore since E is Cauchy complete the sequence must converge to $\bar{e}$.

Now $\bar{e} \notin$ any M_i because $e_1 \notin M_1$ since e_1 is constructed by the action of the function $\bar{f}_1^{-1} : E \longrightarrow E \setminus M_1$ and $e_i \notin M_i$ since e_i is constructed by the action of the function $\bar{f}_i^{-1}\ i : E \setminus M_{i-1} \subset E \longrightarrow E \setminus M_i$, and because e_i is mapped perpendicular to $M_i \supset M_j \cap M_i$ by $\bar{f}_i$ for $j \leqslant i$. Hence the limit cannot be in any M_j for $j \leqslant i\ \forall\ i$. Therefore since $\bigcap^{\infty} M_i \subset \bigcup M_i$ for any infinite intersection, $\bar{e}$ can be contained in only finitely many tubes. Call this number q. Then $e = f(\bar{e}) = \prod_1^q f_i\ (\bar{e})$. If the sequence started out with any other $i > 1$ the same conclusion would result. Hence the map is onto.

Q.E.D.

CHAPTER III

C^k $\mathcal{E}$ APPROXIMATIONS WITH THIN SINGULARITIES

The next step in the plan of this work is to establish the existence of maps which start the foliation. These are to be smooth functions $f_p : E \longrightarrow R^p$ on a geodesically and Cauchy complete Hilbert manifold E. The techniques that are used result from an application of approximating techniques of Eells and MacAlpin [10].

<u>Definition 1</u>. If M and N are finite dimensional differentiable manifolds and $f : M \longrightarrow N$ is a C^k map, then f is a μ_n - Sard map if $k > \max(m-n, o)$, where $m =$ dim M and $n =$ dim N, and μ_n is Lebesque n - measure on N, where the term Sard map means $\mu_n(f(C_f)) = o$ for C_f the set of critical points of f.

If E and F are finite or infinite dimensional manifolds and $f : E \longrightarrow F$ is a smooth map and if $f(C_f)$ has no interior point in F call f a smooth Sard map.

The case exploited here is when E is a Cauchy complete, separable Hilbert manifold without boundary and $F = R^p$, $p = 1, \cdots, m, \cdots$ where R^p is the real Euclidean space of dimension p. The main proposition which must be reworked is the following: (Eells and MacAlpin) Let E be a smooth separable Hilbert manifold without boundary. Then the smooth Sard maps are dense in the fine topology on $C^o(E,R^p)$.

The following theorem is the desired generalized specialization of the above mentioned result.

Theorem 1. Let E be a C^k Cauchy complete separable Hilbert manifold without boundary and let $\psi: E \longrightarrow R^p$ be an open bounded continuous function. Then $\exists$ a smooth Sard ε approximation f to $\psi \ni$ the singular set of f is a closed locally compact subset of E.

Before the proof of this result can be given the necessary machinery for the construction of approximating functions must be established.

Proposition (Eells and MacAlpin). Let X be an open subset of a separable Hilbert space H. For each pair of disjoint closed subsets C_o, C_1 in X $\exists$ a μ_1- Sard function $\varphi: X \longrightarrow [o,1] \ni \varphi^{-1}(o) = C_o, \varphi^{-1}(1) = C_1$.

Since this proposition is of fundamental importance, the proof will be reviewed in the following sequence of lemmas.

Lemma (Eells and MacAlpin). Consider an open subset $X \subset H$. Then for any closed subset $C \subset X$ and a neighborhood V of C $\ni$ $V \subset X$, $\exists$ a countable collection $(U_i)_{i \geqslant 1}$ of open discs of H $\ni$ those with even (resp.odd) subscripts are in and cover V (resp. $X\setminus C$) and $\ni$ the centers of (U_i), $(a_i)_{i\geq 1}$ are linearly independent points in H.

<u>Proof</u> The proof is essentially an application of Lindelöf's theorem. For at each $x \in V$ select an open disc $U(x,r_x)$ of radius r_x which is contained in V. Then Lindelöf's theorem assures the existence of a countable sub cover of V by discs of the form $U(x,r_x/2)$. Similarly $\exists$ a countable cover of $X \setminus C$, by sets of the form $U'(x',r_{x'}/2)$.

Now $\forall$ $U(x_i,r_{x_i}/2)$ select discs $U_{2i} \ni U(x_i,r_{x_i}/2) \subset U_{2i} \subset U(x_i,r_{x_i})$. Select discs U_{2i+1} similarly from the discs $U'(x'_i,r_{x'_i})$. Require also that the centers $(a_i)_{i\geq 1}$ form a linearly independent set.

Now for $A \subset X$ and $r > o$, let $(A,r) = \{x \in X \mid \rho(x,A) < r\}$, where ρ denotes metric. Also let $A^c = x \setminus A$. Then the following lemma is standard [16].

<u>Lemma</u> (<u>Lang</u>) Let $V_1 = U_1$ and for $k \geq 1$ define $V_{2k+1} = U_{2k+1} \cap (U_1^c,1/k) \cap \cdots \cap (U_{2k-1}^c,1/k)$. Similarly let $V_2 = U_2$ and $V_{2k} = U_{2k} \cap (U_2^c,1/k) \cap \cdots \cap (U_{2k-2},1/k)$. In the first case the k come from the cover of $X \setminus C$ and the sets represent an open locally finite refinement of $(U_{2i+1})_{i\geq 1}$. In the second case the k come from the cover of V and the sets represent a locally finite refinement of $(U_{2i})_{i\geq 1}$.

The following is the construction of the function φ according to (Eells and MacAlpin).

Let $C = C_i$ and $V = X \setminus C_o$. Then smooth functions $f_{ij}: X \longrightarrow R$, $j \leqslant i$ and $(-1)^j = (-1)^i$, are constructed by composing smooth functions $r_{ij}: R \longrightarrow R$ with the norm $|x-a_j|$.

$$\begin{cases} f_{ii} > o \text{ on } U_i \\ f_{ii} = o \text{ for } X \setminus U_i \end{cases} \qquad \text{For } j < i \begin{cases} f_{ij} = 1 \text{ outside } U_j = U(a_j, r_j) \\ f_{ij} = o \text{ on } U(a_j, r_j - 1/i) \\ o < f_{ij}(x) < 1 \text{ between} \end{cases}$$

The functions r_{ij} are built upon functions of the form $\int_a^{b} e^{\frac{-1}{b-t}}\, e^{\frac{-1}{t-a}}\, dt$. Using the gradient of the Hilbert structure for H, $\nabla f_{ij}(x) = \alpha_{ij}(|x-a_j|)\,(x-a_j)$, where α_{ij} is a suitable smooth function $\ni \alpha_{ii}(t) = o$ only if $t =$ or or $t \geqslant r_i$.

Then define $f_i(x) = \prod f_{ij}(x)$. It is evident from the construction that $\begin{cases} f_i > o \text{ on } V_i \\ f_i = o \text{ on } X \setminus U_i \end{cases}$ $\nabla f_i(x) = \sum_j \beta_{ij}(x)\,(x-a_j)$.

Now $\nabla f_i(x) = \sum \beta_{ij}(x)\,(x-a_j)$, where β_{ij} are smooth real valued functions $\ni$ $\beta_{ii}(x) = \alpha_{ii}(|x-a_i|) \prod_{j<i} f_{ij}(x) = o$ only if $x = a_i$ or $x \notin V_i$. Then define $f'(x) = \sum f_{2i}(x)$ and $f''(x) = \sum f_{2+1}(x)$. These are locally finite sums and

$$\begin{cases} f' > o \text{ on } V \\ f' = o \text{ on } C_o \end{cases} \quad \text{and} \begin{cases} f'' > o \text{ on } X \setminus C_1 \\ f'' = o \text{ on } C_1 \end{cases}$$

Then define $\varphi : X \longrightarrow [o,1]$ by $\varphi(x) = f'(x)/(f'(x) + f''(x))$. From the definition $\varphi^{-1}(o) = C_o$ and $\varphi^{-1}(1) = C_1$. Also $\nabla\varphi = (f''\nabla f' - f'\nabla f'')/(f' + f'')^2 = \sum K_j \nabla f_j$ for smooth functions K_j.

Determine a countable open covering $(W_p)_{p\geq 1}$ of $X \setminus C_o \cup C_1 = \varphi^{-1}(o,1) \ni$ each W_p meets only finitely many V_j. If $x \in W_p$ is a critical point of φ, then $o = \sum K_j(x) \quad f_j(x) \ni$ the sum is taken over the index set $\{ j | V_j \cap W_p \neq \emptyset \}$, $o = \sum_j K_j(x) \nabla f_j(x) = \sum_{j,k} K_j(x) \beta_{jk}(x) \; (x-a_k)$ where $K_j \neq o$ in W_p. Since $\beta_{jj}(x) = o$ only $x = a_j$ or $x \notin V_j \Longrightarrow$ either $x = a_j$ or $\sum \eta_k(x-a_k) = (\sum \eta_k)x - \sum \eta_k \; a_k = o$. But because $(a_i)_{i\geq 1}$ is linear independent $\sum \eta_k \neq o$. Therefore, x is contained in the linear span $\{ a_k | \exists \, j \geq k \ni V_j \cap W_p \neq \emptyset \}$. If M_p denotes W_p intersected with that span, then M_p is finite dimensional and the finite dimensional form of the Morse-Sard theorem applies. But the important observation for this work is that this set of critical points lies locally in a finite dimensional subset of X. This happens in such a way that the functions f_{ij}, and $\prod f_{ij}$ and hence the locally finite sum $\sum \prod f_{ij}$ along with f are transverse to a smooth path within a finite dimensional span.

The puspose of the functions of the type constructed above is to generate controlled partitions of unity. Bonic and Frampton [4] have shown that if E is a C^k Hilbert manifold for $k \leq \infty$, then E admits C^k partitions of unity. In fact J. Eells was the first to construct C^∞ partitions of unity for a Hilbert manifold. The construction of (Bonic and Frampton) is based upon the following scheme:

Lev V be an open subset of E and $x \in V$. Then use one of the above φ to obtain a function $\varphi : E \longrightarrow R \ni \varphi \geq o$, $\varphi(x) > o$ and $\{x \mid \varphi(x) > o\} \subset V$, where V is the union of such sets.

Consequently if (U_x) is an open covering of $E \exists$ a refinement of (U_x) consisting of sets of the form $\{x \mid \varphi(x) > o \ni \varphi$ is a function of the type described above.$\}$. Since E is separable and hence Lindelöf, $\exists$ a countable subcover (W_j) of this refinement with the representation $W_j = \{x \mid \varphi_j(x) > o, \varphi j \geq o.\}$. Then let $V_1 = \{x \mid \varphi_1(x) > o\}$ and $V_{r+1} = \{x \mid \varphi_{r+1}(x) > o, \ \varphi_1(x) < 1/r, \ \ , \varphi_r(x) < 1/r\}$.
$\forall x \in E \exists$ an integer $n_x \ni \varphi_{n_x}(x) > o$ and for $j < n_x$, $\varphi_j(x) > o$. Then $x \in V_{n_x}$ and consequently (V_n) is a cover of E. Choose $m_x \ni o < m_x < \varphi_{n_x}(x)$ and define $V_x = \{x \mid \varphi_{n_x}(x) > m_x\}$. Now V_x is a neighborhood of $x \ni V_x \cap V_n = \emptyset$ for large n. Select $g_{r+1} \in C^\infty(R^{r+1}, R) \ni g_{r+1}(t_1, \ \ , t_r+1) > o \iff t_i < 1/r$, $i = 1, \ \ , r$ and $t_{r+1} > o$ and also in this neighborhood g_{r+1} has no singularities. Then define $\bar{g}_{r+1} = g_{r+1} \circ (\varphi_1 \times \cdots \times \varphi_{r+1}) \circ$ diag. This is C^k and $\{x \mid \bar{g}_{r+1} > o\} =$

V_{r+1}. Also $\forall x$, all but a finite number of $\bar{g}_j$ vanish on V_x. Then we have the following formula:

$$d\bar{g}_{r+1} = \frac{\sum \partial g_{r+1}}{\partial \varphi_i} d\varphi_i \text{ and using the gradient}$$

$$\nabla \bar{g}_{r+1} = \sum \frac{\partial g_{r+1}}{\partial \varphi_i} \nabla \varphi_i .$$

As before this gives a transversal direction for the gradient structure in such a way so that the critical points are locally within a finite demensional span. Then the final partition of unity is constructed by defining $\bar{g} = \sum_{i \geq 1} \bar{g}_i$ which is a locally finite sum. Also for all x $\bar{g} \mid V_x \in C^k(V_x, R^+)$. Therefore $\bar{g} \in C^k(E, R^+)$ and since inversion in R^+ is a C^∞ map, $\bar{g}^{-1}$ $C^K(E,R^+)$. Therefore, define the collection $(h_j)_{j \geq 1}$ by $h_j = \bar{g}^{-1} \bar{g}_j$. Again, the h_j are maps whose singularities are locally in a finite dimensional span since it is a quotient of maps which satisfy that property. This collection of maps is the required partition of unity.

Proof of Theorem 1. First let X be an open subset of H and consider $\psi | X : X \longrightarrow R^p$. Then for any continuous $\varepsilon \ni \varepsilon : X \longrightarrow R^+$ we make the suitable approximation .

Let $(U_i)_{i \geq 1}$ be a countable cover of X by open discs in

X with centers $(a_i)_{i\geq 1}$ $\ni$ $|\psi(x)-\psi(a_i)| < \varepsilon'(x), \forall x \in U_i$, where $\varepsilon'(x) = \varepsilon(x)/2$. Take a locally finite scalloped refinement, which is simply the refinement of a cover described in the previous lemmas and denoted by $(V_j)_{j\geq 1}$. Then $\exists$ a corresponding partition of unity $(h_j)_{j\geq 1}$. Let $b_j = \psi(a_j)$. Since ψ is an open map we can consider $r_j = \text{dist}(b_j, \text{bdy}\ (U_j))$. Then $r_i \leq |\psi(x)-b_i|\ \forall\ x \in U_i$. Then take a collection of vectors (v_j^i) for all j and $1 \leq i \leq p \ni |v_j^i| < r_j$ and $(b_j+v_j^i)_{1\leq i\leq p}$ are linearly independent vectors and orthogonal with respect to a fixed orthogonal basis for R^p. Since ψ is bounded we can consider a vector $r \in R^p \ni \psi + r$ is bounded away from $o \in R^p$, so that if we assume $\psi' = \psi + r$ is the function we are dealing with then we can assume that $b_j \neq b_k$ and $(\sum b_j+v_j^i)_{1\leq i\leq p}$ is linearly independent for any finite collection of j.

Define $f = \sum_{i=1}^{p}\sum_{j} (b_j+v_j^i)\ h_{j/p}$. Then as before take a countable open cover $(W_p)_{p\geq 1}$ of $X \ni W_p$ meets only finitely many V_i. Then for $x \in W_p$, $df_x = \sum_{i=1}^{p}\sum_{j} (b_j+v_j^i)(dh_j)_x/p$. This then is a finite sum of linearly independent vectors so that the critical set is contained in the union of the critical sets of the h_i, which are locally contained in finite dimensional spans. Hence the critical set for f is locally compact.

Now to see that f is a approximation consider the following:

$$|f(x)-\psi(x)| = \left|\sum_{i=1}^{p}\sum_{j}((b_j+v_j^i)-\psi(x))h_j(x)/p\right|$$

$$\leq \sum_{1}^{p}\sum_{j}|((b_j+v_j^i)-\psi(x))|h_j(x)/p$$

$$\leq 2\varepsilon'(x)\sum_{i=1}^{p}\sum_{j}h_j(x)/p = 2\varepsilon'(x) = \varepsilon(x).$$

Since we assumed that we were approximating for $\psi + r$, then simply take f-r for the approximation.

The extension to the entire manifold E follows (Eells and MacAlpin). Using the smooth Riemannian structure with metric ρ let $V_i \subset \overline{V}_i \subset U_i$ $(i \geq 1)$ be locally finite open coverings of E by charts.

Then by induction choose a continuous $\varepsilon_1 : U_1 \longrightarrow R^+$ with $\varepsilon_1(x) < \min(\varepsilon(x)/2, \rho(x, \mathrm{bdy}U_1))$. Then take a Sard ε_1 approximation to ψ on U_1, call it $f_1 : U_1 \longrightarrow R^p$. Here $|\psi(x)-f_1(x)| \longrightarrow 0$ as $x \longrightarrow \mathrm{bdy}\ U_1 \Longrightarrow f_1$ can be extended to a continuous function by defining $f_1 = \psi$ on $E \setminus U_1$.

Now assume ε_i and f_i are defined and let $W_{i+1} = U_{i+1} \cap \overline{V}_1^c \cap \cdots \cap \overline{V}_i^c$ and let $\varepsilon_{i+1} : W_{i+1} \dashrightarrow R^+$ be a map $\ni \varepsilon_{i+1}(x) < \min(\varepsilon(x)/2^{i+1}, \rho(x, \mathrm{bdy}W_{i+1}))$ $\forall\ x \in W_{i+1}$. Then choose a Sard ε_{i+1} approximation to f_i on

W_{i+1}, call it $f_{i+1}: W_{i+1} \longrightarrow R^p$. Then extend f_{i+1} by letting $f_{i+1} = \psi$ on $E \setminus W_{i+1}$. Define $f(x) = \lim f_i(x) \; \forall x \in E$. $f: E \longrightarrow R^p$ is a Sard map $\ni f|V_i = f_j$ for $i \leq j$. Also $|\psi(x) - f(x)| \leq \sum_{i \geq 1} \varepsilon(x)/2^i = \varepsilon(x)$.

Now since $f|V_i = f_j$ for $i \leq j$, we are assured that the critical set is locally compact. This follows since the only places that the critical set is possibly not finite dimensional is at bdyV_i for some i, but since $f_j|$ is constructed to converge to f_i for $j \geq i$ on bdyV_i then any sequence of critical values within a suitably small chart intersecting bdyV_i must contain a convergent subsequence. Therefore since the critical set is closed by definition the result follows.

Q.E.D.

Remark From this point on, the notion of coordinate projection will refer to a map $f: \longrightarrow R^p$ and p smooth homeomorphisms of R^1, denoted by $(1_i)_{1 \leq i \leq p} \ni 1_i \subset f(E)$ and projections $(p_i)_{1 \leq i \leq p}$, for $p_i: f(E) \longrightarrow 1_i$, where 1_i is the i coordinate space.

It will also be assumed without mention that all the appropriate open functions $f: E \longrightarrow R^p$ satisfy the condition that for a suitably small open neighborhood $O \subset f(E)$, then $(f^{-1}(O), \phi)$ is a connected chart, for a map ϕ.

Definition 2. Call a continuous bounded open function $f: E \longrightarrow R^p$ directionally transverse if the following is satisfied: Consider the collection $(p_i f)_{1 \le i \le p}$, where p_i is the retraction onto R^1_i = the i coordinate projection. Assume for convention that $p_i f(E) = (o,d)$ for all E and also require that for $c < d$, $(p_i f)^{-1}(o,c]$ is bounded.

As an example consider $\bar{e} \in E$ and the function $\psi_{\bar{e}}(e) = \rho(\bar{e},e)$ for the metric ρ. Construct a diffeomorphism $g: E \longrightarrow E \setminus \bar{e}$ which is not the identity only in an arbitrarily small neighborhood of $\bar{e}$. If we assume that this metric function is not bounded above, then using a suitable homeomorphism $r: R^+ \longrightarrow (o,d)$ for $d < \infty$, $\exists$ a composition $r \psi_{\bar{e}}\, g: E \longrightarrow E \setminus \bar{e} \longrightarrow R^+ \longrightarrow (o,d)$ giving a bounded open function so that $(r \psi_{\bar{e}}\, g)^{-1}(o,c]$ is bounded for $c < d$. Now $\mathcal{E}$ approximate as in Theorem 1 for a differentiable $\mathcal{E}$ which satisfies the property that $\mathcal{E}(e) <$ sup($|b-d|$, $|b|$), where $b = r \psi_{\bar{e}}\, g(e)$ and $\mathcal{E}(e) \longrightarrow o$ as $b \longrightarrow o$ or $b \longrightarrow d$. This approximation f is open, bounded above and below, and satisfies the condition that $f^{-1}(o,c]$ for $c < d$ is closed and bounded. From now on we make the assumption that the locally finite open covers $V_i \subset \bar{V}_i \subset U_i$ which are used in the peacing together process of Theorem 1 are constructed by taking two such star finite open covers for $r \psi_{\bar{e}}\, g(E)$ and then taking the inverse images, thus actually getting two star finite covers.

The notion of a maximum solution curve will be recalled in Chapter IV. For now consider this to be a smooth curve $\sigma_x(t)$, where $\sigma_x(o) = x$ and satisfying the property that $f(\sigma_x(t))$ is monotone increasing with the parameter $t \in R^1$. If the above approximation were constructed for a directionally transverse function $f: E \longrightarrow R^p$, then we consider the collection $p_i f(\sigma_x(t))$.

<u>Theorem 2</u>. Let $f: E \longrightarrow R^p$ be a directionally transverse approximation for E satisfying the conditions of Theorem 1. Then if $p_i f(\sigma_x(t)) \uparrow c < d$ as t increases or $p_i f(\sigma_x(t)) \downarrow b > o$ as t decreases, $\exists\ \bar{a} \in K \subset E$ and $\exists\ t_o \in R^1 \ni \sigma_x(t_o) = \bar{a}$, where K is defined below.

<u>Proof</u> Assume $p_i f(\sigma_x(t)) \uparrow c < d$ as t increases. For convenience we use the notation $\bar{f} = p_i f$. Then let $i = \inf(j \mid \bar{f}^{-1}(c) \in \bar{V}_j)$. Since the covers are star finite $\bar{f}^{-1}(c) \in$ only $\bar{V}_j$ for $j \in \pi_i =$ a finite set. So we can choose a sequence $(e_n)_{n\geq 1} \subset E \cap U_i \cap V_{j_o}$ for $j_o \in \pi_i$. Then we can choose an $\mathcal{E}_{i,j_o}$ so that it is differentiable in $W_{j_o} \cap U_i$ and if $x_n =$ coordinate representation of e_n in U_i, then $\nabla \mathcal{E}_{i,j_o}(x_n) \longrightarrow o$ as $n \longrightarrow \infty$ because $\mathcal{E}_{i,j_o}(x) \longrightarrow o$ as $x \longrightarrow \mathrm{bdy} W_{j_o} \cap U_i$ and for the sake of continuity $\mathcal{E}_{i,j_o}$ is smoothed out to be equal to zero on the set $E \setminus W_{j_o}$ and

differentiable everywhere, and so that finally $\overline{f}_{j_o} = \overline{f}_i + \mathcal{E}_{i,j_o}$. Now $\overline{f}_{j_o} = \overline{f} \mid V_{j_o}$ and $d\overline{f}| = \nabla\overline{f}| = \nabla\overline{f}_i + \nabla\mathcal{E}_{i,j_o}$ takes the form $((\sum \eta_i(x_n))x_n - \sum\eta_i(x_n)a_i^n + \nabla\mathcal{E}_{i,j_o}(x_n))$ because $d\overline{f}_i = \sum b_k\, dh_k$, which is a locally finite sum and each dh_k has that representation. This then generates a sequence which converges to o as $n \longrightarrow \infty$. There is a slight ambiguity in referring to convergence to o, for this is understood to be the zero section of the tangent bundle or if the norm is introduced then $o \in R^1$.

Now let K = those points of U_i contained in the locally finite dimensional span which determine the critical points of $\overline{f}_i$. This is a locally compact connected set and if $\overline{f}_i\ (U_i) = (b_i, d_i)$, then for $b_i < b < c < d_i$, $\overline{f}_i^{-1}[b,c]$ is closed and bounded and then $\overline{f}_i^{-1}[b,c] \cap K$ is a bounded closed subset of a locally compact, connected, complete, ANR, hence compact. Now since $(\sum\eta_i(x_n)a_i^n) \subset \overline{f}_i^{-1}[b,c] \cap K$ there must be a convergent subsequence converging to a. Now $\sum\eta_i(x_n) \neq o$ unless x_n is contained in span(a_i^n) and this is what we assume doesn't happen. For then $x_n \in K$. In that case since $(\mathcal{E}_{i,j_o}(x_n)) \longrightarrow o$, then $((\sum \eta_i(x_n))x_n) \longrightarrow -a$. If $\lim_n \sum \eta_i(x_n) = o$, then $-a = o \in U_i \cap H$. Let $\overline{a} =$

$$\begin{cases} -a, \text{ if } \lim \quad {}_i(x_n) = o \\ -a/\lim \quad {}_i(x_n) \text{ otherwise} \end{cases}$$

Hence $x_n \longrightarrow \bar{a} \in K$. Now by continuity and the monotonicity of $\bar{f}(\sigma_x(t))$, and by the fact that a maximum solution curve is defined for every point and any curve which is a restriction of the maximum can be extended through this maximum curve, if we understand that $x_n = \sigma_x(t_n)$, then $\sigma_x(t_n) \longrightarrow \bar{a}$ and $\bar{f}(\sigma_x(t_n))$ must converge to c and attain the value c for some value t_o, i.e., $\bar{f}(\bar{a}) = \bar{f}(\sigma_x(t_o))$.

The case where $\bar{f}(\sigma_x(t)) \downarrow b > o$ as t decreases is true by the same argument.

Q.E.D.

CHAPTER IV

DIFFERENTIABLE FUNCTIONS WITHOUT SINGULARITIES AND LOCAL p- PARAMETER GROUPS OF DIFFEOMORPHISMS

We formulate here the principle that if a differentiable map $f:E \longrightarrow R^p$ is given without singularities in such a way so that the maximum solution curves are defined on all levels, then a local p- parameter group of diffeomorphisms is generated so that for $r, \bar{r} \in f(E)$, $f^{-1}(r) \approx f^{-1}(\bar{r})$, where $\approx$ denotes diffeomorphism. As before $3 \leq k \leq \infty$. This process is important in generating a foliation technique.

We refer to Lang [16] and Palais [18] and [20] for the details concerning the material in the following definitions.

<u>Definition 1</u>. A C^{k-1}-vector field X on a C^k manifold E is a C^{k-1} cross section of the tangent bundle T(E) i.e., $X:E \longrightarrow T(E) \ni \pi X =$ identity, where π is the projection of the bundle. A solution curve of X is a C^1 map $\sigma_e:(b,c) \longrightarrow E \ni \sigma' = X\sigma$. If $o \in (b,c)$, we call $\sigma_e(o) = e$ the initial condition. $\forall\ e \in E$, $\exists$ a solution curve σ_e of X so that every solution curve of X with initial condition e is a restriction of σ_e. This is called a maximum solu-

tion curve.

There are functions $t^+ : E \longrightarrow (0,\infty]$, $t^- : E \longrightarrow [-\infty, 0)$ defined by requiring that the domain of σ_e is $(t^-(e), t^+(e))$. These are the positive and negative escape time functionals for X.

If $t^{-1}(e) < s < t^+(e)$ and $\bar{e} = \sigma_e(s)$, then $\sigma_{\bar{e}} = \sigma_e \tau_s$, where $\tau_s : R \longrightarrow R$ is defined by requiring that $\tau_s(t) = s + t$, and also $t^+(\bar{e}) = t^+(e) - s$ and $t^-(\bar{e}) = t^-(e) - s$. t^+ is upper semi-continuous and t^- is lower semi-continuous.

Let $D = D(X) = \{(e,t) \in E \times R \mid t^-(e) < t < t^+(e)\}$ and $\forall t \in R$ let $D_t = D_t(X) = \{e \in E \mid (e,t) \in D\}$. Define $\varphi : D \longrightarrow E$ by $\varphi(e,t) = \sigma_e(t)$ and let $\varphi_t : D_t \longrightarrow E$ be defined by $\varphi_t(e) = \sigma_e(t)$. The index set (φ_t) is called the maximum local one parameter group generated by X. $D \subset E \times R$ is open and $\varphi : D \longrightarrow E$ is a C^k map. $\forall t \in R$, D_t is open in E and φ_t is a C^k- isomorphism of D_t onto D_{-t} with φ_{-t} as its inverse. If $e \in D_t$ and $\varphi_t(e) \in D_s$, then $e \in D_{t+s}$ and $\varphi_{t+s}(e) = \varphi_s(\varphi_t(e))$.

<u>Definition 2</u>. A C^{k-1}-vector field on a C^k manifold E without boundary is called strongly transverse to a C^k function $f : E \longrightarrow R$ on a closed interval $[b,c]$ if for some $\delta > 0$ the following conditions hold for $V = f^{-1}(b-\delta, c+\delta)$:

1. Xf is of class C^{k-1} and $\neq 0$ on V.

2. If $e \in V$ and σ_e is the maximum solution curve of X with initial condition e, then $\sigma_e(t)$ is defined and not in V for some $t > o$ and also for some $t < o$.

<u>Definition 3</u>. Let E be a C^k manifold and $f: E \longrightarrow R$ a C^k function. Then $X \in T(E)_e$ is called a pseudo-gradient vector for f at e if :

1. $\|X\| \leq 2\|df_e\|$
2. $Xf = df_e(X) \geq \|df_e\|^2$

A vector field X is a pseudo-gradient vector field if X_e is a pseudo-gradient vector. A pseudo-gradient vector field can easily be constructed as follows (Palais):

First a pseudo-gradient vector X_e is constructed at a point e. Then it is extended to a local constant vector field in a neighborhood U_e of e. Then let $O = \{\bar{e} \in U_e \mid X_{\bar{e}}f > \|df_{\bar{e}}\|^2$ and $\|X_{\bar{e}}\| \leq 2\|df_{\bar{e}}\|\}$. Since all of these functions are continuous in U_e, O is open. Now $\forall e \in E$, define such a neighborhood $O_e \ni$ there is a pseudo-gradient vector field X^e for f in O_e. Then take a C^k partition of unity $(v_b)_{b \in B}$ for $E \ni \forall b \in B, \exists e(b) \in E$ with support of $v_b \subset O_{e(b)}$. Then let $X = \sum_{b \in B} v_b X^{e(b)}$. This is a C^{k-1} vector, and since the pseudo-gradient vectors at $T(E)_e$ are convex , X is a pseudo-gradient vector field.

The function $g(t) = f(\sigma(t))$ is a strictly monotone increasing function. This follows because $g'(t) = df_{\sigma(t)}(\sigma'(t))$ $= df_{\sigma(t)}(X_{\sigma(t)}) \geq \| df_{\sigma(t)} \|^2 > o$.

Lemma 1. Let E be a connected, geodesically and Cauchy complete separable C^k Hilbert manifold, and let $f: E \longrightarrow R$ be a C^k bounded function with no singularities so that $f(\sigma(t)): R \longrightarrow f(E)$. Then if $f(E) = (b,d)$ and if $c \in (b,d)$ and $W = f^{-1}(c)$, W is a closed submanifold and $\exists$ a C^k- isomorphism $F: W \times (b,d) \longrightarrow E \ni \forall \bar{c} \in (b,d)$ the map $e \longrightarrow F(e,\bar{c})$ is a C^k isomorphism of $W \longrightarrow f^{-1}(\bar{c})$ which is the identity fof $c = \bar{c}$.

Proof. Let X denote the pseudo-gradient vector field for f, which in this case is really the traditional gradient field. Since $f(\sigma(t))$ maps onto $f(E)$, X is strongly transverse to f and $Xf > \|df\|^2 > o$. Also since $t \longrightarrow 1/t$ is C^∞ for $t \neq o$, we can define $Y = X/Xf$, which is a pseudo-gradient vector field. Then let (φ_t) be the maximum local one parameter group generated by Y. $t \longrightarrow \varphi_t(e)$ is a maximum integral curve of Y with initial condition e. Since $\frac{d}{dt} f(\varphi_t(e)) = Yf = xf/xf = 1$, we have $f(\varphi_t(e)) = f(e) + t$. Since $f(\sigma(t))$ is strictly monotone increasing, $\exists$ a unique t_o for $c \in (b,d)$ so that $f(\sigma(t_o)) = c$.

Therefore, $\varphi_t(e)$ is defined for all t so that we can define $F: W \times (b,d) \longrightarrow E$ by $F(w,t) = \varphi_{t-c}(w)$, which is of class C^k according to the facts in Definition 1. $F(w,c) = \varphi_0(w) = w$ and $f(F(w,t)) = f(\varphi_{t-c}(w)) = f(w) + t - c = c + t - c = t$. Therefore, $F: W \times (b,d) \longrightarrow E$ and $w \longrightarrow F(w,\bar{c})$ maps W into $f^{-1}(\bar{c})$. If we define $\bar{F}: E \longrightarrow W \times (b,d)$ by $\bar{F}(e) = (\varphi_{c-f(e)}(e), f(e))$, then $\bar{F}$ is C^k and $F\bar{F} = \varphi_{f(e)-c}(\varphi_{c-f(e)}(e)) = \varphi_0(e) = e$. Similarly $\bar{F}F$ = identity. Therefore F and $\bar{F}$ are both 1 to 1 and onto.

Q.E.D.

Lemma 2. Consider now a C^k directionally transverse function $f: E \longrightarrow R^p$ so that for every point $e \in E$ there is a coordinate system of the type described in Remark III,1 and Definition III.2 so that the p lines intersect only at $f(e)$. Also assume that $p_i f$ satisfies the hypothesis of Lemma 1 for $1 \leq i \leq p$. Then for $r, \bar{r} \in f(E)$, $f^{-1}(r) \approx f^{-1}(\bar{r}) \approx E^{p,r}$, where $\approx$ denotes diffeomorphism and $E^{p,r}$ is a closed submanifold of codimension p.

Proof. Consider a suitably small disc $O_r \subset f(E)$ about r. Now consider f restricted to $f^{-1}(O_r) = U_r$. Then with respect to the coordinate projections about r, we have $p_i f: U_r \longrightarrow (b_i, d_i)$. For convenience it is assumed that $b_i = b_j$ and $d_i = d_j$. Then since $d(p_i f)_e = (dp_i)_{f(e)} df_e$ and since p_i is surjective and hence $dp_i \neq o$, we have $d(p_if)_e \neq o$. Therefore, if we actually only assume that f is without singularities and p_if have their maximum integral curves defined over the entire manifold, the hypothesis of Lemma 1 is satisfied. Therefore $F_i: W_{r_i} \times (b,d) \longrightarrow U_r$. Now $f^{-1}(r) = \bigcap_1^p W_{r_i}$. We also have the p local one parameter groups of diffeomorphisms (φ_t^i). Then for any $\bar{r} \in O_r$, $f^{-1}(\bar{r}) = \bigcap_1^p W_{\bar{r}_i} = \bigcap_1^p W_{r_i} \times (c_{r_i})$. There are maps $g_1: f^{-1}(r) \longrightarrow f^{-1}(\bar{r})$, $g_2: f^{-1}(\bar{r}) \longrightarrow f^{-1}(r)$ defined by $g_1(w) = \varphi^p_{c_{r_p}} \circ \circ \varphi^i_{c_{r_i}} \circ \varphi^1_{c_{r_1}}(w)$

and $g_2(\bar{w}) = \varphi^1_{-c_{r_1}} \circ \circ \varphi^i_{-c_{r_i}} \circ \circ \varphi^p_{-c_{r_p}}(\bar{w})$. Obviously by the

definition of the collections (φ_t^i), $g_1g_2 =$ identity and $g_2g_1 =$ identity. Therefore each g_j is bijective and since they are C^k maps we have the result that $f^{-1}(r) \approx f^{-1}(\bar{r})$. Now for $\bar{r} \in O_r$ we can construct a finite chain $(O_{r_k})^n \ni$

$O_{r_k} \bigcap O_{r_{k+1}} \neq \emptyset$ and $\ni r_1 = r$ and $r_n = \bar{r}$. Since we then have for $q \in O_{r_k} \bigcap O_{r_{k+1}}$ that $f^{-1}(r_k) \approx f^{-1}(q) \approx f^{-1}(r_{k+1})$ then by induction $f^{-1}(r) \approx f^{-1}(\bar{r})$.

Since the p local groups are determined by p commuting projections in the sense that the derivative of the composition determines a transverse direction, the local p-parameter groups constructed above are locally abelian.

To see that $f^{-1}(r) = E^{p,r}$ is a closed submanifold of codimension p, we observe that $f^{-1}(r) = \bigcap_{1}^{p} W_{r_i}$, where W_{r_i} is a closed submanifold of codimension 1 and where locally the p-parameter groups which determine the W_{r_i} are transverse with respect to each other.

Q.E.D.

<u>Note</u>. Since $f^{-1}(r) = \bigcap_{1}^{p} W_{r_i}$, it may happen that for $p > 1$ the relative topology of $E^{p,r} \times E_p$ is not the same as that determined by the Riemann structure of $E^{p,r}$, which is the structure used in the above result to give diffeomorphisms by following solution curves.

In Chapter V the notation $E \approx E^{p,r} \times E_p$ is used, keeping in mind the fact that the topology of the product is determined by the bijection $E \longleftrightarrow E^{p,r} \times E_p$.

CHAPTER V

AN INVERSE LIMIT REPRESENTATION FOR MANIFOLDS

The machinery has been set down so that a representation theorem for infinite dimensional separable Hilbert manifolds can be formulated in the form of an inverse system of finite dimensional open manifolds, which are induced as the image of smooth maps $f_p: E \longrightarrow R^p$ and with the inverse limit structure being determined by systems of foliations.

<u>Definition 1</u>. A separable infinite dimensional manifold E is foliated by a collection $(E^{p,r} \mid E^{p,r}$ is a closed submanifold without boundary of codimension p and $r \in E_p \subset R^p.)$ if $E = \bigcup_r E^{r,p}$ with $\dim E_p = p$. Also require that $E^{p,r} \cap E^{p,\bar{r}} = \emptyset$ for $r \neq \bar{r}$ and $E^{p,r} \approx E^{p,\bar{r}}$, $\approx$ denotes diffeomorphism in the differentiable case.

Define a saturated system of foliations of E to be a system $((E^{m,r}) \mid m \geqslant p) \ni \forall\, m$, $E = \bigcup_r E^{m,r}$ so that each $E^{m+1,r}$ is contained in a unique $E^{m,s}$ and each $E^{m,s}$ is foliated by a subsystem $(E^{m+1,r} \mid r \in E_{m+1,m} \subset R^{m+1})$, with $\dim E_{m+1,m} = 1$. Require also that $\bigcap_{m \geqslant p} E_e^{m,r} = e$, where $E_e^{m,r}$ is the unique leaf of the foliation of codimension m containing e.

Lemma 1. Under the above conditions $\exists$ a bijection $E \longleftrightarrow \underset{\overleftarrow{m \geq p}}{L} E_m$, for an inverse system determined by the saturated system of foliations.

Proof Construct a function $p_m : E \longrightarrow E_m$ by defining $p_m(e) = r_{e,m}$, where $r_{e,m}$ is the parameter element of $E_e^{m,r}$. Define $p_m^{m+1} ; E_{m+1} \longrightarrow E_m$ by $p_m^{m+1}(r_{e,m+1}) = r_{e,m}$, where both parameter elements are elements of some set $E_{m+1,m}$, which is unique since each $E^{m+1,r}$ is contained in a unique $E^{m,s}$. This map is well defined because e is an element of a unique $E^{m+1,r}$ and a unique $E^{m,s}$ and is clearly onto for the same reasons. Therefore by composing these maps to obtain the collection (p_n^m) for $m \geqslant n \geqslant p$, we obtain an inverse system $(E_m , p_n^m \mid p_n^m$ is onto) with limit $\underset{\overleftarrow{m \geq p}}{L} E_m$. Now define the correspondence $E \longleftrightarrow \underset{\overleftarrow{m \geq p}}{L} E_m$ by mapping $e \longrightarrow (p_m(e) \mid m \geqslant p)$. This is 1 to 1. For if $(p_m(e)) = (p_m(\bar{e}))$, then $r_e = p_m(e) = p_m(\bar{e}) = r_{\bar{e}} \; \forall m$, and $e, \bar{e} \in E_e^{m,r} \cap E_{\bar{e}}^{m,r}$ which is $\emptyset$ unless $r_e = r_{\bar{e}}$. But in that case $e, \bar{e} \in \bigcap_{m \geq p} E_e^{m,r} = \bigcap_{m \geq p} E_{\bar{e}}^{m,r} = e = \bar{e}$.

The map is onto by definition. Therefore the bijection is established.

Q.E.D.

Definition 2. A connected manifold E satisfies condition O if for $e \in E$, the metric function $\rho_e = \rho(e, \cdot)$ satisfies the condition that for $r, \bar{r} \in \rho_e(E)$, chosen arbitrarily close, then $\rho_e^{-1}(r, \bar{r})$ is an open chart and hence considered as an open subset of the Hilbert space H. It is assumed, of course, that we deal only with the separable case.

We can deform the function by taking an arbitrarily small chart about e and then construct a diffeomorphism $f: E \longrightarrow E \setminus e$ which is not the identity only within that small chart. Then the composition $\rho_e f: E \longrightarrow E \setminus e \longrightarrow R$ gives a map $E \longrightarrow (o, \infty)$ if we assume for the moment that the original metric function is unbounded. Then since each point of E is a finite distance from e, we can construct a diffeomorphic retraction $\alpha: (o, \infty) \longrightarrow (o, d) \ni d < \infty$, and then obtain the composition $\alpha \rho_e f: E \longrightarrow (o, d)$. Then by simply denoting the above composition by ρ_e, $\rho_e(o, b]$ for $b < d$ is a bounded closed subset of E. Also if E^m is a closed submanifold contained in $\rho_e^{-1}[r, \bar{r}]$ then a similar metric function $\rho_e: E^m \longrightarrow (o, d]$ for $d < \infty$ can be constructed for $e \in E^m$. Assume that $E^m \subset U = E^m \times R^m \subset \rho_e^{-1}[r, \bar{r}] \subset H$. If d is finite choose r arbitrarily close to d so that we can consider $\rho_e^{-1}(r, d]$. Each component can now be considered as an open subset of H since U has the simple form $E^m \times R^p$. The result of Eells described in Chapter III can be applied to construct a function $\psi : \overline{C} \longrightarrow [r,d]$, where $\overline{C}$ represents the closure of this component, and so that $\psi^{-1}(r) = \rho_e^{-1}(r)$ and $\psi^{-1}(d)$ is a point. Then consider the function $\rho : E^m \longrightarrow (o,d]$ defined so that $\rho \mid \rho_e^{-1}(o,r] = \rho_e$ and $\rho \mid \rho_e^{-1}[r,d] = \psi$. Then $\rho^{-1}(d)$ is a disjoint collection of points, each in a separate component, thereby allowing arbitrarily small disjoint neighborhoods to be constructed about each point which support a diffeomorphism $f_j : E \longrightarrow E \setminus \rho^{-1}(d)_j$. Then the composition gives a map $\rho \prod_j f_j : E \longrightarrow E \setminus \rho^{-1}(d) \longrightarrow (o,d)$. If d was not finite then the above construction can be used. Similarly, if for the original metric function ρ_e the assumption is made that if this function takes on a maximum finite value d, then for r arbitrarily close to d $\rho_e^{-1}(r,d]$ is an open subset or chart of H the latter construction applies. In any case, a metric function can be constructed, deformed slightly, whose image is a finite open interval.

Theorem 1. Let E be a connected, separable, geodesically and Cauchy complete, C^k Hilbert manifold satisfying condition 0 of Definition 2, where $3 \leqslant k \leqslant \infty$. Then $\forall$ q $\in$ Z, $\exists$ a homeomorphism $E \approx \underleftarrow{L}_{m \geq q} E_m$, where $E_m \subset R^m$ and the inverse limit structure is determined by a saturated system of foliations. Also E_m splits in the sense that $\exists$ E_{m+} and E_{m-} $\ni$

$E_{m+} \cup E_{m-} = E_m$ and $E_{m+} \cap E_{m-} = E_{m-1}$.

<u>Proof</u> For some point $\bar{e} \in E$, consider the metric function ρ_e, assumed to be deformed so that $\rho_e : E \longrightarrow (o,d)$ for d finite. Then since $\rho_e^{-1}(o,b]$ is bounded for $b < d$, we can apply Theorem III.2 to obtain an approximation $\bar{f}_1 : E \longrightarrow (o,d)$ so that $\bar{f}_1$ has singularities in a closed locally compact subset K $\ni$ if for any maximum solution curve σ, $(d\bar{f}_1)_{\sigma(t)} \longrightarrow o$, then $\exists$ a t_o so that $\sigma(t_o)$ is a singularity. Now as in the proof of Thoerem II.1 K $= (K \setminus K_c) \cup K_c$ so that $K_c \subset K$ and is closed and $\ni$ $\exists$ diffeomorphisms F_1 and F_2 so that $F_1 : E \setminus \widetilde{K}_c \longrightarrow E$ and $F_2 : E \setminus F_1\widetilde{(K \setminus K_c)} \longrightarrow E$. These maps are constructed by covering first K_c by a countable collection of finite dimensional closed manifolds and then constructing trivial tubes which support a collection of maps. A similar construction takes place for $F_1(K \setminus K_c)$. Now if instead we take these closed finite dimensional manifolds and extend them by taking all the points on the maximum solution curves with initial conditions in these manifolds we obtain two collections which can be covered by finite dimensional manifolds and then extend to trivial tubes, which may constitute a non-star finite collection of tubes. But we still have the case where we want to remove from the manifold E a countable collection

of finite dimensional submanifolds, so Theorem II.2 can be applied for the collection determined by K_c and also $F_1(K \setminus K_c)$ so that we get a diffeomorphism $g_1^{-1}: E \setminus K^t \longrightarrow E$, where K^t denotes the total collection of submanifolds removed and so that all the maximum solution curves in $E \setminus K^t$ are defined on the entire manifold in the sense that for every $c \in (o,d)$, $\exists\, t_o$ so that $f_1(\sigma(t_o)) = c$, for $f_1 = \bar{f}_1 g$. Then Lemma IV.2 applies to give a foliation in terms of a local turned global 1-parameter group of diffeomorphisms. Therefore $E \approx (o,d) \times E^{1,r}$ for any $r \in (o,d)$. where $E^{1,r}$ is a closed bounded submanifold.

Now for any $q \in Z$, we can choose q points e_i so that $(\rho_{e_i})_{1 \leq i \leq q}$ is a collection of directionally transverse functions and have no singularities in a small neighborhood of a point $e \ni$ they transverse q linearly independent directions at e. This is possible by choosing the chart $U_e = \exp T(E)_e$ and then choosing q points e_i in U_e, which are linearly independent and so that the corresponding distance functions transverse q linearly independent directions through e, and so that in a small disc about e there are no singularities. Then define a function $\rho : E \longrightarrow R^q$ by $\rho(e) = (\rho_{e_1}(e), \quad , \rho_{e_q}(e))$. Then since each ρ_{e_i} is directionally transverse in the sense of Definition III. 2 we can apply Theorem III.2

to get an approximation $\overline{f}_q$ $\ni$ for any q coordinate lines, in the sense of Definition III.2, contained in $\overline{f}_q(E)$, then the corresponding projections $(p_i\overline{f}_q)_{1\leq i\leq q}$ satisfy the condition that if $d(p_i\overline{f}_q)_{\sigma(t)} \longrightarrow o$, then $\exists$ a t_o so that $\sigma(t_o)$ is a singularity. Therefore Lemma IV.2 applies to give a local q-parameter group of diffeomorphisms. Now in fact this a global abelian group; because if there is a hole in $\overline{f}_q(E)$, then two locally parallel lines determining coordinate lines in two different systems, which are on different sides or homologically separated from the hole in the sense that these lines may be contained in a neighborhood of some plane passing through the hole which isn't locally connected, determine the same function $p_i\overline{f}_q$ with respect to each co-ordinate system. Therefore the diffeomorphism determined by going around the loop about the hole is actually determined by following solution curves which are parallel with respect to parallel coordinate lines on different sides of the hole. Therefore going around the loop gives the identity.

Now to continue on, we apply a metric function $\rho_{e_j}: E^{q,r} \longrightarrow (o,d)$ constructed as in Definition 2 above and with the condition that in some neighborhood of a point $e \in E^{q,r}$ this function has no singularities and transverses a direction which is linearly independent with respect to the first q directions. Therefore since $E^{q,r}$ is closed, and contained in E, it is Cauchy complete and we get a $\mathcal{E}_q$ approximation $\overline{f}_1 : E^{q,r} \longrightarrow (o,d)$ so that $\mathcal{E}_q$ has no singularities in a small neighborhood about e. This is possible by letting this small neighborhood be the first set V_1 in the cover which determines the dominated locally finite cover by charts used in Theorems III.1 and III.2 and by letting $\overline{f}_1 = \rho_{e_j}$ on $U_1 \supset \overline{V}_1$, where U_1 is a small extension of V_1 so that ρ_{e_j} has no singularities in U_1 and U_1 is the first set in the dominating cover of the above theorems. Then we can define a function $\overline{f}_{q+1} : E \longrightarrow R^{q+1}$ defined by letting $\overline{f}_{q+1}(e) = f_q(e) + \overline{f}_1(xe)$, where x is an appropriate element in the global q-parameter group of diffeomorphisms. Therefore, since $(d\overline{f}_{q+1})_e = (df_{q+1})_e + (d\overline{f}_1\ dx)_{x^{-1}e} \neq$ onto map $\Longleftrightarrow$ $x^{-1}e$ is a critical point in E^q, we can then conclude that the critical set is still closed and locally compact since it is then of the form $K \times E_q$. Then $\exists$ as before a diffeomorphism $g : E \longrightarrow E \setminus K^t$, where K^t contains the critical set and all the points determined by maximum solution curves which contain a critical point. Also as before, if $(d(p_i \overline{f}_{q+1}))_{\sigma(t)} \longrightarrow o$, then there is a $t_o \ni \sigma(t_o)$ is a critical point. Now in order to maintain the notion of a foliation we must take note of the fact that g does not necessarily map $E^{q,r} \longrightarrow$

$E^{q,r}$. But if for $e_r \in E^{q,r}_{e_r}$ we map $e_r \longrightarrow g(e_r)$ and then map $g(e_r) \longrightarrow E$ by the element of the q-parameter group of diffeomorphisms which maps $E^{q,s}_{g(e_r)} \longrightarrow E^{q,r}_{e_r}$ we get a composition which maps $E^{q,r}_{e_r} \xrightarrow{\text{into}} E^{q,r}_{e_r}$. Since the total critical set K^t = critical points and the appropriate maximum solution curves plus the covering collections of finite dimensional manifolds and all the appropriate solution curves is still of the form $K^t_r \times E_q$ under a diffeomorphism, then the last map in the above composition is the same as a parallel projection, and hence e_r cannot be mapped to an element of K^t. Also this is a composition of a diffeomorphism and a projection and since $E \approxeq E_q \times E^{q,r}_{e_r}$, locally $\exists$ a differentiable map $\ni E^{q,r}_{e_r} \xrightarrow{\text{into}} E^{q,r}_{e_r}$, and which simply deforms the solution curves of the appropriate projections of the vector valued function $\overline{f}_{q+1}$ because the map g is constructed by an infinite composition of maps with non identity support within trivial tubular neighborhoods of finite dimensional manifolds containing parallel copies of E_q, chosen so small that in local coordinate charts containing the tubes no leaf is ever orthogonal to a fibre, and since each map of this infinite composition maps within a fibre over the base manifold as in Theorem II.2. Now since the E_q represents an abelian q-parameter group of diffeomorphisms we can combine the newly constructed composite maps by varying the parameter $r \in E_q$ to obtain a new map which by an abuse of notation will be denoted by $g: E \xrightarrow{\text{into}} E \setminus K^t$, which is differentiable, and does not disturb any of the solution curves, plus the additional fact that it maps each leaf into the same leaf and thus preserves the foliation. Actually, all that is happening is that $E \setminus K^t$ is being retracted into itself differentiably by catching some of the critical levels from the various $E^{q,r}$ in fibres of the various trivial tubes and deforming the entire structure to give larger breaks about areas of K^t, thus giving a measure of how any possible geodesically complete structure on a leaf $E^{q,r}$ is not preserved under the diffecmorphisms of the q-parameter group. Then as before $\exists$ a composition $f_{q+1} = \tilde{f}_{q+1}\ g: E \longrightarrow R^{q+1}$ so that there is a global abelian q+1-parameter group of diffeomorphisms and with the property that $f_q(E) = E_q$ is mapped onto by f_{q+1} since there are no singularities in a neighborhood $E_q \times V_1$, where V_1 is determined by some parameter value r_o. Hence the splitting for E_{q+1} is determined by taking $E_{(q+1)+}$ = all of those points above and $\geqslant r_o$ for the q+1-parameter value. Similarly $E_{(q+1)-}$ = those points with parameter value in the q+1 coordinate $\leqslant r_o$.

Now we must proceed inductively with the above process in a maner to get a saturated system of foliations. Therefore the only thing that needs to be determined is that $\bigcap_m E_e^{m,r} = e$. To do this, consider the following:

E is a separable manifold. Therefore we can continue the above inductive process for each integer $m \geqslant q$ and eliminate each coordinate line in a small local chart about some fixed $e_o \in E$ by letting each such coordinate line correspond successively to a new transverse direction in the m-parameter group of diffeomorphisms. This inductive process continues word for word as that described above and gives two systems $(E^{m,r}_{e_o})$ and $(E_m)_{m \geq q}$, where it is understood that $r = r(e_o)$. By definition of the construction $\bigcap_{m \geq q} E^{m,r}_{e_o} = \underset{\overleftarrow{m \geq q}}{L} E^{m,r}_{e_o} = e_o$ because if some $e \neq e_o$ is an element of $\bigcap_{m \geq q} E^{m,r}_{e_o}$, then e has the same coordinate representation as e_o since every local coordinate line is a transverse direction and every parameter in each coordinate line at some stage belongs to a new leaf of one of the foliations. Hence $e = e_o$.

Now to see that $\bigcap_{m \geq q} E^{m,r}_{e} = \underset{\overleftarrow{m \geq q}}{L} E^{m,r}_{e} = e$ we construct a homeomorphism $g: \underset{\overleftarrow{m \geq q}}{L}(E^{m,r}_{e_o}, h_{m-q}) \longrightarrow \underset{\overleftarrow{m \geq q}}{L} E^{m,r}_{e}$. $g_1 : E^{q,r}_{e_o} \longrightarrow E^{q,r}_{e}$ is represented as any element in the abelian q-parameter group of diffeomorphisms which is determined by following a path denoted by $1(f_q(e_o), f_q(e))$. The map $g_2 : E^{q+1,r}_{e_o} \longrightarrow E^{q+1,r}_{e}$ is given by a map in the abelian q+1-parameter group which is determined by following a path in E_{q+1} denoted by $1(f_{q+1}(e_o), f_{q+1}(e))$. Now construct h_1 by following a path $1(f_{q+1}(e_o), r(\bar{e}))$ starting from $f_{q+1}(e_o)$, where $g_1(\bar{e}) = e_o$ and all of these paths are taken in E_{q+1}. This is then given by the following diagram:

$$\begin{tikzcd} E_e^{q+1,r} \arrow[d, "i"'] & E_{e_o}^{q+1,r} \arrow[l, "g_2"'] \arrow[d, "h_1"] \\ E_e^{q,r} & E_{e_o}^{q,r} \arrow[l, "g_1"'] \end{tikzcd}$$

Inclusion = i , therefore obviously $i\ g_2 = g_2$. Now by the construction of the m-parameter groups the q-parameter group can be considered as a subgroup of the q+1-parameter group on intersecting the domains. So we have $g_1\ h_1$, the map determined by starting from $f_{q+1}(e_c)$ and going along the path $l(f_{q+1}(e_o),r(\bar{e}))$ and then along the path $l(r(\bar{e}),f_{q+1}(e))$, where these paths are considered in E_{q+1} and determining elements of the q+1-parameter group. But the union of these paths joins $f_{q+1}(e_o)$ and $f_{q+1}(e)$ and hence gives the same map as that determined by $l(f_{q+1}(e_o),f_{q+1}(e))$, which is g_2. Hence the diagram commutes. We continue the rest of the construction inductively, with the inductive step the same as that above. Hence since each g_{m-q} is a diffeomorphism we get $\underset{\overleftarrow{m\geq q}}{L}(E_{e_o}^{m,r}, h_{m-q}) \approx \underset{\overleftarrow{m\geq q}}{L} E_e^{m,r} \approx \bigcap_m E_e^{m,r}$. But we can map $\underset{\overleftarrow{m\geq q}}{L}(E_{e_o}^{m,r}, h_{m-q}) \longrightarrow \underset{\overleftarrow{m\geq q}}{L} E_{e_o}^{m,r} = \bigcap_m E_{e_o}^{m,r}$ by considering squares of the form:

$$\begin{tikzcd}
E^{q+j,r}_{e_o} \arrow[d, "i"'] & & E^{q+j+1,r}_{e_o} \arrow[ll, "h_1h_2 \quad h_{j+1}"'] \arrow[d, "h_{j+1}"] \\
E^{q+j-1,r}_{e_o} & & E^{q+j,r}_{e_o} \arrow[ll, "h_1h_2 \quad h_j"']
\end{tikzcd}$$

The compositions indicated in the diagram are well defined because each h_i is constructed with respect to a q+i-parameter group, with each q+i-k-parameter group being considered as a subgroup of q+j-parameter group by intersecting the appropriate domains of the maps. Therefore the horizental compositions give diffeomorphisms. Now since the diagram obviously commutes we have $\underleftarrow{L}(E^{m,r}_{e_o}, h_{m-q}) \approx \underset{m\geq q}{\underleftarrow{L}} E^{m,r}_{e_o} = \bigcap_{m\geq q} E^{m,r}_{e_o} = e_o$. Therefore $\bigcap_{m\geq q} E^{m,r}_{e} \approx e_o$, and since $e \in \bigcap_{m\geq q} E^{m,r}_{e}$ then $e = \bigcap_{m\geq q} E^{m,r}_{e}$.

Therefore, the existence of a saturated system of foliations has been established and $E \leftrightarrow \underset{m\geq q}{\underleftarrow{L}} E_m$, where $E_m = f_m(E)$. It is also clear from the construction of the foliations that if we give $\underset{m\geq q}{\underleftarrow{L}} E_m$ the topology consisting of open sets which are the images under the correspondence of open sets of E, then $E \approx \underset{m\geq q}{\underleftarrow{L}} E_m$, and since the maps determining our foliations are onto, the topology of $E_m \subset \underset{m\geq q}{\underleftarrow{L}} E_m$ is the same as the manifold topology of $E_m \subset R^m$. The topology behaves like a modified box topology for $\underset{\overleftarrow{m \geq q}}{LE_m} \subset \prod E_m$.

Since the splitting for E_m has been established the theorem follows.

Q.E.D.

<u>Remark</u> In any representation $E \approx \underset{\overleftarrow{m \geq q}}{LE_m}$, we can choose a point $e \in E$ and take a diffeomorphic image of E_m passing through e, where this is determined as the transverse manifold of the m-parameter group of diffeomorphism.

<u>Definition 3</u>. Let $\underset{\overrightarrow{m \geq p}}{LE_m}$ denote $\bigcup_{m \geq p} E_m$ with the weak topology. That is $A \subset \underset{\overrightarrow{m \geq p}}{LE_m}$ is closed (open) $\Longleftrightarrow A \cap E_m$ is closed (open) $\forall\ m \geq p$. Then a generalized Palais-Svarc lemma [19] can be formulated as follows:

<u>Theorem 2</u>. Given a manifold E with the hypothesis of Theorem 1 $\ni E \approx \underset{\overleftarrow{m \geq q}}{LE_m}$, then $E \simeq \underset{\overrightarrow{m \geq q}}{LE_m}$, where $\simeq$ denotes a strong homotopy equivalence.

The proof will follow after a series of lemmas.

<u>Lemma 2</u>. $\underset{\overrightarrow{m \geq q}}{LE_m}$ is dense E.

<u>Proof</u> For $\bar{e} \in E$ fix a transverse manifold E_m passing through $\bar{e}$, where $h: E \approx \underset{\overleftarrow{m \geq q}}{LE_m}$. Then a neighborhood basis can be given at $e \in E$ by $(U_m)_{m \geq q}$ for $U_m \subset E_e^{m,r} \times D^m$, where $D^m \approx R^m$, chosen so that $\bigcap_{m \geq q} p_i h(U_m) = r_i(e) \in E_i$. Clearly since

$e = \bigcap_{m \geq q} E_e^{m,r}$, $\bigcap_{m \geq q} U_m =$ e and some element of the sequence $(e_m \mid e_m = (r_1(e), \quad , r_m(e), o, o, \quad))$ is contained in every neighborhood U_e of e.

Q.E.D.

Lemma 3. For a compact $K \subset E \approx \underset{\overleftarrow{m \geq q}}{LE_m}$, $\exists$ a homotopy h_t: $K \longrightarrow E \ni h_o = id$, $h_t \mid K \cap E_n = id$ and $h_1(K) \subset E_n$, for large n and for E_n embedded in K as a transverse manifold passing through some point e E as in the remark above.

Proof Since through each point $k \in K$, we can put a closed transverse manifold $E_{m,k}$ and then since E is geodesically complete, then construct a trivial tubular neighborhood $\mathcal{T} E_{m,k}$ as in Chapter II. Now since K is compact we can cover K by a finite number of the tubes ($\mathcal{T} E_{m,k_i}$). But then since $\underset{\overrightarrow{m \geq q}}{LE_m}$ is dense in $\underset{\overleftarrow{m \geq q}}{LE_m}$ and therefore in the homeomorphic image E, we can find a point of $\underset{\overrightarrow{m \geq q}}{LE_m}$ in each $\mathcal{T} E_{m,k_i}$. But since there are only finitely many tubes, we then can assume that $\exists$ some large $n \geq m$ so that E_n contains these points. Also when we construct the tubular neighborhoods with the orientable manifolds $E_{m,k_i} \subset E_{n,k_i}$ we can assume that $\mathcal{T} E_{n,k_i} \supset \mathcal{T} E_{m,k_i}$ by the continuity of the exp function which is the basis for the construction of the trivial tubes. Therefore, we can assume that K is covered by the finite collection ($\mathcal{T} E_{n,k_i}$). Now take the point $e_i \in E_n$ which is in the tube $\mathcal{T} E_{n,k_i}$ and connected to the base manifold by a path in the fibre containing it. Then to each point of this path we can apply the n-parameter group of diffeomorphisms and extend this set to $E_n \times I$. Then by the continuity of exp in the construction of the trivial tubes we can assume the E_n is contained as a section of $\mathcal{T} E_{n,k_i}$. But since these are all trivial tubular neighborhoods each one can then be considered to be a trivial tube with base E_n. Hence, again by the continuity of the exp we can assume that these can be deformed so that there is just one trivial tube $\mathcal{T} E_n$ containing K. Therefore we can define h_t as the retraction in the trivial tube defined so that $h_o =$ id and $h_1 \mid H_e = e$ where H_e is the fibre over $e \quad E_n$.

Q.E.D.

Lemma 4. Let A be a closed subspace of a compact space X and let $f_o : X \longrightarrow E$ be a map $\ni f_o \mid A : A \longrightarrow \underset{\overrightarrow{m \geq q}}{L} E_m$. Then $\exists$ a homotopy $f_t : X \longrightarrow E$ of $f_o \ni f_1 : X \longrightarrow \underset{\overrightarrow{m \geq p}}{L} E_m$ and $f_t \mid A = f_o \mid A$, $o \leq t \leq 1$.

Proof Consider a compact $K \cap \underset{\overrightarrow{m \geq q}}{L} E_m$. Then the intersection is contained in some E_n. For cover $K \cap \underset{\overrightarrow{m \geq q}}{L} E_m$ by sets $(O \cap E_m)$ and take a finite subcover. O can be the trivial tubular neighborhood of Lemma 3. Then there is a maximum m

$=$ n appearing. Then in Lemma 3 let $K = f_o(X)$ with the n chosen here to be equal to n above. Define $f_t = h_t f_o$.

Q.E.D.

The proof of Theorem 2 is now standard (Palais).

For $n \geq o$, let $a \in \pi_n(E)$. Then let $a = [f_o]$ and let $f_t : S^n \longrightarrow E$ be a homotopy of f_o with $f_1 : S^n \longrightarrow \underset{m \geq q}{\overrightarrow{L}E_m}$ by Lemma 4. Then under the injection $i : \underset{m \geq q}{\overrightarrow{L}E_m} \longrightarrow \underset{m \geq q}{\overleftarrow{L}E_m}$ we have $i_* [f_1] = a$. Therefore i_* is onto.

To prove that i_* is 1 to 1 let $b \in \pi_n(\overrightarrow{L}E_m) \ni i_* b = o$. Then let $b = [f_o]$. Now $i(f_o(S^n)) \subset \overrightarrow{L}E_m \subset E$. But since $i_* b = o \ \exists f_t : S^n \times I \longrightarrow E \ni f_1(S^n) =$ base point and f_o is the original f_o. Therefore by identifying $S^n \times 1$ to a point we have $\bar{f} : S^n \times I / S^n \times 1 = D^{n+1} \longrightarrow E$ so that $[\bar{f} \mid S^n] = b$. Then by Lemma 4 $\exists$ a homotopy $\bar{f}_t \ni \bar{f}_o = \bar{f}$ with $\bar{f}_t : D^{n+1} \longrightarrow E$ and $\bar{f}_t : D^{n+1} \longrightarrow \underset{m \geq q}{\overrightarrow{L}E_m}$ and $\bar{f}_t \mid S^n = f_o \mid S^n$. Therefore $b = [\bar{f}_1 \mid S^n]$ and since $\bar{f}_1 : D^{n+1} \longrightarrow \underset{m \geq q}{\overrightarrow{L}E_m}$, then $b = o$ and $i_* \ \pi_n(\overrightarrow{L}E_m) \xrightarrow{\text{onto}} \pi_n(E)$ is onto for all n.

Since E is an ANR we have both E and $\underset{m \geq q}{\overrightarrow{L}E_m}$ dominated by CW complexes. Therefore the weak homotopy equivalence is the same as strong homotopy equivalence in the category.

Q.E.D.

CHAPTER VI

A HOMOLOGY FUNCTOR $H_{\infty-p}(\cdot, Z)$

We now have the machinery of Chapter V at our disposal, which in a sense can be considered to constitute a saturated foliation category, to obtain a homology theory which distinguishes the subsets of cofinite dimension. The theories of Geba [12] , Geba and Granas [13] , Eells [9] , and recently Mukherjea [17] consider the possibility of constructing homotopy and homology functors which are defined with respect to cofinite dimensional sets and which give various duality isomorphisms. Mukherjea uses a strong homotopy equivalence $M \xrightarrow{\simeq} \underrightarrow{\lim} M_n$ along with Poincare duality in finite dimensions to construct a cohomology functor $H_c^{\infty-p}(M,G)$ which is determined by a direct system constructed from the injection maps of the homology of M_n and the dualities. A similar homology functor could be defined, but the connecting homomorphisms would be unnatural in the sense that one must first appeal to a duality to obtain an inverse system which then automatically gives a duality.

In the following, the notions and fundamental facts about sheaves are used and for details we refer to D.G.Bourgin [5] and G. Bredon [6] . It will also always be assumed that the infinite dimensional manifolds E under consideration are those which satisfy the conditions of Theorem V.1. This then gives a manifold $E \approx \underset{\overleftarrow{m \geq p}}{L} E_m \ni \underset{\overrightarrow{m \geq p}}{L} E_m \simeq E$ where E_m is a finite dimensional orientable manifold $\ni E_m \subset R^m$ and so that there is a splitting $E_m = E_{m+} \cup E_{m-}$ and $E_{m-1} = E_{m+} \cap E_{m-}$.

Definition 1. The only sheaves considered will be those with stalks isomorphic to $Z =$ integers and taken over paracompactifying families of supports.

Consider the Mayer-Vietoris sequence $\longrightarrow H^{\varphi|E_{m+}}_{m-p}(E_{m+},Z) \oplus H^{\varphi|E_{m-}}_{m-p}(E_{m-},Z) \longrightarrow H^{\varphi|E_m}_{m-p}(E_m,Z) \xrightarrow{\partial_{m-p}} H_{(m-1)-p}(E_{m-1},Z) \longrightarrow$, where the homology is specified as the Borel-Moore theory. Define $H_{\infty-p}(E,Z) = \underset{\leftarrow}{L}(H^{\varphi|E_m}_{m-p}(E_m,Z), \partial_{m-p})$.

Theorem 1. For any representation $\underset{\overleftarrow{m \geq q}}{L} E_m \approx E, \exists$ a duality isomorphism $H_{\infty-p}(E,Z) \approx H^p_\varphi(E,Z)$, where φ is a paracompactifying family and where we consider this to be the equivalent of the singular cohomology with support in φ since E is locally contractible.

Proof By Theorem V.2 $E \simeq \underset{\overrightarrow{m \geq q}}{L} E_m$. Then $H^p_\varphi(E,Z) \approx H^p_{\varphi|\underset{\rightarrow}{L}E_m}(\underset{\rightarrow}{L}E_m,Z) \approx \underset{\leftarrow}{L} H^p_{\varphi|E_m}(E_m,Z)$. Now for the orientable E_m there is the Poincare duality $H^{\varphi|E_m}_{m-p}(E_m,Z) \overset{\approx}{\longleftrightarrow} H^p_{\varphi|E_m}(E_m,H_m(S \otimes Z)) \approx H^p_{\varphi|E_m}(E_m,Z)$, where $H_m(S \otimes Z) = \Theta_m \otimes Z$ is the orientation sheaf for E_m which in this case is trivial and Θ_m is the sheaf whose presheaf structure for an open $U \subset E_m$ is given by $U \longrightarrow H_*(E_m, E_m \setminus U) \simeq H_m(E_m, E_m \setminus U)$. Since E_m is orientable this sheaf is actually isomorphic to $Z \mid E_m$. Consider the diagram

$$\begin{array}{ccc} H^{\varphi|E_m}_{m-p}(E_m, Z) & \xleftarrow{\quad D_m \quad} & H^p_{\varphi|E_m}(E_m, H_m(S \otimes Z)) \\ \Big\downarrow{\scriptstyle \partial_{m-p}} & & \Big\downarrow{\scriptstyle i^*_m} \\ H^{\varphi|E_{m-1}}_{(m-1)-p}(E_{m-1}, Z) & \xleftarrow{\quad D_{m-1} \quad} & H^p_{\varphi|E_{m-1}}(E_{m-1}, H_{m-1}(S \otimes Z)) \end{array}$$

where i^*_m is induced from the injection and ∂_{m-p} is the connecting boundary of the Mayer-Vietoris sequence. To prove the desired result we only need check for commutativity of the diagram.

Locally with respect to a presheaf structure and with f^p denoting a representative of a cohomology class with respect to that structure we have $D^{\#}_{m-1} i^{\#}_m f^p = f^p \cap i\sigma_{m-1}$ and $\partial^{\#}_{m-p} D_m f^p = f^p(\sigma_p)\partial^{\#}_{m-p}\sigma_{m-p}$. If supp $f^p \subset \sigma_{m-1}$ then these two expressions are equal up to a sign depending upon the orientation of E_m. If supp $f^p \not\subset \sigma_{m-1}$, then $f^p \cap i\sigma_{m-1} = 0$. But then $\partial_{m-p} D^{\#}_m f^p =$ a contribution to a boundary. For consider the following diagram which defines ∂_{m-p};

$$\begin{tikzcd}
 & & C^{\varphi|E_{m+}}_{m-p}(E_{m+},Z)\oplus C^{\varphi|E_{m-}}_{m-p}(E_{m-},Z) \arrow[r, "\eta"] & C^{\varphi|E_m}_{m-p}(E_m,Z) \arrow[r] \arrow[dl, "\partial"'] & 0 \\
0 \arrow[r] & C^{\varphi|E_{m-1}}_{(m-1)-p}(E_{m-1},Z) \arrow[r, "i"] & C^{\varphi|E_{m+}}_{(m-1)-p}(E_{m+},Z)\oplus C^{\varphi|E_{m-}}_{(m-1)-p}(E_{m-},Z) & &
\end{tikzcd}$$

where $i\sigma = (\sigma,-\sigma)$ and $\eta(\sigma^1,\sigma^2)=\sigma^1+\sigma^2$. $\partial^{\#}_{m-p}$ is defined as $i^{-1}\partial\eta^{-1}$. Now $\sigma_{m-p}=\eta(\sigma^1_{m-p}\oplus\sigma^2_{m-p})$. Therefore if supp $f^p \not\subset \sigma_{m-1}$ then $i^{-1}\partial(\,{}^{1}_{m-p}\ 0\ {}^{2}_{m-p}\,)$ must be a boundary element since σ_{m-p} can be considered to be strictly contained in E_{m-1}. Therefore the original diagram commutes up to a sign and if we make the convention that the orientation of E_m is chosen to make the diagram commute we then have $\underleftarrow{LH}^p_{\varphi|E_m}(E_m,H_m(S\otimes Z)) \approx \underleftarrow{LH}^p_{\varphi|E_m}(E_m,Z) \approx H^p_{\varphi|\underrightarrow{L}E_m}(\underrightarrow{L}E_m,Z) \approx H^p_{\varphi}(E,Z) \approx \underleftarrow{LH}^{\varphi|E_m}_{m-p}(E_m,Z) \approx H_{\infty-p}(E,Z)$.

Q.E.D.

<u>Definition 2</u>. In the previous definition and theorem, we had an ∞-p homology theory which is isomorphic to a sheaf cohomology which is equivalent to the singular cohomology with integer coefficients. In view of the use of the Mayer-Vietoris connecting maps in the inverse system which defines the homology, it does distinguish sets of finite codimension to some extent. But since there may be many representations $E \approx \underset{m\geq q}{\underleftarrow{L}}E_m$, the question still

remains that if given two such representations $E \approx \varprojlim_{m\geq q} E_m \approx \varprojlim_{m\geq q} F_m$ and the identity $i: E \longrightarrow E$, which takes the form

$$\begin{tikzcd} \downarrow & \downarrow \\ E_m \arrow[r, "i_m"] \arrow[d] & F_m \arrow[d] \\ E_{m-1} \arrow[r, "i_{m-1}"] & F_{m-1} \end{tikzcd}$$

where $\varprojlim_{m\geq q} i_m = i$ since both representations are homeomorphic to E and i_m is defined by the composition $E_m \longrightarrow E \xrightarrow{i} E \xrightarrow{p_m} F_m$ for the projection p_m of the second system; then does commute.

$$\begin{tikzcd} \varprojlim_{m\geq q} LH^{\varphi|E_m}_{m-p}(E_m, Z) \arrow[r, "\varprojlim L(i_m)_*"] \arrow[d, leftrightarrow] & \varprojlim_{m\geq q} LH^{\varphi|F_m}_{m-p}(F_m, Z) \arrow[d, leftrightarrow] \\ H^p_\varphi(E, Z) & H^p_\varphi(E, Z) \arrow[l, "i^*"'] \end{tikzcd}$$

It would therefore be most natural to unify these functors with a sheaf-theoretic homology with smooth locally closed sets of codimension p of the form $M^p \simeq \varinjlim_{m\geq q} M^p_m$ serving as the building blocks of a chain theory leading to an ∞ -p homology functor. A smooth locally closed subset M^p is understood to be of the form $U \cap M$, where M is a C^k submanifold of codimension p, U a smooth open set with a C^k boundary which is a submanifold without boundary and with the condition that any point set boundary of M^p is of codimension p+1 and of class C^k. We of course use the term submanifold here in the sense of being closed and without boundary. Since any point set boundary above sould be closed it would then be also a C^k submanifold. It is understood thet k refers to the class of E.

For M a submanifold of codimension p and of class C^k let $M \cap E_m = \bigcup M_m^c$, where M_m^c is a component. Now since M and E_m are two closed submanifolds then M_m^c is a manifold but possibly with a boundary. However, this can happen only if M winds about in E so that some collection considered as a subspace of covectors smoothly fits orthogonally to the transverse manifold E_m, where for convenience all of the transverse manifolds of the system $\underset{\overleftarrow{m \geq q}}{L} E_m$, each one corresponding to the m-parameter group of Chapter V is considered to be passing through some fixed $m \in M$. But then cover each point of M_m^c by an open set in E considered to be contained within the unique tubular neighborhood of M. Then we can take a finite subcover since M_m^c with a boundary will be compact. Therefore, we can differentiably deform M by a finite number of deformations each with support within one of the above sets so that $M_m^c \setminus \partial M_m^c$ is pushed off E_m into some E_n for $n > m$. Then we have a new collection $\bigcup N_m^c$ so that each N_m^c is a closed submanifold without boundary. This process can now be carried on inductively and since M has codimension p there are in total only a finite number of local deformation directions possible. Hence we may consider $\underset{\to}{L}M_m \approx \underset{\to}{L}N_m$ so that $N_m = \bigcup N_m^c$, where N_m^c is a component which is a closed finite dimensional manifold without boundary and so that there is a splitting for N_m, with $N_{m+} = N_m \cap E_{m+}$ and $N_{m-} = N_m \cap E_{m-}$. The fact that $E \approx \underset{\overleftarrow{m\geq q}}{L} E_m$ is actually used in the above argument in the sense that we were able to push off some E_m in a direction determined by a covector and map into a new E_n because all the coordinate directions are represented in the system $\underset{\overleftarrow{m\geq q}}{L} E_m$. We will assume then that $\underset{\overrightarrow{m\geq q}}{L} M_m$ satisfies the conditions specified for its homeomorphic image $\underset{\overrightarrow{m\geq q}}{L} N_m$.

We denote the point set boundary of a set X as $\dot{X}$. Now for $M^p = U \cap M$ as given above, we consider $\underset{\overrightarrow{m\geq q}}{L}(U \cap M_m) = \underset{\overrightarrow{m\geq q}}{L} M_m^p$. Since $U \cap M_m^c$ can be considered as an open submanifold or simply an open subset of a finite dimensional manifold without boundary, then the point set boundary, when it exists, is a closed submanifold and can have a boundary only if the boundary of this open set is contained in the interior of M_m^c since all the sets in question are smooth. But U as an open set of E was specified to have a smooth boundary of codimension 1 so that M^p has a boundary which is smooth and of codimension p+1. Hence it would be impossible for the point set boundary of $U \cap M_m^c$ to be contained in its interior, for this would force the smooth boundary of U which is a manifold without boundary to have a boundary. Hence we can consider $\dot{M}^p_m$ to be the union of manifolds without boundries and $\underset{m\geq 1}{\underrightarrow{L}}\dot{M}^p_m$ to satisfy the same properties as those specified for $\underset{m\geq 1}{\underrightarrow{L}}M^p_m$.

<u>Lemma 1</u>. $\underset{m\geq 1}{\underrightarrow{L}}M^p_m \simeq M^p$ and $\underset{m\geq 1}{\underrightarrow{L}}\dot{M}^p_m \simeq \dot{M}^p$.

<u>Proof</u> Consider the first case . $\underset{m\geq 1}{\underrightarrow{L}}M^p_m$ is dense in M^p. For the sequence $(m_i \mid m_i = (r_1(m), r_2(m), \quad , r_i(m), o, o \quad) \in M^p_i$, for $r_i(m)$ designating the parameter corresponding to the i-parameter group.) gives a collection with an element in each arbitrarily small neighborhood of $m \in M^p$. Now the parameter of course, refers to the transverse manifolds E_i. But since M is itself a closed submanifold of codimension p, and also of class C^k, we can join the point of M,which originally determined the positioning of the transverse manifolds, to the point m above. This path will be of finite length and hence compact. So then each point of the path can be covered by a neighborhood so that at most p local transverse directions of the i-parameter groups are not defined. But since a finite subcover can be selected there are only a finite number of total directions of the transverse i-parameter groups not defined within the entire open cover. So then if in the sequence (m_i), we take all values of i large enough, m_i will be a point of $M \cap E_i$. Since M_m is the union

of closed finite dimensional manifolds without boundary, trivial tubular neighborhoods can be constructed and by restricting everything to U, the proof proceeds exactly as that of Theorem V.2.

The case for $\underset{\overrightarrow{m\geq q}}{L}\dot{M}^p_m \simeq \dot{M}^p$ is proved in the same fashion.

Q.E.D.

Remark 1. Some recent work of Herrera [14] suggests the notion of pairing $M^p \simeq \underset{\overrightarrow{m\geq q}}{L}M^p_m$ with an algebraic element $c_{\infty -p}$ which is an element of some functor associated with M^p in order that a presheaf of chains might be constructed. It would seem that in general we would still be confronted with a situation where the definition of a boundary would be impeded because of the notion of an infinite number of faces. But a natural notion of codimension which allows one to canonically drop to a set of 1 higher codemension will overcome any possible trouble.

Definition 3. At this point we define prechain groups for smooth open $V \subset E$. It is without loss of generality to assume that any structure which gives rise to a sheaf is defined over a basis of smooth sets in a smooth manifold, since these sets will form a cofinal sub-collection of any collection of open sets. Below, $M^p = M \cap U \mid V$. This means that consideration is restricted to the manifold V and we say $M \cap V$ as a reminder that M may be the restriction of a closed submanifold of E.

$PJ_{\infty -p}(V,Z)$ is generated by pairs $(M^p, c_{\infty -p})$ where M^p

represents a smooth locally closed set as specified in Definition 2 above and $c_{\infty-p} = \varprojlim_{m \geq q} L(c_{m-p} \mid c_{m-p} \in H^{\varphi \mid M_m^\ell}_{m-p}(M_m^p, Z)$ and the connecting maps ∂_{m-p} are the Mayer-Vietoris boundaries of the exact sequence

$$\longrightarrow H^{\varphi \mid M_m^\ell}_{m-p}(M_m^p, Z) \longrightarrow H^{\varphi \mid M_{m-1}^\ell}_{(m-1)-p}(M_{m-1}^p, Z) \longrightarrow$$

$H^{\varphi \mid M_{m+}^\ell}_{(m-1)-p}(M_{m+}^p, Z) \oplus H^{\varphi \mid M_{m-}^\ell}_{(m-1)-p}(M_{m-}^p, Z) \longrightarrow$, which is induced from the splitting of each component of M_m^p.). Now M_m is the union of components of manifolds without boundary but some may have dimension $<$ m-p. However, $E \approx \varprojlim_{i \geq q} LE_i$ and at some point of a component every covector from M^p beyond some ith stage is eliminated as a transverse direction and hence the dimension of that component at that point is i-p. So consider the original component and the point to be contained in this new component of local dimension i-p which in view of the fact that this new component is a smooth submanifold must globalize so that the whole component has dimension i-p. There is also then a Poincare duality $H^{\varphi \mid M_m^\ell}_{m-p}(M_m^p, Z) \xleftrightarrow{\approx}$ $H^0_{\varphi \mid M_m^\ell}(M_m^p, Z \otimes \theta_m)$, where θ_m is an orientation sheaf with presheaf structure given as follows:

M_m^p is considered as an open manifold without boundary and contained in M_m. The contravariant functor for the presheaf structure is $U \cap M_m^p \longrightarrow H_{m-p}(M_m^p, M_m^p \setminus M_m^p \cap U, Z)$, where U is taken in E so that we may consider $U \cap M_m^p = U_m \cap M_m^p$ for $U_m \subset M_m^p$. This then gives a presheaf structure locally isomorphic to Z if the component has dimension m-p and locally isomorphic to o otherwise.

<u>Lemma 2</u>. $\underset{\overleftarrow{m\geq q}}{L}H^{\varphi|M^p_m}_{m-p}(M^p_m, Z) \approx H^o_{\varphi|M^p}(M^p, Z\otimes\Theta) \approx \underset{\overleftarrow{m\geq q}}{L}H^o_{\varphi|M^p_m}(M^p_m, Z\otimes\Theta_m)$ $\ni Z\otimes\Theta \approx \underset{\overleftarrow{m\geq q}}{L}H_{m-p}(S\otimes Z) \approx \underset{\overleftarrow{m\geq q}}{L}(\Theta_m\otimes Z) \approx (\underset{\overleftarrow{m\geq q}}{L}\Theta_m)\otimes Z$. Also $H^o_{\varphi|M^p}(M^p, Z\otimes\Theta)$ $\approx H^o_{\varphi|M^p}(M^p, Z\otimes\phi^p)$, where ϕ^p is an Eells type orientation sheaf [8] of the pair (E,M) restricted to the open set M^p, considered open in M. Consequently $\underset{\overleftarrow{m\geq q}}{L}H^{\varphi|M^p_m}_{m-p}(M^p_m, Z)$ is independent of the homotopic decomposition $M^p \simeq \underset{\overrightarrow{m\geq q}}{L}M^p_m$.

<u>Proof</u> By Lemma 1 $M \simeq \underset{\overrightarrow{m\geq q}}{L}M^p_m$. Therefore for a sheaf A, $H^o_{\varphi|M^p}(M^p, A) \approx H^o_{\varphi|\underset{\rightarrow}{L}M^p_m}(\underset{\rightarrow}{L}M^p_m, A) \approx \underset{\overleftarrow{m\geq q}}{L}H^o_{\varphi|M^p}(M^p_m, A \mid M^p_m)$. Let $A = \underset{\overleftarrow{m\geq q}}{L}H_{m-p}(S\otimes Z)$. where the inverse system is actually that of $\underset{\overleftarrow{m\geq q}}{L}(\Theta_m\otimes Z)$ with the connecting maps being the boundries of relative Mayer-Vietoris exact sequences

$H^{\varphi|M^p_{m+}}_{m-p}(M^p_{m+}, M^p_{m+}\setminus(M^p_{m+}\cap U_m), Z)\oplus H^{\varphi|M^p_{m-}}_{m-p}(M^p_{m-}, M^p_{m-}\setminus(M^p_{m-}\cap U_m), Z) \longrightarrow$
$\longrightarrow H^{\varphi|M^p_m}_{m-p}(M^p_m, M^p_m\setminus(M^p_m\cap U_m), Z) \xrightarrow{\partial_m} H^{\varphi|M^p_{m-1}}_{(m-1)-p}(M^p_{m-1}, M^p_{m-1}\setminus(M^p_{m-1}\cap U_m), Z) \longrightarrow$. The map is actually defined by mapping $gen_m\otimes Z \longrightarrow \partial_m gen_m\otimes Z$, where gen_n denotes generator, which is permissible since these functors are locally free in the sense of determining locally free groups. By excision the terms in the above exact sequence are either of the form $H_{m-p}(R^j, R^j\setminus o, Z)$ or $H_{m-p}(R^j_+, R^j_+\setminus o, Z)$, where R^j_+ denotes the half space.

Since $R^j_+ \setminus o$ is a deformation retract of R^j_+, the terms in the exact sequence containing these pairs are o. Also

$$H_{m-p}(R^j,Z) \approx \begin{cases} Z \text{ if } j = m-p \\ o \text{ if } j \neq m-p \end{cases} .$$

Therefore, we have exact sequences either of the form $o \longrightarrow o \longrightarrow o$ or $o \longrightarrow Z \longrightarrow o$ or $o \longrightarrow Z \longrightarrow Z$. Hence $\partial_m \mathrm{gen}_m = \mathrm{gen}_{m-1}$. Now since these groups are locally free, we then have the relation $\underset{\overleftarrow{m \geq q}}{L}(\Theta_m \otimes Z) \approx (\underset{\overleftarrow{m \geq q}}{L}\Theta_m) \otimes Z$.

In order to show that $H^o_{\Phi|M^p}(M^p, Z\otimes\Theta) \approx H^o(\underset{\rightarrow}{L}M^p_m, Z\otimes\Theta) \approx \underset{\overleftarrow{m \geq q}}{L} H^o_{\Phi|M^p_m}(M^p_m, Z\otimes\Theta_m)$ we only need to examine the following diagram as in the proof of Theorem 1 above:

$$\begin{array}{ccc} H^{\Phi|M^p_m}_{m-p}(M^p_m, Z) & \xleftarrow{D_m} & H^o_{\Phi|M^p_m}(M^p_m, H_{m-p}(S\otimes Z)) \\ \Big\downarrow \partial_{m-p} & & \Big\downarrow i^* \\ H^{\Phi|M^p_{m-1}}_{(m-1)-p}(M^p_{m-1}, Z) & \xleftarrow{D_{m-1}} & H^o_{\Phi|M^p_{m-1}}(M^p_{m-1}, H_{(m-1)-p}(S\otimes Z)) \end{array}$$

The horizontal maps are given by either the standard Poincare duality or the zero map if the dimension of the corresponding component of M^p_m is not m-p. Since we know that each point is finally included in components which all have the appropriate dimension m-p for $m \geq n_o$, for n_o depending on the particular point, the diagram commutes by the proof of Theorem 1. Therefore the required isomorphism is established.

Φ^p is defined first for M and then restricting to the open M^p, considered as an open set in M. The presheaf structure over $U \cap M$ is given by $U \cap M \longrightarrow H^*(C^*(U, U \setminus (M \cap U), Z))$. $H^*(C^*(U, U \setminus (M \cap U), Z)) \approx H^p(R^p, R^p \setminus o) = Z$. Now we can infer from Lemma 1 that $\underset{\substack{\rightarrow \\ m \geq q}}{L}(U_m, U_m \setminus (M_m^p \cap U_m)) \simeq (U, U \setminus (M^p \cap U))$ since this is true for each term of the pair and all the spaces and relative pairs are dominated by CW complexes. Therefore,

$$H^p(U, U \setminus (M^p \cap U), Z) \quad H^p(\underset{\rightarrow}{L}(U_m, U_m \setminus (M_m^p \cap U_m)), Z)$$

$$\underset{\substack{\leftarrow \\ m \geq q}}{L} H^p(U_m, U_m \setminus (M_m^p \cap U_m), Z).$$

There is also the following sequence of duality isomorphisms, consisting of Poincare and relative Poincare dualities:

$$H^p(U_m, U_m \setminus (M_m^p \cap U_m), Z) \longrightarrow H_{m-p}(M_m^p \cap U_m, Z) \longrightarrow H^o(M_m^p \cap U_m, Z) \xrightarrow{\approx} H_{m-p}(M_m^p, M_m^p \setminus (M_m^p \cap U_m), Z).$$

It is assumed that we are locally at a stage E_m so that the dimenson of the component under consideration is m-p. We then have the commutative diagram:

$$\begin{array}{ccc}
H^p(U_m, U_m \setminus (M_m^p \cap U_m), Z) & \xrightarrow{\approx} & H_{m-p}(M_m^p, M_m^p \setminus (M_m^p \cap U_m), Z) \\
\downarrow & & \downarrow \\
H^p(U_{m-1}, U_{m-1} \setminus (M_{m-1}^p \cap U_{m-1}), Z) & \xrightarrow{\approx} & H_{(m-1)-p}(M_{m-1}^p, M_{m-1}^p \setminus (M_{m-1}^p \cap U_{m-1}))
\end{array}$$

The horizontal maps are given by the above sequence of dualities, the vertical map on the left is that of the inverse system given above and the vertical map on the right is the Mayer-Vietoris connecting map as previously defined. The commutativity follows because all the groups in question are free with one generator for a local component and the maps in question map generators to generators in a unique fashion. It is also clear that all the maps commute with the restriction maps of the sheaf structures since these are only restriction maps of inclusions. Therefore, upon taking the corresponding direct limits in the sheaf structures we have $\underset{\overleftarrow{m \geq q}}{L} H_{m-p}(S \otimes Z) \approx (\underset{\overleftarrow{m \geq q}}{L} \Theta_m) \otimes Z \approx \Phi^p \otimes Z$.

We can conclude that $\underset{\overleftarrow{m \geq q}}{L} H_{m-p}^{\varphi | M_m^p}(M_m^p, Z)$ is independent of any decomposition $M^p \simeq \underset{\overrightarrow{m \geq q}}{L} M_m^p$ because the above relations are actually dependent only upon the covectors of M^p and the support of an element $c_{\infty - p}$ is independent of any homotopic decomposition for M^p.

Q.E.D.

<u>Note</u> We must keep in mind that as in the proof of Theorem 1 the relation $\underset{\overleftarrow{m \geq q}}{L} H_{m-p}^{\varphi | M_m^p}(M_m^p, Z) \approx H^0_{\varphi | M^p}(M^p, Z \otimes \Theta)$ is given by fudging the orientation sign of Θ_m, but in a fashion only dependent upon p, so that this fudge factor is passed on uniformly to $Z \otimes \Phi^p$ since it is a uniform orientation reversal of E_m.

<u>Definition</u> <u>4</u>. Given two pairs $(M^p, c_{\infty - p})$ and $(N^p, d_{\infty - p})$ so that $M^p \simeq \underset{\overrightarrow{m \geq q}}{L} M_m^p$ and $N^p \simeq \underset{\overrightarrow{m \geq q}}{L} N_m^p$ let $L = (\overline{M}^p \setminus M^p) \cup (\overline{N}^p \setminus N^p)$. Now $\overline{M}^p \setminus M^p$ is either empty or the point set boundary $\dot{M}^p$ of M^p, which is a smooth closed submanifold of

codimension p+1 and of course without boundary. Now for a fixed representation $E \approx \varprojlim_{m \geq q} LE_m$ we then have from the discussion in Definition 2 and Lemma 1 that $(M^p \cup N^p) \setminus L \simeq \varinjlim_{m \geq q} L((M^p_m \cap N^p_m) \setminus L_m)$, where $L_m = \dot{M}^p_m \cup \dot{N}^p_m$. Since the boundaries in question arise from at most two smooth closed submanifolds of codimension p-1 and of course without boundary we can assume without loss of generality as in the discussion of Definition 2 that all the summands of the direct limits are the union of components which themselves are unions of finite dimensional manifolds without boundary, although when intersecting with an open set of E, the submanifolds may be open. The only difference above is that we are taking the union of two smooth locally closed sets of codimension p+1 whose covectors at points of their intersections may not determine the same subspace.

Define $(M^p, c_{\infty - p}) + (N^p, d_{\infty - p}) =$

$$= ((M^p \cup N^p) \setminus L), \bar{c}_{\infty - p} + \bar{d}_{\infty - p}).$$

where $\bar{c}_{\infty - p} + \bar{d}_{\infty - p}$ is defined by considering the composition

$$H^{\Phi|M^p_m}_{m-p}(M^p_m, Z) \longrightarrow H^{\Phi|M^p_m \setminus L_m}_{m-p}(M^p_m \setminus L_m, Z) \longrightarrow H^{\Phi|M^p_m \cup N^p_m \setminus L_m}_{m-p}(M^p_m \quad N^p_m \quad L_m, Z).$$

Since $c_{\infty - p} = \varprojlim_{m \geq q} Lc_{m-p}$ we take $\bar{c}_{m-p} =$ the image of c_{m-p} under the above composition. Since the maps of that composition are only a natural restriction and inclusion, they must commute with the Mayer-Vietoris connecting maps of the inverse systems. Also $\underset{\overleftarrow{m\geq q}}{L}H_{m-p}(M_m^p \cup N_m^p \setminus L_m, Z)$, where of course the maps of this inverse system are induced by the Mayer-Vietoris splitting of $(M_m^p \cup N_m^p) \setminus L_m$, is independent of any homotopic decomposition $(M^p \cup N^p) \setminus L \simeq \underset{\overrightarrow{m\geq q}}{L}((M_m^p \cup N_m^p) \setminus L_m)$ because the homology inverse system only picks out locally, cells of dimension m-p. Hence these cells must come from either M_m^p or N_m^p if these sets intersect locally in a fashion that the covectors determine two different subspaces. If this happens then the inverse system must have support in either M^p or N^p, whichever determines the initial m-p dimensional support cell, because then for the system to have support outside this particular codimensional set would force a support cell at some stage n-p for $n > m$ to be contained in E_n, but not in $M_n \cup N_n$ since the codimension would then be determined by some subspace of the span of the two spaces determined by the covectors of M^p and N^p. But then the interior of this support cell would clearly span some points not in $M_n^p \cup N_n^p$. We should take note that we are using the fact that $E \approx \underset{\overleftarrow{m\geq q}}{L}E_m$ in that all coordinate directions at each point are determined by the system $\underset{\overleftarrow{m\geq q}}{L}E_m$. Now if locally the sets M^p and N^p determine the same p dimensional cospace then clearly the support of the ∞-p cocell can be taken in either set M^p or N^p simultaneously and the results of Lemma 2 guarantee that it determines the same algebraic element locally.

We can then add $\bar{c}_{m-p}$ and $\bar{d}_{m-p}$ so that $\bar{c}_{\infty-p} + \bar{d}_{\infty-p}$

$= \underset{\underset{m\geq q}{\leftarrow}}{L}\bar{c}_{m-p} + \underset{\underset{m\geq q}{\leftarrow}}{L}\bar{d}_{m-p} = \underset{\underset{m\geq q}{\leftarrow}}{L}(\bar{c}_{m-p} + \bar{d}_{m-p})$. Now this addition is then well defined and independent of any representation $\underset{\underset{m\geq q'}{\leftarrow}}{L}E_m \approx E$ from the above discussion.

We define an equivalence relation $(M^p, c_{\infty-p}) \sim (N^p, d_{\infty-p})$ $\Longleftrightarrow (M^p, c_{\infty-p}) + (N^p, -d_{\infty-p}) = (M^p \cup N^p \setminus L, o)$, where $o = \underset{\leftarrow}{L}o_m$ for o_m the zero determined by the functor $H_{m-p}(\cdot, Z)$. Then let $J_{\infty-p}(V,Z) = PJ_{\infty-p}(V,Z)/\sim$.

A boundary homomorphism $\partial_{\infty-p}$ is defined as follows:
$\partial_{\infty-p}: J_{\infty-p}(V,Z) \longrightarrow J_{\infty-(p+1)}(V,Z)$, where
$\partial_{\infty-p}(M^p, c_{\infty-p}) = (\dot{M}^p, \underset{\leftarrow}{L}\,\partial_{\dot{M}^p_m}(c_{m-p}))$ for $\partial_{\dot{M}^p_m}$ the connecting homomorphism of the exact sequence $\longrightarrow H^{\varphi|M^p_m}_{m-p}(M^p_m, Z) \longrightarrow H^{\varphi|\dot{M}^p_m}_{m-(p+1)}(\dot{M}^p_m, Z) \longrightarrow H^{\varphi|\bar{M}^p_m}_{m-(p+1)}(\bar{M}^p_m, Z) \longrightarrow$. This is the same form used by Herrera [14] for his semianalytic chains, although his notation is ambiguous due to the omission of a symbol. He uses the notation $\partial_{M^p_m, \dot{M}^p_m}$ for the connecting homomorphism, with the positioning of M^p_m in the symbol forcing it to be the dominating space in the above exact sequence. But since M^p_m does not contain $\dot{M}^p_m$ in general, this would be wrong. The above change is the only possible one. Since $\dot{M}^p$ is a closed submanifold of codimension 1 within $M \cap V$,

where $M^p = M \cap U \mid V$ for $M \cap U$ the smooth locally closed sets specified in Definition 2, and since $\dot{M}^p$ is actually the point set boundary of $M \cap U$ restricted to V, we can assume the existence of a unique tubular neighborhood of $\dot{M}^p$ of codimension 1 considered as a subset of $M \cap V$ by constructing a tubular neighborhood of codimension 1 about the point set boundary of $M \cap U$ and then restricting to V. We must also be careful to remember that $\dot{M}^p$ is always taken as the point set boundary of $M \cap U \mid V$ and taken with respect to the relative topology of $E \cap V$. But as above, we can consider this to be a restriction of the boundary of the set $M \cap U \subset E$ since V is a smooth open set.

Since we ultimately only need to consider components of M_m^p which contribute support cells of dimension m-p, we can assume that we are at a stage where the covector of the trivial tube locally determines a transverse direction which projects onto E_m and hence is deformed into M_m. This then gives locally a tubular neighborhood of $\dot{M}_m^p$ in M_m^p of codimension 1. But since our boundary homomorphism in the above exact sequence is defined by the following diagram:

$$\begin{tikzcd} & C^{\Phi|\bar{M}_m^p}_{m-p}(\bar{M}_m^p, Z) \arrow[r, "\eta"] \arrow[d, "\partial"] & C^{\Phi|M_m^p}_{m-p}(M_m^p, Z) \arrow[r] & 0 \\ 0 \arrow[r] & C^{\Phi|\dot{M}_m^p}_{m-(p+1)}(\dot{M}_m^p, Z) \arrow[r, "i"] & C^{\Phi|\bar{M}_m^p}_{m-(p+1)}(\bar{M}_m^p, Z) & \end{tikzcd}$$

where $\partial^{\#}_{\dot{M}_m^p} = i^{-1} \partial \eta^{-1}$, it then becomes a local restriction map about $\dot{M}^p_m$ within this local tube, where the closures are always understood to take place with respect to the relative topology of V. Hence we can assume that our boundary homomorphism is actually equivalent to the composition

$$H^{\varphi|M^p_m}_{m-p}(M^p_m, Z) \longrightarrow H^{\varphi|M_m\cap V\setminus \dot{M}^p_m}_{m-p}(M_m \cap V \setminus \dot{M}^p_m, Z) \longrightarrow H^{\varphi|\dot{M}^p_m}_{m-(p+1)}(\dot{M}^p_m, Z),$$ where

the first map is an inclusion and the second is a boundary homomorphism in the exact sequence of the pair $(M_m \setminus V, \dot{M}^p_m)$, using the fact that $H^{\varphi|M_m\cap V}_{m-p}(M_m \cap V, \dot{M}^p_m; Z) \approx H^{\varphi|M_m\cap V\setminus \dot{M}^p_m}_{m-p}(M_m \cap V \setminus \dot{M}^p_m, Z)$.

Of course, we must remember that our attention is actually restricted to some component of M_m, but since all coordinate directions are engulfed by the transverse direction of the system $\varprojlim_{m\geq q} LE_m \approx E$, every point of $\varinjlim_{m\geq q} LM^p_m$ is eventually contained in a component of some M^p_m with the properties specified above.

<u>Note</u> Although Borel-Moore homology is specified for the use of sheaves we can treat this homology as being equivalent to the Singular or Čech Theories for manifolds of the class being used.

The next lemma will give a uniqueness for $\partial_{\infty-p}$ with respect to pairs $(M^p, c_{\infty-p}) \in PJ_{\infty-p}(V,Z)$ with respect to any homotopy decomposition for M^p with respect to any representation $E \approx \varprojlim_{m\geq q'} LE_m$. We can then pass to a uniqueness with respect to $J_{\infty-p}(V,Z)$ by applying Herrera's result [14] at each stage of the inverse systems.

Lemma 3. $\partial_{\infty -p}$ is independent of any homotopy decomposition $M^p \simeq \underset{\substack{\rightarrow \\ m\geq q}}{L} M^p{}_m$, where $M^p = M \cap U \mid V = M \cap U \cap V$, for M and U satisfying the conditions of Definition 2 and the total object being considered a submanifold within the manifold V. For a pair $(M^p, c_{\infty -p}) \in PJ_{\infty -p}(V,Z)$, $\partial_{\infty -p}$ is given by the composition

$$H^o_{\varphi|M^p}(M^p, Z\otimes\Phi^p) \xrightarrow{j^*} H^o_{\varphi|}(M\cap V \setminus \dot{M}^p, Z\otimes\Phi^p) \xrightarrow{\delta} H^1_{\varphi|M\cap V}(M\cap V, M\cap V\setminus \dot{M}^p, Z\otimes\Phi^p)$$

$\approx H^o_{\varphi|\dot{M}^p}(\dot{M}^p, Z\otimes\Phi^{p+1})$, where δ is the connecting map of the exact sequence of the pair $(M\cap V, M\cap V\setminus \dot{M}^p)$ and j^* is a natural restriction map.

Proof From the above discussion we know that $\partial_{\infty -p}$ is defined by taking the inverse limit with respect to Mayer-Vietoris connecting maps of the composition

$$H^{\varphi|M^p_m}_{m-p}(M^p_m, Z) \xrightarrow{i_*} H^{\varphi|M_m\cap V\setminus \dot{M}^p_m}_{m-p}(M_m \cap V \setminus \dot{M}^p_m, Z) \longrightarrow H^{\varphi|\dot{M}^p_m}_{m-(p+1)}(\dot{M}^p_m, Z).$$

But as before $\underset{\rightarrow}{L}(M_m \cap V, M_m \cap V \setminus \dot{M}^p_m) = \underset{\rightarrow}{L}(M_m \cap V_m, M_m \cap V_m \setminus \dot{M}^p_m)$ $\simeq (M\cap V, M\cap V\setminus \dot{M}^p)$ from Lemma 1 since $\underset{\substack{\rightarrow \\ m\geq q}}{L} M_m \cap V_m \simeq M\cap V$ and $\underset{\substack{\rightarrow \\ m\geq q}}{L} M_m \cap V_m \setminus \dot{M}^p_m \simeq M\cap V\setminus \dot{M}^p$ and since the relative pairs are dominated by relative pairs of CW complexes.

Then we can again apply the dualities

$$\begin{array}{ccccc}
H^{\varphi|M^p_m}_{m-p}(M^p_m, Z) & \xrightarrow{i_*} & H^{\varphi|}_{m-p}(M_m \cap V_m \setminus \dot{M}^p_m, Z) & \longrightarrow & H^{\varphi|\dot{M}^p_m}_{m-(p+1)}(\dot{M}^p_m, Z) \\
D_m \uparrow & & D_m \uparrow & & D_m \uparrow \\
H^o_{\varphi|M^p_m}(M^p_m, Z\otimes\Theta_m) & \xrightarrow{j^*} & H^o_{\varphi|}(M_m \cap V_m \setminus \dot{M}^p_m, Z\otimes\Theta_m) & \longrightarrow & H^1_{\varphi|M_m\cap V_m}(M_m \cap V_m, M_m \cap V_m \setminus \dot{M}^p_m, Z\otimes\Theta_m)
\end{array}$$

where the commutativity of the square on the right is due to the standard commutativity of the squares formed by relating the dual homology and cohomology sequences by duality maps. The first square commutes since i and j are only natural inclusions and restrictions.

Then by the proof of Theorem 1 the inverse systems with connecting maps which are Mayer-Vietoris boundaries commute with the above maps, including the relative case which is treated in exactly the same fashion as the absolute homology functor. Therefore in passing to the inverse limit we have $\partial_{\infty - p}$ being represented under duality by the composition

$$H^o(M^p, Z\otimes\Phi^p) \xrightarrow{j^*} H^o(M \cap V \setminus \dot{M}^p, Z\otimes\Phi^p) \xrightarrow{\delta} H^1(M \cap V, M \cap V \setminus \dot{M}^p, Z\otimes\Phi^p).$$

But since have the tubular neighborhood of codimension 1 of $\dot{M}^p$ considered to be a neighborhood in the space $M \cap V$, $\exists$ an Eells type orientation sheaf as described in Lemma 2. Hence we can apply the Eells spectral sequence [8] to give $H^1_{\varphi|M\cap V}(M \cap V, M \cap V \setminus \dot{M}^p, Z\otimes\Phi^p) \approx H^o_{\varphi|\dot{M}^p}(\dot{M}^p, Z\otimes\Phi^p\otimes\Phi^1)$, where this is constructed with respect to the manifold V, with $M \cap V$ as a closed submanifold and $\dot{M}^p$ as a closed submanifold of $M \cap V$, and of course all manifolds are without boundary. Then there is a pairing $\Phi^p\otimes\Phi^1 \approx \Phi^{p+1}$ described as follows:

We consider the uniquely determined isomorphism $H^p(R^p, R^p\setminus o, Z)\otimes H^1(R^1, R^1\setminus o, Z) \approx H^{p+1}(R^p\times R^1, R^p\times R^1\setminus o \cup R^p\setminus o\times R^1, Z) \approx H^{p+1}((R^p, R^p\setminus o)\times(R^1, R^1\setminus o), Z)$ which is a form of the relative

Künneth theorem which in this case gives a canonical pairing since the groups are free on a finite number of generators. The terms on the left of this tensor pairing represent the presheaf structures of Φ^p and Φ^1 and $(R^p \times R^1, R^p \times R^1 \setminus o \cup R^p \setminus o \times R^1) \simeq (R^p, R^p \setminus o)$ forces the right terms of the tensor pairing to represent the presheaf structure of Φ^{p+1} which is the Eells orientation sheaf for the pair $(M \cap V, \dot{M}^p)$. Therefore, upon taking direct limits and passing to the appropriate sheaf structures we have the pairing $\Phi^p \otimes \Phi^1 \approx \Phi^{p+1}$ which gives $H^o_{\varphi|\dot{M}^p}(\dot{M}^p, Z \otimes \Phi^p \otimes \Phi^1) \approx H^o_{\varphi|\dot{M}^p}(\dot{M}^p, Z \otimes \Phi^{p+1})$.

Therefore, $\partial_{\infty -p}$ is completely independent of any homotopic decomposition $M^p \simeq \underset{\overrightarrow{m \geq q'}}{L} M^p_m$ arising from some representation $E \approx \underset{\overleftarrow{m \geq q'}}{L} E_m$ since the support of the image under $\partial_{\infty -p}$ is completely determined by the covectors of $\dot{M}^p$ and there is a canonical isomorphism with respect to the algebraic structure.

Q.E.D.

Remark 2. Now since the definition given for addition and the boundary in the groups $PJ_{\infty -p}(V,Z)$ is independent of any homotopic decomposition we can conclude that all of these maps are independent of the equivalence relation by simply performing the operations in each finite dimension and taking inverse limits with respect to the systems we have defined with the Mayer-Vietoris connecting maps and then apply (Herrera's) results, which give the prechain homology theory we have described for semianalytic sets of finite dimension. But since $E_m \subset R^m$ is a relatively simple orientable C^k manifold his results apply. Hence we can pass to the groups $J_{\infty-p}(V,Z)$. Also $\partial_{\infty-(p+1)}\partial_{\infty-p} = o$ and $\partial_{\infty-p}$ commutes with addition since this is true at each finite dimensional stage, although the fact that $\partial_{\infty-(p+1)}\partial_{\infty-p} = o$ is simply due to the fact that $\dot{M}^p$ has no point set boundary. We of course also agree to admit the pair $(\emptyset, o) \in J_{\infty-p}(V,Z)$.

Now for $W \subset V$ there is a natural restriction map $J_{\infty-p}(V,Z) \longrightarrow J_{\infty-p}(W,Z)$ induced by the natural restriction map of (Herrera) at each finite dimensional stage along with the fact that a natural restriction map commutes with a Mayer-Vietoris boundary. Then also the restrictions commute with all of the other maps defined above so that we may pass to an induced sheaf which will simply be denoted by $J_{\infty-*}$. Then define ${}^{q}H_{\infty-p}(E,Z) = H_p(\Gamma J_{\infty-*})$.

<u>Theorem 2</u>. ${}^{q}H_{\infty-p}(E,Z) \approx H_{\infty-p}(E,Z)$ for $H_{\infty-p}(E,Z)$ defined as in Theorem 1 for a specific representation $E \approx \underset{\underset{m \geq q}{\leftarrow}}{L} E_m$. This is then canonical in the sense of Definition 2.

<u>Proof</u> For the fixed representation $E \approx \underset{\underset{m \geq q}{\leftarrow}}{L} E_m$ we can project $J_{\infty-p}(V,Z)$ to each E_m and by definition obtaine the relation $\underset{\leftarrow}{L} J_{m-p}(V_m,Z) = J_{\infty-p}(V,Z)$ where the maps of the in-

verse system are the Mayer-Vietoris maps defined on the pairs. Now, Herrera defines his prechain groups for pairs (M,c) where M is a semianalytic locally closed set of dimension m-p and $c \in H^{\Phi|M}_{m-q}(M,Z)$ with Borel-Moore homology specified. But since E_m is an orientable open C^k submanifold contained in R^m there is no more homology information needed then that contained in the system by restricting our attention to smooth locally closed sets of the type $N^p \cap E_m$, where N^p is described in Definition 2. This is because E_m, can be smoothly triangulated, and since the inverse system $\underleftarrow{L}E_m$ is determined by m-parameter groups of diffeomorphisms, there is a set $N^p \cap E$ obtained by locally extending all transverse directions above a set in E_m which is smooth and locally closed in the above sense. Therefore, by passing to the sheaf structure the Herrera theory guarantees that $H^{\Phi|E_m}_{m-p}(E_m,Z) \approx H^{\Phi|E_m}_{*}(\Gamma J_{m-p})$. Then since the Mayer-Vietoris boundaries of the inverse system commute with the restriction maps of the presheaf structure, we then have $\mathcal{H}_{\infty-p}(E,Z) \approx \underleftarrow{L}H^{\Phi|E_m}_{m-p}(E_m,Z) \approx H_{\infty-p}(E,Z)$. The last isomorphism follows because the Mayer-Vietoris connecting maps in Theorem 1 were actually defined with respect to local support.

Therefore, since $\mathcal{H}_{\infty-p}(E,Z)$has elements with support uniquely specified the following diagram must commute, where $\underleftarrow{L}E_m \approx \underleftarrow{L}F_m$ are two representations of Theorem V.1 of E.

$$\begin{tikzcd}
\underleftarrow{LH}^{\Phi|E_m}_{m-p}(E_m,Z) \arrow[rr, "\underleftarrow{L}(i_m)_*"] \arrow[d, leftrightarrow, "\approx"] & & \underleftarrow{LH}^{\Phi|F_m}_{m-p}(F_m,Z) \arrow[d, leftrightarrow, "\approx"] \\
H^p_\Phi(E\ ,Z) & & H^p_\Phi(E,Z) \arrow[ll, "i^*"']
\end{tikzcd}$$

Hence we have a canonical isomorphism.

Q.E.D.

<u>Remark</u> 3. If $f: E \longrightarrow F$ is a map between two manifolds satisfying the conditions of Theorem V.1 with the properties that f is differentiable and for $e \in E$, $\ker df_e = \operatorname{coker} df_e = o$, then there is a uniquely defined map $\underleftarrow{Lf}^*_m : \underleftarrow{LH}_{m-p}(E_m,Z) \longrightarrow \underleftarrow{LH}_{m-p}(F_m,Z)$ provided f_m defined by the composition $E_m \rightarrow E \xrightarrow{f} F \xrightarrow{p_m} F_m$ respects orientation. This is defined by mapping a support cell of dimension m-p onto a support cell of dimension m-p as a result of the conditions above. Then locally this maps $H^{\Phi|N}_{m-p}(N,Z) \longrightarrow H^{\Phi|f_m(N)}_{m-p}(f_m(N),Z)$. Then by passing to the sheaf structure and taking the homology the required map is defined.

<u>Remark</u> 4. In 1935, J.W. Alexander [1] defined ∞-p-1 cells in a separable Hilbert space so that there is a homology theory and a duality with respect to a compact metric space $K \subset H$. But since the duality is between $H^p(K,Z)$ and the ∞-p-1 homology of $H \setminus K$, and since $H \setminus K$ is diffeomorphic to H at least when K is a manifold, it would seem that these

are homologies constructed in some way so that they are determined by non-cofinal systems of support sets.

CHAPTER VII

CONCLUSION

We have studied separable infinite dimensional manifolds by assuming a smooth local Hilbert structure which allows a diffeomorphism to be constructed between each fibre of the tangent space and a neighborhood of the base point on the manifold. This allows the construction of trivial tubular neighborhoods of closed finite dimensional submanifolds. Then arbitrarily small neighborhoods of this type support diffeomorphisms so that upon taking the infinite product of such diffeomorphisms we have $E \approx E \setminus K$, for K a thin subset.

The whole technique can then be considered as a generalized foliation category, consisting of sequences of foliations which are ordered so that each sequence is a collection of elements so that a higher order element foliates any lower order element with an associated collection of m-parameter groups, with the m-parameter group being a subgroup of the n-parameter group for $n \geqslant m$ by intersecting domains of definition. Each m-parameter group is globally abelian and generates an m dimensional submanifold considered as an open set of R^m and a closed transverse submanifold of E. These groups are generated by approximating distance functions, defined with respect to the metric of E, with smooth functions that have singularities in a thin subset so that any maximum solution curve, defined by composing the approximating function with a projection, is defined globally. Then the above principle of removing thin subsets differentiably allows the group structure to be defined.

In such a foliated category a $H_{\infty-p}(\cdot, Z)$ functor has been established with the use of a Mayer-Vietoris splitting of the manifolds associated with the m-parameter groups. This functor distinguishes sets of finite codimension and satisfies a duality with $H^p(\cdot, Z\otimes\Phi^p)$ in terms of a Poincaré duality isomorphism.

At the center of the principle of the removal of thin subsets differentiably is a renorming process which utilizes different Cauchy completions. It seems possible to generalize the above structure to give a space generated by a collection of sequences of foliations with a topology and associated structure determined by combining the associated collection of functors $H_{\infty-p}(\cdot, Z)$, one for each sequence, and relating them by a functor which is dependent upon the m-parameter groups and local collections of completions.

It also seems reasonable that a decent generalization will take into account Finsler instead of just Hilbert structures.

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