THE ORDER-SUM IN CLASSES OF PARTIALLY ORDERED ALGEBRAS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

Houston, Texas

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

Margret Hesse Höft

May 1973

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Abstract

Let $\mathcal{K}$ be a class of partially ordered algebras, T a partially ordered set. A family of homomorphisms $\phi_t : P_t \longrightarrow P$, where P, $P_t \varepsilon \mathcal{K}$, is called a T-family provided that $\phi_s(x) \leq \phi_t(y)$ in P, for all $x \varepsilon P_s$, $y \varepsilon P_t$, whenever $s < t$ in T. The order-sum in $\mathcal{K}$ is a universal T-family. I.e., the T-family $\phi_t : P_t \longrightarrow P$ is an order-sum if, for each $Q \varepsilon \mathcal{K}$ and each T-family $\psi_t : P_t \longrightarrow Q$, there is a unique homomorphism $\psi : P \longrightarrow Q$ such that $\psi \circ \phi_t = \psi_t$, for all $t \varepsilon T$.

The following existence theorem is derived: The order-sum exists without any restriction if $\mathcal{K}$ is a quasiprimitive class. Simple conditions on the class $\mathcal{K}$ are found, to ensure that the homomorphisms ϕ_t in an order-sum $\phi_t : P_t \longrightarrow P$ are order-embeddings and that their images are pairwise disjoint.

An internal characterization is given for the order-sum in the class of all k-join-semilattices. Also, the order-sum in the class of all partially ordered algebras of a given type is completely described.

Preface

Order-sums of distributive lattices have been defined and studied by Balbes and Horn [1]. The order-sum of lattices appears in Lakser [5]. There, however, it is not refered to as an order-sum, but rather as a partially ordered free product of lattices. [1] and [5] apparently are the only two sources, where order-sums occur in the literature.

In this thesis, we extend the notion of an order-sum to arbitrary classes of partially ordered algebras. The order-sum of partially ordered algebras will be defined as the solution of a universal problem, and as a generalisation of the coproduct.

In Part I , we develop a general theory of the order-sum. In particular, we describe the order-sum for the class of a l l partially ordered algebras (with and without constant operations), and prove a general existence theorem. We also find conditions on a class of partially ordered algebras that force the order-sum to be an (order-) extension of the lexicographic sum. The results of our general theory are then applied to classes of lattices and semilattices.

In the second part, we study the order-sum for one particular class of partially ordered algebras, the class $\mathcal{K}$ of k-join-semilattices. The main result in Part II is an internal characterisation of the order-sum for this class $\mathcal{K}$.

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Part I

General theory

In this first part we define the order-sum for partially ordered algebras, and prove that the order-sum and lexicographic sum coincide for the class of all partially ordered algebras. Furthermore, we show that the order-sum exists without restriction in quasi-primitive classes (cf. Theorem 3.2). Some properties of the order-sum are investigated in section 4.

1. Order-sum and lexicographic sum

We will consider classes of partially ordered algebras. The algebras under consideration will be *partial algebras* $(A,(f_i)_{i\varepsilon I})$ *of* arbitrary finitary or infinitary type $\Delta = (K_i)_{i\varepsilon I}$. I.e., the *index-sets* K_i may be finite or infinite, and f_i is a mapping of a subset of A^{K_i} into A. If the domain of f_i is all of A^{K_i}, for each $i\varepsilon I$, we may call $(A,(f_i)_{i\varepsilon I})$ a *complete algebra* of type Λ. A *partially ordered algebra* is a triple $(A,(f_i)_{i\varepsilon I},\leqslant)$, where $(A,(f_i)_{i\varepsilon I})$ is a partial algebra and $(A,\leqslant)$ a partially ordered set. The algebraic structure might be empty, $I=\emptyset$. In that case, the partially ordered algebra is nothing but a partially ordered set, and any class of partially ordered sets is an example of a class of partially ordered algebras. Since, on the other hand, the partial order may be total disorder, i.e. distinct elements are incomparable, we can also interpret any class of partial algebras as a class of partially ordered algebras.

For the general theory, it would be unwise to require any kind of compatibility postulates to hold between the algebraic structure and the partial order (Actually, it would be near to impossible to merely formulate a satisfying condition of that type in sufficient generality.). This does not mean, of course, that there is any ban on compatibility conditions in applications (as there is no ban on admitting other axioms for the description of the classes $\mathfrak{K}$ we want to consider). Take, for instance, a lattice L - here to be considered as a partially ordered algebra $(L, \wedge, \vee, \leq)$ - as an example for the strongest feasible compatibility interrelations, inasmuch as each of the three data given in L even determines the other two. In other examples, the determination of some parts of the structure by the others may only work one way. Alternatively, we may have, in fact, independent fundamental data still linked together by some reasonable compatibility axioms (e.g. partially ordered semigroups).

A *homomorphism* of the partially ordered algebra $(A,(f_i)_{i \varepsilon I}, \leq)$ into the partially ordered algebra $(B,(g_i)_{i \varepsilon I}, \leq)$ - of the same type Δ - is a mapping $\phi : A \longrightarrow B$ that is *order-preserving*:

$$\text{(1.1)} \qquad \text{if } x \leq y, \text{ then } \phi(x) \leq \phi(y) \ ,$$

for all elements $x, y \varepsilon A$, and at the same time an *algebraic homomorphism*:

$$\text{(1.2)} \qquad \phi(f_i(a_\kappa | \kappa \varepsilon K_i)) = g_i(\phi(a_\kappa) | \kappa \varepsilon K_i),$$

for each index $i \varepsilon I$, for each sequence $(a_\kappa)_{\kappa \varepsilon K_i}$ in the domain of f_i (making the left side exist - it is understood that the right side will then exist too).

In the sequel, $\mathfrak{K}$ will always be a class of partially ordered algebras of the same type Δ. Homomorphisms between $\mathfrak{K}$-algebras will, unless otherwise stated, always be homomorphisms as defined above.

Given a class $\mathfrak{K}$, the order-sum we are going to define now will be a generalization of the coproduct in the class (category) $\mathfrak{K}$. Like the coproduct, it will be the solution of a universal problem. However, whereas the coproduct can be defined in classes not involving any order, it takes partial orders in the objects under consideration, say P_t ($t \in T$), in order to define their order-sum. It even takes a partial order of the index-domain T. Given, in short, a *partially ordered family* of partially ordered algebras $P_t \in \mathfrak{K}$, a family of homomorphisms $\phi_t : P_t \longrightarrow P$, where P is also supposed to be in $\mathfrak{K}$, is called a *T-family* provided that the following condition holds true for all indices $s, t \in T$:

(1.3) $\qquad$ if $s < t$ (in T), then $\phi_s(x) \leq \phi_t(y)$ (in P),

for all elements $x \in P_s$, $y \in P_t$. The *order-sum* is now simply a universal T-family. I.e., the T-family $\phi_t : P_t \longrightarrow P$ is an order-sum if, for each partially ordered algebra $Q \in \mathfrak{K}$ and each T-family $\psi_t : P_t \longrightarrow Q$, there is a unique homomorphism $\psi : P \longrightarrow Q$ such that $\psi \circ \phi_t = \psi_t$, for all indices $t \in T$.

In the special case where the index-set T as well as all algebras P_t are totally disordered, the order-sum coincides with the coproduct. One may even admit, for that matter, that the algebras P_t are, in fact, endowed with some non-trivial partial orders, reinterpreting the latter as partial binary operations, say, the maximum-operations.

Clearly, the order-sum (if it exists) will be unique up to unique isomorphism, and in that sense it is justified to talk about "the" order-sum.

Assume now that $\mathfrak{K}$ is the class of a l l partially ordered algebras of a given type $\Delta = (K_i)_{i\varepsilon I}$, and assume further that the type Δ is without constants, i.e. $K_i \neq \emptyset$ for each $i\varepsilon I$. The algebraic lexicographic sum of partially ordered algebras that we are going to define is a combination of the partial direct sum of partial algebras (cf. Schmidt [8]) and of the well-known lexicographic sum of partially ordered sets (cf. Birkhoff [2], Schmidt [6],[7]).

For a partially ordered family of partially ordered algebras P_t, we define

$$\bigsqcup_{t\varepsilon T} P_t$$

to be the set of all ordered pairs (t,x), where $t\varepsilon T$ and $x\varepsilon P_t$, endowed with the *lexicographic order* :

(1.4) $\qquad (s,x) \leq (t,y)$ iff $s < t$ or $s = t$ and $x \leq y$.

The *natural mappings* $i_t : P_t \longrightarrow \bigsqcup P_t$, defined by $i_t(x) = (t,x)$, are obviously order-preserving, even order-embeddings (cf.4.). On $\bigsqcup P_t$, there exists now the "weakest" algebraic structure $(f_i)_{i\varepsilon I}$ such that the natural mappings i_t become algebraic homomorphisms, i.e. the final algebraic structure for the mappings i_t (cf. Bourbaki

[3], Schmidt [8]). For an index $i \varepsilon I$, the operation f_i is explicitly given by

$$(1.5) \qquad f_i((t_\kappa, x_\kappa) | \kappa \varepsilon K_i) = i_t(f_{ti}(x_\kappa | \kappa \varepsilon K_i)) .$$

Here, f_{ti} denotes the corresponding operation in P_t. The understanding is that the left side of (1.5) exists if and only if the right side does. This means, in particular, that f_i operates only on such sequences of pairs $(t_\kappa, x_\kappa) \varepsilon \sqcup P_t$, all members of which have the same first component $t_\kappa = t$. Thus, dom $f_i = \bigcup_{t \varepsilon T} i_t(\text{dom } f_{ti})$.

$\sqcup P_t$ with the lexicographic order $\leq$ and this algebraic structure $(f_i)_{i \varepsilon I}$ - and with the natural mappings i_t - will be called the *algebraic lexicographic sum*, briefly the *lexicographic sum*, of the partially ordered algebras P_t $(t \varepsilon T)$.

If T and all algebras P_t are totally disordered, then the algebraic lexicographic sum coincides with the partial direct sum of the algebras P_t as in the algebraic part above (cf. Schmidt [8]). On the other hand, if $I = \emptyset$, our algebraic lexicographic sum is nothing but the ordinary lexicographic sum of partially ordered sets P_t.

Theorem 1.1 In the class $\mathfrak{K}$ of all partially ordered algebras of type Δ, the algebraic lexicographic sum $i_t : P_t \longrightarrow \sqcup P_t$ is the order-sum.

Proof. Since the type Δ is without constants, $\sqcup P_t$ is in $\mathfrak{K}$. Each i_t is a homomorphism, even an embedding (cf. 4.), and $i_t : P_t \longrightarrow \sqcup P_t$ clearly is a T-family. Consider an arbitrary

T-family $\psi_t : P_t \longrightarrow Q$. It is well-known that there is exactly one algebraic homomorphism ψ from the partial direct sum $\coprod P_t$ into Q such that $\psi \circ i_t = \psi_t$, for all indices $t \varepsilon T$. All that is left to show: ψ is also order-preserving. So let $(s,x) \leq (t,y)$. If $s < t$, then $\psi_s(x) \leq \psi_t(y)$, i.e. $\psi(s,x) \leq \psi(t,y)$. If $s=t$ and $x \leq y$, then we arrive at the same conclusion since ψ_t is order-preserving.

In particular, if no operations are involved, we have come up with a universal property for the ordinary lexicographic sum of partially ordered sets.

In the class of partially ordered topological spaces, a topological lexicographic sum can be defined in a similar manner as for partially ordered algebras: It will be a combination of the topological sum of the spaces and the lexicographic sum of the partially ordered sets. An exact analogue of Theorem 1.1 holds true.

Unfortunately, we had to restrict ourselves so far to the case where Δ is a type without constants. This is to a good extent due to

Theorem 1.2. Suppose $\psi_t : P_t \longrightarrow P$ is a T-family of order-preserving mappings. Assume that for each $t \varepsilon T$, there is an $a_t \varepsilon P_t$ such that $\psi_t(a_t) = a$, where a is, of course, independent of t. Suppose $s < t$ in T. Then $\max \psi_s(P_s) = \min \psi_t(P_t) = a$.

Proof. By virtue of (1.3), $\psi_s(x) \leq \psi_t(a_t) = a$, for each $x \varepsilon P_s$. Moreover, $\psi_s(a_s) = a$ by hypothesis. Hence $a = \max \psi_s(P_s)$. Dually, one obtains $a = \min \psi_t(P_t)$.

As an immediate consequence, we get

Corollary 1. max $\psi_s(P_s)$ = a if s is not maximal in T, min $\psi_s(P_s)$ = a if s is not minimal in T. $\psi_s(P_s)$ collapses into {a} if s is not extremal in T (neither maximal nor minimal).

Corollary 2. If s is not maximal in T, and min $P_s = a_s$, then again $\psi_s(P_s)$ collapses into {a}.

Proof. a = max $\psi_s(P_s)$ by Theorem 1.2. On the other hand, since ψ_s is order-preserving, $a = \psi_s(a_s) = \psi_s(\min P_s) = \min \psi_s(P_s)$.

Let us show which damage Theorem 1.2 does to the order-sum $\phi_t : P_t \longrightarrow P$ in the - explicit or implicit - presence of constants Suppose $\phi_t(a_t) = a$, for each $t \varepsilon T$; the elements a_t and a might, e.g. be explicitly listed among the constants. Suppose T contains no extremal elements. By Corollary 1, $\phi_t(P_t) = \{a\}$ for each $t \varepsilon T$. Suppose now that {a} is a subalgebra (closed subset) of P. This is, for instance, the case if some (if not each) of the algebras P_t are complete. Anyway, if {a} is a subalgebra, P collapses into {a} provided that the class $\mathfrak{K}$ is closed with respect to taking subalgebras. This is due to

Theorem 1.3. Let the class $\mathfrak{K}$ be closed under taking subalgebras. Let $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathfrak{K}$. Then the union $\bigcup \mathrm{im}\, \phi_t$ generates P.

The proof is a repetition of a standard argument.

In more special classes $\mathcal{K}$, things may even become worse. Let us talk about partially ordered groups. Compatibility assumed here is the isotonicity in each factor. Here it suffices for T to be without isolated elements to make the order-sum $\phi_t : P_t \longrightarrow P$ collapse (t is isolated if it is not comparable with any other element). The elements a_t and a above are, of course, the respective identity elements (even if they are not listed as constants, they have got to be preserved under homomorphisms!). Again, $\phi_t(P_t) = \{a\}$. For if t is not minimal, min $\phi_t(P_t) = a$ by Corollary 1. But $\phi_t(P_t)$ is a subgroup of P, so in fact $\phi_t(P_t) = \{a\}$. The same happens if t is not maximal.

Here is another, in a sense more terrible example. Let $\mathcal{K}$ be the class of partially ordered algebras with least and greatest elements, the latter explicitly listed as constants among the fundamental operations. I.e. the homomorphisms in $\mathcal{K}$ are supposed to preserve both least and greatest elements. We now assume only that T contains a pair of comparable elements $s < t$. Consider a T-family $\psi_t : P_t \longrightarrow P$. Since s is not maximal, $\psi_s(P_s)$ consists of the least element of P only, according to Corollary 2. On the other hand, it contains the greatest element of P. So the latter has to coincide with the least element, thus squeezing P again into one element. In such a class, in other words, the old coproduct will be the only meaningful order-sum. If we give up insisting on the extrema, however, other order-sums become highly meaningful.

The presence of constants is not always as damaging as in the examples above. For certain nice index-sets T, one may still get a non-trivial order-sum. Moreover, the order-sum may even be very easy to describe:

Suppose $\mathfrak{K}$ is a class of partially ordered algebras with least elements, the latter explicitly listed among the constants. Assume furthermore that all constant mappings between $\mathfrak{K}$-algebras are homomorphisms (the latter condition will reappear in 4.).

Theorem 1.4. Suppose $\mathfrak{K}$ is as described. Suppose T has a greatest element e. Then $\phi_t : P_t \longrightarrow P_e$, where

$$\phi_t(x) = \begin{cases} x & \text{if } t = e, \\ \min P_e & \text{otherwise} \end{cases}$$

is the order-sum.

Proof. Clearly, $\phi_t : P_t \longrightarrow P_e$ is a T-family of homomorphisms. We have to show that $\phi_t : P_t \longrightarrow P_e$ is the universal T-family. Suppose Q is a $\mathfrak{K}$-algebra and $\psi_t : P_t \longrightarrow Q$ is a T-family of homomorphisms. We define $\psi : P_e \longrightarrow Q$ to be ψ_e. For t=e, we have $\psi\circ\phi_e = \psi_e$. So suppose $t \neq e$. We get $\psi(\phi_t(x)) = \psi(\min P_e) = \psi_e(\min P_e) = \min Q$. But min Q is the only element of $\psi_t(P_t)$, by Corollary 2 of Theorem 1.2. Hence $\min Q = \psi_t(x)$, for each $x \in P_t$, whence $\psi\circ\phi_t = \psi_t$. The uniqueness of ψ is clear.

2. The algebraic lexicographic sum with constants

The examples at the end of the previous section showed us that in the presence of constants the notions of a T-family and of the order-sum in particular have a strong tendency to collapse into trivialities. So we are prepared to find the notion of the order-sum meaningful almost only in the case of no constants, unless the index-domain T happens to be an anti-chain. On the other hand, some part of the theory (cf. 3.) can be developed for the general case, disregarding the presence or absence of constants.

For that reason, we want to extend the notion of the algebraic lexicographic sum to the general case where the type Δ may now contain some constants, $K_i = \emptyset$ for some indices $i \in I$. This should be done in such a way that Theorem 1.1 remains valid. The construction will be similar to the construction of the partial direct sum of algebras (cf. Schmidt [8]), but somewhat more involved in the presence of partial orders.

Throwing out those indices $i \in I$ standing for constants, we arrive at the reduced index-domain $I^* = \{ i \mid K_i \neq \emptyset \}$ and the corresponding *reduced type* Δ^*, without constants. Correspondingly, forgetting the constants, the partially ordered algebras P_t of type Δ are turned into partially ordered algebras P_t^* of type Δ^*. We can consider the algebraic lexicographic sum of the latter, $\bigsqcup P_t^*$. In order to arrive at an appropriate factorization, we consider quasi-orders ρ of $\coprod P_t^*$ which are *admissible* in the sense that the following three conditions hold:

(i) $\rho \cap \rho^{-1}$ is a congruence relation of the algebra $\coprod P_t^*$;

(ii) ρ contains the lexicographic order of $\coprod P_t^*$;

(iii) ρ takes care of the constants inasmuch as $(s,f_{si})\ \rho\ (t,f_{ti})$, for each $s,t\in T$ and for each $i\in I\setminus I^*$.

It is easy to see that there is a least admissible quasi-order, say σ. The contraction $\coprod P_t^* /\sigma\cap\sigma^{-1}$ is then a partially ordered algebra of type Δ^*, and the natural projection $p : \coprod P_t^* \longrightarrow \coprod P_t^* /\sigma\cap\sigma^{-1}$ is a homomorphism between them. One makes $\coprod P_t^* /\sigma\cap\sigma^{-1}$ an algebra of type Δ by introducing the constants $g_i = p(t,f_{ti})$, for each $i\in I\setminus I^*$, this definition being actually independent of t. (As a dogmatical remark: We may assume, without loss of generality, that there is a t such that the constant $f_{ti}\in P_t$ really exists.) The partially ordered algebra $\coprod P_t^* /\sigma\cap\sigma^{-1}$ so enriched may be called the *algebraic lexicographic sum* of the partially ordered algebras P_t $(t\in T)$ and again be denoted

$$\coprod_{t\in T} P_t .$$

Clearly, in the case without constants, $I^* = I$, $\Delta^* = \Delta$, $P_t^* = P_t$, nothing has happened at all: Condition (iii) becomes meaningless, making σ the lexicographic order itself (note that (i) holds for the lexicographic order, due to its anti-symmetry). Returning to the general case, we introduce the mappings

$$j_t = p \circ i_t : P_t \longrightarrow \coprod P_t \,,$$

which are homomorphisms by construction (in particular, they preserve the constants).

Theorem 2.1. In the class $\mathcal{K}$ of all partially ordered algebras of type Δ, the algebraic lexicographic sum $j_t : P_t \longrightarrow \coprod P_t$ is the order-sum.

Proof. Clearly, $j_t : P_t \longrightarrow \coprod P_t$ is a T-family since $i_t : P_t^* \longrightarrow \coprod P_t^*$ is a T-family. Consider an arbitrary T-family $\psi_t : P_t \longrightarrow Q$. Then there is a unique order-preserving mapping $\psi^* : \coprod P_t^* \longrightarrow Q^*$, a homomorphism between reduced algebras, such that $\psi^* \circ i_t = \psi_t$, for each $t \varepsilon T$. Let ρ be the quasi-order in $\coprod P_t^*$ defined by

$$(s,x)\ \rho\ (t,y) \text{ iff } \psi^*(s,x) \leq \psi^*(t,y).$$

$\rho \cap \rho^{-1} = \ker \psi^*$ is a congruence relation of the algebra $\coprod P_t^*$. Let now $(s,x) \leq (t,y)$. If $s < t$, then $\psi^*(s,x) = \psi_s(x) \leq \psi_t(y) = \psi^*(t,y)$ since $\psi_t : P_t \longrightarrow Q$ is a T-family. If $s=t$ and $x \leq y$, we arrive at the same conclusion since ψ_t is order-preserving. In other words, we get $(s,x)\ \rho\ (t,y)$ in either case, showing that ρ has property (ii) above. To show property (iii), let $K_i = \emptyset$ and $s,t \varepsilon T$ such that the constants $f_{si} \varepsilon P_s$ and $f_{ti} \varepsilon P_t$ do exist. Since ψ_s and ψ_t preserve constants, we get $\psi^*(s,f_{si}) = \psi_s(f_{si}) = \psi_t(f_{ti}) = \psi^*(t,f_{ti})$, whence $(s,f_{si})\ \rho\ (t,f_{ti})$ as wanted. So ρ is an admissible quasi-order, so

$\sigma \subset \rho$ and $\sigma \cap \sigma^{-1} \subset \rho \cap \rho^{-1} = \ker \psi^*$. So there is exactly one algebraic homomorphism ψ from the reduced algebra $(\bigsqcup P_t)^*$ into Q^* such that $\psi \circ p = \psi^*$:

$$\begin{tikzcd} P_t \arrow[r, "i_t"'] \arrow[rr, bend left, "j_t"] \arrow[dr, "\psi_t"'] & \bigsqcup P_t^* \arrow[r, "p"'] \arrow[d, "\psi^*"] & \bigsqcup P_t \arrow[dl, "\psi"] \\ & Q & \end{tikzcd}$$

So $\psi_t = \psi^* \circ i_t = \psi \circ p \circ i_t = \psi \circ j_t$. ψ also preserves the constants, so it is even a homomorphism between the algebras of full type Δ. Finally, ψ is order-preserving. For suppose $p(s,x) \trianglelefteq p(t,y)$ in $\bigsqcup P_t$, then $(s,x)\ \sigma\ (t,y)$ in $\bigsqcup P_t^*$, whence $(s,x)\ \rho\ (t,y)$, i.e. $\psi^*(s,x) \leq \psi^*(t,y)$ or $\psi(p(s,x)) \leq \psi(p(t,y))$. ψ is actually the only homomorphism between the partially ordered algebras (of type Δ) $\bigsqcup P_t$ and Q such that $\psi \circ j_t = \psi_t$, for each $t \in T$. This is easily obtained from the uniqueness of ψ^* and ψ as stated above. This completes the proof of Theorem 2.1.

Note that the homomorphisms $j_t : P_t \longrightarrow \bigsqcup P_t$ need no longer be one-one since p can no longer be expected to be one-one. This throws some new light on the phenomena connected with constants. The examples at the end of the previous section show, indeed, that σ and $\sigma \cap \sigma^{-1}$ may become the universal relation in $\bigsqcup P_t^*$, forcing $\bigsqcup P_t$ to collapse into one element.

In spite of Theorem 2.1, one might not be too happy with the construction of $\bigsqcup P_t$ as given above since one really does not have much information about σ. Here is an alternative approach that one might occasionally find more convenient. Not wanting to factorize, instead of throwing the constants (t,f_{ti}) - with the same i - together, we leave them apart. To be precise, we enrich the algebraic structure of $\coprod P_t^*$ by listing the elements (t,f_{ti}), $i \in I \setminus I^*$, as new constants. The resulting modified algebraic lexicograpic sum,

$$\coprod_{t\in T}^{*} P_t \text{ ,}$$

is, of course, no longer of type Δ or Δ^* unless there are no constants. $\coprod^* P_t$ still has a universal property similar to that of the order-sum. This makes $\coprod^* P_t$ sometimes a useful substitute for the latter.

<u>Theorem 2.2</u>. Suppose $\psi_t : P_t \longrightarrow Q$ is a T-family of homomorphisms. Then there is a unique order-preserving mapping $\psi : \coprod^* P_t \longrightarrow Q$ such that $\psi \circ i_t = \psi_t$ for each $t \in T$, and each restriction $\psi | i_t(P_t)$ is an algebraic homomorphism.

3. A general existence theorem

Even if one is predominantly interested in algebras with complete operations, the coproduct in a class $\mathfrak{K}$ of such algebras will unavoidably lead to the consideration of partial algebras, cf., for instance the free products of semigroups or groups (Schmidt [10],[11]). Analogously, the order-sum in any class $\mathfrak{K}$ of partially ordered algebras can be built up in two steps. The first one of these has been described in sections 1 and 2. In general, of course, the algebraic lexicographic sum of algebras $P_t \varepsilon\, \mathcal{R}$ will not be in $\mathcal{R}$. So the second step will consist in associating with the latter a universal object in $\mathcal{R}$. Let us make that more precise in

<u>Theorem 3.1</u>. Consider a T-family of homomorphisms $\phi_t : P_t \longrightarrow P$ in $\mathcal{R}$ and the associated homomorphism $\phi : \mathrm{L}\, P_t \longrightarrow P$ (which exists according to Theorem 2.1). Then the following two conditions are equivalent:

(i) $\phi_t : P_t \longrightarrow P$ is the order-sum in $\mathcal{R}$;

(ii) $\phi : \mathrm{L}\, P_t \longrightarrow P$ is the universal homomorphism of $\mathrm{L}\, P_t$ into a $\mathcal{R}$-algebra.

Proof. (i) $\Longrightarrow$ (ii): Suppose Q is a $\mathcal{R}$-algebra and $\psi : \mathrm{L}\, P_t \longrightarrow Q$ is a homomorphism.

$$\begin{tikzcd}
\mathrm{L}\, P_t \arrow[r, "\phi"] \arrow[dr, "\psi"] & P \arrow[d, "\bar{\psi}"] \\
P_t \arrow[u, "j_t"] \arrow[ur, "\phi_t"] \arrow[r, "\psi_t"'] & Q
\end{tikzcd}$$

$\psi_t = \psi \circ j_t : P_t \longrightarrow Q$ is a T-family, so by hypothesis, there is a unique homomorphism $\bar{\psi} : P \longrightarrow Q$ such that $\bar{\psi} \circ \phi_t = \psi \circ j_t$ for each $t \varepsilon T$. On the other hand, $\bar{\psi} \circ \phi_t = \bar{\psi} \circ \phi \circ j_t$, thus $\bar{\psi} \circ \phi = \psi$, by Theorem 2.1. Using the uniqueness of $\bar{\psi}$ as stated above, $\bar{\psi}$ is in fact the only homomorphism such that $\bar{\psi}\circ\phi = \psi$. The proof of (ii) $\Longrightarrow$ (i) is similar.

If we prefer to use $\mathsf{L}^* P_t$ instead of $\mathsf{L} P_t$, we have to use an analogue of Theorem 3.1. Its proof, with minor changes, will be essentially the same as that of Theorem 3.1.

<u>Theorem 3.2</u>. Consider a T-family of homomorphisms $\phi_t : P_t \longrightarrow P$ in $\mathfrak{K}$ and the associated mapping $\phi : \mathsf{L}^* P_t \longrightarrow P$ of Theorem 2.2. The following two conditions are equivalent:

(i) $\phi_t : P_t \longrightarrow P$ is the order-sum in $\mathfrak{K}$;

(ii) for every $\mathfrak{K}$-algebra Q and every order-preserving mapping $\psi : \mathsf{L}^* P_t \longrightarrow Q$ such that each restriction $\psi | i_t(P_t)$ is a homomorphism, there is a unique homomorphism $\bar{\psi} : P \longrightarrow Q$ such that $\bar{\psi}\circ\phi = \psi$.

It is time by now to come to the existence of the order-sum in reasonable classes $\mathfrak{K}$. It should be clear what the *direct* (cartesian) *product of partially ordered algebras* P_α is: it is the direct product of the underlying sets, endowed with the product of the algebraic structures (i.e. the "strongest" algebraic structure making all natural projections pr_α homomorphisms) and the product ("cardinal product") of the partial orders. In short, everything is calculated componentwise. For the proof of the following existence theorem, we will actually reinterpret partial orders as the corresponding binary maximum-operations. We should hence observe that this reinterpretation commutes with cartesian multiplication. As far as this reinterpretation is concerned, the same observation applies to taking subalgebras and isomorphic copies. As in universal algebra without partial order, a *quasi-primitive class* of partially ordered algebras will be a class closed under the aforementioned three procedures of taking cartesian products, subalgebras, and isomorphic images. After reinterpretation of the partial orders such a class will become a quasi-primitive class in the ordinary sense of universal algebra. It is understood, by the way, that a quasi-primitive class $\mathfrak{K}$ contains the complete algebra of one element, as the direct product of the empty family (or, if one prefers, by definition).

We are now ready to formulate

Theorem 3.3 (Existence of Order-sums). In a quasi-primitive class all order-sums exist.

Proof. Suppose P_t $(t \in T)$ are $\mathfrak{K}$-algebras, and let $j_t : P_t \longrightarrow \sqcup P_t$ be their algebraic lexicographic sum (cf. Theorem 2.1). By the general existence theorem from universal algebra (Schmidt [9], Theorem 2), there is a universal $\mathfrak{K}$-algebra P for $\sqcup P_t$, to be more precise, a universal homomorphism into a $\mathfrak{K}$-algebra, $\phi : \sqcup P_t \longrightarrow P$. Defining $\phi_t = \phi \circ j_t : P_t \longrightarrow P$, we get a T-family of homomorphisms. ϕ is then the homomorphism associated with that family. By Theorem 3.1, $\phi_t : P_t \longrightarrow P$ is the order-sum.

Another proof of Theorem 3.3 might be based on Theorem 3.2 instead of Theorem 3.1. One then has the simple-minded $\mathbb{L}^* P_t$ instead of $\mathbb{L} P_t$. Trouble comes in, however, through the poorer universal property (Theorem 2.2) of $\mathbb{L}^* P_t$.

4. When is the order-sum an extension of the lexicographic sum ?

By definition, a partially ordered algebra P is an *extension* of Q if Q is a relative algebra of P. The order-sum will interest us preferably if it happens to be essentially an extension of the lexicographic sum, i.e. if the universal mapping of Theorem 3.1 is an embedding or at least an order-embedding. For $\phi : Q \longrightarrow P$ to be an *embedding* means, of course, to be at the same time an algebraic and an order-embedding. To be an *order-embedding* simply means that

$$(4.1) \qquad q_1 \leq q_2 \quad \text{iff} \quad \phi(q_1) \leq \phi(q_2),$$

for each $q_1, q_2 \varepsilon Q$. Due to anti-symmetry, ϕ is then one-one, and the inverse mapping $\phi^{-1} : \text{im } \phi \longrightarrow Q$ is order-preserving too.
$\phi : Q \longrightarrow P$ is an *algebraic embedding* iff it is an isomorphism of the algebra Q onto the relative algebra (not necessarily a subalgebra, i.e. not necessarily closed under the operations) im $\phi \subset P$.
Note: if the algebra Q happens to be complete, it suffices that ϕ, assumed to be a homomorphism anyway, is one-one. In addition, im ϕ will become a subalgebra of P in this case. Hence, if the homomor-

phism ϕ is an order-embedding of the complete algebra Q into P, ϕ is an algebraic embedding also, and it is onto a subalgebra of P.

Suppose now that $\mathfrak{K}$ is a class of partially ordered algebras. Suppose $\phi_t : P_t \longrightarrow P$ to be an order-sum in $\mathfrak{K}$ and $i_t : P_t \longrightarrow \llcorner P_t$ the lexicographic sum. Let $\phi : \llcorner P_t \longrightarrow P$ be the universal homomorphism of Theorem 3.1,

$$\begin{tikzcd} P_t \arrow[rr, "i_t"] \arrow[dr, "\phi_t"'] & & \mathop{\llcorner}\limits_{t\varepsilon T} P_t \arrow[dl, "\phi"] \\ & P & \end{tikzcd}$$

In order to avoid the difficulties connected with the constants (cf. 1. and 2.) - the latter will not occur in the applications below anyway - we shall assume from now on that the type Δ be *without constants* ($K_i \neq \emptyset$ for each $i \varepsilon I$). This agreement plays a role in the following obvious

<u>Theorem 4.1</u>. Equivalent are:

(i) $\phi : \llcorner P_t \longrightarrow P$ is one-one;

(ii) the homomorphisms $\phi_t : P_t \longrightarrow P$ are one-one, and their images are pairwise disjoint;

(iii) there is a $\mathfrak{K}$-algebra Q and a one-one homomorphism $\psi : \llcorner P_t \longrightarrow Q$.

<u>Theorem 4.2</u>. Equivalent are:

(i) $\phi : \coprod P_t \longrightarrow P$ is an order-embedding;

(ii) the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embedding, and the indexed family of their images is not only pairwise disjoint, but a "lexicographic decomposition" of the partially ordered set $\bigcup_{t\varepsilon T} \text{im } \phi_t$ (= imϕ);

(iii) there is a $\mathfrak{K}$-algebra Q and an order-embedding (and algebraic homomorphism) $\psi : \coprod P_t \longrightarrow Q$.

Proof. (i) $\Longrightarrow$ (ii): $\mathcal{D} = \{ \text{ im } \phi_t \mid t\varepsilon T \}$ is already a decomposition of im ϕ by Theorem 4.1. The "pieces" im ϕ_t are even in one-one correspondence with the indices $t\varepsilon T$. That $\mathcal{D}$ is a "lexicographic decomposition" (cf. Schmidt [6], [7]) means exactly that the elements of different pieces compare like the corresponding indices:

(4.2) $$\text{if } s \neq t, \text{ then } \phi_s(x) \leq \phi_t(y) \text{ iff } s \leq t.$$

Since $\phi_t : P_t \longrightarrow P$ is a T-family (even the order-sum), it suffices to postulate only:

(4.3) $$\text{if } \phi_s(x) \leq \phi_t(y), \text{ then } s \leq t.$$

(Note that in this formulation the pairwise disjointness of the images is still included) Suppose now $\phi_s(x) \leq \phi_t(y)$, for some $x\varepsilon P_s$, $y\varepsilon P_t$. So $\phi(s,x) \leq \phi(t,y)$. By assumption, ϕ is an order-embedding, so $(s,x) \leq (t,y)$, whence $s \leq t$, proving (4.3). The proof of (ii) $\Longrightarrow$ (i) is similar, and (i) $\Longrightarrow$ (iii) is trivial. (iii) $\Longrightarrow$ (i) is a consequence of the universality property of the order-sum.

The following two theorems, though they are obviously weaker than the corresponding theorems 4.1 and 4.2, do not follow directly from the latter. We state them without their straightforward proofs.

Theorem 4.3. Equivalent are:

(i) the homomorphisms $\phi_t : P_t \longrightarrow P$ are one-one;

(ii) for all indices $s \in T$ and all elements $x,y \in P_s$ such that $x \neq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t : P_t \longrightarrow Q$ separating x and y, $\psi_s(x) \neq \psi_s(y)$.

Theorem 4.4. Equivalent are:

(i) the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embeddings;

(ii) for all indices $s \in T$ and all elements $x,y \in P_s$ such that $x \not\leq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t : P_t \longrightarrow Q$ such that $\psi_s(x) \not\leq \psi_s(y)$.

Because of the completeness of the algebras to be considered in the applications (cf. 5.), Theorem 4.2 - assisted by Theorem 4.4 - will suffice for our purposes. Let us at least state, however, the general conditions for ϕ to be an embedding.

<u>Theorem 4.5</u>. Equivalent are:

(i) $\phi : \llcorner P_t \longrightarrow P$ is an embedding;

(ii) the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embeddings, the indexed family of their images is a lexicographic decomposition of the partially ordered set im ϕ, and the algebraic structure of im ϕ (as a relative algebra of P) is the final structure for the mappings ϕ_t;

(iii) there is a $\mathcal{R}$-algebra Q and an embedding $\psi : \llcorner P_t \longrightarrow Q$.

Note that condition (ii) essentially repeats the description of the algebraic lexicographic sum as given in section 1.

Whenever the universal homomorphism ϕ is an embedding, we can replace it by the inclusion mapping of the lexicographic sum into an isomorphic copy of P, due to the well-known Zermelo-van der Waerden replacement procedure. I.e., the order-sum can then be considered as a genuine extension of the lexicographic sum. In particular, the partial algebra $\llcorner P_t$ will become a relative algebra of P, obtained by the total restriction of the operations of P to the subset $\llcorner P_t$. In our applications, however, we will usually be content with somewhat less. In fact, we will be happy to know that $\phi : \llcorner P_t \longrightarrow P$ is at least an order-embedding (it is, of course, an algebraic homomorphism anyway). Using that replacement procedure again, $\llcorner P_t$ becomes again a subset of P, the lexicographic order of $\llcorner P_t$ still being the restriction of the partial order of P.

However, the inclusion of the partial algebra $\bigsqcup P_t$ into the algebra P will only be a homomorphism, not necessarily an embedding. I.e., $\bigsqcup P_t$ may only be a *weak* relative algebra of P. We will refer to this situation by saying that P is an *order-extension* of $\bigsqcup P_t$ (algebraically, it may only be a *weak extension*). This is really not too bad if all $\mathfrak{K}$-algebras are complete. In that case, at least the inclusion of the pieces $i_t(P_t)(= \phi_t(P_t))$ into P are *strong*, i.e. the pieces are genuine subalgebras of the complete algebra P.

We now find convenient sufficient conditions on the class $\mathfrak{K}$ to garantee that our mapping $\phi : \bigsqcup P_t \longrightarrow P$ will certainly be an order-embedding. The conditions are the following:

(I) $\mathfrak{K}$ is *non-trivial*, i.e. contains a *non-trivial algebra* Q in the sense that Q contains a pair of distinct comparable elements.

(II) All constant mappings between $\mathfrak{K}$-algebras are homomorphisms.

(III) For every $\mathfrak{K}$-algebra P and all elements $x,y \in P$ such that $x \not\leq y$, there is a $\mathfrak{K}$-algebra Q and a homomorphism $\alpha : P \longrightarrow Q$ such that $\alpha(y) = \min \alpha(P) < \alpha(x) = \max \alpha(P)$ ("separability").

As it turns out, (III) may be replaced, for our purposes, by the following condition:

(III') Every $\mathfrak{K}$-algebra is embeddable into a non-trivial $\mathfrak{K}$-algebra with least and greatest element.

Note that in the class $\mathfrak{K}$ of all distributive lattices, all four conditions hold. In the class of modular lattices, at least (I), (II), (III') hold.

<u>Theorem 4.6</u>. Let again $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathfrak{K}$. Suppose that $\mathfrak{K}$ fulfills the conditions (I), (II), and (III) or, alternatively, (III'). Then the universal homomorphism $\phi : \bigsqcup P_t \longrightarrow P$ is an order-embedding.

Proof. We are going to show that condition (ii) of Theorem 4.2 holds true. In order to prove (4.3), we only need (I) and (II). In fact, let $s,t \varepsilon T$ and $\phi_s(a) \leq \phi_t(b)$, for some $a \varepsilon P_s$, $b \varepsilon P_t$. By virtue of condition (I), there is a $\mathfrak{K}$-algebra Q containing two comparable elements $c < d$. For each $u \varepsilon T$, we define a mapping $\psi_u : P_u \longrightarrow Q$ as follows:

(*) $$\psi_u(z) = \begin{cases} d & \text{if } s \leq u, \\ c & \text{otherwise.} \end{cases}$$

By virtue of (II), these mappings are homomorphisms. We are going to show that they form a T-family. Suppose $u < v$ in T. We have to show that $\psi_u(x) \leq \psi_v(y)$, for each $x \varepsilon P_u$, $y \varepsilon P_v$. Assuming $\psi_u(x) \not\leq \psi_v(y)$, we arrive at $\psi_v(y) = c$ and $\psi_u(x) = d$, whence $s \not\leq v$, but $s \leq u$, contradicting $u < v$.

We are now ready to show $s \leq t$. By the universality property of the order-sum $\phi_t : P_t \longrightarrow P$, there is a unique homomorphism $\psi : P \longrightarrow Q$ such that $\psi_u = \psi \circ \phi_u$, for each $u \varepsilon T$. We get $\psi_s(a)$

$= \psi(\phi_s(a)) \leqslant \psi(\phi_t(b)) = \psi_t(b)$. However, $\psi_s(a) = d$ by definition (*), therefore $\psi_t(b) = d$ as well. But then, again by (*), $s \leqslant t$, completing the proof of (4.3).

It remains to show that the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embeddings. We will show that condition (ii) of Theorem 4.4 holds. For that, it takes conditions (II) and (III) or (II) and (III') respectively. Suppose $s \in T$ and $a \nleq b$ in P_s. Assuming condition (III), there is a $\mathcal{K}$-algebra Q and a homomorphism $\alpha : P_s \longrightarrow Q$ such that $\alpha(b) = \min \alpha(P_s) < \alpha(a) = \max \alpha(P_s)$. For each $t \in T$, we define $\psi_t : P_t \longrightarrow Q$ as follows:

$$(**) \qquad \psi_t(x) = \begin{cases} \alpha(b) & \text{if } t < s, \\ \alpha(x) & \text{if } t = s, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

These mappings are homomorphisms by conditions (II) and (III). Moreover, $\psi_s(a) = \alpha(a) \nleq \alpha(b) = \psi_s(b)$. Condition (ii) of Theorem 4.4 will therefore be fulfilled as soon as we will have proven that $\psi_t : P_t \longrightarrow Q$ is a T-family. Suppose $r < t$ in T. It needs to be shown that $\psi_r(x) \leqslant \psi_t(y)$, for each $x \in P_r$, $y \in P_t$. Clearly, $\psi_r(x) = \alpha(x) \leqslant \alpha(a) = \psi_t(y)$ if $s = r < t$, and $\psi_r(x) = \alpha(b) \leqslant \alpha(x) = \psi_t(y)$ if $r < t = s$. Assuming $r \neq s$, $t \neq s$, and $\psi_r(x) \nleq \psi_t(y)$, we arrive at $\psi_r(x) = \alpha(a)$ and $\psi_t(y) = \alpha(b)$, hence $t < s$ and $r \nleq s$, contradicting $r < t$. Hence $\psi_t : P_t \longrightarrow Q$ is a T-family indeed.

Alternatively, let us assume (II) and (III'). Given again $s \in T$

and $a \not\leq b$ in P_s. By virtue of (III'), there is a $\mathfrak{K}$-algebra Q and an embedding $\alpha : P_s \longrightarrow Q$, where $e = \max Q \neq o = \min Q$. Again, we define $\psi_t : P_t \longrightarrow Q$ as follows:

$$(***) \qquad \psi_t(x) = \begin{cases} e & \text{if } s < t, \\ \alpha(x) & \text{if } s = t, \\ o & \text{otherwise.} \end{cases}$$

These mappings are homomorphisms by (II) and (III'). Moreover, since α is at least an order-embedding, $\psi_s(a) = \alpha(a) \not\leq \alpha(b) = \psi_s(b)$. Again, it remains to show that $\psi_t : P_t \longrightarrow Q$ is a T-family. Suppose $r < t$ in T, $x \in P_r$, $y \in P_t$. We get $\psi_r(x) = \alpha(x) \leq e = \psi_t(y)$ if $r = s < t$, and $\psi_r(x) = o \leq \alpha(x) = \psi_t(y)$ if $r < t = s$. Assuming again $r \neq s$, $t \neq s$ and $\psi_r(x) \not\leq \psi_t(y)$, we arrive at $\psi_r(x) = e$, $\psi_t(y) = o$, which leads to a similar contradiction as above. This completes the proof of Theorem 4.6.

<u>Corollary</u>. Suppose $\mathfrak{K}$ is a class of complete algebras fulfilling conditions (I), (II), and one of (III) or (III'). Then the order-sum P (provided it exists) is an order-extension (and weak algebraic extension) of the lexicographic sum $\textstyle\coprod P_t$, and the pieces $i_t(P_t)$ are subalgebras of P.

Recall that P does exist, if $\mathfrak{K}$ is quasi-primitive.

5. The order-sum extends the lexicographic sum in some nice classes

All applications here will be to certain classes of partially ordered sets with appropriate homomorphisms. To become more specific, let us consider, as the simplest example, the class $\mathfrak{L}$ of all (join-) semilattices. For our purposes, we look at such a semilattice as a partially ordered algebra $(A, \vee, \leq)$, where $(A, \vee)$ is a semilattice, $\leq$ the associated partial order. Here, as in the sequel, we have the remarkable fact that the algebraic structure dtermines the partial order and vice versa. Homomorphisms are just the join-preserving mappings, the latter being order-preserving.

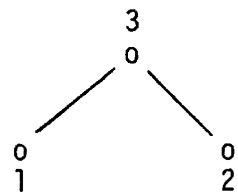
Incidentally, the underlying partial order of a subsemilattice (subalgebra) is the restriction of the global partial order. Likewise, the underlying partial order of a direct product of semilattices is the direct ("cardinal") product of the individual underlying partial orders. So there is no trouble in applying our general theory. In particular, the class of semilattices, in this interpretation, is still quasi-primitive. So each order-sum $\phi_t : P_t \longrightarrow P$ exists without any restriction whatsoever (Theorem 3.2). But every partially ordered set can be order-embedded into a join-semilattice, even a complete lattice, such that all finite joins are preserved; take, e.g., the ideal-completion (cf. Schmidt [12]). Applying this to the algebraic lexicographic sum $\bigsqcup P_t$, we see that condition (ii) of Theorem 4.2 holds. So the universal homomorphism $\phi : \bigsqcup P_t \longrightarrow P$ is an order-embedding, and we are faced with the situation described in detail after Theorem 4.5: The semilattice P may be considered to be an order-extension of $\bigsqcup P_t$, and the pieces $i_t(P_t)$ will be subsemilattices of P. As far as all of $\bigsqcup P_t$ is concerned, the situation is fairly complex insofar as we should now distinguish three structures:

(i) the algebraic structure of $\bigsqcup P_t$ which, according to construction (section 1), admits the operation $\vee$ only in the pieces $i_t(P_t)$;

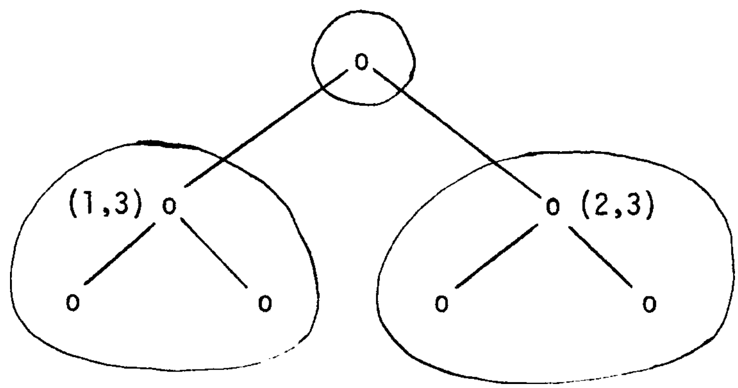
(ii) the total restriction to the set $\bigsqcup P_t$ of the semilattice-operation of P;

(iii) the joins that exist in the lexicographic order of $\bigsqcup P_t$.

The structure (i) may be weaker than both (ii) and (iii). The coincidence of (i) and (ii) would make the algebra $\bigsqcup P_t$ a (strong) relative algebra of the semilattice P and P a full-fledged extension in the sense of section 4. A more detailed study of the structure of the order-sum of semilattices (cf. Part II) will show that for the class $\mathcal{K}$ of all semilattices (i) and (ii), and also (ii) and (iii) do not coincide. Clearly, (i) and (iii) will not coincide, in general, unless the index-set T happens to be an antichain. A good example for the latter is the following: Take T to be



and $P_1 = P_2 = T$, P_3 has just one element. Then $\llcorner P_t$ with its three pieces has the structure



In particular, $(1,3) \vee (2,3)$ exists in the structure (iii), i.e. in the lexicographic order of $\llcorner P_t$, but not in the algebraic structure (i). We have here the extreme case that the partially ordered set $\llcorner P_t$ is a join-semilattice itself. We know that the inclusion of $\llcorner P_t$ into the order-sum P preserves all joins of the algebraic structure (i). We cannot claim, however, that all existing joins of $\llcorner P_t$ will be preserved: the partially ordered set $\llcorner P_t$ is not necessarily join-faithful in P (cf.Part II). In our example above, $\llcorner P_t$ will not be a subsemilattice of P (cf., however, Theorem 5.4 below).

Let us summarize the main result of this discussion in

Theorem 5.1. In the class $\mathcal{R}$ of semilattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

By appropriate interpretations, this result can be extended to *k-(join-) semilattices*, where k is an infinite regular cardinal. By that, we mean partially ordered sets in which all joins of non-empty k-small subsets exist, *k-small* meaning less in cardinality than k. The homomorphisms will then be the k-join-preserving mappings. Note that (in contrast to Schmidt [12]) we want to exclude the preservation of zeros (= joins of the empty set) because of the problems discussed in section 1. In Part II, we will completely describe the structure of the order-sum of k-semilattices P_t.

What we have said for semilattices can be almost literally repeated for lattices. Note that the order-embedding of a partially ordered set into its ideal-completion preserves not only finite joins, but also arbitrary meets. We get

Theorem 5.2. In the class $\mathcal{R}$ of lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Again, the lattices P_t, more precisely, their canonical images $i_t(P_t)$, are sublattices of the order-sum. The homomorphisms we are talking about here are, of course, the lattice-homomorphisms.

Let us now consider classes of distributive or modular lattices respectively. Using Theorem 4.6, we get

Theorem 5.3. In the class $\mathfrak{K}$ of distributive (modular) lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

For distributive lattices, the existence of the order-sum has been proven by Balbes and Horn [1]. They only stated that the mappings $\phi_t : P_t \longrightarrow P$ are (lattice-) embeddings (which for them was part of the very definition of the order-sum). They also considered the special case that T is a chain. In this case, in fact, there is a neat description of the order-sum. This observation made by Balbes and Horn for distributive lattices applies to semilattices, lattices and modular lattices as well, but not to k-semilattices for uncountable k:

Theorem 5.4. In the class $\mathfrak{K}$ of semilattices (lattices, distributive, modular lattices), the order-sum over a chain T coincides with the lexicographic sum of the partially ordered sets.

Proof. Consider the $\mathfrak{K}$-algebras P_t and their lexicographic sum $i_t : P_t \longrightarrow \sqcup P_t$. Note that $\sqcup P_t$ endowed with its proper algebraic structure can still not be expected to be in $\mathfrak{K}$. However, if T is a chain, the lexicographic order of $\sqcup P_t$ makes, indeed, $\sqcup P_t$ a $\mathfrak{K}$-algebra. This can be easily checked. With the universal property of the lexicographic sum of the partially ordered sets P_t (cf. the comment after Theorem 1.1) - again one uses the fact that T is a chain - the proof is complete.

Here is an example that Theorem 5.4 will no longer hold for k-semilattices, where $k = \aleph_1$. Let $T = \mathbb{N} \cup \{\infty\}$, $P_t = \{a\}$ for each $t \in T$. Note that $T = \bigsqcup P_t$ is a k-semilattice, but that it fails to be the order-sum. In fact, let $\mathbf{2} = \{0,1\}$ and define a T-family $\psi_t : P_t \longrightarrow \mathbf{2}$ by

$$\psi_t(a) = \begin{cases} 0 & \text{if } t \in \mathbb{N} \\ 1 & \text{if } t = \infty . \end{cases}$$

Let $\psi : T = \bigsqcup P_t \longrightarrow \mathbf{2}$ be the order-preserving mapping such that $\psi \circ i_t = \psi_t$, for each $t \in T$, i.e.

$$\psi(t) = \begin{cases} 0 & \text{if } t \in \mathbb{N} \\ 1 & \text{if } t = \infty . \end{cases}$$

ψ does not preserve countable joins.

Part II

The order-sum of k-join-semilattices

In this part we will study a special class of partially ordered algebras, the class $\mathcal{K}$ of k-join-semilattices, where k is an infinite regular cardinal number. First (cf. section 6.), we apply the results of our general theory to this class $\mathcal{K}$. In addition, we succeed in obtaining a complete description of the order-sum in $\mathcal{K}$. To simplify matters, we begin with a special case (cf. Theorem 8.3), and then proceed to the general case (cf. Theorem 9.3).

6. Order-sum and lexicographic sum of k-semilattices

We have mentioned already in section 5 what we mean by a *k-(join-)semilattice*, where k is an infinite regular cardinal: It is a partially ordered set in which all joins of non-empty, k-small subsets exist. The corresponding homomorphisms are the k-join-preserving mappings. Recall that we will not require that a k-join-preserving mapping preserve the least element (provided it exists) in a k-semilattice.

The class $\mathcal{K}$ of k-semilattices can be considered a class of partially ordered algebras in the usual sense, if the algebraic structure of a k-semilattice P is appropriately reinterpreted: For each cardinal $i < k$ $(i \neq 0)$ one introduces an operation $f_i : P^i \longrightarrow P$ by $f_i(x_\kappa | \kappa < i) = \sup\{x_\kappa | \kappa < i\}$. The homomorphisms of this structure are the k-join-preserving mappings and vice versa. Everything we said on p. 27 and p. 28 for semi-lattices remains true for k-semilattices. In particular, the class $\mathfrak{K}$ of k-semilattices is quasi-primitive. Moreover, every partially ordered set can be order-embedded into a k-semilattice in such a way that all k-small joins are preserved; take, for instance, the k-ideal-completion (cf. Schmidt [12]). Hence, condition (ii) of Theorem 4.2 holds for the algebraic lexicographic sum $\bigsqcup P_t$. Applying Theorem 3.2 and Theorem 4.2, we get

<u>Theorem 6.1.</u> In the class $\mathfrak{K}$ of k-semilattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Because of the k-join-completeness of our algebras P_t, we also can conclude that the k-semilattices P_t, or rather their images $i_t(P_t)$ are sub-k-semilattices of the order-sum. It should thus be allowed to delete in the order-sum $\phi_t : P_t \longrightarrow P$ the k-join-preserving mappings ϕ_t. They cannot be anything else but the natural inclusions i_t anyway.

Again, in the lexicographic sum $\bigsqcup P_t$ of k-semilattices P_t, we have to distinguish the structures (i), (ii), and (iii) of p. 28.

Suppose $\bigsqcup P_t$ has the algebraic structure (i), i.e. k-small suprema are admitted only in the pieces $i_t(P_t)$. $\bigsqcup P_t$ with this structure is not a k-semilattice, unless $|T| = 1$. Suppose Q is a $\mathcal{K}$-algebra and $\psi : \bigsqcup P_t \longrightarrow Q$ is an order-preserving mapping that preserves also the algebraic structure (i). In order to avoid confusion, we will call such a mapping ψ always a "homomorphism" rather than "k-join-preserving".

The following theorem is a direct consequence of Theorem 3.1 for the class $\mathcal{K}$ of k-semilattices. It is weaker than Theorem 3.1 since we have strengthened the hypothesis somewhat. However, it is still strong enough to fit our purposes, and we will have to use it several times in the sequel.

Theorem 6.2. Suppose P is an order-extension of the algebraic lexicographic sum $\bigsqcup P_t$. Then the following two conditions are equivalent:

(i) P is the order-sum of the k-semilattices P_t;

(ii) for every k-semilattice Q and every homomorphism $\psi : \bigsqcup P_t \longrightarrow Q$, there is a unique k-join-preserving mapping $\bar{\psi} : P \longrightarrow Q$ such that the restriction $\bar{\psi} | \bigsqcup P_t = \psi$.

7. Lower ends and lexicographic k-ideals

A non-empty subset E of a partially ordered set Q is a *lower end*, provided that for each $x,y\varepsilon Q$:

$$(7.1) \qquad \text{if } y\varepsilon E \text{ and } x \leq y, \text{ then } x\varepsilon E.$$

Note that the set $\mathcal{E}(Q) = \{E|E \text{ lower end of } Q\}$ of all lower ends of Q contains all *principal lower ends*

$$(7.2) \qquad (x] = \{y|y \leq x\},$$

for $x\varepsilon Q$. The empty set, however, is not in $\mathcal{E}(Q)$. The latter is a necessary exclusion due to the frequently mentioned difficulties with the constants (cf. Part I). Consequently, $\mathcal{E}(Q)$, with set-theoretical inclusion as partial order, is not necessarily a complete lattice. But all the joins of non-empty subsets still exist. In fact, the join in $\mathcal{E}(Q)$ is simply set-theoretical union.

Suppose $Q \subset P$. P will be called a *k-join-completion* of Q, provided that P is a k-semilattice, and each $x\varepsilon P$ is the join of a k-small, non-empty subset of Q. I.e., for each $x\varepsilon P$,

$$(7.3) \qquad x = \sup_P S, \text{ where } S \subset Q,\ S \neq \emptyset,\ |S| < k.$$

Q will be called *k-join-faithful* in P, provided that the inclusion mapping is k-join-preserving. I.e., for each $x\varepsilon Q$ and all non-empty k-small subsets $S \subset Q$

(7.4) if $x = \sup_Q S$, then $x = \sup_P S$.

With a partially ordered set $P \supset Q$ we associate the *canonical mapping* $\kappa_P : P \longrightarrow \mathcal{L}(Q)$, defined by

$$\kappa_P(x) = Q \cap (x]_P. \tag{7.5}$$

κ_P always is order-preserving, and it is an order-embedding if each $x \in P$ is the join of some subset of P (cf. Schmidt [13]). In particular, κ_P is in the latter case an order-isomorphism onto the *canonical image* $\dot{P} = \operatorname{im} \kappa_P$ of P in $\mathcal{L}(Q)$.

A lower end E of a partially ordered set Q is called *k-small-generated*, if there is a k-small subset $S \subset E$ such that

(7.6) $x \in E$ iff there is $s \in S$ such that $x \leq s$.

Clearly, all principal lower ends are k-small generated. The set of all k-small generated lower ends of Q will be denoted by $\mathcal{L}_k(Q)$.

In introducing the above notions, we have tried to be as brief as possible. For more information and a more general setting, one should consult Schmidt [13]. Arbitrary (order-)extensions rather than k-join-completions are discussed there. But note that in Schmidt [13], $\emptyset$ always is a lower end, which changes the situation enough so that some of the results become false when translated into our setting (e.g., Theorem 2.4 in [13]; $\mathcal{L}_k(Q) = Q^k$ is, in general, not k-join-distributive if $\emptyset \notin \mathcal{L}_k(Q)$!).

Anyway, the following theorem, also contained in Schmidt [13], is still true.

Theorem 7.1. Suppose Q is a partially ordered set. $\mathcal{E}_k(Q)$ is the canonical image of a k-join-completion P of Q.

Proof. A lower end E of Q is k-small generated if it is the set-theoretical union of a non-empty, k-small set of principal lower ends. This is an immediate consequence of the definition (7.6). Moreover, $\mathcal{E}_k(Q) \subset \mathcal{E}(Q)$ with set-theoretical union as join, is k-join-complete because of the inaccessibility of k. Thus, $\mathcal{E}_k(Q)$ is a k-join-completion of $\dot{Q}$. In other words, there is a k-join-completion P of Q, where $P = \mathcal{E}_k(Q)$.

From now on, a k-small set is always supposed to be non-empty.

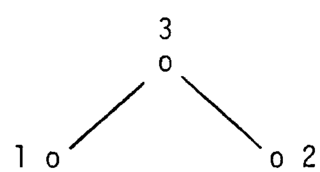
Suppose now that Q is the algebraic lexicographic sum of the k-semilattices P_t. A lower end $E \subset Q$ is a *lexicographic k-ideal* if $\sup_Q S \in E$, for each k-small subset $S \subset E$ such that $\sup_Q S$ exists. Fortunately, we know exactly which k-small joins exist in the algebraic lexicographic sum Q: $\sup_Q S$ exists iff $S \subset i_t(P_t)$ for some $t \in T$. Lexicographic k-ideals can consequently also be characterized in the following way. A lower end $E \subset Q$ is a lexicographic k-ideal, provided the following condition holds for each k-small subset $S \subset E$:

(7.7) $\qquad$ if $S \subset i_t(P_t)$, for some $t \in T$, then $\sup_Q S \in E$.

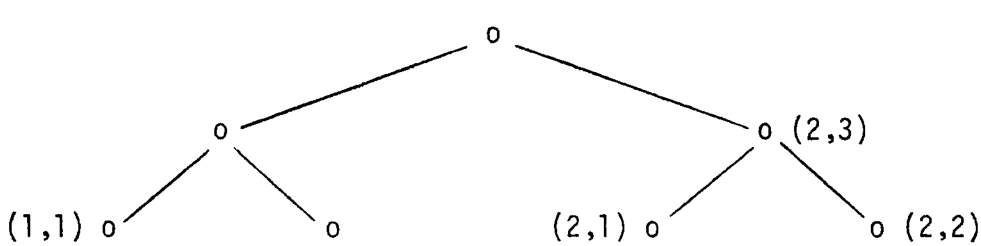
Note that Q itself as well as all principal lower ends of Q are lexicographic k-ideals. The empty set is again not a lexicographic k-ideal. In fact, $\emptyset$ is by our convention not even a lower end.

Let $\mathcal{L}^k(Q)$ be the set of all lexicographic k-ideals in Q. By definition, $\mathcal{L}^k(Q)$ is a subset of $\mathcal{E}(Q)$, partially ordered by set-theoretical inclusion. For $\{L_i \mid i \in I\} \subset \mathcal{L}^k(Q)$, the set-theoretical intersection $L = \bigcap\{L_i \mid i \in I\}$ is again a lexicographic k-ideal, provided that $L \neq \emptyset$. Hence, the smallest lexicographic k-ideal $\overline{L}$ containing $\{L_i \mid i \in I\}$ exists. In fact, $\overline{L} = \bigcap\{L \mid L \supset L_i$, for all $i \in I$, $L \in \mathcal{L}^k(Q)\} = \sup_{\mathcal{L}^k(Q)}\{L_i \mid i \in I\}$ - note that $\overline{L} \supset \bigcup\{L_i \mid i \in I\} \neq \emptyset$, thus $L \neq \emptyset$.

By the previous remarks, arbitrary joins exist in $\mathcal{L}^k(Q)$. But clearly, the join in $\mathcal{L}^k(Q)$ is, in general, not the set-theoretical union; i.e., $\mathcal{L}^k(Q)$ is not join-faithful in $\mathcal{E}(Q)$. As an example for that, take T to be



and $P_1 = P_2 = T$, P_3 has just one element. Then $Q = \mathop{\llcorner} P_t$ has the structure



All principal lower ends are lexicographic k-ideals. But while the union of ((1,1)] and ((2,1)] is a lexicographic k-ideal, the union of ((2,1)] and ((2,2)] is not. Actually, $\sup_{\mathcal{L}^k(Q)}\{((2,1)],((2,2)]\}$ $= ((2,3)]$, and $\sup_{\mathcal{L}^k(Q)}\{((1,1)],((2,1)],((2,2)]\} = ((1,1)] \cup ((2,3)]$. Thus, the join in $\mathcal{L}^k(Q)$ and the join in $\mathcal{E}(Q)$ may coincide, but will not, in general.

For $Q = \coprod P_t$ we consider now in $\mathcal{E}_k(Q)$ the set of all those lower ends that are lexicographic k-ideals. We denote this set by $\mathcal{L}^k_k(Q)$. Hence, $\mathcal{L}^k_k(Q) = \mathcal{L}^k(Q) \cap \mathcal{E}_k(Q)$, or equivalently:

(7.8) $\qquad E \in \mathcal{L}^k_k(Q)$ iff $E \in \mathcal{L}^k(Q)$ and $E = \bigcup\{(u] \mid u \in U\}$,

where U is a k-small subset of Q.

For a subset $U \subset Q = \coprod P_t$, let $U(t) = \{x \in P_t \mid (t,x) \in U\}$. $U(t)$ is a subset of P_t, consequently a partially ordered set, and $|U(t)| < k$, whenever $|U| < k$. Note that $U(t)$ may be empty for some $t \in T$. Let $T(U) = \{t \in T \mid U(t) \neq \emptyset\}$. Again, $|T(U)| < k$, whenever $|U| < k$. Moreover, $U(t) \neq \emptyset$, for each $t \in T(U)$, and $T(U) = \emptyset$ if and only if $U = \emptyset$.

Theorem 7.2. Suppose $E = \bigcup\{(u] \mid u \in U\}$ - where $U \subset Q$ is k-small - is a k-small generated lower end of Q. Then the following two conditions are equivalent:

(i) E is a lexicographic k-ideal;

(ii) if t is maximal in $T(U)$, then $U(t)$ has a greatest element.

Proof. (i) $\Longrightarrow$ (ii): By hypothesis, $U \subset E = \bigcup\{(u] \mid u \in U\}$. Hence $U(t) \subset E(t)$, for all $t \in T$, and $\sup_Q\{(t,x) \mid x \in U(t)\} = (t, \sup_{P_t} U(t))$ exists in Q if $t \in T(U)$, i.e. $U(t) \neq \emptyset$. Moreover, $\sup_Q\{(t,x) \mid x \in U(t)\}$ is an element of E, since E is a lexicographic k-ideal. Thus $(t, \sup_{P_t} U(t)) \in \bigcup\{(u] \mid u \in U\}$, i.e. there is $u_0 \in U$ such that $(t, \sup_{P_t} U(t)) \leq u_0$. Suppose now that t is maximal in T(U). Then $u_0 \in i_t(P_t)$, hence $u_0 = (t, x_0)$, where $x_0 \in U(t)$. In addition, since $(t, \sup_{P_t} U(t)) \leq (t, x_0)$, we get $\sup_{P_t} U(t) \leq x_0 \leq \sup_{P_t} U(t)$, proving $x_0 = \max U(t)$.

(ii) $\Longrightarrow$ (i) is similarly straightforward and is left to the reader.

Theorem 7.3. Suppose $\{E_i \mid i \in I\}$ is a k-small subset of $\mathcal{L}_k^k(Q)$. Suppose for each $i \in I$, $E_i = \bigcup\{(u_i] \mid u_i \in U_i\}$, where $U_i \subset Q$ is k-small. Let $U = \bigcup U_i \subset Q$. Then $E = \bigcup\{((t, \sup_{P_t} U(t))] \mid t \in T(U)\}$ is in $\mathcal{L}_k^k(Q)$ and is the least lexicographic k-ideal containing all E_i. I.e., $E = \sup_{\mathcal{L}_k^k(Q)}\{E_i \mid i \in I\}$.

Proof. We should mention that every E_i can be represented as $E_i = \bigcup\{(u_i] \mid u_i \in U_i\}$ by (7.8). The inaccessibility of k makes U a k-small set, hence T(U) is k-small, and for each $t \in T(U)$, $\sup_{P_t} U(t)$ exists in P_t. So $E = \bigcup\{((t, \sup_{P_t} U(t))] \mid t \in T(U)\}$ is a k-small generated lower end in Q. For $i \in I$ and $u_i \in U_i$, we have $u_i \in U$, hence $u_i = (t,x)$, for some $t \in T(U)$ and $x \in U(t)$. Thus $x \leq \sup_{P_t} U(t)$ and

$u_i = (t,x) \leq (t,\sup_{P_t} U(t))$, i.e. $u_i \varepsilon E$. This proves $E_i \subset E$, for each $i \varepsilon I$.

In order to show that E is a lexicographic k-ideal, we consider $M = \{(t,\sup_{P_t} U(t)) | t \varepsilon T(U)\}$. M is a k-small subset of Q, and $|M(t)| \leq 1$, for all $t \varepsilon T$. Moreover, E is of the form $E = \bigcup \{(m] | m \varepsilon M\}$. Now, if $t \varepsilon T(M)$, then $M(t) \neq \emptyset$, i.e. $|M(t)| = 1$, which trivially implies that max M(t) exists, for all $t \varepsilon T(M)$. We have now verified condition (ii) of Theorem 7.2. Consequently, E is a lexicographic k-ideal.

It remains to show that E is the least lexicographic k-ideal containing all E_i, $i \varepsilon I$. So let E' be a lexicographic k-ideal containing all E_i, and let $t \varepsilon T(U)$. For each $x \varepsilon U(t)$, we have $x \varepsilon U_i(t)$, for some $i \varepsilon I$, i.e. $(t,x) \varepsilon U_i \subset E_i$, for some $i \varepsilon I$. But then $(t,x) \varepsilon \bigcup E_i \subset E'$, for all $x \varepsilon U(t)$. So $\sup_Q \{(t,x) | x \varepsilon U(t)\} = (t,\sup_{P_t} U(t)) \varepsilon E'$, since E' was assumed to be a lexicographic k-ideal. We proved $(t,\sup_{P_t} U(t)) \varepsilon E'$, for all $t \varepsilon T(U)$, hence $((t,\sup_{P_t} U(t))] \subset E'$, for all $t \varepsilon T(U)$. Thus $E \subset E'$, completing the proof of Theorem 7.3.

For a lexicographic k-ideal $E \varepsilon \mathcal{L}_k^k(Q)$ with the representation $E = \bigcup \{(u] | u \varepsilon U\}$, U k-small, we obtain as an immediate consequence of Theorem 7.3 that E can also be represented as $E = \bigcup \{((t,\sup_{P_t} U(t))] | t \varepsilon T(U)\}$. The latter representation will, in general, be strictly "shorter" than the first, but there may, of course, still be a lot of redundant elements in the representation. Nevertheless, Theorem 7.3 completely describes the joins in $\mathcal{L}_k^k(Q)$.

They obviously do not always coincide with the set-theoretical union of principal lower ends.

Let us explain what happens, in more detail, at least for the case $k = \aleph_0$. Then we are talking about semilattices, and $\mathcal{L}_k(Q)$ is the semilattice of finitely generated lower ends of Q. Suppose $x_1,\dots,x_n \in Q$, and let $E = \bigcup\{(x_i] \mid i=1,\dots,n\}$. Assume now that no two different elements $x_i \neq x_j$ are in the same piece $i_t(P_t)$. Then condition (ii) of Theorem 7.2 holds true, and E is a lexicographic k-ideal. Hence $E = \sup_{\mathcal{L}_k^k(Q)}\{(x_i] \mid i \in I\}$. In case there are different elements in the same piece, we get a partition of $\{x_1,\dots,x_n\}$ into subsets, say $S_1,\dots,S_m$, where $S_j \subset i_t(P_t)$, for some $t \in T$ $(j=1,\dots,m)$. For the lower end $E' = (\sup_Q S_1] \cup \dots \cup (\sup_Q S_m]$ condition (ii) of Theorem 7.2 is again fullfilled, and by Theorem 7.3 we have $E' = \sup_{\mathcal{L}_k^k(Q)}\{(x_i] \mid i \in I\} = \bigcup\{(\sup_Q S_j] \mid j=1,\dots,m\}$.

We are now ready to state an analogue to Theorem 7.1:

<u>Theorem 7.4.</u> Suppose Q is the lexicographic sum of the k-semilattices P_t. $\mathcal{L}_k^k(Q)$ is the canonical image of a k-join-completion P of Q. Moreover, $i_t(P_t)$ is k-join-faithful in P, for every $t \in T$.

Proof. Every element in $\mathcal{L}_k^k(Q)$ is the join of a k-small set of principal lower ends. In fact, it is even the set-theoretical union of a k-small set of principal lower ends. By Theorem 7.3, $\mathcal{L}_k^k(Q)$ is itself k-join-complete. Hence $\mathcal{L}_k^k(Q)$ is a k-join-completion of Q. I.e., there is a k-join-completion P of Q, where $P = \mathcal{L}_k^k(Q)$. Assume now $t \in T$ and $S \subset i_t(P_t)$, S k-small. Then $x = \sup_{i_t(P_t)} S$ exists and $x = \sup_Q S$. Hence, the least lexicographic k-ideal containing $\{(s] \mid s \in S\}$ is nothing but the principal lower end $(x]$. This implies that $i_t(P_t)$ is k-join-faithful in P.

It should be no secret by now that $\mathcal{L}_k^k(Q)$, where $Q = \llcorner P_t$, will turn out to be the order-sum of the k-semilattices P_t.

8. The order-sum of one-element k-semilattices, a special case

A careful inspection of Theorem 7.4 reveals immediately that Theorem 7.1 is nothing but a special case of Theorem 7.4. This is due to the fact that every partially ordered set Q can be considered a lexicographic sum of one-element k-semilattices : $Q = \llcorner P_t$ where $T = Q$ and $P_t = \{t\}$, for $t \in T$. In this interpretation, every k-small generated lower end of Q is a lexicographic k-ideal, in fact, $\mathcal{E}_k(Q) = \mathcal{L}_k^k(Q)$ with set-theoretical union as join. Clearly, $i_t(P_t) = \{t\}$ is k-join-faithful in every k-join-completion P of Q. On the other hand, the algebraic lexicographic sum of one-element k-semilattices is just the partially ordered set T, without any additional structure. I.e., all joins in T that might possibly exist, are disregarded.

Our goal in this section is, to give an internal characterization of the order-sum P of one-element k-semilattices P_t. To do this, we refer to Theorem 6.2. First of all, the remarks above make clear that P will have to be an order-extension of the partially ordered set T. Condition (ii) of Theorem 6.2, in this special case, reads as follows:

(*) for every k-semilattice Q and every isotonic mapping $\psi : T \longrightarrow Q$ there is a unique k-join-preserving mapping $\bar{\psi} : P \longrightarrow Q$ such that $\bar{\psi}|T = \psi$.

So, in order to characterize the order-sum, all we have to do is to characterize the order-extension P of T (P necessarily is a k-join-completion of T) having the property (*). For the class of k-semilattices with least elements this has been done already in Schmidt [13], and there are no difficulties in extending the characterization to the class of k-semilattices. We will, however, briefly discuss the sequence of steps leading to the characterization. This will turn out to be helpful for section 9, where we have to deal with the general case of arbitrary k-semilattices.

An element x in a partially ordered set Q is *k-join-primitive*, provided the following condition holds for each k-small subset $S \subset Q$:

(8.1) if $x \leq \sup_Q S$, then $x \leq s$, for some $s \in S$.

Note that in contrast to Schmidt [13], the least element of Q, if it exists, is k-join-primitive.

Suppose now $Q = \bigsqcup P_t$, where $|P_t| = 1$, for all $t \in T$. I.e., Q is (up to order-isomorphism) just the partially ordered set T.

Theorem 8.1. $\mathcal{L}_k(Q)$ is the canonical image of a k-join-completion P of Q, and every element of Q is k-join-primitive in P.

Proof. The first part is a repetition of Theorem 7.1, or, if one prefers, of Theorem 7.4. For the second part, see Schmidt [13], Theorem 1.1.

Theorem 8.2. Suppose P is a k-join-completion of Q. Suppose further that each element of Q is k-join-primitive in P. Then P is the order-sum of the one-element semilattices P_t.

Proof. It suffices to verify condition (*) of the previous page for the k-join-completion P. This can be done as in Schmidt [13], Theorem 1.2 "(i)$\Longrightarrow$(ii)". The k-join-primitivity of the elements of Q is only needed for the proof that the k-join-preserving mapping $\bar{\psi} : P \longrightarrow Q'$ extending $\psi : Q \longrightarrow Q'$ is well-defined. Since this is a crucial point as far as the general case in section 9 is concerned, we will show here in detail, in what way k-join-primitivity enters the proof:

Suppose that $x \in P$ has the representation $x = \sup_P S$ where S is a k-small subset of Q. $\bar{\psi} : P \longrightarrow Q'$ is defined by $\bar{\psi}(x) = \sup_{Q'} \psi(S)$. Let now $x \in P$ have two such representations, $s = \sup_P S = \sup_P R$, where

R and S are k-small subsets of Q. We have to show: $\sup_{Q'}\psi(S) = \sup_{Q'}\psi(R)$. But for all $s \in S$, $s \leq \sup_P R$, thus $s \leq r$, for some $r \in R$, since s is k-join-primitive in P. Isotonicity of ψ implies $\psi(s) \leq \psi(r)$, hence $\psi(s) \leq \sup_{Q'}\psi(R)$, for each $s \in S$. But then $\sup_{Q'}\psi(S) \leq \sup_{Q'}\psi(R)$, and the other inequality can be shown in the same way.

Theorem 8.3 (Internal Characterization).

Let P be an order-extension of $Q = \sqcup P_t$, where $|P_t| = 1$, for each $t \in T$. Then the following statements are equivalent:

(i) P is a k-join-completion of Q and each element of Q is k-join-primitive in P;

(ii) P is the order-sum of the k-semilattices P_t;

(iii) P is a k-join-completion of Q and $\dot{P} = \mathcal{E}_k(Q)$.

Proof. (i)$\Longrightarrow$(ii) by Theorem 8.2. (ii)$\Longrightarrow$(iii): P is a k-join-completion by Theorem 1.3. By Theorem 8.1, there is a k-join-completion P' of Q such that $\dot{P}' = \mathcal{E}_k(Q)$, and every element of Q is k-join-primitive in P'. Now we use Theorem 8.2 to conclude that P' is the order-sum of the k-semilattices P_t. Thus P' and P are isomorphic (as algebras and as partially ordered sets), in particular, $\dot{P}' = \dot{P} = \mathcal{E}_k(Q)$. (iii)$\Longrightarrow$(i) by Theorem 8.1.

Let us take another look at the example at the end of section 5. It is clear now, why the index-set T fails to be the order-sum. The order-sum is the set of all k-small generated lower ends of $\mathbb{N} \cup \{\infty\}$.

The lower end $\mathbb{N}$ itself, however, is k-small generated, if $k = \aleph_1$. Thus the order-sum P is $\{(n] \mid n \in \mathbb{N}\} \cup \mathbb{N} \cup T$ with the appropriate total order.

9. Internal characterization of the order-sum, the general case

In the previous section, the notion of k-join-primitivity served our purposes for the characterization of an extremely special order-sum. For the internal characterization of the general order-sum, k-join-primitivity will have to be replaced by a more general property. This new property should be defined in such a way that it contains k-join-primitivity as a special case.

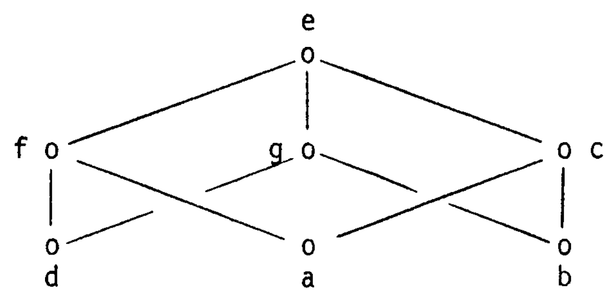
We will say that a subset R of a partially ordered set P is *covered* by a subset S of P, if for every $r \in R$, there is some $s \in S$, such that r s.

Let Q, P be partially ordered sets, Q a subset of P. If $R \subset Q \subset P$, and if we write $x \leqslant \sup_P R \in Q$, then this shall always mean that $\sup_P R$ exists, that $\sup_P R \in Q$, and that $x \leqslant \sup_P R$. An element $x \in P$ will be called *(Q,k)-join-primitive* provided the following condition holds for every k-small subset $S \subset P$:

(9.1) if $x \leqslant \sup_P S$, then Q contains a k-small subset R, covered by S, such that $x \leqslant \sup_P R \in Q$.

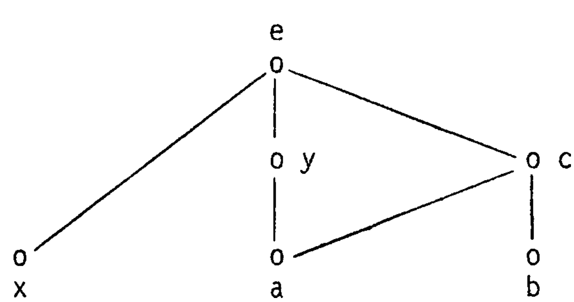
Obviously, every $x \in P$ is (P,k)-join-primitive in P, and no element of P is $(\emptyset,k)$-join-primitive in P.

For a non-trivial example of a (Q,k)-join-primitive element, let P have the following structure:



Suppose $Q = \{a,b,c\} \subset P$. Then $c \in P$ is (Q,k)-join-primitive in P. To show this, we have to verify (9.1). So suppose $S \subset P$ and $c \leq \sup_P S$. (9.1) is trivially fulfilled, if $c \in S$ or if $S \subset Q$. If we eliminate these cases, then $c \leq \sup_P S$ implies $\sup_P S = e$, and either $\{f,g\} \subset S$ or $\{f,b\} \subset S$ or $\{a,g\} \subset S$. In all three cases $R \subset Q$ can be chosen as $R = \{a,b\}$.

The following example shows that for $Q \subset P$, not every $q \in Q$ is (Q,k)-join-primitive in P. Suppose P has the structure



and let $Q = \{a,b,c\} \subset P$. c is not (Q,k)-join-primitive, since

$c \leq \sup_P\{x,y\}$, but there is no $R \subset Q$ such that $c \leq \sup_P R$ and $r \leq x$ or $r \leq y$, for all $r \in R$.

Our first theorem establishes the relationship between k-join-primitivity and (Q,k)-join-primitivity.

Theorem 9.0. Suppose Q and P are partially ordered sets, $Q \subset P$. The following statements are equivalent:

(i) $x \in Q$ is k-join-primitive in P;

(ii) $x \in Q$ is $(\{q\},k)$-join-primitive in P, for some $q \in Q$.

Proof. (i) $\Longrightarrow$ (ii): Assume $x \leq \sup_P S$, where $S \subset P$ and S k-small. Then $x \leq s$, for some $s \in S$, and (9.1) is fulfilled for $R = \{x\} \subset Q$. (ii) $\Longrightarrow$ (i): Let $x \leq \sup_P S$, where $S \subset P$ and S k-small. By hypothesis, $x \leq q$, for some $q \in Q$, where $\{q\}$ is covered by S. Hence $x \leq q \leq s$, for some $s \in S$.

The following three theorems, 9.1, 9.2, and 9.3 are extensions of theorems 8.1, 8.2, and 8.3, respectively.

Let Q be the algebraic lexicographic sum of the k-semilattices P_t.

Theorem 9.1. $\mathcal{L}_k^k(Q)$ is the canonical image of a k-join-completion P of Q. Each piece $i_t(P_t)$ is k-join-faithful in P. Moreover, every element of Q is $(i_t(P_t),k)$-join-primitive in P, for some $t \in T$.

Proof. The first part is a repetition of Theorem 7.4. For notational simplification, let us introduce the mapping $\tau : Q \longrightarrow \mathcal{L}^k_k(Q)$, defined by $\tau(x) = (x]$. To prove the second part of our theorem, we are going to show that every principal lower end $(x]$ of Q is $(\tau i_t(P_t),k)$-join-primitive in $\mathcal{L}^k_k(Q)$, for some $t \varepsilon T$. So assume $(x] \subset \sup_{\mathcal{L}^k_k(Q)}\{E_i | i \varepsilon I\}$, where I is k-small, and let $E_i \varepsilon \mathcal{L}^k_k(Q)$. For each $i \varepsilon I$, $E_i = \bigcup\{(u_i] | u_i \varepsilon U_i\}$, where $U_i \subset Q$ and U_i k-small. Let $U = \bigcup U_i$. Then $U \subset Q$ and U k-small, and by Theorem 7.3. $\sup_{\mathcal{L}^k_k(Q)}\{E_i | i \varepsilon I\} = \bigcup\{((t, \sup_{P_t} U(t))] | t \varepsilon T(U)\}$. Now, $(x] \subset \bigcup\{((t, \sup_{P_t} U(t))] | t \varepsilon T(U)\}$, hence there is $t \varepsilon T(U)$ such that $x \leq (t, \sup_{P_t} U(t)) = \sup_Q i_t(U(t))$. Moreover, $U(t) \neq \emptyset$ since $t \varepsilon T(U)$. Let now $R = i_t(U(t))$. Then $R \subset i_t(P_t)$, and R is k-small. Also, for every $r \varepsilon R$, $r \varepsilon U = \bigcup U_i$, i.e., $r \varepsilon U_i$, for some $i \varepsilon I$. Hence $r \varepsilon E_i$, for some $i \varepsilon I$, thus $(r] \subset E_i$, for some $i \varepsilon I$. So, $\tau(R)$ is covered by $\{E_i | i \varepsilon I\}$, $\tau(R) \subset \tau(i_t(P_t))$, and $\tau(R)$ is k-small. It remains to show: $(x] \subset \sup_{\mathcal{L}^k_k(Q)}\{(r] | r \varepsilon R\} \varepsilon \tau(i_t(P_t))$. But $\tau(i_t(P_t))$ is k-join-faithful in $\mathcal{L}^k_k(Q)$ by Theorem 7.4, hence $(\sup_Q R] = \sup_{\mathcal{L}^k_k(Q)}\{(r] | r \varepsilon R\}$. In addition, $(\sup_Q R] = ((t, \sup_{P_t} U(t))] \varepsilon \tau(i_t(P_t))$, and $(x] \subset ((t, \sup_{P_t} U(t))]$, since $x \leq (t, \sup_{P_t} U(t))$ as we have seen earlier. $\tau(R)$ has therefore all the required properties, and the proof is completed.

Theorem 9.2. Suppose P is a k-join-completion of $Q = \coprod P_t$, such that all the pieces $i_t(P_t)$ are k-join-faithful in P. Suppose further that each element of Q is $(i_t(P_t),k)$-join-primitive in P, for some $t \in T$. Then P is the order-sum of the k-semilattices P_t.

Proof. It is sufficient to verify condition (ii) of Theorem 6.2. So let Q' be a k-semilattice, and suppose $\psi : Q \longrightarrow Q'$ is a homomorphism. P is, by assumption, a k-join-completion of Q. Hence, every $x \in P$ is representable in the form $x = \sup_P S$, for some k-small set $S \subset Q$. We now define $\bar{\psi} : P \longrightarrow Q'$ by $\bar{\psi}(x) = \sup_{Q'} \psi(S)$. Clearly, $\bar{\psi}|Q = \psi$, and the unicity of $\bar{\psi}$ follows from the fact that P is a k-join-completion of Q.

In order to prove that the definition of $\bar{\psi}$ is independent of the representation of x, let $x = \sup_P S = \sup_P S'$, where S and S' are k-small subsets of Q. We have to show: $\sup_{Q'} \psi(S) = \sup_{Q'} \psi(S')$. For each $s \in S$, we have $s \leqslant \sup_P S'$, and each $s \in S$ is, by hypothesis, $(i_t(P_t),k)$-join-primitive, for some $t \in T$. I.e., there is $t \in T$ and a k-small set $R \subset i_t(P_t)$, such that $s \leqslant \sup_P R \in i_t(P_t)$, and R is covered by S'. Now all pieces $i_t(P_t)$ are k-semilattices and k-join-faithful in P, by hypothesis. Thus $s \leqslant \sup_P R = \sup_{i_t(P_t)} R \in i_t(P_t)$. Consequently, $\psi(s) \leqslant \psi(\sup_{i_t(P_t)} R) = \sup_{Q'} \psi(R)$, since ψ was assumed to be a homomorphism. In addition, R is covered by S', so for every $r \in R$ there is $s' \in S'$, such that $r \leqslant s'$, hence $\psi(r) \leqslant \psi(s')$, by isotonicity of ψ. But then $\psi(s) \leqslant \sup_{Q'} \psi(R) \leqslant \sup_{Q'} \psi(S')$, for all $s \in S$. Hence,

$\sup_{Q'}\psi(S) \leq \sup_{Q'}\psi(S')$, and the other inequality, $\sup_{Q'}\psi(S') \leq \sup_{Q'}\psi(S)$, can obviously be shown in the same manner.

It still remains to show that $\bar{\psi}$ is k-join-preserving. That, however, is a straightforward consequence of the inaccessibility of k and the associativity of the supremum-operation. This completes the proof of Theorem 9.2.

In the following theorem, we finally come to our internal characterization of the order-sum of k-semilattices. We know already that if P is the order-sum of the k-semilattices P_t, then the pieces $i_t(P_t)$ have to be k-join-faithful in P. Thus, it suffices to consider only those order-extensions of $\sqcup P_t$ that contain the k-semilattices $i_t(P_t)$ as sub-k-semilattices.

Theorem 9.3 (Internal Characterization)

Let P be an order-extension of $Q = \sqcup P_t$, such that all the pieces $i_t(P_t)$ are k-join-faithful in P. Then the following statements are equivalent:

(i) P is a k-join-completion of Q, and each element of Q is $(i_t(P_t),k)$-join-primitive in P, for some $t \in T$;

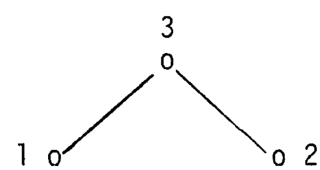
(ii) P is the order-sum of the k-semilattices P_t;

(iii) P is a k-join-faithful completion of Q and $P = \mathcal{L}_k^k(Q)$.

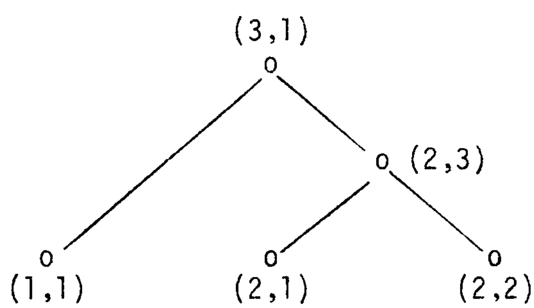
Proof. (i)$\Longrightarrow$(ii) by Theorem 9.2. (ii)$\Longrightarrow$(iii) is a repetition of (ii)$\Longrightarrow$(iii) in Theorem 8.3, just substitute $\mathcal{L}_k^k(Q)$ for $\mathcal{E}_k(Q)$ and $(i_t(P_t),k)$-join-primitivity for k-join-primitivity. (iii)$\Longrightarrow$(i) by Theorem 9.1.

Clearly, the previous three theorems contain their counterparts in section 8 as special cases. This is due to the fact that in section 8 the k-small generated lexicographic k-ideals coincide with the k-small generated lower ends of Q. In addition, $(i_t(P_t),k)$-join-primitivity coincides with k-join-primitivity in this case (cf. Theorem 9.0).

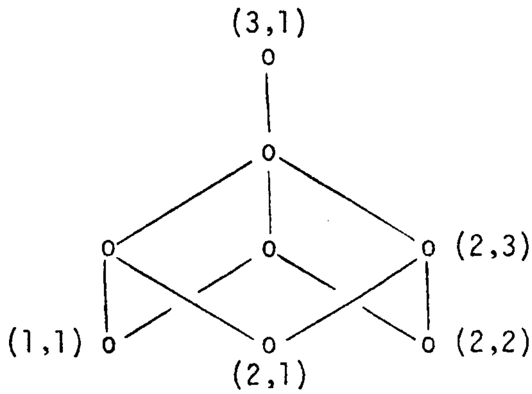
We now give an example for the order-sum P of k-semilattices P_t, where $k = \aleph_0$. Take T to be



and $P_2 = T$; P_1 and P_3 have just one element. Then $\llcorner P_t$ has the structure



and is itself a semilattice. To determine the order-sum of the k-semilattices P_t, we have to find all finitely generated lexicographic k-ideals of $\llcorner P_t$. This is not hard to do in this case: All principal lower ends are lexicographic k-ideals and the only non-principal ones are the lower ends $((1,1)]\vee((2,j)]$, for $j=1,2,3$. P is now the semilattice with the following structure:



Obviously, $\llcorner P_t$ considered as the lexicographic sum of the partially ordered sets P_t (structure (iii) on p. 28), is not k-join-faithful in P. Also, $(1,1)\vee(3,1)$ exists in P, but not in the algebraic lexicographic sum $\llcorner P_t$ (structure (i)). This shows that the three structures discussed on p. 28 can all be different from each other. In particular, the inclusion mapping $i : \llcorner P_t \longrightarrow P$ - where $\llcorner P_t$ has structure (i) - is, in general, not an algebraic embedding.

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