NEARLY REDUCIBLE AND NEARLY DECOMPOSABLE - SPECIAL CLASSES OF IRREDUCIBLE AND FULLY INDECOMPOSABLE MATRICES

A Dissertation

Presented to

the Faculty of the

Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

Mark Blondeau Hedrick

August 1972

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ABSTRACT

The purpose of this dissertation is to examine the structural properties of matrices whose entries are either 0 or 1. There are three main results. In Theorem 1, the author shows that the maximal number of positive entries (arcs) in an $n \times n$ nearly reducible matrix (minimally connected graph with n vertices) is $2(n - 1)$ and the matrix has a canonical form. In Theorem 2, he argues that the maximal number of positive entries in a nearly decomposable $n \times n$ matrix is $3(n - 1)$ and is obtained uniquely at a canonical matrix. In Theorem 3, he examines the structure of those nearly decomposable (0,1)-matrices whose permanent equals $\sigma(A) - 2n + 2$ where $\sigma(A)$ is the number of positive entries in A.

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CHAPTER I

HISTORICAL BACKGROUND AND DEFINITIONS

In order to argue inductively on the number of positive entries in a fully indecomposable matrix, Knopp and Sinkhorn developed in [8] a canonical form for a fully indecomposable matrix which becomes partly decomposable when any positive entry is replaced by a zero. They called such matrices nearly decomposable and when in canonical form, such matrices are easily adaptable to inductive arguments on their dimension. They used their canonical form to prove that if A is a fully indecomposable matrix all of whose positive diagonals are equal, then there is a unique positive matrix B such that it has rank one and $a_{ij} = b_{ij}$ if $a_{ij} > 0$. This author used their form in [7] to prove an analogous theorem for a fully indecomposable matrix each of whose positive diagonals has the same sum. Sinkhorn used the canonical form in [14] to argue that if A is an n x n (0,1)-matrix with exactly three ones in each row and column, then perA $\geqq$ n. Minc used the canonical form in [11] to argue that

Theorem A. <u>If</u> A <u>is an</u> n x n <u>fully indecomposable</u> (0,1)-<u>matrix</u>, <u>then</u>

$$\sigma(A) - 2n + 2 \leqq \text{perA}.$$

where $\sigma(A)$ is the number of positive entries in A and perA is the permanent of A. At this point, Sinkhorn and this author began to study the structure of an irreducible matrix which becomes reducible when any positive entry is replaced by a zero. In analogy to nearly decomposable matrices, they called such an irreducible matrix nearly reducible. They derived in [6] a canonical form for a nearly reducible matrix which plays the same role in the class of irreducible matrices as the canonical form for a nearly decomposable matrix does in the class of fully indecomposable matrices. Using this canonical form, they showed in [6] that if A is a nearly reducible (0,1)-matrix, then perA $\leqq 1$. This author then discovered that by

Theorem B. _If_ A _is an_ n x n (0,1)-_matrix with a positive main diagonal, then_ $A = A' + I_n$ _is fully indecomposable if and only if_ A' _is irreducible_.

(where I_n is the n x n identity matrix) of Brualdi, Parter, and Schneider [2], he could trivially prove

Theorem C. _If_ A _is an_ n x n _nearly decomposable_ (0,1)-_matrix with a positive main diagonal, then_ $A = A' + I_n$ _where_ A' _is nearly reducible_.

At this point, Hartfiel found in the literature that an irreducible matrix was the same thing as a strongly connected graph [15, p. 20]. He also found that nearly reducible matrices had been studied using graph techniques under the name of minimally connected graphs [1, pp. 122-4]. Using a theorem on minimally connected graphs, Hartfiel derived a canonical form for a nearly reducible matrix. Using this canonical form and Theorem C, he derived a canonical form for a nearly decomposable matrix. Both of these forms in [5] were refinements of the previously mentioned ones. Using his forms, he simplified in [5] the proofs of the previously mentioned theorems in [14] and [6].

Thus the author studies the structural properties of these matrices in the hope that such information will inhance their usefulness. There are three main results. In Theorem 1, the author shows that the maximal number of positive entries (arcs) in an n x n nearly reducible matrix (minimally connected graph with n vertices) is 2(n - 1) and the matrix has a canonical form. In Theorem 2, he argues that the maximal number of positive entries in a nearly decomposable n x n matrix is 3(n - 1) and is obtained uniquely at a canonical matrix. In Theorem 3, he examines the structure of those nearly decomposable matrices for which Minc's inequality in Theorem A is equality.

Most of the matrix terminology can be found in [10] and the graph terminology in [1]. Since the author does not make use of graph theoretic terms in the proofs of his main results and since the dissertation can be read by simply suppressing the references to graph theory, the author will not explicitly define the graph terminology. He has included these references in order to acquaint the reader with the connections that do exist. As the author explains his terminology, he will indicate certain facts which are trivial but crucial.

A (0,1)-matrix is a matrix all of whose entries are 0 or 1. Let E_{ij} denote the n x n matrix which has 1 in the (i,j) position and zeros elsewhere. Let A be an n x n matrix and let α be a nonempty, proper subset of $\{1,\ldots,n\}$ ordered increasingly. Then $A[\alpha|\alpha)$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by the complement of α in $\{1,\ldots,n\}$ ordered increasingly. The transpose of A, denoted by A^T, is the n x n matrix whose (i,j) entry is the (j,i) entry of A.

A diagonal of a square matrix is a set of entries from the matrix, one from each row and one from each column. If β is a permutation of $\{1,\ldots,n\}$, then the diagonal associated with β is $a_{1\beta(1)},\ldots,a_{n\beta(n)}$. Every diagonal corresponds to a permutation. The permutation matrix corresponding to β is the matrix which has ones in the $(1,\beta(1)),\ldots,(n,\beta(n))$ positions and zeros elsewhere. A positive diagonal is a diagonal in which $a_{i\beta(i)} > 0$ for all i.

The product of the diagonal β is the product of the $a_{1\beta(1)}$ through $a_{n\beta(n)}$. The permanent of A, denoted by perA, is obtained by taking the product of each diagonal in A and then taking the sum of these products.

An n x n nonnegative matrix A is partly decomposable if there is an s x t zero submatrix where s + t = n. By convention, the 1 x 1 zero matrix (0) is partly decomposable. An n x n nonnegative matrix A is fully indecomposable if A is not partly decomposable. An n x n fully indecomposable matrix A is <u>nearly decomposable</u> if $A - a_{ij}E_{ij}$ is partly decomposable for each $a_{ij} > 0$.

An n x n nonnegative matrix A is reducible if there is a nonempty, proper subset α of $\{1,\ldots,n\}$ such that $A[\alpha|\alpha) = 0$. An n x n nonnegative matrix A is irreducible if A is not reducible. An n x n irreducible matrix A is <u>nearly reducible</u> if $A - a_{ij}E_{ij}$ is reducible for each $a_{ij} > 0$. (In graph theory, minimally connected graph with n vertices corresponds to an n x n nearly reducible matrix and arc corresponds to positive entry.)

Let $A = (a_{ij})$ be an n x n nonnegative matrix. The nonzero matrix A has <u>doubly stochastic pattern</u> if every positive entry lies on a positive diagonal. Two positive entries $a_{i_1j_1}$ and $a_{i_kj_k}$ are <u>chainable</u> if there is a sequence $a_{i_1j_1},\ldots,a_{i_kj_k}$ of distinct positive entries of A such that for $1 \leqq r \leqq k$, $i_r = i_{r+1}$ or $j_r = j_{r+1}$; and if $1 < h+1 < p \leqq k$, then $i_h \neq i_p$ and $j_h \neq j_p$. Such a sequence is called a <u>chain</u>. Two different chains are <u>disjoint chains</u> if, with the possible exception of the endpoints, it is true that a_{km} is a term of one if and only if k is not the row index and m is not the column index of any term of the other. The concept can be visualized by the movements of a rook with stationary positions on positive entries of A. Observe that the rook moves only once in each row (column) and that once it leaves a row (column) it cannot return to that row (column).

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. There is a path which connects $a_{i_1 j_1} > 0$ to $a_{i_k j_k} > 0$ if there is a sequence $a_{i_1 j_1}, \ldots, a_{i_k j_k}$ of positive entries of A such that $j_r = i_{r+1}$ for $1 \leqq r < k$. We now make an observation which we shall use below and in the proof of Theorem 3. Assume that there is a nonempty, proper subset α of $\{1,\ldots,n\}$ such that $A[\alpha|\alpha) = 0$ and $a_{ij}, \ldots, a_{pq}$ is a path. Then if $i \in \alpha$, $q \in \alpha$. The result follows immediately by induction on the number of terms in the path.

Let A be an $n \times n$ nonnegative matrix. The i-th row can be connected by a path to the j-th column if there is a path $a_{ip}, \ldots, a_{qj}$. With the above definitions, one can argue that for $n > 1$, (1) A is irreducible if and only if (2) every row and column contains at least one positive entry and every two positive entries can be connected by a path if and only if (3) every row can be connected to any column. That (2) implies (3) is immediate. To show (3) implies (2), consider $a_{ij} > 0$ and $a_{pq} > 0$. By (3), there is a path $a_{jk}, \ldots, a_{mp}$. Hence $a_{ij}, a_{jk}, \ldots, a_{mp}, a_{pq}$ is a path. That (2) implies (1) follows from our above observation. Assume (1) and that $a_{ij} > 0$ cannot be connected to $a_{pq} > 0$ by a path. Let $\alpha = \{k \mid \text{there is some } a_{rk} > 0$ which can be connected to a_{ij} by a path$\}$. Observe that $j \in \alpha$ and $p \notin \alpha$. Let $k \in \alpha$ and $m \notin \alpha$ and assume $a_{km} > 0$. By definition, there is a path $a_{ij}, \ldots, a_{rk}$. Hence $a_{ij}, \ldots, a_{rk}, a_{km}$ is a path. Thus $m \in \alpha$ which is a contradiction. Thus $a_{km} = 0$ or $A[\alpha|\alpha) = 0$ which contradicts (1). Hence (1) implies (2). Because of (2) and (3), an irreducible matrix is said to be strongly connected. (This condition is usually stated by saying that a matrix is irreducible if and only if the directed graph corresponding to it is strongly connected.) We shall use the above equivalences in the proofs of Propositions 3 and 4.

CHAPTER II

REPHRASING OF CURRENT RESULTS IN THE LITERATURE, TERMINOLOGY, AND PRELIMINARY RESULTS

Since we will be concerned with structural properties of matrices, we shall assume that for the remainder of this dissertation all of the matrices are (0,1)-matrices.

We shall use

Theorem D. <u>Assume</u> A <u>is an</u> $n \times n$ $(n > 1)$ <u>matrix which has the form</u>

$$A = \begin{bmatrix} a_1^1 & 0 & \dots & 0 & E_1^1 \\ e_2^1 & a_2^1 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & a_p^1 & 0 \\ 0 & 0 & \dots & E_{p+1}^1 & A_1 \end{bmatrix}$$

<u>where</u> $a_i^1 = 0$ <u>for</u> $1 \leqq i \leqq p$, $e_i^1 > 0$ <u>for</u> $2 \leqq i \leqq p$, E_1^1 <u>and</u> E_{p+1}^1 <u>are</u> $1 \times (n-p)$ <u>and</u> $(n-p) \times 1$ <u>matrices, respectively, each of which contains a single positive entry, and</u> A_1 <u>is an</u> $(n-p) \times (n-p)$ <u>irreducible matrix. Then</u> A <u>is irreducible</u> [5].

Observe that if A is an $n \times n$ matrix with $n > 1$ which has the form of Theorem D and a positive main diagonal, then it follows by Theorem B that A is fully indecomposable. We shall make use of this observation in the proof of Theorem 2.

We shall need the following two theorems and notation.

Theorem E. <u>If</u> A <u>is an</u> $n \times n$ <u>nearly reducible matrix with</u> $n \geqq 2$, <u>then there is a permutation matrix</u> P <u>such that</u>

$$PAP^T = \begin{bmatrix} a_1^1 & 0 & \dots & 0 & E_1^1 \\ e_2^1 & a_2^1 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & a_{p_1}^1 & 0 \\ 0 & 0 & \dots & E_{p_1+1}^1 & A_1 \end{bmatrix}$$

<u>where</u> $a_i^1 = 0$ <u>for</u> $1 \leqq i \leqq p_1$; $e_i^1 > 0$ <u>for</u> $2 \leqq i \leqq p_1$; E_1^1 <u>and</u> $E_{p_1+1}^1$ <u>are</u> $1 \times (n-p_1)$ <u>and</u> $(n-p_1) \times 1$ <u>matrices, respectively, each of which contains a single positive entry; and</u> A_1 <u>is an</u> $(n-p_1) \times (n-p_1)$ <u>nearly reducible matrix</u> [5].

If A_1 is 1×1, we shall call A the <u>trivial nearly reducible</u> matrix and say that A has 0 <u>decompositions</u>. If A_1 is not 1×1, then we can place A_1 in the form of Theorem E without destroying the form of PAP^T. Assume A_1 has the form of PAP^T and is indexed as PAP^T is with 2 replacing the superscript 1, the subscript 1 on p, and the subscript 1 on A_1, etc., for $A_2, \dots, A_k$ where A_k is a trivial matrix. If we assume that each of these matrices, $A_1, \dots, A_k$, has been placed in the form of Theorem E, then we shall say that A is in <u>canonical form with</u> k <u>decompositions</u>.

Because of

Theorem F. *If* A *is an* $n \times n$ *nearly decomposable matrix* $(n \geqq 2)$, *then there are permutation matrices* P *and* Q *such that* $PAQ = A' + I_n$ *where* A' *is a nearly reducible matrix which has the form of Theorem* E *and* A_1 *in* PAQ *is nearly decomposable* [5].,

we shall also use the terminology which follows Theorem E for nearly decomposable matrices.

Let us outline a proof of Theorem F. Assume $a_{pq} > 0$. By

Theorem G. *A matrix* A *is fully indecomposable if and only if it is chainable and has doubly stochastic pattern* [8].,

it follows that there is a positive diagonal d which contains a_{pq}. Let P^T be the $n \times n$ permutation matrix corresponding to d. Then PA has a positive main diagonal (which contains a_{pq}). Since $n \geqq 2$, by Theorems C and E, there is an $n \times n$ permutation matrix Q such that $Q(PA - I_n)Q^T$ has the form of Theorem E. Thus we need only to argue that A_1 is nearly decomposable in $Q(PA)Q^T$. By Theorem B, it is fully indecomposable. If there is some a_{ij} in A_1 such that the matrix obtained by replacing $a_{ij} > 0$ with zero is fully indecomposable, then by our remarks after Theorem D, $A - a_{ij}E_{ij}$ is fully indecomposable which contradicts our assumption that A is nearly decomposable. Hence A_1 is nearly decomposable.

We shall let $\sigma(A)$ denote the number of positive entries in the matrix A. If A is an $n \times n$ irreducible matrix with $n > 1$, then each row and column must contain at least one positive entry. Hence $\sigma(A) \geqq n$. If $\sigma(A) = n$, then A is nearly reducible. Thus there is an $n \times n$ permutation matrix Q such that QAQ^T has the form of Theorem E. Since there is only one positive entry in each row and column, A_1 must be 1×1. Hence if $\sigma(A) = n$, QAQ^T is the trivial matrix. If A is an $n \times n$ fully indecomposable matrix with $n \geqq 2$, then every row and column has at least two positive entries. Hence $2n \leqq \sigma(A)$. If $\sigma(A) = 2n$, then A is nearly decomposable. Thus by a similar argument to the one above, if $\sigma(A) = 2n$, then PAQ is the trivial nearly decomposable matrix.

In examining Theorem G, this author wondered how the two properties behaved in $A - a_{ij}E_{ij} = B$ for $a_{ij} > 0$. By using the Frobenius-Konig Theorem that every positive entry of an $n \times n$ matrix lies on a positive diagonal if and only if there does not exist an $s \times t$ zero submatrix such that $s + t = n + 1$ and the Birkhoff Theorem that a matrix has doubly stochastic pattern if and only if every positive entry lies on a positive diagonal (for proofs see [10, pp. 97-8]), he could trivially prove that if A is an $n \times n$ nearly decomposable matrix, then B does not have doubly stochastic pattern. However he discovered (Proposition 2) that chainability was invariant. First let us prove a preliminary result which is of interest in its own right.

Proposition 1. _If_ A _is an_ $n \times n$ _fully indecomposable matrix, then every two positive entries_ a_{ij} _and_ a_{pq} _are chainable by two disjoint chains._

Proof: The proposition is true for $n = 1$. Assume $n \geqq 2$. We shall prove the conjecture by induction on $\sigma(A)$. By our previous remarks, $\sigma(A) \geqq 2n$. If $\sigma(A) = 2n$, A is the trivial matrix. In this case, the choice is obvious. Assume the conjecture is true for all fully indecomposable $n \times n$ matrices B such that $2n \leqq \sigma(B) < \sigma(A)$. We need to consider two cases - (1) there is some $a_{ij} > 0$ such that $A - a_{ij}E_{ij}$ is fully indecomposable or (2) A is nearly decomposable. Let us consider (1). By the induction hypothesis, the proposition is true for $A - a_{ij}E_{ij}$. Thus we need only to argue that for $a_{pq} > 0$

where $(p,q) \neq (i,j)$, there are two disjoint chains which connect a_{ij} and a_{pq}. Since A is fully indecomposable, there are positive entries a_{is} and a_{rj}, $s \neq j$ and $r \neq i$. By the induction hypothesis, there are two disjoint chains a_1 and a_2 which connect a_{is} and a_{pq}. By Theorem G, there is a chain b which connects a_{rj} and a_{pq}. If a_1 and b or a_2 and b are disjoint chains, then we are finished. If neither a_1 nor a_2 are disjoint from b, then b must cross one of them first. Assume b crosses a_1 first. Let a_{tu} and a_{vw} be the first terms in a_1 and b, respectively, such that $t = v$ or $u = w$. The chain formed by taking the terms $a_{rj},\ldots,a_{vw}$ of b and the terms $a_{tu},\ldots,a_{pq}$ of a_1 is disjoint from a_2.

In case (2), we use induction on n. If A is 2×2, the proposition is true. Assume it is true for $1 \leqq m < n$. By Theorem F, we can assume that A is in canonical form. Since $\sigma(A) > 2n$, A cannot be the trivial matrix. Hence $n-p_1 > 1$. Thus A_1 satisfies the induction hypothesis. Hence if a_{ij} and a_{pq} are both in A_1, we are finished. Let $e^1_{p_1+1}$ and e^1_1 be the positive entries in $E^1_{p_1+1}$ and E^1_1, respectively. Assume $e^1_{p_1+1}$ lies in the r-th row and e^1_1 lies in the s-th column. Let a_{rt} and a_{us} be two positive entries in A_1. If a_{ij} and a_{pq} are both not in A_1, one takes a chain in A_1 which joins a_{rt} and a_{us} (which we know to exist by Theorem G) and the appropriate elements outside of A_1 (the choice is uniquely determined by the canonical form). Assume a_{ij} is not in A_1 and a_{pq} is. Then by a similar argument to the one in (1), one can find two disjoint chains - one which connects a_{pq} and a_{rt} and one which connects a_{pq} and a_{us}. Then by choosing the obvious elements not in A_1, one obtains the desired disjoint chains.

The reader should observe that one can use Proposition 1 and the method of proof in case (1) with the a_1, a_2, and b to show that for any three, different positive entries c_1, c_2, and c_3 of a fully indecomposable matrix, one can find a chain which connects c_1 and c_3 and contains c_2.

One might conjecture from the above paragraph that there are two disjoint chains which connect c_1 and c_3, one of which contains c_2. Unfortunately such a conjecture is false. Consider a_{11}, a_{24}, and a_{32} in the following fully indecomposable matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} .$$

Proposition 2. Let A be an $n \times n$ fully indecomposable matrix. Then $A - a_{ij}E_{ij}$ is chainable for $a_{ij} > 0$.

Proof: Since every two positive entries of A are the endpoints of two disjoint chains, the removal of any other positive entry of A cannot destroy their chainability.

The author now proves a proposition which he could not find in the literature. He will not use the proposition in the remainder of the dissertation. It arose from an analogous conjecture for nearly decomposable matrices which he will mention at the end of the dissertation.

Proposition 3. If B is an $m \times m$ $(m > 1)$ irreducible, principal submatrix of an $n \times n$ nearly reducible matrix A, then B is nearly reducible.

Proof: Assume that there is some positive a_{pq} such that the matrix B' obtained from B by replacing a_{pq} by 0 is irreducible. Let $A' = A - a_{pq}E_{pq}$.

Let a_1 and c be two positive entries in A'. If they are in B', then since B' is strongly connected, there is a path in A' which connects a_1 to c. Assume that a_1 and c are not in B'. Since A is strongly connected, there is a path $a_1,\ldots,a_k = c$ in A. If no entry of the path is in B, a_1 and c can be connected in A'. Assume that some term of the path is in B. Let a_r and a_s be the first and last members of the path in B, respectively, Assume that a_r lies in the i-th row and a_s lies in the j-th column. Since B' is strongly connected, there is a path $b_1',\ldots,b_t'$ in B' which connects the i-th row to the j-th column. Thus $a_1,\ldots,a_{r-1},b_1',\ldots,b_t',a_{s+1},\ldots,a_k = c$ is a path in A' which connects a_1 to c. By a similar argument, if a_1 or c is in B' and the other is not, then there is a path in A' which connects them. Thus if B is not nearly reducible, neither is A.

The author now proves two propositions which help to delineate the structure of nearly reducible and nearly decomposable matrices.

Proposition 4. Let A be an $n \times n$ nearly reducible matrix in canonical form with k decompositions where $k > 0$. The row containing $e^i_{p_i+1}$ and the column containing e^i_1 for $1 \leqq i \leqq k$ do not intersect in any E^j_t block.

Proof: Assume that the p-th row which contains $e^i_{p_i+1}$ and the q-th column which contains e^i_1 intersect in E^j_t. There are three cases which can exist. (1) Assume $1 < t \leqq p_j$. Since A is nearly reducible, there is a proper, nonempty subset α of $\{1,\ldots,n\}$ such that

$$(A - a_{pq}E_{pq})[\alpha|\alpha) = (A - e^j_tE_{pq})[\alpha|\alpha) = 0.$$

Since A is irreducible, $p \in \alpha$ and $q \notin \alpha$. However $e^i_{p_i+1},\ldots,e^i_1$ is a path which connects the p-th row and q-th column. Thus $q \in \alpha$ (by remarks in Section 2 of Chapter I) which is a contradiction. (2) Assume $t = 1$. Assume the r-th column contains e_1^j. Then, as above, there is an α such that

$$(A - a_{pr}E_{pr})[\alpha|\alpha) = (A - e_1^j E_{pr})[\alpha|\alpha) = 0$$

and $p \in \alpha$ and $r \notin \alpha$. Recall that A_j is in canonical form. Thus since A_{j+1} is strongly connected and the r-th column of A passes through A_{j+1}, every column index of A which passes through A_{j+1} is not in α. Hence q is not in α. However $e^i_{p_i+1},\ldots,e^i_1$ is a path which connects the p-th row to the q-th column. Hence $p \in \alpha$ implies $q \in \alpha$ which is a contradiction. (3) Assume $t = p_j+1$. As in (2), one uses the fact that A_{j+1} is strongly connected to show that $p \in \alpha$ implies $q \in \alpha$.

Proposition 5. Let A be an $n \times n$ nearly decomposable matrix in canonical form with k decompositions where $k > 0$. The row which contains $e^i_{p_i+1}$ and the column which contains e^i_1 intersect neither in any E^j_t block nor on the main diagonal of A for $1 \leqq i \leqq k$.

Proof: By Theorem C and Proposition 4, we need only to argue that if $e^i_{p_i+1}$ lies in the p-th row and e^i_1 lies in the q-th column, then $p \neq q$.

Since $k > 0$, A_1 cannot be 1×1. Let us show the above is true when A_1 is the trivial matrix. Let

$$A_1 = \begin{bmatrix} b_1 & 0 & .. & 0 & c_1 \\ c_2 & b_2 & .. & 0 & 0 \\ . & . & .. & . & . \\ 0 & 0 & .. & c_m & b_m \end{bmatrix} .$$

Assume $p = q$ where $i = 1$. We shall argue that $A' = A - a_{pp}E_{pp}$ is fully indecomposable. By Theorem G and Proposition 2, it is enough to argue that A' has doubly stochastic pattern. Observe that $a_1^1,\ldots,a_{p_1}^1, c_1,\ldots,c_m$ is a positive diagonal in A' and $e_1^1,\ldots,e_{p_1+1}^1$ plus the b_i except for a_{pp} is a positive diagonal in A'. Hence A' is fully indecomposable which is a contradiction. Thus $p \neq q$.

We now use induction on n. By the above paragraph, the dimension of A_1 is greater than or equal to three. Thus $n \geqq 4$. If $n = 4$, then $n-p_1 = 3$. Thus A_1 is the trivial matrix and by the above paragraph the proposition is true for $n = 4$. Assume the proposition is true for $4 \leqq m < n$. By the above paragraph, we can assume that A_1 is not the trivial matrix. Thus $4 \leqq n-p_1$ and A_1 satisfies the induction hypothesis. Hence we need only to argue that $p \neq q$ for $i = 1$. Assume $p = q$. As in the second paragraph, we shall obtain a contradiction by showing that A' has doubly stochastic pattern. Let d_1 be a positive diagonal in $A_1(p|p)$(the submatrix of A_1 obtained by deleting the p-th row and p-th column) which we know to exist by Theorem G. Thus $e_1^1,\ldots,e_{p_1+1}^1$, d_1 is a positive diagonal in A'. Since $n-p_1 > 1$, there is a positive entry a_{pi}, $i \neq p$, in A_1 which lies on a positive diagonal d_2 in A_1. Hence $a_1^1,\ldots,a_{p_1}^1$, d_2 is a positive diagonal in A'. Let a_{ij}, $i \neq p$ or $j \neq p$, be a positive entry of A_1 which is contained in a positive diagonal which contains a_{pp}. Let d_3 be the portion of such a diagonal in $A_1(p|p)$. Thus $e_1^1,\ldots,e_{p_1+1}^1$, d_3 is a positive diagonal in A' which contains a_{ij}. Hence A' has doubly stochastic pattern.

CHAPTER III

MAIN RESULTS

Theorem 1. <u>Let</u> A <u>be an</u> $n \times n$ <u>nearly reducible matrix in canonical form with</u> k <u>decompositions and with</u> $n \geqq 2$. <u>Then</u> $\sigma(A) \leqq 2(n-1)$ <u>with equality if and only if</u>

$$A_{j-1} = \begin{bmatrix} a_1^j & E_1^j \\ E_2^j & A_j \end{bmatrix}$$

<u>where</u> $j = 1$ <u>for</u> $k = 0$ $(A_o = A)$ <u>or</u> $1 \leqq j \leqq k$ <u>for</u> $k > 0$ <u>and</u> A_k <u>is</u> 2×2.

Proof: We shall argue by induction on n. If $n = 2$,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus assume the proposition is true for $2 \leqq m < n$. If A is the trivial matrix, then $\sigma(A) = n$. However the $n \times n$ matrix

$$B = \left[\begin{array}{cccc|cc} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \hline 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

which is irreducible by Theorem D and is nearly reducible by inspection has $\sigma(B) = 2(n-1) > n = \sigma(A)$ since $n > 2$. If A is not the trivial matrix, then the dimension of A_1 is greater than or equal to two. Let C be the $n \times n$ nearly reducible matrix in canonical form with $(n-p_1)-1$ decompositions,

$$C = \begin{bmatrix} 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 1 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 \\ 0 & 1 & . & 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 \\ . & . & . & . & . & . & . & .. & . & . & . \\ 0 & 0 & . & 1 & 0 & 0 & 0 & .. & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\ . & . & . & . & . & . & . & .. & . & . & . \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 0 & 0 & . & 0 & 1 & 1 & 1 & .. & 1 & 0 & 1 \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \end{bmatrix},$$

(which is nearly reducible for the same reasons as B) where the matrix C_1 in the right-hand bottom corner has the same dimension $n-p_1$ as A_1. By the induction hypothesis, $\sigma(A_1) \leqq 2(n-p_1) - 2 = \sigma(C_1)$. Thus $\sigma(A) = p_1 + 1 + \sigma(A_1) \leqq p_1 + 1 + \sigma(C_1) = \sigma(C) = p_1 + 1 + [2(n-p_1) - 2] \leqq 2(n-1) = \sigma(B)$. If $p_1 \neq 1$, then $\sigma(A) \leqq \sigma(C) < \sigma(B)$. Hence if equality is achieved, the matrix A must have the form of the proposition.

Since $\sigma(A) = 2(n-1)$ when A has the form of Theorem 1, we are finished.

Theorem 2. Let A be an $n \times n$ nearly decomposable matrix in canonical form with k decompositions and with $n \geqq 3$. Then $\sigma(A) \leqq 3(n-1)$ with equality if and only if

$$A_{j-1} = \begin{bmatrix} a_1^j & E_1^j \\ E_2^j & A_j \end{bmatrix},$$

<u>the</u> $e_2^j(e_1^j)$ <u>lie in the same row</u> (<u>column</u>) <u>where</u> $j = 1$ <u>for</u> $k = 0$ $(A_o = A)$ <u>or</u> $1 \leqq j \leqq k$ <u>for</u> $k > 0$ <u>and</u> A_k <u>is</u> 3×3.

Proof: We shall prove the proposition by induction on n. Observe that if A has the form of the theorem, $\sigma(A) = 3(n-1)$. Since the only 3×3 nearly decomposable matrix in canonical form is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

the induction step holds. Assume Theorem 2 is true for $3 \leqq m < n$. If A is the trivial matrix, the nearly decomposable $n \times n$ matrix

$$B = \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ \hline 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{array}\right]$$

(which is fully indecomposable by our remarks after Theorem D and nearly decomposable since whenever $b_{ij} > 0$, the i-th row or the j-th column has exactly two positive entries) has the property that $2n = \sigma(A) < \sigma(B) = 3(n-1)$ since $n > 3$. If A is not the trivial matrix, then the dimension $n-p_1$ of A_1 is greater than one. If $n-p_1 = 2$, the column which contains e_1^1 and the row which contains $e_{p_1+1}^1$ intersect in a positive entry which is impossible by Proposition 5. Thus $n-p_1 \geqq 3$. Let C be the nearly decomposable $n \times n$ matrix in canonical form with $(n-p_1) - 2$ decompositions

$$C = \begin{bmatrix} 1 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\ 1 & 1 & . & 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 \\ 0 & 1 & . & 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 \\ . & . & . & . & . & . & . & .. & . & . & . \\ 0 & 0 & . & 1 & 1 & 0 & 0 & .. & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 1 & 0 & .. & 0 & 1 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 1 & .. & 0 & 1 & 0 \\ . & . & . & . & . & . & . & .. & . & . & . \\ 0 & 0 & . & 0 & 1 & 1 & 1 & .. & 1 & 0 & 1 \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 1 & 1 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 & .. & 0 & 1 & 1 \end{bmatrix},$$

(which is nearly decomposable for the same reasons as B is) where the submatrix C_1 of C is $(n-p_1) \times (n-p_1)$. Thus by the induction hypothesis, $\sigma(A_1) \leqq \sigma(C_1)$. Hence $\sigma(A) = 2p_1 + 1 + \sigma(A_1) \leqq 2p_1 + 1 + \sigma(C_1) = \sigma(C)$. However $\sigma(C) = 2p_1 + 1 + 3(n-p_1) - 3 \leqq 3(n-1) = \sigma(B)$. If $p_1 > 1$, then $\sigma(A) \leqq \sigma(C) < \sigma(B)$. Hence equality implies $p_1 = 1$ and A_1 has the form of the proposition. Suppose e_1^1 lies in the q_1-th column, $e_{p_1+1}^1$ lies in the q_2-th row, $e_1^2,\ldots,e_1^k$ lie in the r_1-th column, and $e_2^2,\ldots,e_2^k$ lie in the r_2-th row. If $q_1 = r_1$ and $q_2 \neq r_2$ or $q_1 \neq r_1$ and $q_2 = r_2$, then the entry in the (q_2,q_1) position is positive which contradicts Proposition 5. If $q_1 \neq r_1$ and $q_2 \neq r_2$, then (q_2,q_1) lies in an E_t^j block which contradicts Proposition 5. Hence $q_1 = r_1$ and $q_2 = r_2$.

It is interesting to observe that while the form of Theorem 2 is unique up to independent permutations on the rows and columns, the form of Theorem 1 is not. For instance, each of the matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

has the form of Theorem 1.

Lemma. *If* A *is an* $n \times n$ $(n \geqq 2)$ *nearly decomposable matrix in canonical form such that* $\text{per}A = \sigma(A) - 2n + 2$, *then* $e_1^1, \ldots, e_{p_1+1}^1$ *lie on exactly one positive diagonal*.

Proof: Let e_1^1 lie in the q-th column and $e_{p_1+1}^1$ lie in the p-th row. By Theorem G, $\text{per}A_1 + 1 \leqq \text{per}A_1 + \text{per}A(p|q) = \text{per}A$. Since A_1 is fully indecomposable, it follows by Theorem A that $[\sigma(A_1) - 2(n-p_1) + 2] + 1 \leqq \text{per}A_1 + 1$. Thus by the canonical form, $\sigma(A) - 2n + 2 = (2p_1 + 1) + \sigma(A_1) - 2n + 2 = [\sigma(A_1) - 2(n-p_1) + 2] + 1 \leqq \text{per}A_1 + 1 \leqq \text{per}A_1 + \text{per}A(p|q) \leqq \sigma(A) - 2n + 2$ which implies that $\text{per}A(p|q) = 1$.

Theorem 3. *Let* A *be an* $n \times n$ *nearly decomposable matrix in canonical form with* k *decompositions and with* $n \geqq 2$. *A necessary and sufficient condition for* $\text{per}A = \sigma(A) - 2n + 2$ *is that* $a_{ij} = 1$ *implies that the* i-*th row or* j-*th column has exactly two positive entries in it and* $e_1^i, \ldots, e_{p_i+1}^i$ *lie on exactly one positive diagonal in* A_{i-1} *where* $A_0 = A$ *for* $1 \leqq i \leqq k + 1$.

Proof: We shall prove the condition is sufficient by induction on n. Since the only 2×2 fully indecomposable matrix is a positive matrix, the assertion is true for $n = 2$. Assume the condition is sufficient for $2 \leqq m < n$. Since the assertion is true for the trivial matrix, assume A is not the trivial matrix. Thus $(n-p_1) \geqq 2$. Hence A_1 satisfies the induction hypothesis. Thus $\text{per}A_1 = \sigma(A_1) - 2(n-p_1) + 2$. However, by the canonical form, $\sigma(A) = 2p_1 + 1 + \sigma(A_1)$. Thus by the second part of the condition and the last two sentences, $\text{per}A = \text{per}A_1 + 1 = [\sigma(A_1) + 2p_1 + 1] - 2n + 2 = \sigma(A) - 2n + 2$.

Let us now prove the necessity of the condition. By using the Lemma on $A_0, \ldots, A_k$, we see that the second part of the condition is true. Thus we need only to argue the first condition holds, which we shall do by induction on n. If $n = 2$, A is the positive matrix and the assertion is true. Assume the condition is necessary for $2 \leqq m < n$. If A is the trivial matrix, the assertion is true. Thus assume $n-p_1 \geqq 2$. Thus A_1 satisfies the induction hypothesis. Since A is in canonical form, we need only to examine the i-th row which contains $e^1_{p_1+1}$ and the q-th column which contains e^1_1. Assume that $a_{ij} = 1$ and the j-th column contains at least three positive entries. Observe that since A_1 is fully indecomposable and $n-p_1 \geqq 2$, the i-th row of A must contain at least three positive entries. If a_{ij} is on the main diagonal, we can choose $a_{it} = 1$ where $t \neq j$. Then proceeding as in the proof of Theorem F, we can place A in canonical form with a_{ij} not on the main diagonal. Thus we shall assume that a_{ij} is not on the main diagonal of A. Since A is in canonical form, the fact that the j-th column contains at least three positive entries implies that there must be some e^r_1 for $2 \leqq r < k$ which lies in the j-th column. Since $A - I_n$ is nearly reducible, there must be a nonempty, proper subset α of $\{1,\ldots,n\}$ such that $(A - a_{ij}E_{ij})[\alpha|\alpha) = 0$. By our remarks in Section II of Chapter I, we can assume α is the set of all integers h such that either there is a path which connects the i-th row to the h-th column or there is no path which connects the h-th row to the j-th column. Since A and A_r have doubly stochastic pattern by Theorem G, there is a positive cycle g_1 in A which contains $e^1_1,\ldots,e^1_{p_1+1}$ and a positive cycle g_2 in A_r which contains $e^r_1,\ldots,e^r_{p_r+1}$ (where cycle is to be taken in the permutation sense). Thus every member of $g_1(g_2)$ can be connected by a path to $e^1_{p_1+1}$ (e^r_1). Hence if a_{tu} is a member of g_1, then $u \in \alpha$. Likewise, since g_1 is a cycle, there is some a_{vt} in g_1. Thus $t \in \alpha$ if a_{tu} is a term of g_1. Similarly, if a_{tu} is a member of g_2, then t and u are not in α. Let d_1 be the positive diagonal of A composed of g_1 and those entries on the main diagonal whose indices do not occur as row indices in g_1. Let d_2 be the positive diagonal of A composed of g_1, g_2, and the entries on the main diagonal whose indices do not occur as row indices in g_1 or g_2. Thus $e_1^1, \ldots, e_{p_1+1}^1$ lie on two positive diagonals which contradicts the Lemma. By a similar argumemt, we can prove the theorem for the q-th column.

It is interesting to observe that a nearly docomposable matrix which has the form of Theorem 2 satisfies the two conditions of Theorem 3. Hence the matrix which has the maximal number of positive entries has minimal permanent.

CHAPTER IV

CONJECTURES AND CONNECTIONS BETWEEN THIS RESEARCH AND OTHER INVESTIGATIONS

Proposition 3 was motivated by

<u>If</u> B <u>is an</u> $m \times m$ <u>fully indecomposable</u>, <u>principal submatrix of an</u> $n \times n$ <u>nearly decomposable matrix</u> A, <u>then</u> B <u>is nearly decomposable</u>.

which the author could not prove.

Indeed, E. J. Roberts has found the following counterexample:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Observe that A is chainable and the permutations (14376,10), (9,10), (9876,10), (1432), and (37654)(9,10) correspond to positive diagonals (where by (...6,10), we mean 6 goes into 10, etc.). Hence A is fully indecomposable by Theorem G. With the exception of a_{37}, if $a_{ij} = 1$, then either the i-th row or j-th column has exactly two positive entries. In the case of a_{37}, the 5 x 5 submatrix of $A - a_{37}E_{37}$ composed of rows 1 through 5 and columns 1 and 6 through 9 is zero. Thus A is nearly decomposable. Observe that the principal submatrix $A(1|1)$ is fully indecomposable by Theorem D since it is the trivial matrix with one additional entry. Thus the additional entry is removable.

The author believes

Conjecture. *Assume* A *is an* $n \times n$ *nearly decomposable* (0,1)*-matrix. Then* A *has rank* n-1 *if and only if* A *is the trivial matrix and* n *is even. Otherwise* A *has rank* n.

is true but has been unable to prove it.

In light of Theorem C, the author would like to know what conditions must be placed on an $n \times n$ nearly reducible matrix B in order that $A = B + I_n$ be nearly decomposable. The reader might conjecture that if B is an $n \times n$ nearly reducible matrix in canonical form such that $A = B + I_n$ satisfies the conclusion of Proposition 5, then A is nearly decomposable. However such a conjecture is false. Consider the following 5×5 nearly reducible (0,1)-matrix in canonical form with 2 decompositions:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is irreducible by Theorem D and nearly reducible by inspection. The matrix $A = B + I_n$ is fully indecomposable by Theorem B. However $A - a_{55}E_{55}$ is chainable and has doubly stochastic pattern. Thus by Theorem G, A is not nearly decomposable

As mentioned in Chapter I, nearly decomposable and nearly reducible matrices have recently come under scrutiny.

E. J. Roberts of NASA's Manned Spacecraft Center has independently discovered Proposition 2 [13, p. 35], Theorem 2 [13, pp. 67-68], and Theorem A [13, p. 73]. He called the author's attention to the fact that if a matrix has a positive diagonal or equivalently, if the associated bipartite graph G has no isolated vertices, then the matrix is chainable if and only if the graph G is connected. (A graph is bipartite if its vertex set can be written as the union of two disjoint sets S and T such that each edge of G has one endpoint in S and the other in T. See [4].) When the author told E. J. Roberts about Proposition 1, Doctor Roberts observed that one could also prove it by using his result that the associated bipartite graph of a fully indecomposable matrix is 2-connected [13, pp. 28-30] and the result in [1, p. 201] that if a graph contains neither loops nor isolated vertices, then it is 2-connected if and only if every two edges lie on an elementary cycle (in the graph sense of cycle). One should note that by using the above result in [1, p. 201] and Proposition 1, he has another proof that the associated bipartite graph of a fully indecomposable matrix is 2-connected.

It is interesting to note that by using Theorems B and G, one can quickly prove that if v is an arc from i to j where $i \neq j$ in a strongly connected graph H, then v lies on an elementary circuit in H. Let B be the irreducible matrix corresponding to H. By Theorem B, $A = B + I_n$ is fully indecomposable. By Theorem G, a_{ij} lies on a positive diagonal in A. Since $i \neq j$, a_{ij} lies on a positive cycle (in the permutation sense) in B. Hence v lies on an elementary circuit in H.

D. Hartfiel and J. Crosby have independently proven Theorem A [3]. All of the authors of [3], [11], and [13] were motivated to consider and in their methods of proof of Theorem A by Professor Sinkhorn's work (mentioned in the introduction) in [14].

After the author had finished his research, he received word that Minc has independently discovered Theorem 2 [12].

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NOTATION

SYMBOL

$\sigma(A)$	p. 1, line 20; p. 8, line 17
$A_1(p\|p)$	p. 14, line 15