# MANIPULATOR GUIDANCE STRATEGIES

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A Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

---

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Shyam J. Nadkarni

May 1977

TO MY PARENTS

Mr. Jagannath V. Nadkarni

Mrs. Tara J. Nadkarni

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## ABSTRACT

This thesis presents several different numerical methods for guidance of a mechanical manipulator by simultaneous control of all its joint variables. The emphasis is on avoiding unwanted displacements of the manipulator hand while transfering it from one position to another. Algorithms are developed for computing changes in all the joint variables for a desired small displacement of the manipulator hand. Basically two different types of computer controls are discussed. One is that of guiding the manipulator when the initial and the final positions of the hand are specified. The other is that of controlling the manipulator by means of a computer which receives commands from an external operator.

The linear and angular velocities of the manipulator hand are equated to the sums of the linear and angular velocities produced at the manipulator hand by motion about individual joints of the manipulator. For a very small interval of time the same equations generate small changes in the joint variables to give specific small displacements of the manipulator hand. Six linear equations in the six unknown joint variable increments are obtained, for a known small change in position of the manipulator hand. If we know the path by which the manipulator hand should move,

intermediate positions of the manipulator hand can be assumed at small intervals and the changes in joint variables necessary for the displacement from one position to the next position can be computed.

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# I. INTRODUCTION

A number of methods, well suited for the analysis of manipulator displacements are known. The basic idea in uniform motion control is that of computing simultaneous changes in all joint variables so as to guide the manipulator hand in a well defined fashion. Usually it is desired that the manipulator hand should follow the shortest path with a uniform change in its orientation.

A need for a good algorithm for the control of manipulators is realized in some research fields. One of them is the manipulator interference problem. Pieper [6], Loeff and Soni [3] have presented algorithms for obstacle avoidance for particular manipulator geometries while Waldron [11] has presented a general theory for the same problem. For two important reasons, the interference problem requires that the manipulator should follow a well defined path, preferably the shortest. One reason is, if the path of the manipulator movement is not known, one cannot predict in advance the possibility of interference. The second reason is that the unwanted movement of the manipulator can increase the possibility of interference.

Uicker, Denavit and Hartenberg [8] developed an iterative method for the displacement analysis of a closed spatial mechanism with the help of 4x4 matrix notation developed by Denavit and Hartenberg [5]. Seth and Uicker [7] modified the notation in the reference [5]. The computer program IMP (Integrated Mechanism Program) developed by Uicker [9] uses the references [7,8] for its displacement analysis. A mechanical manipulator being an open mechanism, the closed loop equations alone cannot be used for its analysis.

Pieper [6] has discussed position analysis for a few manipulator geometries. Also he has modified the Newton-Raphson algorithm of the reference [8] for application to manipulator problems. He has also developed another method called the "Iterative Velocity Method." He claims that the "Iterative Velocity Method" is faster and more accurate than the Newton-Raphson method. An iterative method can be used to compute the final configuration of the mechanical manipulator but it does not give the intermediate configurations needed to guide the manipulator hand uniformly to that position.

Resolved motion rate control of manipulators was discussed by Whitney [12,13,14] and Gavrilovic and Maric [5]. The objective of coordinated control is to command rates of movement of the manipulator hand along coordinate axes which are convenient and visible to the operator. Moe [4] has discussed the rate control of the Rancho arm.

Chapter II of this thesis contains a review of notation for the description of a manipulator. We have also written transformations for describing positions of the manipulator in a global coordinate frame. In Chapter III, an algorithm for locating the screw axis of a finite displacement is described. In Chapter IV, a general algorithm for displacement analysis which is referred to as Analysis Type I is developed. For the section referred to as Analysis Type II, we have shown that Pieper's [5] Iterative Velocity Method is a particular case of the general displacement analysis. Next in the section referred to as Analysis Type III, we describe a method of obtaining the movement of the manipulator hand by the shortest path. In the section referred to as Analysis Type IV, we have described how we can modify the Analysis Type II to obtain coordinated control of manipulators. A computer program DISP is developed based on most of the analytical work done in this thesis, the use and the test results of which are given in Appendices I and II.

# II. MATHEMATICAL PRELIMINARIES

Notation:

In this chapter, we will be reviewing a notation for describing the configuration of a spatial mechanism. We will be using a modified form of the notation for mechanisms developed by Denavit and Hartenburg [1].

Figure 2-1 shows a closed spatial mechanism while Figure 2-2 shows a mechanical manipulator. The different parameters shown in the figures are as follows:

$a_i$ The length of the normal between the ith and (i+1)th axis. It is always positive.

$\theta_i$ The angle between the (i-1)th and the ith normal. This angle is determined as follows:

Rotate the (i-1)th normal by an angle $\theta_i$ about the ith axis to make it parallel to the ith normal and on the same side of the ith axis. We use the sign convention that $-180^\circ < \theta_i \leq 180^\circ$ and, if the rotation is clockwise in the positive direction of the axis, $\theta_i$ is positive.

$r_i$ The distance between the intersection of the ith axis with the (i-1)th normal and the intersection of the ith axis with the ith normal. If a' is the intersection of the ith axis with the (i-1)th normal and b' is the intersection of the ith axis with the ith normal then $r_i$ is positive if the direction of the ith axis is same as the direction of the vector $\underline{a'b'}$.

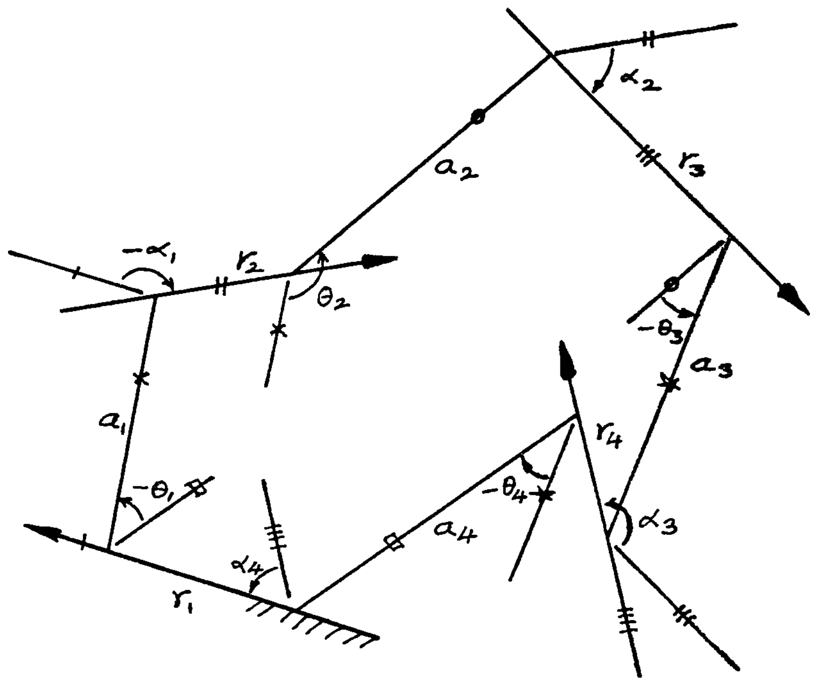


FIGURE 2.1

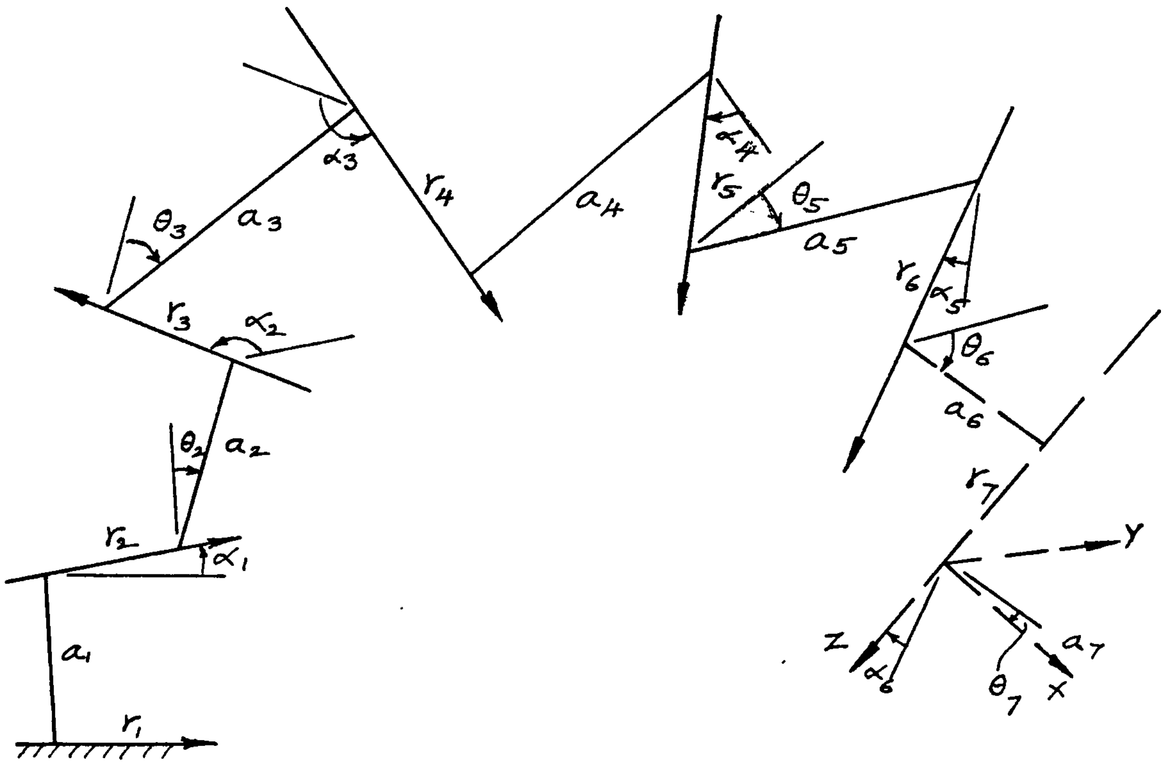


FIGURE 2.2

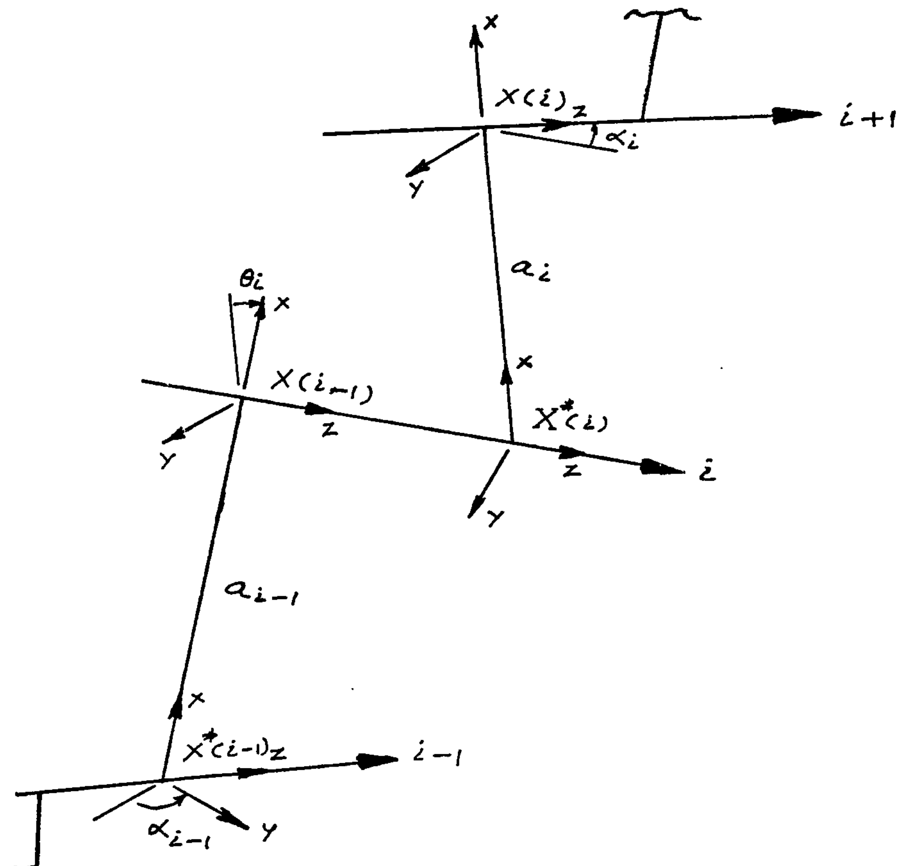


FIGURE 2-3

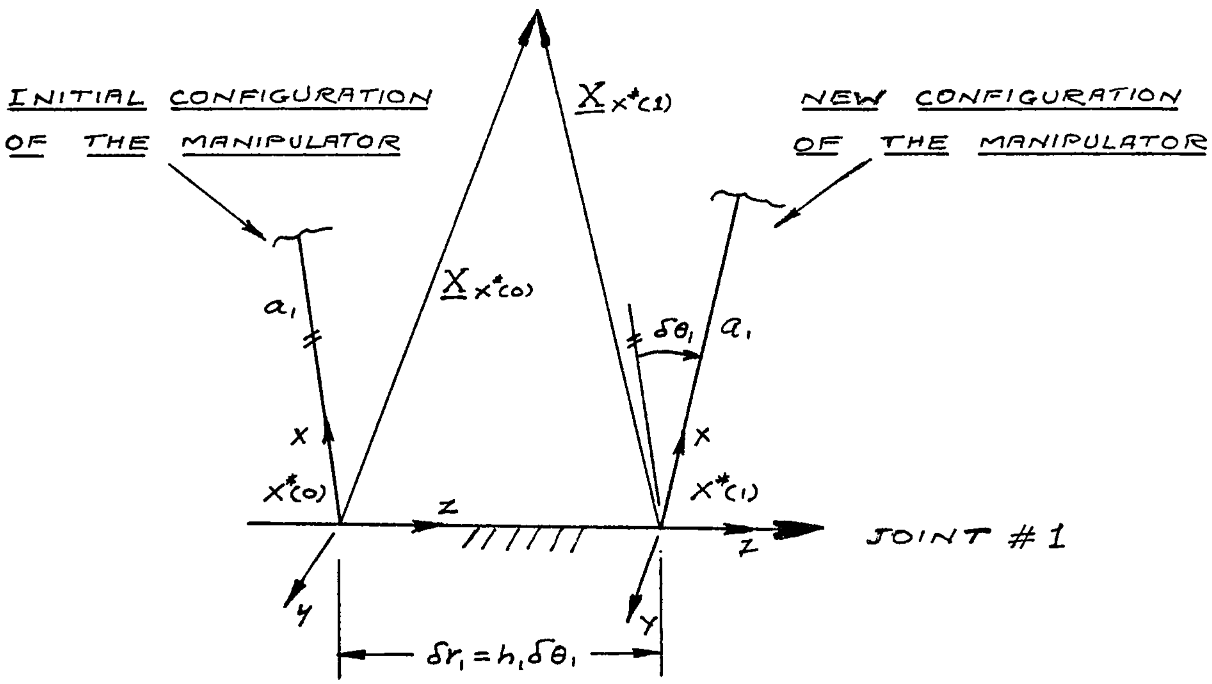


FIGURE 2.4

$\alpha_i$ The angle between the ith axis and the (i+1)th axis. This angle is decided as follows:

i) Rotate the ith axis by an angle $\alpha_i$ about the normal $a_i$ such that the direction of the ith axis becomes same as that of the (i+1)th axis. The direction of rotation should be such that $-180° < \alpha_i \leq 180°$.

ii) The angle $\alpha_i$ is positive if the rotation is clockwise looking from the intersection of the normal with the ith axis towards the intersection of the normal with the (i+1)th axis.

NOTE: The directions of all the axes are decided arbitrarily. It is convenient to take positive directions of the axes such that all $r_i$'s become positive.

In the representation of the mechanism described above, the common normals represent the members (sometimes called links) connected by the joints. On each link we will fix two coordinate frames. The coordinate frame x(i) will be fixed, with its origin at the intersection of the ith normal with the (i+1)th joint axis, with its z-axis along the positive direction of the (i+1)th joint axis and its x-axis along the normal extended beyond the (i+1)th joint axis. The coordinate frame x*(i) will be fixed, with its origin at the intersection of the ith normal with the ith joint axis, with its z-axis along the positive direction of the ith joint axis and the x-axis along the ith normal. In both the coordinate frames, the y-axis will be such as to give a right handed cartesian coordinate frame. For further details, please see Figure 2-3.

Any general vector referred to a particular frame will be represented by a subscript giving the frame name [e.g., $\underline{X}_{x(i)}$, would represent a vector in the coordinate frame x(i)]. All matrices will be denoted by a capital letter and may have subscripts [e.g., $U_i$]. All scalars will be in lower case letters, with or without subscripts [e.g., $a_i$, $s_i$]. The above notation may not be followed strictly, where mechanisms are not involved. Notation will be clarified whenever necessary.

We can write the transformation of the position vector of a point from coordinate frame x(i) to coordinate frame x*(i) as

$$\underline{X}_{x^*(i)} = V_i \underline{X}_{x(i)} + [a_i \;\; 0 \;\; 0]^T \tag{2-1}$$

where

$$V_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix} \tag{2-2}$$

[<u>Note</u>: In the above matrix $c\alpha_i$ and $s\alpha_i$ stand for cos $\alpha_i$ and sin $\alpha_i$. The same notation will be used most of the time.]

A similar transformation from the coordinate frame x*(i) to coordinate the frame x(i-1) can be written as

$$\underline{X}_{x(i-1)} = U_i \; \underline{X}_{x^*(i)} + [0 \;\; 0 \;\; r_i]^T \tag{2-3}$$

where

$$U_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-4}$$

Combining (2-2) and (2-4), we can write the transformation of the vector from the coordinate frame x(i) to the coordinate frame x(i-1) as

$$\underline{X}_{x(i-1)} = U_i \; [V_i \; \underline{X}_{x(i)} + \underline{S}_i] \tag{2-5}$$

where

$$\underline{S}_i = [a_i \quad 0 \quad r_i]^T \tag{2-6}$$

In the Denavit and Hartenberg form of representation, we can write (2-5) as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}_{x(i-1)} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\theta_i & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}_{x(i)} \tag{2-7}$$

In the above 4x4 matrix, the upper left 3x3 submatrix is the matrix product $U_iV_i$ while in the last column, the upper 3x1 vector is the product $U_iS_i$. We will use the transformation of the form given in equation (2-5).

We can also write a transformation of position vectors from the x*(i) coordinate frame to the x*(i-1) coordinate frame as

$$\underline{X}_{x^*(i-1)} = V_{i-1}[U_i\underline{X}_{x^*(i)} + \underline{T}_i] \qquad (2\text{-}8)$$

where

$$\underline{T}_i = [a_i \quad 0 \quad r_i]^T \qquad (2\text{-}9)$$

Description of a Mechanical Manipulator in the Global Cartesian Coordinate Frame.

It is sometimes necessary to describe the position of the mechanism in a global cartesian coordinate frame. What we need, in fact, are all intersections of normals with the joint axes, directions of all the joint axes, etc. We can find these details by transforming the position vectors of all intersection points, and also the unit vectors along the joint axes, to the coordinate frame x*(1) and then to the global coordinate frame.

The vector $\underline{X}_{x(i-1)}$ obtained by transformation (2-5) can be further transformed to the coordinate frame x(i-2) by resubstituting in equation (2-5) with the subscript i replaced by (i-1). In this manner, by successive transformations, any vector in the x(i) coordinate frame can be transformed to the x(1) coordinate frame. The transformation can be written as

$$\underline{X}_{x(1)} = U_2(V_2U_3(V_3U_4(---(V_{i-1}U_i(V_i\underline{X}_{x(i)}$$

$$+ \underline{S}_i) + \underline{S}_{i-1}) + ---) + \underline{S}_2 + \underline{S}_1) \qquad (2\text{-}10)$$

The same vector can be further transformed to the x*(1) coordinate frame as

$$\underline{X}_{x^*(1)} = V_1\, \underline{X}_{x(1)} + [a_1 \quad 0 \quad 0]^T \qquad (2\text{-}11)$$

The intersection of the ith normal with the (i+1)th joint axis lies at the origin of the x(i) coordinate frame. Substituting $\underline{X}_{x(i)} = [0 \quad 0 \quad 0]^T$ in the equation (2-10) and the transformed vector to the equation (2-11) gives us the position of the intersection point in the x*(1) coordinate frame.

Transformation of the position vector in the coordinate frame x*(i) to the coordinate frame x*(1) can be obtained using the transformation (2-8) successively:

$$\underline{X}_{x^*(1)} = V_1(U_2V_2(U_3V_3(---(U_{i-1}V_{i-1}(U_i\underline{X}_{x^*(i)}$$

$$+ \underline{T}_i) + \underline{T}_{i-1}) + \underline{T}_{i-2}) + ---) + \underline{T}_2) \qquad (2\text{-}12)$$

As the intersection of the ith axis with the ith normal lies at the origin of the x*(i) coordinate frame, substituting $\underline{X}_{x^*(i)} = 0$ in the equation (2-12) will give us the position vector of the intersection point in the x*(1) coordinate frame.

Once we get the position vectors of all the intersection points in the x*(1) coordinate frame we have to transform them to the global coordinate frame. The coordinate frame x*(1) changes as per the configuration of the manipulator. The variable that decides the position of x*(1) is $\theta_1$, which is arbitrary. We will fix a coordinate frame on the joint No. 1 fixed in the global coordinate frame. This frame should have its z-axis along the positive direction of the joint axis. A suitable choice for this frame is the frame x*(1) in the initial position of the manipulator. Let us call this frame x*(0). Now if the joint variable changes from $\theta_1$ to $\theta_1+\delta\theta_1$, we can transform all the vectors from the x*(1) coordinate frame to the x*(0) coordinate frame as follows (see figure 2-4):

$$\underline{X}_{x*(0)} = U_0\underline{X}_{x*(1)} + [0 \quad 0 \quad \delta r_1] \tag{2-13}$$

where

$$U_0 = \begin{bmatrix} c\delta\theta_1 & -s\delta\theta_1 & 0 \\ s\delta\theta_1 & c\delta\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-14}$$

and

$$\begin{aligned} \delta r_1 &= h_1\delta\theta_1 && \text{for general screw joint} \\ &= 0 && \text{for revolute joint} \\ &= \delta r_1 && \text{for presmatic joint.} \end{aligned}$$

As the x*(0) coordinate frame is fixed in the global coordinate frame, we can finally transform all the position vectors of the points to the global coordinate frame by a simple transformation of the form

$$\underline{X}_g = Q'\underline{X}_{x^*(0)} + \underline{q}' \tag{2-15}$$

where Q′ is a 3x3 rotational transformation matrix and $\underline{q}'$ is the vector joining the origin of the global coordinate frame to the origin of the x*(0) coordinate frame.

Let $\underline{U}_{x^*(i)}$ be the unit vector along the ith joint axis in the x*(i) coordinate frame, then

$$\underline{U}_{x^*(i)} = [0 \quad 0 \quad 1]^T$$

$\underline{U}_{x^*(i)}$ can be transformed to the x*(1) coordinate frame by using only the rotational part of the transformation (2-12).

$$\underline{U}_{x^*(1)} = V_1U_2V_2U_3\text{-----}V_{i-1}U_i[\underline{U}_{x^*(i)}] \tag{2-16}$$

Similarly using only the rotational part of the transformations in (2-13) and (2-15), we can get the unit vectors in the global coordinate frame

$$\underline{U}_{x^*(0)} = U_0\underline{U}_{x^*(1)} \tag{2-17}$$

and

$$\underline{U}_j = Q'\underline{U}_{x^*(0)} \tag{2-18}$$

A computer subroutine TRANSM was developed based on the above transformations. The use of this subroutine is explained in Appendix I.

## III. THE SCREW AXIS OF A FINITE DISPLACEMENT

### Preliminaries

A body can be moved from any one position to another by a rotation about a fixed axis combined with a translation about the same axis. This is called a screw displacement. The screw axis can be uniquely found if we have the following restriction on it:

$$-180 < \theta \leq 180$$

Here, the term screw has very general and broad meaning. A pure rotational motion can be considered as a screw motion of zero pitch, while a pure translational motion can be considered as a screw motion of infinite pitch.

We will present a method to find the screw axis, given the locations of the body in its initial and final positions. Before we discuss the method it is proper to explain how we locate a body in space.

There are many different ways of specifying the position of a body in space. One of the ways is to give the positions of three non-collinear points fixed on the body in a global cartesian system. The body can also be located if a separate cartesian coordinate frame is fixed on the body and the origin of the frame and direction of two of its axes are given in the global cartesian coordinate system.

We will locate the position of the body with the help of position vectors to three points fixed in the body. Let these three position vectors be $\underline{X}_{01}$, $\underline{X}_{02}$ and $\underline{X}_{03}$, giving the positions of three points, 1, 2 and 3 respectively (please see figure 3-1). Now we will define a coordinate frame fixed on the body with the help of these three points. The point 1 we will consider as the origin of the frame and let the z-axis pass through the point 2. Hence the unit vector along z-axis is given by

$$\hat{z}_i = \frac{\underline{X}_{02} - \underline{X}_{01}}{|\underline{X}_{02} - \underline{X}_{01}|} \tag{3-1}$$

Now the unit vectors along the y-axis and the x-axis are defined as follows. Let us define a vector $\underline{b}$ as

$$\underline{b} = \frac{(\underline{X}_{02} - \underline{X}_{01}) \times (\underline{X}_{03} - \underline{X}_{01})}{|\underline{X}_{02} - \underline{X}_{01}| \quad |\underline{X}_{03} - \underline{X}_{01}|}$$

then

$$\hat{y}_i = \frac{\underline{b}}{|\underline{b}|} \tag{3-3}$$

There is something to note at this point. If we select any other point 2′ instead of the point 2, that lies on the $z_i$-axis on its positive side and a point 3′ instead of the

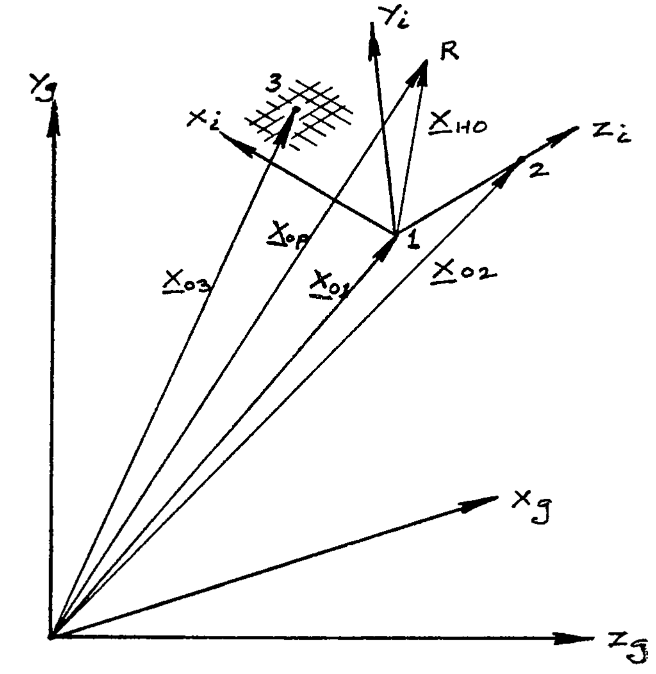


FIGURE 3.1

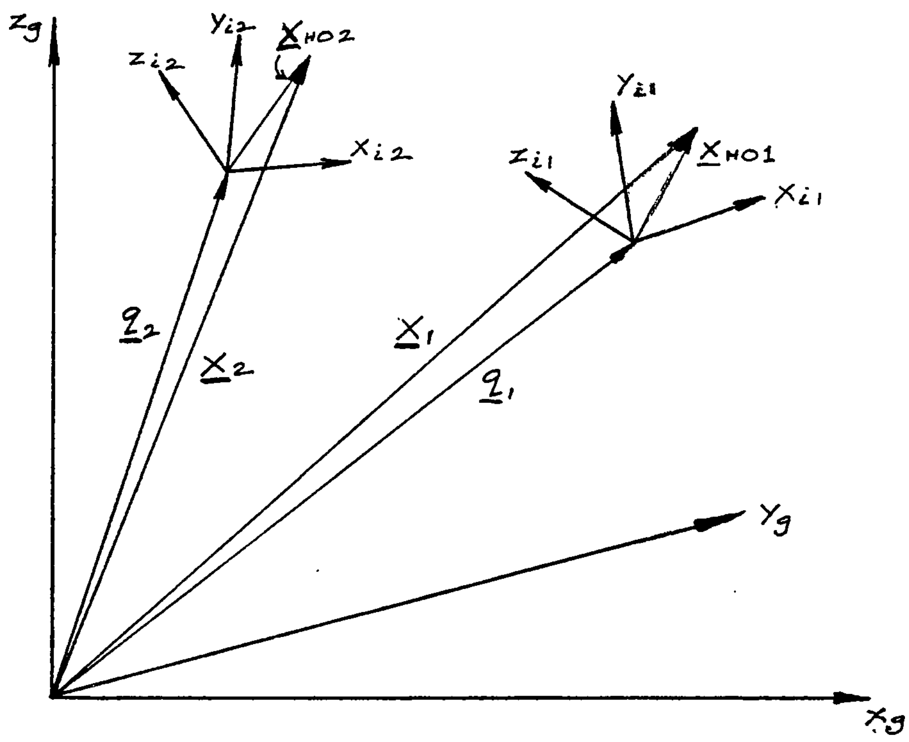


FIGURE 3.2

point 3 that lies in the plane formed by points 1, 2 and 3, and on the same side of the $z_i$-axis, the coordinate frame that will be defined by 1, 2′ and 3′ will be the same. In other words, the set of the points 1, 2 and 3 and the set of the points 1, 2′ and 3′ locate the body in the same position.

Let us denote the coordinate frame fixed in the body as the frame i and let us denote the global coordinate frame as the frame g (please see figure 3-1). Let $\underline{X}_H$ be the vector in the frame i, the tip of which gives the position of a point R. Let the vector $\underline{X}_{op}$ give the position of the same point in the frame g. Then

$$\underline{X}_{op} = \underline{X}_{HO} + \underline{X}_{01} \tag{3-4}$$

The vector $\underline{X}_{HO}$ is the same as the vector $\underline{X}_H$ transformed to the coordinate frame g. Therefore, we can write

$$\underline{X}_{op} = [Q_i]\underline{X}_H + \underline{X}_{01} \tag{3-5a}$$

where

$$[Q_i] = \begin{bmatrix} q_{11i} & q_{12i} & q_{13i} \\ q_{21i} & q_{22i} & q_{23i} \\ q_{31i} & q_{32i} & q_{33i} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x_i,x_g) & \cos(y_i,x_g) & \cos(z_i,x_g) \\ \cos(x_i,y_g) & \cos(y_i,y_g) & \cos(z_i,y_g) \\ \cos(x_i,z_g) & \cos(y_i,z_g) & \cos(z_i,z_g) \end{bmatrix} \qquad (3\text{-}5b)$$

Since $\hat{x}_i$, $\hat{y}_i$ and $\hat{z}_i$ are the unit vectors along the axes $x_i$, $y_i$ and $z_i$ (in figure 3-1) in the global coordinate frame g, we have

$$\hat{x}_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos(x_i,x_g) \\ \cos(x_i,y_g) \\ \cos(x_i,z_g) \end{bmatrix} = \begin{bmatrix} q_{11i} \\ q_{21i} \\ q_{31i} \end{bmatrix} \qquad (3\text{-}6a)$$

$$\hat{y}_i = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos(y_i,x_g) \\ \cos(y_i,y_g) \\ \cos(y_i,z_g) \end{bmatrix} = \begin{bmatrix} q_{12i} \\ q_{22i} \\ q_{32i} \end{bmatrix} \qquad (3\text{-}6b)$$

and

$$\hat{z}_i = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos(z_i,x_g) \\ \cos(z_i,y_g) \\ \cos(z_i,z_g) \end{bmatrix} = \begin{bmatrix} q_{13i} \\ q_{23i} \\ q_{33i} \end{bmatrix} \qquad (3\text{-}6c)$$

Let $\hat{x}'_i$, $\hat{y}'_i$ and $\hat{z}'_i$ be the unit vectors along coordinate axes in the global coordinate frame, transformed to the coordinate frame i then

$$\hat{x}'_i = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos(x_i, x_g) \\ \cos(y_i, x_g) \\ \cos(z_i, x_g) \end{bmatrix} = \begin{bmatrix} q_{11i} \\ q_{12i} \\ q_{13i} \end{bmatrix} \tag{3-6d}$$

$$\hat{y}'_i = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(x_i, y_g) \\ \cos(y_i, y_g) \\ \cos(z_i, y_g) \end{bmatrix} = \begin{bmatrix} q_{21i} \\ q_{22i} \\ q_{23i} \end{bmatrix} \tag{3-6e}$$

$$\hat{z}'_i = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos(x_i, z_g) \\ \cos(y_i, z_g) \\ \cos(z_i, z_g) \end{bmatrix} = \begin{bmatrix} q_{31i} \\ q_{32i} \\ q_{33i} \end{bmatrix} \tag{3-6f}$$

This gives us

$$\underline{X}_{op} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \underline{X}_H + \underline{X}_{01} \tag{3-7a}$$

The same equation we will write in a general form

$$\underline{X}_p = [Q_i]\underline{X}_H + \underline{q}_i \tag{3-7b}$$

Observation No. 1

$$[Q_i][Q_i]^T = I$$

Because the matrix $[Q_i]$ is formed of three orthonormal row vectors. Hence $[Q_i]^T = [Q_i]^{-1}$, i.e., $[Q_i]$ is an orthogonal matrix.

Let two different positions of a body be known. We will correlate the position of a point fixed in the body in its first position to the position of the point in its second position (please see figure 3-2). Let us denote vectors in these two positions by subscripts 1 and 2 respectively. The position vector of the point fixed in the body in its first position can be given using equation (3-7b) as

$$\underline{X}_1 = [Q_1]\underline{X}_H + \underline{q}_1, \tag{3-8}$$

Here vectors $\underline{X}_1$ and $\underline{q}_1$ are in the coordinate frame g, while vector $\underline{X}_H$ is in the coordinate frame i. In the second position of the body, the vector $\underline{X}_H$ remains the same because the point is static in the coordinate frame fixed in the body. Hence the position vector of the point in the second position can be given using equation (3.7b) as

$$\underline{X}_2 = [Q_2]\underline{X}_H + \underline{q}_2 \tag{3-9}$$

We want $\underline{X}_2$ in terms of $\underline{X}_1$, so we will eleminate $\underline{X}_H$ from equations (3-8) and (3-9). From equation (3-8) we have

$$\underline{X}_H = [Q_1]^{-1}\underline{X}_1 - [Q_1]^{-1}\underline{q}_1 \quad (3\text{-}10a)$$

From Observation No. 1

$$[Q_1]^{-1} = [Q_1]^T$$

$$\underline{X}_H = [Q_1]^T\underline{X}_1 - [Q_1]^T\underline{q}_1 \quad (3\text{-}10b)$$

Substituting (3-10b) in (3-9) we get

$$\underline{X}_2 = [Q_2][Q_1]^T\underline{X}_1 - [Q_2][Q_1]^T\underline{q}_1 + \underline{q}_2 \quad (3\text{-}11)$$

or

$$\underline{X}_2 = [Q]\underline{X}_1 + \underline{q} \quad (3\text{-}12)$$

where

$$[Q] = [Q_2][Q_1]^T \quad (3\text{-}13a)$$

and

$$\underline{q} = \underline{q}_2 - [Q]\underline{q}_1 \quad (3\text{-}13b)$$

<u>Observation No. 2</u>

Substituting (3-13a) in equation (3-11) we get

$$\underline{X}_2 - \underline{q}_2 = [Q]\underline{X}_1 - [Q]\underline{q}_1$$

i.e.,

$$\underline{X}_{HO2} = [Q]\underline{X}_{HO1} \quad (3\text{-}14)$$

The vectors $\underline{X}_{HO1}$ and $\underline{X}_{HO2}$ are same as vector $\underline{X}_H$ transformed to the frame g in the first and the second position of the body respectively, i.e.,

$$\underline{X}_{HO1} = [Q_1]\underline{X}_H \quad \text{and}$$

$$\underline{X}_{HO2} = [Q_2]\underline{X}_H \tag{3-15}$$

In equation 3-14, the matrix $[Q_2]$ purely rotates the vector $\underline{X}_{HO1}$ to the vector $\underline{X}_{HO2}$. Hence the matrix [Q] gives the rotational transformation of a vector parallel to a straight line fixed in the body in its first position (please see figure 3-2) to the same straight line in the body in its second position. In terms of unit vectors defined in (3-6d) - (3-6f) with subscripts defining the position of the coordinate frame i, we can write

$$[Q] = \begin{bmatrix} \hat{x}'_{i2}\cdot\hat{x}'_{i1} & \hat{x}'_{i2}\cdot\hat{y}'_{i1} & \hat{x}'_{i2}\cdot\hat{z}'_{i1} \\ \hat{y}'_{i2}\cdot\hat{x}'_{i1} & \hat{y}'_{i2}\cdot\hat{y}'_{i1} & \hat{y}'_{i2}\cdot\hat{z}'_{i1} \\ \hat{z}'_{i2}\cdot\hat{x}'_{i1} & \hat{z}'_{i2}\cdot\hat{y}'_{i1} & \hat{z}'_{i2}\cdot\hat{z}'_{i1} \end{bmatrix} \tag{3-16}$$

Since the vector sets $[\hat{x}'_{i1}, \hat{y}'_{i1}, \hat{z}'_{i1}]$ and $[\hat{x}'_{i2}, \hat{y}'_{i2}, \hat{z}'_{i2}]$ are orthonormal sets, all row vectors or column vectors of the matrix [Q] also form an orthonormal vector set. Hence the matrix [Q] is orthogonal, i.e.,

$$[Q]^T = [Q]^{-1}$$

## The Direction of the Screw Axis

Any straight line in the body should make the same angle with the axis of the screw in both positions of the body. If $\underline{u}$ is the unit vector along the axis of the screw then

$$\underline{u} \cdot \underline{X}_{HO1} = \underline{u} \cdot \underline{X}_{HO2} \tag{3-18}$$

Equation (3-14) gives us

$$\underline{X}_{HO2} = [Q]\underline{X}_{HO1} \tag{3-19}$$

Eliminating $\underline{X}_{HO2}$ we get

$$[\underline{u}]^T\underline{X}_{HO1} = [\underline{u}]^T[Q]\underline{X}_{HO1}$$

or

$$[\underline{u}]^T[Q-I]\underline{X}_{HO1} = 0 \tag{3-20}$$

If the equation (3-20) has to be true for all $\underline{X}_{HO1}$ then

$$[\underline{u}]^T[Q-I] = 0$$

or

$$[Q-I]^T[\underline{u}] = 0$$

or

$$[Q^T-I][\underline{u}] = 0 \tag{3-21}$$

i.e.,

$$[Q]^T[\underline{u}] = [\underline{u}] \tag{3-22}$$

If this has to be true, one of the eigen values of $[Q]^T$ must be equal to 1. We will prove that this is true in all conditions. Let $\lambda$ be the eigen value of the matrix $[Q]^T$. Then

$$\lambda[\underline{u}] = [Q]^T[\underline{u}] \tag{3-23}$$

If we dot multiply equation (3-23) with the same equation we get

$$\begin{aligned}
\lambda^2[\underline{u}]^T[\underline{u}] &= \{[Q]^T[\underline{u}]\}^T \{[Q]^T[\underline{u}]\} \\
&= [\underline{u}]^T[Q][Q]^T[\underline{u}] \\
&= [\underline{u}]^T[Q][Q]^{-1}[\underline{u}] \\
&\qquad\qquad \text{(as } [Q] \text{ is orthogonal)} \\
&= [\underline{u}]^T I[\underline{u}] \\
&= [\underline{u}]^T[\underline{u}]
\end{aligned}$$

So we have

$$\lambda = \pm 1$$

Hence equations (3-21) and (3-22) hold good. Let us write the matrix [Q] as

$$[Q] = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

The equation (3-21) can be written as

$$\begin{bmatrix} q_{11}-1 & q_{21} & q_{31} \\ q_{12} & q_{22}-1 & q_{32} \\ q_{13} & q_{23} & q_{33}-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \tag{3-24}$$

Besides equation (3-24) we have one more equation to satisfy, i.e.,

$$u_1^2 + u_2^2 + u_3^2 = 1 \tag{3-25}$$

As $[Q]^T$ has one of its eigen values as +1, we have

$$\det[[Q]^T-I] = 0 \tag{3-26}$$

This condition holds true because only two of the three equations in (3-24) are independent. Hence equations (3-24) and (3-25) give us three independent equations in three unknowns $u_1$, $u_2$, $u_3$.

One of the ways to solve equations (3-24) and (3-25) is to find all the eigen values and eigen vectors of the matrix $[Q]^T$ and select the vector with eigen value +1 and then normalize it to get the vector $\underline{u}$.

Due to the simplicity of the equations (3-24) and (3-25) it is really not necessary to use expensive eigen-value, eigen vector routines if we have to solve this problem on the computer. If we write

$$u_1' = \frac{u_1}{u_3} \text{ and } u_2' = \frac{u_2}{u_3}$$

we can write equation (3-24) as

$$(q_{11}-1)u_1' + q_{21}u_2' = -q_{31} \tag{3-27a}$$

$$q_{12}u_1' + (q_{22}-1)u_2' = -q_{32} \tag{3-27b}$$

$$q_{13}u_1' + q_{23}u_2' = -(q_{33}-1) \tag{3-27c}$$

If we multiply equation (3-27c) with an arbitrary constant $\beta$ and add this equation to (3-27a), we will get two entirely independent equations in $u_1'$ and $u_2'$. Once we get $u_1$ and $u_2$ we can get $u_1$, $u_2$ and $u_3$ as follows:

$$u_3 = 1 \;/\; (u_1'^2 + u_2'^2 + 1)$$

$$u_1 = u_1'u_3$$

and

$$u_2 = u_2'u_3 \tag{3-28}$$

## Singularities

1) When the displacement of the body is a pure translation we have

$$[Q_1] = [Q_2]$$

hence

$$[Q] = [Q_2][Q_1]^T$$

$$= [Q_1][Q_1]^{-1}$$

$$= I$$

In equation (3-21) we get

$$[I-I][\underline{u}] = 0$$

In the present situation we fortunately do not need to find $\underline{u}$ by this method. Displacement of any one point on the body is enough to give the vector $\underline{u}$. If the two positions of the body are given as in figure 3-1 we can write $\underline{u}$ as

$$\underline{u} = (\underline{X}_{02} - \underline{X}_{01})/|\underline{X}_{02} - X_{01}| \tag{3-29}$$

2) If in equation (3-27)

$$\beta = -\frac{(q_{11}-1)}{q_{13}} = \frac{-q_{21}}{q_{23}} = \frac{q_{31}}{(q_{33}-1)} \tag{3-30}$$

By avoiding geometrically special choices of $\beta$, this possibility can be made almost nil.

3) When $u_3$=0 we cannot solve for $\frac{u_1}{u_3}$ and $\frac{u_2}{u_3}$. In this situation we have

$$(q_{11}-1)u_1 + q_{21}u_2 = 0$$

$$q_{12}u_1 + (q_{22}-1)u_2 = 0$$

$$q_{13}u_1 + q_{23}u_2 = 0 \tag{3-31}$$

Equations (3-31) give us

$$\frac{q_{11}-1}{q_{21}} = \frac{q_{12}}{q_{22}-1} = \frac{q_{13}}{q_{23}} \tag{3-32}$$

It can be shown that these equalities satisfy the condition

$$\det[Q^T - I] = 0$$

In this condition we can find $u_1$ and $u_2$ from the following equations:

$$u_2/u_1 = -(q_{11}-1)/q_{21} \tag{3-33}$$

and

$$u_1 2 + u_2^2 = 1 \tag{3-34}$$

We must also consider the third possibility when

$$q_{21} = q_{22}-1 = q_{23} = 0 \tag{3-35}$$

If we observe carefully the definition of Q in (3-16) we find that Q takes the following form when $u_1$=0 and $u_3$=0:

$$[Q] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3-36}$$

In this condition

$u_1=0,\ u_2=1,\ u_3=0$

The only singularity arises when the body displacement is a pure translation in which case [Q]=I which satisfies (3-35).

## The Location of the Screw Axis

Since we already know the unit vector along the screw axis, the position of any point on the axis is enough to uniquely locate it. The point on the body at the tip of the normal vector will travel a distance d when the body moves from the first position to the second. Hence for this point we have

$$\underline{X}_2 - \underline{X}_1 = d\ \underline{u} \tag{3-37}$$

where

$$\underline{X}_1 = \underline{\rho}_n \tag{3-38}$$

But from equation (3-12) we have

$$\underline{X}_2 = [Q]\underline{X}_1 + \underline{q} \quad .$$

($\underline{\rho}_n$ is the normal vector to the screw axis).

Substituting equations (3-12) and (3-38) in (3-37) we get

$$[Q]\underline{\rho}_n + \underline{q} - \underline{\rho}_n = d\,\underline{u}$$

$$[Q-I]\underline{\rho}_n - d\,\underline{u} = -\underline{q} \tag{3-39}$$

Also we have

$$\underline{\rho}_n \cdot \underline{u} = 0 \tag{3-40}$$

Equations (3-39) and (3-40) give us, in all, four equations with four unknowns which can be written in a matrix form as follows:

$$\begin{bmatrix} q_{11}-1 & q_{12} & q_{13} & -u_1 \\ q_{21} & q_{22}-1 & q_{23} & -u_2 \\ q_{31} & q_{32} & q_{33}-1 & -u_3 \\ u_1 & u_2 & u_3 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ d \end{bmatrix} = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ 0 \end{bmatrix} \tag{3-41}$$

Here $\rho_1$, $\rho_2$ and $\rho_3$ are the components of $\underline{\rho}_n$ along the x, y and z axes in the global coordinate system. We can find $\rho_1$, $\rho_2$, $\rho_3$ and d uniquely from equations (3-41).

## The Angle of Rotation About the Screw Axis

From equation (3-41) we have

$$\underline{X}_{HO2} = [Q]\underline{X}_{HO1}$$

Here $\underline{X}_{HO1}$ is a vector parallel to a straight line in the body in its first position while $\underline{X}_{HO2}$ is a vector parallel to the same straight line in the body in its second position. This implies that if $\underline{X}_{HO1}$ is parallel to the axis then $\underline{X}_{HO2}$ will also be parallel to the axis, i.e.,

$$\underline{u} = [Q]\underline{u} \tag{3-42}$$

This is another form of the equation (3-22).

Also, if $\underline{X}_{HO1}$ is perpendicular to the screw axis, then $\underline{X}_{HO2}$ will also be perpendicular to the screw axis and the angle between $\underline{X}_{HO1}$ and $\underline{X}_{HO2}$ will be the total angle of rotation, $\theta$, about the screw axis.

Now to find the angle, $\theta$, we will select a vector $\underline{X}_{HO1}$ that is perpendicular to the screw axis. Let

$$\underline{X}_{HO1} = \underline{a} - (\underline{a}\cdot\underline{u})\underline{u} \tag{3-43}$$

wjere $\underline{a}$ is any arbitrary vector. Then

$$\underline{X}_{HO2} = [Q][\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}]$$

$$= [Q]\underline{a} - (\underline{a}\cdot\underline{u})[Q]\underline{u}$$

Using the identity (3-42)

$$\underline{X}_{HO2} = [Q]\underline{a} - (\underline{a}\cdot\underline{u})\underline{u} \tag{3-44}$$

From the definition of $\underline{X}_{HO1}$ and $\underline{X}_{HO2}$ in equations (3-15), we have

$$\underline{X}_{HO1} = \underline{X}_{HO2} \quad . \tag{3-45}$$

Now

$$\cos\theta = \frac{\underline{X}_{HO1} \cdot \underline{X}_{HO2}}{|\underline{X}_{HO1}|\ |\underline{X}_{HO2}|} \tag{3-46}$$

Substituting (3-43) through (3-45) in equation (3-46) we get

$$\cos\theta = \frac{[\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}] \cdot [[Q]\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}]}{|\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}|^2}$$

$$= \frac{\underline{a}^T[Q]\underline{a} - (\underline{a}^T\underline{u})^2 - (\underline{a}^T\underline{u})\underline{u}^T[Q]\underline{a} + (\underline{a}^T\underline{u})^2}{\underline{a}^2 - 2(\underline{a}^T\underline{u})^2 + (\underline{a}^T\underline{u})^2} \qquad (3\text{-}47)$$

Now from equation (3-18) we have

$$\underline{u}\cdot\underline{X}_{HO1} = \underline{u}\cdot\underline{X}_{HO2}$$

Substituting for $\underline{X}_{HO1}$ and $\underline{X}_{HO2}$ from (3-43) and (3-44) we have

$$\underline{u} \cdot [\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}] = \underline{u}[[Q]\underline{a} - (\underline{a}\cdot\underline{u})\underline{u}]$$

Converting into the matrix form and solving we get [in matrix form]

$$\underline{u}^T\underline{a} - \underline{a}^T\underline{u} = \underline{u}^T[Q]\underline{a} - (\underline{a}^T\underline{u})$$

i.e.,

$$\underline{u}^T\underline{a} = \underline{u}^T[Q]\underline{a} \qquad (3\text{-}48)$$

Substituting (3-48) in (3-47) and solving, we get

$$\cos\theta = \frac{\underline{a}^T[Q]\underline{a} - (\underline{a}^T\underline{u})^2}{\underline{a}^2 - (\underline{a}^T\underline{u})^2} \qquad (3\text{-}49)$$

As (3-49) is true for any vector $\underline{a}$ we will substitute

$\underline{a} = [1 \quad 0 \quad 0]^T$

Then

$\underline{a}^T[Q]\underline{a} = q_{11}, \quad \underline{a}^T\underline{u} = u_1{}^2$

Therefore

$$\cos\theta = \frac{q_{11} - u_1{}^2}{1 - u_1{}^2} \tag{3-50}$$

Similarly if we substitute $\underline{a} = [0 \quad 1 \quad 0]^T$ and $\underline{a} = [0 \quad 0 \quad 1]^T$ we can get different expressions for $\cos\theta$. They are

$$\cos\theta = \frac{q_{22} - u_2{}^2}{1 - u_2{}^2} = \frac{q_{33} - u_3{}^2}{1 - u_3{}^2} \tag{3-51}$$

The angle found from the equation (3-51) will be the magnitude of the angle $\theta$. The sign of the angle can only be known if we know $\sin\theta$. Without trying to prove the formulas for $\sin\theta$ we will list them here.

If $u_3 \neq 0$

$$\sin\theta = [u_1u_2(1-\cos\theta) - q_{12}]/u_3 \tag{3-52}$$

If $u_3 = 0$ and $u_2 \neq 0$

$$\sin\theta = q_{13}/u_2 \tag{3-53}$$

If $u_3 = 0$ and $u_2 = 0$

$$\sin\theta = q_{32} \tag{3-54}$$

A computer subroutine SCREWM was developed based on the theory in this chapter, the details and some results of which are given in Appendices I and II.

# IV. THE ALGORITHM FOR THE DISPLACEMENT ANALYSIS

## Introduction

In Chapter II, we discussed how we can describe a configuration of a mechanical manipulator in space. We know that we need to fix one joint variable at every joint (considering that all joints are single degree of freedom joints) in order to fix the manipulator in space. In this chapter, we will discuss paths that the manipulator hand may follow in order to reach its destination, and we will generate positions for the manipulator hand on these paths at small intervals. Then we will describe an algorithm to compute increments in all the joint variables in order to achieve a small known displacement in the position of the manipulator hand. The first type will be a general displacement analysis. The second type of displacement we will consider will be the screw displacement which we have already discussed extensively in Chapter III. We will observe later on that choosing the screw path for the displacement analysis simplifies the algorithm that we are going to discuss. In the third type of displacement analysis, we will generate the positions for the manipulator hand such that one point on the manipulator hand will move on a straight line while the hand will change its orientation gradually along its path so as to finally coincide with the desired final position. Finally, we will consider an operator guided motion.

## Analysis Type I: A General Analysis of the Displacement

The linear and angular velocity of a point on the manipulator hand can be given as a sum of the linear and angular velocities produced at that point by all the joints. We can write the velocity of a point $O_H$ on the manipulator shown in figure 4-1 as

$$\underline{V}_H = v_H \underline{u}_v = \sum_{j=1}^{6} (h_j \omega_j \underline{u}_j + \omega_j \underline{u}_j x \underline{\rho}_{jH}) \tag{4-1}$$

and

$$\underline{W}_H = \omega_H \underline{u}_\omega = \sum_{j=1}^{6} \omega_j \underline{u}_j \tag{4-2}$$

All the vectors above are given in the global coordinate frame.

$\underline{V}_H$ is the velocity of the point $O_H$.

$\underline{u}_v$ is the unit vector along $\underline{V}_H$.

$\underline{W}_H$ is the angular velocity of the hand.

$\underline{u}_\omega$ is the unit vector along $\underline{W}_H$.

$\underline{W}_j$ is the angular velocity of the jth joint.

$\underline{u}_j$ is the unit vector along $\underline{W}_j$.

$\underline{\rho}_o$ is the vector from the origin of the x*(1) coordinate frame (already described in Chapter II) to $O_H$.

$\underline{\rho}_{jH}$ is the vector from the origin of the x*(j) coordinate frame to $O_H$.

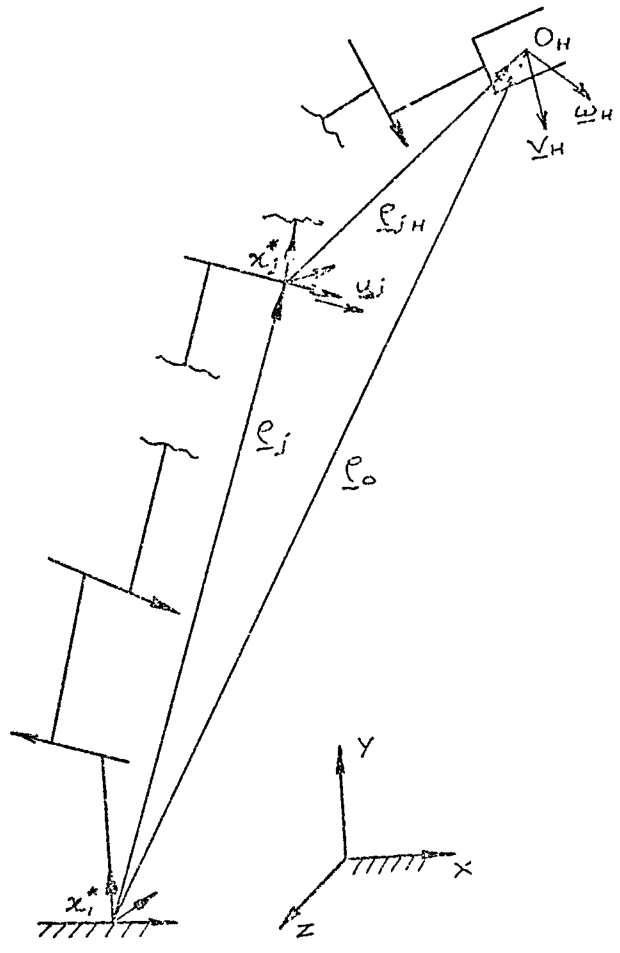


FIGURE 4-1

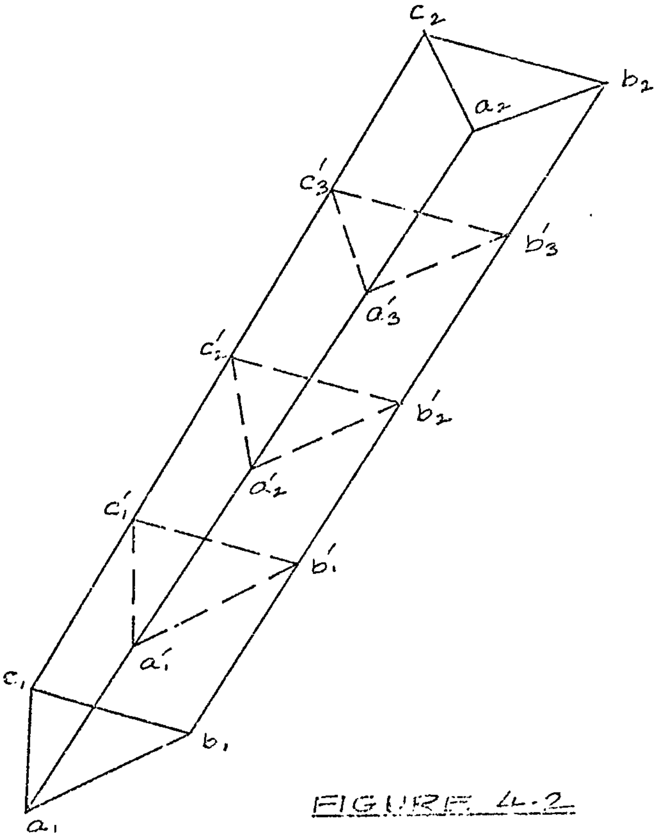


FIGURE 4-2

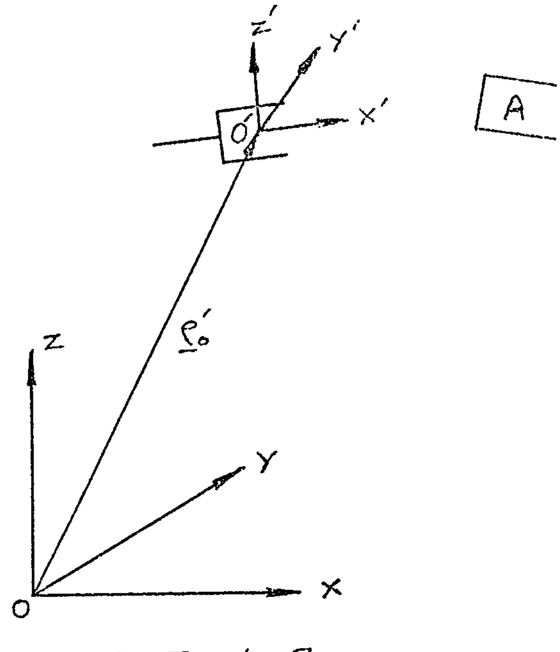


FIGURE 4-3

$\underline{\rho}_j$ is the vector from the origin of the x*(1) coordinate frame to the origin of the x*(j) coordinate frame.

$h_j$ is the pitch of the joint j.

$h_j = 0$ for the revolute joint.

$h_j = \infty$ for the prismatic joint.

If the kth joint is prismatic then the equation (4-1) we can write as

$$v_H\underline{u}_v = \sum_{j=1}^{k-1}(h_j\omega_j\underline{u}_j + \omega_j\underline{u}_j x \underline{\rho}_{jH}) + v_K\underline{u}_K$$

$$+ \sum_{j=K+1}^{6}(h_j\omega_j\underline{u}_j + \omega_j\underline{u}_j x \underline{\rho}_{jH}) \tag{4-3a}$$

and the equation (4-2) we can write as

$$\omega_H\underline{u}_\omega = \sum_{j=1}^{k-1}\omega_j\underline{u}_j + \sum_{j=K+1}^{6}\omega_j\underline{u}_j \tag{4-3b}$$

We can write

$$\underline{\rho}_{jH} = \underline{\rho}_o - \underline{\rho}_j \tag{4-4}$$

Substituting (4-4) in (4-1) we get

$$v_H\underline{u}_v = \sum_{j=1}^{6}[h_j\omega_j\underline{u}_j + \omega_j\underline{u}_j x(\underline{\rho}_o - \underline{\rho}_j)]$$

$$= \sum_{j=1}^{6}[h_j\omega_j\underline{u}_j - \omega_j\underline{u}_j x\underline{\rho}_j] + \sum_{j=1}^{6}\omega_j\underline{u}_j x\underline{\rho}_o$$

In the second summation, the vector $\underline{\rho}_o$ is a constant vector. Hence we can write

$$v_H\underline{u}_v = \sum_{j=1}^{6}(h_j\omega_j\underline{u}_j - \omega_j\underline{u}_j x\underline{\rho}_j) + [\sum_{j=1}^{6}\omega_j\underline{u}_j]x\underline{\rho}_o$$

Substituting from (4-2) we get

$$v_H\underline{u}_v = [\sum_{j=1}^{6}(h_j\omega_j\underline{u}_j - \omega_j\underline{u}_j x\underline{\rho}_j)] + \omega_H\underline{u}_\omega x\underline{\rho}o$$

or

$$v_H\underline{u}_v - \omega_H\underline{u}\omega x\underline{\rho}_o = \sum_{j=1}^{6}(h_j\omega_j\underline{u}_j - \omega_j\underline{u}_j x\underline{\rho}_j) \qquad (4\text{-}5)$$

If the kth joint of the manipulator is a prismatic joint, the kth term of the summation on the right side of (4-5) becomes $v_j\underline{u}_j$.

The equations (4-2) and (4-5) are equating the components of the motor of the manipulator hand with the sum of the respective components of the motors of the manipulator joints. The theory of motors is well discussed in reference [10].

In the equations (4-2) and (4-5) $\underline{W}_H$, $\underline{V}_H$ and $\underline{W}_j$ ($\underline{V}_j$ in the case of prismatic joints) are all instantaneous velocities. But for a very small interval of time we can approximate them as

$$\underline{W}_H = \omega_H \underline{u}_\omega = \frac{\delta\theta}{\delta t} \underline{u}_\omega \tag{4-6a}$$

$$\underline{V}_H = v_H \underline{u}_V = \frac{\delta r}{\delta t} \underline{u}_V \tag{4-6b}$$

and

$$\underline{W}_j = \omega_j \underline{u}_j = \frac{\delta\theta_j}{\delta t} \underline{u}_j \tag{4-6c}$$

and for prismatic joints

$$\underline{V}_j = v_j \underline{u}_j = \frac{\delta r_j}{\delta t} \underline{u}_j \tag{4-6d}$$

Where

$\delta\theta$ is a small angular change in the manipulator hand position in the time $\delta t$.

$\delta r$ is a small linear displacement of the manipulator hand at $O_H$ in the time $\delta t$.

$\delta\theta_j$ is a small angular change in the joint angle at the joing j in the time $\delta t$.

$\delta r_j$ is a small linear sliding displacement at the jth joint if the joint is a prismatic joint.

After substituting the relations (4-6), the equations (4-2) and (4-5) become the equations of approximations as follows:

$$\frac{\delta\theta}{\delta t} \underline{u}_\omega = \sum_{j=1}^{6} \frac{\delta\theta j}{\delta t} \underline{u}_j \tag{4-7}$$

and

$$\frac{\delta r}{\delta t}\,\underline{u}_v - \frac{\delta\theta}{\delta t}\,\underline{u}_\omega x \underline{\rho}_o = \sum_{j=1}^{6}\left(h_j \frac{\delta\theta_j}{\delta t}\,\underline{u}_j - \frac{\delta\theta_j}{\delta t}\,\underline{u}_j x \underline{\rho}_j\right) \tag{4-8}$$

Cancelling the common factor $\delta t$ from both sides of (4-7) and (4-8) we get

$$\delta\theta\underline{u}_\omega = \sum_{j=1}^{6}\delta\theta_j\,\underline{u}_j \tag{4-9}$$

and

$$\delta r\underline{u}_v - \delta\theta\underline{u}_\omega x\underline{\rho}_o = \sum_{j=1}^{6}\left(h_j\delta\theta_j\underline{u}_j - \delta\theta_j\underline{u}_j x\underline{\rho}_j\right) \tag{4-10}$$

In (4-9) and (4-10) above we get six linear equations in six unknowns $\delta\theta_j$ which can be written in the form

$$\left[\begin{array}{c|c|c|c|c}
\underline{u}_1 & -- & \underline{u}_j & -- & \underline{u}_6 \\
\hline
h_1\underline{u}_1 - \underline{u}_1 x\underline{\rho}_1 & -- & h_j\underline{u}_j - \underline{u}_j x\underline{\rho}_j & -- & h_6\underline{u}_6 - \underline{u}_6 x\underline{\rho}_6
\end{array}\right]
\begin{bmatrix}\delta\theta_1 \\ \vdots \\ \delta\theta_6\end{bmatrix}$$

$$= \left[\begin{array}{c}
\delta\theta\underline{u}_\omega \\
\hline
\delta r\underline{u}_v - \delta\theta\underline{u}_\omega x\underline{\rho}_o
\end{array}\right] \tag{4-11a}$$

The relations in (4-11a) we will write in a simpler form as

$$[A]\underline{D} = \underline{B} \tag{4-11b}$$

In (4-11b), [A] is a 6x6 matrix which is totally dependent on the configuration of the manipulator. The vector $\underline{D}$ is a 6x1 vector of increments of the six joint variables. The vector $\underline{B}$ is a 6x1 vector giving a small change in the position of the manipulator hand along the six degree of freedom of its movements in the global coordinate frame.

In the case where the kth joint is a prismatic joint, the kth column of [A] becomes

$$\text{The kth column of } [A] = \left[\begin{array}{c} 0 \\ \hline \underline{u}_k \end{array}\right]$$

Also the kth component of the vector $\underline{D}$ becomes $\delta r_k$.

A small displacement of the manipulator hand can be described as a small linear displacement $\delta r$ of a point on the manipulator hand along a unit vector $\underline{u}_v$ plus a small angular rotation $\delta\theta$ about a unit vector $\underline{u}_\omega$. Hence for a known small displacement of the manipulator hand we know $\underline{B}$. Also we know the matrix [A] which is dependent on the configuration of the manipulator. Now we can compute the vector $\underline{D}$ by solving the six linear equations (4-11) in the six unknowns in $\underline{D}$. Let us consider some simple examples.

Example 1: A pure rotation of the manipulator hand about an axis parallel to the z-axis and passing through $O_H$ (please see figure 4-1) in the global coordinate frame.

If the total angular rotation is $\theta$ we can divide it into k small increments

$$\delta\theta = \frac{\theta}{k}$$

We have

$$\underline{u}_\omega = [0 \quad 0 \quad 1]^T \quad \text{and} \quad \delta r = 0$$

Let

$$\underline{\rho}_o = [\rho_1 \quad \rho_2 \quad \rho_3]^T$$

Substituting all these values in $\underline{B}$, we get

$$\underline{B} = [0 \quad 0 \quad \delta\theta \quad -\rho_2\delta\theta \quad \rho_1\delta\theta \quad 0]^T$$

Now we can solve the six linear equations in (4-11) for $\delta\theta_1$, $\delta\theta_2$ ---, $\delta\theta_6$. Updating all the joint angles, we get the new position of the manipulator. Now, using the transformations explained in Chapter II, we can find the new values of $\underline{u}_j$, $\underline{\rho}$ , $\underline{\rho}_o$, etc. Repeating the above procedure k times, we can find all the intermediate positions the manipulator has to go through to get to its destination.

Example 2: A pure linear displacement parallel to the x-axis in the global coordinate frame.

We have

$$\delta r = \frac{r}{k}\,, \quad \underline{u}_V = [1 \quad 0 \quad 0]^T \quad \text{and} \quad \delta\theta = 0$$

Hence in equation (4-11)

$$\underline{B} = [0 \quad 0 \quad 0 \quad \delta r \quad 0 \quad 0]^T \tag{4-13}$$

The procedure from here on will be the same as explained in Example 1.

Example 3: A rotation at $O_H$ about an axis parallel to the z-axis, in addition to a linear displacement parallel to the x-axis.

We have

$$\delta\theta = \frac{\theta}{k}\,, \qquad \delta r = \frac{r}{k}$$

$$\underline{u}_\omega = [0 \quad 0 \quad 1]^T, \quad \underline{u}_V = [1 \quad 0 \quad 0]^T \tag{4-14}$$

Hence we get

$$\underline{B} = [0 \quad 0 \quad \delta\theta \quad -\rho_2\delta\theta+\delta r \quad \rho_1\delta\theta \quad 0]^T$$

The procedure from here on will be the same as explained in Example 1.

Example 4: A displacement from one known position to another known position of the manipulator hand.

In all the three examples discussed above, the direction of the rotation $\underline{u}_\omega$ and the direction of the linear displacement $\underline{u}_V$ remain the same throughout the analysis. This simplifies the situation to some extent. This method can be applied to all general cases, as follows:

We have explained in Chapter III the way in which we can define a cartesian coordinate frame on the body (the manipulator hand) in any one position. In Chapter III, we have also seen that we can find the screw axis of rotation and the amount of rotation about the screw axis in order to transfer the body from one position to another. If $\underline{U}$ is the unit vector along the screw axis, and $\theta$ is the angle of rotation about the screw axis, then the rotation of the manipulator hand about $\underline{U}$ in its first position by an angle $\theta$ will make it identical in orientation to that of the second position.

Hence we can write

$$\underline{u}_\omega = \underline{U} \tag{4-16}$$

If $\underline{R}$ is the vector joining the origin of the cartesian coordinate system on the body, in its first position, to the origin, in its second position, then if

$$\underline{U}_r = \frac{\underline{R}}{|\underline{R}|}$$

then

$$\underline{u}_V = \underline{U}_r$$

So we have

$$\delta\theta = \frac{\theta}{k}, \qquad \delta r = \frac{|\underline{R}|}{k}$$

$$\underline{u}_\omega = \underline{U} \quad \text{and} \quad \underline{u}_V = \underline{U}_r \tag{4-17}$$

Substituting these values we can get the vector $\underline{B}$. Solving (4-11) we can get $\underline{D}$. Updating the configuration of the manipulator and resubstituting the above values (k-1) more times we can get to the final position of the manipulator.

*Analysis Type II: Screw Displacement Analysis

We have seen in Chapter III that any body can be transferred from one position to another by a screw displacement. We will develop a method that will give the manipulator hand a screw displacement. Consider the point on the screw axis lying at the tip of the normal to the axis from the origin, with the manipulator hand in its first position. This point will move along the screw axis when the manipulator hand executes the screw motion. If $\underline{U}$ is the unit vector along the screw axis and if $\underline{\rho}_n$ is the vector perpendicular to the screw axis then we have

---

*This method is a modified form of the "Iterative Velocity Method" developed by Pieper [6].

$\underline{u}_\omega = \underline{u}_V = \underline{U}$ and

$\underline{\rho}_o = \underline{\rho}_n$ (constant throughout the analysis)

$$\delta\theta = \frac{\theta}{K} \quad \text{and} \quad \delta r = h\delta\theta \tag{4-18}$$

Substituting (4-18), the vector $\underline{B}$ becomes

$$\underline{B} = \left[\begin{array}{c} \delta\theta\underline{U} \\ \hline h\delta\theta\underline{U} - \delta\theta\underline{U}x\underline{\rho}_n \end{array}\right] \tag{4-19a}$$

In the case of pure linear translation, we have

$$\underline{B} = \left[\begin{array}{c} 0 \\ \hline \delta r\underline{U} \end{array}\right] \tag{4-19b}$$

The simplicity in this type of the analysis is that, in the general analysis, one, or more than one, variables such as $\underline{u}_\omega$, $\underline{u}_V$, $\underline{\rho}_o$ may be changing at every intermediate position of the manipulator hand while in the screw motion analysis most of the values remain constant for all the k steps. On the other hand, in the general analysis, in many cases, $\underline{u}_\omega$ and $\underline{u}_V$ can be kept constant (as already illustrated by few examples) and, since for every intermediate position we need to compute $\underline{u}_i$ and $\underline{\rho}_i$, it does not require extra effort to compute $\underline{\rho}_o$. This makes it simpler to use the general form of the analysis.

The method of analysis is as follows.

1) Locate the screw axis of the displacement between the given two positions of the manipulator hand and compute h, $\theta$, $\underline{U}$ and $\underline{\rho}_n$ as explained in Chapter III.

2) Divide $\theta$ into k equal parts

$$\delta\theta = \frac{\theta}{K}$$

3) Substitute h, $\delta\theta$, $\underline{U}$ and $\underline{\rho}_j$ in the equation (4-19a) and compute the vector $\underline{B}$.

4) For the present configuration of the manipulator compute the vectors $\underline{u}_j$, $\underline{\rho}_j$, etc. using the transformations given in Chapter II.

5) Substitute all the values above in equations (4-11b) and solve for $\delta\theta_1$, $\delta\theta_2$-----$\delta\theta_6$.

6) Upgrade the joint variables

$$\theta_j(\text{new}) = \theta_j(\text{old}) + \delta\theta_j$$

7) Repeat cycle 4-5-6 k times to get displacements of joints throughout the motion.

## Analysis Type III: Displacement with One Point Moving on a Straight Line

We have explained in Chapter III how we can completely specify a position of the body (the manipulator hand) in space. We have also explained how we will define a cartesian coordinate frame fixed on the body, e.g., at the position of the body defined by $a_1$, $b_1$, $c_1$ in figure (4-2), we will have origin at $a_1$ and the axes defined by the unit vectors

$$\hat{z} = \frac{\underline{a_1 b_1}}{|\underline{a_1 b_1}|}, \qquad \hat{y} = \frac{\underline{a_1 b_1} \times \underline{a_1 c_1}}{|\underline{a_1 b_1} \times \underline{a_1 c_1}|}$$

and

$$\hat{x} = \hat{y} \times \hat{z} \tag{4-20}$$

Let $a_2 b_2 c_2$ be the final position after the displacement. Let us divide $a_1 a_2$, $b_1 b_2$ and $c_1 c_2$ into k equal parts. Let the triangle formed by the first division on each line be $a_1' b_1' c_1'$. If k is large, the triangles $a_1 b_1 c_1$ and $a_1' c_1' b_1'$ will be close enough and there will be a very small change in orientation as well as displacement. If the screw axis of the displacement between these positions is found we can compute all the joint variables by Method I or II.

Analysis Type IV: Operator Controlled Motion

In all the above discussed analysis, we need to know the final position of the manipulator before we start the analysis. Alternatively an operator may desire to control the manipulator hand by a trial and error method with visual inspection. Operators usually find it easy to observe and give directions with respect to a cartesian coordinate system, e.g., in figure 4-3, in order to fix the manipulator hand on the block A, one may prefer to move the manipulator hand in the direction of the x-axis by a distance say 5" and then rotate the hand by about 30° clockwise about the y-axis and finally adjust the hand on the block by moving it 1" along the x-axis. This type of displacement can be achieved as follows:

1) Displacement in the global coordinate system.

Let $\underline{R}$ be the vector parallel to which linear motion is desired and let all the angular rotations of the manipulator hand be required about the axis parallel to $\underline{R}$ and passing through 0. Let the total displacement desired be 'd' and the total angular rotation desired be 'θ'. This problem is no different than the analysis type II. The only difference is we do not know the final position of the manipulator hand and we do not compute the screw axis of displacement; instead we manipulate it from the input data as follows. In (4-19) we have

$$\underline{U} = \frac{\underline{R}}{|\underline{R}|} \quad , \qquad \delta\theta = \frac{\theta}{k}$$

$$h = \frac{d}{\theta} \quad \text{and} \qquad \underline{\rho_n} = \underline{\rho_o'} \tag{4-21}$$

$$[\text{as } \underline{U} \times \underline{\rho_o'} = \underline{U} \times (\underline{\rho_n} + \underline{\rho_o'} \cdot \underline{U}\ \underline{U})$$

$$= \underline{U} \times \underline{\rho_n}]$$

For displacement of the manipulator hand parallel to the x-axis

$$\underline{R} = [1 \quad 0 \quad 0]^T$$

For stretching of the manipulator arm along $OO'$ in figure 4-3

$$\underline{R} = \underline{\rho_o'}$$

The procedure will be similar to that in the analysis type II.

2) Displacement with reference to the coordinate frame fixed on the manipulator hand.

If $R'$ is the vector parallel to which the motion is desired then

$$\underline{R} = Q'\underline{R}'$$

where Q is the transformation matrix for transforming vectors from the manipulator hand coordinate frame to the global coordinate frame. The procedure will be the same as that in (1).

## Singularities

1) A situation can arise when the degree of freedom of the manipulator may drop down below six. In this condition the matrix [A] in the equation (4-11b) does not have an inverse and hence (4-11b) cannot be solved. One simple way of overcoming this problem is to assume a new configuration of the manipulator by giving small increments to the joint variables till the singularity becomes non-existent.

2) When one of the joints reaches its limit, the manipulator cannot move in the desired direction. There is no solution to this problem because we are confined to manipulators with six joints.

Several subroutines were developed based on the theory developed in this chapter, the function and test details of which are given in Appendices I and II.

# V. CONCLUDING REMARKS

In this thesis, several strategies for manipulator guidance are presented. We will list here a few comments on the merits and demerits of these strategies.

1) Displacement analysis type I is also capable of performing displacement analysis of types III and IV whereas analysis type II is a modification of the analysis type I. Hence the displacement analysis type I is called "A General Analysis of the Displacement."

2) Examples 1, 2 and 3 given in the Displacement Analysis Type I illustrate the type of analysis discussed in analysis type IV. Both of these types of analysis give a coordinated control of manipulators. In the analysis type IV the axis of rotation and the axis of displacement of the manipulator hand has to be the same, whereas in the analysis type I, these axes can be different. This is an advantage of the analysis type I over the analysis type IV.

3) Example 4 given in the Displacement Analysis Type I illustrates the type of analysis discussed in the analysis type III. Analysis type III gives a straight path displacement as a result of many small screw displacements. Hence it is an approximated analysis. Analysis type I is free of this type of approximation.

4) In the analysis type III, the screw axis has to be located at every stage of the analysis whereas in the analysis type I the screw axis has to be located only once. Keeping this in mind, that is, that the algorithm for locating the screw axis of displacement is much more involved than the algorithm for the displacement analysis, the analysis type I has an advantage over analysis type III.

5) The analysis type II does not give us the short path displacement of the manipulator hand but as is evident, it is useful in the analysis types III and IV.

A few suggestions:

1) Development:velocity analysis from the displacement analysis we have presented.

From equation (4-6a) through (4-6d) we have

$$\frac{|\underline{W}_j|}{|\underline{W}_H|} = \frac{\delta\theta_j}{\delta\theta} \tag{5-1}$$

and

$$\left|\frac{\underline{V}_j}{\underline{W}_H}\right| = \frac{\delta r_j}{\delta\theta} \quad \text{(for prismatic joints)} \tag{5-2}$$

or

$$|\underline{W}_j| = \frac{\delta\theta_j}{\delta\theta} |\underline{W}_H| \tag{5-3}$$

and

$$|\underline{V}_j| = \frac{\delta r_j}{\delta \theta}|\underline{W}_H| \quad \text{(for prismatic joints)} \tag{5-4}$$

As we can compute $\delta\theta_j$, $\delta r_j$ and $\delta\theta$ in the displacement analysis, if we know the velocity of the manipulator hand and the position of the manipulator hand, we can compute the joint velocities. We know that

$$\underline{W}_j = |\underline{W}_j|\ \underline{u}_j$$

and

$$\underline{V}_j = |\underline{V}_j|\ \underline{u}_j \quad \text{(for prismatic joints)}$$

If the displacement of the manipulator hand is pure translational, $\underline{W}_H$ in (5-1) through (5-4) has to be replaced by $\underline{V}_H$, and $\delta\theta$ by $\delta r$.

2) Improvement of the accuracy of the algorithm for the displacement analysis.

The equations (4-9) and (4-10) are in fact the equations of approximations with all the factors of second order and higher eliminated. If we include the second order terms we can write the linear displacement of the manipulator hand in terms of the linear displacements produced by each joint at the manipulator hand as

$$\delta r\underline{u}_V = \sum_{j=1}^{6} [h_j \delta\theta_j \underline{u}_j + \delta\theta_j \underline{u}_j \times \underline{\rho}_{jH}$$

$$+ \frac{\delta\theta_j^{\,2}}{2} \underline{u}_j \times (\underline{u}_j \times \underline{\rho}_{jH})] \tag{5-5}$$

Equations (4-9) and (5-5) give us six non-linear simultaneous equations in the six unknowns $\delta\theta_1$, $\delta\theta_2$, $---\delta\theta_6$. The solution obtained from (4-11) can be considered as the first approximation and the equations (4-9) and (5-5) can be solved by the Parameter-Perturbation Procedure developed by Freudenstein and Roth [2].

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# APPENDIX I

## Computer Program Structure

A computer program DISP is developed basically for displacement analysis of a mechanical manipulator. Only manipulators with six, single degree of freedom joints are considered for the analysis. For manipulators with more than six joints, only the first six joints from the ground will be considered. Remaining joints will be considered as structural parts of the manipulator hand. The type of joints considered are revolute, prismatic and screw joints. Since the use of the screw joints is seldom practical, only problems with prismatic and revolute joints are considered for testing. Besides displacement analysis based on types II, III and IV described in Chapter IV, DISP has a few other capabilities which can be independently called.

1) Given two or more positions of a body (in the manner explained in Chapter III), locate the screw axis of the displacements between every pair of consecutive positions. (Limit on the number of positions: 99.)

2) Given a configuration of a spatial mechanism (closed or open type) in the Denavit-Hartenberg type of representation, compute the details of the mechanism in a global cartesian

coordinate system. (Limit on the number of joints: 15.)

The basic structure of DISP is shown in figure A-1. Each rectangular block in the figure represents a subroutine. The functions of these subroutines may be summarized as follows:

I) SUBROUTINE READM:

It reads and writes all the input data, rearranges it, and stores it in COMMON storage for use at the proper time in other subroutines. The input data cards are read in the following order:

MS Decides the type of the analysis.

= 1 to locate the screw axis of the displacement.

= 2 to calculate mechanism position data in the global cartesian coordinate system.

= 3 combination of MS = 1 and MS = 2.

= 4 for displacement analysis of type III explained in Chapter IV.

= 5 for displacement analysis of type IV explained in Chapter IV.

= 0 for displacement analysis of type II explained in Chapter IV.

= 6 for the displacement analysis of type I in the [Ex. 4] explained in Chapter IV.

NP i) For MS = 1, it gives the number of positions of a body for which the screw axis of the displacement is required.

ii) For MS = 0, 3, 4 and 6, it gives the number of positions of the manipulator hand.

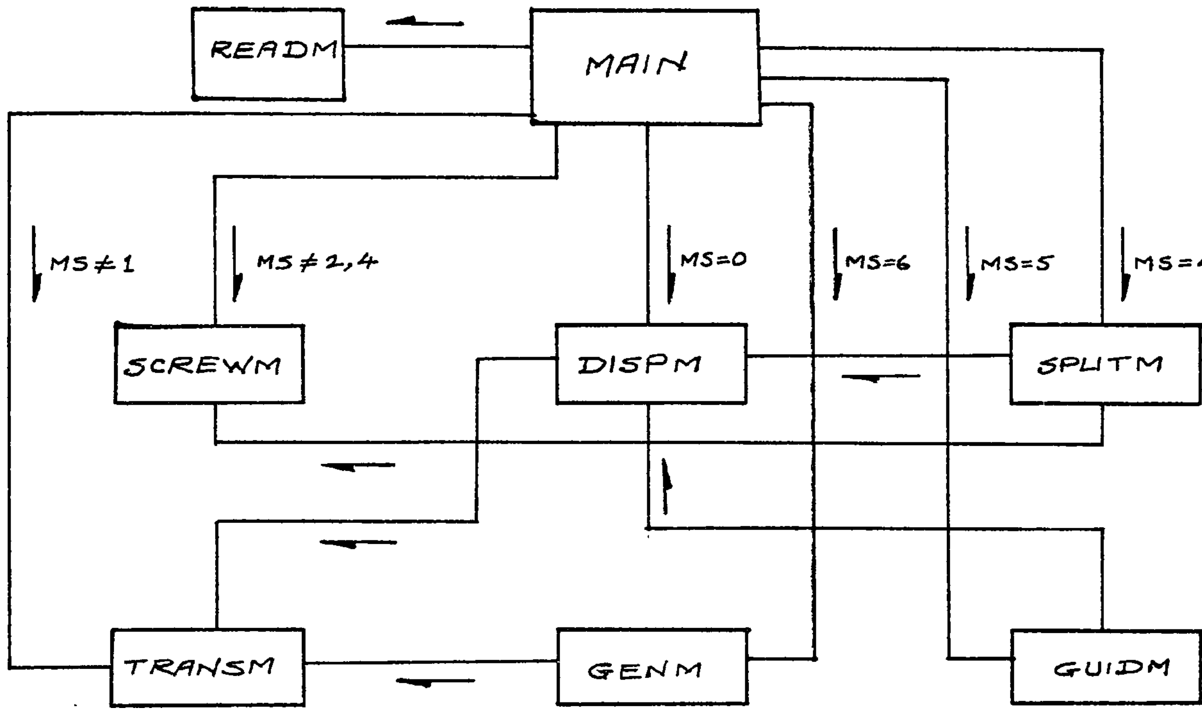


FIGURE A.1

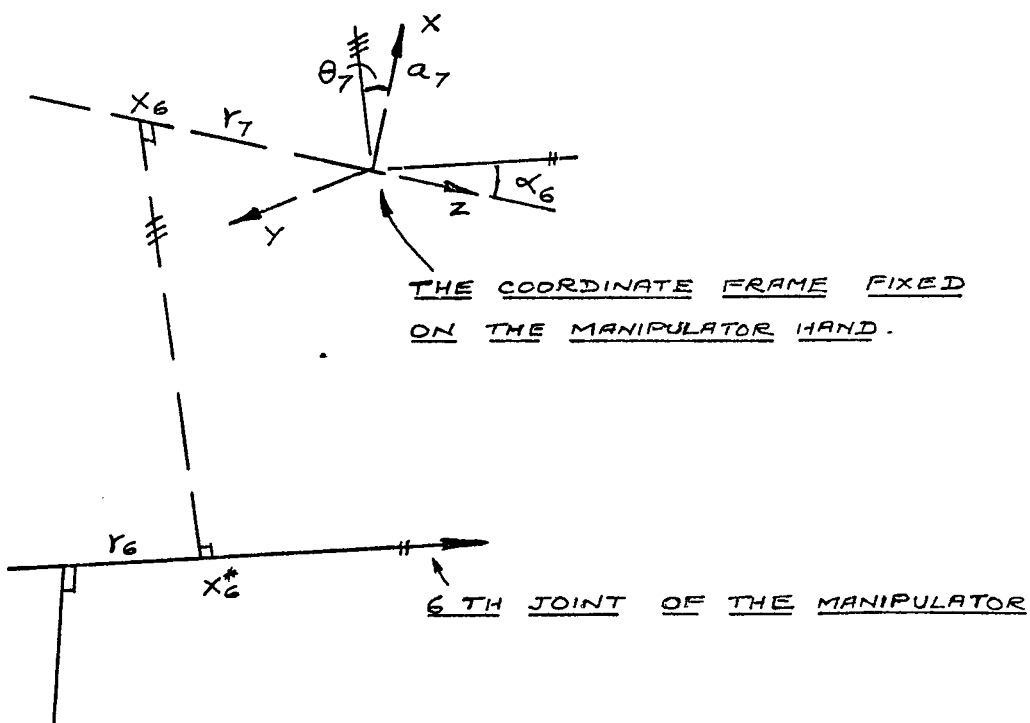


FIGURE A.2

iii) For MS = 5, it gives the number of the operator's requested steps.

iv) For MS = 2, the value is ignored.

IT The number of steps of the analysis. The value of IT may sometimes be changed for some specific reasons. if IT = 0, DISP decides the value of IT on its own.

IO The amount of information desired from a displacement analysis.

= 0 for complete information. It gives complete information of the mechanical manipulator at every step of the analysis.

= 1 for restricted information. It gives complete information of the manipulator at its initial and at its final position and it gives only joint variable changes in its intermediate positions.

2) SET TWO: (3XNP) cards

Input Variables: X(I, J, K)

FORMAT: 3F10.2

1) Each card gives a position of a point in a global coordinate frame.

2) Every set of three cards, in order, define a position of a body (the manipulator hand) in the global cartesian coordinate frame by defining a cartesian coordinate frame on the body. The first card in the set gives the position of a point which is considered the origin of the coordinate system. The second card gives a point on the positive side of the z-axis of the system. The third card gives a point

in the plane formed by the z-axis and the positive side of the x-axis.

The above convention of giving different positions of a body gives flexibility to some extent. However, for simplicity, one may prefer to give the position of the same three points on the body for every position of the body.

There are two options for specifying the first position of the manipulator hand: 1) The first three cards in SET TWO giving the first position of the body in the manner explained above. This option is usually inferior to the one explained below. 2) The first three cards in SET TWO giving all the zero values. In this case the coordinate frame defining the first position of the body has to be specified in terms of the seventh (imaginary) joint parameters. This can be done as follows.

Consider the z-axis of the coordinate frame defining the position of the body as the seventh joint of the manipulator. Consider the x-axis of the coordinate frame as the seventh normal of the manipulator. Now all the parameters of the (imaginary) seventh joint can be specified in SET FOUR in the manner explained in Chapter II. For better understanding please see figure A-2. A Restriction: The parameter $a_7$ must have a positive, non-zero (arbitrary) value. The advantage in using this option is that DISP uses the actual position of the manipulator hand defined by the seventh joint parameters for any intermediate stage of the analysis rather than using computed values from the previous stage of the analysis. This gives better accuracy in the results.

3) SET THREE: NP cards.

This set of data cards are required only for displacement analysis of the type IV (MS = 5), explained in Chapter IV. This set totally replaces SET TWO above.

Input Variables: NTY(I),(DC(J,I),J=1,3),DD(I),THA(I)

FORMAT: (I10,5F10.3)

NTY indicates the coordinate frame in relation to which the manipulator hand movements are desired.

= 0 for global coordinate frame.

= 1 for the coordinate frame fixed on the manipulator hand.

DC gives the three components of

i) $\underline{R}$ in the equations (4-21) if NTY = 0.

ii) $\underline{R}'$ in the equation (4-22) if NTY = 1.

DD gives the value of d in the equations (4-21).

THA gives the value of $\theta$ in the equations (4-21).

4) SET FOUR:

i) First card

Input variable: $\overset{\circ}{N}$

FORMAT: I5

N is the number of joints of the mechanical manipulator.

ii) N cards

Input variables: R(I),ALPHA(I),A(I),THETA(I),PH(I)

FORMAT: 5F10.3

R (I) is same as $r_i$ in Chapter II.

ALPHA(I) is same as $\alpha_i$ in Chapter II.

A(I) is same as $a_i$ in Chapter II.

THETA(I) is same as $\theta_i$ in Chapter II.

PH(I) is the pitch of the joint. If the joint is prismatic $PH(I) \geq 10^5$.

5) SET FIVE: three cards

FORMAT: 3F10.3

This set gives position of three points on the coordinate frame x*(1) in the initial position of the manipulator. The first point is at the origin of the coordinate frame X*(1), the second is on the z-axis on its positive side and the third is on the normal $a_1$ (i.e., on the x-axis of the frame x*(1)).

All the five sets of the data cards are not required for any one type of analysis. The table A1-1 indicates the data sets to be provided for a specific type of analysis.

TABLE A1-1: Data Arrangements

| | SET ONE | SET TWO | SET THREE | SET FOUR | SET FIVE |
|---|---|---|---|---|---|
| MS = 0 | Yes | Yes | No | Yes | Yes |
| MS = 1 | Yes | Yes | No | No | No |
| MS = 2 | Yes | No | No | Yes | Yes |
| MS = 3 | Yes | Yes | No | Yes | Yes |
| MS = 4 | Yes | Yes | No | Yes | Yes |
| MS = 5 | Yes | No | Yes | Yes | Yes |
| MS = 6 | Yes | Yes | No | Yes | Yes |

II) CONTROL ROUTINE MAIN:

The section MAIN controls all the functions of DISP. It works as follows:

1) It calls READM and stores the value of MS and all the positions through which the manipulator hand has to pass.
2) According to the value of MS it calls the appropriate subroutine.

III) SUBROUTINE SCREWM:

This subroutine receives information about only two positions of a body at a time and locates the screw axis of the displacement. The subroutine is developed using the theory discussed in Chapter III. It writes the data it computes.

IX) SUBROUTINE TRANSM:

This subroutine receives details of a mechanism in the Denavit-Hartenberg form of notation and computes the details in a global coordinate system. It writes the data it computes. The subroutine is developed, based on the theory discussed in Chapter II.

V) SUBROUTINE DISPM:

This subroutine does the displacement analysis of type II explained in the Chapter IV. It computes the small changes in the joint variables for the desired type of displacement and then it calls TRANSM for updating the manipulator configuration in the global coordinate frame. DISPM writes down the changes in the joint variables.

VI) SUBROUTINE SPLITM:

This subroutine receives information about two positions of the manipulator hand at a time and divides the displacement between them into many small displacements as explained in the displacement analysis type III in Chapter IV. Then it calls DISPM for the displacement analysis.

VII) SUBROUTINE GUIDM:

This subroutine receives all the control commands from the subroutine READM and executes them one after another. It follows the displacement analysis type IV in Chapter IV.

VIII) SUBROUTINE GENM:

This subroutine does the displacement analysis of type I [Ex. 4] explained in Chapter IV. It works similar to DISPM.

# APPENDIX II

## Numerical Illustrations

1) MS = 1

Locate the screw axis of the displacement. Three positions of a body are given (please see figure 3-1).

First Position of the Body

$$\underline{X}_{01} = [\ 25.00 \quad 0.00 \quad 20.00]^T$$

$$\underline{X}_{02} = [\ 30.00 \quad 0.00 \quad 20.00]^T$$

$$\underline{X}_{03} = [\ 25.00 \quad 0.00 \quad 10.00]^T$$

Second Position of the Body

$$\underline{X}_{01} = [-20.30 \quad -7.83 \quad -6.66]^T$$

$$\underline{X}_{02} = [-20.65 \quad -8.29 \quad -5.85]^T$$

$$\underline{X}_{03} = [-21.06 \quad -7.18 \quad -6.62]^T$$

Third Position of the Body

$$\underline{X}_{01} = [-24.50 \quad 5.30 \quad 4.00]^T$$

$$\underline{X}_{02} = [-24.67 \quad 5.47 \quad 4.97]^T$$

$$\underline{X}_{03} = [-25.00 \quad 6.00 \quad 4.00]^T$$

Data cards were arranged as shown in Table A2-1. (For further details, please see Appendix I.)

TABLE A2-1: The Data Arrangement for Illustration 1.

| | | | | |
|---|---|---|---|---|
| SET ONE<br>FORMAT 4I5 | 1 | 3 | 0 | 0 |

| | | | |
|---|---|---|---|
| SET TWO | 25.00 | 0.00 | 20.00 |
| FORMAT 3F10.2 | 30.00 | 0.00 | 20.00 |
| | 25.00 | 0.00 | 10.00 |
| | -20.30 | -7.83 | -6.66 |
| | -20.65 | -8.29 | -5.85 |
| | -21.06 | -7.18 | -6.62 |
| | -24.50 | 5.30 | 4.00 |
| | -24.67 | 5.47 | 4.97 |
| | -25.00 | 6.00 | 4.00 |

SUMMARY OF THE RESULTS:

The screw axis between the first position and the second position.

The unit vector along the screw axis

$$\underline{u} = [0.5699 \quad -0.4433 \quad 0.6918]^T$$

The normal vector to the screw axis from the origin

$\underline{\rho}_n = [-1.758 \quad -0.762 \quad 0.960]^T$

The axial displacement

$d = -40.79$ units

The angular rotation about the axis

$\theta = 176.36^\circ$

The pitch of the screw

$h = -13.25$ unit length per radian.

The screw axis between the second position and the third position.

The unit vector along the screw axis

$\underline{u} = [0.8381 \quad -0.3386 \quad 0.4276]^T$

The normal vector to the screw axis from the origin

$\underline{\rho}_n = [5.942 \quad 6.862 \quad -6.212]^T$

The axial displacement

$d = -3.407$ units

The angular rotation about the axis

$\theta = -40.54^\circ$

The pitch of the screw

h = 4.815 unit length per radian.

2) MS = 2

Calculate mechanism position data in the global cartesian coordinate system. Details of a seven joint mechanical manipulator shown in Table A2-2 are given in the Denavit-hartenberg notation.

TABLE A2-2: The Manipulator Configuration #1

| $i$ | $r_i$ | $\alpha_i$ | $a_i$ | $\theta_i$ | JOINT |
|---|---|---|---|---|---|
| 1 | 0.00 | 90.00 | 0.00 | 70.00 | REVOLUTE |
| 2 | 0.00 | -90.00 | 0.375 | 80.00 | REVOLUTE |
| 3 | 12.20 | 90.00 | 0.375 | 40.00 | REVOLUTE |
| 4 | 0.00 | -90.00 | 0.00 | 0.00 | REVOLUTE |
| 5 | 9.50 | 90.00 | 0.00 | 60.00 | REVOLUTE |
| 6 | 0.00 | 0.00 | 5.90 | 30.00 | REVOLUTE |
| 7 | 0.00 | 0.00 | 1.00 | 0.00 | --- |

(For notation, please see Chapter II)

The position of the x*(1) coordinate frame is given by three points on the axes of the x*(1) coordinate frame. (The position of the x*(0) coordinate frame coincides with x*(1) coordinate frame in the initial position of the manipulator. For further information, please see Chapter II).

The origin of the coordinate frame x*(1) is given by the position vector

$\underline{a} = [0 \quad 0 \quad 0]^T$

A point on the axis of the joint #1 which defines the positive z-axis is given by the position vector

$\underline{b} = [0 \quad 0 \quad 1]^T$

A point on the normal $a_1$ which defines the x-axis is given by

$\underline{c} = [1 \quad 0 \quad 0]^T$

[Note: In this specific case, the global coordinate frame is defined to coincide with the x*(1) coordinate frame. It is not necessary to define the global coordinate frame coinciding with the x*(1) coordinate frame.]

Data cards were arranged as shown in Table A2-3.

Table A2-3: The Data Arrangement for Illustration 2.

| | | | | | |
|---|---|---|---|---|---|
| SET ONE<br>4I5 | 2 | 0 | 0 | 0 | |
| i) 1I5 | 7 | | | | |
| | 0.00 | 90.00 | 0.00 | 70.00 | 0.00 |
| SET FOUR | 0.00 | -90.00 | 0.375 | 80.00 | 0.00 |
| | 12.20 | 90.00 | 0.375 | 40.00 | 0.00 |
| ii) 5F10.3 | 0.00 | -90.00 | 0.00 | 0.00 | 0.00 |
| | 9.50 | 90.00 | 0.00 | 60.00 | 0.00 |
| | 0.00 | 0.00 | 5.90 | 30.00 | 0.00 |
| | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 |
| SET FIVE | 0.00 | 0.00 | 0.00 | | |
| 3F10.3 | 0.00 | 0.00 | 1.00 | | |
| | 1.00 | 0.00 | 0.00 | | |

SUMMARY OF THE RESULTS:

The position vectors of the origins of the coordinate frames x*(1), x(1), x*(2), x(2), etc. are given as follows:

$$\underline{X}^*(1) = [0.00 \quad 0.00 \quad 0.00]^T$$

$$\underline{X}(1) = [0.00 \quad 0.00 \quad 0.00]^T$$

$$\underline{X}^*(2) = [0.00 \quad 0.00 \quad 0.00]^T$$

$$\underline{X}(2) = [0.065 \quad 0.00 \quad 0.369]^T$$

$\underline{X}^*(3) = [-11.95 \quad 0.00 \quad 2.488]^T$

$\underline{X}(3) = [-11.90 \quad 0.241 \quad 2.771]^T$

$\underline{X}^*(4) = [-11.90 \quad 0.241 \quad 2.771]^T$

$\underline{X}(4) = [-11.90 \quad 0.241 \quad 2.771]^T$

$\underline{X}^*(5) = [-21.255 \quad 0.241 \quad 4.42]^T$

$\underline{X}(5) = [-21.255 \quad 0.241 \quad 4.42]^T$

$\underline{X}^*(6) = [-21.255 \quad 0.241 \quad 4.42]^T$

$\underline{X}(6) = [-24.315 \quad 5.273 \quad 4.059]^T$

$\underline{X}^*(7) = [-24.315 \quad 5.273 \quad 4.059]^T$

$\underline{X}(7) = [-24.833 \quad 6.126 \quad 3.998]^T$

The unit vectors along the joint axes are given as

$\underline{U}(1) = [0.00 \quad 0.00 \quad 1.00]^T$

$\underline{U}(2) = [0.00 \quad -1.00 \quad 0.00]^T$

$\underline{U}(3) = [-0.985 \quad 0.00 \quad 0.174]^T$

$\underline{U}(4) = [0.112 \quad -0.766 \quad 0.633]^T$

$\underline{U}(5) = [-0.985 \quad 0.00 \quad 0.174]^T$

$\underline{U}(6) = [0.171 \quad 0.174 \quad 0.970]^T$

$\underline{U}(7) = [0.171 \quad 0.174 \quad 0.970]^T$

The normal vectors to the joint axes from the origin are given as

$$\underline{\rho_n}(1) = [0.00 \quad 0.00 \quad 0.00]^T$$

$$\underline{\rho_n}(2) = [0.00 \quad 0.00 \quad 0.00]^T$$

$$\underline{\rho_n}(3) = [0.65 \quad 0.00 \quad 0.369]^T$$

$$\underline{\rho_n}(4) = [-11.927 \quad 0.426 \quad 2.618]^T$$

$$\underline{\rho_n}(5) = [0.115 \quad 0.241 \quad 0.652]^T$$

$$\underline{\rho_n}(6) = [-21.374 \quad 0.121 \quad 3.747]^T$$

$$\underline{\rho_n}(7) = [-24.433 \quad 5.152 \quad 3.386]^T$$

3) MS = 3

The results obtained in the illustrations 1 and 2 can be obtained in a single run of the computer with MS = 3. As the input and output data is already given in the illustrations 1 and 2, we have given here, in Table A2-4, only the data card arrangement.

TABLE A2-4: The Data Arrangement for Illustration 3.

| | | | | |
|---|---|---|---|---|
| SET ONE | 3 | 3 | 0 | 0 |
| SET TWO | same as in Table A2-1 | | | |
| SET FOUR | same as in Table A2-3 | | | |
| SET FIVE | same as in Table A2-3 | | | |

4) MS = 0

This does the displacement analysis of type II explained in Chapter IV.

The initial configuration of the manipulator is given in Table A2-2.

The seventh joint information given in the table defines the initial position of the manipulator hand.

The initial position of the manipulator hand can also be given by position vectors of three points on it:

$$\underline{a}_1 = [-24.315 \quad 5.273 \quad 4.059]^T$$

$$\underline{b}_1 = [-24.144 \quad 5.447 \quad 5.029]^T$$

$$\underline{c}_1 = [-24.833 \quad 6.126 \quad 3.998]^T$$

Final position of the manipulator hand is given by

$\underline{a}_2 = [-20.30 \quad -7.83 \quad -6.66]^T$

$\underline{b}_2 = [-20.65 \quad -9.29 \quad -5.85]^T$

$\underline{c}_2 = [-21.06 \quad -7.18 \quad -6.62]^T$

Data cards were arranged as shown in Table A2-5.

TABLE A2-5: The Data Arrangement For Illustration 4.

| | | | | |
|---|---|---|---|---|
| SET ONE 4I5 | 0 | 2 | 0 | 0 |
| SET TWO 3F10.2 | 0.00 | 0.00 | 0.00 | |
| | 0.00 | 0.00 | 0.00 | |
| | 0.00 | 0.00 | 0.00 | |
| | -20.30 | -7.83 | -6.66 | |
| | -20.65 | -8.29 | -5.85 | |
| | -21.06 | -7.18 | -6.62 | |
| SET FOUR | same as that in Table A2-3 | | | |
| SET FIVE | same as that in Table A2-3 | | | |

SUMMARY OF THE RESULTS:

The joint variable changes are given in Table A2-6.

The computed final position of the manipulator hand is

$\underline{a}_2$ = [-20.30 -7.83 -6.65]$^T$

$\underline{b}_2$ = [-20.65 -8.29 -5.83]$^T$

$\underline{c}_2$ = [-21.06 -7.18 -6.61]$^T$

TABLE A2-6: The Joint Variable Changes in Illustration 4.

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|---|---|---|---|---|---|---|
| Initial Position | 70.000 | 80.000 | 40.000 | 0.000 | 60.000 | 30.000 |
| | Singularity encountered A new configuration assumed. | | | | | |
| | 67.135 | 82.865 | 54.324 | 14.324 | 45.676 | 15.676 |
| | 68.106 | 83.55 | 61.919 | 17.688 | 36.391 | 12.618 |
| | 69.611 | 84.265 | 63.399 | 20.212 | 33.161 | 10.420 |
| | 71.373 | 85.001 | 63.108 | 22.239 | 31.714 | 8.608 |
| | 73.303 | 85.772 | 61.990 | 23.914 | 31.133 | 7.039 |
| | 75.358 | 86.589 | 60.404 | 25.317 | 31.073 | 5.666 |
| | 77.512 | 87.459 | 58.524 | 26.499 | 31.371 | 4.450 |
| | 79.746 | 88.388 | 56.445 | 27.495 | 31.941 | 3.37 |
| | Total 22 steps | | | | | |
| Final Position | 102.843 | 100.014 | 30.213 | 29.926 | 49.794 | -0.034 |

5) MS = 4

This does the displacement analysis of type III explained in Chapter IV.

The initial configuration of the manipulator is given in Table A2-7. The seventh joint information given in the table defines the initial position of the manipulator hand.

TABLE A2-7: The Manipulator Configuration #2

| i | $r_i$ | $\alpha_i$ | $a_i$ | $\theta_i$ | JOINT |
|---|---|---|---|---|---|
| 1 | 0.00 | 90.00 | 0.00 | 0.00 | PRISMATIC |
| 2 | 25.00 | 90.00 | 0.00 | 90.00 | PRISMATIC |
| 3 | 8.00 | 0.00 | 0.00 | 180.00 | PRISMATIC |
| 4 | 0.00 | 90.00 | 0.00 | 0.00 | REVOLUTE |
| 5 | 0.00 | 90.00 | 0.00 | 90.00 | REVOLUTE |
| 6 | 10.00 | 90.00 | 0.00 | 0.00 | REVOLUTE |
| 7 | 0.00 | 0.00 | 1.00 | 0.00 | --- |

The initial position of the manipulator hand can also be given by position vectors of three points on it:

$$\underline{a}_1 = [15.00 \quad -25.00 \quad 12.00]^T$$

$$\underline{b}_1 = [15.00 \quad -24.00 \quad 12.00]^T$$

$$\underline{c}_1 = [15.00 \quad -25.00 \quad 11.00]^T$$

Final position of the manipulator hand is given by

$\underline{a}_2 = [10.00 \quad -10.00 \quad 20.00]^T$

$\underline{b}_2 = [15.00 \quad -24.00 \quad 12.00]^T$

$\underline{c}_2 = [10.00 \quad -10.00 \quad 19.00]^T$

Data cards were arranged as shown in Table A2-8.

TABLE A2-8: The Data Arrangement for Illustration 5.

| | | | | | |
|---|---|---|---|---|---|
| SET ONE 4I5 | 4 | 2 | 0 | 0 | |
| | 0.00 | 0.00 | 0.00 | | |
| | 0.00 | 0.00 | 0.00 | | |
| SET TWO 3F10.2 | 0.00 | 0.00 | 0.00 | | |
| | 10.00 | -10.00 | 20.00 | | |
| | 11.00 | -10.00 | 20.00 | | |
| | 10.00 | -10.00 | 19.00 | | |
| 1I5 | 7 | | | | |
| | 0.00 | 90.00 | 0.00 | 0.00 | 999999.00 |
| SET FOUR 5F10.3 | 25.00 | 90.00 | 0.00 | 90.00 | 999999.00 |
| | 8.00 | 0.00 | 0.00 | 180.00 | 999999.00 |
| | 0.00 | 90.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 90.00 | 0.00 | 90.00 | 0.00 |
| | 10.00 | 90.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | |
| | 25.00 | 0.00 | 20.00 | | |
| SET FIVE 3F10.3 | 30.00 | 0.00 | 20.00 | | |
| | 25.00 | 0.00 | 10.00 | | |

SUMMARY OF THE RESULTS:

The joint variable changes are given in Table A2-9.

The computed final position of the manipulator hand was found to coincide with the desired final position of the manipulator hand with three decimal accuracy.

TABLE A2-9: The Joint Variable Changes in Illustration 5.

| | $r_1$ | $r_2$ | $r_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|---|---|---|---|---|---|---|
| Initial Position | 0.000 | 25.000 | 8.000 | 0.000 | 90.000 | 0.000 |
| | -0.225 | 26.496 | 7.704 | 10.000 | 90.000 | 0.000 |
| | -0.948 | 27.775 | 7.407 | 21.801 | 90.000 | 0.000 |
| | -1.960 | 28.790 | 7.111 | 32.005 | 90.000 | 0.000 |
| | -3.108 | 29.464 | 6.815 | 41.009 | 90.000 | 0.000 |
| | -4.229 | 29.864 | 6.519 | 48.652 | 90.000 | 0.000 |
| | -5.317 | 29.953 | 6.222 | 55.008 | 90.000 | 0.000 |
| | -6.254 | 29.919 | 5.926 | 60.255 | 90.000 | 0.000 |
| | -7.138 | 29.674 | 5.630 | 64.592 | 90.000 | 0.000 |
| | -7.918 | 29.343 | 5.333 | 68.199 | 90.000 | 0.000 |
| | Total 38 steps | | | | | |
| Final Position | -15.000 | 20.000 | 0.000 | 90.000 | 90.000 | 0.000 |

6) MS = 5

This does the displacement analysis of type IV explained in Chapter IV.

The initial configuration of the manipulator is given in Table A2-7. The seventh joint information in the table defines the initial position of the manipulator hand. The initial position of the manipulator hand can also be given by position vectors of three points on it:

$$\underline{a}_1 = [15.00 \quad -25.00 \quad 12.00]^T$$

$$\underline{b}_1 = [15.00 \quad -24.00 \quad 12.00]^T$$

$$\underline{c}_1 = [15.00 \quad -25.00 \quad 11.00]^T$$

i) <u>Displacement parallel to z-axis in the global coordinate frame</u>.

$$\underline{R} = [0 \quad 0 \quad 1]^T, \qquad d = 4.00, \qquad \theta = -30^o$$

$$\underline{\rho}_o' = [15 \quad -25 \quad 12]^T$$

[For the notation, please see the analysis type IV in Chapter IV.]

From the final computed position of the manipulator continue with

ii) <u>Displacement parallel to x-axis in the coordinate frame fixed on the manipulator hand</u>.

$\underline{R}' = [1 \quad 0 \quad 0]^T, \quad d = -1.00, \quad \theta = 15^o$

Data cards were arranged as shown in Table A2-10.

TABLE A2-10: The Data Arrangement For Illustration 6.

| | | | | | | |
|---|---|---|---|---|---|---|
| SET ONE 4I5 | 5 | 2 | 0 | 0 | | |
| SET THREE (I10,5F10.3) | 0 | 0.00 | 0.00 | 1.00 | 4.00 | -30.00 |
| | 1 | 1.00 | 0.00 | 0.00 | -1.00 | 15.00 |
| SET FOUR | | | same as in Table A2-8 | | | |
| SET FIVE | | | same as in Table A2-8 | | | |

SUMMARY OF THE RESULTS:

The joint variable changes are given in Table 2-11.

At the end of the first state

$\underline{\rho}_0'$ [computed] = $[15.14 \quad -25.07 \quad 16.00]^T$

$\underline{\rho}_0'$ [expected] = $[15.00 \quad -25.00 \quad 16.00]^T$

At the end of the second stage

$\underline{\rho}_0'$ [computed] = $[15.19 \quad -25.12 \quad 17.00]^T$

$\underline{\rho}_0'$ [expected] = $[15.00 \quad -25.00 \quad 17.00]^T$

TABLE A2-11: The Joint Variable Change in Illustration 6.

| | $r_1$ | $r_2$ | $r_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|---|---|---|---|---|---|---|
| Initial Position | 0.00 | 25.00 | 8.00 | 0.00 | 90.00 | 0.00 |
| | 0.00 | 25.58 | 7.56 | 3.33 | 90.00 | 0.00 |
| | -0.03 | 26.16 | 7.11 | 6.67 | 90.00 | 0.00 |
| | -0.10 | 26.74 | 6.67 | 10.00 | 90.00 | 0.00 |
| | | | Total 9 steps | | | |
| End of Stage 1 | -1.20 | 30.07 | 4.00 | 30.00 | 90.00 | 0.00 |
| | -1.53 | 30.64 | 3.75 | 33.75 | 90.00 | 0.00 |
| | -1.90 | 31.18 | 3.50 | 37.50 | 90.00 | 0.00 |
| | -2.29 | 31.70 | 3.25 | 41.25 | 90.00 | 0.00 |
| End of Stage 2 | -2.73 | 32.20 | 3.00 | 45.00 | 90.00 | 0.00 |

7) MS = 6

This does the displacement analysis of type I explained in Chapter IV.

The initial configuration of the manipulator is given in Table A2-7.

The seventh joint information given in the table defines the initial position of the manipulator hand.

The initial position of the manipulator hand can also be given by position vectors of three points on it:

$\underline{a}_1$ = [15.00 -25.00 12.00]$^T$

$\underline{b}_1$ = [15.00 -24.00 12.00]$^T$

$\underline{c}_1$ = [15.00 -25.00 11.00]$^T$

Final position of the manipulator hand is given by

$\underline{a}_2$ = [10.00 -10.00 20.00]$^T$

$\underline{b}_2$ = [11.00 -10.00 20.00]$^T$

$\underline{c}_2$ = [10.00 -10.00 19.00]$^T$

Data cards were arranged as shown in Table A2-12.

TABLE A2-12: The Data Arrangement For Illustration 7.

| | | | | |
|---|---|---|---|---|
| SET ONE<br>4I5 | 6 | 0 | 0 | 0 |
| SET TWO | Same as in Table A2-8 | | | |
| SET FOUR | Same as in Table A2-8 | | | |
| SET FIVE | Same as in Table A2-8 | | | |

SUMMARY OF THE RESULTS:

The joint variable changes are given in Table A2-13.

The computed final position of the manipulator hand is given by:

$$\underline{a}_2 = [10.00 \quad -10.01 \quad 20.00]^T$$

$$\underline{b}_2 = [11.00 \quad -10.01 \quad 20.00]^T$$

$$\underline{c}_2 = [10.00 \quad -10.01 \quad 19.00]^T$$

TABLE A2-13: The Joint Variable Changes in Illustration 7.

| | $r_1$ | $r_2$ | $r_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|---|---|---|
| Initial Position | 0.000 | 25.000 | 8.000 | 0.000 | 90.000 | 0.000 |
| | -0.143 | 25.020 | 7.771 | 2.571 | 90.000 | 0.000 |
| | -0.306 | 25.040 | 7.543 | 5.143 | 90.000 | 0.000 |
| | -0.490 | 25.058 | 7.314 | 7.714 | 90.000 | 0.000 |
| | -0.694 | 25.075 | 7.086 | 10.286 | 90.000 | 0.000 |
| | -0.918 | 25.087 | 6.857 | 12.857 | 90.000 | 0.000 |
| | -1.162 | 25.096 | 6.629 | 15.429 | 90.000 | 0.000 |
| | -1.427 | 25.100 | 6.400 | 18.000 | 90.000 | 0.000 |
| | -1.710 | 25.098 | 6.171 | 20.571 | 90.000 | 0.000 |
| | -2.013 | 25.089 | 5.943 | 23.143 | 90.000 | 0.000 |
| | -2.336 | 25.073 | 5.714 | 25.714 | 90.000 | 0.000 |
| | Total number of steps = 35 | | | | | |
| Final Position | -15.000 | 20.010 | 0.000 | 90.000 | 90.000 | 0.000 |

## APPENDIX III

### The Program Usage

The following control cards are necessary for executing DISP. on Univac 1108 at the University of Houston.

```
@RUN   SJN,______, ______, S25, 100.
@ASG   DCT*PROGRAMS.
@UNPKG   DCT*PROGRAMS.MATHSTAT,TPF$.
@FOR   LSIMEQ
@FOR   MXMLT
@FOR   MXSUB
@FOR   GJR
@ASG,A   DISP.
@MAP
IN DISP.
IN .LSIMEQ,.MXMLT,.MXSUB,.GJR
@XQT
Data Cards
@FIN
```