INVESTIGATION OF A GENERAL TECHNIQUE TO SOLVE ELECTROMAGNETIC WAVE PROPAGATION AND RADIATION PROBLEMS IN BOUNDED MAGNETOPLASMA

A Thesis

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

by

Richard Fong Chew, Jr.

February 1967

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ABSTRACT

Electromagnetic wave propagation and transmission in a bounded magnetoplasma with an arbitrary orientation of the imposed magnetic field has been investigated by solving the resulting wave equation using the separation method. For a given orientation class of the magnetic field, the wave equation was found to be solvable by the separation method in at least two but not more than four of the cylindrical coordinate systems. Solutions of the electric field can be obtained in the form of three scalar functions each composed of a complete set of eigenfunctions. Investigation has also been carried out for problems involving source radiations in the bounded magnetoplasma. Solutions to the inhomogeneous wave equation by the Green's function method are derived. The usual complex surface integral arising from the Green's function method was forced to vanish by demanding the Green's function to satisfy a given set of boundary conditions. It will be shown that the technique outlined in this thesis is adequate to solve problems involving wave propagations or radiations in a bounded magnetoplasma, providing that a dispersion relation for the given orientation of the magnetic field can be found.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In the course of the investigation of electromagnetic wave interaction with shock generated plasma imbedded in a strong magnetic field in a shock tube configuration, very little analysis was found on electromagnetic waves in anisotropic plasma with certain degree of freedom in the orientation of its imposed magnetic field. Analysis is lacking even in the case of stationary plasma. The purpose of this thesis is to investigate solutions of the wave equation in a finite domain, stationary, homogeneous, anisotropic plasma with the static magnetic field expressed in terms of a vector which has constant components in each of the orthogonal directions. The knowledge gained in this work will be applicable to shock tube investigation. Since finite domain problems are under consideration, it is desirable that the solutions of the wave equation be separated into three vectorial components which fit the boundary configuration.

The three separated vectorial functions are then constructed from some vectorial operations on three scalar functions, each of which satisfies the scalar Helmholtz equation. The three vectorial functions so obtained are seen to be either perpendicular or parallel to a surface described by the coordinate which has unity scale factor. A substitution of these functions into the wave equation yields the dispersion relation.

The wave in anisotropic plasma depends on the orientation of the magnetic fields. Under certain restrictions on the functional form of the static magnetic field, a case of arbitrarily oriented static magnetic field has been analyzed here. The technique presented here may be used for cases involving other magnetic field orientations under the same restrictions. The selection of the coordinate system is dictated by the configuration of the boundaries; therefore, the exact number of coordinate systems in which the wave equation can be solved by the separation method is determined by the ease in obtaining the dispersion relation.

Whenever a radiation source is present in a bounded anisotropic plasma, the wave field is calculated by two integrals. Each involves the Green's function. One of the integrals is a surface integral. Evaluation of this surface integral is difficult as it requires the knowledge of the surface charge density and the surface current density which are not usually given in the statement of the problem. This surface integral can be made to vanish if the Green's function satisfies the same boundary condition as the wave field. The wave field may be the electric or the magnetic field. The second integral is a volume integral involving the volume source distribution function and the Green's function. For a vanishing surface integral, evaluation of the volume integral suffices in order to obtain the wave field. The general solutions of the homogeneous wave equation must satisfy the boundaty conditions; therefore, its appropriate Green's function can be constructed from the general solution of the homogeneous wave equation. Since the general solution of the homogeneous wave equation is obtained by the separation method, the solutions are in the form of eigenfunctions. The properties of such eigenfunctions greatly facilitate the construction of the Green's function. The Green's function is found to be a dyadic and is reciprocal with respect to the source and the observer coordinates. The homogeneous wave equation is composed of purely transverse terms and therefore the Green's function so constructed satisfies only a pure transverse source function. If the source contains a longitudinal component, an additional longitudinal term of the Green's function can be derived from an algebraic equation (Appendix C).

The method described here has been used for a fixed magnetic field oriented in the direction of the unity scale factor coordinate (Seto & Dougal, 1964). There has been little work of this nature done (see Bibliography).

CHAPTER II

HOMOGENEOUS WAVE EQUATION

The wave equation obtained from Maxwell's equations for the electric field intensity $\bar{E}$ in a bounded, homogeneous, uniform magnetoplasma volume without a source present is

$$\nabla\times\nabla\times\bar{E} - \hat{\hat{k}}\cdot\bar{E} = 0 \tag{2.1}$$

where $\hat{\hat{k}}$ is a dyadic determined by the medium (See Appendix A). Equation (2.1) can be expressed in the following form including the orientation of the magnetic field as

$$\nabla\times\nabla\times\bar{E} - k_{\perp}\bar{E} - (k_{\parallel} - k_{\perp})(\bar{E}\cdot\bar{i}_b)\bar{i}_b - k_T(\bar{E}\times\bar{i}_b) = 0 \tag{2.2}$$

where $\bar{i}_b$ is the unit vector in the direction of the static magnetic field, and $k_{\perp}$, k_T, and $k_{\parallel}$ are entries in $\hat{\hat{k}}$.

In general the direction of the static magnetic field can be written in a generalized coordinate system as

$$\bar{i}_b = A_1\bar{\epsilon}_1 + A_2\bar{\epsilon}_2 + A_3\bar{\epsilon}_3 \tag{2.3}$$

where $\bar{\epsilon}_i$, $i = 1, 2, 3$ are the three right handed orthogonal unit vectors in the generalized coordinate system. The coefficients A_j, $j = 1, 2, 3$ are in general functions of the coordinates.

In this work only $\bar{i}_b$ with constant A_j was studied. This restriction on A_j may seem to be too severe, but only the cases where $A_i \neq 0$, $A_j = A_k = 0$ where $i = 1, 2, 3$ and j, k permutated have been discussed in the literature (Allis, Buchsbaum, & Bers, 1963).

A different orientation of the static magnetic field yields different forms of the dyadic $\hat{\hat{k}}$. In the past, different techniques were required to solve wave equations with different forms of $\hat{\hat{k}}$ (Allis, Buchsbaum, & Bers, 1963). An attempt to derive a uniform technique to solve the wave equation of different static field orientations with constant A_j, $j = 1, 2, 3$ is made here.

For infinite domains, solutions can be expressed in terms of plane waves with $e^{i\bar{k}\cdot\bar{r}}$ space variations; but in a finite domain problem, boundary conditions play an important role in determining the closed form of the solutions, which is more difficult to calculate. Since the coordinate system employed is dictated by the boundary configurations of the problem and since the solvability of the wave equation by the separation method is dictated by the coordinate system employed, it is advisable to investigate the number of coordinate systems in which the wave equation is separable. The first term in Eq. (2.2) is the curl of curl and is only separable in six coordinate systems, all of which have a unity scale factor for one coordinate axis. The ratio of the remaining scale factors are independent of the variable of the unity scale factor. The six coordinate systems are the four cylindrical, the spherical, and the conical coordinate systems (Morse & Feshbach, 1953). It is anticipated that the number of coordinate systems in which Eq. (2.2) is solvable may be less than six since the second to the fourth terms in Eq. (2.2) may impose additional restrictions on the separation of the wave equation. Following this technique to separate $\nabla\times\nabla\times\bar{E}$, the solutions of Eq. (2.2) may be separated into three vectors (Morse & Feshbach, 1953).

$$\bar{E} = \bar{L} + \bar{M} + \bar{N} \tag{2.4}$$

with
$$\bar{L} = \nabla\varphi$$
$$\bar{M} = \nabla\times\psi\bar{a}_3$$
$$\bar{N} = \nabla\times\nabla\times\chi\bar{a}_3$$

where φ , ψ , and χ are scalar functions to be determined; and $\bar{a}_3$ is the unit vector along the coordinate axis which has unity scale factor. The separation of $\bar{E}$ into $\bar{L}$, $\bar{M}$, and $\bar{N}$ does facilitate the fitting of the class of boundaries $\bar{S} = \sum_{\ell} \bar{S}_{\ell}$ such that at least one of the $\bar{S}_{\ell}$'s is $\bar{S}_{\ell}\times\bar{a}_3 = 0$. In matrix form, Eq. (2.4) becomes

$$\bar{E} = \begin{bmatrix} \nabla_{\perp}\psi\times\bar{a}_3 \\ \nabla_{\perp}(\nabla_{\parallel}\cdot\bar{a}_3\chi - \varphi) \\ -\bar{a}_3(\nabla_{\perp}^2\chi - \nabla_{\parallel}\cdot\bar{a}_3\varphi) \end{bmatrix} = \begin{bmatrix} \nabla_{\perp}\psi\times\bar{a}_3 \\ \nabla_{\perp} A \\ -\bar{a}_3 B \end{bmatrix} \tag{2.5}$$

where $A = \nabla_{\parallel} \cdot \bar{a}_3 \, \chi - \varphi$, and

$B = \nabla_{\perp}^2 \chi - \nabla_{\parallel} \cdot \bar{a}_3 \, \varphi$.

The elements of Eq. (2.5) are mutually orthogonal vectors with

$$\bar{\epsilon}_1 = \frac{\nabla_{\perp} \psi \times \bar{a}_3}{|\nabla_{\perp} \psi|}$$

$$\bar{\epsilon}_2 = \frac{\nabla_{\perp} A}{|\nabla_{\perp} A|}$$

$$\bar{\epsilon}_3 = \bar{a}_3$$

The separability of Eq. (2.2) for a given is therefore determined by the solvability of the equation resulting from the substitution of Eq. (2.5) into Eq. (2.2). As an example, take the case of the unit vector expressed as

$$\bar{1}_b = \cos \alpha_{\parallel} \, \bar{a}_3 + \cos \alpha_{\perp} \frac{\nabla_{\perp} f}{|\nabla_{\perp} f|} + \cos \alpha_{\times} \frac{\nabla_{\perp} f \times \bar{a}_3}{|\nabla_{\perp} f \times \bar{a}_3|} \tag{2.6}$$

where f is either ψ , φ , χ , A , or B ; and the cosines are directional cosines and are assumed to be constant. Substitution of Eq. (2.5) into Eq. (2.2) yields three orthogonal equations.

$$\left[-T_m^2 - k_{\perp} - (k_{\parallel} - k_{\perp}) \cos^2 \alpha_{\times}\right] \nabla_{\perp} \psi \cdot \frac{\nabla_{\perp} f}{|\nabla_{\perp} f|} + \left[-k_T \cos \alpha_{\parallel} - (k_{\parallel} - k_{\perp}) \cos \alpha_{\perp} \cos \alpha_{\times}\right] \nabla_{\perp} A \cdot \frac{\nabla_{\perp} f}{|\nabla_{\perp} f|}$$
$$+ \left[-k_T \cos \alpha_{\perp} + (k_{\parallel} - k_{\perp}) \cos \alpha_{\times} \cos \alpha_{\parallel}\right] B = 0 \tag{2.7}$$

$$\left[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x\right] \nabla_\perp\psi \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} + \left[k_m^2 - k_\perp - (k_\parallel - k_\perp)\cos^2\alpha_\perp\right] \nabla_\perp A \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} +$$
$$\left[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel\right] B = 0 \qquad (2.8)$$

$$\left[-k_T \cos\alpha_\perp - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x\right] \nabla_\perp\psi \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} + \left[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_\perp\right] \nabla_\perp A \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} +$$
$$\left[-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel\right] B = 0 \qquad (2.9)$$

In Eq. (2.7) to Eq. (2.9), the following arguments have been employed. It is desirable that the solutions of $|\nabla_\perp\psi|$, $|\nabla_\perp A|$, and B be in the forms that satisfy the scalar Helmholtz equations such that

$$\nabla^2(|\nabla_\perp\psi|, |\nabla_\perp A|, B) = T_m^2 (|\nabla_\perp\psi|, |\nabla_\perp A|, B) \qquad (2.10)$$

$$\nabla_\perp^2(|\nabla_\perp\psi|, |\nabla_\perp A|, B) = K_m^2 (|\nabla_\perp\psi|, |\nabla_\perp A|, B) \qquad (2.11)$$

$$\nabla_\parallel^2(|\nabla_\perp\psi|, |\nabla_\perp A|, B) = k_m^2 (|\nabla_\perp\psi|, |\nabla_\perp A|, B) \qquad (2.12)$$

$$|\nabla_\perp \nabla_\parallel \cdot \vec{a}_3 (|\nabla_\perp A|, B)| = K_m k_m (|\nabla_\perp A), B) \qquad (2.13)$$

$$T_m^2 = K_m^2 + k_m^2 \qquad (2.14)$$

Equations (2.10) to (2.14) imply that the scalar function be solved by the separation of variable method, and that their solutions are the desired eigenfunction forms subject to the boundary conditions. The subscript m in T_m^2 , K_m^2 , and k_m^2 is an index denoting that these separation constants are eigenvalues. They are subject to the indexed values resulting from the boundary requirements. Observing that Eq. (2.13) demands $\nabla^2\nabla_\perp(A,\Psi) = \nabla_\perp\nabla^2(A,\Psi)$, and one has replaced $-T_m^2$ for ∇^2 , $-K_m^2$ for $\nabla_\perp^2$, $-k_m^2$ for $\nabla_\parallel^2$, and $-K_m k_m$ for $|\nabla_\perp \nabla_\parallel|$ whenever desired to obtain Eq. (2.7) through Eq. (2.9).

For a nontrivial solution of Eq. (2.7) through Eq. (2.9), the determinant of the coefficients must vanish; thus

$$\begin{vmatrix} -T_m^2 - k_\perp - (k_\parallel - k_\perp)\cos^2\alpha_x & -k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_x & -k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\parallel \\ k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_x & k_m^2 - k_\perp - (k_\parallel - k_\perp)\cos^2\alpha_\perp & K_m k_m + k_T\cos\alpha_x \;(k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel \\ -k_T\cos\alpha_\perp + -(k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x & -k_m K_m + k_T\cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_\perp & -K_m^2 + k_\perp \;(k_\parallel - k_\perp)\cos^2\alpha_\parallel \end{vmatrix} = 0$$

Using Eq. (2.14), the resulting dispersion relation is

$$T_m^8(-a_1) + k_m^4(a_4^2 + a_5^2) - k_m^2(k_m^2[a_1^2 - 2a_4a_5] - 2a_4a_8 + +2a_5a_8) +$$

$$T_m^6\left(k_m^2[2a_1^2 + a_6^2 - 2a_1a_2] - 2a_1[a_3 + a_4]\right) + 2k_m^2 a_4^2 - a_8^2 + k_m^2 a_3^2$$
$$T_m^4\left(-k_m^4 a_1^2 - k_m^2[k_m^2 a_6^2 - 2k_m^2 a_1a_2 - 2a_1a_4] + 2k_m^2 a_2a_3 - a_2^2 k_m^4\right.$$
$$\left. - a_3^2 + 4k_m^2 a_1a_4 + 2k_m^2[a_6a_7 - a_1a_5 - a_2a_4] + 2a_4a_3 - a_4^2\right) + 2k_m^2 a_4a_5$$
$$T_m^2\left(-k_m^4 2a_1a_4 - 2k_m^4 a_2a_4 - 2a_4a_8 + k_m^2[2k_m^2\{a_6a_7 - a_1a_5 - a_2a_4\} + 2a_4a_3 + 2k_m^2[a_3a_5 - a_2a_8]\}\right) = 0 \quad (2.16)$$

where

$$a_1 = b_1$$
$$a_2 = b_2$$
$$a_3 = b_6b_7 + b_1b_2$$
$$a_4 = -b_1b_3 - b_4b_5$$
$$a_5 = -b_2b_3 + b_6^2$$
$$a_6 = b_6 - b_7$$
$$a_7 = b_4b_6 + b_7b_5 + (b_7 - b_6)b_3$$
$$a_8 = b_8b_4b_6 - b_9b_7b_5 - b_1b_6^2 + b_2b_5b_4 + b_1b_2b_3 + b_3b_8b_9$$

and,

$$b_1 = k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp$$
$$b_2 = k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel$$
$$b_3 = k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x$$
$$b_4 = k_T\cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\perp$$
$$b_5 = k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\perp$$
$$b_6 = k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x$$
$$b_7 = k_T\cos\alpha_\perp - (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x$$
$$b_8 = k_T\cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel$$
$$b_9 = k_T\cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel$$

As seen, the dispersion relation is a very complicated expression. As long as this determinant is solvable, the separation of $\bar{E}$ in terms of φ , χ , and ψ which are in turn solutions of the Helmholtz equation is possible.

The solutions of ψ , χ , and ψ represent an infinite series of orthogonal eigenfunctions which have the coefficients to be determined, and the combinations $\nabla_\perp \psi \times \bar{a}_3$, $\nabla_\perp (\nabla_\parallel \cdot \bar{a}_3 \chi - \varphi)$, and $\bar{a}_3 (\nabla_\perp^2 \chi + \nabla_\parallel \cdot \bar{a}_3 \varphi)$ each form a complete set so that so obtained is also a complete set (Seto & Dougle, 1964). Because of the curl of curl in the wave equation, the solution is only valid in six coordinate systems due to separability. If only physically realizable magnetic fields are considered, the solution is only valid in the four cylindrical coordinate systems.

This technique is general enough to be applicable to problems where their magnetic field direction can be expressed as a constant vector in a coordinate system.

CHAPTER III

INHOMOGENEOUS WAVE EQUATION

The solutions obtained through the general technique outlined in the last chapter are the free-wave solutions. When radiation sources are present in the bounded magneto-plasma volume, the wave excited by the radiation sources will be only a part of the free wave. Mathematically, the wave equation with sources present is inhomogeneous (Appendix A).

$$\nabla \times \nabla \times \bar{E} - \hat{\hat{k}} \cdot \bar{E} = \bar{J}_s \tag{3.1}$$

One approach in obtaining a solution for Eq. (3.1) is by the Green's function method. By this method, the wave field is

$$\bar{E} = \int_{VOL} \bar{\bar{G}} \cdot \bar{J}_S \, dv \tag{3.2}$$

where $\bar{\bar{G}}$ is the Green's function and is in dyadic form (Appendix B).

The equation which the Green's function must satisfy is

$$\nabla \times \nabla \times \bar{\bar{G}} - k_\perp \bar{\bar{G}} - (k_\parallel - k_\perp)(\bar{\bar{G}} \cdot \bar{i}_b)\bar{i}_b + k_T \bar{\bar{G}} \times \bar{i}_b = \bar{\bar{I}} \, \delta(\bar{r} - \bar{r}_o) \tag{3.3}$$

where $\bar{\bar{I}}$ is the idemefactor. It is noted that this equation is not identical to the wave equation but differs in the source function and the constants (Appendix B). If $\bar{\bar{G}}$ satisfies Eq. (3.4) as well as the same boundary conditions $\bar{E}$ satisfies, then Eq. (3.2) suffices in evaluating $\bar{E}$. On the other hand, if $\bar{\bar{G}}$ does not satisfy the same boundary conditions $\bar{E}$ satisfies, then an additional surface integral must be evaluated.

Another property of the Green's function is the reciprocity with respect to the source coordinate $\bar{r}_0$ and the observer coordinate $\bar{r}$. This is the direct result from the reciprocity theorem.

One method of obtaining a Green's function that satisfies the same boundary conditions $\bar{E}$ satisfies is to construct it from the eigenfunction solutions of the free-wave. The eigenfunctions are complete sets, are solutions of the homogeneous wave equation, and have orthogonal characteristics which facilitates the construction of the Green's function.

Taking the exemplified case of Chapter II with the assumed $\bar{J}_b$, the Green's function may be assumed to be

$$\bar{\bar{G}} = \bar{\bar{G}}_L + \bar{\bar{G}}_N + \bar{\bar{G}}_M \tag{3.4}$$

with

$$\bar{\bar{G}}_L = \sum \nabla_\perp \Psi(\bar{r}) \times \hat{a}_3 \, \bar{F}(\bar{r}_0) \tag{3.5}$$

$$\bar{\bar{G}}_M = \sum \nabla_\perp A(\bar{r}) \, \bar{G}(\bar{r}_0) \tag{3.6}$$

$$\bar{\bar{G}}_N = \sum \hat{a}_3 \, B(\bar{r}) \, \bar{H}(\bar{r}_0) \tag{3.7}$$

where $\bar{F}(\bar{r}_0)$, $\bar{G}(\bar{r}_0)$, and $\bar{H}(\bar{r}_0)$ are the source coordinate functions (Seto & Dougal, 1964). Substituting Eq. (3.4) into Eq. (3.3) and demanding that $\bar{F}$, $\bar{H}$, and $\bar{G}$ take the form that would satisfy Eq. (3.3). This results in a set of three dyadic equations which have the following form since satisfy the same equations and boundary conditions:

$$- \nabla_\perp \times \nabla^2 \psi \bar{a}_3 \bar{F} - k_\perp \nabla \psi \times \bar{a}_3 \bar{F} + k_T \left[-\cos\alpha_\perp \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f|} B\bar{H} - \cos \nabla_\perp A \times \bar{a}_3 \bar{G} \right] + \\ - [k_\parallel - k_\perp] \cos\alpha_x \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f|} \left[\nabla_\perp \psi \bar{F} \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \cos\alpha_x + \nabla_\perp A \bar{G} \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \cos\alpha_\perp + \\ - B\bar{H} \cos\alpha_\parallel \right] = \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} \delta(\bar{r} - \bar{r}_0) \tag{3.8}$$

$$- \nabla_\perp \nabla_\parallel \cdot \bar{a}_3 B\bar{H} - \nabla_\perp \nabla_\parallel^2 A\bar{G} - k_\perp \nabla_\perp A \bar{G} + k_T \left[\cos\alpha_x B \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{H} + \nabla_\perp \psi \bar{F} \cos\alpha_\parallel \right] + \\ - [k_\parallel - k_\perp] \cos\alpha_\perp \frac{\nabla_\perp f}{|\nabla_\perp f|} \left[\nabla_\perp \psi \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{F} \cos\alpha_x + \nabla_\perp A \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{G} \cos\alpha_\perp + \\ - B\bar{H} \cos\alpha_\parallel \right] = \frac{\nabla_\perp f}{|\nabla_\perp f|} \frac{\nabla_\perp f}{|\nabla_\perp f|} \delta(\bar{r} - \bar{r}_0) \tag{3.9}$$

$$\nabla_\perp^2 B \bar{a}_3 \bar{H} + \nabla_\perp^2 \nabla_\parallel A \bar{G} + k_\perp B \bar{a}_3 \bar{H} - k_T \left[\nabla_\perp \psi \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \cos\alpha_\perp + \nabla_\perp A \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{G} \cos\alpha_\perp \cos\alpha_x \right] + \\ - [k_\parallel - k_\perp] \cos\alpha \bar{a}_3 \left[\nabla_\perp \psi \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{F} \cos\alpha_x + \nabla_\perp A \cdot \frac{\nabla_\perp f}{|\nabla_\perp f|} \bar{G} \cos\alpha_\perp - B\bar{H} \cos\alpha_\parallel \right] = \\ - \bar{a}_3 \bar{a}_3 \delta(\bar{r} - \bar{r}_0) \tag{3.10}$$

Since $\bar{F}$, $\bar{G}$, and $\bar{H}$ may have components in all three orthogonal directions, they can be expressed as:

$$\bar{F}_{mn} = F_{mn}^{x} \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} + F_{mn}^{\perp} \frac{\nabla_\perp f}{|\nabla_\perp f|} + F_{mn}^{\parallel} \bar{a}_3 \tag{3.11}$$

$$\bar{G}_{mn} = G_{mn}^{x} \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} + G_{mn}^{\perp} \frac{\nabla_\perp f}{|\nabla_\perp f|} + G_{mn}^{\parallel} \bar{a}_3 \tag{3.12}$$

$$\bar{H}_{mn} = H_{mn}^{x} \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} + H_{mn}^{\perp} \frac{\nabla_\perp f}{|\nabla_\perp f|} + H_{mn}^{\parallel} \bar{a}_3 \tag{3.13}$$

The three equations, Eq. (3.8) through Eq. (3.10), are each reduced to a vector equation by vectorially multiplying each with an appropriate unit vector so that only the constants G^i_{mn} , F^i_{mn} , and H^i_{mn} remain as vectors. After some regrouping, the three equations are:

$$
\begin{aligned}
&\left[T_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x \right] \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} \cdot \nabla_\perp \Psi \times \bar{a}_3 \bar{F} \; + \\
&-\left[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\perp \right] \frac{\nabla_\perp f}{|\nabla_\perp f|} \cdot \nabla_\perp A \bar{G} \; + \\
&\left[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\parallel \right] B \bar{H} = \frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} \; \delta(\bar{r} - \bar{r}_0)
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\left[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x \right] \frac{\nabla_\perp f}{|\nabla_\perp f|} \cdot \nabla_\perp \Psi \bar{F} \; + \\
&-\left[-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp \right] \frac{\nabla_\perp f}{|\nabla_\perp f|} \cdot \nabla_\perp A \bar{G} \; + \\
&\left[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel \right] B \bar{H} = \frac{\nabla_\perp f}{|\nabla_\perp f|} \; \delta(\bar{r} - \bar{r}_0)
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&-\left[k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x \right] \frac{\nabla_\perp f}{|\nabla_\perp f|} \cdot \nabla_\perp \Psi \bar{F} \; + \\
&\left[-k_m K_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_\perp \right] \frac{\nabla_\perp f}{|\nabla_\perp f|} \cdot \nabla_\perp A \bar{G} \; + \\
&\left[-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel \right] B \bar{H} = \bar{a}_3 \; \delta(\bar{r} - \bar{r}_0)
\end{aligned}
\tag{3.16}
$$

Now each of the vector equations has components in the three orthogonal directions and each can be separated by dot product of a vector in each of the orthogonal directions; and this results in nine equations:

$$[T_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x]\,|\nabla_\perp\psi|\,F^x_{mn} f^x_m - [k_T\cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\perp]\,|\nabla_\perp A|\,G^x_{mn} g^x_m + [-k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\parallel]\,B H^x_{mn} h^x_m = \delta(\bar{v} - \bar{v}_0)$$

$$[T_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x]\,|\nabla_\perp\psi|\,F^\perp_{mn} f^\perp_m - [k_T\cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\perp]\,|\nabla_\perp A|\,G^\perp_{mn} g^\perp_m + [-k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\parallel]\,B H^\perp_{mn} h^\perp_m = 0$$

$$[T_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x]\,|\nabla_\perp\psi|\,F^\parallel_{mn} f^\parallel_m - [k_T\cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\perp]\,|\nabla_\perp A|\,G^\parallel_{mn} g^\parallel_m + [-k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x\cos\alpha_\parallel]\,B H^\parallel_{mn} h^\parallel_m = 0$$

$$[k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_x]\,|\nabla_\perp\psi|\,F^x_{mn} f^x_m - [-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp]\,|\nabla_\perp A|\,G^x_{mn} g^x_m + [K_m k_m + k_T\cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel]\,B H^x_{mn} h^x_m = 0$$

$$[k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_x]\,|\nabla_\perp\psi|\,F^\perp_{mn} f^\perp_m - [-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp]\,|\nabla_\perp A|\,G^\perp_{mn} g^\perp_m + [K_m k_m + k_T\cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel]\,B H^\perp_{mn} h^\perp_m = \delta(\bar{v} - \bar{v}_0)$$

$$[k_T\cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_x]\,|\nabla_\perp\psi|\,F^\parallel_{mn} f^\parallel_m - [-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp]\,|\nabla_\perp A|\,G^\parallel_{mn} g^\parallel_m + [K_m k_m + k_T\cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp\cos\alpha_\parallel]\,B H^\parallel_{mn} h^\parallel_m = 0$$

$$-[k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x]\,|\nabla_\perp\psi|\,F^x_{mn} f^x_m + [-k_m K_m + k_T\cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_\perp]\,|\nabla_\perp A|\,G^x_{mn} g^x_m + [-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel]\,B H^x_{mn} h^x_m = 0$$

$$-[k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x]\,|\nabla_\perp\psi|\,F^\perp_{mn} f^\perp_m + [-k_m K_m + k_T\cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_\perp]\,|\nabla_\perp A|\,G^\perp_{mn} g^\perp_m + [-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel]\,B H^\perp_{mn} h^\perp_m = 0$$

$$-[k_T\cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_x]\,|\nabla_\perp\psi|\,F^\parallel_{mn} f^\parallel_m + [-k_m K_m + k_T\cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel\cos\alpha_\perp]\,|\nabla_\perp A|\,G^\parallel_{mn} g^\parallel_m + [-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel]\,B H^\parallel_{mn} h^\parallel_m = 0$$

Equations (3.17)-(3.25)

From Eq. (3.17)-Eq.(3.25), the $F^i_{mn}f_s$'s, $G^i_{mn}g^i_s$'s, and $H^i_{mn}h^i_m$'s are solved by Cramer's method (Appendix D). It is noted that the determinant of the coefficient is exactly the dispersion relation given in Eq. (2.15). The determinant can be written in its factored form:

$$(\nabla^2 + {}_1T^2_m)(\nabla^2 + {}_2T^2_m)(\nabla^2 + {}_3T^2_m)(\nabla^2 + {}_4T^2_m) \tag{3.26}$$

where ${}_iT^2_m$, = 1, 2, 3, 4 are the roots of Eq. (2.15). Since the solution is only valid if the dispersion relation is satisfied, the solution will be valid only in the neighborhood of each root. This means that Eq. (3.26) is of the form

$$(\nabla^2 + {}_1T^2_m)(-{}_1\nabla^2_m + {}_2T^2_m)(-{}_1T^2_m + {}_3T^2_m)(-{}_1T^2_m + {}_4T^2_m) = \eta_i(\nabla^2 + {}_1T^2_m) \tag{3.27}$$

if evaluated in the neighborhood of the ${}_1T^2_m$ root and has a similar form when evaluated in the neighborhood of the other roots. The right hand side of Eq. (3.27) is just one form of the Helmholtz equation with a constant η_i which depends on the root in consideration.

$$\eta_i = (-{}_iT^2_m + {}_jT^2_m)(-{}_iT^2_m + {}_kT^2_m)(-{}_iT^2 + {}_lT^2_m) \tag{3.28}$$

for the i^{th} root. Equation (3.17)-Equation (3.25) leads to the following forms:

$$\sum \eta_i (\nabla^2 + i T_m^2) |\nabla_\perp \Psi| F^{\parallel}_{mn} f^{\parallel}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ \big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \big[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp) \cos\alpha_\perp \cos\alpha_\parallel \big]$$
$$+ \big[-k_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\perp \big] \big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\parallel \big] \Big\} \qquad (3.29)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) |\nabla_\perp A| G^{\parallel}_{mn} g^{\parallel}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ \big[-T_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_x \big] \times$$
$$\big[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp) \cos\alpha_\perp \cos\alpha_\parallel \big] + \big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\parallel \big] \times$$
$$\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp) \cos\alpha_\perp \cos\alpha_x \big] \Big\} \qquad (3.30)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) B H^{\parallel}_{mn} h^{\parallel}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ \big[-T_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_x \big] \times$$
$$\big[-k_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\perp \big] + \big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \times$$
$$\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp) \cos\alpha_\perp \cos\alpha_x \big] \Big\} \qquad (3.31)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) |\nabla_\perp \Psi| F^{\perp}_{mn} f^{\perp}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ \big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \times$$
$$\big[-K_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\parallel \big] - \big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp) \cos\alpha_\parallel \cos\alpha_\perp \big] \times$$
$$\big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\parallel \big] \Big\} \qquad (3.32)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) |\nabla_\perp A| G^{\perp}_{mn} g^{\perp}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ - \big[T_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_x \big] \times$$
$$\big[-K_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\parallel \big] + \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_\parallel \cos\alpha_x \big] \times$$
$$\big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\parallel \big] \Big\} \qquad (3.33)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) B H^{\perp}_{mn} h^{\perp}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ \big[-T_m^2 + k_\perp + (k_\parallel - k_\perp) \cos\alpha_x \big] \times$$
$$\big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp) \cos\alpha_\parallel \cos\alpha_\perp \big] + \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp) \cos\alpha_\parallel \cos\alpha_x \big] \times$$
$$\big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \Big\} \qquad (3.34)$$

$$\sum \eta_i (\nabla^2 + i T_m^2) |\nabla_\perp \Psi| F^{x}_{mn} f^{x}_m = \delta(\bar{r} - \bar{r}_0) \Big\{ - \big[-K_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\perp \big] \times$$
$$\big[-K_m^2 + k_\perp + (k_\parallel - k_\perp) \cos^2\alpha_\parallel \big] - \big[K_m k_m + \cos\alpha_x k_T + (k_\parallel - k_\perp) \cos\alpha_\perp \cos\alpha_\parallel \big] \times$$
$$\big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp) \cos\alpha_\parallel \cos\alpha_x \big] \Big\} \qquad (3.35)$$

$$\Sigma \eta_i (\nabla^2 + {}_iT_m^2)|\nabla_\perp A| G^x_{mn} g^x_m = \delta(\bar{v}-\bar{v}_0)\Big\{\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x\big] \times \big[-K_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel\big] + \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x\big] \times \big[-K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel\big]\Big\} \quad (3.36)$$

$$\Sigma \eta_i (\nabla^2 + {}_iT_m^2) B\, H^x_{mn} h^x_m = \delta(\bar{v}-\bar{v}_0)\Big\{\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x\big] \times \big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_\perp\big] - \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x\big] \times \big[-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp\big]\Big\} \quad (3.37)$$

Now $F^i_{mn} f^i_m$'s, $G^i_{mn} g^i_m$'s, and $H^i_{mn} h^i_m$'s can be determined since the functions $|\nabla_\perp\psi|$, $|\nabla_\perp A|$, and B are orthogonal eigen-functions. Utilizing the orthogonality properties,

$$\int \nabla_\perp\psi \cdot \nabla_\perp\psi^* \, dv = {}_\psi\Lambda^2_{mnl} \quad (3.38)$$

$$\int \nabla_\perp A \cdot \nabla_\perp A^* \, dv = {}_A\Lambda^2_{mnl} \quad (3.39)$$

$$\int B B^* \, dv = {}_B\Lambda^2_{mnl} \quad (3.40)$$

where $*$ indicates complex conjugate and the Λ 's are the normalization factors. $F^i_{mn} f^i_m$'s, $G^i_{mn} g^i_m$'s, and $H^i_{mn} h^i_m$ found for the ${}_iT_m^2$ root to be

$$F''_{mnl} f''_m = \frac{|\nabla_\perp \psi_l|^*}{\Lambda^2_{mnl}\,\eta_i(-T_m^2 + {}_iT_m^2)}\Big\{\big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\perp\big] \times \big[K_m k_m + k_\perp \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel\big] - \big[-k_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp\big] \times \big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\parallel\big]\Big\} \quad (3.41)$$

$$G^{\parallel}_{mn}\, g^{\parallel}_{m} = \frac{|\nabla_{\perp}^{\circ} A|^*}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ \big[-T^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x \big] \times$$
$$\big[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel \big] - \big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\parallel \big] \times$$
$$\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x \big] \Big\} \qquad (3.42)$$

$$H^{\parallel}_{mn}\, h^{\parallel}_{m} = \frac{B^{\circ *}}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ \big[-T^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x \big] \times$$
$$\big[-k^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp \big] + \big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\perp \big] \times$$
$$\big[k_T \cos\alpha_\parallel - (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_x \big] \Big\} \qquad (3.43)$$

$$F^{\perp}_{mn}\, f^{\perp}_{m} = \frac{|\nabla_{\perp}^{\circ} \psi|^*}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ \big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\perp \big] \times$$
$$\big[-K^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel \big] + \big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_\perp \big] \times$$
$$\big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\parallel \big] \Big\} \qquad (3.44)$$

$$G^{\perp}_{mn}\, g^{\perp}_{m} = \frac{|\nabla_{\perp}^{\circ} A^*|}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ - \big[-T^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x \big] \times$$
$$\big[-K^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel \big] + \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x \big] \times$$
$$\big[-k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\parallel \big] \Big\} \qquad (3.45)$$

$$H^{\perp}_{mn}\, h^{\perp}_{m} = \frac{B^{\circ *}}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ \big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_\perp \big] \times$$
$$\big[-T^2_m + k_\perp + (k_\parallel - k_\perp)\cos\alpha_x \big] + \big[k_T \cos\alpha_\perp + (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x \big] \times$$
$$\big[k_T \cos\alpha_\parallel + (k_\parallel - k_\perp)\cos\alpha_x \cos\alpha_\perp \big] \Big\} \qquad (3.46)$$

$$F^{x}_{mn}\, f^{x}_{m} = \frac{|\nabla_{\perp}^{\circ} \psi|^*}{\Lambda^2_{mnl}\,\eta_i(-T^2_m + iT^2_m)} \Big\{ - \big[K^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\perp \big] \times$$
$$\big[-K^2_m + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_\parallel \big] + \big[K_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel \big] \times$$
$$\big[-K_m k_m + k_T \cos\alpha_x - (k_\parallel - k_\perp)\cos\alpha_\parallel \cos\alpha_x \big] \Big\} \qquad (3.47)$$

$$G^{x}_{mn}\, g^{x}_{m} = \frac{|\nabla_\perp^{\circ} A|^{*}}{\Lambda^2_{mn\ell}\, \eta_i\, (-T^2_m + iT^2_m)} \Big\{ \big[k_T \cos\alpha_{\parallel} - (k_{\parallel} - k_{\perp}) \cos\alpha_\perp \cos\alpha_x \big] \times$$
$$\big[-K^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_{\parallel} \big] + \big[k_T \cos\alpha_\perp + (k_{\parallel} - k_\perp) \cos\alpha_{\parallel} \cos\alpha_x \big] \times$$
$$\big[-K_m k_m + k_T \cos\alpha_x + (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_{\parallel} \big] \Big\} \tag{3.48}$$

$$H^{x}_{mn}\, h^{x}_{m} = \frac{B^{*}}{\Lambda^2_{mn\ell}\, \eta_i\, (-T^2_m + iT^2_m)} \Big\{ \big[k_T \cos\alpha_{\parallel} - (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_x \big] \times$$
$$\big[-K_m k_m + k_T \cos\alpha_x - (k_{\parallel} - k_\perp) \cos\alpha_{\parallel} \cos\alpha_\perp \big] - \big[-k^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_\perp \big] \times$$
$$\big[k_T \cos\alpha_\perp + (k_{\parallel} - k_\perp) \cos\alpha_{\parallel} \cos\alpha_x \big] \Big\} \tag{3.49}$$

The entire Green's function is therefore:

$$\overline{\overline{G}} = \sum_{\ell mn} \frac{\nabla_\perp \psi \times \bar{a}_3\, |\nabla_\perp^{\circ} \psi|^{*}}{\Lambda^2_{mn\ell}\, \eta_i (-T^2_m + iT^2_m)} \Big\{ \bar{a}_3 \Big[\big[k_T \cos\alpha_{\parallel} + (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \times$$
$$\big[K_m k_m + k_\perp \cos\alpha_x + (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_{\parallel} \big] + \big[k^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_\perp \big] \times$$
$$\big[-k_T \cos\alpha_\perp + (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_{\parallel} \big] \Big] +$$
$$\frac{\nabla_\perp f \times \bar{a}_3}{|\nabla_\perp f \times \bar{a}_3|} \Big[- \big[K^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_\perp \big] \big[-K^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_{\parallel} \big] +$$
$$- \big[K_m k_m + k_T \cos\alpha_x + (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_{\parallel} \big] \big[K_m k_m + k_T \cos\alpha_x - (k_{\parallel} - k_\perp) \cos\alpha_{\parallel} \cos\alpha_\perp \big] \Big] +$$
$$\frac{\nabla_\perp f}{|\nabla_\perp f|} \Big[\big[k_T \cos\alpha_{\parallel} - (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \big[-K^2_m + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_{\parallel} \big] +$$
$$\big[-K_m k_m + k_T \cos\alpha_x - (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_\perp \big] \big[-k_T \cos\alpha_\perp + (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_{\parallel} \big] \Big] \Big\} +$$

$$\sum_{lmn} \frac{\nabla_\perp A\,|\nabla_\perp A|^*}{\Lambda^2_{mnl}\,\eta_i\,(-T_m^2 + \mathcal{T}_m^2)} \Big\{ \bar a_3 \Big[[-T_m^2 + k_\perp + (k_\parallel - k_\perp)\cos^2\alpha_x][k_m k_m + k_T \cos\alpha_x + (k_\parallel - k_\perp)\cos\alpha_\perp \cos\alpha_\parallel] +$$

$$[k_T \cos\alpha_\perp + (k_\parallel-k_\perp)\cos\alpha_x\cos\alpha_\parallel][k_T\cos\alpha_\parallel - (k_\parallel-k_\perp)\cos\alpha_\perp\cos\alpha_x]\Big] +$$

$$\frac{\nabla_\perp f \times \bar a_3}{|\nabla_\perp f\times \bar a_3|}\Big[[k_T\cos\alpha_\parallel - (k_\parallel-k_\perp)\cos\alpha_\perp\cos\alpha_x][-K_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_\parallel] +$$

$$[k_T\cos\alpha_\perp + (k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_x][-K_m k_m + k_T\cos\alpha_x + (k_\parallel-k_\perp)\cos\alpha_\perp\cos\alpha_\parallel]\Big] +$$

$$\frac{\nabla_\perp f}{|\nabla_\perp f|}\Big[-[-T_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_x][-K_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_\parallel] +$$

$$[k_T\cos\alpha_\perp + (k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_x][-k_T\cos\alpha_\perp + (k_\parallel-k_\perp)\cos\alpha_x\cos\alpha_\parallel]\Big]\Big\} +$$

$$\sum_{lmn} \frac{\bar a_3\, B\, B^{*}}{\Lambda^2_{mnl}\,\eta_i\,(-T_m^2+\mathcal{T}_m^2)}\Big\{\bar a_3\Big[[-T_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_x][-k_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_\perp] +$$

$$[k_T\cos\alpha_\parallel+(k_\parallel-k_\perp)\cos\alpha_x\cos\alpha_\perp][k_T\cos\alpha_\parallel-(k_\parallel-k_\perp)\cos\alpha_\perp\cos\alpha_x]\Big]+$$

$$\frac{\nabla_\perp f\times\bar a_3}{|\nabla_\perp f\times \bar a_3|}\Big[[k_T\cos\alpha_\parallel-(k_\parallel-k_\perp)\cos\alpha_\perp\cos\alpha_x][-K_m k_m+k_T\cos\alpha_x-(k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_x]$$

$$+ - [k_T\cos\alpha_\perp+(k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_x][-k_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_\perp]\Big]+$$

$$\frac{\nabla_\perp f}{|\nabla_\perp f|}\Big[[-T_m^2+k_\perp+(k_\parallel-k_\perp)\cos^2\alpha_x][-K_m k_m + k_T - (k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_\perp]+$$

$$[k_T\cos\alpha_\perp+(k_\parallel-k_\perp)\cos\alpha_\parallel\cos\alpha_x][k_T\cos\alpha_\parallel+(k_\parallel-k_\perp)\cos\alpha_x\cos\alpha_\perp]\Big]\Big\}$$

(3.50)

Since from boundary considerations it can be shown that $\nabla_\perp \Psi \times \bar{a}_3$, $\nabla \times \nabla \times \chi \bar{a}_3$, and $\nabla \varphi$ are not independent of each other, it is reasoned that for a given set of boundary conditions

$$
\begin{aligned}
\nabla_\perp \Psi \times \bar{a}_3 &\sim \nabla_\perp f(\xi_1, \xi_2)\, g(\xi_3) \times \bar{a}_3 \\
\nabla_\perp A &\sim \nabla_\perp f(\xi_1, \xi_2)\, g(\xi_3) \\
\bar{B} \bar{a}_3 &\sim f(\xi_1, \xi_2)\, h(\xi_3)
\end{aligned}
$$

where $f(\xi_1,\xi_2)$ is a two variable function of (ξ_1,ξ_2) and $g(\xi_3)$, and $h(\xi_3)$ are one variable functions of ξ_3 only. Furthermore,

$$
\begin{aligned}
\frac{\partial g}{\partial \xi_3} &= \pm k_m h(\xi_3) \\
\frac{\partial h}{\partial \xi_3} &= \mp k_m g(\xi_3)
\end{aligned}
$$

must hold. By this observation, the Green's dyadic can be written as:

$$
\bar{\bar{G}} = \left|
\begin{array}{ccc}
a_{11}\, \nabla_\perp f \times \bar{a}_3 \ \nabla_\perp^0 f^* \times a_3 & a_{12}\, \nabla_\perp f \times a_3\ \nabla_\perp^0 f^* & a_{13}\, \nabla_\perp f \times \bar{a}_3 f^* a_3 \\
a_{21}\, \nabla_\perp f\ \nabla_\perp^0 f^* \times \bar{a}_3 & a_{22}\, \nabla_\perp f\ \nabla_\perp^0 f^* & a_{23}\, \nabla_\perp f \cdot f^{0*} \bar{a}_3 \\
a_{31}\, f \bar{a}_3\ \nabla_\perp^0 f^* \times \bar{a}_3 & a_{32}\, f \bar{a}_3\ \nabla_\perp^0 f^* & a_{33}\, f \bar{a}_3\ f^{*0} \bar{a}_3
\end{array}
\right|
$$

where

$$
\begin{aligned}
a_{11} = &-\left[K_m^2 + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_\perp \right]\left[-K_m^2 + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_{\parallel} \right] + \\
&-\left[K_m k_m + k_T \cos\alpha_x + (k_{\parallel} - k_\perp) \cos\alpha_\perp \cos\alpha_{\parallel} \right]\left[K_m k_m + k_T \cos\alpha_x - (k_{\parallel} - k_\perp) \cos\alpha_{\parallel} \cos\alpha_\perp \right]
\end{aligned}
$$

$$
\begin{aligned}
a_{12} = &\left[k_T \cos\alpha_{\parallel} - (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_\perp \right]\left[-K_m^2 + k_\perp + (k_{\parallel} - k_\perp) \cos^2\alpha_{\parallel} \right] + \\
&\left[K_m k_m + \cos\alpha_x - (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_\perp \right]\left[-k_T \cos\alpha_\perp + (k_{\parallel} - k_\perp) \cos\alpha_x \cos\alpha_{\parallel} \right]
\end{aligned}
$$

$$a_{13} = \left[k_T \cos\alpha_{\parallel} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\perp} \right]\left[K_m l_m + k_{\perp}\cos\alpha_x + (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_{\parallel} \right] +$$
$$\left[k_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_{\perp} \right]\left[-k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\parallel} \right]$$

$$a_{21} = \left[k_T \cos\alpha_{\parallel} - (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_x \right]\left[-K_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_{\parallel} \right] +$$
$$\left[k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_x \right]\left[-K_m l_m + k_T \cos\alpha_x + (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_{\parallel} \right]$$

$$a_{22} = -\left[-T_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_x \right]\left[-K_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_{\parallel} \right] +$$
$$-\left[k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_x \right]\left[-k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\parallel} \right]$$

$$a_{23} = \left[-T_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_x \right]\left[K_m l_m + k_T \cos\alpha_x + (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_{\parallel} \right] +$$
$$\left[k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\parallel} \right]\left[k_T \cos\alpha_{\parallel} - (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_x \right]$$

$$a_{31} = \left[k_T \cos\alpha_{\parallel} - (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_x \right]\left[-K_m l_m + k_T \cos\alpha_x - (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_x \right] +$$
$$-\left[k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_x \right]\left[-k_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_{\perp} \right]$$

$$a_{32} = \left[-T_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_x \right]\left[-K_m l_m + k_T - (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_{\perp} \right] +$$
$$\left[k_T \cos\alpha_{\perp} + (k_{\parallel}-k_{\perp}) \cos\alpha_{\parallel} \cos\alpha_x \right]\left[-k_T \cos\alpha_{\parallel} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\perp} \right]$$

$$a_{33} = \left[-T_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_x \right]\left[-k_m^2 + k_{\perp} + (k_{\parallel}-k_{\perp}) \cos^2\alpha_{\perp} \right] +$$
$$\left[k_T \cos\alpha_{\parallel} + (k_{\parallel}-k_{\perp}) \cos\alpha_x \cos\alpha_{\perp} \right]\left[k_T \cos\alpha_{\parallel} - (k_{\parallel}-k_{\perp}) \cos\alpha_{\perp} \cos\alpha_x \right]$$

Equation (3.50) is in a dyadic form and is reciprocal with respect to the source coordinate and the observer coordinate.

The Green's dyadic is subjected to the same restrictions that the free-wave solution is subjected to.

CHAPTER V

DISCUSSION

The application of the solutions obtained from the wave equation for a bounded homogeneous, anisotropic plasma are valid only in the coordinate systems in which a dispersion relation can be obtained for the given orientation of the static magnetic field. The number of such coordinate systems is limited to not more than four. In spite of this limitation, the technique described is found to be more flexible than other techniques previously available in terms of the relaxation of the restrictions in the magnetic field orientation and boundary configurations.

The mathematical description of the magnetic field direction was restricted to those cases which can be transformed to the chosen orthogonal coordinate system with constant coefficients. The form of dispersion relation would be different for different magnetic field orientation with respect to the boundaries.

In view of Eq. (2.10)-Eq.(2-14) the forms of solutions are required to be of certain form, this might not be desirable or possible for certain problems. Such limitation could be a most serious drawback for the application of this technique. The technique proposed in this paper is applicable to electromagnetic wave radiation or propagation in such physically realizable configurations as the mirror region of the Magnetic Mirror machine, the magnetic field of the Stellarator, the Cusp machine, and such similar confinement devices (Bishop, 1958). With some modifications, it can also possibly be applicable to non-laboratory configurations, such as the electromagnetic interaction in the finite volume of a meteor trail. Furthermore, since the work reported here is a part of the electromagnetic driven shock tube investigation, the technique with some extension is also applicable to the investigation of electromagnetic wave interaction with anisotropic plasma shock wave.

APPENDIX A

REDUCTION OF MAXWELL EQUATIONS TO THE WAVE EQUATION

The Maxwell's equations are

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial D}{\partial t} \tag{A.1}$$

$$\nabla \times \bar{E} = -\frac{\partial B}{\partial t} \tag{A.2}$$

$$\nabla \cdot \bar{D} = \rho \tag{A.3}$$

$$\nabla \cdot \bar{B} = 0 \tag{A.4}$$

and the Constitutive equations are

$$\bar{D} = \hat{K} \cdot \bar{E} \tag{A.5}$$

$$\bar{H} = \frac{1}{\mu} \bar{B} \tag{A.6}$$

where μ is a scalar and

$\hat{K}$ is a dyadic.

Performing the curl operation on Eq. (A.2), substituting the Constitutive equations into Eq. (A.1) and Eq. (A.2), and eliminating the field from the two resulting equations will result in the equation (A.7).

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial \bar{J}}{\partial t} - \mu \hat{K} \cdot \frac{\partial^2 \bar{E}}{\partial t^2} \tag{A.7}$$

If only monochromatic waves are of interest, Eq. (A.7) reduces to

$$\nabla \times \nabla \times \bar{E} - \mu \omega^2 \hat{K} \cdot \bar{E} = -j\omega\mu \bar{J} \tag{A.8}$$

If new variables are chosen such that $-j\omega\mu\bar{J}$ is $\bar{J}_s$, and $\mu\omega^2\hat{\hat{K}}$ is $\hat{\hat{k}}$; then Eq. (A.8) will result in the wave equation of the form

$$\nabla\times\nabla\times\bar{E} - \hat{\hat{k}}\cdot\bar{E} = \bar{J}_s \tag{A.9}$$

The $\hat{\hat{K}}$ dyadic for the magnetic field oriented along the coordinate axis of the unity scale factor will have the form

$$\hat{\hat{K}} = \begin{bmatrix} K_\perp & K_T & 0 \\ -K_T & K_\perp & 0 \\ 0 & 0 & K_{\parallel} \end{bmatrix} \tag{A.10}$$

where

$$K_T = i\,\frac{N_i e_i^2}{\omega^2 \epsilon_0 m_i}\left[\frac{\Omega_i/\omega}{(1-i\nu_{EN}/\omega)^2 - \frac{\Omega_i^2}{\omega^2}}\right] \tag{A.11}$$

$$K_\perp = 1 + \frac{i}{\omega^3\epsilon_0}\left\{\frac{i N_i e_i^2}{m_i}\,\frac{[1+i\,\nu_{EN}/\omega]}{[(1+\nu_{EN}/\omega)^2 - \frac{\Omega_i^2}{\omega^2}]}\right\} \tag{A.12}$$

$$K_{\parallel} = 1 - \frac{e_i^2 N_i}{\omega^2\epsilon_0 m_i}\,\frac{\Omega_i/\omega}{[(1-i\frac{\nu_{EN}}{\omega})^2 - \frac{\Omega_i^2}{\omega^2}]} \tag{A.13}$$

where e_i = charge of particle

N_i = concentration of the particle

m_i = mass of the particle

ω = frequency of the wave

B = magnetic field

ν_{EN} = collision frequency

$\Omega_i = \frac{e_i B}{m_i}$ (Holt & Haskell, 1965).

APPENDIX B

SOLUTION OF THE WAVE EQUATION IN TERMS OF THE GREEN'S FUNCTION

The wave equation is

$$\nabla\times\nabla\times\bar{E} - k_\perp \bar{E} - (k_{\parallel}-k_{\perp})(\bar{E}\cdot\bar{i}_b) - k_T \bar{E}\times\bar{i}_b = \bar{J}_s \qquad (B.1)$$

where
$$\bar{i}_b = \cos\alpha_\times \frac{\nabla_\perp f\times\bar{a}_3}{|\nabla_\perp f\times\bar{a}_3|} + \cos\alpha_\perp \frac{\nabla_\perp f}{|\nabla_\perp f|} + \cos\alpha_\parallel\, \bar{a}_3 \qquad (B.2)$$

for the coordinate

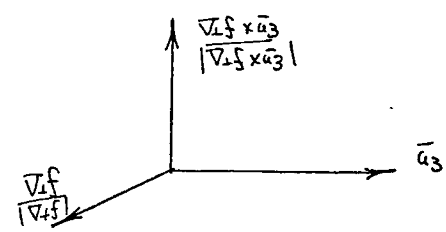

Fig. B.1.

The Green's equation is assumed in the form

$$\nabla\times\nabla\times\bar{\bar{G}} - k'_\perp \bar{\bar{G}} - (k'_\parallel - k_\perp)(\bar{\bar{G}}\cdot\bar{i}_b)\,\bar{i}_b - k'_T\, \bar{\bar{G}}\times\bar{i}_b = \delta(\bar{r}-\bar{r}_0)\,\bar{\bar{I}} \qquad (B.3)$$

with the constants $k'_\perp$, k'_T, and $k'_\parallel$ to be determined. $\bar{\bar{I}}$ is the idemefactor. Pre-multiplying Eq. (B.1) by $\bar{\bar{G}}$ and post-multiplying Eq. (B.2) by $\bar{E}$ and subtracting the two equations will result in equation

$$\bar{\bar{G}}\cdot\nabla\times\nabla\times\bar{E} - k_\perp\,\bar{\bar{G}}\cdot\bar{E} - (k_\parallel-k_\perp)\,\bar{\bar{G}}\cdot\bar{i}_b\,(\bar{E}\cdot\bar{i}_b) - k_T\,\bar{\bar{G}}\cdot\bar{E}\times\bar{i}_b - \bar{\bar{G}}\cdot\bar{J}_s + \delta(\bar{r}-\bar{r}_0)\,\bar{\bar{I}}\cdot\bar{E}$$

$$-\nabla\times\nabla\times\bar{E}\circ\bar{\bar{G}} + k'_{\perp}\bar{\bar{G}}\cdot\bar{E} + (k'_{\parallel} - k_{\perp})(\bar{\bar{G}}\cdot\bar{i}_b)(\bar{E}\cdot\bar{i}_b) + k'_T\,\bar{\bar{G}}\times\bar{i}_b\circ\bar{E} = 0 \qquad \text{(B.4)}$$

If $k'_{\perp} = k_{\perp}$ and if $k'_{\parallel} = k_{\parallel}$, Eq. (B.4) reduces to

$$\bar{\bar{G}}\cdot\nabla\times\nabla\times\bar{E} - \nabla\times\nabla\times\bar{\bar{G}}\cdot\bar{E} - k_T\bar{\bar{G}}\circ\bar{E}\times\bar{i}_b + k'_T\,\bar{\bar{G}}\times\bar{i}_b\cdot\bar{E} = \bar{\bar{G}}\cdot\bar{J}_s - \delta(\bar{r}-\bar{r}_0)\bar{\bar{I}}\cdot\bar{E} \qquad \text{(B.5)}$$

Now expressing $k_T\bar{\bar{G}}\times\bar{i}_b\circ E$ in matrix form will result in

$$\begin{bmatrix} k_T\left[(g_{12}\cos\alpha_x - g_{13}\cos\alpha_{\parallel})E_1 + (g_{13}\cos\alpha_{\perp} - g_{11}\cos\alpha_x)E_2 + (g_{11}\cos\alpha_{\parallel} - g_{12}\cos\alpha_{\perp})E_3\right] \\ k_T\left[(g_{22}\cos\alpha_x - g_{23}\cos\alpha_{\parallel})E_1 + (g_{23}\cos\alpha_{\perp} - g_{21}\cos\alpha_x)E_2 + (g_{21}\cos\alpha_{\parallel}\ \ g_{22}\cos\alpha_{\perp})E_3\right] \\ k_T\left[(g_{32}\cos\alpha_x - g_{33}\cos\alpha_{\parallel})E_2 + (g_{33}\cos\alpha_{\perp} - g_{31}\cos\alpha_x)E_2 + (g_{31}\cos\alpha_{\parallel}\ \ g_{32}\cos\alpha_{\perp})E_3\right] \end{bmatrix}$$

where $$\bar{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \qquad \text{(B.6)}$$

$$\bar{i}_b = \begin{bmatrix} \cos\alpha_{\perp} \\ \cos\alpha_{\parallel} \\ \cos\alpha_x \end{bmatrix} \qquad \text{(B.7)}$$

$$\bar{\bar{G}} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad \text{(B.8)}$$

Similarly $k_T\bar{\bar{G}}\cdot\bar{E}\times\bar{i}_b$ in matrix form is

$$\begin{bmatrix} k_T\left[g_{11}\left(E_2 \cos\alpha_x - E_3 \cos\alpha_\perp \right) + g_{12}\left(E_3 \cos\alpha_\parallel - E_1 \cos\alpha_x \right) + g_{13}\left(E_1 \cos\alpha_\parallel - E_2 \cos\alpha_\perp \right) \right] \\ k_T\left[g_{21}\left(E_2 \cos\alpha_x - E_3 \cos\alpha_\perp \right) + g_{22}\left(E_3 \cos\alpha_\parallel - E_1 \cos\alpha_x \right) + g_{23}\left(E_1 \cos\alpha_\parallel - E_2 \cos\alpha_\perp \right) \right] \\ k_T\left[g_{31}\left(E_2 \cos\alpha_x - E_3 \cos\alpha_\perp \right) + g_{32}\left(E_3 \cos\alpha_\parallel - E_1 \cos\alpha_x \right) + g_{33}\left(E_1 \cos\alpha_\parallel + E_2 \cos\alpha_\perp \right) \right] \end{bmatrix}$$

If $k_T = -k_T'$, Eq. (B.5) will be

$$\bar{\bar{G}} \cdot \nabla\times\nabla\times\bar{E} - \nabla\times\nabla\times\bar{\bar{G}} \cdot \bar{E} = \bar{\bar{G}} \cdot \bar{J}_S - \delta(\bar{r}-\bar{r}_0)\,\bar{\bar{I}} \cdot \bar{E} \tag{B.9}$$

If Eq. (B.9) is integrated over the volume of interest, the first two terms in the integral can be changed into surface integral by Green's theorem. This surface integral will vanish if both $\bar{\bar{G}}$ and $\bar{E}$ satisfy the same boundary conditions. The resulting integral is (Seto & Dougal, 1964)

$$\bar{E} = \int_{VoL} \bar{\bar{G}} \cdot \bar{J}_S \, dv \tag{B.10}$$

APPENDIX C

WAVE EQUATION WITH LONGITUDINAL AND TRANSVERSE SOURCE

The wave equation is

$$\nabla \times \nabla \times \bar{E} - \hat{\hat{k}} \cdot \bar{E} = \bar{J} \tag{C.1}$$

If the source is composed of both longitudinal $\bar{J}_\ell$ and transverse $\vec{J}_T$ components, the resulting wave equation will have the following form:

$$\nabla \times \nabla \times \bar{E} - \hat{\hat{k}} \cdot \vec{E} = \bar{J}_\ell + \bar{J}_t \tag{C.2}$$

The electric field intensity $\bar{E}$ is assumed to have the form

$$\bar{E} = \bar{E}_\ell + \bar{E}_t \tag{C.3}$$

where $\bar{E}_\ell$ is the longitudinal part and $\bar{E}_t$ is the transverse part. Substituting Eq. (C.3) into Eq. (C.2) results in two equations, one longitudinal

$$- \hat{\hat{k}} \cdot \bar{E}_\ell = \bar{J}_\ell \tag{C.4}$$

and one transverse

$$\nabla \times \nabla \times \bar{E}_T + \hat{\hat{k}} \cdot \bar{E}_t = \bar{J}_t \tag{C.5}$$

Two Green's functions can be derived from the two associated equations Eq. (C.4) and Eq. (C.5).

APPENDIX D

SOLUTION OF A NONHOMOGENEOUS SYSTEM OF n LINEAR EQUATIONS BY CRAMER'S METHOD

For a nonhomogeneous system of n linear equations in n unknowns

$$\begin{aligned} \alpha_{11} X_1 + \cdots + \alpha_{1n} X_n &= \beta_1 \\ &\vdots \\ \alpha_{n1} X_1 + \cdots + \alpha_{nn} X_n &= \beta_n \end{aligned} \tag{D.1}$$

has a unique solution if and only if the determinant of the coefficient matrix is not zero, i.e.

$$D(\alpha_{ii}) \neq 0 \tag{D.2}$$

If $D(\alpha_{ii}) \neq 0$, the solution is given by

$$X_i = \frac{D(C_1, \ldots, C_{i-1}, b, C_{i+1}, \ldots, C_n)}{D(a_{ii})} \tag{D.3}$$

where $C_1, \ldots, C_n$ are the columns of a_{ii} and $b = \langle \beta_1, \ldots, \beta_n \rangle$. (J. C. Curtis, 1963, p. 79).

APPENDIX E

SEPARABILITY OF THE VECTOR WAVE EQUATION

A scalar wave equation

$$\nabla^2 E_i + k^2 E_i = 0 \tag{E.1}$$

is separable into three independent differential equations in three different space variables, x_1, x_2, and x_3. This separability condition is (Mose & Feshbach, 1953)

$$\begin{aligned} g_{ij} &= S / m_{i1} \\ g^{1/2} &= f_1(x_1) f_2(x_2) f_3(x_3) \end{aligned} \tag{E.2}$$

where S = Stekel determinant and m_{i1} = i^{th} Minor. Similarly, the vector wave equation

$$\nabla \times \nabla \times \bar{E} - k \bar{E} = 0 \tag{E.3}$$

is simply separable if the term $\nabla \times \nabla \times \bar{E}$ can be written as $k^2 \bar{E}$. This can be accomplished by assuming $\bar{E}$ in the form $\nabla \times \nabla \times \chi \bar{a}_1$. Upon substituting this form for $\bar{E}$ into Eq. (E.3), $\nabla \times \bar{E}$ becomes in generalized curvilinear coordinate system

$$\begin{aligned} &\frac{1}{g_{11}} \nabla \left(\frac{\partial}{\partial \xi_1} g_{11}^{1/2} \chi \right) - \bar{a}_1 \left[\frac{1}{g_{11}^{3/2}} \left(\frac{\partial^2}{\partial \xi_1^2} g_{11}^{1/2} \chi \right) + \frac{1}{(g_{22} g_{33})^{1/2}} \frac{\partial}{\partial \xi_2} \left(\frac{g_{33}^{1/2}}{g_{11}^{1/2} g_{22}^{1/2}} \frac{\partial}{\partial \xi_2} g_{11}^{1/2} \chi \right) \right. \\ &\qquad \left. + \frac{1}{g_{11}^{1/2} g_{33}^{1/2}} \frac{\partial}{\partial \xi_3} \left(\frac{g_{22}^{1/2}}{g_{11}^{1/2} g_{33}^{1/2}} \frac{\partial}{\partial \xi_3} g_{11}^{1/2} \chi \right) \right] \end{aligned} \tag{E.4}$$

This has been shown to become separable (Mose & Feshbach, 1953) if

$$g_{11}^{1/2} = 1 \tag{E.5}$$

$$\left(g_{33} / g_{22} \right)^{1/2} = f^1 (x_2, x_3) \tag{E.6}$$

and this implies that x satisfies

$$\nabla^2 \chi - k^2 \chi \tag{E.7}$$

Then $\nabla \times \nabla \times \bar{E} = \nabla \times [k^2 \chi_{\bar{a}_1}]$ and thus the field is said to be separable in terms of $\nabla \times \chi_{\bar{a}_1}$. Similar analysis shows that $\bar{E}$ equal to $\nabla \times \nabla \times \psi_{\bar{a}_\rho}$ and $\nabla \times \chi_{\bar{a}_3}$ satisfy the same separability criteria.

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