A REAL TIME SPACECRAFT GUIDANCE FORMULATION BASED UPON OPTIMIZATION THEORY

A Thesis

Presented to

the Faculty of the Graduate School

The University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Industrial Engineering

by

A. David Long

May, 1972

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ABSTRACT

This thesis is concerned with the development of a spacecraft guidance which will solve the problem associated with the optimum transfer of a spacecraft between two states. The theoretical development of an existing guidance formulation is shown and this formulation is extended to include a more general mission capability. Specifically, the guidance formulation presented is extended to an operational capability for low-thrust maneuvers.

Numerical results are presented which compare the guidance solution and a near optimal solution to the same low-thrust transfer problem. These results indicate that the guidance procedure can be extended to an operational capability for low-thrust maneuvers with performance (propellant expenditure) comparable to an optimum transfer.

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LIST OF SYMBOLS

Symbol	
$a, a(t)$	thrust acceleration magnitude
$\bar{a}$	instantaneous spacecraft thrust acceleration vector
c_1, c_2, c_3	components of a parameterized control vector
$\bar{c}$	control vector
d_1, d_2, d_3	components of a parameterized control vector
$\bar{d}$	control vector
f	subscript indicating final state or time
G	path dependent loss function for an orbital transfer maneuver
g	gravitational acceleration magnitude
$\bar{g}$	instantaneous gravitational acceleration vector
g_x, g_y, g_z	components of the instantaneous gravity vector expressed in the guidance coordinate system
H	Hamiltonian
h	altitude
L	integral of the loss function over time
$\dot{L}$	instantaneous loss rate function
m	instantaneous spacecraft mass
$\dot{m}$	instantaneous mass flow rate of the rocket engine
$\bar{P}$	three component Lagrange multiplier vector associated with the velocity state equations

Symbol	
$\ddot{\bar{P}}$	second time derivative of the Lagrange multiplier vector
R	return function for an orbital transfer maneuver
r	instantaneous radius vector magnitude
$\bar{r}$	instantaneous spacecraft position vector
$\dot{\bar{r}}$	time derivative of the spacecraft position vector
$\ddot{\bar{r}}$	second time derivative of the spacecraft position vector
$\bar{S}$	state variable vector
$\dot{\bar{S}}$	time derivative of the state variable vector
$T, T_1, T_2, \ldots, T_n$	components of t
t	independent variable for an orbital transfer maneuver, time
U, V', W	components of the instantaneous spacecraft velocity vector
$\dot{U}, \dot{V}', \dot{W}$	components of the instantaneous spacecraft acceleration vector
V, v	instantaneous velocity magnitude
$\bar{V}, \bar{v}$	instantaneous spacecraft velocity vector
V_{ex}	exhaust velocity of the rocket engine
V_{ex} L, J, P, S', Q, U'	successive integrals of thrust acceleration over time
X, Y, Z	components of the instantaneous spacecraft position vector
$\dot{X}, \dot{Y}, \dot{Z}$	components of the instantaneous spacecraft velocity vector

Symbol	
$\ddot{X}, \ddot{Y}, \ddot{Z}$	components of the instantaneous spacecraft acceleration vector
α	angle of attack measured between the thrust acceleration vector and the velocity vector
β	component of a parameterized control vector
γ	flight path angle
ΔV	change in velocity required for a transfer maneuver
η	component of a parameterized control vector
θ	control angle
θ_p, θ_y	control angles, pitch, and yaw
$\bar{\theta}_p$, $\bar{\theta}_y$	pitch and yaw control angles which achieve the desired velocity in a transfer maneuver
$\lambda_1 \ldots \lambda_6$	components of the Lagrange multiplier vector
$\bar{\lambda}$, $\bar{\lambda}(t)$	the six component Lagrange multiplier vector
$\dot{\lambda}_1 \ldots \dot{\lambda}_6$	time derivative of the Lagrange multiplier vector
μ	gravitational constant
ξ	component of a parameterized control vector
$\bar{\rho}$	control vector
Φ	return function which is a function of the final state, S_f
ϕ	central angle traversed during a transfer maneuver

Symbol	
$\dot{\phi}$	time derivative of the central angle
$\dot{\phi}_m$	average angular rate during a transfer maneuver
$\ddot{\phi}$	second time derivative of the central angle
ψ	component of a parameterized control vector

Chapter 1

INTRODUCTION

The recent concept of an earth orbital space shuttle has given impetus to the development of new and more general guidance concepts and programs. Unlike previous manned vehicles, the shuttle has advanced features such as reusable stages, high maneuverability, multiple thrust levels, and throttlable engines. While the shuttle concept significantly enhances earth orbital mission capability, it also poses special guidance problems (i.e., low thrust maneuvers, constant acceleration maneuvers, etc.) The shuttle concept provides motivation for the guidance formulation developed in this thesis.

In past manned and unmanned space missions, real time spacecraft guidance and control have often been based upon principles of optimization theory. Although optimization can be directly applied to most guidance problems, the computation is usually lengthy and requires much computer storage. Thus the process is impractical for real time guidance and control systems.

The purpose of a spacecraft guidance and control system is to solve the two-point boundary value problem of orbital transfer. The approximate solution to the guidance problem is normally in closed form and is always executable in real time (at a recurring frequency) during a guided maneuver.

Because of its special nature and importance, the guidance problem has received considerable attention in the recent literature and will continue to do so as more complex space hardware and missions are planned. This thesis is concerned with the development of a specific guidance formulation based upon principles of optimization theory. The problem considered is limited to the transfer of a spacecraft between two states with minimum propellant consumption when two external forces, thrust and gravity, are considered. The problem will be concerned with single rather than multiple burn arcs. A brief review of the guidance problem will be given at this point to familiarize the reader with past and current concepts.

The complete optimization problem can be stated as one of transferring from state ($\bar{r}_0$, $\bar{v}_0$) to state ($\bar{r}_f$, $\bar{v}_f$) while minimizing time of powered flight (this is equivalent to minimizing propellant usage).

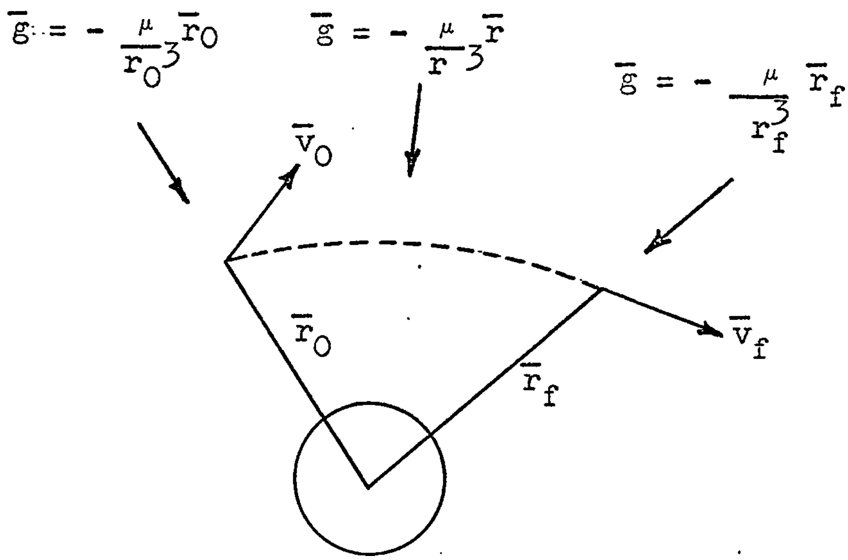


Figure 1.1

Orbit Transfer For
Inverse Squared Gravity

The dynamical equations governing motion of the spacecraft (for burn arcs only) can be shown from figure 1.1. If $\bar{r}$, $\bar{g}(r)$, $a(t)$, and $\bar{\rho}$ are, respectively, the radius vector, the gravitational acceleration vector, the engine thrust acceleration, and the thrust direction unit vector, then

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} + a(t)\,\bar{\rho} \tag{1-1}$$

where $\bar{r}(0) = \bar{r}_0$

$\dot{\bar{r}}(0) = \bar{v}_0$.

Application of the calculus of variations to this problem yields one of the necessary conditions for an extremum stated in the form of an auxilliary equation as follows:

$$\ddot{\overline{P}} = f(\overline{r},\overline{P},t) = \overline{P}\cdot\nabla\overline{g} \qquad (1\text{-}2)\ ^{a}$$

$$\text{where}\quad \overline{\rho} = \frac{\overline{P}}{|\overline{P}|}$$

This result is stated by Lawden (1). $\overline{P}$ is the classical Lagrange multiplier vector which adjoins the velocity equations in the Hamiltonian expression. To find a solution to this problem one must successively guess values for both $\overline{P}(0)$ and $\dot{\overline{P}}(0)$ and numerically integrate equations (1-1) and (1-2) until the boundary conditions ($\overline{r}_f$, $\overline{v}_f$) are satisfied.(this procedure is illustrated in appendix A). If the boundary conditions are satisfied, then a solution to the optimization problem has been found, and the solution is a local minimum or maximum.

The solution to the optimization problem can normally be simplified for the purpose of spacecraft guidance and control. This can be done by making assumptions such that, although the problem is simplified, its solution approximates the solution to the original problem. For instance, if the problem can be formulated as one of changing only the velocity vector and the acceleration of the spacecraft can be considered infinite, the solution to the problem can be computed readily from equation (1-3).

[a] $\overline{P}$ corresponds to the three component Lagrange multiplier vector ($\lambda_4\ \lambda_5\ \lambda_6$). See appendix A.

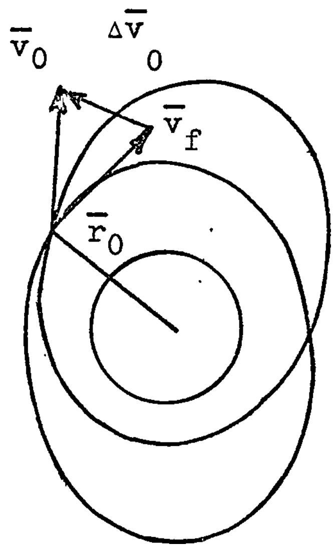


Figure 1.2

Impulsive Orbit Change

The result from equation (1-3) is often referred to as an impulsive solution and constitutes a lower bound cost to the transfer. If the spacecraft acceleration were infinite, a velocity increment, $\Delta\bar{v}_0$, could be added instantaneously at $\bar{r}_0$ to effect the transfer from the initial to the desired ellipse. The velocity increment magnitude is computed from

$$\Delta v_0^2 = v_0^2 + v_f^2 - 2v_0\, v_f \cos\delta\gamma \quad . \tag{1-3}$$

It is assumed that the position vector, $\bar{r}_0$, does not change during the maneuver; if the acceleration were infinite, this would be the case. Guidance and control based upon this impulsive solution have been used successfully for many small powered flight maneuvers. Compensations must be made in thrust direction because of the finite length of the maneuver.

The major limitation of this impulsive approximation is that it does not explicitly control position and is therefore limited to short burn arcs. (Robbins (2) derives analytic results for multiple impulsive maneuvers.)

A more general approach (to guidance and control) that is applicable to longer burn arcs than the previous impulsive approach is discussed by McAllister, Grier, and Wagner (3). If the problem remains one of changing the velocity vector, it can be shown that the optimal thrust policy is given by $\overline{a} \times \overline{V}_g = c\ \overline{b} \times \overline{V}_g$ (1-4) [a]

during the finite thrust maneuver. The terms $\overline{a}$, $\overline{V}_g$, and c are, respectively, the thrust acceleration vector, the velocity change vector, and a scalar constant. The $\overline{b}$ vector can be computed as follows:

$$\overline{b} = \dot{\overline{V}}_f - \overline{g} \tag{1-5}$$

$$\overline{V}_g = \overline{V}_f - \overline{V} \tag{1-6}$$

[a] This equation represents what is often called cross product guidance.

If the scalar constant c is chosen to be 0, then $\bar{a} \times \bar{V}_g = 0$, which implies that the thrust vector should be directed parallel to $\bar{V}_g$. If, however, c is chosen to be 1, then $\bar{a} \times \bar{V}_g = \bar{b} \times \bar{V}_g$, or $\bar{a} \times \bar{V}_g = (\dot{\bar{V}}_f - \bar{g}) \times \bar{V}_g$.

By substituting for $\dot{\bar{V}}_f$ the expression becomes

$\bar{a} \times \bar{V}_g = (\dot{\bar{V}}_g + \bar{a}) \times \bar{V}_g$. This reduces to $\dot{\bar{V}}_g \times \bar{V}_g = 0$ and implies that the thrust should be directed to maintain $\dot{\bar{V}}_g$ parallel to $\bar{V}_g$. Use of equation (1-4) (as a guidance and control equation) with an appropriate value of c $(0 \leqq c \leqq 1)$ will achieve the desired velocity, V_f, in minimum time. The difficulty with this guidance procedure, is that position cannot be directly controlled and therefore the range of applicability is limited to small burn arcs. It is, however, an improvement over the previously discussed impulsive approximation since it is applicable over larger burn arcs. In various forms it has produced excellent results for limited orbital transfer problems.

A still more general approach to the orbital transfer guidance and control problem is discussed by Smith (4) and Jezewski and Stoolz (5). This approach simplifies the solution to the original optimization problem to one that is solvable in closed form when gravity is assumed to be strictly a function of time (or constant).

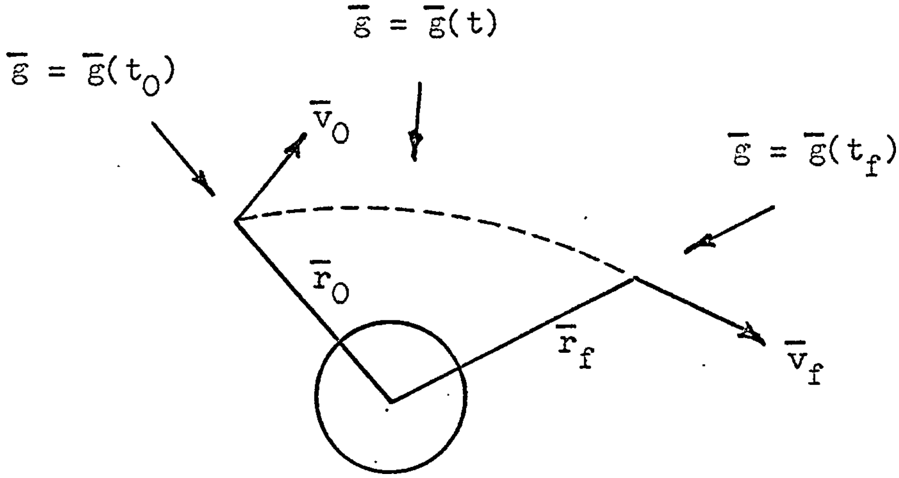


Figure 1.3

Orbit Transfer For
Time-Varying Gravity

The original state and auxilliary equations (1-1 and 1-2) then become

$$\ddot{\bar{r}} = -\bar{g}(t) + a(t)\,\bar{\rho} \tag{1-7}$$

where $\bar{r}(0) = \bar{r}_0$

$\dot{\bar{r}}(0) = \bar{v}_0$

$$\text{and} \quad \ddot{\bar{P}} = f(\bar{r},\bar{P},t) = \bar{P}\cdot\nabla\bar{g} \tag{1-8}$$

where $\bar{\rho} = \dfrac{\bar{P}}{|\bar{P}|}$.

In this case, however, $\nabla\bar{g}$ is equal to zero, which insures that $\ddot{\bar{P}} = 0$ and therefore that $\bar{P} = \bar{c} + \bar{d}t$. Substitution of this control vector into the dynamical equation yields the result

$$\ddot{\bar{r}} = -\bar{g}(t) + a(t)\,\frac{(\bar{c} + \bar{d}t)}{|\bar{c} + \bar{d}t|} \quad . \tag{1-9}$$

Integrations of this vector equation (1-9) yield six independent scalar equations for position and velocity. The six scalar equations are transcendental in terms of the control variables $\overline{c}, \overline{d}$, and t_f (of which only six are independent). A multivariable search method (gradient, Newton-Raphson, etc.) can be used to vary these parameters and achieve the desired final state ($\overline{r}_f$, $\overline{v}_f$). In lieu of using a search procedure, the formulation by Smith (4) makes added assumptions such that the control parameters can be evaluated explicitly.

This solution controls three components of velocity ($\dot{X}_f$, $\dot{Y}_f$, and $\dot{Z}_f$) and two components of position (X_f and Y_f). The Z_f- component measures position in the spacecraft flight. plane and is not controlled.

A distinct advantage of this guidance formulation is that both the position (two components) and velocity can be controlled and that the control constants ($\overline{c}, \overline{d}$, and t_f) can be evaluated explicitly. A major limitation of this approach is apparent, however. As the burn arc becomes increasingly large the gravity assumption becomes increasingly worse. For certain problems convergence cannot be attained due to the size of the burn arc. Additionally, orbital transfer problems which involve rendezvous maneuvers cannot be solved unless all six components of position and velocity are controlled as well as time, t_f. Since only five components of position and velocity can be controlled this formulation will not work for rendezvous guidance problems.

Another general approach to the orbital transfer guidance problem is considered by Jezewski (6). This formulation reduces the original optimization problem to one that is solvable in closed form. The assumption is made that the gravity vector is a linear function of the position vector (on burn arcs only).

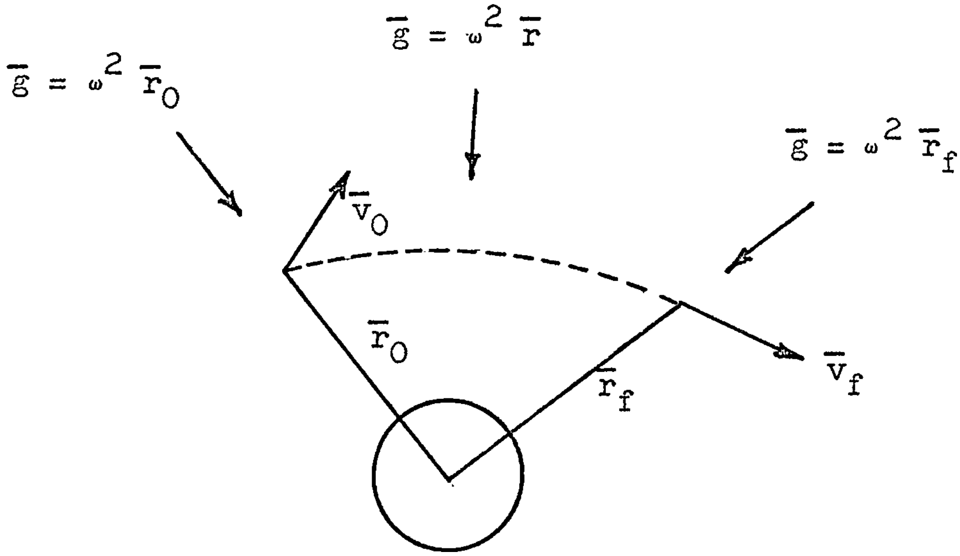


Figure 1.4

Orbit Transfer For Linear Gravity

The original state and auxilliary equations then become

$$\ddot{\bar{r}} = -\omega^2 \bar{r} + a(t)\,\bar{\rho} \tag{1-10}$$

where $\bar{r}(0) = \bar{r}_0$

$\dot{\bar{r}}(0) = \bar{v}_0$

and $\ddot{\bar{P}} = f(\bar{r},\bar{P},t) = \bar{P} \cdot \nabla \bar{g}$

$$\text{where} \quad \bar{\rho} = \frac{\bar{P}}{|\bar{P}|} \quad . \tag{1-11}$$

In this case equation (1-11) reduces to $\ddot{\overline{P}} = -\omega^2\overline{P}$, which represents the motion of a harmonic oscillator without damping and without a forcing function. Its solution is given by

$$\overline{P} = \overline{c}\sin\omega t + \overline{d}\cos\omega t. \tag{1-12}$$

The solution to the dynamical equation (1-10) follows:

$$\overline{r} = \overline{p}\sin\omega t + \overline{q}\cos\omega t \tag{1-13}$$

$$\overline{v} = \omega(\overline{p}\cos\omega t - \overline{q}\sin\omega t) \tag{1-14}$$

$$\dot{\overline{p}}\sin\omega t + \dot{\overline{q}}\cos\omega t = 0 \tag{1-15}$$

$$\omega(\dot{\overline{p}}\cos\omega t - \dot{\overline{q}}\sin\omega t) = a(t)\frac{\overline{P}}{|\overline{P}|} \tag{1-16}$$

Since the values of $\overline{p}$ and $\overline{q}$ can be evaluated as integral functions of $a(t)\dfrac{\overline{P}}{|\overline{P}|}$, the optimal burn arc for the orbit change can be solved in closed form.

This procedure could also be used to consider multiple burn arcs. Since the state variables and the Lagrange multipliers can be propagated across coast arcs in closed form, this complete problem with multiple burn and coast arcs can also be solved in closed form. By implicitly solving a set of non-linear equations which are transcendental in the control variables ($\overline{c}$, $\overline{d}$, and t_f), a solution for the optimal burn arc is found. The addition of multiple burn arcs increases the dimensionality of the problem (t_f then has multiple components); however, the solution method remains the same.

This guidance procedure has a very general formulation and can explicitly control all position and velocity components.

The previous discussion has outlined the optimization problem and some past and more recent guidance formulations. The original optimization problem cannot be solved in closed form (it requires numerical integration and iteration) and, therefore, has limited or no applicability as a guidance and control system. Varying degrees of complexity are also involved in the different guidance formulations depending upon the assumptions which are made. Relatively simple procedures, such as the impulsive and cross product procedure, are limited to short burn arcs and therefore will not solve a large number of orbital transfer problems. The time-varying gravity formulation ($\nabla \bar{g} = 0$) has a fairly general capability. However, it will not work for long burn arcs or lengthy low-thrust maneuvers. For low-thrust maneuvers the gravitational acceleration becomes a much more significant term and introduces convergence problems in the computation of the control parameters ($\bar{c}$, $\bar{d}$, and t_f). Also, since this formulation can only control two components of position, it is unsuitable for a rendezvous guidance.

The linear gravity formulation has none of the above problems; however, use of the more sophisticated approach (multiple burn arcs) may introduce problems concerning its use in repetitive guidance solutions. Also, since this formulation requires iteration, its speed of execution would require investigation.

The guidance formulation considered in this thesis is limited to a time-varying gravity formulation and a single burn solution. Specifically, the formulation will be similar to Smith (6) in that the principal closed-form computations will be retained. The formulation will be extended to an operational capability for long burn arcs and low-thrust maneuvers. Numerical results will be presented for comparing the solution of a low-thrust problem with that of an extremized solution to the same problem. In addition, the formulation will include an extension such that all six components of position and velocity can be controlled, although implementation and numerical results are not within the scope of this thesis.

The remainder of the thesis will proceed with the development of the necessary conditions for an extremum, development of the control law where $\nabla \bar{g} = 0$, and development of the guidance equations to be used for evaluation of the control parameters. Final chapters will be devoted to numerical results and the extension of the original formulation to control all final state variables. Development of a loss function (appendix B), effective gravity equations (appendix C), and intermediate boundary value equations (appendix D) are included in the appendix as guidance related improvements.

The loss function is considered as a switching function (to determine engine on-off time) and the intermediate boundary value equations are used to extend the guidance formulation capability to large burn arcs. The effective gravity computation will be used with the guidance equations to approximate gravity over each burn arc. Appendix E considers the optimal control law under conditions of constant thrust acceleration and constant gravity. Appendix F develops the required guidance integrals and appendix G develops the time-to-go computation. These are both necessary inputs for the guidance formulation.

Chapter 2

NECESSARY CONDITIONS

Consider the problem of minimizing or maximizing some return function,

$$R = \Phi(\overline{S}_f) + \int_{t_0}^{t_f} G\ (t,\overline{S},\overline{\rho}\ )\ dt\ . \qquad (2\text{-}1)$$

The term $\Phi(\overline{S}_f)$ corresponds to a penalty for not attaining the final state, while the integral function is a path-dependent value and depends upon the state history $\overline{S}$ and the control function $\overline{\rho}$. This return function is subject to the state equations of form

$$\dot{\overline{S}} = f(t,\overline{S},\overline{\rho}) \qquad (2\text{-}2)$$

where $\overline{S}\ (t_0) = \overline{S}_0$.

The state equations are adjoined as an equality constraint similar to an ordinary non-time-varying minimization problem (where $\overline{\lambda}\ (t)$ is an unknown Lagrange multiplier).

Therefore, $$R = \Phi(\overline{S}_f) + \int_{t_0}^{t_f} G + \overline{\lambda}\ (t)^T\ (\overline{f}-\dot{\overline{S}})\ dt\ . \qquad (2\text{-}3)$$

The Hamiltonian is defined and substituted into the return function.

$$H\ (t,\overline{S},\overline{\lambda},\overline{\rho}) = G\ (t,\overline{S},\overline{\rho}) + \overline{\lambda}^T\ \overline{f}\ (t,\overline{S},\overline{\rho}) \tag{2-4}$$

$$R = \Phi(\overline{S}_f) + \int_{t_0}^{t_f} (H - \overline{\lambda}^T\ \dot{\overline{S}})\ dt \tag{2-5}$$

The return function, R, is expanded to first order in a Taylor series expansion about $\overline{\rho}^*$ (where $\overline{\rho}^*$ minimizes R).

$$R\ (\overline{\rho}^* + \Delta\overline{\rho}) = \Phi(\overline{S}_f)^* + \left.\frac{\partial\Phi}{\partial\overline{S}_f}\right|^* \Delta\overline{S}_f$$

$$+ \int_{t_0}^{t_f} \Big(H^* + H_{\overline{S}}^*\ \Delta\overline{S} + H_{\overline{\lambda}}^*\ \Delta\overline{\lambda} + H_{\overline{\rho}}^*\ \Delta\overline{\rho}$$

$$- \overline{\lambda}^{T*}\ \dot{\overline{S}}^* - \overline{\lambda}^{T*}\Delta\dot{\overline{S}} - \Delta\overline{\lambda}^T\ \dot{\overline{S}}^* \Big)\ dt \tag{2-6}$$

The necessary conditions can then be established.

$$R\ (\overline{\rho}^* + \Delta\overline{\rho}) = \Phi(\overline{S}_f)^* + \int_{t_0}^{t_f} (H^* - \overline{\lambda}^{T*}\ \dot{\overline{S}}^*\ )\ dt$$

$$+ \left.\frac{\partial\Phi}{\partial\overline{S}_f}\right|^* \Delta\overline{S}_f + \int_{t_0}^{t_f} (H_{\overline{S}}^*\ \Delta\overline{S} + H_{\overline{\lambda}}^*\ \Delta\overline{\lambda} - \dot{\overline{S}}^{T*}\Delta\overline{\lambda} + H_{\overline{\rho}}^*\ \Delta\overline{\rho})\ dt$$

$$- \int_{t_0}^{t_f} (\overline{\lambda}^{T*}\Delta\dot{\overline{S}})\ dt$$

But $H_{\overline{\lambda}}^*\ \Delta\overline{\lambda} - \dot{\overline{S}}^{T*}\Delta\overline{\lambda} = (H_{\overline{\lambda}}^* - \dot{\overline{S}}^{T*}\ )\ \Delta\overline{\lambda}$

and $H_{\overline{\lambda}}^* = \overline{f}^{T*} = \dot{\overline{S}}^{T*}$

$$\int_{t_0}^{t_f} \overline{\lambda}^{T*}\Delta\dot{\overline{S}}\ dt = \left.\overline{\lambda}^{T*}\Delta\overline{S}\ \right|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\overline{\lambda}}^{T*}\Delta\overline{S}\ dt$$

$$R\,(\overline{\rho}^{*} + \Delta\overline{\rho}) = R(\overline{\rho}^{*}) + \left.\frac{\partial \Phi}{\partial \overline{S}_f}\right|^{*} \Delta\overline{S}_f - \overline{\lambda}_f^{T*} \Delta\overline{S}_f$$

$$+ \int_{t_0}^{t_f} (H_{\overline{S}}^{*}\ \Delta\overline{S} + \dot{\overline{\lambda}}^{T*}\ \Delta\overline{S} + H_{\overline{\rho}}^{*}\ \Delta\overline{\rho})\, dt \qquad (2\text{-}8)$$

For $\overline{\rho}$ to be minimizing $R = R(\overline{\rho}^{*} + \Delta\overline{\rho}) - R(\overline{\rho}\ )$ must be $\leq$ 0 for all $\Delta\overline{\rho}$ and therefore the following necessary conditions follow for a minimizing control $\overline{\rho}^{*}$.

$$\dot{\overline{\lambda}}^{T} = -H_{\overline{S}} \qquad (2\text{-}9)\ ^{a}$$

$$\overline{\lambda}^{T}(t_f) = \frac{\partial \Phi}{\partial \overline{S}_f} \qquad (2\text{-}10)\ ^{a}$$

$$H_{\overline{\rho}} = 0 \qquad (2\text{-}11)\ ^{a}$$

Necessary conditions (2-9) and (2-11) must be satisfied at every point along the trajectory, while (2-10) represents a necessary condition at the terminal boundary.

[a] These results are derived by Powers (7).

Chapter 3

OPTIMAL FORM OF CONTROL

To deduce an optimal control history for the orbital transfer problem, the Hamiltonian can be constructed and the necessary conditions applied. For the problem under consideration it is desired to minimize the time of powered flight (or to minimize propellant consumed). In this case the return function is of the form

$$R = \Phi(X_f Y_f Z_f \ \dot{X}_f \dot{Y}_f \dot{Z}_f) + \int_{t_0}^{t_f} 1 \, dt \ .$$

This return function is subject to the state equations already introduced; subsequently, the Hamiltonian can be defined as

$$H = 1 + \lambda_1 U + \lambda_2 V + \lambda_3 W + \lambda_4 (a \cos\theta_y \sin\theta_p - g_x)$$
$$+ \lambda_5 (a \sin\theta_y - g_y) + \lambda_6 (a \cos\theta_y \cos\theta_p - g_z) \qquad \text{(3-1) a}$$

[a] The previously defined $\overline{P}$ vector is a three vector composed of λ_4, λ_5, and λ_6.

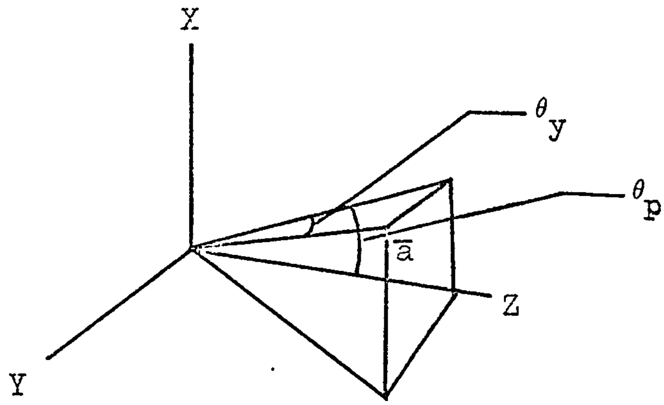


Figure 3.1

Thrust Control Angle
Coordinate System

where

$$\dot{U} = a \cos \theta_y \sin \theta_p - g_x \tag{3-2}$$

$$\dot{V}' = a \sin \theta_y - g_y \tag{3-3}$$

$$\dot{W} = a \cos \theta_y \cos \theta_p - g_z \tag{3-4}$$

$$\dot{X} = U \tag{3-5}$$

$$\dot{Y} = V' \tag{3-6}$$

$$\dot{Z} = W \tag{3-7}$$

$$F = \dot{m} V_{ex} \text{ (const)} \qquad \text{(3-8) a}$$

As previously introduced, the necessary conditions are

a F/m may also be constant. See appendix E.

$$\dot{\lambda}^T = -H_{\overline{S}} \qquad (2\text{-}9)$$

$$\lambda^T (t_f) = \frac{\partial \Phi}{\partial \overline{S}_f} \qquad (2\text{-}10)$$

$$H_{\overline{\rho}} = 0 \qquad (2\text{-}11)$$

The state variables, $\overline{S}$, in this problem are U, V', W, X, Y, Z, and the control, $\overline{\rho}$, corresponds to the two vector control variables θ_p and θ_y .

Applying the first necessary condition yields the following results:

$$\dot{\lambda}_1 = -H_x = \lambda_4 \frac{\partial g_x}{\partial x} + \lambda_5 \frac{\partial g_y}{\partial x} + \lambda_6 \frac{\partial g_z}{\partial x} \qquad (3\text{-}9)^{a}$$

$$\dot{\lambda}_2 = -H_y = \lambda_4 \frac{\partial g_x}{\partial y} + \lambda_5 \frac{\partial g_y}{\partial y} + \lambda_6 \frac{\partial g_z}{\partial y} \qquad (3\text{-}10)^{a}$$

$$\dot{\lambda}_3 = -H_z = \lambda_4 \frac{\partial g_x}{\partial z} + \lambda_5 \frac{\partial g_y}{\partial z} + \lambda_6 \frac{\partial g_z}{\partial z} \qquad (3\text{-}11)^{a}$$

$$\dot{\lambda}_4 = -H_u = -\lambda_1 \qquad (3\text{-}12)^{a}$$

$$\dot{\lambda}_5 = -H_{v'} = -\lambda_2 \qquad (3\text{-}13)^{a}$$

$$\dot{\lambda}_6 = -H_w = -\lambda_3 \qquad (3\text{-}14)^{a}$$

[a] This system of equations corresponds to equation (1-2), $\dot{\overline{P}} = P \cdot \nabla \overline{g}$.

If the approximation is made that the gravitational acceleration, g , is independent of position ($\nabla \bar{g} = 0$), then the Lagrange multipliers can be determined.

$$\dot{\lambda}_1 = 0 \qquad \lambda_1 = d_1 \tag{3-15}$$

$$\dot{\lambda}_2 = 0 \qquad \lambda_2 = d_2 \tag{3-16}$$

$$\dot{\lambda}_3 = 0 \qquad \lambda_3 = d_3 \tag{3-17}$$

$$\dot{\lambda}_4 = - \lambda_1, \text{ or } \lambda_4 = c_1 - d_1 t \tag{3-18}$$

$$\dot{\lambda}_5 = - \lambda_2, \text{ or } \lambda_5 = c_2 - d_2 t \tag{3-19}$$

$$\dot{\lambda}_6 = - \lambda_3, \text{ or } \lambda_6 = c_3 - d_3 t \tag{3-20}$$

Applying the third necessary condition (differentiating the Hamiltonian with respect to the control variables θ_p and θ_y) yields the following result:

$$H_{\theta_p} = \lambda_4 \,(a \cos \theta_y \cos \theta_p) - \lambda_6 \,(a \cos \theta_y \sin \theta_p) = 0 \tag{3-21}$$

$$\lambda_4 \cos \theta_p = \lambda_6 \sin \theta_p$$

$$\tan \theta_p = \frac{\lambda_4}{\lambda_6}$$

$$\tan \theta_p = \frac{c_1 - d_1 t}{c_3 - d_3 t} \tag{3-22}$$

$$H_{\theta_y} = \lambda_5 \, (a \cos \theta_y) - \lambda_6 \, (a \sin \theta_y) = 0 \qquad \text{(3-23) a}$$

$$\lambda_5 \cos \theta_y = \lambda_6 \sin \theta_y$$

$$\tan \theta_y = \frac{\lambda_5}{\lambda_6}$$

$$\tan \theta_y = \frac{c_2 - d_2 t}{c_3 - d_3 t} \qquad \text{(3-24)}$$

[a] This is true only when $\theta_p = 0$ since θ_p and θ_y are coupled angles.

Chapter 4

CLOSED FORM SOLUTION FOR THE TWO-POINT BOUNDARY VALUE PROBLEM

The previous analysis has shown that, under certain assumptions ($\nabla \bar{g} = 0$ and constant thrust) the optimal form of control for an orbital transfer maneuver is of a bi-linear tangent form (equations 3-22 and 3-24).

$$\tan = \frac{\psi + \beta t}{\xi + \eta t} \qquad (4\text{-}1)$$

Guidance formulations based upon this approximation have been used successfully for limited transfer problems. Smith (4) presents such a formulation, in which the variables θ_p and θ_y can be determined explicitly from a system of algebraic equations.

A major limitation of this formulation results from the assumption that $\nabla \bar{g} = 0$. As the powered flight burn arc increases in length, the gravity assumption becomes worse and convergence cannot be obtained. The size of the burn arc for which convergence can be insured is also a function of the thrust acceleration of the maneuvering spacecraft. As this acceleration level decreases, the maximum size of the burn arc (for which convergence can be obtained) also decreases.

The following analysis will be concerned with application of this bi-linear tangent control law (in an abbreviated form) to include the more extreme problems of long burn arcs and low-thrust maneuvers. The following guidance formulation will be limited to single burn arcs and minimization of powered flight burn time considering gravitational and thrust acceleration forces only.

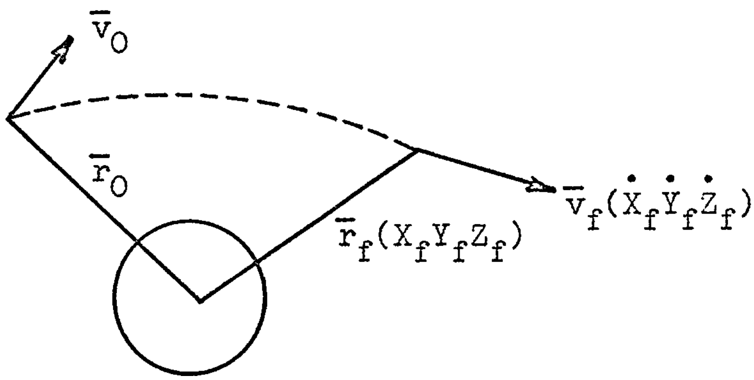


Figure 4.1

Final State Components
For Orbit Change

Temporarily the solution will be limited to the control of five components of position and velocity ($X_f, Y_f, \dot{X}_f, \dot{Y}_f$, and $\dot{Z}_f$). The basic assumption, $\nabla \bar{g} = 0$, will be made such that the form of control can be expressed as tan $= \dfrac{\psi + \beta t}{\xi + \eta t}$.

However, a piecing procedure will be used to form large burn arcs from smaller burn arcs. As illustrated in figure 4.2, the large burn arcs may be subdivided into smaller burn arcs and the approximations made need only be valid for the much smaller burn arcs.

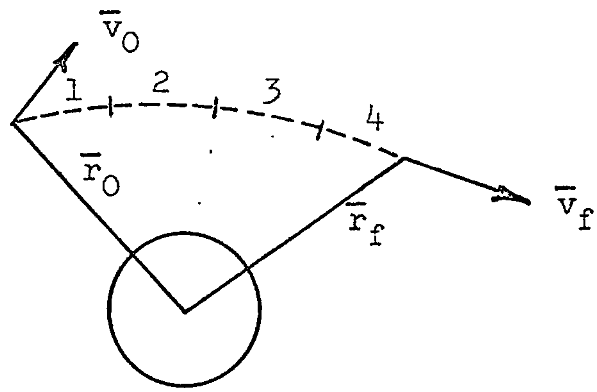


Figure 4.2

Piecing Procedure For Orbit Change

With such a piecing procedure, a guidance formulation can be extended to very large burn arcs and low-thrust maneuvers. However, an additional problem is introduced concerning intermediate boundary values for each of these smaller arcs. While each individual burn arc may be near-optimal, the sum of these burn arcs cannot be near-optimal unless the boundary values $\bar{r}$ and $\bar{v}$ are properly selected. A procedure for selecting near-optimal boundary values is contained in appendix D. This procedure assumes that the Lagrange multiplier vector, $\bar{P}$, is piecewise linear and continuous. Experience has shown this to be a good approximation for a variety of orbital missions.

The solution across any individual burn arc can now be developed and coupled with this piecing procedure. The solution will be generalized to multiple performance periods (to account for change of thrust level, constant acceleration periods, mixture ratio changes, etc.), although only one may be required during a particular small burn arc. The particular solution to the optimal thrust direction has been shown to be (equations 3-22 and 3-24).

$$\tan \theta_p = \frac{\lambda_4}{\lambda_6} = \frac{c_1 - \lambda_2 t}{c_3 - \lambda_3 t} \tag{4-2}$$

$$\tan \theta_y = \frac{\lambda_5}{\lambda_6} = \frac{c_2 - \lambda_2 t}{c_3 - \lambda_3 t} \tag{4-3}$$

From the necessary condition, $\lambda^T(t_f) = \frac{\partial \Phi}{\partial \overline{S}_f}$, it is shown that $\lambda_3(t_f) = \lambda_3 = \frac{\partial \Phi}{\partial Z_f} = 0$ if the Z_f - component of position is not controlled (see figure 4-3).

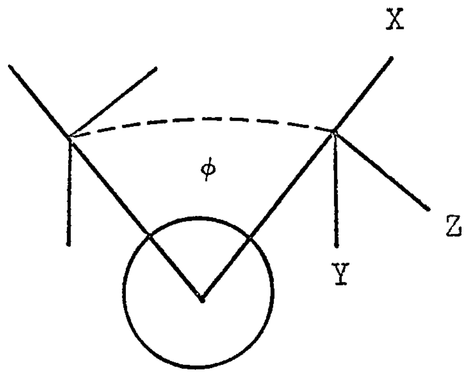


Figure 4.3
Thrust Control Angle
Coordinate System
(Inertially Fixed)

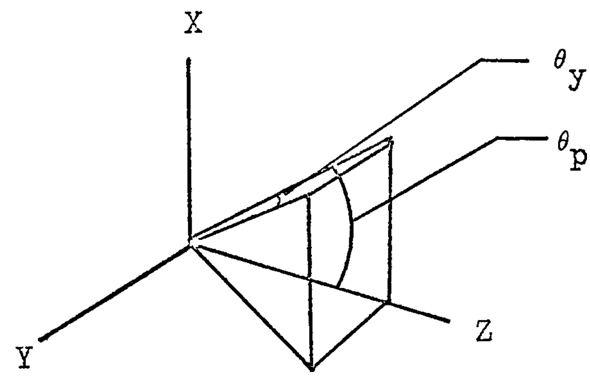


Figure 4.4
Thrust Control Angle
Coordinate System

Additionally,

$$\lambda_1(t_f) = \lambda_1 = \frac{\partial \Phi}{\partial X_f} \tag{4-4}$$

$$\lambda_2(t_f) = \lambda_2 = \frac{\partial \Phi}{\partial Y_f} \tag{4-5}$$

$$\lambda_4(t_f) = \frac{\partial \Phi}{\partial \dot{X}_f} \tag{4-6}$$

$$\lambda_5(t_f) = \frac{\partial \Phi}{\partial \dot{Y}_f} \tag{4-7}$$

$$\lambda_6(t_f) = \frac{\partial \Phi}{\partial \dot{Z}_f} \tag{4-8}$$

which reduce the control functions to the following form:

$$\tan \theta_p = \frac{\partial \Phi}{\partial \dot{X}_f} \Big/ \frac{\partial \Phi}{\partial \dot{Z}_f} - \frac{\partial \Phi}{\partial X_f} \Big/ \frac{\partial \Phi}{\partial Z_f} \; (t) \tag{4-9}$$

$$\tan \theta_y = \frac{\partial \Phi}{\partial \dot{Y}_f} \Big/ \frac{\partial \Phi}{\partial \dot{Z}_f} - \frac{\partial \Phi}{\partial Y_f} \Big/ \frac{\partial \Phi}{\partial Z_f} \; (t) \tag{4-10}$$

Explicit evaluation of these constant partial derivatives is possible if the following approximations are made.

$$\tan \theta_p \approx \theta_p \approx (\bar{\theta}_p - k_1) + k_2 t \tag{4-11}$$

$$\tan \theta_y \approx \theta_y \approx (\bar{\theta}_y - k_3) + k_4 t \tag{4-12}$$

$$\sin(-k_1 + k_2 t) \approx -k_1 + k_2 t \tag{4-13}$$

$$\cos(-k_1 + k_2 t) \approx 1.0 \tag{4-14}$$

$$\sin(-k_3 + k_4 t) \approx -k_3 + k_4 t \tag{4-15}$$

$$\cos(-k_3 + k_4 t) \approx 1.0 \tag{4-16}$$

At first these approximations appear to be restrictive, but this is not the case. As the burn arc is segmented into more pieces, these become very good approximations. The $\bar{\theta}_p$, $\bar{\theta}_y$ terms are constant control angle terms which will solve the velocity required part of the transfer; however, since the control is linear, compensating terms k_1 and k_3 are added to achieve the velocity. The terms k_2 and k_4 directly correspond to the position partial derivative terms. The solution to this problem is now found by determining values for k_1, k_2, k_3, and k_4 which satisfy the boundary conditions ($\Phi(S_f) = 0$).

This solution can be implemented by introducing the appropriate dynamical equations (the Z equation is not needed since Z_f is free) and taking their first and second integrations.

$$\ddot{X} = a \cos \theta_y \sin \theta_p - g_x \tag{4-17}$$

$$\ddot{Y} = a \sin \theta_y - g_y \tag{4-18}$$

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \theta_y \sin \theta_p \, dt - \int_{t_0}^{t_f} g_x \, dt \tag{4-19 a}$$

$$\dot{Y}_f = \dot{Y} + \int_{t_0}^{t_f} a \sin \theta_y \, dt - \int_{t_0}^{t_f} g_y \, dt \tag{4-20}$$

The following trigonometric substitutions may be made to introduce the control constants $\bar{\theta}_p$, $\bar{\theta}_y$, k_1, k_2, k_3, and k_4.

$$\sin \theta_p \approx \sin \bar{\theta}_p + \cos \bar{\theta}_p \, (-k_1 + k_2 t) \tag{4-21}$$

$$\sin \theta_y \approx \sin \bar{\theta}_y + \cos \bar{\theta}_y \, (-k_3 + k_4 t) \tag{4-22}$$

$$\cos \theta_p \approx \cos \bar{\theta}_p \tag{4-23}$$

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \bar{\theta}_y \left(\sin \bar{\theta}_p + \cos \bar{\theta}_p \, (-k_1 + k_2 t)\right) dt - \int_{t_0}^{t_f} g_x \, dt \tag{4-24}$$

[a] The value of t_f comes from solution of an explicit equation in appendix G.

$$\dot{Y}_f = \dot{Y} + \int_{t_0}^{t_f} a\ (\sin \bar{\theta}_y + \cos \bar{\theta}_y\ (-k_3 + k_4 t))\ dt$$

$$- \int_{t_0}^{t_f} g_y\ dt \tag{4-25}$$

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \bar{\theta}_y\ (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\ dt$$

$$+ \int_{t_0}^{t_f} a\ k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\ dt$$

$$- \int_{t_0}^{t_f} g_x\ dt \tag{4-26}$$

$$\dot{Y}_f = \dot{Y} + \int_{t_0}^{t_f} a\ (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y)\ dt$$

$$+ \int_{t_0}^{t_f} a\ k_4 t \cos \bar{\theta}_y\ dt - \int_{t_0}^{t_f} g_y\ dt \tag{4-27}$$

Generalization of this integration to several components of t_f will yield a general form.

$$\dot{X}_f = \dot{X} + \int_{0}^{T_1} a_1 \cos \bar{\theta}_y\ (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\ dt$$

$$+ \int_{0}^{T_1} a_1 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\ dt$$

$$+ \int_{0}^{T_2} a_2 \cos \bar{\theta}_y\ (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\ dt$$

$$+\int_0^{T_2} a_2 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p \, dt$$

$$+\int_0^{T_2} a_2 k_2 (T_1+T_c) \cos \bar{\theta}_y \cos \bar{\theta}_p \, dt$$

$$-\int_0^{T_1+T_c+T_2} g_x \, dt \tag{4-28}$$

$$\dot{Y}_f = \dot{Y} + \int_0^{T_1} a_1 (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y) \, dt$$

$$+\int_0^{T_1} a_1 k_4 t \cos \bar{\theta}_y \, dt$$

$$+\int_0^{T_2} a_2 (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y) \, dt$$

$$+\int_0^{T_2} a_2 k_4 t \cos \bar{\theta}_y \, dt$$

$$+\int_0^{T_2} a_2 k_4 (T_1+T_c) \cos \bar{\theta}_y \, dt$$

$$-\int_0^{T_1+T_c+T_2} g_y \, dt \tag{4-29}$$

Integral values derived in appendix F are now substituted into the expressions.

$$\dot{X}_f = \dot{X} + V_{ex_1} L_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)$$

$$+ k_2 J_1 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ V_{ex_2} L_2 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)$$

$$+ k_2 J_2 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 V_{ex_2} L_2 (T_1 + T_c) \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$- g_x (T_1 + T_c + T_2) \tag{4-30}$$

$$\dot{Y}_f = \dot{Y} + V_{ex_1} L_1 (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y)$$

$$+ k_4 J_1 \cos \bar{\theta}_y + V_{ex_2} L_2 (\sin \bar{\theta}_y - K_3 \cos \bar{\theta}_y)$$

$$+ k_4 J_2 \cos \bar{\theta}_y + k_4 (T_1 + T_c) V_{ex_2} L_2 \cos \bar{\theta}_y$$

$$- g_y (T_1 + T_c + T_2) \tag{4-31}$$

Equations (4-17) and (4-18) must now be integrated similarly using $\bar{\theta}_p$ and $\bar{\theta}_y$. Integration using these constant control angles insures that the required velocity is obtained during the transfer.

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \bar{\theta}_y \sin \bar{\theta}_p \, dt$$

$$- \int_{t_0}^{t_f} g_x \, dt \tag{4-32}$$

$$\dot{\bar{Y}}_f = \dot{Y} + \int_{t_0}^{t_f} a \sin \bar{\theta}_y \, dt - \int_{t_0}^{t_f} g_y \, dt \tag{4-33}$$

$$\dot{\bar{X}}_f = \dot{X} + \int_0^{T_1} a_1 \cos \bar{\theta}_y \sin \bar{\theta}_p \, dt$$

$$+ \int_0^{T_2} a_2 \cos \bar{\theta}_y \sin \bar{\theta}_p \, dt$$

$$- \int_0^{T_1+T_c+T_2} g_x \, dt \tag{4-34}$$

$$\dot{\bar{Y}}_f = \dot{Y} + \int_0^{T_1} a_1 \sin \bar{\theta}_y \, dt + \int_0^{T_2} a_2 \sin \bar{\theta}_y \, dt$$

$$- \int_0^{T_1+T_c+T_2} g_y \, dt \tag{4-35}$$

$$\dot{\bar{X}}_f = \dot{X} + V_{ex_1} L_1 \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$+ V_{ex_2} L_2 \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$- g_x (T_1+T_c+T_2) \tag{4-36}$$

$$\dot{\bar{Y}}_f = \dot{Y} + V_{ex_1} L_1 \sin \bar{\theta}_y + V_{ex_2} L_2 \sin \bar{\theta}_y$$

$$- g_y (T_1+T_c+T_2) \tag{4-37}$$

If the conditions $\dot{\bar{X}}_f - \dot{X}_f = 0$ and $\dot{\bar{Y}}_f - \dot{Y}_f = 0$ are enforced, integration with the linear control law will enforce the velocity condition.

$$\dot{\bar{X}}_f - \dot{X}_f = 0 = -k_1 V_{ex_1} L_1 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 J_1 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$- k_1 V_{ex_2} L_2 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 V_{ex_2} L_2 (T_1+T_c) \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 J_2 \cos \bar{\theta}_y \cos \bar{\theta}_p \qquad (4\text{-}38)$$

$$\dot{\bar{Y}}_f - \dot{Y}_f = 0 = -k_3 V_{ex_1} L_1 \cos \bar{\theta}_y + k_4 J_1 \cos \bar{\theta}_y$$

$$- k_3 V_{ex_2} L_2 \cos \bar{\theta}_y$$

$$+ k_4 V_{ex_2} L_2 (T_1+T_c) \cos \bar{\theta}_y$$

$$+ k_4 J_2 \cos \bar{\theta}_y \qquad (4\text{-}39).$$

$$\dot{\bar{X}}_f - \dot{X}_f = 0 = -A_p k_1 + B_p k_2 \qquad (4\text{-}40)$$

$$\dot{\bar{Y}}_f - \dot{Y}_f = 0 = -A_y k_3 + B_y k_4 \qquad (4\text{-}41)$$

A second integration of equations (4-17) and (4-18) will now satisfy the position requirement in X and Y.

$$X_f = X + \int_0^{T_1+T_c+T_2} \dot{X}\, dt$$

$$+ \int_0^{T_1} \int_0^{T_1} a_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt^2$$

$$+\int_0^{T_1}\int_0^{T_1} a_1 k_2 t \cos\bar{\theta}_y \cos\bar{\theta}_p \, dt$$

$$+ (T_2+T_c) \int_0^{T_1} a_1 \cos\bar{\theta}_y (\sin\bar{\theta}_p - k_1 \cos\bar{\theta}_p) \, dt$$

$$+ (T_2+T_c) \int_0^{T_1} a_1 k_2 t \cos\bar{\theta}_y \cos\bar{\theta}_p \, dt$$

$$+\int_0^{T_2}\int_0^{T_2} a_2 \cos\bar{\theta}_y (\sin\bar{\theta}_p - k_1 \cos\bar{\theta}_p) \, dt^2$$

$$+\int_0^{T_2}\int_0^{T_2} a_2 k_2 t \cos\bar{\theta}_y \cos\bar{\theta}_p \, dt^2$$

$$+\int_0^{T_2}\int_0^{T_2} a_2 k_2 (T_1+T_c) \cos\bar{\theta}_y \cos\bar{\theta}_p \, dt^2$$

$$-\int_0^{T_1+T_c+T_2} g_x \, dt^2 \qquad (4\text{-}42)$$

$$X_f = X + \dot{X} (T_1+T_c+T_2) - S_1' \cos\bar{\theta}_y (\sin\bar{\theta}_p - k_1 \cos\bar{\theta}_p)$$

$$- Q_1 k_2 \cos\bar{\theta}_p \cos\bar{\theta}_y$$

$$+ V_{ex_1} L_1 (T_2+T_c) \cos\bar{\theta}_y (\sin\bar{\theta}_p - k_1 \cos\bar{\theta}_p)$$

$$+ J_1 (T_2+T_c) k_2 \cos\bar{\theta}_y \cos\bar{\theta}_p$$

$$- S_2' \cos\bar{\theta}_y (\sin\bar{\theta}_p - k_1 \cos\bar{\theta}_p)$$

$$- Q_2 k_2 \cos\bar{\theta}_y \cos\bar{\theta}_p - S_2' k_2 (T_1+T_c) \cos\bar{\theta}_y \cos\bar{\theta}_p$$

$$- g_x \; (T_1+T_c+T_2)^2/2 \qquad (4\text{-}43)$$

$$Y_f = Y + \int_0^{T_1+T_c+T_2} \dot{Y}\, dt$$

$$+ \int_0^{T_1}\int_0^{T_1} a_1 \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y) \; dt^2$$

$$+ \int_0^{T_1}\int_0^{T_1} a_1 \; k_4 t \cos \bar{\theta}_y \; dt^2$$

$$+ \; (T_2+T_c)\int_0^{T_1} a_1 \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y) \; dt$$

$$+ \; (T_2+T_c)\int_0^{T_1} a_1 \; k_4 t \cos \bar{\theta}_y \; dt$$

$$+ \int_0^{T_2}\int_0^{T_2} a_2 \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y) \; dt^2$$

$$+ \int_0^{T_2}\int_0^{T_2} a_2 \; k_4 (T_1+T_c) \cos \bar{\theta}_y \; dt^2$$

$$- \int_0^{T_1+T_c+T_2}\int_0 g_y \; dt^2 \qquad (4\text{-}44)$$

$$Y_f = Y + \dot{Y} \; (T_1+T_c+T_2) - S_1' \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y)$$

$$- \; Q_1 \; k_4 \cos \bar{\theta}_y$$

$$+ \; (T_2+T_c) \; (V_{ex_1} \; L_1) \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y)$$

$$+ \; (T_2+T_c) \; J_1 \; k_4 \cos \bar{\theta}_y - S_2' \; (\sin \bar{\theta}_y - k_3 \cos \bar{\theta}_y)$$

$$- S_2' k_4 (T_1+T_c) \cos \bar{\theta}_y - Q_2 k_4 \cos \bar{\theta}_y$$

$$- g_y (T_1+T_c+T_2)^2/2 \tag{4-45}$$

Equations (4-44) and (4-45) are each a function of known integrals and the constants k_1, k_2, k_3, and k_4.

$$0 = C_p k_1 - D_p k_2 + E_p \tag{4-46}$$

$$0 = C_y k_3 - D_y k_4 + E_y \tag{4-47}$$

The control constants k_1, k_2, k_3, and k_4 may now be evaluated by solving equations (4-40) (4-41), (4-46), and (4-47) simultaneously. The control angles (θ_p θ_y) can be determined as follows:

$$\theta_p = \bar{\theta}_p - k_1 + k_2 t_f \tag{4-48}$$

$$\theta_y = \bar{\theta}_y - k_3 + k_4 t_f \tag{4-49}$$

$$\Delta\dot{X} = \dot{X}_f - \dot{X} - g_x t_f \tag{4-50}$$

$$\Delta\dot{Y} = \dot{Y}_f - \dot{Y} - g_y t_f \tag{4-51}$$

$$\Delta\dot{Z} = \dot{Z}_f - \dot{Z} - g_z t_f \tag{4-52}$$

$$\bar{\theta}_p = \tan^{-1} \left(\frac{\Delta \dot{X}}{\Delta \dot{Z}} \right) \tag{4-53}$$

$$\bar{\theta}_y = \tan^{-1} \left(\frac{\Delta \dot{Y}}{\sqrt{\Delta \dot{Y}^2 + \Delta \dot{Z}^2}} \right) \tag{4-54}$$

A generalization of the above equations to multiple components of t_f (n components) is possible by inspection. The general equations which can be used for the solution of $\bar{\theta}_p$, $\bar{\theta}_y$, k_1, k_2, k_3, and k_4 are shown as follows:

$$\Delta\dot{X} = \dot{X}_f - \dot{X} - g_x \sum_{i=1}^{n} T_i \tag{4-55}$$

$$\Delta\dot{Y} = \dot{Y}_f - \dot{Y} - g_y \sum_{i=1}^{n} T_i \tag{4-56}$$

$$\Delta\dot{Z} = \dot{Z}_f - \dot{Z} - g_z \sum_{i=1}^{n} T_i \tag{4-57}$$

$$\bar{\theta}_p = \tan^{-1}\left(\frac{\Delta\dot{Y}}{\Delta\dot{Z}}\right) \tag{4-58}$$

$$\bar{\theta}_y = \tan^{-1}\left(\frac{\Delta\dot{Y}}{\sqrt{\Delta\dot{Y}^2+\Delta\dot{Z}^2}}\right) \tag{4-59}$$

$$A_y = \sum_{i=1}^{n} V_{ex_i} L_i \tag{4-60}$$

$$B_y = \sum_{i=1}^{n} J_i + \sum_{i=1}^{n-1} V_{ex_{i+1}} L_{i+1} \sum_{j=1}^{i} T_j \tag{4-61}$$

$$C'_y = \sum_{i=1}^{n} S_i - \sum_{i=1}^{n-1} T_{i+1} \sum_{j=1}^{i} V_{ex_j} L_j \tag{4-62}$$

$$C_y = C'_y \cos \bar{\theta}_y \tag{4-63}$$

$$D'_y = \sum_{i=1}^{n} Q_i + \sum_{i=1}^{n-1} S'_{i+1} \sum_{j=1}^{i} T_j -$$

$$\sum_{i=1}^{n-1}\left((T_{i+1}) \sum_{j=1}^{i} J_j\right) -$$

$$\sum_{i=l+1}^{n-1}\left(\sum_{i}^{n-1} (T_{i+1}) V_{ex_i} L_i \sum_{j=1}^{j-1} T_j\right) \qquad (4\text{-}64)$$

$$D_y = D_y' \cos \bar{\theta}_y \qquad (4\text{-}65)$$

$$E_y = Y - Y_f + \dot{Y} \sum_{i=1}^{n} T_i + \frac{1}{2} g_y \left(\sum_{i=1}^{n} T_i\right)^2 -$$

$$C_y' \sin \bar{\theta}_y \qquad (4\text{-}66)$$

$$k_3 = \frac{B_y E_y}{A_y D_y - B_y C_y} \qquad (4\text{-}67)$$

$$k_4 = \frac{k_3 A_y}{B_y} \qquad (4\text{-}68)$$

$$\bar{\theta}_p = \tan^{-1}\left(\frac{\Delta \dot{X}}{\Delta \dot{Z}}\right) \qquad (4\text{-}69)$$

$$A_p = A_y \cos \bar{\theta}_y \qquad (4\text{-}70)$$

$$B_p = B_y \cos \bar{\theta}_y \qquad (4\text{-}71)$$

$$C_p = C_y' \cos \bar{\theta}_y \cos \bar{\theta}_p \qquad (4\text{-}72)$$

$$E_p = X - X_f + \dot{X} \sum_{i=1}^{n} T_i + \frac{1}{2} g_x \left(\sum_{i=1}^{n} T_i\right)^2 -$$

$$C_y' \cos \bar{\theta}_y \sin \bar{\theta}_p \qquad (4\text{-}73)$$

$$D_p = D'_y \cos \bar{\theta}_y \cos \bar{\theta}_p \tag{4-74}$$

$$k_1 = \frac{B_p E_p}{A_p D_p - B_p C_p} \tag{4-75}$$

$$k_2 = \frac{k_1 A_p}{B_p} \tag{4-76}$$

Chapter 5

NUMERICAL INVESTIGATION

To illustrate the workability of the closed form solution which has been developed, an example problem has been selected and solved. Additionally, the same problem is also solved by an optimization procedure and the characteristic velocities [a] compared as a measure of performance. The problem selected consists of a transfer from a 50 by 100 - nautical mile ellipse to a coplanar 400 - nautical mile circular orbit with minimum propellant usage. This is a typical shuttle transfer from low earth orbit to a space station orbit.

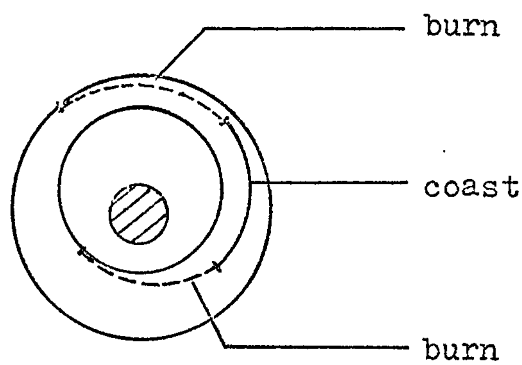


Figure 5.1

Two-Burn Orbit Transfer

[a] Characteristic velocity is the integral of thrust acceleration, $\int_{t_0}^{t_f} a(t)\,dt$.

This transfer is accomplished for a relatively low thrust-to-weight ratio (.05 g's) and thus provides the type of problem which is most sensitive to the guidance formulation. The following analysis will be concerned with both solutions to this problem and a comparison of the results.

To achieve the numerical results for this comparison, two digital program simulations were required. These two simulations are a simulation for an optimal orbit transfer and a simulation for the guidance formulation previously shown. These two simulations will be briefly discussed to familiarize the reader with each solution procedure.

The simulation result for an optimal orbit transfer was achieved by using an existing gradient search parameter optimization program.[a] The program constructs a return function $R = t_f + \Phi(\overline{S}_f)$ where $\Phi(\overline{S}_f)$ is a penalty for not attaining the desired final state and t_f is the total time of powered flight. It is desired to minimize R. The solution procedure then requires that the initial control variables $(\overline{\rho})$ be individually perturbed and a trajectory numerically integrated (using a fourth-order Runga-Kutta scheme) to find the value of the gradient vector $\frac{\partial R}{\partial \overline{\rho}}$.

After finding this vector, a one dimensional search scheme is used to find a step size value along the gradient direction

[a] This program was developed by the Guidance and Dynamics Branch at the Manned Spacecraft Center.

which minimizes R. This multi-step process is then repeated until the gradient magnitude becomes less than some small value (convergence is attained.) The gradient procedure uses a double precision state ($\overline{r}$ and $\overline{v}$) and, therefore, achieves the final state with a good deal of accuracy. Good initial values for the control vector are necessary for this procedure to attain convergence. Once these good initial values are provided, however, the solution procedure provides a near optimal orbit transfer trajectory.

The simulation result for the guidance formulation is achieved by implementing the set of closed form equations in chapter 4 (equation 4-55 through 4-76) into a digital simulation program. Additional equations used to evaluate gravity, the piecing procedure, the thrust acceleration integrals, and burn time are taken from appendices C,D,F, and G, respectively. From these equations the control constants $\theta_p = \overline{\theta}_p - k_1 + k_2\ t_f$ and $\theta_y = \overline{\theta}_y - k_3 + k_4\ t_f$ are then evaluated explicitly at every two second interval in the powered flight trajectory simulation. The resulting control history (θ_p and θ_y) is then used and the dynamical equations (3-2 through 3-8) numerically integrated using a fourth order Runga-Kutta integration scheme. This guidance procedure uses a single precision state ($\overline{r}$ and $\overline{v}$) in the integration process.

The extremized solution to this problem, which is obtained from the gradient search parameter optimization procedure, will now be considered. The solution is formulated as a burn-coast-burn in which the parameters ($\bar{\rho}$) appear as engine on-off time and constants in some assumed control law. This assumed control law takes the form $\theta_p = \psi^* + \xi^* t + \eta^* t^2$ during each burn. The total parameters for this problem are therefore t_0, t_f, t_0', t_f', ψ^*, ξ^*, η^*, ψ^{**}, ξ^{**}, η^{**} (figure 5.2). The parameters are varied to achieve the final orbital conditions while minimizing total time of powered flight.

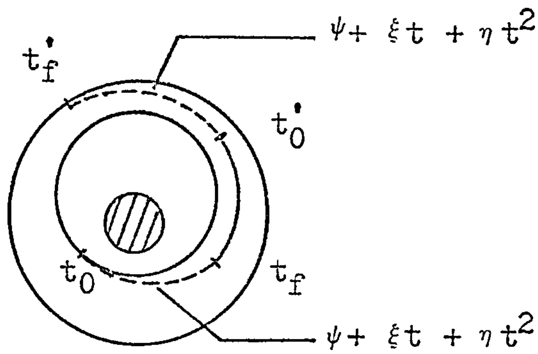


Figure 5.2

Two-Burn Orbit Transfer
Using Parameter Optimization

The solution to this problem is a 319 second burn and a 367 second burn separated by a 2440 second coast. The first burn is initiated at a true anomaly of -18 degrees and the second burn at a true anomaly of 173 degrees. The loss function [a] and control history for this transfer are shown in figures 5.3 through 5.6.

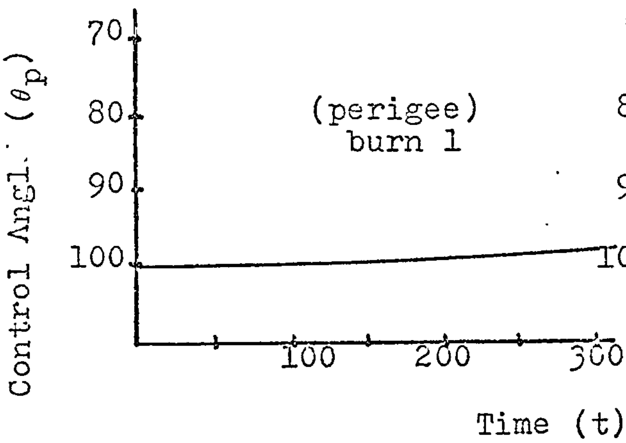


Figure 5.3

Control Angle History
Using Parameter
Optimization

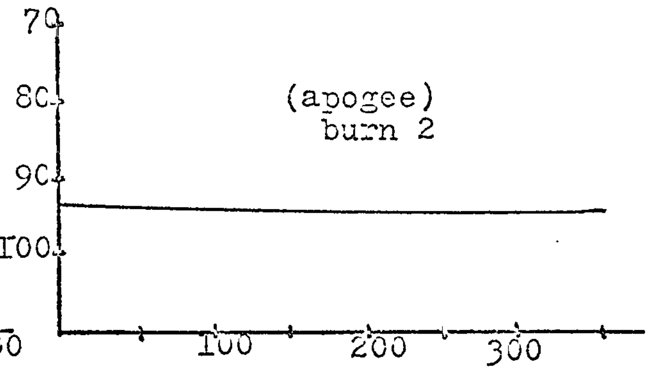
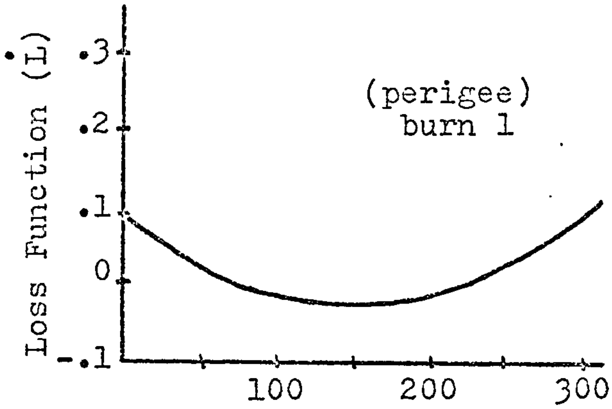
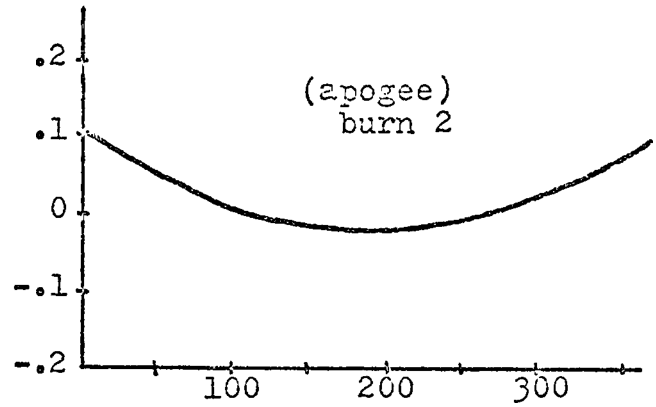


Figure 5.4

Control Angle History
Using Parameter
Optimization

[a] This loss function is derived in appendix B. Its integral value represents the difference between the characteristic velocity and the relative velocity change during a powered flight maneuver. Its integral value is therefore a measure of the efficiency of the maneuver.





Time (t)

Figure 5.5

Velocity Loss Function
Using Parameter
Optimization

Figure 5.6

Velocity Loss Function
Using Parameter
Optimization

The control angle during both burns is nearly linear with time and the slope is minimal. The loss function is approximately symmetric around the midpoint of the burn arc for both burns.

The guidance solution to this problem is now considered. The solution is posed as a burn-coast- burn; however, the procedure must handle each burn individually. The first burn is targeted to achieve the desired apogee radius and the second burn to achieve the desired perigee radius. Neither of the two burns could be made to converge as single arcs and, therefore, a two-burn arc piecing procedure is used.

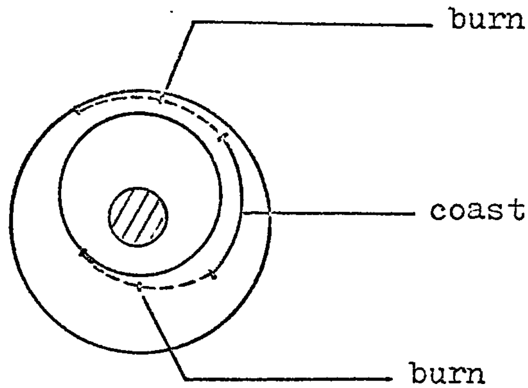


Figure 5.7

Two-Burn Orbit Transfer
Using The Guidance Formulation

The solution to the problem is a 324 second burn and a 380 second burn separated by approximately a 2400 second coast. The initial thrust maneuver is initiated at a true anomaly of -13 degrees and the second at a true anomaly of 173 degrees. The loss function and control history for this transfer are shown in figures 5.8 through 5.11 .

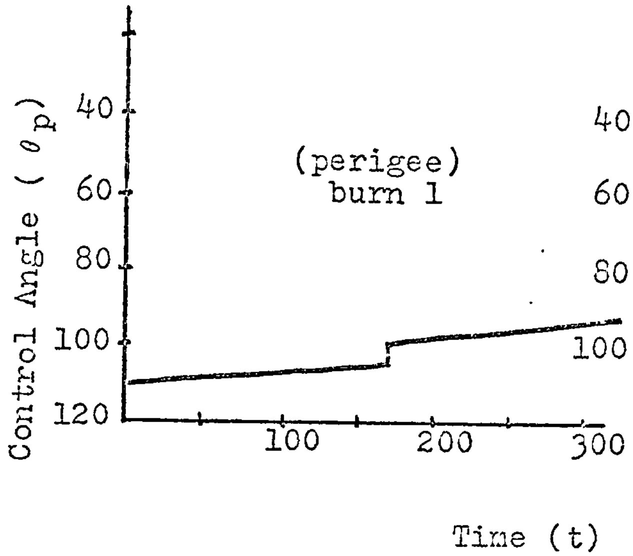


Figure 5.8

Control Angle History
Using The Guidance
Formulation

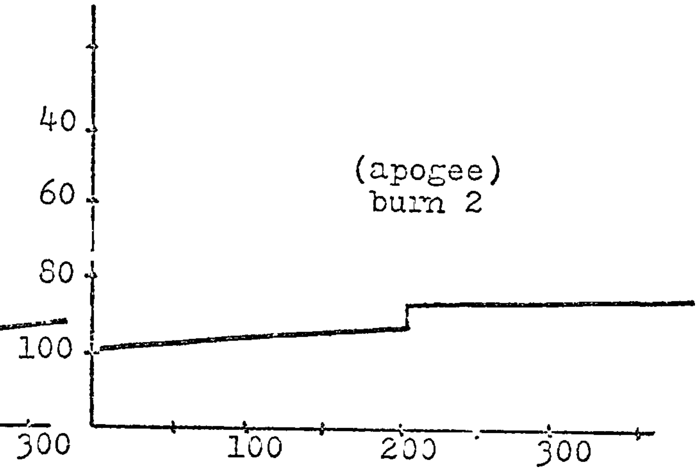


Figure 5.9

Control Angle History
Using The Guidance
Formulation

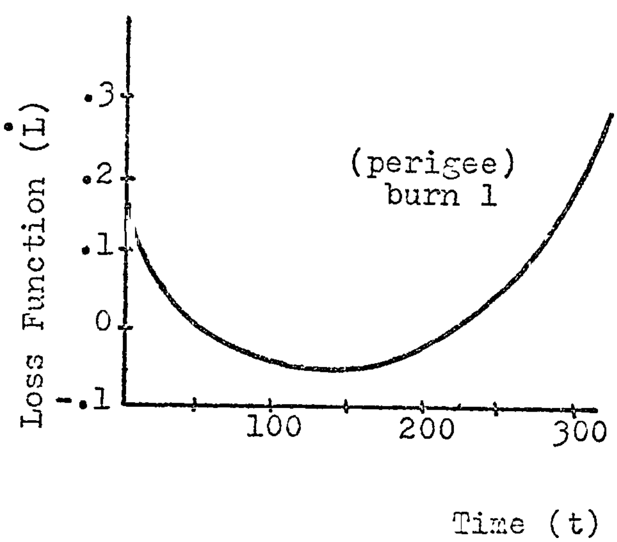


Figure 5.10

Velocity Loss Function
Using The Guidance
Formulation

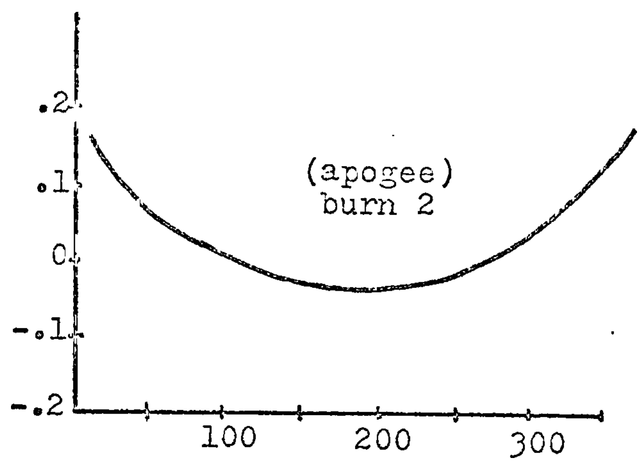


Figure 5.11

Velocity Loss Function
Using The Guidance
Formulation

It is observed that the control angle histories show some difference between the extremized and guidance solution, although this difference is not large in terms of burn time or propellant usage. It is also observed that the burn arc piecing procedure causes discontinuities in the control angle (θ_p, figures 5-8 and 5-9) and, therefore, some loss of performance. The discontinuities result from the inability of the multi-arc algorithm [a] to predict intermediate boundary values perfectly.

As a matter of interest, the first burn of this transfer was segmented into five burn pieces during which only a velocity control was used ($\theta_p = \bar{\theta}_p$). The control history and loss function in figures 5.12 and 5.13 correspond to this transfer.

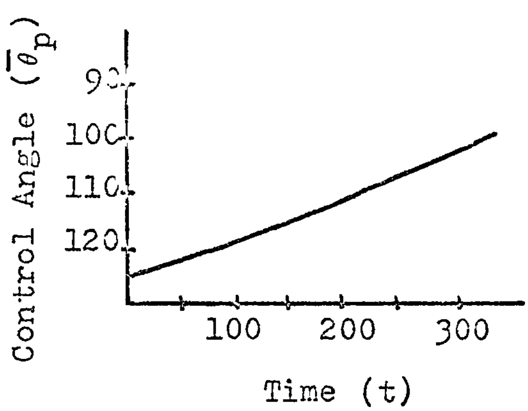
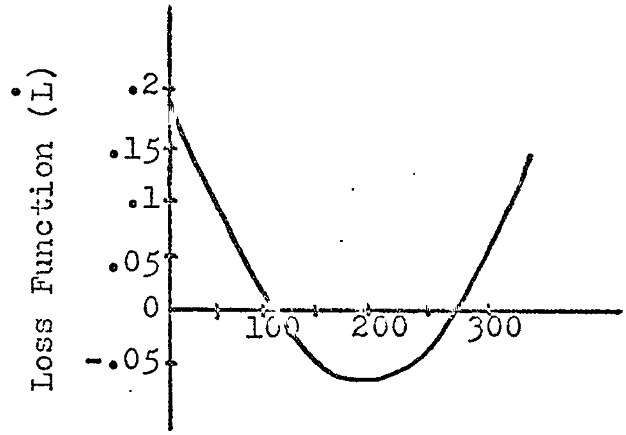


Figure 5.12

Control Angle History Using Velocity Control Only In The Guidance Formulation

[a] See appendix D.



Time (t)

Figure 5.13

Velocity Loss Function Using Velocity Control
Only In The Guidance Formulation

It is observed that the control angle history does not have discontinuities as seen in figures 5-8 and 5-9 and is approximately linear in time. Although only the velocity control is used ($\theta_p = \overline{\theta}_p$), the position boundary conditions ($\overline{r}_f$) are almost achieved for the transfer.

The significance of the velocity control option is that it may be useful for lengthy powered maneuvers during which position control ($\overline{r}_f$) is not required. It is, therefore, an alternate procedure which could be used in place of cross product of impulsive guidance procedures.

Chapter 6

EXTENSIONS

The guidance solution to the orbital transfer problem has solved for only five components of position and velocity ($X_f, Y_f, \dot{X}_f, \dot{Y}_f$, and $\dot{Z}_f$).

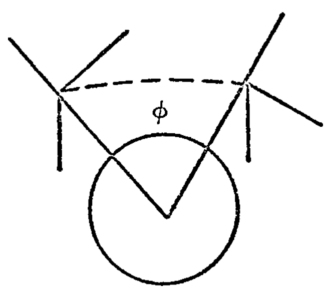


Figure 6.1

Control Angle
Coordinate System

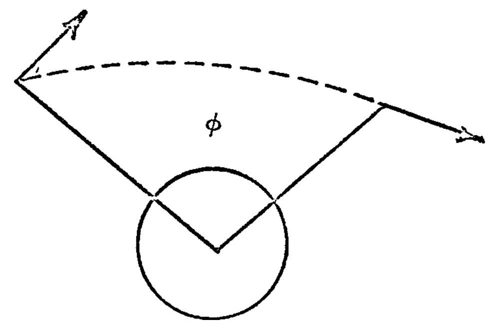


Figure 6.2

Final State Components
For Orbit Change

The general solution to this problem requires control of all six components of state; however, explicit control of all these variables has not been implemented in the framework of a parameterized control law. If the Z component of position is not free, then the optimál control (as established previously) is of a bilinear tangent form, $\tan\theta = \dfrac{\psi + \beta t}{\xi + \eta t}$. Under appropriate conditions (relatively small control angles) this form of control may be expressed as $\theta = \dfrac{\psi + \beta t}{\xi + \eta t}$ and may be expanded to $\theta = \psi^* + \xi^* t + \eta^* t^2$. Subsequent evaluation of the parameters in this quadratic control law will yield a solution which controls all final components of state. The following formulation will be concerned with the explicit solution of these control parameters (ψ , ξ , and η). (Numerical simulation of this formulation is not within the scope of this paper but may be implemented in the future.)

Explicit evaluation of the control parameters is possible if the following approximations are made.

$$\tan \theta_p \approx \theta_p \approx (\overline{\theta}_p - k_1) + k_2 t + k_5 t^2 \qquad (6-1)$$

$$\tan \theta_y \approx \theta_y \approx (\overline{\theta}_y - k_3) + k_4 t \qquad (6-2)$$

$$\sin (-k_1 + k_2 t + k_5 t^2) \approx - k_1 + k_2 t + k_5 t^2 \qquad (6-3)$$

$$\cos (-k_1 + k_2 t + k_5 t^2) \approx 1.0 \qquad (6-4)$$

$$\sin (-k_3 + k_4 t) \approx -k_3 + k_4 t \qquad (6-5)$$

$$\cos (-k_3 + k_4 t) \approx 1.0 \qquad (6-6)$$

The control parameters can now be introduced into the dynamical equations and integrated as before. Since the control for θ_y has not changed, the y dynamical equation will not be reintroduced.

$$\ddot{X} = a \cos \theta_y \sin \theta_p - g_x \qquad (6-7)$$

$$\ddot{Z} = a \cos \theta_y \cos \theta_p - g_z \qquad (6-8)$$

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \theta_y \sin \theta_p \, dt - \int_{t_0}^{t_f} g_x \, dt \tag{6-9}$$

$$\dot{Z}_f = \dot{Z} + \int_{t_0}^{t_f} a \cos \theta_y \cos \theta_p \, dt - \int_{t_0}^{t_f} g_z \, dt \tag{6-10}$$

The following trigonometric substitutions are made to introduce the control constants $\overline{\theta}_p$, $\overline{\theta}_y$, k, k_2, k_3, k_4, and k_5.

$$\sin \theta_p \approx \sin \overline{\theta}_p + \cos \overline{\theta}_p (-k_1 + k_2 t + k_5 t^2) \tag{6-11}$$

$$\sin \theta_y \approx \sin \overline{\theta}_y + \cos \overline{\theta}_y (-k_1 + k_2 t) \tag{6-12}$$

$$\cos \theta_p \approx \cos \overline{\theta}_p \tag{6-13}$$

$$\dot{X}_f = \dot{X} + \int_{t_0}^{t_f} a \cos \overline{\theta}_y \left(\sin \overline{\theta}_p + \cos \overline{\theta}_p (-k_1 + k_2 t + k_5 t^2) \right) dt - \int_{t_0}^{t_f} g_x \, dt \tag{6-14}$$

$$\dot{Z}_f = \dot{Z} + \int_{t_0}^{t_f} a \cos \overline{\theta}_y \left(\cos \overline{\theta}_p - \sin \overline{\theta}_p (-k_1 + k_2 t + k_5 t^2) \right) dt - \int_{t_0}^{t_f} g_z \, dt \tag{6-15}$$

This integration is now extended to multiple performance periods.

$$\dot{X}_f = \dot{X} + \int_0^{T_1} a_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt$$

$$+ \int_0^{T_1} a_1 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt \quad +$$

$$\int_0^{T_1} a_1 k_5 t^2 \cos \bar{\theta}_y \cos \bar{\theta}_y\, dt$$

$$+ \int_0^{T_2} a_2 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt$$

$$+ \int_0^{T_2} a_2 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt \quad +$$

$$\int_0^{T_2} a_2 k_5 t^2 \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt$$

$$+ \int_0^{T_2} a_2 k_2 (T_1 + T_c) \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt$$

$$+ \int_0^{T_2} a_2 k_5 (T_1 + T_c)^2 \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt \quad -$$

$$\int_0^{T_1+T_c+T_2} g_x\, dt \tag{6-16}$$

$$\dot{Z}_f = \dot{Z} + \int_0^{T_1} a_1 \cos \bar{\theta}_y (\cos \bar{\theta}_p + k_1 \sin \bar{\theta}_p)\, dt$$

$$- \int_0^{T_1} a_1 k_2 t \cos \bar{\theta}_y \sin \bar{\theta}_p\, dt \quad -$$

$$- \int_0^{T_1} a_1 k_5 t^2 \cos \bar{\theta}_y \sin \bar{\theta}_p\, dt$$

$$- \int_0^{T_2} a_2 k_2 t \cos \bar{\theta}_y \sin \bar{\theta}_p\, dt \quad -$$

$$- \int_0^{T_2} a_2 k_5 t^2 \cos \bar{\theta}_y \sin \bar{\theta}_p\, dt$$

$$-\int_0^{T_2} a_2 k_2 (T_1+T_c) \cos \bar{\theta}_y \sin \bar{\theta}_p \, dt \; -$$

$$\int_0^{T_2} a_2 k_5 (T_1+T_c)^2 \cos \bar{\theta}_y \sin \bar{\theta} \; dt$$

$$+\int_0^{T_2} a_2 \cos \bar{\theta}_y (\cos \bar{\theta}_p + k_1 \sin \bar{\theta}_p) \; dt$$

$$-\int_0^{T_1+T_c+T_2} g_z \, dt \tag{6-17}$$

Integral values as derived in Appendix F are now substituted into these expressions.

$$\dot{X}_f = \dot{X} + V_{ex_1} L_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)$$

$$+ k_2 J_1 \cos \bar{\theta}_y \cos \bar{\theta}_p + k_5 P_1 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ V_{ex_2} L_2 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)$$

$$+ k_2 J_2 \cos \bar{\theta}_y \cos \bar{\theta}_p + k_5 P_2 \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 V_{ex_2} L_2 (T_1+T_c) \cos \bar{\theta}_y \cos \bar{\theta}_p \; +$$

$$k_5 V_{ex_2} L_2 (T_1+T_c)^2 \cos \bar{\theta}_y \cos \bar{\theta}_p \; -$$

$$g_x (T_1+T_c+T_2)^2 /2 \tag{6-18}$$

$$\dot{Z}_f = \dot{Z} + V_{ex_1} L_1 \cos \bar{\theta}_y (\cos \bar{\theta}_p + k_1 \sin \bar{\theta}_p)$$

$$-k_2 J_1 \cos \bar{\theta}_y \sin \bar{\theta}_p - k_5 P_1 \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$-k_2 J_2 \cos \bar{\theta}_y \sin \bar{\theta}_p - k_5 P_2 \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$-k_2 V_{ex_2} L_2 (T_1+T_c) \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$-k_5 V_{ex_2} L_2 (T_1+T_c)^2 \cos \bar{\theta}_y \sin \bar{\theta}_p$$

$$+ V_{ex_2} L_2 \cos \bar{\theta}_y (\cos \bar{\theta}_p + k_1 \sin \bar{\theta}_p)$$

$$- g_z (T_1 + T_c + T_2) \qquad (6\text{-}19)$$

If the condition $\dot{\bar{X}}_f - \dot{X}_f = 0$ is enforced, the quadratic control law will achieve the desired velocity, $\dot{X}_f$. The velocity equation for $\dot{\bar{X}}_f$ has been shown previously (equation 4-36) and does not change.

$$\dot{\bar{X}}_f = \dot{X} + V_{ex_1} L_1 \cos \bar{\theta}_y \sin \bar{\theta}_p +$$

$$V_{ex_2} L_2 \cos \bar{\theta}_y \sin \bar{\theta}_p - g_x (T_1+T_c+T_2) \qquad (6\text{-}20)$$

Enforcing this velocity condition yields the following equation in terms of the control parameters k_1, k_2, and k_5.

$$0 = k_1 (V_{ex_1} L_1 + V_{ex_2} L_2) \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_2 (-J_1 - J_2 - V_{ex_2} L_2 (T_1+T_c)) \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$+ k_5 (-P_1 - P_2 - V_{ex_2} L_2 (T_1+T_c)^2) \cos \bar{\theta}_y \cos \bar{\theta}_p$$

$$(6\text{-}21)$$

$$0 = - A_p k_1 + B_p k_2 + F_p k_5 \tag{6-22}$$

Equations (6-7) and (6-8) must now be integrated a second time to yield two more equations involving k_1, k_2, and k_5.

$$X_f = X + \int_0^{T_1+T_c+T_2} \dot{X}\, dt \quad +$$

$$\int_0^{T_1}\int_0^{T_1} a_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt^2$$

$$\int_0^{T_1}\int_0^{T_1} a_1 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt^2 \quad +$$

$$\int_0^{T_1}\int_0^{T_1} a_1 k_5 t^2 \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt^2 \quad +$$

$$\int_0^{T_2}\int_0^{T_2} a_2 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt^2 \quad +$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt^2 \quad +$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_5 t^2 \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt^2 \quad +$$

$$(T_c + T_2)\int_0^{T_1} a_1 \cos \bar{\theta}_y (\sin \bar{\theta}_p - k_1 \cos \bar{\theta}_p)\, dt$$

$$(T_2 + T_c)\int_0^{T_1} a_1 k_2 t \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt \quad +$$

$$(T_2 + T_c)\int_0^{T_1} a_1 k_5 t^2 \cos \bar{\theta}_y \cos \bar{\theta}_p\, dt$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_2(T_1+T_c) \cos \overline{\theta}_y \cos \overline{\theta}_p \, dt^2 \; +$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_5(T_1+T_c)^2 \cos \overline{\theta}_y \cos \overline{\theta}_p \, dt^2 \; -$$

$$\int\int_0^{T_1+T_c+T_2} g_x \, dt^2 \qquad (6\text{-}23)$$

$$Z_f = Z + \int_0^{T_1+T_c+T_2} \dot{Z} \, dt \; +$$

$$\int_0^{T_1}\int_0^{T_1} a_1 \cos \overline{\theta}_y (\cos \overline{\theta}_p + k_1 \sin \overline{\theta}_p) \, dt^2 \; -$$

$$\int_0^{T_1}\int_0^{T_1} a_1 k_2 t \cos \overline{\theta}_y \sin \overline{\theta}_p \, dt^2 \; -$$

$$\int_0^{T_1}\int_0^{T_1} a_1 k_5 t^2 \cos \overline{\theta}_y \sin \overline{\theta}_p \, dt \; +$$

$$\int_0^{T_2}\int_0^{T_2} a_2 \cos \overline{\theta}_y (\cos \overline{\theta}_p + k_1 \sin \overline{\theta}_p) \, dt^2 \; -$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_2 t \cos \overline{\theta}_y \sin \overline{\theta}_p \, dt^2 \; -$$

$$\int_0^{T_2}\int_0^{T_2} a_2 k_5 t^2 \cos \overline{\theta}_y \sin \overline{\theta}_p \, dt^2 \; +$$

$$(T_c+T_2)\int_0^{T_1} a_1 \cos \overline{\theta}_y (\cos \overline{\theta}_p + k_1 \sin \overline{\theta}_p) \, dt \; -$$

$$(T_2+T_c)\int_0^{T_1} a_1\ k_2 t \cos\overline{\theta}_y \sin\overline{\theta}_p\ dt\quad -$$

$$(T_2+T_c)\int_0^{T_1} a_1\ k_5 t^2 \cos\overline{\theta}_y \sin\overline{\theta}_p\ dt\quad -$$

$$\int_0^{T_2}\int_0^{T_2} a_2\ k_2\ (T_1+T_c) \cos\overline{\theta}_y \sin\overline{\theta}_p\ dt^2\quad -$$

$$\int_0^{T_2}\int_0^{T_2} a_2\ k_5\ (T_1+T_c)^2 \cos\overline{\theta}_y \sin\overline{\theta}_p\ dt^2\quad -$$

$$\int_0^{T_1+T_c+T_2}\int_0 \varepsilon_z\ dt^2 \tag{6-24}$$

$$X_f = X + \mathring{X}\ (T_1+T_c+T_2) - S_1' \cos\overline{\theta}_y \sin\overline{\theta}_p\quad +$$

$$S_1'\ k_1 \cos\overline{\theta}_y \cos\overline{\theta}_p - Q_1\ k_2 \cos\overline{\theta}_y \cos\overline{\theta}_p\ -$$

$$U_1'\ k_5 \cos\overline{\theta}_y \cos\overline{\theta}_p - S_2' \cos\overline{\theta}_y \sin\overline{\theta}_p\quad +$$

$$S_2'\ k_1 \cos\overline{\theta}_y \cos\overline{\theta}_p - Q_2\ k_2 \cos\overline{\theta}_y \cos\overline{\theta}_p\ -$$

$$U_2'\ k_5 \cos\overline{\theta}_y \cos\overline{\theta}_p + (T_c+T_2)\ V_{ex_1}\ L_1 \cos\overline{\theta}_y \sin\overline{\theta}_p$$

$$-\ (T_c+T_2)\ V_{ex_1}\ L_1 \cos\overline{\theta}_y \cos\overline{\theta}_p\quad +$$

$$(T_c+T_2)\ J_1\ k_2 \cos\overline{\theta}_y \cos\overline{\theta}_p\quad -$$

$$(T_c+T_2)\ P_1\ k_5 \cos\overline{\theta}_y \cos\overline{\theta}_p\quad -$$

$$
\begin{aligned}
&(T_1+T_c)\ S_2'\ k_2 \cos \bar{\theta}_y \cos \bar{\theta}_p \ - \\
&(T_1+T_c)^2\ S_2'\ k_5 \cos \bar{\theta}_y \cos \bar{\theta}_p \ - \\
&g_x\ (T_1+T_c+T_2)^2/2 \qquad (6\text{-}25)
\end{aligned}
$$

$$
\begin{aligned}
Z_f = Z + \dot{Z}\ (T_1+T_c+T_2) - S_1' \cos \bar{\theta}_y \cos \bar{\theta}_p \ - \\
&S_1'\ k_1 \cos \bar{\theta}_y \sin \bar{\theta}_p + Q_1\ k_2 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&U_1'\ k_5 \cos \bar{\theta}_y \sin \bar{\theta}_p - S_2' \cos \bar{\theta}_y \cos \bar{\theta}_p \ - \\
&S_2'\ k_1 \cos \bar{\theta}_y \sin \bar{\theta}_p + Q_2\ k_2 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&U_2'\ k_5 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&(T_c+T_2)\ V_{ex_1}\ L_1 \cos \bar{\theta}_y \cos \bar{\theta}_p \ + \\
&(T_c+T_2)\ V_{ex_1}\ L_1\ k_1 \cos \bar{\theta}_y \sin \bar{\theta}_p \ - \\
&(T_c+T_2)\ J_1\ k_2 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&(T_c+T_2)\ P_1\ k_5 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&(T_1+T_c)\ S_2'\ k_2 \cos \bar{\theta}_y \sin \bar{\theta}_p \ + \\
&(T_1+T_c)^2\ S_2'\ k_5 \cos \bar{\theta}_y \sin \bar{\theta}_p \ - \\
&g_z\ (T_1+T_c+T_2)^2/2 \qquad (6\text{-}26)
\end{aligned}
$$

$$X_f = C_p\, k_1 - D_p\, k_2 - G_p\, k_5 + E_p' \tag{6-27}$$

$$Z_f = -C_p\, k_1 \tan \bar{\theta}_p + D_p\, k_2 \tan \bar{\theta}_p + G_p\, k_5 \tan \bar{\theta}_p + A_p \tag{6-28}$$

It is observed at this point that equations (6-27) and (6-28) are not independent and cannot, therefore, be solved simultaneously for constants k_1, k_2, and k_5. A solution may be possible, however, if the effect of k_5 upon equation (6-27) is assummed small. This assumption appears to be justified for small desired changes in Z_f. The result of of this assumption is equation (6-29).

$$X_f = C_p\, k_1 - D_p\, k_2 + E_p' \tag{6-29}$$

Equations (4-41), (4-47), (6-22), (6-28), and (6-29) may now be solved simultaneously for k_1, k_2, k_3, k_4, and k_5.

Chapter 7

CONCLUSIONS AND RECOMMENDATIONS

Chapters 1 through 4 have developed the theory for a spacecraft guidance and control system based upon principles of optimization theory. Chapter 5 illustrates some numerical results from the implementation of this theory and chapter 6 extends the guidance equations to control all final state variables. Some specific conclusions and recommendations may now be reached concerning the development in these preceding chapters.

As illustrated in chapter 1, most guidance formulations which exist in closed form experience problems with long burn arcs. These problems result since, to achieve a closed form solution, some assumption must be made concerning gravity. The formulation introduced in this thesis makes the assumption that $\nabla \bar{g} = 0$ and, consequently, has convergence difficulties with long burn arcs ($\nabla \bar{g} = 0$ is certainly not true for long burn arcs). Low thrust maneuvers introduce similar convergence problems because the gravity acceleration becomes large relative to the thrust acceleration. The development of the burn arc piecing procedure in appendix D is intended to circumvent these convergence problems.

Chapter 5 illustrates a numerical comparison of the guidance solution and an extremized solution to a low thrust transfer maneuver. It is noted that the guidance formulation could not be made to converge with a single burn arc; however, the burn arc piecing procedure (with two burn arc pieces) was used and the solution to the problem obtained. The boundary conditions for the transfer were met accurately. A comparison of the burn times for the two solutions shows that the guidance solution is relatively efficient. The only difficulty with the piecing procedure appears to be discontinuities that occur at terminal points of each burn arc piece. While some efficiency seems to be sacrificed as a result of these discontinuities, they do not constitute a major problem. Some additional work on the piecing procedure may eliminate these discontinuities.

The burn arc piecing procedure was also used with five burn arc pieces during which only a velocity control ($\bar{\theta}_p$) was used. The velocity boundary conditions were achieved precisely and in this case no discontinuities were observed. The burn time is also very close to that of the extremized solution. This velocity control procedure should provide much better results than the cross product and impulsive formulations since it has a much wider range of applicability (larger burn arcs).

Although execution time has not been an explicit part of the guidance evaluation, some analysis can be made. The guidance formulation is analytic and closed form and, therefore, one would suspect very fast. Execution time on the Univac 1108 computer has been of the order .01 seconds per guidance evaluation. It is obvious, therefore, that this guidance formulation could be easily implemented by a digital flight computer.

In conclusion, the guidance formulation in conjunction with the piecing procedure seems to work reasonably well and the piecing procedure appears to be an adequate means of evaluating gravity over long burn arcs. This procedure should also work for burn arcs much larger than the example problem and should have applicability to a large class of orbital transfer problems. The guidance formulation presented in this thesis should, therefore, have applicability to a wide range of orbital transfer problems.

As a result of the analysis concluded in this thesis, further study can be recommended. Implementation of the results in chapter 6 (Extensions) and extension of the piecing procedure to out-of-plane maneuvers (θ_y concerns the out-of-plane control) would seem worthwhile.

Appendix A

THE ADJOINT METHOD

The following derivation shows the complete solution of the optimal orbital transfer problem by the adjoint method. Since the intent has been to solve this problem explicitly the solution here is primarily tutorial in nature.

The dynamical equations are repeated here where only the planar case is considered.

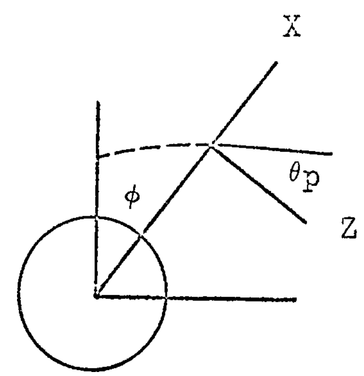


Figure A.1

Orbit Transfer For The Planar Case

$$\dot{U} = a \sin \theta_p - \frac{g_0 r_0^2 X}{(\sqrt{X^2+Z^2})^3} \tag{A-1}$$

$$\dot{W} = a \cos \theta_p - \frac{g_0 r_0^2 Z}{(\sqrt{X^2+Z^2})^3} \tag{A-2}$$

$$\dot{X} = U \tag{A-3}$$

$$\dot{Z} = W \tag{A-4}$$

The Hamiltonion and necessary conditions are also repeated.

$$H = 1 + \lambda_1 U + \lambda_3 W + \lambda_4 \left(a \sin \theta_p - \frac{g_0 r_0^2 X}{(\sqrt{X^2+Z^2})^3} \right)$$

$$+ \lambda_6 \left(a \cos \theta_p - \frac{g_0 r_0^2 Z}{(\sqrt{X^2+Z^2})^3} \right) \tag{A-5}$$

$$\dot{\lambda}_1 = -H_x = \lambda_4 g_0 r_0^2 \left(\frac{(X^2+Z^2)^{3/2}}{(X^2+Z^2)^{6/2}} - \frac{X(3/2)(X^2+Z^2)^{1/2}(2X)}{(X^2+Z^2)^{6/2}} \right) +$$

$$\lambda_6 g_0 r_0^2 \left(\frac{-Z\ (3/2)(X^2+Z^2)^{1/2}(2X)}{(X^2+Z^2)^{6/2}} \right) \tag{A-6}$$

$$\dot{\lambda}_1 = \lambda_4 g_0 r_0^2 \left(\frac{1}{(\sqrt{X^2+Z^2})^3} \right) - \lambda_4 g_0 r_0^2 \left(\frac{3X^2}{(\sqrt{X^2+Z^2})^5} \right)$$

$$- \lambda_6 g_0 r_0^2 \left(\frac{3XZ}{(\sqrt{X^2+Z^2})^5} \right) \tag{A-7}$$

$$\dot{\lambda}_1 = g_0 r_0^2 \left(\frac{-3 \lambda_6 XZ - 3 \lambda_4 X^2}{(\sqrt{X^2+Z^2})^5} + \frac{\lambda_4}{(\sqrt{X^2+Z^2})^3} \right) \tag{A-8}$$

$$\dot{\lambda}_3 = -H_z = \lambda_4 g_0 r_0^2 \left(\frac{-X(3/2)(X^2+Z^2)^{1/2}(2Z)}{(X^2+Z^2)^{6/2}} \right)$$

$$+ \lambda_6 g_0 r_0^2 \left(\frac{(X^2+Z^2)^{3/2} - Z(3/2)(X^2+Z^2)^{1/2}(2Z)}{(X^2+Z^2)^{6/2}} \right) \quad \text{(A-9)}$$

$$\dot{\lambda}_3 = g_0 r_0^2 \left(\frac{-3\lambda_6 Z^2 - 3\lambda_4 XZ}{(\sqrt{X^2+Z^2})^5} + \frac{\lambda_6}{(\sqrt{X^2+Z^2})^3} \right) \quad \text{(A-10)}$$

$$\dot{\lambda}_4 = -H_u = -\lambda_1 \quad \text{or} \quad \dot{\lambda}_4 + \lambda_1 = 0 \quad \text{(A-11)}$$

$$\dot{\lambda}_6 = -H_w = -\lambda_3 \quad \text{or} \quad \dot{\lambda}_6 + \lambda_3 = 0 \quad \text{(A-12)}$$

$$H_{\theta_p} = 0 = a\,\lambda_4 \cos\theta_p - a\,\lambda_6 \sin\theta_p \quad \text{(A-13)}$$

$$\text{or} \quad \tan\theta_p = \frac{\lambda_4}{\lambda_6}$$

These equations may be combined to form five differential equations in five unknowns.[a]

$$\ddot{X} = a \sin\theta_p - \frac{g_0 r_0^2 X}{(\sqrt{X^2+Z^2})^3} \quad \text{(A-14)}$$

$$\ddot{Z} = a \cos\theta_p - \frac{g_0 r_0^2 Z}{(\sqrt{X^2+Z^2})^3} \quad \text{(A-15)}$$

$$\tan\theta_p = \frac{\lambda_4}{\lambda_6} \quad \text{(A-16)}$$

[a] These five second order differential equations are formed from the ten first order differential equations.

$$\ddot{\lambda}_4 = g_0 r_0^2 \left(\frac{-3\lambda_6 XZ - 3\lambda_4 X^2}{(\sqrt{X^2+Z^2})^5} + \frac{\lambda_4}{(\sqrt{X^2+Z^2})^3} \right) \qquad \text{(A-17)}$$

$$\ddot{\lambda}_6 = g_0 r_0^2 \left(\frac{-3\lambda_6 Z^2 - 3\lambda_4 XZ}{(\sqrt{X^2+Z^2})^5} + \frac{\lambda_6}{(\sqrt{X^2+Z^2})^3} \right) \qquad \text{(A-18)}$$

The set of five equations (A-14 through A-18) can be solved numerically to define a time minimizing trajectory and meet the prescribed final conditions. It is observed that a set of four initial conditions must be found to solve this problem. These initial conditions are the initial values associated with the Lagrange multipliers, $\lambda_4(t_0)$, $\dot{\lambda}_4(t_0)$, $\lambda_6(t_0)$, $\dot{\lambda}_6(t_0)$, and must be known in order to meet the prescribed final conditions. The final conditions for this trajectory are completely specified by altitude, velocity, flight path angle, and range (h, v, γ, ϕ) and if all these quantities are specified a unique set of values exist for the initial Lagrange multipliers values to solve the problem. If however, fewer than these four final conditions are specified (such as h, v, γ) then these initial values are not unique and a further minimization problem can be done to select the optimum range. An equation can be introduced which relates the Lagrange multipliers and state variables at the terminal time and this equation implicitly selects the optimum range. This equation is known as a transversality equation and is illustrated below for the range free case (equation A-31).

It is noted that one transversality equation is introduced for every free final condition.

The transversality equation can be derived from the determinant of the following matrix of partial derivatives.

	t_0	X_0	Z_0	U_0	W_0	θ_0	t_f	X_f	Z_f	U_f	W_f	θ_f
t_f	0	0	0	0	0	0	1	0	0	0	0	0
X_0-e_1	0	1	0	0	0	0	0	0	0	0	0	0
Z_0-e_2	0	0	1	0	0	0	0	0	0	0	0	0
U_0-e_3	0	0	0	1	0	0	0	0	0	0	0	0
W_0-e_4	0	0	0	0	1	0	0	0	0	0	0	0
t_0-e_5	1	0	0	0	0	0	0	0	0	0	0	0
$X_f^2+Z_f^2-(r_0+h)^2$	0	0	0	0	0	0	0	$2X_f$	$2Z_f$	0	0	0
$\dot{X}_f^2+\dot{Z}_f^2-V_f^2$	0	0	0	0	0	0	0	0	0	$2\dot{X}_f$	$2\dot{Z}_f$	0
$\dot{X}_f X_f+\dot{Z}_f Z_f-\gamma_f$	0	0	0	0	0	0	0	$\dot{X}_f$	$\dot{Z}_f$	X_f	Z_f	0
	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}

0 .... Initial Time

f .... Final Time

Where

$$M_1 = H_0 = 1 + \lambda_{10} U + \lambda_{30} W + \lambda_{40} \left(a \sin \theta_p - \frac{g_0 r_0^2 X}{(\sqrt{X^2+Z^2})^3} \right) + \lambda_{60} \left(a \cos \theta_p - \frac{g_0 r_0^2 Z}{(\sqrt{X^2+Z^2})^3} \right) \quad \text{(A-19)}$$

$$M_2 = \left.\frac{\partial H}{\partial \dot{X}}\right)_0 = \lambda_{10} \tag{A-20}$$

$$M_3 = \left.\frac{\partial H}{\partial \dot{Z}}\right)_0 = \lambda_{30} \tag{A-21}$$

$$M_4 = \left.\frac{\partial H}{\partial \dot{U}}\right)_0 = \lambda_{40} \tag{A-22}$$

$$M_5 = \left.\frac{\partial H}{\partial \dot{W}}\right)_0 = \lambda_{60} \tag{A-23}$$

$$M_6 = \left.\frac{\partial H}{\partial \dot{\theta}_p}\right)_0 = 0 \tag{A-24}$$

$$M_7 = H_f = 1 + \lambda_{1f}\, U + \lambda_{3f} W + \lambda_{4f} \left(a \sin \theta_p - \frac{g_0 r_0^{\,2} X}{\left(\sqrt{X^2+Z^2}\right)^3} \right) + \lambda_{6f} \left(a \cos \theta_p - \frac{g_0 r_0^{\,2} Z}{\left(\sqrt{X^2+Z^2}\right)^3} \right) \tag{A-25}$$

$$M_8 = \left.\frac{\partial H}{\partial \dot{X}}\right)_f = \lambda_{1f} \tag{A-26}$$

$$M_9 = \left.\frac{\partial H}{\partial \dot{Z}}\right)_f = \lambda_{3f} \tag{A-27}$$

$$M_{10} = \left.\frac{\partial H}{\partial \dot{U}}\right)_f = \lambda_{4f} \tag{A-28}$$

$$M_{11} = \left.\frac{\partial H}{\partial \dot{W}}\right)_f = \lambda_{6f} \tag{A-29}$$

$$M_{12} = \left.\frac{\partial H}{\partial \theta_p}\right)_f = 0 \qquad \text{(A-30)}$$

Working through the algebra and setting the determinant equal to 0 yields the equation (A-31).

$$\lambda_{4f}\,\dot{Z}_f - \lambda_{6f}\,\dot{X}_f + \lambda_{1f}\,Z_f - \lambda_{3f}\,X_f = 0 \qquad \text{(A-31)}$$

Once the transversality equations have been determined (if any), then the two point boundary value problem can be solved by iterating the initial values for the Lagrange multipliers until the boundary conditions and transversality equations are satisfied.

Appendix B

DERIVATION OF A LOSS FUNCTION

For any powered flight maneuver the difference between the earth relative velocity gained, V, and the characteristic velocity, ΔV, spent represents a measurable quantity which can be used to evaluate different trajectories. The relative velocity change cannot be achieved by an equal expenditure of characteristic velocity since retarding accelerations are present. It is of interest, therefore, to analyze the difference between these two velocities and to identify the source of the velocity loss.

Consider a rotating system with one axis instantaneously along the earth relative velocity vector, $\overline{V}$, and another normal to this direction and in the plane of motion.

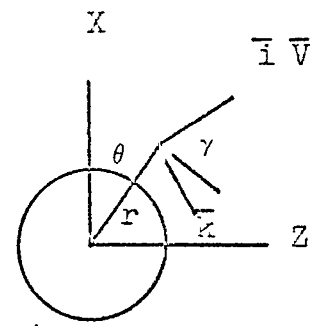


Figure B.1

Loss Function Coordinate System

The rotating system (for planar motion) moves at the rate $\dot{\theta}$ minus $\dot{\gamma}$ and the acceleration of a particle referenced to this rotating system can be shown.

$$\bar{V} = V\bar{i} \qquad \text{(B-1)}$$

$$\dot{\bar{V}} = \dot{V}\bar{i} + V\dot{\bar{i}} \qquad \text{(B-2)}$$

$$\dot{\bar{V}} = \dot{V}\bar{i} - V\,(\dot{\theta}-\dot{\gamma})\bar{k} \qquad \text{(B-3)}$$

where

$$\dot{\bar{i}} = (\dot{\theta}-\dot{\gamma})\bar{j} \times \bar{i}$$

The summation of accelerations in this system can then be equated to the kinematic acceleration to yield the equations of motion.

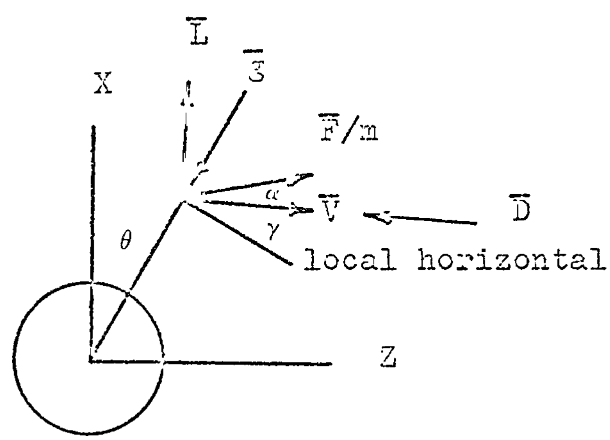


Figure B.2

Loss Function Coordinate System

where

$\bar{D}$ = drag acceleration vector

$\bar{L}$ = lift acceleration vector

$\bar{g}$ = gravitational acceleration vector

$\frac{\bar{F}}{m}$ = thrust acceleration vector

$$\dot{V} = \frac{F}{m}\cos\alpha - g\sin\gamma - \frac{D}{m} \qquad \text{(B-4)}$$

$$0 = -\frac{F}{m}\sin + g \cos\gamma - \frac{L}{m} + V(\dot{\theta} - \dot{\gamma}) \qquad \text{(B-5)}$$

The first scalar equation represents the change in velocity along the $\overline{V}$ direction while the change normal to this direction is $V(\dot{\theta} - \dot{\gamma})$. $\dot{\theta} - \dot{\gamma}$ represents the turning rate of the coordinate system and is usually very small for all powered maneuvers. This $\dot{V}$ equation can be used to determine velocity losses when $V(\dot{\theta} - \dot{\gamma})$ is small.

Loss = characteristic velocity - relative velocity (B-6)

$$L = \int_0^T \frac{F}{m}\, dt - \int_0^T \left(\frac{F}{m}\cos\alpha - g \sin\gamma - \frac{D}{m}\right) dt \qquad \text{(B-7)}$$

$$L = \int_0^T \left(\frac{F}{m}(1 - \cos\alpha) + g \sin\gamma + \frac{D}{m}\right) dt \qquad \text{(B-8)}$$

and $$\dot{L} = \frac{F}{m}(1 - \cos\alpha) + g \sin\gamma + \frac{D}{m} \qquad \text{(B-9)}$$

This loss function can be used as a switching function (to determine engine on and off time) since it is a function of not only control during the maneuver, but of time to initiate the maneuver (i.e., $L = g(\gamma_0\ \gamma_f\ \rho)$).

The components of the loss function ideally take the following form around pericenter.

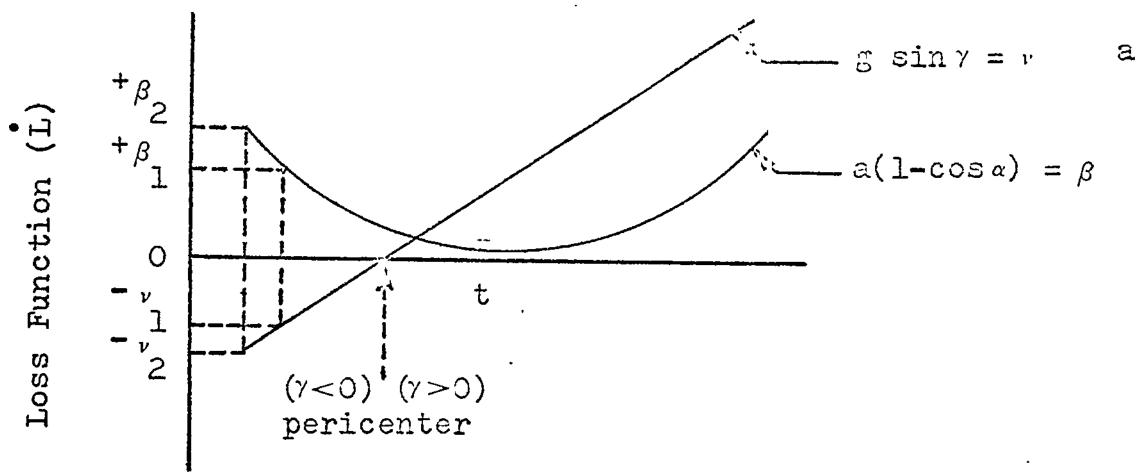


Figure B.3

Loss Function

It is observed that a reduction of the loss function occurs when $g \sin\gamma\,(\nu)$ becomes negative. Since this function becomes negative when γ is negative, a maneuver should be biased to the $-\gamma$ side of pericenter. The same is true for a maneuver centered around apocenter except in this case the slope of $g \sin\gamma$ is negative. Thus one should not center a maneuver around pericenter or apocenter geometrically but bias these maneuvers to the negative γ side of the line of apsides. Since the gravity loss function ($g \sin\gamma$) is ideally linear, it serves to displace the loss, $a\,(1-\cos\alpha)$, while retaining the original form. Therefore, the total loss function during a transfer maneuver will tend to have the form, $1-\cos\delta$, and an approximate method for minimizing this

[a] D/m is not present for exo-atmospheric maneuvers.

function is to make it symmetric around the midpoint of the burn arc.

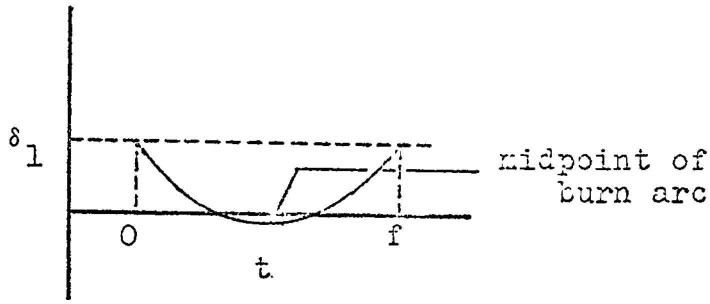


Figure B.4

Loss Function Form

The loss function can be used as a switching function (for single burn arcs) by implementing such a procedure for minimizing the form, 1-cos δ . The guidance procedure discussed previously can solve in closed form for values of this loss function and can therefore minimize this function by insuring that $\dot{L}_0 = \dot{L}_f$.

Appendix C

DERIVATION OF AN EFFECTIVE GRAVITY COMPUTATION

The previous guidance equations have a strong dependence upon the gravity computation. For small burn arcs the magnitude and direction of the gravity vector do not change substantially, however, as the burn arc is increased both the magnitude and direction may change substantially. If however, a gravity computation can be introduced to yield "effective" gravity values the performance of the guidance equations can be improved.

The following equations compute values which estimate the effect of gravity over burn arcs. g represents the average gravity magnitude, g^* is the effective gravity direction, and ϕ is the central angle (or range angle).

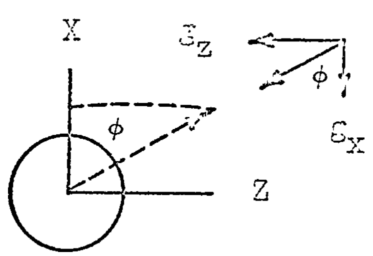


Figure C.1

Effective Gravity Coordinate System

Assume that $\phi(t)$ can be approximated by $\phi = \dot{\phi}_m t + \frac{\ddot{\phi} t}{2}(t-T)$.

The first integrals for g_X^* and g_Z^* follow.

$$g_x^* = \frac{1}{T}\int_0^T \cos\phi \, dt \qquad g_z^* = \frac{1}{T}\int_0^T \sin\phi \, dt \qquad \text{(C-1)}$$

For small $\ddot{\phi}$

$$\sin\phi \simeq \sin\dot{\phi}_m t + \frac{1}{2}\ddot{\phi} t\,(t-T)\cos\dot{\phi}_m t \qquad \text{(C-2)}$$

$$\cos\phi \simeq \cos\dot{\phi}_m t - \frac{1}{2}\ddot{\phi} t\,(t-T)\sin\dot{\phi}_m t \qquad \text{(C-3)}$$

$$g_z^* = \frac{1}{T}\int_0^T \sin\phi \, dt = \frac{1}{T}\left(-\frac{1}{\dot{\phi}_m}\cos\dot{\phi}_m t - \frac{1}{2}\ddot{\phi} T\left(\frac{t}{\dot{\phi}_m}\sin\dot{\phi}_m t\right.\right.$$

$$\left. + \frac{1}{\dot{\phi}_m^2}\cos\dot{\phi}_m t\right) + \frac{1}{2}\ddot{\phi}\left(\left(\frac{t^2}{\dot{\phi}_m} - \frac{2}{\dot{\phi}_m^3}\right)\sin\dot{\phi}_m t\ +\right.$$

$$\left.\left.\frac{2t}{\dot{\phi}_m^2}\cos\dot{\phi}_m t\right)\right)\Bigg|_0^T \qquad \text{(C-4)}$$

$$= -\frac{1}{\dot{\phi} T}\left((1 - \frac{1}{2}\frac{\ddot{\phi} T}{\dot{\phi}_m})\cos\dot{\phi}_m T + \frac{\ddot{\phi}}{\dot{\phi}_m^2}\sin\dot{\phi}_m T - \right.$$

$$\left.(1 + \frac{1}{2}\frac{\ddot{\phi} T}{\dot{\phi}_m}\right) \qquad \text{(C-5)}$$

Substituting the identities yields the following results

$$\cos\dot{\phi}_m t = 1 - 2\sin^2\left(\frac{\dot{\phi}_m t}{2}\right) \qquad \text{(C-6)}$$

$$\sin\dot{\phi}_m t = 2\sin\frac{\dot{\phi}_m t}{2}\cos\frac{\dot{\phi}_m t}{2} \qquad \text{(C-7)}$$

$$g_z^* = \frac{1}{T}\int_0^T \sin\phi\, dt = \left(\left(1 - \frac{\ddot{\phi}\, T}{2\,\dot{\phi}_m}\right)\sin\frac{\dot{\phi}_m T}{2} - \right.$$

$$\left.\frac{\ddot{\phi}}{\dot{\phi}_m^2}\cos\frac{\dot{\phi}_m T}{2}\right)\frac{\sin(\dot{\phi}_m T/2)}{(\dot{\phi} T/2)} + \frac{\ddot{\phi}}{\dot{\phi}_m^2} \qquad \text{(C-8)}$$

In a similar manner the other integral may be obtained.

$$g_x^* = \frac{1}{T}\int_0^T \cos\phi\, dt = \left(\left(1 - \frac{1}{2}\frac{\ddot{\phi}\, T}{\dot{\phi}_m}\right)\cos\frac{\dot{\phi}_m T}{2} + \right.$$

$$\left.\frac{\ddot{\phi}}{\dot{\phi}_m^2}\sin\frac{\dot{\phi}_m T}{2}\right)\frac{\sin \dot{\phi}_m T/2)}{(\dot{\phi}_m T/2)} \qquad \text{(C-9)}$$

The second integrals for g_{xx}^* and g_{zz}^* follow in an analogous manner.

$$g_{zz}^* = \frac{2}{T^2}\iint_0^T \sin\phi\, dt = \frac{2}{\dot{\phi}_m T}\left(1 + \frac{\ddot{\phi}\, T}{2\,\dot{\phi}_m} - \right.$$

$$\frac{\sin(\dot{\phi}_m T/2}{(\dot{\phi}_m T/2)}\left(\left(1 - \frac{\ddot{\phi}\, T}{\dot{\phi}_m}\right)\cos\frac{\dot{\phi}_m T}{2} + \right.$$

$$\left.\left.\frac{3\ddot{\phi}}{\dot{\phi}_m^2}\sin\frac{(\dot{\phi}_m T)}{2}\right)\right) \qquad \text{(C-10)}$$

$$g_{xx}^* = \frac{2}{T^2}\iint_0^T = \frac{2}{\dot{\phi}_m T}\left(\frac{3\ddot{\phi}}{\dot{\phi}_m^2} + \right.$$

$$\frac{\sin(\dot{\phi}_m T/2)}{(\dot{\phi}_m T/2)} \left(\left(1 - \frac{\ddot{\phi} T}{\dot{\phi}_m}\right) \left(\sin \frac{\dot{\phi}_m T}{2}\right) - \frac{3\ddot{\phi}}{\dot{\phi}_m^2} \cos \frac{\dot{\phi}_m T}{2} \right) \qquad \text{(C-11)}$$

First and second gravity integrals may now be expressed and used as follows.

$$\int_0^T g_x \, dt = T\,(g \cdot g_x^*) \qquad \text{(C-12)}$$

$$\int_0^T g_z \, dt = T\,(g \cdot g_z^*) \qquad \text{(C-13)}$$

$$\iint_0^T g_x \, dt = \frac{T^2}{2}\,(g \cdot g_{xx}^*) \qquad \text{(C-14)}$$

$$\iint_0^T g_z \, dt = \frac{T^2}{2}\,(g \cdot g_{zz}^*) \qquad \text{(C-15)}$$

These equations provide a means for expressing gravity as a constant value both in magnitude and direction and may be used with the guidance equations introduced in chapter 4.

Appendix D

AN ALGORITHM FOR MULTI-ARC BOUNDARY CONDITIONS

As previously indicated, the derived guidance formulation will not work for large burn arcs or relatively low-thrust maneuvers. The formulation, however, does show very good results for small burn arcs and can be made to work under both large and small arcs if a burn arc piecing procedure is used. This is a procedure by which a problem can be segmented into a series of small burn arcs. If such a procedure is used, however, one cannot be assured that the entire burn is near optimal. The following algorithm is formulated to select sets of intermediate boundary values such that a piecing procedure can be near optimal.

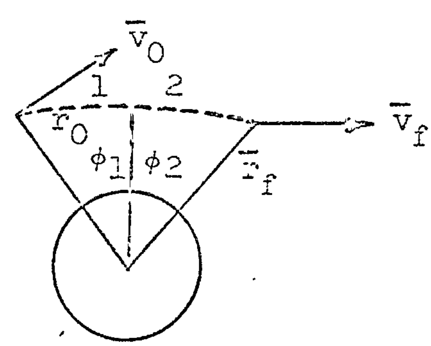


Figure D.1

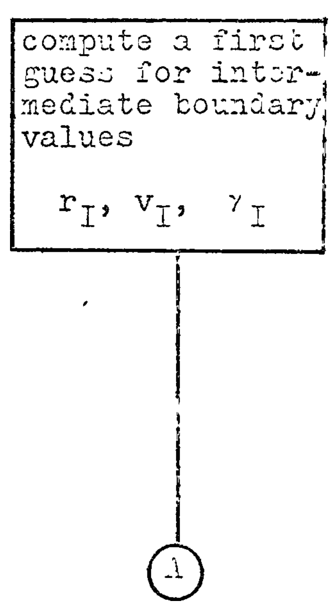
Piecing Procedure

The general procedure employed will be to assume that the Lagrange multipliers ($\bar{P}$) are piecewise linear and continuous. For instance, if the above maneuver is divided into two separate burn arcs and the control function is desired to be continuous, then the following equation is true (for the planar case only).

$$\frac{\lambda_{4_f}^{(1)}}{\lambda_{6_f}^{(1)}} = \frac{\lambda_{4_0}^{(2)}}{\lambda_{6_0}^{(2)}}$$

The term $\lambda_6(t)$ is constant ($\lambda_{6_0} = \lambda_6 = \lambda_{6_f}$), equal to $\frac{\partial \Phi}{\partial \dot{Z}_f}$

and is therefore proportional to the velocity required, $\Delta\dot{Z} = \dot{Z}_f - \dot{Z} - g_z T$. Therefore, $\lambda_{6_f}^{(1)}$ can be made equal to $\lambda_{6_0}^{(2)}$ if $\Delta\dot{Z}^{(1)} = \Delta\dot{Z}^{(2)}$. These two conditions then insure that the control function and multipliers are continuous. An algorithm to successfully implement this procedure follows.



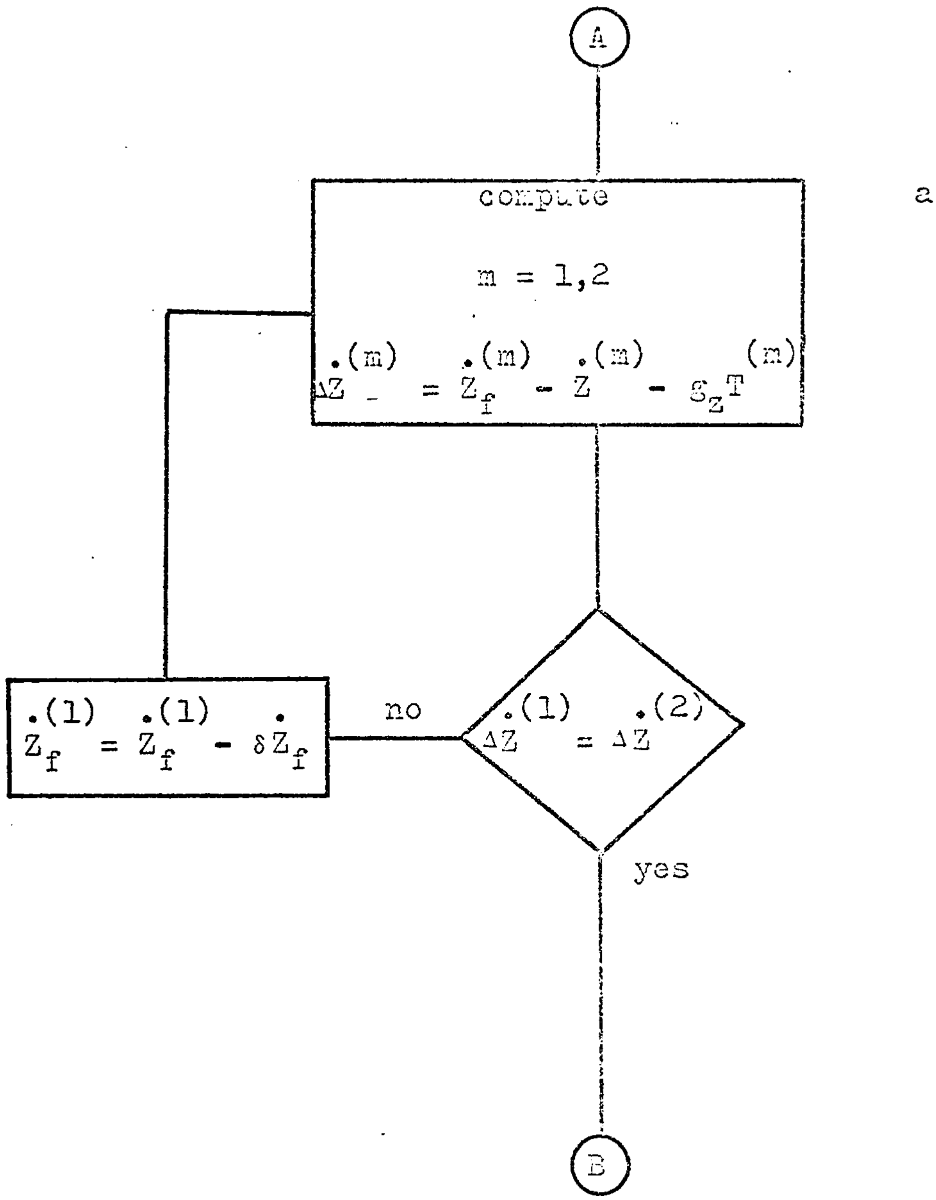


Figure D.2

Multi-Arc Algorithm

[a] $T^{(1)}$ and $T^{(2)}$ are solved using the recursive equation in Appendix G.

B

$m = 1,2$

$$\Delta\dot{X}^{(m)} = \dot{X}_f^{(m)} - \dot{X}^{(m)} - g_x T^{(m)}$$

$$\bar{\theta}_p^{(m)} = \tan^{-1} \frac{\Delta\dot{X}^{(m)}}{\Delta\dot{Z}^{(m)}}$$

$$B_p^{(m)} = \left(\sum_{i=1}^{n} J_i^{(m)} + \sum_{i=1}^{n-1} \left(V_{ex_{i+1}} L_{i+1}^{(m)} \sum_{j=1}^{i} T_j^{(m)}\right) \right) \cos \bar{\theta}_p^{(m)}$$

$$A_p^{(m)} = \sum_{i=1}^{n} V_{ex_i} L_i^{(m)} \cos \bar{\theta}_p^{(m)}$$

$$C_p^{(m)} = \left(\sum_{i=1}^{n} S_i^{(m)} - \sum_{i=1}^{n-1} \left(T_{i+1}^{(m)} \sum_{j=1}^{i} V_{ex_j} L_j^{(m)}\right) \right) \cos \bar{\theta}_p^{(m)}$$

$$D_p^{(m)} = \left(\sum_{i=1}^{n} Q_i^{(m)} + \sum_{i=1}^{n-1} \left(S_{i+1}^{(m)} \sum_{j=1}^{i} T_j^{(m)}\right) - \sum_{i=1}^{n-1} \left((T_{i+1}^{(m)}) \sum_{j=1}^{i} J_j^{(m)} \right) \right) \cos \bar{\theta}_p^{(m)}$$

$$E_p^{(m)} = X^{(m)} - X_f^{(m)} + \dot{X}^{(m)} \sum_{i=1}^{n} T_i^{(m)} + g_x \left(\sum_{i=1}^{n} T_i^{(m)}\right)^2 / 2 - C_p^{(m)}$$

$$k_1^{(m)} = (B_p^{(m)} E_p^{(m)}) / (A_p^{(m)} D_p^{(m)} - B_p^{(m)} C_p^{(m)})$$

$$k_2^{(m)} = A_p^{(m)} k_1^{(m)} / B_p^{(m)}$$

$$\theta_p^{(1)} = \bar{\theta}_p^{(1)} + k_1^{(1)}$$

$$\theta_p^{(2)} = \bar{\theta}_p^{(2)} - k_1^{(2)}$$

C

Figure D.2 continued

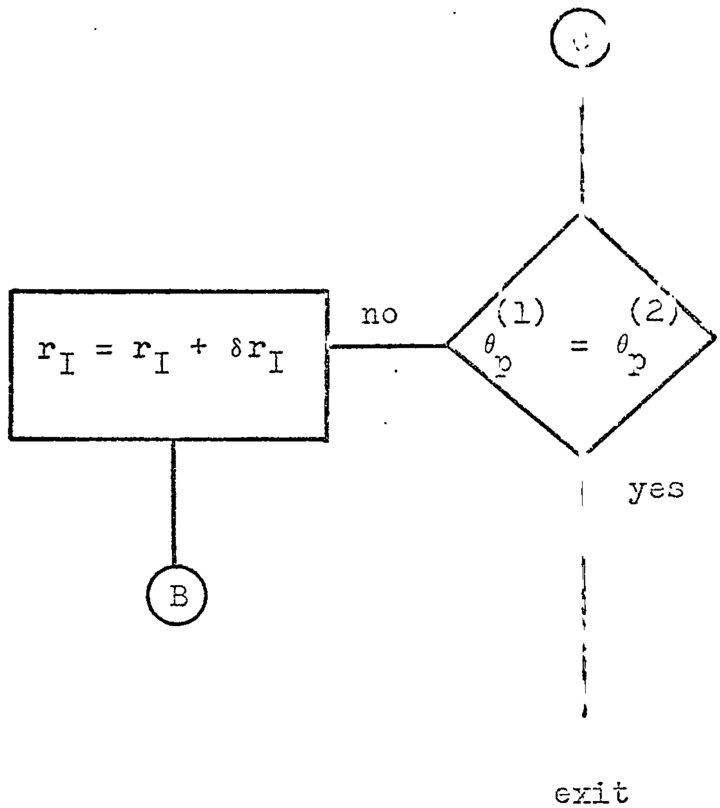


Figure D.2

continued

Appendix E

DERIVATION OF AN OPTIMAL FORM OF CONTROL UNDER CONDITIONS OF CONSTANT GRAVITY AND CONSTANT THRUST ACCELERATION

A derivation of the control equation for a constant gravity, constant mass flow rate guidance problem has been previously introduced. The assumptions made in this derivation were as follows.

1. No atmospheric forces are present
2. Gravity is constant or a fixed function of time
3. Vehicle mass flow rate is constant or a fixed function of time.

A variation of this problem is introduced if one considers the effect of a constant thrust acceleration constraint upon the transfer problem. Consider the effect upon the control of a trajectory which consists of an unconstrained acceleration phase followed by a constant acceleration phase (which is implemented by throttling the spacecraft engines). During the first trajectory phase both the vehicle flow rate and thrust are constants and the acceleration may be determined as follows.

$$a(t) = F / (m_0 - \dot{m}t) \qquad \text{(E-1)}$$

The time at which the engine throttles may also be expressed where a_m is the acceleration level to be maintained.

$$t_T = m_0/\dot{m} - F/\dot{m}\, a_m \qquad \text{(E-2)}$$

After this time (t_T) the vehicle engines are throttled to maintain the acceleration, a_m, and the mass flow rate is some function of engine thrust.

$$F_{(t)} = a_m\, m(t) \qquad \text{(E-3)}$$

$$\dot{m}(t) = E\,(F\,(t)\,) \qquad \text{(E-4)}$$

$$\dot{m}(t) = X\,(m\,(t)\,) \qquad \text{(E-5)}$$

The mass of the vehicle after t_T is therefore described by afirst order differential equation with initial condition $m\,(t_T) = F / a_m$. Since X and $\frac{\partial X}{\partial m}$ can be assumed to be continuous with respect to m and t then a unique solution $m^*(t)$ to equation E-5 exists. The vehicle mass is then a fixed function of time described as follows.

$$m\,(t) = m_0 - \dot{m}t \qquad t \leq t_T \qquad \text{(E-6)}$$

$$m\,(t) = m^*\,(t) \qquad t > t_T \qquad \text{(E-7)}$$

Since F (t) is also a fixed function of time the transfer problem containing constant acceleration phases belongs to the class of problem for which the bilinear tangent control law is optimal.

Appendix F

CLOSED FORM INTEGRALS FOR THRUST ACCELERATION

The thrust acceleration integrals for a spacecraft are developed for use in the guidance formulation. Consider a spacecraft propelled by its rocket thrust in a vacuum.

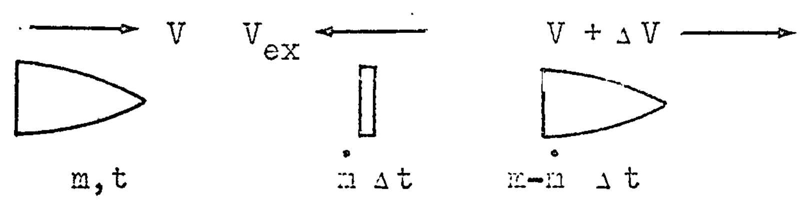


Figure F.1

Rocket System

Where

m	=	mass of the rocket at time t
$\dot{m}\,\Delta t$	=	mass increment from the rocket engine
V	=	velocity of the rocket at time t
$m-\dot{m}\,\Delta t$	=	mass of the rocket at time $t + \Delta t$
$V+\Delta V$	=	velocity of the rocket at time $t + \Delta t$
V_{ex}	=	exhaust velocity of the rocket
a	=	thrust acceleration of the rocket

The acceleration equation results from application of the conservation of momentum principle.

$$mV = (m - \dot{m}\,\Delta t)(V + \Delta V) + \dot{m}\,\Delta t\,(V + \Delta V - V_{ex}) \quad (F-1)$$

$$mV = mV + m\,\Delta V - \dot{m}\,\Delta t\,V_{ex} \quad (F-2)$$

$$m\,\frac{\Delta V}{\Delta t} = V_{ex}\,\dot{m} \quad (F-3)$$

as $\Delta t \longrightarrow 0$

$$a = V_{ex}\,\frac{\dot{m}}{m} \quad (F-4)$$

The rocket mass may be expressed as a linear function of time ($m_0 - \dot{m}t$) for a constant mass flow rate ($\dot{m}$). This is true for non-throttled rocket engines which operate in a vacuum.

$$\therefore \; a = V_{ex}\,\frac{\dot{m}}{m_0 - \dot{m}t} \quad \text{or} \quad a = \frac{V_{ex}}{m_0/\dot{m} - t} \quad (F-5)$$

$$\text{letting} \; \frac{m_0}{\dot{m}} = \tau \qquad a = \frac{V_{ex}}{\tau - t} \quad (F-6)$$

Successive evaluation of the integrals can now be made.

$$\int_0^T a\,dt = \int_0^T \frac{V_{ex}}{\tau - t}\,dt = V_{ex}\ln\left(\frac{\tau}{\tau - T}\right) = V_{ex}\,L \quad (F-7)$$

$$\int_0^T at\,dt = \int_0^T \frac{V_{ex}}{\tau - t}\,t\,dt = \tau L - V_{ex}\,T = J \quad (F-8)$$

$$\int_0^T at^2\,dt = \int_0^T \frac{V_{ex}\,t^2}{\tau - t}\,dt = -\frac{1}{2}\,V_{ex}\,T^2 + \tau J = P \quad (F-9)$$

$$\iint_0^T a \, dt^2 = \int_0^T \frac{V_{ex}}{\tau - t} \, dt^2 = V_{ex} \int_0^T \Big(\ln \quad - \ln(\tau - T) \Big) dt$$

$$= \; -J + TL = \; -S \qquad \text{(F-10)}$$

$$\iint_0^T at \, dt^2 = \iint_0^T \frac{V_{ex} t}{\tau - t} \, dt^2 =$$

$$- V_{ex} \int_0^T \Big(\tau \ln (\tau) - \tau \ln (\tau - t) - t \Big) \, dt =$$

$$- \tau S - \frac{V_{ex}}{2} T^2 = - Q \qquad \text{(F-11)}$$

$$\iint_0^T at^2 \, dt^2 = \iint_0^T \frac{V_{ex} \, t^2}{\tau - t} \, dt^2 =$$

$$V_{ex} \int_0^T \Big(\tau^2 \ln (\tau) - \tau^2 \ln (\tau - t) - \tau t - \frac{t^2}{2} \Big) dt =$$

$$- \tau Q - \frac{V_{ex}}{6} T^3 = -U \qquad \text{(F-12)}$$

Appendix G

DERIVATION OF A RECURSIVE EQUATION FOR BURN TIME, t_f

The value of t_f is necessary to solve the closed form equations presented in the guidance formulation. Instead of evaluating t_f directly, however, it is more convenient to develop a recursive relationship relating the current t_f to the previous t_f. A first guess for t_f is then sufficient to yield a starting solution and then updating of this value can follow from the recursive equation. Assume that the value of t_f can be expressed as the sum of an estimate, t_f', and a small perturbation.

$$t_f = t_f' + \delta t \tag{G-1}$$

The velocity to be gained over the interval, t_f, can be determined as follows.

$$(\Delta V')^2 = (\Delta \dot{X}')^2 + (\Delta \dot{Y}')^2 + (\Delta \dot{Z}')^2 \tag{G-2}$$

$$\Delta \dot{X}' = \dot{X}_f - \dot{X} - g_x \, t_f' \tag{G-3}$$

$$\Delta \dot{Y}' = \dot{Y}_f - \dot{Y} - g_y \, t_f' \tag{G-4}$$

$$\Delta \dot{Z}' = \dot{Z}_f - \dot{Z} - g_z \, t_f' \tag{G-5}$$

Introducing equation (G-1) into (G-2) yields the following results.

$$\Delta V^2 = (\Delta \dot{X}' - g_x \, \delta t)^2 + (\Delta \dot{Y}' - g_y \, \delta t)^2$$

$$+ (\Delta \dot{Z}' - g_z \, \delta t)^2 \qquad \text{(G-6)}$$

The velocity change, ΔV, resulting from the engine thrust follows.

$$\Delta V = V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} + V_{ex_2} \ln \frac{\tau_2}{\tau_2 - (T_2' + \delta T_2)} \qquad \text{(G-7)}$$

$$\Delta V = V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} +$$

$$V_{ex_2} \left(\ln \frac{\tau_2 / (\tau_2 - T_2')}{(\tau_2 - (T_2' + T_2)) / (\tau_2 - T_2')} \right) \qquad \text{(G-8)}$$

$$\Delta V = V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} +$$

$$V_{ex_2} \left(\ln \frac{\tau_2}{\tau_2 - T_2'} - \ln \left(1 - \frac{\delta T_2}{\tau_2 - T_2'} \right) \right) \qquad \text{(G-9)}$$

But $\ln (1 + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} \; . \; . \; .$ (G-10)

Or $\ln (1 + X) \cong X$ (G-11)

$$\therefore \quad \Delta V \cong V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} +$$

$$V_{ex_2} \ln \frac{\tau_2}{\tau_2 - T_2'} + \frac{\delta T}{\tau_2 - T_2'} \qquad \text{(G-12)}$$

The velocity change achieved by the engine (G-12) must be equivalent to the required velocity change (G-6).

$$\Delta V^2 = \left(V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} \right)^2 +$$

$$2V_{ex_1} V_{ex_2} \ln \frac{\tau_1}{\tau_1 - T_1} \ln \frac{\tau_2}{\tau_2 - T_2'} +$$

$$2V_{ex_1} V_{ex_2} \ln \frac{\tau_1}{\tau_1 - T_1} \frac{\delta T_2}{\tau_2 - T_2'} + \left(V_{ex_2} \ln \frac{\tau_2}{\tau_2 - T_2'}\right)^2$$

$$+ 2V_{ex_2}^2 \ln \frac{\tau_2}{\tau_2 - T_2'} \frac{\delta T_2}{\tau_2 - T_2'} + \left(V_{ex_2} \frac{\delta T_2}{\tau_2 - T_2'}\right)^2$$

$$= (\Delta \dot{X}')^2 + (\Delta \dot{Y}')^2 + (\Delta \dot{Z}')^2$$

$$- 2 \delta T (\Delta \dot{X}' g_x + \Delta \dot{Y}' g_y + \Delta \dot{Z}' g_z)$$

$$+ \delta T_2^2 (g_x^2 + g_y^2 + g_z^2) \qquad \text{(G-13)}$$

This equation further reduces if it is assumed that the velocity gained by the thrust acceleration for δT is equal to the gravity loss over the same interval.

$$V_{ex_2}^2 \left(\frac{\delta T}{\tau_2 - T_2'} \right)^2 = -2\,\delta T (\Delta \dot{X}' S_x + \Delta \dot{Y}' S_y + \Delta \dot{Z}' S_z)$$

$$+ \ \delta T_2^{\,2} (S_x^{\,2} + S_y^{\,2} + S_z^{\,2}) \quad \text{(G-14)}$$

$$\therefore \quad \frac{\delta T_2}{\tau_2 - T_2'} \left(2\, V_{ex_1} V_{ex_2} \ln \frac{\tau_1}{\tau_1 - T_1} + 2\, V_{ex_2}^2 \ln \frac{\tau_2}{\tau_2 - T_2'} \right.$$

$$= (\Delta \dot{X}')^2 + (\Delta \dot{Y}')^2 + (\Delta \dot{Z}')^2 - V_{ex_1}^2 \left(\ln \frac{\tau_1}{\tau_1 - T_1} \right)^2$$

$$- 2\, V_{ex_1} V_{ex_2} \ln \frac{\tau_1}{\tau_1 - T_1} \ln \frac{\tau_2}{\tau_2 - T_2'} - V_{ex_2}^2 \left(\ln \frac{\tau_2}{\tau_2 - T_2'} \right)^2$$

$$\text{(G-15)}$$

$$\delta T_2 = \frac{\tau_2 - T_2'}{2\, V_{ex_2}} \left(\frac{(\Delta \dot{X}')^2 + (\Delta \dot{Y}')^2 + (\Delta \dot{Z}')^2}{V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} + V_{ex_2} \ln \frac{\tau_2}{\tau_2 - T_2'}} \right.$$

$$\left. - V_{ex_1} \ln \frac{\tau_1}{\tau_1 - T_1} - V_{ex_2} \ln \frac{\tau_2}{\tau_2 - T_2'} \right) \quad \text{(G-16)}$$

It can easily be seen that a generalization of this integral is as follows.

$$\delta T_n = G \left(\frac{\tau_2 - T_n'}{V_{ex_n}} \right) \qquad \text{(G-17)}$$

$$G = \frac{1}{2} \left(\frac{(\Delta \dot{X}')^2 + (\Delta \dot{Y}')^2 + (\Delta \dot{Z}')^2}{\sum_{i=k}^{n} V_{ex_i} L_i} - \sum_{i=k}^{n} V_{ex_i} L_i \right) \qquad \text{(G-18)}$$

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