CONCERNING COUNTABLE PARACOMPACTNESS AND PSEUDO NORMALITY IN TOPOLOGICAL SPACES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Carolyn India Kerr

May, 1973

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Abstract

This paper extends the results of several mathematicians, notably Professor C. H. Dowker, Phillip Zenor, and John Mack, in the study of countably paracompact spaces. The concept of open set summability is introduced and the following theorem is proved.

THEOREM. A space is perfectly normal if, and only if, it is both countably paracompact and open set summable.

Also included in this paper is an example of a pseudo-normal, non-normal screenable Moore space.

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CHAPTER I

INTRODUCTION

Ever since C. H. Dowker defined countable paracompactness in 1951 (4), one of the outstanding problems in topology has been that of deciding that class of spaces in which countable paracompactness and normality are equivalent.

In Chapter II of this paper, we extend the study of countable paracompact spaces by first introducing the concept of open set summability. It is then proved that in the class of spaces having the open set summability property, normality and countable paracompactness are equivalent.

Chapter III provides an example of a non-normal, pseudo-normal screenable Moore space. A Moore space is one which satisfies Axiom 0 and the first three parts of Axiom 1 of (12). Both pseudo-normality and screenability are defined in the chapter.

One of the central problems in topology for the last half century has been the question of the existence of a normal, non-metrizable Moore space. One consequence of the key role this problem has played is the relative neglect of the search for non-normal Moore spaces. There are, it seems, relatively few examples available of non-normal Moore spaces. Some excellent ones are found in (1), (6) and most recently Professor Proctor (14) has provided us with a beautiful example of one which is also separable and pseudo-normal.

It is hoped that the example described herein is both easy to comprehend and that it will provide insight into the study of non-normal Moore spaces.

CHAPTER II

ON COUNTABLE PARACOMPACTNESS

In 1951 C. H. Dowker defined countable paracompactness. A space S is said to be countably paracompact provided every countable open covering of the space has a locally finite refinement. Recently several mathematicians, notably Professors Zenor (21) and Mack (11) have studied the properties of countably paracompact spaces in depth.

While it is known that there is a countably paracompact, non-normal space (10) and that perfect normality implies countable paracompactness (4), the question of whether or not countable paracompactness is equivalent to normality remains open in spaces as strong as Moore spaces. The purpose of this paper is to define a property (called open set summability) which bridges the apparent gap between countable paracompactness and normality.

Definitions. A set M is an inner limiting set (a strong inner limiting set) provided that M is the common part (respectively, the common part of the closures) of a countable collection of domains. A space is perfectly normal provided it is normal and each closed set in the space is an inner limiting set.

To say that a set M is an inner limiting set strong with respect to a set N means that M is the common part of a sequence of domains D_1, D_2, ... such that N does not intersect the common part of the closures of the domains in the sequence.

To say that a space S is open set summable means that each open set O in S is the union of a countable collection of inner limiting sets strong with respect to the complement of that open set.

The following two examples are chosen to illustrate the open set summability property.

EXAMPLE 1. Professor Heath's (6) well-known example of a screenable non-normal Moore space is open set summable.

The space S consists of the upper half plane including the x - axis where the topology on S is the discrete topology above the x - axis, vertical line segments with lower end point an irrational number, and segments with slope 1 and end point a rational number. The space is readily seen to be open set summable in consideration of the following two properties.

1. The set of all rational numbers is countable and each rational is by itself a strong inner limiting set.

2. Any subset of the irrationals is a strong inner limiting set. For suppose Q is a subset of the irrationals and $r_1, r_2, \ldots$ is an enumeration of the rationals. Since Q is a closed subset of a Moore space, there exists a monotonic sequence $D_1, D_2, \ldots$ of domains whose common part is Q. For each positive integer n, the rational r_n is not a limit point of Q so there exists an open set, call it d_n, containing r_n, whose closure does not intersect Q. For each positive integer n, define $O_n = D_n - \sum_{k=1}^{n} \text{cl } d_k$. Then $O_1, O_2, \ldots$ is a sequence of domains, each containing Q, such that if r is a rational number, then $r = r_n$ for some positive inter n. In which case, d_n is an open set about $r = r_n$ which does not intersect O_n so that r does not belong to cl O_n. Therefore, Q

is the common part of the closures of the elements of the sequence of domains O_1, O_2, ...

EXAMPLE 2. A countably paracompact, normal space which does not have the open set summability property is the space S consisting of all countable ordinals together with the first uncountable ordinal w_1, with the order topology. S is even compact but fails to be open set summable since $S - \{ w_1 \}$ is open but is not the union of countably many inner limiting sets strong with respect to $\{ w_1 \}$.

Proposition 1. Each closed set in an open set summable space S is an inner limiting set.

Proof: Let M denote a closed subset of S. Then $S - M$ is open and hence is the union of a sequence K_1, K_2, ... of inner limiting sets strong with respect to $S - (S - M) = M$.

Thus for each positive integer n there exists a sequence D_{n1}, D_{n2}, ... of domains with common part K_n such that $\bigcap_i \text{cl } D_{ni}$ does not intersect M.

For each positive integer n, let $O_n = S - \bigcap_i \mathrm{cl}\ D_{ni}$. It follows that $O_1, O_2, \ldots$ is a sequence of domains with common part M. That is, M is an inner limiting set.

<u>Proposition 2</u>. Each perfectly normal space has the open set summability property.

<u>Proof</u>: Let S denote a perfectly normal space and O be an open set in S. Then $M = S - O$ is closed and there exists a sequence $D_1, D_2, \ldots$ of domains such that $M = \bigcap_n D_n$.

Since S is normal and for each n, $S - D_n$ and M are two mutually exclusive closed sets there exists a domain O_n containing $S - D_n$ whose closure does not intersect M.

Then $D = \bigcup_n O_n$, and for each n, O_n is an inner limiting set strong with respect to $M = S - O$ since $\mathrm{cl}\ O_n$ does not intersect $M = S - O$.

Therefore, S is open-set summable.

MAIN THEOREM. A space is perfectly normal if, and only if, it is both countably paracompact and open set summable.

LEMMA 1. Suppose S is countably paracompact, M is a subset of a domain D and there exists a sequence D_1, D_2, ... of domains covering S - D such that for each positive integer n, cl D_n does not intersect M. Then there exists a domain O about M whose closure is a subset of D.

Proof: The collection D' consisting of D together with the elements of the sequence D_1, D_2, ... is a countable open cover of S. Hence there exists a locally finite refinement V of D' which covers S.

Let p denote a point of M. There exists an open set about p which intersects an element of V only in case that element is a subset of D. Since V is locally finite at p, there exists an open set O_p about p such that O_p intersects only finitely many elements of V. If each element of V which intersects O_p is a subset of D, then O_p is the desired open set, so define $V_p = O_p$. Otherwise, there must exist some element v in V which intersects O_p but is not a subset of D. It follows that each such element must be a subset of some element of the sequence D_1, D_2, ... and since for each n, cl D_n does not intersect M, the closure of v does not intersect M.

Since there are only finitely many such elements the set V' of closures of elements of the set, to which v belongs if and only if v is an element of V which is not a subset of O but does intersect O_p, is closed. Define then $V_p = O_p \cap (S - V'^*)$.

Define $O = \{ V_p : p \text{ in } M \}$. Then clearly O is open set containing M.

In order to show cl O is a subset of D, suppose v is an element of V intersecting O. Then there exists a point p of M such that v intersects V_p. By definition then of V_p, v is a subset of D. Thus if v is an element of V which intersects O then v is a subset of D. Clearly then cl O is a subset of D since each point of p of S - D must be contained in some element v of V and so that element v is an open set containing p but no points of O.

LEMMA 2. Suppose S is countably paracompact and M is a subset of a domain D such that M is an inner limiting set strong with respect to S - D. Then there exists an open set O about M such that cl O is a subset of D.

Proof: Since M is an inner limiting set strong with respect to S - D, there exists a sequence $D_1, D_2, \ldots$ of domains with common part M such that $\bigcap_n \text{cl } D_n$ does not intersect S - D.

For each positive integer n let $O_n = S - \text{cl } D_n$. Then $O_1, O_2, \ldots$ is a sequence of domains covering S - D, no one of which intersects M in its closure.

Thus, by Lemma 1, there does indeed exist a domain O containing M whose closure is a subset of D.

PROOF OF THEOREM

Necessity. This follows from Dowker's proof that perfectly normal implies countably paracompact and Proposition 2 above.

<u>Sufficiency</u>. Assume S is a countably paracompact, open set summable space. By Proposition 1, each closed set in S is an inner limiting set. It remains to be shown then that S is normal.

Let H and K denote two mutually exclusive closed sets. Then S - K is an open set and so is the union of countably many inner limiting sets strong with respect to K = S - (S - K), say H_1, H_2, ... By Lemma 2, for each positive integer n, there exists an open set O_n about H_n such that cl O_n does not intersect K.

Now K is a subset of the domain S - H and O_1, O_2, ... is a sequence of domains covering S - (S - H) = H such that no one of them intersects K. By Lemma 1, there exists a domain about K whose closure does not intersect H. Therefore S is normal.

CHAPTER III

A PSEUDO-NORMAL, NON-NORMAL SCREENABLE MOORE SPACE

Professor R. H. Bing (see (1)) introduced concepts of screenability and strong screenability. A space S is said to be screenable (strongly screenable) provided that for each open cover of the space, there exists a sequence V_1, V_2, ... such that

1. For each positive integer n, V_n is a (discrete) collection of mutually exclusive open sets refining O, and

2. $\sum_n V_n^* = S$.

In the same paper, Professor Bing gave an example of a screenable, non-normal Moore space. This space, which he referred to as example B, provided the inspiration for the example presented herein. The space which we will describe has all the topological properties possessed by example B. The additional property that our space possesses is pseudo-normality. A space S is said to be pseudo-normal provided S is normal with respect to each pair of mutually exclusive closed sets, one of which is countable.

Before proceeding with the description of the space, we first give some additional terminology.

Let W denote the set of all countable ordinals and A denote the subset of W consisting of all non limit ordinals. Define B to be W - A.

For each ordinal b in B, since b is a countable limit ordinal there exists a monotonic sequence of distinct elements of A which converges to b in order topology on W. Let $A(b) = A_1(b), A_2(b), \ldots$ denote one such sequence converging to b in the order topology on W.

DESCRIPTION OF THE SPACE

Points of the space S are of two types:

(i) countable ordinals and

(ii) ordered triples (a, t, b) where a is an element of A, b is an element of B, $a < b$, and t is a positive number less than 1.

For each positive integer n, define a covering H_n of S as follows. The elements of H_n are of three types depending upon whether they are centered at a point of A, a point of B or a point of type (ii).

(1) An element of H centered at a point (a, t, b) of type (ii) is the set of all points (a, r, b) such that $|t - r| < 1/n$.

(2) An element of H_n centered at a point $\underline{a}$ of A is the set consisting of $\underline{a}$ together with all points of the form ($\underline{a}$, t, b) where $t < 1/n$.

(3) An element of H_n centered at a point b of B is the set consisting of

(i) b,

(ii) all points of the form (a, t, b) where a, in the order topology on W, precedes or is equal to the n^{th} term, in the sequence A(b) and $1 - t < 1/n$, and

(iii) all points of the form ($\underline{a}$, t, b) where $\underline{a}$ follows, in the order of topology on W, the n^{th} term in the sequence A(b) and $1 - t < \frac{1 + n(1 - 1/j)}{1 + n}$ where j denotes the unique integer (necessarily greater than n) such that, in the order topology on W, A_{j-1} precedes $\underline{a}$ which precedes or is equal to $A_j(b)$.

NOTE: For each point x of S and each positive integer n, $H_{n+1}(x)$ is a subset of $H_n(x)$. Hence if x is a point of S and m and n are positive integers such that $m < n$, then $H_m(x)$ is a subset of $H_n(x)$.

Regions of the space S are then defined by a development G_1, G_2 ... where for each positive integer n, G_n is the collection of all elements g with the property that for some positive integer i greater than or equal to n, g is an element of H_i.

It is convenient to think of the space as being the collection of all countable ordinals plus all open unit intervals connecting each non limit ordinal a to all the countable limit ordinals which follow a in the order topology on W. In this context an element of G_n is either

(i) a point of an open interval plus the collection of all points of that open interval which are nearer than 1/n to the point.

(ii) an element of A plus all points that can be joined to it by intervals of length less than 1/n, and

(iii) a point b of B plus all points (a, t, b) that can be joined to b by an interval of length less that x(a) where

(1) x(a) = 1/n provided a precedes or is equal to the n^{th} term in the sequence A(b) converging to b, or

(2) $x(a) = \dfrac{1 + n\,(1 - 1/j)}{n + 1}$ where j is the unique positive integer such that $A_n(b) \leq A_{j-1}(b) < a < A_j(b)$.

BASIC PROPERTIES OF THE SPACE

PROPOSITION 1.1. Assume a is a point of A. Then the following hold.

1. If z is a point of A distinct from a, $H_1(a)$ and $H_1(z)$ are disjoint.

2. If z is a point of B which precedes a in the order topology on W, then $H_1(a)$ and $H_1(z)$ are disjoint.

3. If z is a point of B which follows a in the order topology on W, then there exists a positive integer n such that a precedes the n^{th} term in the sequence A(z). In this case, $H_2(a)$ and $H_{n+1}(z)$ are disjoint. For if (a, t, z) is in $H_2(a)$, then $t < 1/2$ and so $1 - t > 1/2 \geq 1/n+1$. Since $a < A_n(z) < A_{n+1}(z)$ and $1 - t > 1/n+1$, it follows that (a, t, z) does not belong to $H_{n+1}(z)$.

4. If z is a point of type (ii) connecting a point of A distinct from a to B, then

$H_1(a)$ and $H_1(z)$ are disjoint.

5. If z is a point of type (ii) such that $z = (a, t, b)$ then, letting n denote a positive integer such that $1/n < t$, it follows that $H_{2n}(a)$ and $H_{2n}(z)$ are disjoint. This is true since if (a, r, b) is in both $H_{2n}(a)$ and $H_{2n}(z)$, then $r < 1/2n$ and $|t - r| < 1/2n$ giving us the contradiction that
$t = |t - r + r| = |t - r| + r < 1/2n$
$+ 1/2n = 1/n$.

PROPOSITION 1.2. Assume b is a point of B. then the following hold:

1. If z is a point of B distinct from b, then $H_1(b)$ and $H_1(z)$ are disjoint.

2. If z is a point of A which follows b in the order topology on W, then $H_1(b)$ and $H_1(z)$ are disjoint.

3. If z is a point of A which precedes b in

the order topology on W, then there exists a positive integer n such that $H_n(z)$ does not intersect $H_2(b)$.

For suppose $z \leq A_2(b)$. Then if (z, t, b) is in $H_2(z)$, $t < 1/2$. Hence $1 - t > 1/2$ and so (z, t, b) is not in $H_2(b)$. In this case $n = 2$ is a positive integer such that $H_n(z)$ and $H_n(b)$ are disjoint.

And supposing $z > A_2(b)$. There exists a unique positive integer k such that $A_{k-1}(b) < a \leq A_k(b)$.

Then $1 - 1/k < 1$ and $\frac{1 + 2\ (1 - 1/k)}{1 + 2} < 1$ so there exists a positive integer m such that $\frac{1 + 2\ (1 - 1/k)}{1 + 2} < 1 - 1/m$. Then if (z, t, b) is in $H_m(z)$ it follows that $t < 1/m$.

Then $1 - t > 1 - 1/m > \frac{1 + 2\ (1 - 1/k)}{1 + 2}$

and so by definition of H_m, (z, t, b) is not in $H_m(b)$. Thus $n = m$ is a positive integer such that $H_m(z)$ and $H_2(b)$ are disjoint.

4. If z is a point of type (ii) connecting A to a point of B distinct from b, then $H_1(b)$ and $H_1(z)$ are disjoint.

5. If z is a point of type (ii) such that $z = (a, t, b)$ then there exists a positive integer n such that both a precedes the n^{th} term in the sequence $A(b)$ and $1/n \leq 1 - t$. In this case $H_{2n}(b)$ and $H_{2n}(z)$ are disjoint. For is a point (a, r, b) were in both $H_{2n}(b)$ and $H_{2n}(z)$ then we would have that $1 - r < 1/2n$ and $|t - r| < 1/2n$, giving the contradiction that $1 - t = 1 - r + r - t \leq 1 - r + |t - r| < 1/2_n + 1/2_n = 1/n \leq 1 - t$.

PROPOSITION 2. If A' is a subset of A and b is an order-limit point of A', then for each positive integer n, b is a limit point of $St(A', H_n)$.

Proof: Assume A' is a subset of A and b is an order-limit point of A'. Suppose n is a positive integer and k is an integer greater than n. We wish to show that

$H_k(b)$ intersects $St(A', H_n)$.

Since b is an order-limit point of A', there exists a point of A' such that $A_k(b) < a < b$.

The point $(a, 1/(k+1), b)$ is in $St(A', H_n)$ since $1/(k+1) < 1/n$. To show that this point is also in $H_k(b)$ consider the following:

(a) There exists a unique positive integer j such that $A_{j-1} < a \leq A_j(b)$. Since $A_k(b) < a$, it is clear that $j > k$.

(b) Since $j > k$ we have

$1/j < 1/k$,

$1 - 1/j > 1 - 1/k$, and

$1 + k\ (1 - 1/j) > 1 + k\ (1 - 1/k) =$

$1 + k - 1 = k$.

Thus

$$1 - 1/(k + 1) = k/(k + 1) < \frac{1 + k(1 - 1/j)}{k + 1}$$

By definition of $H_k(b)$, the point $(a, 1/(k + 1), b)$ is in $H_k(b)$.

Therefore, b is a limit point of A'.

PROPOSITION 3. The subspace $S' = S - W$ of S is metrizable.

Proof: Define a function d on $S' \times S'$ as follows:

If each of $x = (a, t, b)$ and $y = (c, r, d)$ is in S', define

1) $d(x, y) = 2$ if $a \neq c$ or $b \neq d$, and

2) $d(x, y) = |t - r|$ if $(a, b) = (c, d)$

It is clear that d is a metric function on $S' \times S'$. In order to show then that the subspace S' is metrizable we need to show that the topology induced on S' by the metric d agrees with the subspace topology on S'.

To this end, suppose that $p = (a, t, b)$ is a point of S' and M is a subset of S'. We must show that p is a limit point of M if, and only if, for each positive number e, there exists a point z of M, distinct from p, such that $d(p, z) < e$.

Necessity. Suppose p is a limit point of M

and e is a positive number. There exists an integer n such that $1/n < e$. Since $H_n(p)$ is an open set about p, there exists a point z of M, distinct from p, in $H_n(p)$. By definition then of $H_n(p)$, z = (a, r, b) and $|t - r| < 1/n$. It follows from the definition of d, that $d(p,z) = |t - r| < 1/n < e$.

Sufficiency. Suppose conversely that for each positive number e, there exists a point z of M distinct from p such that $d(p, z) < e$. Suppose R is a domain. We wish to show R contains a point of M distinct from p.

There exists a positive integer n such that $H_n(p)$ is a subset of R. Since $1/n$ is a positive number, there exists a point z of M distinct from p such that $d(p, z) < 1/n < e$. Then by definition of d, z = (a, r, b) and $d(p, z) = |t - r| < 1/n$. It follows from definition of $H_n(p)$, that z is in $H_n(p)$.

PROPERTIES OF THE SPACE

S is a Moore space.

LEMMA 1 Assume a is a point of A and n is a positive integer. Then cl $H_{n+1}(a)$ is a subset of $H_n(a)$.

Proof:

If z is a point such that

(1) z is a point of A distinct from a,

(2) z is a point of B which precedes a in the order topology or W, or

(3) z is a point of type (ii) which is not on an interval connecting a to B, then z cannot belong to cl $H_{n+1}(a)$ since by Proposition 1.1 (1), (2) and (4), $H_1(z)$ is an open set containing z but no point of $H_1(a)$, which contains $H_{n+1}(a)$.

Further if z is a point of B which follows a in the order topology on W, then z also cannot be in cl $H_{n+1}(a)$ since by Proposition 1.1(3), there exists a positive integer k such that $H_k(z)$ does not intersect $H_2(a)$, which contains $H_{n+1}(a)$.

Thus the only points of cl $H_{n+1}(a)$ distinct from a are those points of type (ii) which belong to an interval connecting a to B. Consider the case that $z = (a, t, b)$ where $t \geq 1/n$.

Then $t > 1/(n+1)$ so there exists a positive integer i such that $t > 1/(n+1) + 1/i$. Suppose (a, p, b) is in both $H_i(z)$ and $H_{n+1}(a)$. Then $| t - p | < 1/i$ and $p < 1/(n+1)$, giving the contradiction that

$$\begin{aligned} t &\leq |t - p| + p \\ &< 1/i + 1/n+1 < t. \end{aligned}$$

Thus $H_i(z)$ is an open set containing z but no point of $H_{n+1}(a)$ and hence z is not in cl $H_{n+1}(a)$.

Therefore, the only points of cl $H_{n+1}(a)$ distinct from a are those points $z = (a, t, b)$ such that $t < 1/n$. That is, cl $H_{n+1}(a)$ is a subset of $H_n(a)$.

<u>LEMMA 2</u>. Assume b is a point of B and n is a positive integer. Then cl $H_{n+1}(b)$ is a subset of $H_n(b)$.

Proof:

If z is a point such that

(1) z is a point of B distinct from b,

(2) z is a point of A which follows b in the order topology on W, or

(3) z is a point of type (ii) which is not on an interval connecting b to A,

then z cannot belong to cl $H_{n+1}(b)$ since by Proposition 1.2 (1), (2), and (4), $H_2(z)$ is an open set containing z but no point of $H_2(b)$ which contains $H_{n+1}(b)$.

Suppose z is a point of A which precedes b in the order topology on W. By Proposition 1.2 (3) there exists a positive integer k such that $H_k(z)$ does not intersect $H_2(b)$, which contains $H_{n+1}(b)$.

Thus the only points of cl $H_{n+1}(b)$ distinct from b are those points of type (ii) which belong to an interval connecting b to A.

Consider the case that z = (a, t, b) and z is not in $H_n(b)$. Either $a \leq A_n(b)$, $A_n(b) < a \leq A_{n+1}(b)$ or $A_{n+1}(b) < a$.

In case $a \leq A_n(b)$, then $1 - t \geq 1/n$ since $z = (a, t, b)$ is not in $H_n(b)$. In this case $1 - t \geq 1/n > 1/(n+1)$ and so there exists a positive integer m such that $1 - t > 1/(n+1) + 1/m$. Then $H_m(z)$ and $H_{n+1}(b)$ are disjoint for if (a, r, b) were in $H_m(z)$ and in $H_{n+1}(b)$ then we would have $|t - r| < 1/m$ and $1 - r < 1/(n+1)$, giving the contradiction to the choice of m that

$$1 - t = 1 - r + r - t \leq |1 - r| + |t - r|$$
$$< 1/(n+1) + 1/m.$$

Thus in this case $H_m(z)$ is an open set containing z but no point of $H_{n+1}(b)$.

In case of $A_n < a \leq A_{n+1}$, then since $z = (a, t, b)$ is not in $H_n(b)$, it follows that

$$1 - t \geq \frac{1 + n\ (1 - 1/(n+1)}{1 + n} > \frac{1 + n\ (1 - 1/n)}{1 + n}$$

$$= \frac{1 + n - 1}{n + 1} = n/(n+1) > 1/(n+1)$$

Thus there exists a positive integer m such that $1 - t > 1/(n+1) + 1/m$. In this case $H_m(z)$ and $H_{n+1}(b)$ are disjoint. For if (a, r, b) were in $H_m(z)$ and $H_{n+1}(b)$ then we would have $|t - r| < 1/m$ and $1 - r < 1/(n+1)$ giving the contradiction that

$$1 - t = 1 - r + r - t \leq 1 - r + |t - r|$$
$$< 1/(n+1) + 1/m < 1 - t.$$

Finally if $A_{n+1}(b) < a$, then there exists a unique positive integer k (necessarily greater than n + 1 such that $A_{k+1}(b) < a \leq A_k(b)$.

Thus since $z = (a, t, b)$ is not in $H_n(b)$, it follows that $1 - t \geq \frac{1 + n\,(1 - 1/k)}{1 + n}$. Now we have

$$\frac{1 + n\,(1 - 1/k)}{1 + n} - \frac{1 + (n+1)\,(1 - 1/k)}{1 + n + 1} = 1/k > 0$$

and hence $1 - t \geq \frac{1 + n\,(1 - 1/k)}{1 + n} > \frac{1 + (n+1)\,(1-1/k)}{1 + (n+1)}$

so there exists a positive integer m such that

$$1 - t > \frac{1 + (n+1)\,(1 - 1/k)}{1 + (n+1)} + 1/m.$$

In this case, $H_m(z)$ and $H_{n+1}(b)$ are disjoint. For if (a, r, b) were in $H_m(z)$ and $H_{n+1}(b)$ then we would have $|t - r| < 1/m$ and $1 - r < \frac{1 + (n+1)\,(1 - 1/k)}{1 + (n+1)}$, giving the contradiction that

$$\begin{aligned} 1 - t = 1 - r + r - t &\leq 1 - r + |t - r| \\ &< \frac{1 + (n+1)\,(1-1/k)}{1 + (n+1)} + 1/m \\ &< 1 - t \end{aligned}$$

Thus it follows that cl $H_{n+1}(b)$ is a subset of $H_n(b)$.

Proof that S is a Moore Space

It is clear that the sequence G_1, G_2 ... satisfies the first two parts of Axiom 1_3. In order to show S is a Moore space, we must then show the sequence G_1, G_2, ... satisfies the following:

If x and y are two points of a region R, then there exists a positive integer n such that if g is an element of G_n containing x then cl g is a subset of R not containing y.

Suppose then that x and y are two points of a region R. Then there exists a positive integer m_1 such that $H_{m_1}(x)$ is a subset of R and applying Propositions 1.1 and 1.2, there exists a positive integer m_2 such that $H_{m_2}(x)$ does not contain y. Letting $m = m_1 + m_2$, then m is a positive integer such that $H_m(x)$ is a subset of R not containing y.

Either x is a point of W or x is a point of type (ii). If x is a point of W, then by lemmas 1 and 2, cl $H_{m+1}(x)$ is a subset of $H_m(x)$. Since no element of G_1 contains the point x of W except those elements centered at x, it follows that (m + 1) is the desired positive integer in this case.

Consider then the case that x is a point of type (ii). Then x = (a, t, b). Let k denote an integer such that

(a) $k > m$,

(b) $a \leq A_k$,

(c) $1/k < 1 - t$, and

(d) $1/k < t$

Then by choice of k, applying Propositions 1.1(5) and 1.2(5), cl $H_{2k}(x)$ is a subset of S' = S - W, and the closure of each element of G_{2k} which contains x is a subset of S'.

By Proposition 3, S' is metrizable and so clearly there does exist a positive integer j such that if g is an element of G_j containing x then cl g is a subset of $H_{2k}(x)$, which is in turn a subset of R not containing y.

S is Pseudo Normal

LEMMA 1. Every Moore space is normal with respect to each pair of mutually exclusive closed countable point sets.

Proof: Let H and K denote two mutually exclusive closed subsets of a Moore space X, each of which is countable. Let $h_1, h_2, \ldots$ and $k_1, k_2, \ldots$ denote enumerations of H and K respectively.

Define inductively as follows monotonic sequences $V_1, V_2, \ldots$ and $W_1, W_2, \ldots$ such that for each positive integer n:

1. V_n is an open set containing h_n and V_i for each positive integer $i \leq n$,
2. W_n is an open set containing k_n and W_i for each positive integer $i \leq n$, and
3. cl $V_n + H$ and cl $W_n + K$ are disjoint.

For the case that n = 1 we define V_n and W_n as follows. Since h_1 is not a limit point of K, there exists an open set, call it V_1, containing h_1 whose closure does not intersect K. Since k_1 of K does not belong to cl $(H + V_1)$, there exists an open set, call it W_1, containing k_1, whose closure does not intersect either H or cl V_1.

Assuming that m is a positive integer such that $V_1, \ldots, V_m$ and $W_1, \ldots, W_m$ have been defined so as to satisfy conditions (1), (2), and (3) for each positive integer $n \leq m$, we then proceed to define V_{m+1} and W_{m+1}.

Since by condition 3, h_{m+1} of H does not belong to cl $W_m + K$, there exists an open set v containing h_{m+1} such that cl v does not intersect cl $W_m + K$. Define V_{m+1} to be $v + V_m$. Then clearly V_{m+1} is an open set containing h_{m+1} and V_i for each positive integer $i \leq m + 1$.

Further k_{m+1} of K does not belong to either cl v or to cl $V_m + H$ by condition 2. Thus since k_{m+1} does not belong to

cl $V_{m+1} + H =$ cl$(v + V_m) + H =$ cl v + cl $V_m + H$

there exists an open set w about K_{m+1} such that cl w does not intersect cl $V_{m+1} + H$.

Define $W_{m+1} = w + W_m$. Then clearly W_{m+1} is an open set containing k_{m+1} and W_i for each positive integer $i \leq m + 1$. Further, neither cl w nor cl $W_m + K$ intersects cl $V_{m+1} + H$ does not intersect cl $W_{m+1} + K = \text{cl } w + \text{cl } W_m + K$.

Thus (1), (2), and (3) are satisfied for each positive integer $n \leq m + 1$.

It follows from the induction principle that there do exist such sequences $V_1, V_2, \ldots$ and $W_1, W_2, \ldots$

Let $O(H) = \sum_n V_n$ and $O(K) = \sum_n W_n$. Then $O(H)$ and $O(K)$ are domains containing H and K respectively. Further $O(H)$ and $O(K)$ are disjoint, for assuming the contrary we reach the following contradiction. If there exists a point p in $O(H) \cap O(K)$, then there exist positive integers m and n such that p is in both V_m and W_n. Then p is in V_{m+n} and W_{m+n}, contradicting condition (3) for the positive integer m + n.

Therefore, S is normal with respect to H and K and the lemma is proved.

Proof that S is Pseudo-Normal

Let H and K denote two mutually exclusive closed sets such that one, say H, is countable.

There exists an ordinal c of A such that if z is a point of H then either

(1) z is a point of W which precedes c in the order topology on W, or

(2) z is a point of type (ii) such that z = (a, t, b) where b precedes c in the order topology on W.

Define K^- to be the set of all elements z of K such that z is an element of W which precedes c in the order topology on W.

Then each of H and K^- is countable and by the preceding lemma, there exist disjoint domains

$O(H)$ and $O(K^-)$ containing H and K^- respectively.

For each point x of H, x is not in the closed set K, so there exists an integer $n_x > 1$ such that cl H_{n_x} contains no point of K. Let V denote the set to which an open set g belongs if, and only if, for some element x of H, $g = H_{2n_x}(x)$.

Let $D(H) = O(H) \cap V$. Then we wish to show that cl $D(H)$ does not intersect $K^+ = K - K^-$.

Suppose z is a point of K^+. Then z is either a point of W which follows or is equal to the ordinal c of A or z is a point of type (ii).

If z is a point of A, then no interval connects z to any point of H so $H_1(z)$ and $St(H,H_1)$, which contains $D(H)$, are disjoint.

If z is in B, then by proposition 1.2(1), (3) and (4), there exists an integer $n > 1$ such that $H_n(z)$ contains no point of $St(H,H_2)$, which contains D(H).

If z is a point of type (ii) then z = (a, t, b) where either only a is in H, only b is in H, both a and b are in H or neither a nor b is in H. Let n denote a positive integer which is greater than

(1) n_a if a is in H,

(2) n_b if b is in H, and

(3) 2.

Since z is not in the closed set H, there exists an integer $m \geq n$ such that cl H (z) does not intersect H. Then $H_{2m}(z)$ does not intersect D(H). For suppose (a, r, b) were in both $H_{2m}(z)$ and D(H). Then there exists a point x of H such that (a, r, b) is in $H_{2n_x}(x)$, which is a subset of D(H). By choice of n, x must be a point of type (ii); that is, x = (a, p, b). Then $|t - r| < 1/2n_x$. Either $m \geq n_x$ or $n_x > m$. If $m \geq n_x$, then we reach the contradiction that z = (a, t, b) of K is in $H_{n_x}(x)$ since $|t - p| < |t - r| + |r - p| < 1/2m + 1/2n_x \leq 1/2n_x + 1/2n_x = 1/n_x$. And if $n_x > m$ we reach the contradiction that x = (a, p, b) of H is in $H_m(z)$ since $|t - p| \leq 1/2m + 1/2n_x \leq 1/2m + 1/2m = 1/m$. Thus no point of K^+ is in cl D(H). Hence D(H) and D(K) = S - cl D(H) are two disjoint domains, one containing H and the other K.

Therefore, S is pseudo-normal.

S is Not Normal

Suppose A' is an uncountable subset of A and $D(A')$ and $D(B)$ are two domains containing A' and B respectively. We wish to show that there is a point common to $D(A')$ and $D(B)$.

Since A' is uncountable there exists a positive integer n such that for each element a of an uncountable subset A'' of A', $H_n(a)$ is a subset of $D(A')$.

Since A'' is uncountable there exists an element b of B which is, in order topology on W, a limit point of A''. Further b is a point of $D(B)$ so there exists a positive integer m such that $H_m(b)$ is a subset of $D(B)$.

Let $k = m + n$. Since b is a limit of A'' in the order topology on W there exists an element a of A'' such that $A_k(b) < a < b$. We wish to show that the point $(a, 1/(k+1), b)$ is in $H_k(b)$. Consider the following, where j denotes that unique integer such that $A_{j-1}(b) < a \leq A_j(b)$. Since j must necessarily be greater than k we have:

$$1 + k\ (1 - 1/j) > 1 + k\ (1 - 1/k) = 1 + k - 1 = k.$$

Hence:

$$1 - 1(k+1) = \frac{k + 1 - 1}{k + 1} = \frac{k}{k + 1} < \frac{1 + k(1 - 1/j)}{1 + k}$$

and by definition of $H_k(b)$, (a, 1(k+1), b) is in $H_k(b)$ which is a subset of $H_m(b)$ and hence of D(B).

Since (a, 1/k+1, b) clearly is in $H_n(a)$ which is a subset of D(A), the two domains do have a point in common.

It is clear that S is not normal since the uncountable sets A and B are then two mutually exclusive closed sets having the property that there do not exist disjoint domains, one containing A and the other B.

S is Screenable

Suppose O is an open cover of S. For each point p of S there exists an element g(p) of G_1 containing p such that g(p) is a subset of some element of the collection O. Let O(A), O(B) and O(S') denote the collections to which an element g of G_1 belongs if, and only if, for some point p of A or of B or of S' = S - W respectively, g = g(p).

It follows from Propositions 1.1(1) and 1.2(1) above that each of O(A) and O(B) is a collection of mutually exclusive domains refining O. Further O(A) covers A and O(B) covers B.

By Proposition 3, S' is metrizable and hence strongly screenable (1). Since O(S') is an open cover of the open subset S' of S, there exists a sequence D_1, D_2, ... of collections of mutually exclusive domains refining O(S') such that $\sum_n D_n^*$ covers S'.

Thus O(A) and O(B) together with the sequence D_1, D_2, ... form a countable sequence of collections of mutually exclusive domains refining O and covering S.

S is Not Countably Paracompact

For each positive integer n, let A_n denote the set of all elements a of A with the property that there exists a point b of B such that $a = b + n$. It is clear that for each positive integer n, A_{n+1} is the set consisting of all immediate successors to elements of A_n.

LEMMA. Suppose $D_1, D_2, \ldots$ is a sequence of domains such that for each positive integer n, D_n contains A_n. There is a point common to the closures of the elements of the sequence $D_1, D_2, \ldots$.

Proof: Define inductively as follows sequences $A'_1, A'_2, \ldots; k_1, k_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integer n,

1. A'_n is an uncountable subset of A_n,
2. k_n is a positive integer such that $St(A'_n, H_{k_n})$ is a subset of D_n,
3. b_n is a limit point of A'_n in the order topology on W,
4. each element in A'_{n+1} follows b_n and is the immediate successor of some element of A'_n, and
5. $b_{n+1} \geq b_n$.

For the case that $n = 1$, define A'_n, k_n and b_n thusly. Since A_1 is uncountable and each point a of A_1 has the property that there exists a positive integer k such that $H_k(a)$ is a subset of D_1, it follows that there exists an uncountable subset A'_1 and a positive integer k_1 such that $S(A'_1, H_{k_1})$ is a subset of D_1. Further since A'_1 is uncountable, there exists a point b_1 of B which is a limit point of A'_1 in the order topology on W.

Assume that m is a positive integer such that $A_1 \ldots, A_m$; $k_1, \ldots k_m$ and $b_1, \ldots, b_m$ have been defined so that conditions (1) through (5) have been satisfied for each positive integer $n \leq m$. We then define A_{m+1}, k_{m+1} and b_{m+1} as follows.

Since A'_n is uncountable, that subset C of A_{n+1} consisting of all elements of A_{n+1} which both follow b_n and are the immediate successors to elements of A'_n is uncountable. Thus there exists a positive integer k_{m+1} and an uncountable subset A'_{m+1} of C such that if a is in A'_{m+1} then $H_{k_{m+1}}(a)$ is a subset of D_{m+1}. That is, $St(A'_{m+1}, H_{k_{m+1}})$ is a subset of D_{m+1}.

Since A'_{m+1} is uncountable and each element of A'_{m+1} follows the point b_m, there exists a point b_{m+1}, necessarily greater than b_m, which is a limit point of A'_{m+1} in the order topology on W.

It is clear then that $A'_1 \ldots, A'_{m+1}$; $k_1, \ldots k_{m+1}$; and $b_1, \ldots, b_{m+1}$ so defined satisfy conditions (1) through (5) for each positive integer $n \leq m + 1$. Therefore, by the induction principle there do exist such sequences $A_1, A_2, \ldots$; $k_1, k_2, \ldots$ and $b_1, b_2, \ldots$

Since $b_1, b_2, \ldots$ is a monotonic sequence in W there exists an ordinal b which is the order-limit point of the sequence. We wish to show that b is an order-limit point of A'_n for each positive integer n.

Suppose n is a positive integer and a is an ordinal less than b. Since b is an order-limit point of the sequence $b_1, b_2, \ldots$ there exists an integer $k > n$ such that $a < b_k$. Then $a < b_k < b_{k+1} < b$.

Since each point of A'_{k+1} follows b_k and b_{k+1} is a limit point of A'_{k+1}, it follows that there are infinitely many points of A'_{k+1} between b_k and b_{k+1}. Let c denote a point of A'_{k+1} such that $b_k < c < b_{k+1}$.

Since c is in A'_{k+1} there exists an element d of B such that $c = d + (k+1)$. Then we have $b_k \leq d < d + (k+1) = c$. Further, since for each positive integer t, each element of A'_{t+1} is the immediate successor of some element of A'_t, it follows from the fact that $n < k$ that $d + n$ is in A'_n.

Thus $d + n$ is a point of A'_n such that

$$a < b_k \leq d < d + n < d + (k+1) = c < b_{k+1} < b.$$

We have shown that between b and each ordinal less than b there exists an element of A'_n. Thus b is a limit of A'_n in the order topology on W.

Therefore, for each positive integer n, b is an order-limit point of A'_n and hence applying Proposition 4, b is a limit point of $St(A'_n, H_{k_n})$ and hence of D_n which contains $St(A'_n, H_{k_n})$. The point b is thus common to the closures of the elements of the sequence $D_1, D_2, \ldots$ and the lemma is proved.

Proof that S is not Countably Paracompact

In order to show that S is not countably paracompact, we will demonstrate a monotonic sequence $F_1, F_2, \ldots$ of closed sets whose common part is void having the property that if $D_1, D_2, \ldots$ is a sequence of domains such that for each positive integer n, D_n contains F_n, then the common part of the closures of the elements of the sequence $D_1, D_2, \ldots$ is non-void.

For each positive integer n, let $F_n = \sum_{i \geq n} A_i$. Then $F_1, F_2, \ldots$ is a monotonic sequence of closed sets with a void intersection. Supposing that $D_1, D_2, \ldots$ is a sequence of domains such that for each positive integer n, D_n contains F_n, it follows that for each positive integer n, D_n contains A_n which is a subset of $F_n = \sum_{i \geq n} A_1$. Thus by the preceding lemma, there is a point common to the closures of the elements of the sequence D_1, D_2, ...

It follows from a theorem of Ishikawa (8) that S is not countably paracompact.

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