UNIQUENESS RESULTS FOR A CLASS OF HOLOMORPHIC MAPPINGS ON A COMPLEX DISC

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Doctor of Philosophy

By

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Abstract

This dissertation gives a brief exposition of the history of Value Distribution Theory, often times referred to as Nevanlinna theory, and studies the case for Nevanlinna theory for holomorphic maps where the source is a disc developed by Ru-Sibony [16]. We start with a motivation into the subject and lay out some classical formulations with a focus on applications to the shared value problems. These problems are also referred to as uniqueness theorems.

It is known that if two complex polynomials P and Q share two values without counting multiplicities, then they are the same. Such problem is called the shared value problem. In this dissertation, we focus on the study of the shared value problem for holomorphic maps where the source is a disc. There are derivations for several new unicity results for a class of holomorphic mappings from the disc into compact Riemann surfaces and n-dimensional complex projective space.

Contents

Chapter 1

Introduction

Consider a polynomial of one complex variable $P(z)$. We can consider the zeroes of this polynomial and their multiplicities. These zeroes have a correspondence to the degree of the given polynomial. This correspondence is known as the Fundamental Theorem of Algebra:

> If $P(z)$ is a non-constant complex polynomial of degree n, then $P(z)$ will have n complex zeroes, provided the zeroes are counted with multiplicity.

If we consider $z = re^{i\theta}$, then $\max\{|P(re^{i\theta})|\}$ essentially grows like r^n as r approaches infinity. The Fundamental Theorem of Algebra can be reformulated with the growth rate in mind as this:

> Let $a \in \mathbb{C}$. The number of z's, counting multiplicity, satisfying the

equation $P(z) - a = 0$ is the order of the growth rate of the polynomial.

Nevanlinna Theory generalizes this reformulation of the Fundamental Theorem of Algebra to holomorphic and meromorphic functions on the complex plane $\mathbb{C}$.

The question is how do we study this growth rate. For a given entire function, J. Hadamard in [11] gave two notions for measuring the growth rate: the maximum modulus on a disc of radius r and the maximum number of times a image point is attained by the function in the disc. It turns out that that these two rates of growth are essentially the same, with the former roughly being the exponential of the latter. Instead of using the maximum modulus to measure the growth, in 1929, R. Nevanlinna [13] found a more suitable substitute. He introduced the characteristic function $T_f(r)$ to measure the growth of a meromorphic function f. Starting with the classical formula from Complex Analysis, Jensen formula, R. Nevanlinna was able to derive a more subtle growth estimate for meromorphic functions in what he called the Second Main Theorem. This theorem gives a quantitative version of the classical versions of Picard's theorem for meromorphic functions.

R. Nevenlinna's Theory can be surmised in the following way. There are three functions that are studied with respect to a meromorphic function f on the complex disc $\triangle(R)$.

Definition 1.0.1. Let f be a meromorphic function on $\triangle(R)$, where $0 \leq R \leq \infty$ and $r < R$. Denote the number of poles of f on the closed disc $\overline{\triangle(R)}$ by $n_f(r, \infty)$, counting multiplicity. We then define the *counting function* $N_f(r, \infty)$ to be

$$N_f(r, \infty) = n_f(\infty, 0)\log r + \int_0^r [n_f(t, \infty) - n_f(0, \infty)]\frac{dt}{t},$$

For each complex number a, we then define the counting function $N_f(r, a)$ to be

$$N_f(r, a) = N_{\frac{1}{f-a}}(r, \infty). \tag{1.1}$$

The counting function counts, as a logarithmic average, the number of times f takes the value of a in a disc of radius r. In this paper, $\overline{N}_f(a, r)$ is the counting function not counting multiplicity. This is often referred to as the truncated counting function.

Definition 1.0.2. The *Proximity function* of f is defined by

$$m_f(\infty, r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta, \tag{1.2}$$

where $\log^+(x) = \max\{0, \log x\}$. For any complex number a, let

$$m_f(a, r) = m_{\frac{1}{f-a}}(\infty, r)$$

The proximity function $m_f(a, r)$ measures how close f is, on average, to a on the circle of radius r.

Definition 1.0.3. The *Characteristic Function* of f is given by

$$T_f(r) = N_f(\infty, r) + m_f(\infty, r). \tag{1.3}$$

$T_f(r)$ measures the growth of f. For example, $T_f(r) = \mathcal{O}(1)$ if and only if f is a constant. Also, $T_f(r) = \mathcal{O}(\log r)$ if and only if f is a rational function. With the above definitions, Nevanlinna established the First Main Theorem.

Theorem 1.0.4 (First Main Theorem)**.** *If f is a non-constant meromorphic function on $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$, then*

$$N_f(a, r) + m_f(a, r) = T_f(r) + O(1) \tag{1.4}$$

Here $T_f(r)$ takes the place of degree n in the polynomial case. Note that $m_f(a, r) \geq 0$. Thus, the First Main Theorem says the function f cannot take on the value a too often in the sense that the frequency with which f takes on the value a cannot be so high that the Counting function $N_f(a, r)$ grows faster than $T_f(r)$ i.e $N_f(a, r) < T_f(r)$ for all $a \in \mathbb{C} \cup \{\infty\}$. This compares to the polynomial case where a polynomial of degree n takes on every value a at most n times. Looking more closely at the First Main Theorem, it says that $T_f(r)$ is actually independent of of any point a.

The next question is whether we can bound the characteristic function above with respect to the counting function. It turns out that this can be done. This is established by the much deeper Theorem.

Theorem 1.0.5 (Second Main Theorem)**.**

$$(q-2)T_f(r) + N_{f,ram}(r) \leq \sum_{j=1}^{q} N_f(a_j, r) + O(\log^+(T_f(r)) + \delta \log r \quad ||_E \tag{1.5}$$

holds for all r outside a set E with finite Lebesgue measure.

The term $N_{f,ram}(r)$ is positive and measures how often the function f is ramified. We can define the deficiency of $a \in \mathbb{C} \cup \{\infty\}$ with respect to f to be

$$\delta_f(a) = \liminf_{r \to \infty} \frac{m_f(a, r)}{T_f(r)} = 1 - \limsup_{r \to \infty} \frac{N_f(a, r)}{T_f(r)}.$$

It is clear from the definition that $0 \leq \delta_f(a) \leq 1$ and if f omits the value a then, $\delta_f(a) = 1$. From the Second Main Theorem, we have that the set of deficient values of a transcendental meromorphic function f is at most countable and

$$\sum \delta_f(a) \leq 2,$$

where the sum is taken over all the deficient values. Thus, the classical theorem of Picard is recovered by Nevanlinna Theory.

Theorem 1.0.6 (Little Picard Theorem)**.** *Let f be a meromorphic function on $\mathbb{C}$. If f omits three distinct points in $\mathbb{C} \cup \{\infty\}$, then f must be constant.*

R. Nevanlinna's work was further extended by H. Cartan for holomorphic mappings into n-th dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$. S.S. Chern followed the trend and was able to extend Nevanlinna's theory for holomorphic mappings from $\mathbb{C}$ into compact Riemann surfaces. Nevanlinna Theory has been a very active research area for the past century.

Chapter 2 briefly covers the primary motivation for the strategies in the proof of one of the main results presented in this paper. A. Sauer in [19], provides an alternate view of the Riemann-Hurwitz formula that can be easily utilized in addressing the shared value problem for mappings between compact Riemann surfaces. The formal statement and its applications are provided in detail in order to showcase his strategy.

Among the many applications to Nevanlinna Theory, the focus of Chapter 3 are the uniqueness results and there historical development. In each setting, we show how the Second Main Theorem is can be used to achieve the desired results. The shared value problems in this chapter motivate the primary results in this dissertation.

Chapter 4 discusses the theory of holomorphic curves from a disc of radius R into a compact Riemann surface. This theory has been recently established by Min Ru and Nessim Sibony in [16]. The comprehensive theory is presented. The chapter concludes with a brand new uniqueness result, that not only is a new contribution to the results in the previous chapter, but recovers some of the prior results as well.

In Chapter 5, the theory for holomorphic curves from a disc of radius R into n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ are discussed. The most current theory, due to Ru and Sibony in [16], is presented in full. The conclusion of the chapter is another brand new uniqueness result. This is the second main result of the paper. This result is also recovers a prior result and therefore extends the list of theorems regarding the shared value problem.

Chapter 2

Historical Motivation

The Second Main Theorem is the primary result in Nevanlinna theory. In [2], Ahlfors made precise the view that the second main theorem can be seen as a trancedental version of the Riemann-Hurwitz for compact Riemann surfaces. An application to the Second Main Theorem, which is the focus of our paper, is investigating shared value problems. The spirit of our main results start with the Riemann-Hurwitz theorem, its reformulation, and its applications to shared value problems presented by A. Sauer [19].

2.1 Riemann-Hurwitz Theorem and The Algebraic Second Main Theorem

Let $f : X \to Y$ be a non-constant holomorphic map, where X and Y are compact Riemann surfaces. We call $\nu_f(p)$ the *multiplicity* of f at $p \in X$ if there are local coordinates z for X at $p \in X$ and $w \in Y$ at $f(p)$ such that $w = z^{\nu_f(p)}$.

Theorem 2.1.1 (Riemann-Hurwitz Theorem [12]). *Let X and Y be compact Riemann surfaces with genera g_X and g_Y, respectively, and $f : X \to Y$ be a non-constant holomorphic map. Then*

$$2g_X - 2 = deg(f)(2g_Y - 2) + r_f(X),$$

where $deg(f)$ is the number of pre-images of a point $y \in Y$, counted with multiplicity and $r_f(X) := \sum_{p\in X}(\nu(p) - 1)$.

There is a well-suited version of the above equation that can be easily utilized to deal with shared value problems. Let f be defined as in the above theorem and $\{y_1, ..., y_n\} \subseteq Y$ be distinct points. Set $M := f^{-1}(\{y_1, ..., y_n\})$. It follows from the definition of $\deg(f)$ that $n \deg(f) = |M| + r_f(M)$, where $|M|$ is the cardinality of M. By noticing that $r_f(X) = r_f(M) + r_f(M^c)$, we can rewrite the Riemann-Hurwitz as

$$(n + 2g_Y - 2) \deg(f) = |M| + 2g_X - 2 - r_f(M^c).$$

This yields the below inequality, which is formally equivalent to Nevanlinna's Second Main Theorem.

$$(n + 2g_Y - 2) \deg(f) \le |M| + 2g_X - 2$$

When Y is taken to be $\mathbb{P}^1(\mathbb{C})$, the inequality is known as the Algebraic Second Main Theorem. For the sake of formality, we provide a statement with proof.

Theorem 2.1.2 (Algebraic Second Main Theorem)**.** *Let X be a compact Riemann surface with genus g_X, $f : X \to \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map, and $a_1, ..., a_n \in \mathbb{P}^1(\mathbb{C})$ be distinct points. Define $M := f^{-1}(\{a_1, ..., a_n\})$. Then*

$$(n-2)\deg(f) \leq |M| + 2g_X - 2,$$

where $|M|$ is the cardinality of M.

Proof. From the definition of degree, we have $n \deg(f) = |M| + r(M)$, where $r(M) = \sum_{x \in X}(\nu(x) - 1)$. On the other hand, by the Riemann-Hurwitz, we have $r_f(M) \leq r_f(X) \leq 2\deg(f) + 2g_X - 2$. Combining this inequality with the definition of degree yields $(n-2)\deg(f) \leq |M| + 2g_X - 2$ $\square$

2.2 Motivating Uniqueness Strategy

The proof of the following result is the essence in which the primary result of this paper was founded upon. The key observation is utilizing a Lemma to fashion a meromorphic function from Y into $\mathbb{P}^1(\mathbb{C})$ such that its degree is related to the genus of Y. We provide the following Lemma and its proof to motivate our proof strategy in our main result.

Lemma 2.2.1 ([19])**.** *Let X and Y be two compact Riemann surfaces and $f, g : X \to Y$ be two distinct non-constant holomorphic maps that share n values. Then*

$$(n-4)(deg(f) + deg(g)) \leq 4(g_X - 1)$$

Proof. Let f and g share the following distinct points $y_1, ..., y_n$ in Y. Let $M := f^{-1}(\{y_1, ..., y_n\}) = g^{-1}(\{y_1, ..., y_n\})$. Choose $x_0 \in X$ such that $f(x_0) \neq g(x_0)$ and set $y_0 := f(x_0)$. It follows from Lemma 4.2.3 that there exists a non-constant meromorphic function $\varphi : Y \to \mathbb{P}^1(\mathbb{C})$ with a single pole at y_0 of multiplicity less than or equal to $g_Y + 1$. This implies $\deg(\varphi) \leq g_Y + 1$. Now consider the following two compositions, both of which are mappings $X \to \mathbb{P}^1(\mathbb{C})$, $F := \varphi \circ f$ and $G := \varphi \circ g$. Now the difference $F - G$ can be defined and is non-constant since $F(x_0) = \infty$ and $G(x_0) \in \mathbb{C}$. By the definition M, for all $x \in M$ we have that $F(x) - G(x) = 0$. By counting the zeros of $F - G$ we have that

$$\begin{aligned} |M| \leq \deg(F - G) \leq \deg(F) + \deg(G) &= \deg(\varphi)(\deg(f) + \deg(g)) \\ &\leq (g_Y + 1)(\deg(f) + \deg(g)). \end{aligned}$$

By the Algebraic Second Main Theorem, we have

$$(n + 2g_Y - 2)\deg(f) \leq |M| + 2g_X - 2.$$

This yields

$$(n + g_Y - 3)\deg(f) \leq (g_Y + 1)\deg(g) + 2g_X - 2.$$

By symmetry, we have

$$(n + g_Y - 3)\deg(g) \leq (g_Y + 1)\deg(f) + 2g_X - 2.$$

Finally, by adding the above two inequalities, we have that

$$(n - 4)(\deg(f) + \deg(g)) \leq 4(g_X - 1).$$

$\square$

A. Sauer([19]) used Lemma 2.1.1 and the Algebraic Second Main Theorem for the next set of results. The following uniqueness theorems really provide the spirit and motivation for the primary results in this paper. The first is a bound of shared values for rational functions on $\mathbb{C}$ (which can be viewed as a holomorphic map from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$). The second provides a bound of shared values for functions between compact Riemann surfaces.

Theorem 2.2.2 ([19])**.** *Let $f, g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be non-constant holomorphic maps. If f and g share four values, then $f = g$.*

Proof. Suppose f and g are distinct. Since the genus of $\mathbb{P}^1(\mathbb{C})$ is 0, by Lemma 2.1.1, we have

$$(n-4)(\deg(f) + \deg(g)) \leq -4. \tag{2.1}$$

Suppose f and g share four values i.e. $n = 4$, then the above inequality gives a contradiction $0 \leq -4$. □

Theorem 2.2.3 ([19])**.** *Let X and Y be two compact Riemann surfaces with $g_X > 0$ and $f, g : X \to Y$ be distinct non-constant holomorphic maps that share n values. Then*

$$n \leq 2 + 2\sqrt{g_X + g_X g_Y - g_Y}.$$

Proof. Since the result is trivial if $n < 4$, consider the case where $4 \leq n$. By the proof in Lemma 2.1.3, we have that $\deg(f) + \deg(g) \geq |M|/(g_Y + 1)$. On the other hand, it is clear that $n \leq |M|$. Using (2.1) gives $(n-4)n \leq 4(g_X - 1)(g_Y + 1)$. Modifying the inequality gives

$$n \leq \sqrt{4 + 4(g_X - 1)(g_Y + 1)} \leq 2 + 2\sqrt{g_X + g_X g_Y - g_Y}.$$

$\square$

We can consider f and g to be meromorphic functions on X (which can be viewed as a holomorphic map from X into $\mathbb{P}^1(\mathbb{C})$) if we take $Y = \mathbb{P}^1(\mathbb{C})$. Then we have the following result.

Corollary 2.2.4 ([19])**.** *Let X be a compact Riemann surface with $g_X > 0$ and $f, g : X \to \mathbb{P}^1(\mathbb{C})$ be distinct non-constant holomorphic maps that share n values. Then,*

$$n \leq 2 + 2\sqrt{g_X}.$$

Proof. The result is proven by Theorem 2.2.3 when $g_Y = 0$. $\square$

Corollary 2.2.4 is meaningful since for all compact Riemann surfaces the existence distinct meromorphic f and g that share three values can easily be constructed. For example, consider

$$F(z) := \frac{(z+i)^3(z-2)}{(z-i)^3(z+2)}$$

and

$$G(z) := \frac{(z+i)(z-2)}{(z-i)(z+3)}.$$

It is clear that F and G share the values $0, 1$, and ∞ on $\mathbb{P}^1(\mathbb{C})$. By the Riemann-Roch theorem, there exists a non-constant meromorphic function $\varphi : X \to \mathbb{P}^1(\mathbb{C})$. If we set $f := F \circ h$ and $g := G \circ h$, we have that f and g share the values $0, 1$, and ∞ and are distinct since $\deg(f) \neq \deg(g)$.

Chapter 3

Functions Sharing Values

In 1925, Rolf Nevanlinna [13] extended the classical theorems of Picard and Borel by developing the value distribution theory of meromorphic functions on the complex plane $\mathbb{C}$, which is now called Nevanlinna theory. As an application of his Second Main Theorem, Nevanlinna derived his celebrated five-point theorem: *If two non-constant meromorphic functions f and g defined on $\mathbb{C}$ share five distinct values in $\mathbb{C} \cup \{\infty\}$ (without counting multiplicities), then $f \equiv g$.* Later, Fujimoto [8] extended the result to holomorphic mappings from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and, recently, Chen-Yan [4] improved Fujimoto's result.

In 1960, S.S. Chern [5] extended Nevanlinna's Second Main Theorem to holomorphic maps from the complex plane $\mathbb{C}$ into compact Riemann surfaces. By using Chern's Second Main Theorem, E. M. Schmid [20] proved the following result: *Let M be a compact Riemann surface of genus 1 and $f, g : \mathbb{C} \to M$ be two non-constant* *holomorphic maps. Suppose that f, g share five distinct values in M (without counting multiplicities), then $f \equiv g$.* However, nothing can be said for compact Riemann surface of genus greater than 1, since, in this case, every holomorphic mapping $f : \mathbb{C} \to M$ must be constant.

Recently, Ru-Sibony [16] developed Nevanlinna's theory for a class of holomorphic maps where the source is a disc by introducing the notion of a *growth index* of a holomorphic map $f : \triangle(R) \to M$ denoted by $c_{f,\omega}$, where M is a complex manifold and ω is a positive (1,1) form of finite volume on M. The purpose of this chapter is to recall prior uniqueness results for holomorphic maps. These prior results then motivate the case for studying holomorphic maps from the disc $\triangle(R)$ into compact Riemann surfaces and $\mathbb{P}^n(\mathbb{C})$. The case for compact Riemann surfaces is especially interesting if the genus is greater than 1.

3.1 Uniqueness Results for Polynomials

The following theorem provides the initial motivation and determines our overall method for addressing shared value problems.

Theorem 3.1.1 ([1])**.** *Let P and Q be two non-constant complex (or any algebraically closed field of characteristic zero) polynomials. Assume that there are two distinct complex values $a_j \in \mathbb{C}$, for $j = 1, 2$, such that $P(z) = a_j$ if and only if $Q(z) = a_j$ without counting multiplicities (we say that P and Q share the values a_j). Then $P \equiv Q$.*

Proof. Without loss of generality, we can assume that P and Q share the two values 0 and 1. So $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Suppose $n = \deg(P) \geq \deg(Q) > 0$ and $P \not\equiv Q$. Since every zero of P is a zero of Q, they are also zeros of $P - Q$. Similarly, since every zero of $P - 1$ is a zero of $Q - 1$, they are also zeros of $P - 1 - (Q - 1) = P - Q$. Considering P', we have $P | P'(P - Q)$ and $P - 1 | P'(P - Q)$. Since P and $P - 1$ are relatively prime, $P(P - 1) | P'(P - Q)$. Thus, we have $\deg(P(P - 1)) = 2n$ and $\deg(P'(P - Q)) = 2n - 1$. For $P(P - 1) | P'(P - Q)$ to hold, it must be the case that $P - Q = 0$. $\square$

3.2 Nevanlinna Five Values Theorem

In 1929, R. Nevanlinna [13] started the theory of value distribution of meromorphic functions with his development of Nevanlinna Theory. As an application of his early work, he proved the following celebrated theorem.

Definition 3.2.1. Consider the set

$$S_f(a) = \{z \in \mathbb{C} \mid f(z) = a\}.$$

Let f and g be two meromorphic functions and $a \in \mathbb{P}^1(\mathbb{C})$. We say that f and g **share the value** a, if $S_f(a) = S_g(a)$.

Theorem 3.2.2 (Nevanlinna's Five Values Theorem)**.** *Let f and g be two non-constant meromorphic functions on $\mathbb{C}$. If $S_f(a_j) = S_g(a_j)$ for five distinct values $a_1, ..., a_5$, then $f \equiv g$.*

Proof. Let f and g be two non-constant meromorphic functions such that $f \neq g$ and $a_1, ..., a_q$ be distinct points in $\mathbb{P}^1(\mathbb{C})$. Then by the Second Main Theorem (1.5), properties of the Characteristic function, and The First Main Theorem (1.4), we have that for f

$$(q-2)T_f(r) \leq \sum_{j=1}^{q} \overline{N}_f(a_j, r) + O(\log^+(T_f(r)) + \delta \log r \quad ||_E.$$

Similarly for g, we have that

$$(q-2)T_g(r) \leq \sum_{j=1}^{q} \overline{N}_g(a_j, r) + O(\log^+(T_f(r)) + \delta \log r \quad ||_E.$$

Adding both inequalities above yields,

$$(q-2)(T_f(r) + T_g(r)) \leq \sum_{j=1}^{q} \overline{N}_f(a_j, r) + \overline{N}_g(a_j, r)$$
$$+O(\log^+(T_f(r)) + O(\log^+(T_g(r)) + \delta \log r \quad ||_E.$$

Then notice, by a counting argument, that $\sum_{j=1}^{q} \overline{N}_f(a_j, r) \leq N_{f-g}(0, r)$. This inequality holds similarly for g for the same reasoning. With the above considerations and the First Main Theorem we have that

$$\begin{aligned}
(q-2)(T_f(r) + T_g(r)) &\leq \sum_{j=1}^{q} \overline{N}_f(a_j, r) + \overline{N}_g(a_j, r) \\
&+ O(\log^+(T_f(r)) + O(\log^+(T_g(r)) + \delta \log r \quad ||_E. \\
&\leq 2N_{f-g}(0, r) + O(\log^+(T_f(r)) + O(\log^+(T_g(r)) + \delta \log r \quad ||_E. \\
&\leq 2T_{f-g}(r) + O(\log^+(T_f(r)) + O(\log^+(T_g(r)) + \delta \log r \quad ||_E. \\
&\leq 2(T_f(r) + T_g(r)) \\
&+ O(\log^+(T_f(r)) + O(\log^+(T_g(r)) + \delta \log r \quad ||_E.
\end{aligned}$$

Thus, as r gets large, if $q = 5$ we would arrive at a contradiction. □

Note that $q = 5$ is a sharp bound. As an example, consider the meromorphic functions $f(z) = \frac{e^z+1}{(e^z-1)^2}$ and $g(z) = \frac{(e^z+1)^2}{8(e^z-1)}$. They share the values $\infty, 0, 1, -\frac{1}{8}$.

3.3 Five Values Theorem for Torus

E. Schmid in [20] extended uniqueness results, like the ones above, using S.S Chern [5] Nevanlinna theory for holomorphic maps from $\mathbb{C}$ into compact Riemann surfaces.

Theorem 3.3.1 (See Theorem 6.1 in [20])**.** *Let $f, g : \mathbb{C} \to T$ be two non-constant holomorphic mappings, where T is a 1-dimensional complex Torus. Then if $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct points $a_j \in T$, $j = 1, ..., 5$, it follows that $f \equiv g$.*

Proof. See Theorem 4.2.4. □

3.4 Uniqueness Theorem's for Holomorphic Maps into $\mathbb{P}^n(\mathbb{C})$

The theory of holomorphic maps on the disc into $\mathbb{P}^n(\mathbb{C})$ is presented in full in chapter 5. This theory generalizes Cartan's original results in [3]. From Theorem 5.1.11, if we take $R = \infty$, we recover Cartan's Second Main Theorem.

Theorem 3.4.1 ([3])**.** *let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve(i.e. its image is not* *contained in any proper subspaces). Then, for any $\delta > 0$, the inequality*

$$\begin{aligned}&\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0)\\ &\leq (n+1)T_f(r) + O(\log^+ T_f(r)) + \delta \log\ r + O(1)\end{aligned}$$

holds for all r outside a set E with finite Lebesgue measure.

Using the Theorem 5.1.1 (First Main Theorem), the inequality in Theorem 3.4.1 can be written as

$$\begin{aligned}&qT_f(r) - \sum_{j=1}^{q} N_f(r, H_j) + N_W(r, 0)\\ &\leq (n+1)T_f(r) + O(\log^+ T_f(r)) + \delta \log\ r + O(1) \quad \|_E.\end{aligned}$$

We use the following reformulation in the uniqueness proofs in this setting.

Theorem 3.4.2. *let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Then, for any $\delta > 0$, the inequality*

$$\begin{aligned}(q-(n+1))T_f(r) &\leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j)\\ &+ \quad +O(\log^+ T_f(r)) + \delta \log\ r + O(1) \quad \|_E\end{aligned}$$

holds for all r outside a set E with finite Lebesgue measure.

Like in the previous sections, we can now establish a unicity result. The following Theorem is due to L.Smiley in 1983.

Theorem 3.4.3 ([22])**.** *Let $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerated holomorphic curves and $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Assume the following:*

i.) $f^{-1}(H_j) = g^{-1}(H_j)$ *for* $j = 1, ..., q$*;*

ii.) $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$*, for each* $i \neq j$*;*

iii.) $A = \cup_{j=1}^{q} f^{-1}(H_j)$*.*

If $f(z) = g(z)$ *for all* $z \in A$ *and* $q > 3n + 1$*, then* $f \equiv g$*.*

Proof. Assume $f \not\equiv g$. We then apply Cartan's Second Main Theorem to f and g to get

$$\begin{aligned}(q - (n+1))T_f(r) &\leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) \\ &+ O(\log^+ T_f(r)) + \delta \log\ r + O(1) \quad \|_E\end{aligned}$$

and

$$\begin{aligned}(q - (n+1))T_g(r) &\leq \sum_{j=1}^{q} N_g^{(n)}(r, H_j) \\ &+ O(\log^+ T_g(r)) + \delta \log\ r + O(1) \quad \|_E.\end{aligned}$$

Adding these two inequalities yields,

$$\begin{aligned}(q - (n+1))(T_f(r) + T_g(r)) &\leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) + N_g^{(n)}(r, H_j) \\ &+ O(\log^+ T_f(r)) + O(\log^+ T_g(r)) \\ &+ \delta \log\ r + O(1) \quad \|_E.\end{aligned}$$

Since $f \not\equiv g$, by fixing the reduced representation of f and g such that $T_{f_i}(r) \leq T_f(r)$ and $T_{g_i}(r) \leq T_g(r)$, there exist $1 \leq i_0 \leq j_0 \leq q$ such that we can define an auxiliary function

$$\chi := f_{i_0} g_{j_0} - f_{j_0} g_{i_0} \not\equiv 0.$$

Then, we can see that

$$\sum_{j=1}^{q} N_f^{(1)}(r, H_j) \leq N_\chi(r, 0).$$

By Cartan's First Main Theorem and the fact that $T_{f-g}(r) \leq T_f(r) + T_g(r)$, we have

$$\begin{aligned}
N_\chi(r,0) &= N_{f_{i_0}g_{j_0} - f_{j_0}g_{i_0}}(r,0) \\
&\leq T_{f_{i_0}g_{j_0} - f_{j_0}g_{i_0}}(r) \\
&\leq T_{f_{i_0}g_{j_0}}(r) + T_{f_{j_0}g_{i_0}}(r) \\
&\leq T_f(r) + T_g(r).
\end{aligned}$$

This gives

$$\begin{aligned}
(q-(n+1))(T_f(r)+T_g(r)) &\leq 2n(T_f(r)+T_g(r)) \\
&+ O(\log^+ T_f(r)) + O(\log^+ T_g(r)) + \delta \log\ r + O(1) \quad \|_E.
\end{aligned}$$

This contradicts the assumption that $q > 3n+1$. $\square$

Observe that if $n = 1$, then Nevanlinna's Five Point Theorem is recovered by the result.

Theorem 3.4.3 can be improved with the consideration of another auxiliary function. In the following theorem take $q = 2n+3$.

Theorem 3.4.4 ([4]). *Let $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerated holomorphic curves and $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Assume the following:*

i.) $f^{-1}(H_j) = g^{-1}(H_j)$ *for* $1 \leq j \leq q$*;*

ii.) $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$*, for each* $i \neq j$*;*

Let $A = \cup_{j=1}^{q} f^{-1}(H_j)$*. If* $f(z) = g(z)$ *for all* $z \in A$ *and* $q > 2n+2$*, then* $f \equiv g$*.*

Proof. Assume $f \not\equiv g$. We construct an auxiliary function as in [4] (see also [9]). By lemma 3.1 in [18], there exists a hyperplane

$$H_c = \{c_0 x_0 + \cdots + c_n x_n = 0\}$$

such that $f^{-1}(H_c \cap H_j) = \emptyset$ and $g^{-1}(H_c \cap H_j) = \emptyset$ for $j = 1, ..., q$. Fix such an H_c. Let $H_j = \{a_{j0} x_0 + \cdots + a_{jn} x_n = 0\}$. Then define $(f, H_j) := \frac{a_{j0} f_0 + \cdots + a_{jn} f_n}{c_0 f_0 + \cdots + c_n f_n}$, where $(f_1, ..., f_n)$ is a local reduced representation of f. We define (g, H_j) similarly. We arrange the hyperplanes $H_1, .., H_q$ into groups:

Group 1:

$$\frac{(f, H_1)}{(g, H_1)} \equiv \cdots \equiv \frac{(f, H_{k_1})}{(g, H_{k_1})} \not\equiv \frac{(f, H_{k_1+1})}{(g, H_{k_1+1})}$$

Group 2:

$$\frac{(f, H_{k_1+1})}{(g, H_{k_1+1})} \equiv \cdots \equiv \frac{(f, H_{k_2})}{(g, H_{k_2})} \not\equiv \frac{(f, H_{k_2+1})}{(g, H_{k_2+1})}$$

$\cdots$

Group s:

$$\frac{(f, H_{k_{s-1}+1})}{(g, H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f, H_{k_s})}{(g, H_{k_s})}$$

where $k_s = q$. The assumption of "in general position" implies that the number of each group does not exceed n. For each $1 \leq i \leq q$, we set $\sigma(i) = i + n$ if $i + n \leq q$ and $\sigma(i) = i + n - q$ if $i + n > q$. Then it can be seen that σ is bijective and $|\sigma(i) - i| \geq n$

since $q \geq 2n$. Define P_i as follows:

$$P_i = (f, H_i)(g, H_{\sigma(i)}) - (g, H_i)(f, H_{\sigma(i)}).$$

Since $f \not\equiv g$, we have that $P_i \not\equiv 0$. We define the auxiliary function

$$P := \prod_{i=1}^{q} P_i. \tag{3.1}$$

Fix an index i with $1 \leq i \leq q$. For $z \notin \cup_{s \neq t} f^{-1}(H_s \cap H_t)$, notice first, if z is a zero of (f, H_i), then z is a zero of P_i with multiplicity at least $\min\{(f, H_i), (g(, H_i)\}$. Similarly, if z is a zero of $(f, H_{\sigma(i)})$, then z is a zero point of P_i with multiplicity at least $\min\{(f, H_{\sigma(i)}), (g(, H_{\sigma(i)})\}$. Secondly, if z is a zero of (f, H_v) with $v \in \{i, \sigma(i)\}$, then z is a zero of P_i since in this case $f(z) = g(z)$. Denote by $\nu_P(z)$, the zero order of P at the point z and $\nu^n_{F(H_i)}(z) = \min\{n, \nu_{F(H_i)}(z)\}$. Then,

$$\nu_{P_i}(z) \geq \min\{\nu_{(f,H_i)}(z), \nu_{(g,H_i)}(z)\} + \min\{\nu_{(f,H_{\sigma(i)})}(z), \nu_{(g,H_{\sigma(i)})}(z)\}$$

for $z \notin \cup_{s \neq t} f^{-1}(H_s \cap H_t)$. Since $\min\{a, b\} \geq \min\{a, n\} + \min\{b, n\} - n$ for all positive integers a and b, it follows from the above inequality

$$\nu_{P_i}(z) \geq \sum_{v=i,\sigma(i)} \min\{\nu_{(f,H_v)}(z), n\} + \min\{\nu_{(g,H_v)}(z), n\} - n \min\{\nu_{(f,H_v}(z), 1\}$$

for $z \notin \cup_{s\neq t} f^{-1}(H_s \cap H_t)$. Using the fact that $f^{-1}(H_s \cap H_t) = \emptyset$ for $s \neq t$, we have

$$\begin{aligned}
\nu_P(z) &\geq 2\sum_{v=1}^{q}\Big(\min\{\nu_{(f,H_v)}(z), n\} + \min\{\nu_{(g,H_v)}(z), n\} - n\min\{\nu_{(f,H_v}(z), 1\}\Big) \\
&+ (q-2)\sum_{v=1}^{q}\min\{\nu_{(f,H_v}(z), 1\} \\
&= 2\sum_{v=1}^{q}(\nu^n_{(f,H_v)}(z) + \nu^n_{(g,H_v)}(z)) + \Big(\frac{q-2-2n}{2}\Big)\sum_{j=1}^{q}(\nu^1_{(f,H_j)}(z) + \nu^1_{(g,H_j)}(z)) \\
&\geq 2\sum_{j=1}^{q}(\nu^n_{(f,H_j)}(z) + \nu^n_{(g,H_j)}(z)) + \Big(\frac{q-2-2n}{2}\Big)\sum_{j=1}^{q}(\nu^n_{(f,H_j)}(z) + \nu^n_{(g,H_j)}(z)) \\
&\geq \Big(\frac{q-2+2n}{2n}\Big)\sum_{j=1}^{q}(\nu^n_{(f,H_j)}(z) + \nu^n_{(g,H_j)}(z)).
\end{aligned}$$

By integrating both sides of the above inequality, we have

$$N_P(r) \geq \Big(\frac{q-2+2n}{2n}\Big)\sum_{j=1}^{q}(N^{(n)}_f(r, H_j) + N^{(n)}_g(r, H_j)).$$

By the definition of the characteristic function and the First Main Theorem, we have

$$N_P(r) \leq q(T_f(r) + T_g(r)) + O(1).$$

This implies

$$q(T_f(r) + T_g(r)) \geq \Big(\frac{q-2+2n}{2n}\Big)\sum_{j=1}^{q}(N^{(n)}_f(r, H_j) + N^{(n)}_g(r, H_j)). \tag{3.2}$$

Applying the Second main Theorem to f and g, we have

$$\begin{aligned}
(q-(n+1))T_f(r) &\leq \sum_{j=1}^{q} N^{(n)}_f(r, H_j) \\
&+ O(\log^+ T_f(r)) + \delta\log\ r + O(1) \quad \|_E
\end{aligned}$$

and

$$\begin{aligned}
(q-(n+1))T_g(r) &\leq \sum_{j=1}^{q} N^{(n)}_g(r, H_j) \\
&+ O(\log^+ T_g(r)) + \delta\log\ r + O(1) \quad \|_E.
\end{aligned}$$

Adding these two inequalities yields,

$$\begin{aligned} (q-(n+1))(T_f(r)+T_g(r)) &\le \sum_{j=1}^{q} N_f^{(n)}(r,H_j)+N_g^{(n)}(r,H_j) \\ &+ O(\log^+ T_f(r))+O(\log^+ T_g(r)) \\ &+ \delta \log\ r + O(1) \quad \|_E. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3),

$$\Big(\frac{q-2+2n}{2n}\Big)(q-(n+1))(T_f(r)+T_g(r)) \le q(T_f(r)+T_g(r))+O(1).$$

Thus,

$$\Big(\frac{q-2+2n}{2n}\Big)(q-(n+1)) \le q.$$

We arrive at a contradiction if $q > 2n+2$.

□

In this dissertation, we will use the auxiliary function given above to study uniqueness results in the disc case.

Chapter 4

Holomorphic Mappings into Compact Riemann Surfaces

4.1 Value distribution of Holomorphic Mappings into Compact Riemann Surfaces

In 2017, Min Ru and Nessim Sibony [16] developed Nevanlinna's theory for a class of holomorphic maps where the source is a disc of radius R. There work was motivated first by Nevenlinna's work in [13], establishing the Second Main Theorem for meromorphic functions on the complex plane $\mathbb{C}$. Shortly after, H. Cartan [3] developed the value distribution theory for holomorphic mappings from $\mathbb{C}$ into the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ and studied the intersection with hyperplanes in general position. In 1960, S.S. Chern [5] was able to extend Nevanlinna's result to holomorphic mappings from the complex plane $\mathbb{C}$ into compact Riemann surfaces. At the time Cartan was developing his theory, it was observed (first by Nevanlinna) that similar results hold for meromorphic functions from the complex unit disc $\triangle(1)$ to $\mathbb{P}^n(\mathbb{C})$, under the condition that

$$\lim_{r\to 1}\frac{T_f(r)}{\log\frac{1}{1-r}}=\infty.$$

This observation motivates the notion of the *growth index*, defined below.

We first introduce some notations. For a complex variable z, let

$$\partial u=\frac{\partial u}{\partial z}dz,\quad \bar{\partial}u=\frac{\partial u}{\partial \bar{z}}d\bar{z}.$$

Let $d=\partial+\bar{\partial}, d^c=\frac{\sqrt{-1}}{4\pi}(\bar{\partial}-\partial)$. Then we have that $dd^c=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

Let M be a compact Riemann surface. Let $\omega=a(z)\frac{\sqrt{-1}}{2\pi}dz\wedge d\bar{z}$ be a positive smooth $(1,1)$ form on M. Let $\mathrm{Ric}(\omega):=dd^c\log a$. Then we have

$$\mathrm{Ric}(\omega)=-K\omega,$$

where K is the Gauss curvature of the metric form ω.

Let $\triangle(R)$ denote a disc of radius R with the convention that $\triangle(\infty)=\mathbb{C}$. Let M be a Hermitian manifold and ω be a positive (1,1) form of finite mass on M.

Definition 4.1.1. The **Characteristic (or height) function of f with respect to** ω of a non-constant holomorphic map $f:\triangle(R)\to M$, for $0<r<R$, is given by

$$T_{f,\omega}(r)=\int_0^r\frac{dt}{t}\int_{|z|<t}f^*\omega.$$

For each $c < \infty$, let

$$\mathcal{E}_c = \Big\{ f | \int_0^R exp(cT_{f,\omega}(r))dr = \infty \Big\}, \tag{4.1}$$

$$\mathcal{E} = \cup_{c<\infty} \mathcal{E}_c \text{ and } \mathcal{E}_0 = \cap_{c>0} \mathcal{E}_c. \tag{4.2}$$

Notice that the set $\mathcal{E}_c$ contains maps from the unit disc to M which satisfy, for r close to 1,

$$\frac{T_{f,\omega}(r)}{\log \frac{1}{1-r}} \leq \frac{1}{c}.$$

Definition 4.1.2. Let M be a complex manifold and ω be a positive (1,1) form of finite volume on M. Let $0 < R \leq \infty$ and $f : \triangle(R) \to M$ be a holomorphic map. We define the **growth index** of f with respect to ω as

$$c_{f,\omega} := \inf\{c > 0 | \int_0^R \exp(cT_{f,\omega}(r)dr = \infty\}.$$

We always assume that the set $\{c > 0 | \int_0^R \exp(cT_{f,\omega}(r)dr = \infty\}$ is non-empty. If f is of bounded characteristic (hence $R < \infty$), then $c_{f,\omega} = \infty$. In the case where $R = \infty$, noticing that $\int_o^R \exp(\epsilon T_{f,\omega}(r)dr = \infty$ for any arbitrary small ϵ if f is not constant, $c_{f,\omega} = 0$ and f is in $\mathcal{E}_0$. Thus, the following results also include the classical results for mappings on the whole complex plane $f : \mathbb{C} \to M$.

We introduce some examples of holomorphic maps on the unit disc which are in the class we will study.

Example 1. Let N be a compact Riemann surface of genus ≥ 2. Then N has a smooth metric form ω_P whose Gauss curvature is -1. We take $\varphi : \triangle(1) \to N$ as the uniformizing map. Then

$$T_{\varphi,\omega_P}(r) = \log \frac{1}{1-r} + O(1).$$

In this case, $c_{\varphi,\omega_P} = 1$. Thus $\varphi \in \mathcal{E}_1$.

Example 2. Let M be a compact Kobayashi manifold and let ω be a metric form. Then, by Brody's theorem (see [15]), there is a constat $C > 0$ such that for any holomorphic map $\varphi : \triangle(1) \to M$, we have that $|f'(0)|_\omega \le C$. Hence $|f'(z)|_\omega \le \frac{C}{1-|z|}$ on $\triangle(1)$. Which implies that we have $T_{f,\omega}(r) \le C \log \frac{1}{1-r}$. So the space $\mathcal{E}_0$ is empty. However, $c_{f,\omega}$ is not necessarily finite since it requires an estimate on the lower bound on $T_{f,\omega}(r)$.

We are ready to begin establishing the preliminaries to the theory of holomorphic mappings from the disc. We proceed in the usual way.

Lemma 4.1.3 (Calculus Lemma)**.** *Let* $0 < R \le \infty$ *and let* $\gamma(r)$ *be a non-negative function defined on* $(0, R)$ *with* $\int_0^R \gamma(r)dr = \infty$*. Let* h *be a non-decreasing function of class* C^1 *defined on* $(0, R)$*. Assume that* $\lim_{r\to R} h(r) = \infty$ *and* $h(r_0) \ge c > 0$*. Then for every* $0 < \delta < 1$*, the inequality*

$$h'(r) \le h^{1+\delta}(r)\gamma(r)$$

holds for all $r \in (0, R)$ *outside a set* E *with* $\int_E \gamma(r)dr < \infty$*.*

Proof. Let $E \subset (r_o, R)$ be the set of r such that $h'(r) \ge h^{1+\delta}(r)\gamma(r)$. Then

$$\int_E \gamma(r)dr \le \int_{r_0}^R \frac{h'(r)}{h^{1+\delta}(r)}dr = \int_c^\infty \frac{dt}{t^{1+\delta}} < \infty.$$

This proves the lemma. $\square$

Lemma 4.1.4. *Let* $0 < R \le \infty$ *and let* $\gamma(r)$ *be a function defined on* $(0, R)$ *with* $\int_0^R \gamma(r)dr = \infty$*. Let* h *be a function of class* C^2 *defined on* $(0, R)$ *such that* rh' *is a* *non-decreasing function. Assume that* $\lim_{r\to R} h(r) = \infty$. *Then*

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{dh}{dr}\Big) \le r^{\delta} \cdot \gamma^{2+\delta}(r) \cdot h^{(1+\delta)^2}(r)$$

holds outside a set $E \subset (0, R)$ *with* $\int_E \gamma(r)dr < \infty$.

Proof. We apply the Calculus lemma twice. First to the function $rh'(r)$ and then to the function $h(r)$. $\square$

The use of the Calculus lemma will be the following. Let Γ be a non-negative on $\triangle(R)$ with $0 < R \le \infty$. Define

$$T_\Gamma(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} \Gamma \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$$

and

$$\lambda(r) := \int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi}.$$

By using polar coordinates,

$$\frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = 2rdr \wedge \frac{d\theta}{2\pi}.$$

Hence

$$r\frac{dT_\Gamma}{dr} = 2\int_0^{2\pi} \Big(\int_0^r \Gamma(te^{i\theta})tdt\Big)\frac{d\theta}{2\pi},$$

$$\frac{d}{dr}\Big(r\frac{dT_\Gamma}{dr}\Big) = 2r\int_0^{2\pi} \Gamma(re^{i\theta})\frac{d\theta}{2\pi} = 2r\lambda(r).$$

Thus, from Lemma 4.1.3, we have

$$\lambda(r) \le \frac{1}{2} r^{\delta} \cdot \gamma^{2+\delta}(r) \cdot T_\Gamma^{(1+\delta)^2}(r) \tag{4.3}$$

holding for all $r \in (0, R)$ outside a set E with $\int_E \gamma(r)dr < \infty$. In this paper, we will always use the inequality (4.3) with a proper chosen $\gamma(r)$.

In order to proceed forward, we recall some definition's regarding Divisors.

Definition 4.1.5. let M be a compact complex manifold. A *Weil Divisor* on M, denoted by D, is a formal sum

$$D = \sum n_j Z_j,$$

where $n_j \in \mathbb{Z}$ and $Z_j \subset M$ are smooth irreducible subvarieties of co-dimension 1. If $n_j \geq 0$ for all j, then we call D an *effective divisor*.

Definition 4.1.6. Let X be a irreducible analytic subset of co-dimension 1 in M. Let $x \in X$ and f be a meromorphic function defined around x. The *order* $\mathrm{ord}_{X,x}(f)$ of f is the largest integer a such that in the local ring $\mathcal{O}_{X,x}$ we have $f = h^a g$. It turns out that $\mathrm{ord}_{X,x}(f)$ is independent of $x \in X$, so we just write $\mathrm{ord}_X(f)$.

For our case, we will need to use a version of the Green-Jensen formula using the notion of a current. Let g be a sub-harmonic function and let Z be the set of singularities in $\triangle(t)$. Denote by $S(Z, \epsilon)(t)$, the union of small circles around the singularities in $\triangle(t)$.

Definition 4.1.7. The *current* $dd^c[g]$ is the functional such that

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g] = \int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c g + \mathrm{Sing}_g(r),$$

where

$$\mathrm{Sing}_g(r) = \int_0^r \frac{dt}{t} \lim_{\epsilon \to 0} \int_{S(Z,\epsilon)(t)} d^c g.$$

Theorem 4.1.8 (Green-Jensen formula [15])**.** *Let g be a function on $\overline{\triangle(r)}$ such that $dd^c[g]$ is of order zero and $g(0)$ is finite. Then*

$$\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g] = \frac{1}{2} \int_0^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} - g(0)).$$

If we take $g = \log |f|^2$ for some holomorphic function f, we have the following Lemma.

Lemma 4.1.9. *Let f be a holomorphic function on $\triangle(r)$, then for any t with $0 < t < r$,*

$$\lim_{\epsilon \to 0} \int_{S(Z,\epsilon)(t)} d^c \log |f|^2 = n_f(t, 0) - n_f(t, \infty).$$

Proof. Let p be a singular point of f. It suffices to prove that

$$\lim_{\epsilon \to 0} \int_{S(p,\epsilon)} d^c \log |f|^2 = \mathrm{ord}_p(f),$$

where $S(p, \epsilon)$ is the circle centered at p with radius ϵ. Without loss of generality, we may assume that $p = 0$. Let $k = ord_0(f)$. We can write $f\bar{f} = r^{2k} h(r, \theta)$, where $h(r, \theta)$ is positive and smooth. So

$$\lim_{\epsilon \to 0} \int_{S(p,\epsilon)} d^c \log |f|^2 = \lim_{\epsilon \to 0} \int_{S(p,\epsilon)} d^c \log r^{2k}.$$

Observe the following,

$$d^c \log r^{2k} = \frac{1}{4\pi} r \frac{\partial(\log r^{2k})}{\partial r} d\theta = \frac{1}{2\pi} k d\theta.$$

Thus,

$$\lim_{\epsilon \to 0} \int_{S(p,\epsilon)} d^c \log |f|^2 = \lim_{\epsilon \to 0} \int_0^{2\pi} k \frac{d\theta}{2\pi} = k.$$

□

In particular, we now have the following

$$
\begin{aligned}
\mathrm{Sing}_{\log|f|^2}(r) &= \int_0^{2\pi} \frac{dt}{t} \lim_{\epsilon\to 0} \int_{S(Z,\epsilon)(t)} d^c \log|f|^2 \\
&= N_f(r,0) - N_f(r,\infty). \qquad (4.4)
\end{aligned}
$$

We now can rewrite Lemma 4.1.9 as the next theorem.

Theorem 4.1.10 (Poincare-Lelong Formula)**.** *Let f be holomorphic on $\triangle(r)$, then*

$$\int_0^{2\pi} \frac{dt}{t} \int_{|\zeta|<t} dd^c[\log|f|^2] = N_f(r,0) - N_f(r,\infty),$$

or we write in the sense of a current,

$$dd^c[\log|f|^2] = D_f,$$

where $D_f = \sum_p (\mathrm{ord}_p f) \cdot p$ is the divisor associated to f.

Proof. Applying the definition of a current to $dd^c[\log|f|^2]$, we have

$$\int_0^{r} \frac{dt}{t} \int_{|\zeta|<t} dd^c[\log|f|^2] = \int_0^{r} \frac{dt}{t} \int_{|\zeta|<t} dd^c|f|^2 + \mathrm{Sing}_{\log|f|^2}(r).$$

However, from (4.4),

$$\mathrm{Sing}_{\log|f|^2}(r) = N_f(r,0) - N_f(r,\infty),$$

and since f is holomorphic, $dd^c[\log|f|^2 = 0$. Thus,

$$\int_0^{r} \frac{dt}{t} \int_{|\zeta|<t} dd^c[\log|f|^2] = N_f(r,0) - N_f(r,\infty).$$

$\square$

Let M be a compact Riemann surface and let ω be a positive smooth (1,1) form of class C^1 on M such that $\int_M \omega = 1$. Consider the equation, in the sense of currents,

$$dd^c u = \omega - \delta_a, \tag{4.5}$$

where δ_a is the Dirac measure at a.

Theorem 4.1.11. *let U be an open set in a compact Riemann surface M such that $M \backslash U$ consists of at most a finite number of points.*

(i). Let ω be a positive (1,1) form of volume 1 on M. Let $a \in M$. Then equation (4.5) *admits a positive solution u_a, smooth in $M \backslash \{a\}$, with a log singularity at the point a*

(ii). If $M \backslash U$ is non-empty and ω is proportional to the Poincare$'$ form of M so that it is of volume 1, then equation (4.5) *admits a positive solution u_a, smooth in $U = \backslash \{a\}$, with a log singularity at the point a.*

Proof. (i). Since the cohomology class of the right hand side is zero, equation (4.5) always has a solution. The regularity in the complement of a and the behavior at a imply that u_a is smooth in $M \backslash \{a\}$, with a log singularity at the point a. By adding a constant if necessary, we have the desired positivity of u_a. This proves case (i).

(ii). In the case of (ii), the proof is similar to the above. Note that the Poincare' metric at the points in $M \backslash U$ behaves like $c dz \wedge d\bar{z}/(|z|^2(\log|z|)^2)$, which has finite volume. Using the fact that the Poincare' metric of the pointed disc has curvature -1, we can by comparison establish that the solution u_a goes to ∞ when approaching the points at the boundary. This gives the positivity of u_a.

$\square$

Let $a \in U$ and u_a be the solution of the equation (4.5). We define the **proximity function** by

$$m_{f,\omega}(r,a) = \frac{1}{2}\int_0^{2\pi} u_a(f(re^{i\theta}))\frac{d\theta}{2\pi}. \tag{4.6}$$

The **counting function** is defined in the standard way as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a)}{t}dt, \tag{4.7}$$

where $n(r,a)$ is the number of elements of $f^{-1}(a)$ inside $|z| < r$, counting multiplicities (for simplicity we assume 0 is not in $f^{-1}(a)$). The **truncated counting function** is defined as

$$\overline{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a)}{t}dt, \tag{4.8}$$

where $\bar{n}(r,a)$ is the number of elements of $f^{-1}(a)$ inside $|z| < r$, without counting multiplicities.

Theorem 4.1.12 (First Main Theorem)**.**

$$m_{f,\omega}(r,a) + N_f(r,a) = T_{f,\omega}(r) + O(1) \tag{4.9}$$

Proof. Observe, by applying the integral operator

$$\int_0^r \frac{dt}{t}\int_{|\zeta|<t}.$$

to equation (4.5) and using the Green-Jensen formula yields the desired equation. $\square$

Theorem 4.1.13 (The Second Main Theorem [16]). *Let M be a compact Riemann surface. Let ω be a smooth positive (1,1) form on M. Let $f : \triangle(R) \to M$ be a holomorphic map with $c_{f,\omega} < +\infty$, where $0 < R \leq \infty$. Let $a_1, \ldots, a_q$ be distinct points on M. Then, for every $\epsilon > 0$, the inequality*

$$\begin{aligned}&\sum_{j=1}^{q} m_{f,\omega}(r, a_j) + T_{f,Ric(\omega)}(r) + N_{f,ram}(r) \\ &\leq (1+\epsilon)(c_{f,\omega} + \epsilon)T_{f,\omega}(r) + O(\log T_{f,\omega}(r)) + \epsilon \log r\end{aligned} \tag{4.10}$$

holds for all $r \in (0, R)$ outside a set E with $\int_E \exp((c_{f,\omega} + \epsilon)T_{f,\omega}(r))dr < \infty$. Here $N_{f,ram}(r)$ is the counting function for the ramification divisor of f.

Note that, by the First Main Theorem and the fact that

$$N_f(r, a) - N_{f,ram}(r) \leq \overline{N}_f(r, a), \tag{4.11}$$

the above inequality (4.10) can be written as

$$\begin{aligned}&(q - (1+\epsilon)(c_{f,\omega} + \epsilon))T_{f,\omega}(r) + T_{f,\mathrm{Ric}(\omega)}(r) \leq \sum_{j=1}^{q} \overline{N}_f(r, a_j) \\ &+ O(\log T_{f,\omega}(r)) + \epsilon \log r \quad \|_E,\end{aligned}$$

where $\|_E$ means the inequality holds for all $r \in (0, R)$ outside a set E with $\int_E \exp((c_{f,\omega} + \epsilon)T_{f,\omega}(r))dr < \infty$.

Proof. Consider

$$\Psi = C\Big(\prod_{j=1}^{q}(u_{a_j}^{-2}\exp(u_{a_j}))\Big)\omega$$

where C is chosen such that $\int_M \Psi = 1$. Write

$$f^*\Psi = \Gamma\frac{\sqrt{-1}}{2\pi}d\zeta \wedge d\bar{\zeta}.$$

Then, by the Poincaré-Lelong formula,

$$dd^c[\log\Gamma] = \sum_{j=1}^{q} dd^c[u_{a_j}\circ f] + [f^*\mathrm{Ric}(\omega)] + D_{f,ram} - 2\sum_{j=1}^{q} dd^c[\log u_{a_j}\circ f]. \quad (4.12)$$

Applying the integral operator

$$\int_0^r \frac{dt}{t}\int_{|\zeta|<t}.$$

to the above identity and using the Green-Jensen formula, we get that

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\log\Gamma(re^{i\theta})\frac{d\theta}{2\pi} + O(1) &= \sum_{j=1}^{q} m_f(r,a_j) + T_{f,\mathrm{Ric}(\omega)}(r) + N_{f,ram}(r)\\ &- 2\sum_{j=1}^{q}\int_0^{2\pi}\frac{dt}{t}\int_{|\zeta|<t} dd^c[\log u_{a_j}\circ f].\end{aligned}$$

Using the Green-Jensen formula, the concavity of log, and the First Main Theorem, we have that

$$\begin{aligned}2\int_0^{2\pi}\frac{dt}{t}\int_{|\zeta|<t} dd^c[\log u_{a_j}\circ f] &= \int_0^{2\pi}\log\, u_{a_j}(f(re^{i\theta}))\frac{d\theta}{2\pi}\\ &\le \log\int_0^{2\pi} u_{a_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + O(1)\\ &= \log\, m_{f,\omega}(r,a_j) + O(1)\\ &\le \log T_{f,\omega}(r) + O(1).\end{aligned}$$

By using the concavity of log, equation (4.3), taking $\gamma(r) := \exp((c_{f,\omega}+\epsilon)T_{f,\omega}(r))$, and $\delta = 2\epsilon$, we have that,

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\log\,\Gamma(re^{i\theta})\frac{d\theta}{2\pi} &\le \frac{1}{2}\log\int_0^{2\pi}\Gamma(re^{i\theta})\frac{d\theta}{2\pi} + O(1)\\ &\le \frac{1}{2}((2+2\epsilon)(c_{f,\omega}+\epsilon)T_{f,\omega}(r)\\ &+ (1+2\epsilon)^2\log^+T_\Gamma(r) + 2\epsilon\log r) + O(1)\end{aligned}$$

holds for all $r \in (0, R)$ outside a set E with $\int_E \exp(c_{f,\omega} T_{f,\omega}(r)) < \infty$.

It remains to estimate

$$T_\Gamma(r) = \int_0^r \frac{dt}{t} \int_{|\zeta| \le t} \Gamma \frac{d\theta}{2\pi} d\zeta \wedge d\bar{\zeta} = \int_0^r \frac{dt}{t} \int_{|\zeta| \le t} f^*\Psi.$$

We follow the approach by Alhors-Chern. By the change of variables formula, we have

$$\int_M n_f(r, a)\Psi(a) = \int_{|\zeta| \le r} f^*\Psi.$$

Then by using the First Main Theorem, we have

$$\int_0^r \frac{dt}{t} \int_{|\zeta| \le t} f^*\Psi = \int_M N_f(r, a)\Psi(a) \le \int_M T_{f,\omega}(r)\Psi(a) + \mathcal{O}(1) = T_{f,\omega}(r) + O(1).$$

This finishes the proof.

□

4.2 A Uniqueness Result for Holomorphic Mappings from a Complex Disc into a Compact Riemann Surface

As a application of the Second Main Theorem (4.10), we are able to derive a new unicity result that recovers previous results dealing with holomorphic mappings from $\mathbb{C}$ to a compact Riemann surface M. In particular, note that if f and g are non-constant and $R = \infty$, then $c_{f,\omega} = c_{g,\omega} = 0$ (see [16]). So Theorem 4.2.4 extends the results of R. Nevanlinna and E. M. Schmid mentioned.

Let M be a complex manifold. A *holomorphic line bundle* over M is a complex manifold L together with a surjective holomorphic map $\pi : L \to M$ having the following properties

i.) Locally trivial: For all $p \in M$ there is a neighborhood U of p and a map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ is a biholomorphic.

ii.) Global linear structure: For each pair of such neighborhoods U_α and U_β there is a holomorphic map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ such that $\phi_\alpha \circ \phi_\beta^{-1}(x, \lambda) = (x, g_{\alpha\beta}(x)\lambda)$.

The map ϕ_U is called the local trivialization of the line bundle and the maps $g_{\alpha\beta}$ are called the transition functions. The set $L_x := \pi^{-1}(x), x \in M$ is called the *fiber of the line bundle at x.*

For a holomorphic line bundle L over M, a *holomorphic section s* of L is a collection $\{s_\alpha\}$ where each s_α is a holomorphic function defined on U_α and satisfying $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$.

A *metric* on a line bundle L is collection of positive smooth functions $h_\alpha : U_\alpha \to \mathbb{R}_{>0}$ such that on $U_\alpha \cap U_\beta$, we have $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$.

For a given metric $\{h_\alpha\}$ on L, we can define a $(1,1)$-form $\theta_L := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha$ on U_α. Since $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$ on $U_\alpha \cap U_\beta$, we have locally $\log h_\beta = \log g_{\alpha\beta} + \log \bar{g}_{\alpha\beta} + \log h_\alpha$ for some local branch of $\log g_{\alpha\beta}$. Since $\bar{\partial}\log g_{\alpha\beta} = 0$ and $\partial \log \bar{g}_{\alpha\beta} = 0$, we have that $-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\beta$ on $U_\alpha \cap U_\beta$. Hence, the form θ_L is well-defined.

The *Chern form* of L with respect to the metric $\{h_\alpha\}$ is defined on U_α to be

$$\theta_L := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_\alpha.$$

θ_L is denoted by $c_1(L,h)$ or $c_1(L)$.

Let L be a line bundle with metric $\{h_\alpha\}$. Given two sections of L s_i and s_j, we define the *inner product* of the two sections by

$$< s_i, s_j >= s_{i\alpha}\bar{s}_{j\alpha}h_\alpha.$$

In particular, $||s||^2 = |s_\alpha|^2 h_\alpha$. This is well-define due to the transition properties of s_α and h_α.

Let $\triangle(R)$ denote the disc of radius R with the convention that $\triangle(\infty) = \mathbb{C}$. For a complex manifold M, let $0 < R \leq \infty$ and $f : \triangle(R) \to M$ be a holomorphic map. Let ω be a positive $(1,1)$ form on M. Recall that the *characteristic (or height) function of f with respect to ω* is defined, for $0 < r < R$, as

$$T_{f,\omega}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\omega.$$

In general, for a complex projective variety M and an hermitian line bundle (L,h) where h is an Hermitian metric on the fibers of L, we define

$$T_{f,L}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L,h),$$

where $c_1(L,h)$ is the first Chern form. It is independent, up to a bounded term, of the choice of h. We also denote by $c_1(L) \in H^2(M,\mathbb{Z})$ the first Chern class of L.

For a compact Riemann surface M, we make the usual identification of $H^2(M,\mathbb{Z})$ with $\mathbb{Z}$ (by fixing a generator). With this identification we can also regard $c_1(L)$ as an integer, which is called the *Chern number* of L (so we use $c_1(L)$ to denote it as the Chern class, as well as the Chern number in the Riemann surface case).

Indeed the Chern number is equal to $\int_M c_1(L,h)$, which is (see the Lemma below) the difference of the number of zeros of σ minus the number of poles of σ on M,counting multiplicities, where σ is any meromorphic section of L.

Lemma 4.2.1. *Let (L,h) be a Hermitian holomorphic line bundle over a compact (without boundary) Riemann surface M and assume that there exists a non-trivial meromorphic section σ of L. Then*

$$\int_M c_1(L,h) = \deg(\sigma = 0) - \deg(\sigma = \infty).$$

Proof. The proof uses the following elementary result

$$\lim_{\epsilon \to 0} \int_{|z|=\epsilon} d^c log|z|^2 = 1.$$

Indeed, write $z = re^{i\theta}$, since $d^c = \frac{1}{4\pi}(\frac{r\partial}{\partial r} \otimes d\theta - r^{-1}\frac{\partial}{\partial \theta} \otimes dr)$, $d^c \log|z|^2 = r\frac{\partial}{\partial r}\log r \frac{\partial \theta}{2\pi} = \frac{d\theta}{2\pi}$. Hence

$$\lim_{\epsilon \to 0} \int_{|z|=\epsilon} d^c log|z|^2 = \int_0^{2\pi} \frac{d\theta}{2\pi} = 1.$$

We can now prove the lemma. For simplicity, we assume that σ is holomorphic (otherwise, we just need to include all the poles as well). The zero set $[\sigma = 0]$ consists of a finite number of points $a_1, ..., a_N$ with vanishing order $\mu_1, ..., \mu_N$ respectively. Chose local coordinates for discs $\triangle_{i,\epsilon}$ centered at each a_i with radius $\epsilon > 0$. At a point $a \in M$, there exists a local coordinate neighborhood U of a and we may write $||\sigma||^2 = h_U|s_U|^2$ with s_U being holomorphic on U. Thus $\log|s_U|^2$ is harmonic (outside the set of zeros of s_U), so that $dd^c \log||\sigma||^2 = dd^c \log h_U + dd^c \log|s_U|^2 = -c_1(L,h)|_U$(outside the set of zeros of s_U). From this we deduce from Stokes' theorem, by writing

$s_U = z^{\mu_i}\phi_i$ where ϕ_i is nowhere vanishing,

$$\begin{aligned}
\int_M c_1(L,h) &= -\lim_{\epsilon\to 0}\int_{M\setminus\cup_i\triangle_{i,\epsilon}} dd^c \log ||\sigma||^2 \\
&= \lim_{\epsilon\to 0}\int_{\partial\triangle_{i,\epsilon}} d^c \log h_U + \sum \lim_{\epsilon\to 0}\int_{\partial\triangle_{i,\epsilon}} d^c \log |s|_U^2 \\
&= \sum \lim_{\epsilon\to 0}\int_{\partial\triangle_{i,\epsilon}} d^c \log |s|_U^2 = \sum \mu_i \lim_{\epsilon\to 0}\int_{\partial\triangle_{i,\epsilon}} d^c \log |z|^2 \\
&= \sum \mu_i = \deg(\sigma = 0).
\end{aligned}$$

This proves the theorem.

□

Theorem 4.2.2 (Riemann-Roch Theorem)**.** *For any divisor D on a compact Riemann surface M of genus g, then*

$$\dim L(D) - \dim L(K-D) = \deg(D) - g + 1,$$

where K is the canonical divisor on M and $L(D)$ is the vector space of meromorphic functions f on M such that $f \equiv 0$ or $(f) + D \geq 0$.

From the Riemann-Roch theorem, we establish the following lemma that is used in the proof of one of our main results.

Lemma 4.2.3. *Let M be a compact Riemann surface with genus g. Let $y_0 \in M$. Then there exists a non-constant meromorphic function ϕ on M with a single pole at y_0 of multiplicity less than or equal to $g+1$.*

Proof. Taking $D = (g+1)y_0$, the above Riemann-Roch theorem implies that $\dim L(D) \geq 2$. Thus, there exists a non-constant meromorphic function ϕ on M with a single pole at y_0 of multiplicity less than or equal to $g+1$. □

We can now prove a new result.

Theorem 4.2.4. *Let M be a compact Riemann surface. Let ω be a smooth positive (1,1) (metric) form on M. Let $f, g : \triangle(R) \to M$ be two holomorphic maps with $c_{f,\omega} < +\infty$ and $c_{g,\omega} < +\infty$, where $0 < R \leq \infty$. Assume that there are q distinct points $a_1, \ldots, a_q$ on M such that $f(z) = a_j$ if and only if $g(z) = a_j$ for $j = 1, \ldots, q$. If $q > 4 + c_{f,\omega} + c_{g,\omega}$, then $f \equiv g$.*

Proof. Let ω be the metric form on M whose Gauss curvature is of constant K, i.e., $\mathrm{Ric}(\omega) = -K\omega$. We also assume that $\int_M \omega = 1$. Then, $T_{f,\mathrm{Ric}(\omega)}(r) = -KT_{f,\omega}(r)$. By the Gauss-Bonnet theorem, $K = 2 - 2g_M$, where g_M is the genus of M. So $T_{f,\mathrm{Ric}(\omega)}(r) = (2g_M - 2)T_{f,\omega}(r)$. Thus, applying Theorem 4.1.13, we get

$$(q + 2g_M - 2 + (1+\epsilon)(c_{f,\omega} + \epsilon))T_{f,\omega}(r) \leq \sum_{j=1}^{q} \overline{N}_f(r, a_j) + O(\log T_{f,\omega}(r)) + \epsilon \log r \quad \|_E$$

and

$$(q + 2g_M - 2 - (1+\epsilon)(c_{g,\omega} + \epsilon))T_{g,\omega}(r) \leq \sum_{j=1}^{q} \overline{N}_g(r, a_j) + O(\log T_{g,\omega}(r)) + \epsilon \log r \quad \|_E.$$

Now assume that $f \not\equiv g$. Then there is a point $z_0 \in \mathbb{C}$ such that $f(z_0) \neq g(z_0)$. Set $y_0 = f(z_0) \in M$. It follows from Lemma 4.2.3 that there exists a non-constant meromorphic function $\phi : M \to \mathbb{C} \cup \{\infty\}$ (which can be regarded as a holomorphic map $\phi : M \to \mathbb{P}^1(\mathbb{C})$) with a single pole at y_0 of multiplicity less than or equal to $g_M + 1$. Consider $F := \phi(f)$ and $G := \phi(g)$. Then $F - G$ is non-constant since $F(z_0) = \infty$ and $G(z_0) \in \mathbb{C}$. Also by the assumptions and the First Main Theorem, we have $\sum_{j=1}^{q} \overline{N}_f(r, a_j) \leq N_{F-G}(r, 0) \leq (T_F(r) + T_G(r))$ and $\sum_{j=1}^{q} \overline{N}_g(r, a_j) \leq$

$N_{F-G}(r,0) \le (T_F(r) + T_G(r))$. Thus, we have

$$(q + 2g_M - 2 - (1+\epsilon)(c_{f,\omega} + c_{g,\omega} + \epsilon))(T_{f,\omega}(r) + T_{g,\omega}(r))$$

$$\le 2(T_F(r) + T_G(r)) + O(\log T_{f,\omega}(r) + \log T_{g,\omega}(r)) + 2\epsilon \log r \quad \|_E.$$

We now estimate $T_F(r)$ and $T_G(r)$. We only consider F (as G is similar). From our definition above, $T_F(r) = T_{F,\mathcal{O}_{\mathbb{P}^1}(1)}(r)$ where $\mathcal{O}_{\mathbb{P}^1}(1)$ is the hyperplane line bundle over $\mathbb{P}^1(\mathbb{C})$. For a compact Riemann surface, we make the usual identification of $H^2(M,\mathbb{Z})$ with $\mathbb{Z}$ (with ω being the generator). With this identification, we know that $c_1(\phi^*\mathcal{O}_{\mathbb{P}^1}(1))$ is an integer, and from Lemma 4.2.1, it is equal to the number of zeros (counting multiplicities) of any holomorphic section of $\phi^*\mathcal{O}_{\mathbb{P}^1}(1)$ on M. In particular, if we take the holomorphic section as $\phi^*\sigma$ with $\sigma \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $[\sigma = 0] = \{[z_0 : z_1] \mid z_1 = 0\}$, then we get, by the fact that ϕ has a single pole at y_0 of multiplicity less than or equal to $g_M + 1$,

$$c_1(\phi^*\mathcal{O}_{\mathbb{P}^1}(1)) \le g_M + 1.$$

Thus,

$$\begin{aligned} T_F(r) &= T_{F,\mathcal{O}_{\mathbb{P}^1}(1)} = \int_1^r \frac{dt}{t}\int_{|z|<t} F^*c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \\ &= \int_1^r \frac{dt}{t}\int_{|z|<t} f^*(c_1(\phi^*\mathcal{O}_{\mathbb{P}^1}(1))) \\ &\le (g_M+1)\int_1^r \frac{dt}{t}\int_{|z|<t} f^*\omega \\ &= (g_M+1)T_{f,\omega}(r). \end{aligned}$$

Similarly, $T_G(r) \le (g_M+1)T_{g,\omega}(r)$. Hence we get,

$$(q + 2g_M - 2 - (1+\epsilon)(c_{f,\omega} + c_{g,\omega} + \epsilon))(T_{f,\omega}(r) + T_{g,\omega}(r))$$

$$\leq 2(g_M+1)(T_{f,\omega}(r)+T_{g,\omega}(r))) + O(\log T_{f,\omega}(r) + \log T_{g,\omega}(r)) + 2\epsilon \log r \quad \|_E,$$

i.e.

$$\begin{aligned} & (q-4-(1+\epsilon)(c_{f,\omega}+c_{g,\omega}+\epsilon))(T_{f,\omega}(r)+T_{g,\omega}(r)) \\ \leq \; & O(\log T_{f,\omega}(r)+\log T_{g,\omega}(r)) + 2\epsilon \log r \quad \|_E. \end{aligned}$$

As r gets large, we have a contradiction if $q > 4 + c_{f,\omega} + c_{g,\omega}$. The theorem is thus proved.

$\square$

Chapter 5

Holomorphic Mappings into $\mathbb{P}^n(\mathbb{C})$

5.1 Value Distribution of Holomorphic Mappings into $\mathbb{P}^n(\mathbb{C})$

In this section, we establish the theory of Nevanlinna for a class of holomorphic maps from a disc of radius R into $\mathbb{P}^n(\mathbb{C})$. We follow Ahlfors' method with some simplifications (see $[6], [15], [21],$ and $[23]$). However, we treat differently the error term. The key is to use the methods in [6] by choosing a suitable γ to allow the usage of the Calculus lemma, lemma 4.1.3. We can choose $\gamma(r) := \exp((c_f + \epsilon)T_f(r))$ for any given ϵ, where $c_f := c_{f,\omega_{FS}}$ and $T_f(r) := T_{f,\omega_{FS}}(r)$. In the following, we use the notation " $\leq ||$" to denote the inequality holds for all $r \in (0, R)$ except for a set E with $\int_E \exp((c_f + \epsilon)T_f(r))dr < \infty$. We always assume that the holomorphic map $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ is linearly non-degenerate (that is, $f(\triangle(R))$ is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$).

Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $L = \mathcal{O}_{\mathbb{P}^n}(1)$ be the hyperplane line bundle with transition functions $g_{\alpha\beta} = \frac{w_\alpha}{w_\beta}$, where $U_\alpha = \{w_\alpha \neq 0\}$. the sections of L are given by $s_H = \{\frac{<\mathbf{a},\mathbf{w}>}{w_\alpha}\}$ with $H = [s_H = 0] = \{a_0 w_0 + \cdots + a_n w_n = 0\}$. The metric on L is given by $h_\alpha = \frac{|w_\alpha|^2}{\|\mathbf{w}\|^2}$. The first Chern form of this line bundle is

$$c_1(L,h) = -dd^c \log\, h_\alpha = dd^c \log\, \|\mathbf{w}\|^2.$$

This local expression is called the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$. We have, by the Green-Jensen formula, an expression for the characteristic function for f in this setting by

$$T_f(r) = \int_0^r \int_{|\zeta|\leq t} dd^c \log\, \|\mathbf{f}\|^2 = \int_0^{2\pi} \log\, \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1).$$

Here $\mathbf{f} = (f_0, ..., f_n)$ is a reduced representation of f, i.e., $f_0, ..., f_n$ have no common zeros. For reduced form f, it can easily be shown that

$$T_{\frac{f_i}{f_j}}(r) \leq T_f(r). \tag{5.1}$$

The **proximity function** is defined by

$$m_f(r,H) = \int_0^{2\pi} \log\, \frac{1}{\|s_H \circ f(re^{i\theta})\|} \frac{d\theta}{2\pi} = \int_0^{2\pi} \log\, \frac{\|\mathbf{f}(re^{i\theta})\| \|H\|}{| < \mathbf{a}, \mathbf{f}(re^{i\theta}) > |} \frac{d\theta}{2\pi}.$$

The **Counting function of f with respect to H** is given by

$$N_f(r,H) = \int_0^r (n_f(t,H) - n_f(0,H)) \frac{dt}{t} + n_f(0,H) \log\, r,$$

where $n_f(t,H)$= number of points of $< \mathbf{a}, \mathbf{f} >= 0$ in the disc $|z| < t$, counting multiplicity. By Jensen's formula, the counting function can be written as

$$N_f(r,H) = \int_0^{2\pi} = \log\, | < \mathbf{a}, \mathbf{f}(re^{i\theta}) > | \frac{d\theta}{2\pi} + O(1).$$

Applying the above definitions, we have the following Theorem.

Theorem 5.1.1 (First Main Theorem)**.**

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1).$$

Let $\mathbf{f} : \triangle(R) \to \mathbb{C}^{n+1} - \{0\}$ be a reduced representation of f. Consider the holomorphic map $\mathbf{F}_k$ defined by

$$\mathbf{F}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)} : \triangle(R) \to \bigwedge^{k+1} \mathbb{C}^{n+1}.$$

Evidently $\mathbf{F}_{n+1} \equiv 0$. Since f is linearly non-degenerate, $\mathbf{F}_k \not\equiv 0$ for $0 \leq k \leq n$. The map $F_k = \mathbb{P}(\mathbf{F}_k) : \triangle(R) \to \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1}) = \mathbb{P}^{N_k}(\mathbb{C})$, where $N_k = \frac{(n+1)!}{(k+1)!(n-k)!} - 1$ and $\mathbb{P}$ is the natural projection, is called **the k-th associated map**. Let $\omega_k = dd^c \log ||Z||^2$ be the Fubini-Study form on $\mathbb{P}^{N_k}(\mathbb{C})$, where $Z = [x_0 : \cdots : x_{N_k}] \in \mathbb{P}^{N_k}(\mathbb{C})$. Let

$$\Omega_k = F_k^* \omega_k = \frac{\sqrt{-1}}{2\pi} h_k dz \wedge d\bar{z},\ 0 \leq k \leq n, \tag{5.2}$$

be the pull-back via the k-th associated curve. Observe that since F_k has no indeterminacy points, $\Omega = F_k^* \omega_k$ is smooth and h_k is non-negative.

We recall the following lemma (see [10], [15], [23], and [21]).

Lemma 5.1.2.

$$h_k = \frac{||\boldsymbol{F}_{k-1}||^2 ||\boldsymbol{F}_{k+1}||^2}{||\boldsymbol{F}_k||^4}.$$

We now turn to the Plücker Formula. By Lemma 5.1.2 and the Poincaré-Lelong formula, we get

$$dd^c \log h_k = \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k + [h_k = 0], \tag{5.3}$$

where $[h_k = 0]$ is the zero divisor of h_k. We recall a few facts on the geometric meaning of this divisor(see [10], [21]). We consider the point z_0 with $\mathbf{F}_k(z_0) = 0$. Without loss of generality, we assume that $z_0 = 0$ and $f(z_0) = [1 : 0 : \cdots : 0]$ and that the reduced representation $\mathbf{f}$ of f in a neighborhood of 0 has the form

$$\mathbf{f}(z) = (1 + \cdots, z^{\nu_1} + \cdots, \cdots, z^{\nu_n} + \cdots),$$

with $1 \leq \nu_1 \leq \cdots \leq \nu_n$. Then it is easy to set that

$$\mathbf{F}_k(z) = z^{m_k}(1 + \cdots, z^{\nu_{k+1}-\nu_k} + \cdots, \cdots),$$

where $m_k = \nu_1 + \cdots + \nu_k - \frac{k(k+1)}{2}$. On the other hand, if we write in a neighborhood of 0, $h_k(z_) = z^{2\mu_k} b(z)$ with $b(0) > 0$, then we get that $\mu_k = m_{k+1} - 2m_k + m_{k-1}$. Define the k-th characteristic function

$$T_{F_k}(r) = \int_0^r \frac{dt}{t} \int_{|z|\leq t} F_k^* \omega_k.$$

Denote by

$$N_{d_k}(r) = \int_0^r n_{d_k}(t) \frac{dt}{t},$$

where $n_{d_k}(t)$ is the number of zeros of h_k in $|z| < t$, counting multiplicities. Note that $N_{d_k}(r)$ does not depend on the choice of reduced representation. Define

$$S_k(r) = \frac{1}{2} \int_0^{2\pi} \log\, h_k(re^{i\theta}) \frac{d\theta}{2\pi}. \tag{5.4}$$

Then, by applying the integral operator

$$\int_0^{2\pi} \frac{dt}{t} \int_{|\zeta|\leq t} .$$

to (5.2) and using Green-Jensen's formula, we get the following lemma.

Lemma 5.1.3 (Plücker's Formula)**.** *For any integers k with $0 \le k \le n$,*

$$N_{d_k}(r) + T_{F_{k-1}}(r) - 2T_{F_k}(r) + T_{F_{k+1}}(r) = S_k(r) + \mathcal{O}(1),$$

where $T_{F_{-1}}(r) \equiv 0$ and $T_{F_0}(r) = T_f(r)$.

Plücker's formula implies the following lemma which gives the estimate of $T_{F_k}(r)$ in terms of $T_f(r)$. We use our estimate of the error term.

Lemma 5.1.4. *For $0 \le k \le n-1$ and every $\delta > 0$,*

$$T_{F_k}(r) \le (n+2)^3(1 + (2+\delta)c_f)T_f(r) + n(n+1)^2\delta \ \log \ r + \mathcal{O}(1) \ \|.$$

Proof. Write $T(r) = \sum_{k=0}^{n-1} T_{F_k}(r)$. Observe that

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{dT_{F_k}(r)}{dr}\Big) = 2\int_0^{2\pi} h_k(re^{i\theta})\frac{d\theta}{2\pi}.$$

Applying the Calculus Lemma (see (4.3)) with $\gamma(r) = \exp((c_f+\delta)T_f(r))$,we get that

$$\int_0^{2\pi} h_k(re^{i\theta})\frac{d\theta}{2\pi} \le r^{2\delta}e^{c_f(4+2\delta)T_f(r)}T_{F_k}^{(1+2\delta)^2}(r) \ \|.$$

This implies

$$\begin{aligned} S_k(r) &= \frac{1}{2}\int_0^{2\pi} \log \ h_k(re^{i\theta})\frac{d\theta}{2\pi} \\ &\le \frac{1}{2}\log\int_0^{2\pi} h_k(re^{i\theta})\frac{d\theta}{2\pi} \\ &\le (2+\delta)c_fT_f(r) + \frac{1}{2}(1+2\delta)^2\log \ T(r) + \delta\log \ r \ \|. \end{aligned} \tag{5.5}$$

From Lemma 5.1.3, we claim that, for $0 \le q \le p$,

$$T_{F_p}(r) + (p-q)T_{Fq-1}(r) \le (p-q+1)T_{F_q}(r) + \sum_{j=q}^{p-1}(p-j)(S_j(r) + O(1).$$

In fact, the claim is true for $p = q$. Assume that the claim is true for $q, q+1, ..., p$. If $p = q$, then the proof is done. If $p < q$, we proceed, by using Lemma 5.1.3,

$$\begin{aligned}
T_{F_{q-1}}(r) - T_{F_q}(r) + T_{F_{p+1}}(r) - T_{F_p}(r) &= \sum_{j=1}^{p}(T_{F_{j-1}}(r) - 2T_{F_j}(r) + T_{F_{j+1}}(r)) \\
&= \sum_{j=q}^{p} S_j(r) - \sum_{j=q}^{q} N_{d_j}(r) + O(1) \\
&\leq \sum_{j=q}^{p} S_j(r) + O(1).
\end{aligned}$$

So

$$T_{F_{p+1}}(r) + T_{F_{q-1}}(r) \leq T_{F_q}(r) + T_{F_{p(r)}} + \sum_{j=q}^{p} S_j(r) + O(1).$$

Thus,

$$\begin{aligned}
(p+1-q)T_{F_{q-1}}(r) + T_{F_{p+1}}(r) &= (p-q)T_{F_{q-1}}(r) + T_{F_{q-1}}(r) + T_{F_{p+1}}(r) \\
&\leq (p-q)T_{F_{q-1}}(r) + T_{F_q}(r) + T_{F_p}(r) \\
&+ \sum_{j=q}^{p} S_j(r) + O(1).
\end{aligned}$$

On the other hand, from Lemma 5.1.3 again, we have that

$$\begin{aligned}
T_{F_p}(r) - (p-q+1)T_{F_q}(r) &+ (p-q)T_{F_{q-1}}(r) \\
&= \sum_{j=q}^{p}(p-j)(T_{F_{j-1}}(r) - 2T_{F_j}(r) + T_{F_{j+1}}(r)) \\
&\leq \sum_{j=q}^{p}(p-j)S_j(r) + O(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
T_{F_p}(r) + T_{F_q}(r) + (p-q)T_{F_{q-1}}(r) &\leq (p-q+2)T_{F_q}(r) \\
&+ \sum_{j=q}^{p}(p-j)S_j(r) + O(1).
\end{aligned}$$

Therefore,

$$\begin{aligned} T_{F_{p+1}}(r) + (p+1-q)T_{F_{q-1}}(r) &\leq (p-q+2)T_{F_q}(r) \\ &+ \sum_{j=q}^{p}(p-j)S_j(r) + O(1). \end{aligned}$$

This proves our claim. Now take $q = 0$ and $p = k$ and notice that $T_{F_{-1}}(r) \equiv 0$, then

$$T_{F_k}(r) \leq (k+1)T_f(r) + \sum_{j=0}^{k-1}(k-j)S_j(r) + O(1).$$

This, together with (5.5), gives for $0 \leq k \leq n$,

$$\begin{aligned} T_{F_k}(r) &\leq (k+1)T_f(r) \\ &+ \frac{1}{2}k(k+1)((2+\delta)c_f T_f(r) + (1+2\delta)^2 \log T(r) + \delta \log r + O(1)\|. \end{aligned}$$

Therefore,

$$\begin{aligned} T(r) &\leq (n+1)^2 T_f(r) \\ &+ \frac{1}{2}n(n+1)^2\Big((2+\delta)c_f T_f(r) + \frac{1}{2}(1+2\delta)^2 \log T(r) + \delta \log r + O(1)\Big) \quad \|. \end{aligned}$$

Since $\frac{1}{2}n(n+1)^2(1+2\delta)^2 \log T(r) \leq \frac{1}{2}T_f(r)$, where r is close enough to R, we have that

$$T(r) \leq (n+2)^3(1+(2+\delta)c_f)T_f(r) + n(n+1)^2\delta \log r + O(1) \quad \|.$$

$\square$

For intgers $1 \leq q \leq p \leq n+1$, the **interior product** $\xi \lfloor \alpha \in \bigwedge^{p-q} \mathbb{C}^{n+1}$ of vectors $\xi \in \bigwedge^{p+1} \mathbb{C}^{n+1}$ and $\alpha \in \bigwedge^{q+1} (\mathbb{C}^{n+1})^*$ is defined by

$$\beta(\xi \lfloor \alpha) = (\alpha \wedge \beta)(\xi)$$

for any $\beta \in \bigwedge^{p-q}(\mathbb{C}^{n+1})^*$. Let

$$H = \{[x_0 : \cdots : x_n] | a_0 x_0 + \cdots + a_n x_n = 0\}$$

be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ with unit normal vector $\mathbf{a} = (a_0, \cdots, a_n)$. In the rest of the section, we regard $\mathbf{a}$ as a vector in $(\mathbb{C}^{n+1})^*$ which is defined by $\mathbf{a}(\mathbf{x}) = a_0 x_0 + \cdots + a_n x_n$ for each $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1}$, where $(\mathbb{C}^{n+1})^*$ is the dual space of $\mathbb{C}^{n+1}$. Let $x \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$. The **projective distance** is defined by

$$\|x; H\| = \frac{\|\xi \lfloor \mathbf{a}\|}{\|\xi\| \|\mathbf{a}\|}, \tag{5.6}$$

where $\xi \in \bigwedge^{k+1} \mathbb{C}^{n+1}$ with $\mathbb{P}(\xi) = x$. Define

$$m_{F_k}(r, H) = \int_0^{2\pi} \log \frac{1}{\|F_k(re^{i\theta}); H\|} \frac{d\theta}{2\pi}. \tag{5.7}$$

We have the following weak form of the First Main Theorem for F_k.

Theorem 5.1.5 (Weak First Main Theorem)**.**

$$m_{F_k}(r, H) \leq T_{F_k}(r) + \mathcal{O}(1).$$

Proof. Let $\mathbf{f}_k : \triangle(R) \to \bigwedge^{k+1} \mathbb{C}^{n+1}$ be a reduced representation of F_k, and we consider the holomorphic map

$$F_k \lfloor \mathbf{a} : \triangle \to \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1}),$$

which is given by $F_k \lfloor \mathbf{a} := \mathbb{P}(G)$, where $G = \mathbf{f}_k \lfloor \mathbf{a}$. Note that G is a representation of the holomorphic map $F_k \lfloor \mathbf{a}$, but not reduced. We denote by ν_G the divisor of G on $\triangle(R)$, and $N_G(r, 0)$ the counting function associated to ν_G (which is independent of the choices of the reduced representation of F_k). We have

$$(F_k \lfloor \mathbf{a})^* \omega_k + \nu_G = dd^c \log \|G\|^2.$$

Applying the integral operator

$$\int_0^r \frac{dt}{t}\int_{|\zeta|\le t}.$$

to the above identity and by using Green-Jensen's formula, we get

$$\begin{aligned} T_{F_k\lfloor \mathbf{a}}(r) + N_G(r,0) &= \int_0^{2\pi} \log \|G(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1) \\ &= \int_0^{2\pi} \log \|\mathbf{f}_k\lfloor \mathbf{a}\|(re^{i\theta})\frac{d\theta}{2\pi} + O(1). \end{aligned}$$

On the other hand, from the definition (notice that $\mathbf{f}_k$ is a reduced representation of F_k),

$$T_{F_k}(r) = \int_0^{2\pi} \log \|\mathbf{f}_k\|(re^{i\theta})\frac{d\theta}{2\pi} + O(1).$$

Hence, from the definition of $m_{F_k}(r,H)$,

$$\begin{aligned} T_{F_k\lfloor \mathbf{a}}(r) + N_G(r,0) + m_{F_k}(r,H) &= \int_0^{2\pi} \log \|\mathbf{f}_k\lfloor \mathbf{a}\|(re^{i\theta})\frac{d\theta}{2\pi} + O(1) \\ &+ \int_0^{2\pi} \log \frac{\|\mathbf{f}_k\|\|\mathbf{a}\|}{\|\mathbf{f}_k\lfloor \mathbf{a}\|}(re^{i\theta})\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \|\mathbf{f}_k\|(re^{i\theta})\frac{d\theta}{2\pi} + O(1) \\ &= T_{F_k}(r) + O(1). \end{aligned}$$

□

We will need the following product to sum estimate. It is an extension of the estimate of the geometric mean by arithmetic mean.

Lemma 5.1.6 (See Theorem 3.5.7 [15]). *Let $H_1, ..., H_q$ (or linear forms $\boldsymbol{a}_1, ..., \boldsymbol{a}_q$) be hyperplanes in $\mathbb{P}^n\mathbb{C}$ in general position. Let $k \in \mathbb{Z}[0, n-1]$ with $n-k \le q$. Then there exists a constant $c_k > 0$ such that for every $0 < \lambda < 1$, $x \in \mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$ with*

$x \not\subset H_j$, *for* $1 \leq j \leq q$ *and* $y \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$ *we have that*

$$\prod_{j=1}^{q} \frac{\|y; H_j\|^2}{\|x; H_j\|^{2-2\lambda}} \leq c_k \Big(\sum_{j=1}^{q} \frac{\|y; H_j\|^2}{\|x; H_j\|^{2-2\lambda}} \Big)^{n-k}.$$

Let $\phi_k(H) = \|F_k; H\|^2$. Define

$$h_k(H) = \frac{\phi_{k-1}(H)\phi_{k+1}(H)}{\phi_k^2(H)} \Omega_k. \tag{5.8}$$

The function $\phi_k(H)$ is defined out of the stationary points, however analysis near those points shows $\phi_k(H)$ can be extended smoothly at those points (see [21]). The key is to use the following so-called Ahlfors' estimate. We include the proof here.

Theorem 5.1.7 (Ahlfors' estimate (see [15] or [23])). *Let* H *be a hyperplane in* $\mathbb{P}^n(\mathbb{C})$. *Then for any* $0 < \lambda < 1$, *we have*

$$\int_s^r \frac{dt}{t} \int_{|z|<t} \frac{\phi_{k+1}(H)}{\phi_k(H)^{1-\lambda}} \Omega_k \frac{dt}{t} \leq \frac{1}{\lambda^2} (8T_{F_k}(r) + O(1)).$$

To prove Ahlfors' estimate, the following lemma plays a crucial role (see [15], [21], or [23]). The proof of the lemma is based on a standard but lengthy computation. For the details of the proof, see Lemma A3.5.10 in [15].

Lemma 5.1.8 (Lemma A3.5.10 in [15]). *Let* H *be a hyperplane in* $\mathbb{P}^n(\mathbb{C})$ *and* λ *be a constant with* $0 < \lambda < 1$. *Then, for* $0 \leq k \leq n$, *the following inequality holds on* $\triangle(R) - \{z \mid \phi_k(H)(z) = 0\}$,

$$\frac{\lambda^2}{4} \frac{\phi_{k+1}(H)}{\phi_k^{1-\lambda}(H)} \Omega_k - \lambda(1+\lambda)\Omega_k \leq dd^c \log \ (1 + \phi_k(H)^\lambda).$$

We now prove Theorem 5.1.7 (Ahlfors' estimate).

Proof. By Lemma 5.1.7,

$$\frac{\lambda^2}{4}\frac{\phi_{k+1}(H)}{\phi_k^{1-\lambda}(H)}\Omega_k - \lambda(1+\lambda)\Omega_k \le dd^c\log\,(1+\phi_k(H)^\lambda).$$

Thus,

$$\frac{\lambda^2}{4}\frac{\phi_{k+1}(H)}{\phi_k^{1-\lambda}(H)}\Omega_k \le dd^c\log\,(1+\phi_k(H)^\lambda) + \lambda(1+\lambda)\Omega_k. \tag{5.9}$$

By Green-Jensen's formula,

$$\int_s^r \frac{dt}{t}\int_{|z|<t} dd^c\log\,(1+\phi_k(H)^\lambda) = \frac{1}{2}\int_0^{2\pi} dd^c\log\,(1+\phi_k(H)^\lambda)\frac{d\theta}{2\pi} + O(1).$$

This, together with (5.9) implies that

$$\begin{aligned}
\frac{\lambda^2}{4}\int_s^r \frac{dt}{t}\int_{|z|<t}\frac{\phi_{k+1}(H)}{\phi_k^{1-\lambda}(H)}\Omega_k &\le \int_0^r\int_{|z|<t} dd^c\log\,(1+\phi_k(H)^\lambda) + \lambda(1+\lambda)T_{F_k}(r)\\
&= \frac{1}{2}\int_0^{2\pi} dd^c\log\,(1+\phi_k(H)^\lambda)\frac{d\theta}{2\pi} + \lambda(1+\lambda)T_{F_k}(r) + O(1)\\
&\le \lambda(1+\lambda)T_{F_k}(r) + \frac{1}{2}\log 2 + O(1)\\
&\le 2T_{F_k}(r) + O(1),
\end{aligned}$$

using $0 \le \phi_k(H) \le 1$. □

We prove the following general version of H. Cartan's theorem.

Theorem 5.1.9 (A General Form of SMT)**.** *Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenrate holomorphic curve (i.e. its image is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$) with $c_f < \infty$, where $c_f := c_{f,\omega_{FS}}$ and $0 < R \le \infty$. Let $H_1, ..., H_q$ (or linear forms $\boldsymbol{a}_1, ..., \boldsymbol{a}_q$) be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then, for any $\epsilon > 0$, we have* *that*

$$
\begin{aligned}
&\int_0^{2\pi} \max_K \sum_{j\in K} \log \ \frac{1}{\|f(re^{i\theta}); H_j\|} \frac{d\theta}{2\pi} + N_W(r,0) \\
\leq \quad & (n+1)T_f(r) + \frac{n(n+1)}{2}(1+\epsilon)(c_f+\epsilon)T_f(r) \\
+ \quad & O(\log \ T_f(r)) + \frac{n(n+1)}{2}\epsilon \log \ r \quad \|,
\end{aligned}
$$

where the max is taken over all subsets K of $\{1,..,q\}$ such that the linear forms $\boldsymbol{a}_j$, $j \in K$ are linearly independent.

Proof. Without loss of generality, we may assume $q \geq n+1$ and that $\#K = n+1$. Let T be the set of all injective maps $\mu : \{0,...,n\} \to \{1,...,q\}$ such that $\mathbf{a}_{\mu(0)},...,\mathbf{a}_{\mu(n)}$ are linearly independent. Take

$$
\lambda := \Lambda(r) = \min_k \Big\{ \frac{1}{T_{F_k}(r)} \Big\}. \tag{5.10}
$$

For any $\mu \in T$, by Lemma 5.1.6 with $\lambda = \Lambda(r)$ and $\phi_k(H) = \|F_k, H\|$, we have for $0 \leq k \leq n-1$,

$$
\prod_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \leq c_k \Big(\sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \Big)^{n-k}
$$

for some constant $c_k > 0$. Since $\phi_n(H_{\mu(j)})$ is a constant for any $0 \leq j \leq n$ and $F_0 = f$, the above inequality implies that

$$
\prod_{j=0}^{n} \frac{1}{\|f; H_{\mu(j)}\|^2} \leq c \prod_{k=0}^{n-1} \Big(\sum_{j=0}^{n-1} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \Big)^{n-k} \cdot \prod_{k=0}^{n-1} \prod_{j=0}^{n} \frac{1}{\phi_k(H_{\mu(j)})^{2\Lambda(r)}}
$$

for some constant $c > 0$. Therefore

$$\int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{1}{\|f(re^{i\theta}); H_j\|^2}\frac{d\theta}{2\pi} = \int_0^{2\pi} \max_{\mu\in T} \ \log \prod_{j=0}^{n} \frac{1}{\|f(re^{i\theta}); H_{\mu(j)}\|^2}\frac{d\theta}{2\pi}$$

$$\begin{aligned}
\leq \ & \sum_{k=0}^{n-1} \int_0^{2\pi} \max_{\mu\in T} \ \log \Big(\sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}}(re^{i\theta}) \Big)^{n-k} \frac{d\theta}{2\pi} \\
+ \ & \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_0^{2\pi} \max_{\mu\in T} \ \log \Big(\sum_{j=0}^{n} \frac{1}{\phi_k(H_{\mu(j)})^{2\Lambda(r)}(re^{i\theta})} \Big) \frac{d\theta}{2\pi} + O(1) \\
= \ & \sum_{k=0}^{n-1} (n-k) \int_0^{2\pi} \max_{\mu\in T} \ \log \Big(\sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}}(re^{i\theta}) \cdot h_k(re^{i\theta} \Big) \frac{d\theta}{2\pi} \\
- \ & \sum_{k=0}^{n-1} (n-k) S_k(r) + \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_0^{2\pi} \max_{\mu\in T} \ \log \Big(\sum_{j=0}^{n} \frac{1}{\phi_k(H_{\mu(j)})^{2\Lambda(r)}}(re^{i\theta}) \Big) \frac{d\theta}{2\pi} + O(1),
\end{aligned}$$

where h_k is defined in (5.2). Observing that $N_W(r, 0) = N_{d_n}(r)$, we have by Plückers formula

$$\begin{aligned}
& \sum_{k=0}^{n-1} (n-k) S_k(r) = \sum_{k=0}^{n-1} (n-k) N_{d_n}(r) \\
+ \ & \sum_{k=0}^{n-1} (n-k)(T_{F_{k-1}}(r) - 2T_{F_k}(r) + T_{F_{k+1}}(r)) + O(1) \\
= \ & N_{d_n}(r) - (n+1)T_f(r) + O(1) \\
= \ & N_W(r, 0) - (n+1)T_f(r) + O(1).
\end{aligned}$$

By Theorem 5.1.5 (The Weak First Main Theorem) and by Lemma 5.1.4,

$$\begin{aligned}
& \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_0^{2\pi} \max_{\mu\in T} \ \log \Big(\sum_{j=0}^{n} \frac{1}{\phi_k(H_{\mu(j)})^{2\Lambda(r)}}(re^{i\theta}) \Big)^{n-k} \frac{d\theta}{2\pi} \\
= \ & \sum_{\mu\in T} \sum_{k=0}^{n-1} \sum_{j=0}^{n} 2\Lambda(r) m_{F_k}(r, H_{\mu(j)}) + O(1) \\
\leq \ & \sum_{k=0}^{n-1} \sum_{j=0}^{n} 2q\Lambda(r) T_{F_k}(r) + O(1) \\
\leq \ & O(1).
\end{aligned}$$

So

$$\int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{1}{\|f(re^{i\theta}); H_j\|} \frac{d\theta}{2\pi} \leq (n+1)T_f(r) - N_W(r,0) + G(r), \tag{5.11}$$

where

$$G(r) = \frac{1}{2}\sum_{k=0}^{n-1}(n-k)\int_0^{2\pi} \max_{\mu\in T} \log\Big(\sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}}(re^{i\theta})\cdot h_k(re^{i\theta})\Big)\frac{d\theta}{2\pi}.$$

We now estimate $G(r)$. Let

$$\hat{T}(r) := \int_0^r \Big(\int_{|z|<t} \frac{\phi_{k+1}(H)}{\phi_k(H)^{2-2\Lambda(r)}} h_k \frac{\sqrt{-1}}{2\pi} dz\wedge d\bar{z}\Big)\frac{dt}{t}.$$

Then, from Ahlfor's estimate, equation (5.10), and lemma 5.1.3, we get

$$\hat{T}(r) \leq O(T^3_{F_k}(r)) = O(T^3_f(r)). \tag{5.12}$$

Then, by the inequality (4.1) with $\gamma(r) = e^{(c_f+\epsilon)T_f(r)}$, for every hyperplane H,

$$\int_0^{2\pi} \frac{\phi_{k+1}(H)(re^{i\theta})}{\phi_k(H)^{2-2\Lambda(r)}(re^{i\theta})} h_k(re^{i\theta})\frac{d\theta}{2\pi} \leq r^{2\epsilon}e^{(c_f+\epsilon)(2+2\epsilon)T_f(r)}\cdot \hat{T}^{(1+2\epsilon)^2}(r) \quad \|.$$

This, together with the concavity of log and (5.13), gives

$$\begin{aligned}
G(r) &= \frac{1}{2}\sum_{k=0}^{n-1}(n-k)\int_0^{2\pi} \max_{\mu\in T} \log \sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}}(re^{i\theta})\cdot h_k(re^{i\theta})\frac{d\theta}{2\pi} \\
&\leq \sum_{k=0}^{n-1}\frac{n-k}{2}\log\int_0^{2\pi}\sum_{j=1}^{q}\frac{\phi_{k+1}(H_j)(re^{i\theta})}{\phi_k(H_j)^{2-2\Lambda(r)}(re^{i\theta})}h_k(re^{i\theta})\frac{d\theta}{2\pi} \\
&\leq ((c_f+\epsilon)(2+2\epsilon)T_f(r) + 2\epsilon\log r)\sum_{k=0}^{n-1}\frac{n-k}{2} + O(\log T(r)) \\
&= \frac{n(n+1)}{2}((1+\epsilon)(c_f+\epsilon)T_f(r) + \epsilon\log r) + O(\log T_f(r)) \quad \|.
\end{aligned}$$

Combining the above with inequality (5.11), we have that

$$
\begin{aligned}
& \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{1}{\|f(re^{i\theta}); H_j\|} \frac{d\theta}{2\pi} + N_W(r,0) \\
\leq & \quad (n+1)T_f(r) + \frac{n(n+1)}{2}(1+\epsilon)(c_f+\epsilon)T_f(r) \\
+ & \quad \frac{n(n+1)}{2}\epsilon \ \log r + O(\log \ T_f(r)) \quad \|_E.
\end{aligned}
$$

Thus, the theorem is proved. $\square$

Definition 5.1.10. Given hyperplanes $H_1, ..., H_q$. We say that $H_1, ..., H_q$ are **in general position** if for any injective map $\mu : \{0, 1, ..., n\} \to \{1, ..., q\}$, $H_{\mu(0)}, ..., H_{\mu(n)}$ are linearly independent.

Lemma 5.1.11 (see Lemma A3.1.6 in [15]). *Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. denote by T the set of all injective maps $\mu : \{0, 1, ..., n\} \to \{1, ..., q\}$. Then*

$$
\sum_{j=1}^{q} m_f(r, H_j) \leq \int_0^{2\pi} \max_K \sum_{j\in K} \log \ \frac{1}{\|f(re^{i\theta}); H_j\|} \frac{d\theta}{2\pi} + O(1)
$$

We can now establish The Second Main Theorem for maps into $\mathbb{P}^n(\mathbb{C})$.

Theorem 5.1.12 ([16]). *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve (i.e. its image is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$) with $c_f < \infty$, where $c_f = c_{f,\omega_{FS}}$ and $0 < R \leq \infty$. Then, for any $\epsilon > 0$, the inequality*

$$
\begin{aligned}
& \sum_{j=1}^{q} m_f(r, H_j) + N_W(r,0) \leq (n+1)T_f(r) + \frac{n(n+1)}{2}(1+\epsilon)(c_f+\epsilon)T_f(r) \\
& + \frac{n(n+1)}{2}\epsilon \log r + O(\log T_f(r))
\end{aligned}
$$

holds for all $r \in (0, R)$ outside a set E with $\int_E \exp((c_f + \epsilon)T_f(r))dr < \infty$. Here W denotes the Wronskian of f.

Proof. By Theorem 5.1.9 and Lemma 5.1.11 we have

$$\begin{aligned}
& \sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \\
\leq & \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{1}{\|f(re^{i\theta}); H_j\|} \frac{d\theta}{2\pi} + N_W(r, 0) \\
\leq & (n+1)T_f(r) + \frac{n(n+1)}{2}(1+\epsilon)(c_f + \epsilon)T_f(r) \\
+ & \frac{n(n+1)}{2}\epsilon \log\ r + \mathcal{O}(\log\ T_f(r)) \ \ \|_E. \qquad (5.13)
\end{aligned}$$

□

We are able to reformulate the above theorem in terms of the truncated counting function by introducing the following Lemma.

Lemma 5.1.13 ([15]). *Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a non-degenerate holomorphic map. Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Then, for $0 < r < R$,*

$$\sum_{j=1}^{q} N_f(r, H_j) - N_W(r, 0) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j).$$

Proof. Without loss of generality, for each $z \in \mathbb{C}$, we assume that $< \mathbf{f}, \mathbf{a}_j >$ vanishes at z for $1 \leq j \leq q_1$ and $< \mathbf{f}, \mathbf{a}_j >$ does not vanish at z for $j > q_1$. There are integers $k_j \geq 0$ and nowhere vanishing holomorphic functions g_j in a neighborhood U of z such that

$$< \mathbf{f}, \mathbf{a}_j >= (\zeta - z)^{k_j} g_j$$

for $j = 1, ..., q$. Here $k_j = 0$ if $q_1 < j \leq q$. Also we can assume that $k_j \geq n$ if $1 \leq j \leq q_0$ and $1 \leq k_j < n$, where $0 \leq q_0 \leq q_1$. Using a property of the Wronskian, we have

$$W = W(f_0, ..., f_n) = CW(<\mathbf{f}, \mathbf{a}_{\mu(1)}>, ..., <\mathbf{f}, \mathbf{a}_{\mu(n+1)}>)$$

and

$$W(<\mathbf{f}, \mathbf{a}_{\mu(1)}>, ..., <\mathbf{f}, \mathbf{a}_{\mu(n+1)}>=\prod_{j=1}^{q_0}(\zeta - z)^{k_j - n}h(\zeta),$$

where $h(\zeta)$ is a holomorphic function defined on U and $\mu : \{0, 1, ..., n\} \to \{1, ..., q\}$ is an injective map such that $\mathbf{a}_{\mu(0)}, ..., \mathbf{a}_{\mu(n)}$ are linearly independent. Thus W vanishes at z with order at least $\sum_{j=1}^{q_0} k_j - q_0 n$. This, together with the definitions of $N_f(r, H_j)$, $N_W(r, 0)$, and $N_f^{(n)}(r, H_j)$, proves the lemma. $\square$

Theorem 5.1.14 ([16])**.** *Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained in any proper subspace, with $c_f < \infty$, where $c_f = c_{f,\omega_{FS}}$ and $0 < R \leq \infty$. Then, for any $\epsilon > 0$, the inequality*

$$\begin{aligned}(q-(n+1))T_f(r) &\leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) \\ &+ \frac{n(n+1)}{2}(1+\epsilon)(c_f+\epsilon)T_f(r) \\ &+ \frac{n(n+1)}{2}\epsilon \log r + O(\log T_f(r)) \quad \|_E\end{aligned}$$

holds for all $r \in (0, R)$ outside a set E with $\int_E \exp((c_f+\epsilon)T_f(r))dr < \infty$. Here W denotes the Wronskian of f.

5.2 Uniqueness Results for Holomorphic Mappings from Complex Discs into $\mathbb{P}^n(\mathbb{C})$

In this setting, we provide another uniqueness result using Theorem 5.1.12. The next Theorem is a brand new result and the second main result of this dissertation.

Theorem 5.2.1. *Let $f, g : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic curves with $c_f < +\infty$ and $c_g < +\infty$, where $0 < R \leq +\infty$. Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position. Assume that $f^{-1}(H_j) = g^{-1}(H_j), 1 \leq j \leq q$, and that there is a k such that $\cap_{j=1}^{k} f^{-1}(H_{i_j}) = \emptyset$. Let $A = \cup_{i=1}^{q} f^{-1}(H_i)$ and assume that for every point $z \in A$, $f(z) = g(z)$. Then we have the following conclusions.*

(a) If $q \geq 2(n+1)k$ and

$$q - (n+1) - \frac{n(n+1)}{2}(c_f + c_g) - \frac{2knq}{q - 2k + 2kn} > 0, \tag{5.14}$$

then $f \equiv g$.

(b) If $q < 2(n+1)k$ and

$$q > (n+1)(k+1) + \frac{n(n+1)}{2}(c_f + c_g), \tag{5.15}$$

then $f \equiv g$.

Note that (5.14) holds if

$$\begin{aligned} q &> k + \frac{1}{2}(n+1) + \frac{1}{2}n(n+1)(c_f + c_g) + \big(2k + kn(c_f + c_g)(n^2 - 1) \\ &+ \ (k + \frac{n+1}{2})^2 + (8kn + 4n^2 + 4n)(n+1)(c_f + c_g) + n^2(n+1)^2(c_f + c_g)^2\big)^{1/2}. \end{aligned}$$

Proof. Assume $f \neq g$ and $q \geq 2n$. We construct an auxiliary function as in [4] (see also [9]). By lemma 3.1 in [18], there exists a hyperplane

$$H_c = \{c_0x_0 + \cdots + c_nx_n = 0\}$$

such that $f^{-1}(H_c \cap H_j) = \emptyset$ and $g^{-1}(H_c \cap H_j) = \emptyset$ for $j = 1, ..., q$. Fix such an H_c. Let $H_j = \{a_{j0}x_0 + \cdots + a_{jn}x_n = 0\}$. Then define $(f, H_j) := \frac{a_{j0}f_0+\cdots+a_{jn}f_n}{c_0f_0+\cdots+c_nf_n}$, where $(f_1, ..., f_n)$ is a local reduced representation of f. We define (g, H_j) similarly. We arrange the hyperplanes $H_1, .., H_q$ into groups:

Group 1:

$$\frac{(f,H_1)}{(g,H_1)} \equiv \cdots \equiv \frac{(f,H_{k_1})}{(g,H_{k_1})} \not\equiv \frac{(f,H_{k_1+1})}{(g,H_{k_1+1})}$$

Group 2:

$$\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})} \not\equiv \frac{(f,H_{k_2+1})}{(g,H_{k_2+1})}$$

...

Group s:

$$\frac{(f,H_{k_{s-1}+1})}{(g,H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_s})}{(g,H_{k_s})}$$

where $k_s = q$. The assumption of "in general position" implies that the number of each group does not exceed n. For each $1 \leq i \leq q$, we set $\sigma(i) = i+n$ if $i+n \leq q$ and $\sigma(i) = i+n-q$ if $i+n > q$. Then it can be seen that σ is bijective and $|\sigma(i) - i| \geq n$ since $q \geq 2n$. Define P_i as follows:

$$P_i = (f, H_i)(g, H_{\sigma(i)}) - (g, H_i)(f, H_{\sigma(i)}).$$

Since $f \not\equiv g$, we have that $P_i \not\equiv 0$. We define the auxiliary function

$$P := \prod_{i=1}^{q} P_i. \tag{5.16}$$

We will use the following Lemma to study the the zero orders of P.

Lemma 5.2.2 ([17])**.** *Let $f, g : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic curves. Let $F = (f_0, .., f_n)$ be a reduced representation of f and G be a reduced representation of g. Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position with $q \geq 2n$. Let $0 < R \leq +\infty$. Assume the following:*

i.) $f^{-1}(H_j) = g^{-1}(H_j)$ *for* $1 \leq j \leq q$*;*

ii.) *Let $k \leq n$ be a positive integer such that $f^{-1}(\cap_{j=1}^{k+1} H_{i_j}) = \emptyset$ for $1 \leq i_1 \leq \cdots \leq i_{k+1} \leq q$;*

iii.) $f = g$ *on* $\cup_{i=1}^{q} f^{-1}(H_i)$.

Then the following holds on $\triangle(R)$:

(a) If $q \geq 2(n+1)k$, then

$$\nu_P(z) \geq \Big(\frac{q - 2k + 2kn}{2kn}\Big) \sum_{i=1}^{q} (\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)),$$

where $\nu_P(z)$ is the zero order of P at the point z and $\nu^n_{F(H_i)}(z) = min\{n, \nu_{F(H_i)}(z)\}$;

(b) for $q < 2(n+1)k$, we have

$$\nu_P(z) \geq \Big(\frac{q}{(n+1)k}\Big) \sum_{i=1}^{q} (\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)).$$

Proof. We first prove (a). Denote by $P_i := F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})$. Observe, if $z \notin \cup_{i=1}^q f^{-1}(H_i)$, then $\nu^n_{F(H_i)}(z) = 0$. In this case the lemma is true. Thus, we consider the case where $z \in \cup_{i=1}^q f^{-1}(H_i)$. Define $I := \{i : F(H_i)(z) = 0, 1 \leq i \leq q\}$ and denote by $t := \#(I)$ the number of elements of I. Then, $t \leq k$ since $\bigcup_{1 \leq i_1 < \ldots < i_{k+1} \leq q} f^{-1}(\cap_{j=1}^{k+1} H_{i_j}) = \emptyset$. If $i \in I$, then z is a zero of $F(H_i)$, and hence z is a zero of P_i with multiplicity at least $\min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\}$. We now define $\sigma^{-1}(I) = \{i : \sigma(i) \in I\}$. If $l \in \{1, \ldots q\} \backslash (I \cup \sigma^{-1}(I))$, then z is a zero of P_l with multiplicity of at least 1 because of the assumption $f(z) = g(z)$ on $\cup_{i=1}^q f^{-1}(H_i)$. Hence $\nu_{P_l} \geq 1$. Then

$$
\begin{aligned}
\nu_P(z) &\geq \sum_{i \in I, i \in \sigma^{-1}(I)} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + \sum_{l \in \{1,\ldots,q\} \backslash (I \cup \sigma^{-1}(I))} \nu_{P_l} \\
&\geq \sum_{i \in I, i \in \sigma^{-1}(I)} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + \sum_{l \in \{1,\ldots,q\} \backslash (I \cup \sigma^{-1}(I))} 1 \\
&\geq 2 \sum_{i \in I} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + q - \#(I \cup \sigma^{-1}(I)) \\
&= 2 \sum_{i \in I} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + q - 2t. \qquad (5.17)
\end{aligned}
$$

Since $t \leq k$, we have

$$\nu_P(z) \geq 2 \sum_{i \in I} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + q - 2k.$$

Using the fact that $\min\{a, b\} \geq \min\{a, n\} + \min\{b, n\} - n$ for all positive integers

a and b, it follows from the above inequality

$$\begin{aligned}\nu_P(z) &\geq 2\sum_{i\in I}[\min\{\nu_{F(H_i)}(z), n\} + \min\{\nu_{G(H_i)}(z), n\} - n] + q - 2k \\ &\geq 2\sum_{i\in I}[\min\{\nu_{F(H_i)}(z), n\} + \min\{\nu_{G(H_i)}(z), n\} - n\cdot\min\{\nu_{F(H_i)}(z), 1\}] \\ &\quad +q - 2k \\ &= 2\sum_{i\in I}(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z) - n\cdot\nu^1_{F(H_i)}(z)) + \frac{q-2k}{2k}\cdot 2k.\end{aligned}$$

Since $k \geq \sum_{i=1}^q \min\{1, \nu_{F(H_i)}(z)\} = \sum_{i=1}^q \nu^1_{F(H_i)}(z)$, we get

$$\begin{aligned}\nu_P(z) &\geq 2\sum_{i\in I}(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z) - n\cdot\nu^1_{F(H_i)}(z)) \\ &\quad +\frac{q-2k}{2k}\sum_{i=1}^q(\nu^1_{F(H_i)}(z) + \nu^1_{G(H_i)}(z)) \\ &\geq 2\sum_{i\in I}(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)) \\ &\quad +(\frac{q-2k}{2k} - n)\sum_{i=1}^q(\nu^1_{F(H_i)}(z) + \nu^1_{G(H_i)}(z)) \\ &\geq 2\sum_{i\in I}(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)) \\ &\quad +(\frac{q-2k}{2kn} - 1)\sum_{i=1}^q(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)) \\ &\geq (\frac{q-2k}{2kn} + 1)\sum_{i=1}^q(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)) \\ &\geq (\frac{q-2k+2kn}{2kn})\sum_{i=1}^q(\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)).\end{aligned}$$

We now prove (b). From (5.14), we have

$$\nu_P(z) \geq 2\sum_{i\in I}\min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + q - 2t.$$

Again, using the fact that $\min\{a,b\} \geq \min\{a,n\} + \min\{b,n\} - n$, we have

$$
\begin{aligned}
\nu_P(z) &\geq 2\sum_{i\in I} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} + q - 2k \\
&\geq \Big(2 - \frac{q}{(n+1)k}\Big) \sum_{i\in I} \min\{\nu_{F(H_i)}(z), \nu_{G(H_i)}(z)\} \\
&+ \frac{q}{(n+1)k} \sum_{i\in I} (\min\{\nu_{F(H_i)}(z), n\} + \min\{\nu_{G(H_i)}(z), 1\} - n) + q - 2t \\
&\geq \Big(2 - \frac{q}{(n+1)k}\Big) t - \frac{qnt}{(n+1)k} + q - 2t \\
&+ \frac{q}{(n+1)k} \sum_{i\in I} (\min\{\nu_{F(H_i)}(z), n\} + \min\{\nu_{G(H_i)}(z), n\}) \\
&\geq \Big(\frac{q}{(n+1)k}\Big) \sum_{i\in I} (\min\{\nu_{F(H_i)}(z), n\} + \min\{\nu_{G(H_i)}(z), n\}) \\
&= \Big(\frac{q}{(n+1)k}\Big) \sum_{i=1}^{q} (\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)).
\end{aligned}
$$

□

Continuing the proof, from the above Lemma we have,

$$\nu_P(z) \geq \Big(\frac{q - 2k + 2kn}{2kn}\Big) \sum_{i=1}^{q} (\nu^n_{F(H_i)}(z) + \nu^n_{G(H_i)}(z)),$$

where $\nu_P(z)$ is the zero order of P at the point z and $\nu^n_{F(H_i)}(z) = min\{n, \nu_{F(H_i)}(z)\}$.

In the case when $q \geq 2(n+1)k$, the inequality in part (a) of the Lemma 5.2.2, can be integrated on both sides to yield, for $0 < r < R$,

$$N_P(r, 0) \geq \Big(\frac{q - 2k + 2kn}{2kn}\Big) \sum_{i=1}^{q} (N^{(n)}_f(r, H_i) + N^{(n)}_g(r, H_i)).$$

Notice, by Jensen's formula and the definition of the characteristic function, we have that

$$N_P(r, 0) \leq q(T_f(r) + T_g(r)) + O(1).$$

Thus, the above Lemma implies that

$$\sum_{i=1}^{q}(N_f^{(n)}(r,H_i)+N_g^{(n)}(r,H_i)) \leq \frac{2knq}{q-2k+2kn}(T_f(r)+T_g(r))+O(1).$$

Applying Theorem 5.1.14 to f and g, and together with the Lemma above, we get

$$\begin{aligned}
& (q-(n+1)-\frac{n(n+1)}{2}(1+\epsilon)(c_f+c_g+\epsilon))(T_f(r)+T_g(r)) \\
\leq & \sum_{j=1}^{q}(N_f^{(n)}(r,H_j)+N_g^{(n)}(r,H_j))+O(\log T_f(r)+\log T_g(r)) \\
& +\frac{n(n+1)}{2}\epsilon \log r \quad \|_E.
\end{aligned}$$

Taking the above into consideration, we have

$$\begin{aligned}
& (q-(n+1)-\frac{n(n+1)}{2}(1+\epsilon)(c_f+c_g+\epsilon))(T_f(r)+T_g(r)) \\
\leq & \frac{2knq}{q-2k+2kn}(T_f(r)+T_g(r))+O(\log T_f(r)+\log T_g(r))+\frac{n(n+1)}{2}\epsilon \log r.
\end{aligned}$$

Thus

$$q-(n+1)-\frac{n(n+1)}{2}(c_f+c_g)-\frac{2knq}{q-2k+2kn} \leq 0,$$

which contradicts with the assumption of (5.14). Thus, the theorem is proved for the case when $q \geq 2(n+1)k$. In the case that $q < 2(n+1)k$, we use part (b) of Lemma 5.2.2 to get

$$N_P(r,0) \geq \left(\frac{q}{(n+1)k}\right)\sum_{i=1}^{q}(N_f^{(n)}(r,H_i)+N_g^{(n)}(r,H_i)).$$

Thus, using $N_P(r,0) \leq q(T_f(r)+T_g(r))+O(1)$, we get

$$\sum_{i=1}^{q}(N_f^{(n)}(r,H_i)+N_g^{(n)}(r,H_i)) \leq (n+1)k(T_f(r)+T_g(r))+O(1).$$

Taking the above into consideration, we have,

$$\begin{aligned}
& (q-(n+1)-\frac{n(n+1)}{2}(1+\epsilon)(c_f+c_g+\epsilon))(T_f(r)+T_g(r)) \\
\leq \ & (n+1)k(T_f(r)+T_g(r))+O(\log T_f(r)+\log T_g(r))+\frac{n(n+1)}{2}\epsilon \log r.
\end{aligned}$$

Thus

$$q \leq (n+1)(k+1)+\frac{n(n+1)}{2}(c_f+c_g),$$

which contradicts with the assumption in (5.16).

$\square$

Chapter 6

Conclusions

Due to the theory by Min Ru and Nessim Sibony, we are able to provide new uniqueness results that recover prior well known theorems. In turn, we are able to shorten the proofs as well. The arguments for the main results in Theorem 4.2.4 and Theorem 5.2.2 follow the same sense of spirit as the prior uniqueness results.

Min Ru and Nessim Sibony in ([16]) established another Second main theorem for holomorphic mappings from a disc into Abeilan varieties.

Theorem 6.0.1 (See Theorem 6.6 in [16])**.** *Let A be an Abelian variety, and let D be an ample divisor on A. Let $f : \triangle(R) \to A$ be a holomorphic map with Zariski dense image. Assume that $f \in \mathcal{E}_0$. Then there is a positive integer k_0 such that, for any $\epsilon > 0$,*

$$T_{f,D}(r) \leq N_f^{(k_0)}(r, D) + \epsilon T_{f,D}(r) + O(\log T_{f,D}(r)) + \epsilon \log \ r$$

holds for $r \in (0, R)$ except for a set E with $\int_E \exp(\epsilon T_{f,D}(r))dr < \infty$.

It is natural to try to apply the strategies in the proofs for the uniqueness theorems in this dissertation to acquire another unicity result. This would require Theorem 6.0.1 to hold for $k_0 = 1$. Indeed, following the work of Dulock-Ru [7], a path that could be utilized is extending the below result from K. Yamanoi's paper to the case of $\triangle(R)$.

Theorem 6.0.2 ([26])**.** *Let A be an Abelian variety and let $D \subset A$ be a reduced effective divisor. Let L be an ample line bundle on A. Let $f : \mathbb{C} \to A$ be a holomorphic curve such that the image of f is Zariski dense. Then we have*

$$T_f(r, D) \leq N_f^{(1)}(r, D) + \epsilon T_f(r, L) \quad ||_\epsilon,$$

for all $\epsilon > 0$.

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