THE CONSTRUCTION OF PHASE ANGLE AND GAIN LOCI BY $j\omega$-AXIS TRANSFORMATION

A Thesis

Presented to

the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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by

Emin M. Ozyazici

August 1968

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ABSTRACT

Some formulas were derived to construct the phase angle loci and the gain loci by the shifting technique. These formulas give the angular frequencies and corresponding gains for the phase angle loci and the angular frequencies for the gain loci for each shift of the $j\omega$ -axis.

The oscillatory conditions for the phase angle loci were determined by applying the Continued Fraction Expansion to the ratio of the polynomials obtained from the real and imaginary part of the phase angle loci equation. From the Continued Fraction Expansion, two sets of formulas were obtained; one set for $\phi = \nu\pi$, $(\nu = 0,1,2,3, \ldots)$ and one set for $\phi \neq \nu\pi$. Each set consists of two formulas to determine the angular frequency and gain. By these formulas the angular frequencies and corresponding gains for a particular value of phase angle ϕ are determined.

Two formulas were derived for construction of the gain loci by the shifting technique; one for a polynomial form and one for a factored form of the phase angle loci equation. By this formula the angular frequencies, for a particular value of gain K , are determined for each shift of the $j\omega$ -axis.

TABLE OF CONTENTS

CHAPTER PAGE

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 The Phase Angle Loci

The subject of this thesis is to develop a technique for constructing the phase angle and the gain loci of a system from its open loop transfer function.

The technique applied for constructing the phase angle loci was previously developed by C. F. Chen and C. Hsu (2) for the construction of the root locus of a system. They have compact formulas for obtaining the angular frequency ω and gain K respectively.

The shifting technique was also applied by V. Krishnan for root locus (9). In his approach, he expresses the numerator and denominator polynomials in Taylor series for determining the angular frequency ω and the gain K.

To construct the phase angle loci by this technique it is necessary to develop the oscillatory conditions for a phase angle loci equation. Thus first effort was to find equations for determining the $j\omega$-axis frequencies and corresponding gains from the open loop transfer function.

The procedure is the application of the invariance principle of the geometrical configuration of the phase angle loci in the complex plane under $j\omega$-axis transformation. The principle can be stated as follows: The geometrical shape of the curve of an algebraic equation in the complex plane does not change under a linear coordinate transformation (6).

The technique consists of a parallel shifting the $j\omega$ -axis by "γ" and then determining the $j\omega$ -axis frequencies and gains from the new phase angle loci equation. Thus a family of curves can be obtained for different values of the phase angle ϕ .

An equation was also developed to determine the new coefficients of the numerator and denominator polynomials of the open loop transfer function after $j\omega$ -axis transformation. This equation gives the coefficients of a polynomial by taking consecutive derivatives of the constant term of the new polynomial with respect to "γ" . This constant term is obtained from the given polynomial replacing "s" by "α" .

The most important information available from the phase angle loci is the relative effects of gain changes along the various root loci. It is also important that the family of constant gain curve (gain loci) is orthogonal to the phase angle loci. The phase angle loci provide the possibility of extending the locus plot to obtain more complete picture of system behavior.

The general concept associated with the phase angle loci is important, because the constant gain counters can often be sketched roughly to indicate the approximate location of corresponding points on the various root loci. The phase angle loci may be considered as a convenient means for re-shaping the root locus of systems to obtain more desirable performance characteristics.

A practical digital computer program can be used to calculate the phase angle loci and the gain loci from the equations derived in this thesis.

CHAPTER II

DETERMINATION OF PHASE ANGLE LOCI

2.1 Phase Angle Loci Equation

On the complex plane, for a given open loop transfer function, $K\,G(s)$ shown in Figure 1, a family of curves can be obtained for various values of ϕ from the following equation:

$$K\,G(s) = \frac{K\,P(s)}{Q(s)} = \mathrm{Exp}\,(\pm j\phi) = \mathrm{Cos}\,\phi \pm j\,\mathrm{Sin}\,\phi \qquad (2.1\text{-}1)$$

where

ϕ - Phase angle (constant)

K - Gain factor (gain constant)

G(s) - Forward transfer function of the system

P(s) - Numerator polynomial of transfer function, G(s)

Q(s) - Denominator polynomial of transfer function, G(s)

s - Complex variable.

Equation (2.1-1) is called the phase angle loci equation.

The phase angle loci have two important properties(1):

a) The phase angle loci are symmetrical with respect to the real axis.

b) The shape of the phase angle loci depends on the relative positions of the poles and zeros, and is independent of the axes of the s-plane.

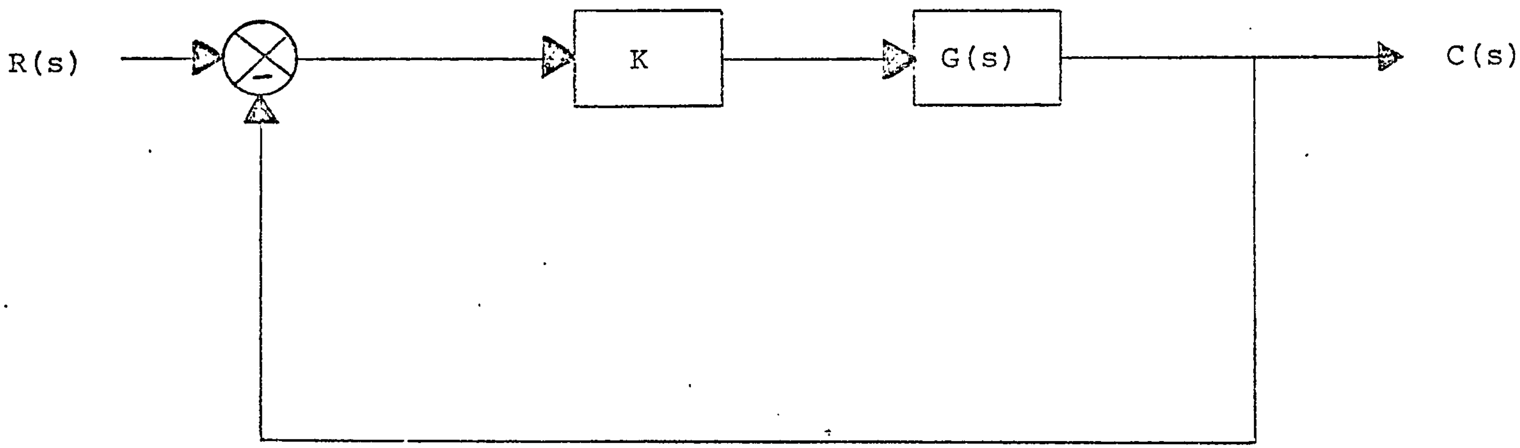


Figure 1. Block Diagram of System

Therefore, according to these two properties, a parallel shifting of the $j\omega$ -axis does not disturb the symmetry and the shape of the phase angle loci. If the amount of shift is "γ", the new complex variable s' in terms of the complex variable s and γ is:

$$s' = (s + \gamma) \tag{2.1-2}$$

Substituting $(s' + \gamma)$ for s in Eq. (2.1-1), one obtains the following result:

$$K\ G(s',\gamma) = \frac{K\ P(s',\gamma)}{Q(s',\gamma)} = \text{Cos}\ \phi \pm j\ \text{Sin}\ \phi \tag{2.1-3}$$

The coefficients of $P(s',\gamma)$ and $Q(s',\gamma)$ in Eq. (2.1-3) are functions of γ and they will be determined in Chapter II, Section 2.4, in convenient forms for any degree polynomials.

By substituting $j\omega$ for s' in Eq. (2.1-3), two polynomials are obtained as functions of the angular frequency, one from real part and one from imaginary part. Then the oscillatory condition can be determined from these two equations for any value of the phase angle ϕ .

The coefficients of these two phase angle loci polynomials are, in general, functions of shift "γ" , gain K and phase angle ϕ . For given values of γ and ϕ , these two polynomials must be satisfied by ω and K simultaneously. This means that the phase angle loci polynomials must have common roots for common values of K . Therefore, the possibilities for these polynomials to have common roots for common K's must be determined.

Common angular frequency ω and gain K can be determined by the Continued Fraction Expansion (8) applied to the ratio of the phase angle loci polynomials. The Continued Fraction Expansion ends prematurely when these two polynomials have common roots for common values of gain K. Thus the solutions for gain K and angular frequency ω can be obtained by forcing the Continued Fraction Expansion of the phase angle loci polynomials to end prematurely. Hence the phase angle loci for each particular value of phase angle ϕ_i can be constructed by this technique, first by shifting the $j\omega$ -axis, and then by determining gain K and corresponding angular frequency ω on the $j\omega$ -axis for each shift. The detail of the procedure for determining the oscillatory condition for a phase angle loci will be given in Chapter II, Section 2.2.

2.2 Determination of Oscillatory Conditions of Phase Angle Loci Equation

The detail for determining the gain K and the angular frequency ω by the Continued Fraction Expansion will be given for six degree phase angle loci polynomials. However, this technique can be applied similarly to the any degree phase angle loci polynomials.

The phase angle equation, Eq. (2.1-1), after shifting the $j\omega$ -axis by ω, is rewritten as follows:

$$K\ G(s') = \frac{K\ P(s')}{Q(s')} = \mathrm{Cos}\ \phi - j\ \mathrm{Sin}\ \phi \qquad (2.2\text{-}1)$$

where

$s = s' + \gamma$

Assuming that $Q(s')$, (or both $P(s')$ and $Q(s')$), is a six degree polynomial of s'. After substituting $j\omega$ for s' in Eq. (2.2-1), the following two phase angle loci polynomials, one from real part and one from imaginary part, are obtained:

$$F_1(\omega) = D_{11}\omega^6+D_{12}\omega^5+D_{13}\omega^4+D_{14}\omega^3+D_{15}\omega^2+D_{16}\omega+D_{17} = 0\ , \qquad (2.2\text{-}2)$$

$$F_2(\omega) = D_{21}\omega^6+D_{22}\omega^5+D_{23}\omega^4+D_{24}\omega^3+D_{25}\omega^2+D_{26}\omega+D_{27} = 0\ , \qquad (2.2\text{-}3)$$

where

$D_{11}, D_{12}, D_{13}, D_{14}, D_{15}, D_{16}, D_{17},$

and

$D_{21}, D_{22}, D_{23}, D_{24}, D_{25}, D_{26}, D_{27}$

are functions of γ, K and ϕ.

The Continued Fraction Expansion is:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}\omega^6 + D_{12}\omega^5 + D_{13}\omega^4 + D_{14}\omega^3 + D_{15}\omega^2 + D_{16}\omega + D_{17}}{D_{21}\omega^6 + D_{22}\omega^5 + D_{23}\omega^4 + D_{24}\omega^3 + D_{25}\omega^2 + D_{26}\omega + D_{27}} =$$

$$\frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{21}}{C_{31}}\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\omega + \frac{M_{61}}{C_{51}} + \cfrac{1}{\frac{C_{51}}{C_{71}}\omega + \frac{M_{81}}{C_{71}} + \cfrac{1}{\frac{C_{71}}{C_{91}}\omega + \frac{M_{10,1}\omega^2 + M_{10,2}\omega + M_{10,3}}{C_{91}\omega^2 + C_{92}\omega + C_{93}}}}}} \tag{2.2-4}$$

The last fraction, $\frac{M_{10,1}\omega^2 + M_{10,2}\omega + M_{10,3}}{C_{9,1}\omega^2 + C_{9,2}\omega + C_{9,3}}$ provides the oscillatory condition for $\phi = \nu\pi$ $(\nu = 0,1,2,3 \ldots)$ when Eq. (2.2-4) ends prematurely.

For the phase angles, $\phi \neq \nu\pi$, the Continued Fraction Expansion is carried on until the following fraction is obtained:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}\omega^6 + D_{12}\omega^5 + D_{13}\omega^4 + D_{14}\omega^3 + D_{15}\omega^2 + D_{16}\omega + D_{17}}{D_{21}\omega^6 + D_{22}\omega^5 + D_{23}\omega^4 + D_{24}\omega^3 + D_{25}\omega^2 + D_{26}\omega + D_{27}} =$$

$$= \frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{21}}{C_{31}}\,\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\,\omega + \frac{M_{61}}{C_{51}} + \cfrac{1}{\frac{C_{51}}{C_{71}}\,\omega + \frac{M_{81}}{C_{71}} + \cfrac{1}{\quad}}}}$$

$$\frac{C_{51}}{C_{71}}\,\omega + \frac{M_{81}}{C_{71}} + \cfrac{1}{\frac{C_{71}}{C_{91}}\,\omega + \frac{M_{10,1}}{C_{91}} + \cfrac{1}{\frac{C_{91}}{C_{11,1}}\,\omega + \frac{M_{12,1}}{C_{11,1}} + \frac{C_{11,1}\omega + C_{11,2}}{C_{13,1}}}}$$

(2.2-5)

The last fraction, $\frac{C_{11,1}\omega + C_{11,2}}{C_{13,1}}$, provides the oscillatory condition for the phase angle loci for $\phi \neq \nu\pi$, when it ends prematurely.

It is assumed that $D_{21} \neq 0$, D_{21} and D_{11} cannot be zero simultaneously. Only one of these (D_{21} or D_{11}) can be zero when $\phi = \nu\pi$. If D_{21} is zero the Continued Fraction Expansion must be applied to $\{F_2(\omega)/F_1(\omega)\}$ instead of $\{F_1(\omega)/F_2(\omega)\}$.

The last fraction of Eq. (2.1-4) and Eq. (2.2-5) provide the equations to calculate the gain K and angular frequency of the phase angle loci for $\phi = \nu\pi$ and $\phi \neq \nu\pi$ respectively.

Case No. 1: $\phi = \nu\pi$, Root Locus

For $\phi = \nu\pi$, $M_{10,1}$, $M_{10,3}$, $C_{9,2}$ are simultaneously zero. Then the last fraction of Eq. (2.1-4) is reduced to

$$\frac{M_{10,2}}{C_{71}\omega^2 + C_{73}} \tag{2.2-6}$$

By forcing $M_{10,2}$ to be zero, the Continued Fraction Expansion can be ended prematurely; this provides the following result:

$$C_{71}\omega^2 + C_{73} = 0 \tag{2.2-7}$$

$$\omega^2 = -\frac{C_{73}}{C_{71}} \tag{2.2-8}$$

$$M_{10,2} = 0 \tag{2.2-9}$$

$M_{10,2}$ is only function of K for a given value of γ . Thus the solution of Eq. (2.2-9) gives common values of K . Substitute K and γ in Eq. (2.2-8), the $j\omega$ -axis angular frequency is then determined.

Case No. 2: $\phi \neq \nu\pi$

The last fraction of Eq. (2.1-5) gives

$$\frac{C_{11,1}\omega + C_{11,2}}{C_{13,1}}$$

The Continued Fraction Expansion, Eq. (2.2-5), ends prematurely when $C_{13,1}$ is zero. Thus by forcing $C_{13,1}$ to be zero the following result is obtained:

$$C_{11,1}\omega + C_{11,2} = 0 \qquad (2.2\text{-}11)$$

$$\omega = -\frac{C_{11,2}}{C_{11,1}} \qquad (2.2\text{-}12)$$

$$C_{13,1} = 0 \qquad (2.2\text{-}13)$$

As in Case No. 1, $C_{13,1}$ is only function of K for a given γ. Thus the solution of $C_{13,1}$ gives common values of K.

Example

The phase angle loci equation is:

$$K\,G(s) + \frac{K(s + 10)}{s(s + 3)(s + 6)} = \text{Cos}\,\phi + j\,\text{Sin}\,\phi \qquad (2.2\text{-}14)$$

For $\gamma = 0$, $G(s') = G(s)$

In this example, the gain K and angular frequency ω will be determined for only $\phi = -30°$ and $\phi = 180°$.

$\phi = -30°$:

$\text{Cos}\,\phi = 0.866$

$\text{Sin}\,\phi = -.500$

$$K(s + 10) = (0.866 - j\,0.500)(s^3 + 9s^2 + 18s) \qquad (2.2\text{-}15)$$

Substituting $j\omega$ for s in Eq. (2.1-15), the following are obtained:

$$K(j\omega + 10) = (0.866 - j\,0.500)(-j\omega^3 - 9\omega^2 + 18j\omega) \qquad (2.2\text{-}16)$$

From real part of Eq. (2.1-16)

$$-\omega^3 \operatorname{Sin}\phi - 9\omega^2 \operatorname{Cos}\phi + 18\omega \operatorname{Sin}\phi - 10K = 0 \qquad (2.2\text{-}17)$$

From imaginary part of Eq. (2.1-16)

$$-\omega^3 \operatorname{Cos}\phi + 9\omega^2 \operatorname{Sin}\phi + (18 \operatorname{Cos}\phi - K) = 0 \qquad (2.2\text{-}18$$

Applying the Continued Fraction Expansion Technique to the ratio of Eq. (2.2-17) and Eq. (2.2-18), the gain K and angular frequency are obtained as follows:

$$\begin{array}{r|l} & \dfrac{0.5}{0.866} \\ \hline +0.866\omega^3 + 4.5\omega^2 - (15.6 + K)\omega & -0.5\omega^3 + 7.8\omega^2 + 9\omega - 10K \\ & -0.5\omega^3 - 2.6\omega^2 + (9+0.577K) \\ \hline & + 10.4\omega^2 - 0.577K\omega - 10K \end{array}$$

$$\begin{array}{r|l} & \dfrac{0.866}{10.4}\omega \\ \hline +10.4\omega^2 - 0.577K\omega - 10K & +0.866\omega^3 + 4.5\omega^2 - (15.6+K)\omega \\ & +0.866\omega^3 - 0.048K\omega^2 - 0.832K\omega \\ \hline & +(4.5+0.048K)\omega^2-(15.6+0.168K)\omega \end{array}$$

$$+\frac{(4.5 - 0.048K)}{10.4}$$

$$+10.4\omega^2 - 0.577K\omega - 10K \;\Big|\; \begin{array}{l} (4.5 + 0.048K)\omega^2 - (15.6 + 0.168K)\omega \\ (4.5 - 0.048K)\omega^2 - (0.25K + 2.68 \times 10^{-3}K^2)\omega - (4.32K + 4.6 \times 10^{-2}K^2) \end{array}$$

$$- (15.6 - 0.082K - 2.68 \times 10^{-3}K^2)\omega + (4.32K + 4.6 \times 10^{-2}K^2)$$

$$\frac{-10.4\omega}{(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)}$$

$$-(15.6-0.082K-2.68 \times 10^{-3}K^2)\omega+(4.32K+4.6 \times 10^{-2}K^2) \;\Big|\; \begin{array}{l} +10.4\omega^2 - 0.577K\omega - 10K \\ +10.4\omega^2 \quad \dfrac{10.4\omega(4.32K + 4.6 \times 10^{-2}K^2)\omega}{(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)} \end{array}$$

$$- \left(0.577K - \frac{10.4\,(4.32K + 4.6 \times 10^{-2}K^2)}{(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)}\right)\omega - 10K$$

$$\left\{0.577K - \frac{10.4\,(4.32K+4.6 \times 10^{-2}K^2)}{(15.6-0.082K-2.68 \times 10^{-3}K^2)}\right\} \Big/ (15.6 - 0.082K - 2.68 \times 10^{-3}K^2)$$

$$-(15.6-0.082K\;2.68 \times 10^{-3}K^2)\omega+(4.32K+4.6 \times 10^{-2}K^2) \;\Big|\; -\left\{0.577K - \frac{10.4\,(4.32K+4.6 \times 10^{-2}K^2)}{(15.6 - 0.082K-2.68 \times 10^{-3}K^2)}\right\}\omega - 10K$$

$$-\left\{0.577K - \frac{10.4\,(4.32K-4.6 \times 10^{-2}K^2)}{(15.6-0.082K-2.68 \times 10^{-3}K^2)}\right\}\omega + \frac{\left\{0.577K-\dfrac{10.4\,(4.32K+4.6\times10^{-2}K^2)}{(15.6-0.082K-2.68\times10^{-3}K^2)}\right\}(4.32K + 4.6 \times 10^{-2}K^2)}{(15.6-0.082K-2.68\times10^{-3}K^2)}$$

$$-10K - \frac{\left\{0.577K - \dfrac{10.4(4.32K+4.6\times10^{-2}K^2)}{(15.6-0.082K-2.68\times10^{-3}K^2)}\right\}(4.32K+4.6\times10^{-2}K^2)}{(15.6\;0.082K-2.68\times10^{-3}K^2)} \qquad (2.1\text{-}19)$$

Forcing the last term to be zero, one obtains

$$-10K(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)^2 + \left\{0.577K(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)-10.4(4.32K+4.6 \times 10^{-2}K^2)\right\}(4.32K + 4.6 \times 10^{-2}K^2) = 0 \qquad (2.1\text{-}20)$$

$$-(15.6-0.082K-2.68 \times 10^{-3}K^2)\omega + (4.32K-4.6 \times 10^{-2}K^2) = 0 \qquad (2.2\text{-}21)$$

or

$$\omega = \frac{(4.32K + 4.6 \times 10^{-2}K)}{(15.6 - 0.082K - 2.68 \times 10^{-3}K^2)} \qquad (2.2\text{-}22)$$

Solution of Eq. (2.1-21) gives

$$K = 13.4$$

Substituting K in Eq. (2.1-22), the angular frequency is obtained:

$$\omega = 4.7$$

<u>$\phi = 180°$: (Root Locus)</u>

$\sin \phi = 0$

$\cos \phi = -1$

For $\phi = 180°$, Eq. (2.2-17) and Eq. (2.2-18) give

$$-0\,\omega^3 + 9\omega^2 - 0\,\omega - 10\,K = 0 \qquad (2.2\text{-}23)$$

$$\omega^3 + 0\,\omega^2 - (18 + K)\omega + 0 = 0 \qquad (2.2\text{-}24)$$

Applying the Continued Fraction Expansion to the ratio of Eq. (2.2-23) and Eq. (2.2-24), the gain K and ω^2 are obtained as follows:

$$\begin{array}{r|l} & \quad \frac{-0}{1} \\ \cline{2-2} \omega^3 + 0\omega^2 - (18 + K)\omega + 0 & -0\omega^3 + 9\omega^2 - 0\omega - 10K \\ & -0\omega^3 + 0\omega^2 - 0\omega - 0 \\ \cline{2-2} & \qquad 9\omega^2 + 0\omega - 10K \end{array}$$

$$
\begin{array}{r|l}
 & \frac{1}{9}\,\omega \\
\cline{2-2}
9\omega^2 + 0\omega - 10K & \omega^3 + 0\omega^2 - (18 + K)\omega + 0 \\
 & \omega^3 + 0\omega^2 - \frac{10K}{9}\,\omega \\
\cline{2-2}
\end{array}
$$

$$0\omega^2 - \left\{(18 + K) - \frac{10K}{9}\right\}\omega + 0 \qquad (2.2\text{-}25)$$

Forcing last term, $\left\{(18 + K) - \frac{10K}{9}\right\}$, to be zero, the following result is obtained:

$$\left\{(18 - K) - \frac{10K}{9}\right\} = 0 \qquad (2.2\text{-}26)$$

$$9\,\omega^2 - 10K = 0$$

$$\omega^2 = \frac{10K}{9} \qquad (2.2\text{-}27)$$

Solution of Eq. (2.2-26) and Eq. (2.2-27) gives:

$$K = 162$$

$$\omega^2 = 180$$

2.2.1 Determination of M_{ij} and C_{ij} of the Continued Fraction in Eqs. (2.1-4) and (2.1-5)

To determine M_{ij} and C_{ij} in terms of D_{11}, D_{12}, D_{13} $D_{14}, D_{15}, D_{16}, D_{17}$ and $D_{21}, D_{22}, D_{23}, D_{24}, D_{25}, D_{26}, D_{27}$ for every value of ϕ, it is assumed that for $\phi = \nu\pi$:

$D_{11} = D_{13} = D_{15} = D_{17} = 0$ and

$D_{22} = D_{24} = D_{26} = 0$

The procedure is as follows:

$$
\begin{array}{r|l}
 & \dfrac{D_{11}}{D_{21}} \\
\hline
D_{21}\omega^6 + D_{22}\omega^5 + D_{23}\omega^4 + D_{24}\omega^3 + D_{25}\omega^2 + D_{26}\omega + D_{27} & D_{11}\omega^6 + D_{12}\omega^5 + D_{13}\omega^4 + D_{14}\omega^3 + D_{15}\omega^2 + D_{16}\omega + D_{17} \\
 & D_{11}\omega^6 + D_{22}\dfrac{D_{11}}{D_{21}}\omega^5 + D_{23}\dfrac{D_{11}}{D_{21}}\omega^4 + D_{24}\dfrac{D_{11}}{D_{21}}\omega^3 + D_{25}\dfrac{D_{11}}{D_{21}}\omega^2 + D_{26}\dfrac{D_{11}}{D_{21}}\omega + D_{27}\dfrac{D_{11}}{D_{21}} \\
\hline
 & +\left(D_{12}-D_{22}\dfrac{D_{11}}{D_{21}}\right)\omega^5 + \left(D_{13}-D_{23}\dfrac{D_{11}}{D_{21}}\right)\omega^4 + \left(D_{14}-D_{24}\dfrac{D_{11}}{D_{21}}\right)\omega^3 + \left(D_{15}-D_{25}\dfrac{D_{11}}{D_{21}}\right)\omega^2 + \left(D_{16}-D_{26}\dfrac{D_{11}}{D_{21}}\right)\omega + \left(D_{17}-D_{27}\dfrac{D_{11}}{D_{21}}\right)
\end{array}
$$

$\left(D_{12}-D_{22}\dfrac{D_{11}}{D_{12}}\right)$, $\left(D_{13}-D_{23}\dfrac{D_{11}}{D_{21}}\right)$, $\left(D_{14}-D_{24}\dfrac{D_{11}}{D_{21}}\right)$, $\left(D_{15}-D_{25}\dfrac{D_{11}}{D_{21}}\right)$, $\left(D_{16}-D_{26}\dfrac{D_{11}}{D_{21}}\right)$ and $\left(D_{17}-D_{27}\dfrac{D_{11}}{D_{21}}\right)$ are indicated by $C_{31}, C_{32}, C_{33}, C_{34}, C_{35}, C_{36}$ and C_{37} respectively.

$$
\begin{array}{r|l}
 & \dfrac{D_{21}}{C_{31}}\omega \\
\hline
C_{31}\omega^5 + C_{32}\omega^4 + C_{33}\omega^3 + C_{34}\omega^2 + C_{35}\omega + C_{36} & D_{21}\omega^6 + D_{22}\omega^5 + D_{23}\omega^4 + D_{24}\omega^3 + D_{25}\omega^2 + D_{26}\omega + D_{27} \\
 & D_{21}\omega^6 + C_{32}\dfrac{D_{21}}{C_{31}}\omega^5 + C_{33}\dfrac{D_{21}}{C_{31}}\omega^4 + C_{34}\dfrac{D_{21}}{C_{31}}\omega^3 + C_{35}\dfrac{D_{21}}{C_{31}}\omega^2 + C_{36}\dfrac{D_{21}}{C_{31}}\omega \\
\hline
 & +\left(D_{22}-C_{32}\dfrac{D_{21}}{C_{31}}\right)\omega^5 + \left(D_{23}-C_{33}\dfrac{D_{21}}{C_{31}}\right)\omega^4 + \left(D_{24}-C_{34}\dfrac{D_{21}}{C_{31}}\right)\omega^3 + \left(D_{25}-C_{35}\dfrac{D_{21}}{C_{31}}\right)\omega^2 + \left(D_{26}-C_{36}\dfrac{D_{21}}{C_{31}}\right)\omega + D_{27}
\end{array}
$$

$(D_{22}-C_{32}\frac{D_{21}}{C_{31}})$, $(D_{23}-C_{33}\frac{D_{21}}{C_{31}})$, $(D_{24}-C_{34}\frac{D_{21}}{C_{31}})$, $(D_{25}-C_{35}\frac{D_{21}}{C_{31}})$, $(D_{26}-C_{36}\frac{D_{21}}{C_{31}})$ and $(D_{27}-C_{37}\frac{D_{21}}{C_{31}})$ are indicated by $M_{41}, M_{42}, M_{43}, M_{44}, M_{45}$ and M_{46} respectively.

$$
\begin{array}{r|l}
 & \frac{M_{41}}{C_{31}} \\
\hline
C_{31}\omega^5 + C_{32}\omega^4 + C_{33}\omega^3 + C_{34}\omega^2 + C_{35}\omega + C_{36} & M_{41}\omega^5 + M_{42}\omega^4 + M_{43}\omega^3 + M_{44}\omega^2 + M_{45}\omega + M_{46} \\
 & M_{41}\omega^5 + C_{32}\frac{M_{41}}{C_{31}}\omega^4 + C_{33}\frac{M_{41}}{C_{31}}\omega^3 + C_{34}\frac{M_{41}}{C_{31}}\omega^2 + C_{35}\frac{M_{41}}{C_{31}}\omega + C_{36}\frac{M_{41}}{C_{31}} \\
\hline
 & +(M_{42}-C_{32}\frac{M_{41}}{C_{31}})\omega^4+(M_{43}-C_{33}\frac{M_{41}}{C_{31}})\omega^3+(M_{44}-C_{34}\frac{M_{41}}{C_{31}})\omega^2+(M_{45}-C_{35}\frac{M_{41}}{C_{31}})\omega+(M_{46}-C_{36}\frac{M_{41}}{C_{31}})
\end{array}
$$

$(M_{42}-C_{32}\frac{M_{41}}{C_{31}})$, $(M_{43}-C_{33}\frac{M_{41}}{C_{31}})$, $(M_{44}-C_{34}\frac{M_{41}}{C_{31}})$, $(M_{45}-C_{35}\frac{M_{41}}{C_{31}})$ and $(M_{46}-C_{36}\frac{M_{46}}{C_{31}})$ are indicated by $C_{51}, C_{52}, C_{53}, C_{54}$ and C_{55}

$$
\begin{array}{r|l}
 & \frac{C_{31}}{C_{51}}\omega \\
\hline
C_{51}\omega^4 + C_{52}\omega^3 + C_{53}\omega^2 + C_{54}\omega + C_{55} & C_{31}\omega^5 + C_{32}\omega^4 + C_{33}\omega^3 + C_{34}\omega^2 + C_{35}\omega + C_{36} \\
 & C_{31}\omega^5 + C_{52}\frac{C_{31}}{C_{51}}\omega^4 + C_{53}\frac{C_{31}}{C_{51}}\omega^3 + C_{54}\frac{C_{31}}{C_{51}}\omega^2 + C_{55}\frac{C_{31}}{C_{51}}\omega + C_{56}\frac{C_{31}}{C_{51}} \\
\hline
 & +(C_{32}-C_{52}\frac{C_{31}}{C_{51}})\omega^4+(C_{33}-C_{53}\frac{C_{31}}{C_{51}})\omega^3+(C_{34}-C_{54}\frac{C_{31}}{C_{51}})\omega^2+(C_{35}-C_{55}\frac{C_{31}}{C_{51}})\omega+(C_{36}-C_{56}\frac{C_{31}}{C_{51}})
\end{array}
$$

$(C_{32}-C_{52}\frac{C_{31}}{C_{51}})$, $(C_{33}-C_{53}\frac{C_{31}}{C_{51}})$, $(C_{34}-C_{54}\frac{C_{31}}{C_{51}})$, $(C_{35}-C_{55}\frac{C_{31}}{C_{51}})$ and $(C_{36}-C_{56}\frac{C_{31}}{C_{51}})$ are indicated by $M_{61}, M_{62}, M_{63}, M_{64}$ and M_{65} respectively.

$$\begin{array}{r|l}
 & \frac{M_{61}}{C_{51}} \\ \hline
C_{51}\omega^4 + C_{52}\omega^3 + C_{53}\omega^2 + C_{54}\omega + C_{55} & M_{61}\omega^4 + M_{62}\omega^3 + M_{63}\omega^2 + M_{64}\omega + M_{65} \\
 & M_{61}\omega^4 + C_{52}\frac{M_{61}}{C_{51}}\omega^3 + C_{53}\frac{M_{61}}{C_{51}}\omega^2 + C_{54}\frac{M_{61}}{C_{51}}\omega + C_{55}\frac{M_{61}}{C_{51}} \\ \hline
 & +(M_{62}-C_{52}\frac{M_{61}}{C_{51}})\omega^3+(M_{63}-C_{53}\frac{M_{61}}{C_{51}})\omega^2+(M_{64}-C_{54}\frac{M_{61}}{C_{51}})\omega+(M_{65}-C_{55}\frac{M_{61}}{C_{51}})
\end{array}$$

$(M_{62}-C_{52}\frac{M_{61}}{C_{51}})$, $(M_{63}-C_{53}\frac{M_{61}}{C_{51}})$, $(M_{64}-C_{54}\frac{M_{61}}{C_{51}})$ and $(M_{65}-C_{55}\frac{M_{61}}{C_{51}})$ are indicated by C_{71}, C_{72}, C_{73} and C_{74} .

$$\begin{array}{r|l}
 & \frac{C_{51}}{C_{71}}\omega \\ \hline
C_{71}\omega^3 + C_{72}\omega^2 + C_{73}\omega + C_{74} & C_{51}\omega^4 + C_{52}\omega^3 + C_{53}\omega^2 + C_{54}\omega + C_{55} \\
 & C_{51}\omega^4 + C_{72}\frac{C_{51}}{C_{71}}\omega^3 + C_{73}\frac{C_{51}}{C_{71}}\omega^2 + C_{74}\frac{C_{51}}{C_{71}}\omega + C_{75}\frac{C_{51}}{C_{71}} \\ \hline
 & +(C_{52}-C_{72}\frac{C_{51}}{C_{71}})\omega^3 + (C_{53}-C_{73}\frac{C_{51}}{C_{71}})\omega^2 + (C_{54}-C_{74}\frac{C_{51}}{C_{71}})\omega + (C_{55} - C_{75}\frac{C_{51}}{C_{71}})
\end{array}$$

$(C_{52}-C_{72}\frac{C_{51}}{C_{71}})$, $(C_{53}-C_{73}\frac{C_{51}}{C_{71}})$, $(C_{54}-C_{74}\frac{C_{51}}{C_{71}})$ and C_{55} are indicated by M_{81}, M_{82}, M_{83} and M_{84} respectively.

$$
\begin{array}{r|l}
 & \frac{M_{81}}{C_{71}} \\
\cline{2-2}
C_{71}\omega^3 + C_{72}\omega^2 + C_{73}\omega + C_{74} & M_{81}\omega^3 + M_{82}\omega^2 + M_{83}\omega + M_{84} \\
 & M_{81}\omega^3 + C_{72}\frac{M_{81}}{C_{71}}\omega^2 + C_{73}\frac{M_{81}}{C_{71}}\omega + C_{74}\frac{M_{81}}{C_{71}} \\
\cline{2-2}
 & + (M_{82} - C_{72}\frac{M_{81}}{C_{71}})\omega^2 + (M_{83} - C_{73}\frac{M_{81}}{C_{71}})\omega + (M_{84} - C_{74}\frac{M_{81}}{C_{71}})
\end{array}
$$

$(M_{82} - C_{72}\frac{M_{81}}{C_{71}})$, $(M_{83} - C_{73}\frac{M_{81}}{C_{71}})$ and $(M_{84} - C_{74}\frac{M_{81}}{C_{71}})$ are indicated by C_{91}, C_{92} and C_{93} .

$$
\begin{array}{r|l}
 & \frac{C_{71}}{C_{91}}\omega \\
\cline{2-2}
C_{91}\omega^2 + C_{92}\omega + C_{93} & C_{71}\omega^3 + C_{72}\omega^2 + C_{73}\omega + C_{74} \\
 & C_{71}\omega^3 + C_{92}\frac{C_{71}}{C_{91}}\omega^2 + C_{93}\frac{C_{71}}{C_{91}}\omega \\
\cline{2-2}
 & + (C_{72} - C_{92}\frac{C_{71}}{C_{91}})\omega^2 + (C_{73} - C_{93} - \frac{C_{71}}{C_{91}})\omega + C_{74}
\end{array}
$$

$(C_{72} - C_{92}\frac{C_{71}}{C_{91}})$, $(C_{73} - C_{93}\frac{C_{71}}{C_{91}})$ and C_{74} are indicated by $M_{10,1}, M_{10,2}$ and $M_{10,3}$ respectively.

$$
\begin{array}{r|l}
 & \dfrac{M_{10,1}}{C_{91}} \\
\cline{2-2}
C_{91}\omega^2 + C_{92}\omega + C_{93} & M_{10,1}\omega^2 + M_{10,2}\omega + M_{10,3} \\
 & M_{10,1}\omega^2 + C_{92}\dfrac{M_{10,1}}{C_{91}}\omega + C_{93}\dfrac{M_{10,1}}{C_{91}} \\
\cline{2-2}
 & +\left(M_{10,2} - C_{92}\dfrac{M_{10,1}}{C_{91}}\right)\omega + \left(M_{10,3} - C_{93}\dfrac{M_{10,1}}{C_{91}}\right)
\end{array}
$$

$\left(M_{10,2} - C_{92}\dfrac{M_{10,1}}{C_{91}}\right)$ and $\left(M_{10,3} - C_{93}\dfrac{M_{10,1}}{C_{91}}\right)$ are indicated by $C_{11,1}$ and $C_{11,2}$ respectively.

$$
\begin{array}{r|l}
 & \dfrac{C_{91}}{C_{11,1}}\omega \\
\cline{2-2}
C_{11,1}\omega + C_{11,2} & C_{91}\omega^2 + C_{92}\omega + C_{93} \\
 & C_{91}\omega^2 + C_{11,2}\dfrac{C_{91}}{C_{11,1}}\omega \\
\cline{2-2}
 & + \left(C_{92} - C_{11,2}\dfrac{C_{91}}{C_{11,1}}\right)\omega + C_{93}
\end{array}
$$

$\left(C_{92} - C_{11,2}\dfrac{C_{91}}{C_{11,1}}\right)$ and C_{93} are indicated by $M_{12,1}$ and $M_{12,2}$ respectively.

$$
\begin{array}{r|l}
 & \dfrac{M_{12,1}}{C_{11,1}} \\
\hline
C_{11,1}\omega + C_{11,2} & M_{12,1}\omega + M_{12,1} \\
 & M_{12,1}\omega + C_{11,2}\,\dfrac{M_{12,1}}{C_{11,1}} \\
\hline
 & \qquad + \left(M_{12,1} - C_{11,2}\,\dfrac{M_{12,1}}{C_{11,1}}\right)
\end{array}
$$

$\left(M_{12,1} - C_{11,2}\,\dfrac{M_{12,1}}{C_{11,1}}\right)$ is indicated by $C_{13,1}$.

To keep the sequence of the rows, even numbers for "i" $(i = 2,4,6, \ldots)$ are assigned to M_{ij} and odd numbers for "i" $(i = 1,3,5,7, \ldots)$ are assigned to C_{ij} .

Similarly the Continued Fraction Expansion is applied to 5 degree and 4 degree phase angle loci polynomials. The results are given in Appendix A and Appendix B.

From the results of the Continued Fraction Expansion of 6 degree, 5 degree and 4 degree phase angle loci polynomials, the following equations for angular frequency ω and gain K can be found by a mathematical intuition:

Two cases must be considered:

a) n is an even number

b) n is an odd number

where n is the degree of the phase angle loci polynomials.

Case No. 1: n is an even number

For $\phi = \nu\pi$, $(\nu = 0,1,2,3, \ldots)$

$$\omega^2 = - \frac{C_{(2n-3),3}}{C_{(2n-3),1}} \tag{2.2-28}$$

$$M_{(2n-2),2} = 0 \tag{2.2-29}$$

For $\phi \neq \nu\pi$

$$\omega = - \frac{C_{(2n-1),2}}{C_{(2n-1),1}} \tag{2.2-30}$$

$$C_{(2n+1),1} = 0 \tag{2.2-31}$$

Case No. 2: n is an odd number

For $\phi = \nu\pi$, $(\nu = 0,1,2,3, \ldots)$

$$\omega^2 = - \frac{C_{(2n-3),3}}{C_{(2n-3),1}} \tag{2.2-32}$$

$$M_{(2n-2),2} = 0 \tag{2.2-33}$$

For $\phi \neq \nu\pi$

$$\omega = - \frac{C_{(2n-1),2}}{C_{(2n-1),1}} \tag{2.2-34}$$

$$C_{(2n+1),1} = 0 \tag{2.2-35}$$

Solution of Eq. (2.2-29) or (2.2-32) and (2.2-31) or (2.2-35), for $\phi = \nu\pi$ and $\phi \neq \nu\pi$ respectively, determines the values of K's, as many as the degree of polynomials. Only some of these K's which give positive real values for ω are valid values of K .

2.2.2 General Formulas for Determining M_{ij} and C_{ij}

The general formulas for determining M_{ij} and C_{ij} in terms of $D_{11}, D_{12}, D_{13}, \dots D_{1n}$ and $D_{21}, D_{22}, D_{23}, \dots D_{2n}$ are written by an intuition as follows:

$$M_{ij} = C_{(i-3),(j+1)} - C_{(i-3),1} \frac{C_{(i-1),(j+1)}}{C_{(i-1),1}} \qquad (2.2\text{-}36)$$

where

$i = 4,6,8,10, \dots,$

$j = 1,2,3,4,5, \dots$

and

$$M_{21} = D_{11},\ M_{22} = D_{12},\ M_{23} = D_{13},\ \dots\dots\dots\ M_{2j} = D_{2j}$$

$$M_{41} = (D_{22} - D_{21} \frac{C_{32}}{C_{31}}), \dots,\ M_{4j} = (D_{2,(j+1)} - D_{21} \frac{C_{3,(j+1)}}{C_{3,1}}$$

$$C_{ij} = M_{(i-1),(j+1)} - C_{(i-2),(j+1)} \frac{M_{(i-1),1}}{C_{(i-2),1}} \qquad (2.2\text{-}37)$$

where

$i = 3,5,7,9, \dots$

$j = 1,2,3,4,5, \dots$

and

$$C_{11} = D_{21},\ C_{12} = D_{22},\ C_{13} = D_{23},\ \dots\dots,\ C_{1j} = D_{2j}$$

$$C_{31} = (D_{12} - D_{11}\frac{D_{22}}{D_{21}}), \ldots, C_{3j} = D_{1,(j+1)} - D_{11}\frac{D_{2,(j+1)}}{D_{21}}$$

Equations (2.2-36) and (2.2-37) are applicable to a computer programming.

Example No. 2

The result, (2.2-32), (2.2-33), (2.2-34) and (2.2-35), will be applied to the same example given in (2.2-14). For convenience it is rewritten again:

$$K\ G(s)\ \frac{K(s + 10)}{s(s+3)(s+6)}\ \ \text{Cos}\ \phi + j\ \text{Sin}\ \phi \tag{2.2-38}$$

For ϕ 30°, the phase angle loci polynomials are:

$$-0.5\ \omega^3 - 7.8\ \omega^2 + 9\ \omega - 10K = 0 \tag{2.2-39}$$

$$-0.866\ \omega^3 + 4.5\ \omega^2 + (15.6 - K)\omega = 0 \tag{2.2-40}$$

Therefore

$$D_{11} = -\ 0.5 \qquad D_{21} = 0.866$$

$$D_{12} = 7.8 \qquad D_{22} = 4.5$$

$$D_{13} = 9 \qquad D_{23} = -\ (15.6 + K)$$

$$D_{14} = -\ 10K \qquad D_{24} = 0$$

From Eqs. (2.2-34) and (2.2-35), one obtains:

$$\omega = -\frac{C_{5,2}}{C_{5,1}} \tag{2.2-41}$$

$$C_{7,1} = 0$$

Then, from Eq. (2.2-36) and Eq. (2.2-37), M's and C's are obtained as follows:

$M_{21} = -\ 0.5$ $\qquad C_{11} = 0.866$

$M_{22} = 7.8$ $\qquad C_{12} = 4.5$

$M_{23} = 9$ $\qquad C_{13} = -\ (15.6 + K)$

$M_{24} = -\ 10K$ $\qquad C_{14} = 0$

$M_{41} = (4.5 - 0.048K)$ $\qquad C_{31} = 10.4$

$M_{42} = -\ (15.6 + 0.168K)$ $\qquad C_{32} = -\ 0.577K$

$C_{33} = -\ 10K$

$$M_{61} = \left\{-0.577 + 10.4\ \frac{(4.33K + 4.6 * 10^{-2}\ K^2)}{(15.6-0.082K-2.66*10^{-3}K^2)}\right\}$$

$M_{62} = -\ 10K$ $\qquad C_{51} = -(15.6-0.082-2.66*10^3 10^2)$

$C_{52} = +(4.33 + 4.6 * 10^{-2}K^2)$

$$C_{71} = -10K-(4.33+4.6*10^{-2}K^2)\ \frac{\left\{0.577-10.4\ \frac{(4.33+4.6*10^{-2}K^2)}{(15.6-0.082K-2.66*10^{-3}K^2)}\right\}}{(15.6-0.082K-2.66*10^{-3}K^2)}$$

Substituting C_{51} and C_{52} in Eq. (2.2-41), and solving C_{71} the same results are obtained:

$$\omega = \frac{(4.33K + 4.6*10^{-2}K^2)}{(15.6 - 0.082K - 2.66*10^{-3}K^2)}$$

$$-10K(15.6-0.082K-2.66*10^{-3}K^2)^2-(4.33K+4.6*10^{-2}K^2)\left\{0.577(15.6 - -0.082K-2.66x10^{-3}K^2)-10.4(4.33+4.6x10^{-2}K^2)\right\} = 0$$

$\phi = 180°$

Phase angle polynomials are:

$$- 0\ \omega^3 + 9\ \omega^2 - 0\omega - 10K = 0 \qquad (2.2\text{-}43)$$

$$\omega^3 - 0\ \omega^2 - (18 + K)\omega - 0 = 0 \qquad (2.2\text{-}44)$$

From Eqs. (2.2-32) and (2.2-33), one obtains:

$$\omega^2 = - \frac{C_{33}}{C_{31}} \qquad (2.2\text{-}45)$$

$$M_{42} = 0$$

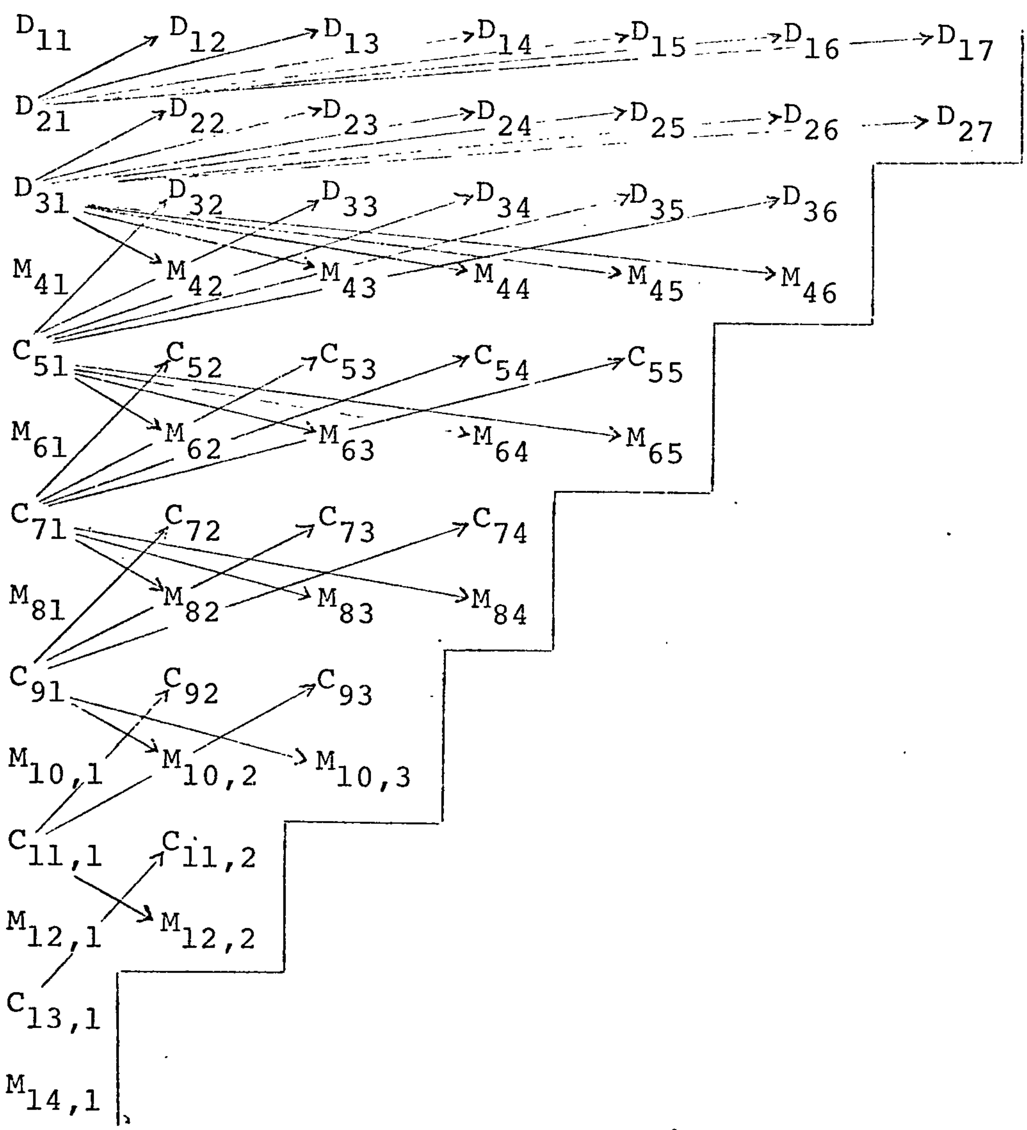
$D_{11} = 0$	$D_{21} = 1$
$D_{12} = 9$	$D_{22} = 0$
$D_{13} = 0$	$D_{23} = -(18 - K)$
$D_{14} = - 10K$	$D_{24} = 0$
$C_{31} = 9$	$M_{42} = -(18 - K) - \frac{10K}{9} = 0$
$C_{32} = 0$	
$C_{33} = - 10K$	

Substituting C_{33} and C_{31} and solving M_{42}, one obtains the same results as in Eq. (2.2-26) and Eq. (2.2-27):

$$K = 168, \qquad \omega^2 = 180$$

2.2.3 Phase Angle Loci Array--Determination of M_{ij} and C_{ij} by an Array

M_{ij} and C_{ij} can also be determined with a typical array. The array for 6 degree phase angle polynomials, Eq. (2.2-2) and Eq. (2.2-3), is as follows:



The arrows indicate the direction of the multiplication, sign of the product of the components of the rows involved to the procedure. The components at the beginning and at the end of an arrow are multiplied, next the product of other two corresponding components is substracted. Then the difference of these two products is divided by the component at which the arrow starts.

Two examples for M_{ij} and C_{ij} are given as follows:

$$M_{61} = C_{32} - C_{52} \frac{C_{31}}{C_{51}}$$

$$M_{62} = C_{33} - C_{53} \frac{C_{31}}{C_{51}}$$

$$C_{71} = M_{62} - M_{61} \frac{C_{52}}{C_{51}}$$

$$C_{72} = M_{63} - M_{61} \frac{C_{53}}{C_{51}}$$

2.3 Phase Angle Loci on the Real-Axis

The phase angle loci may have some points on the real axis. Since the frequency ω is zero on the real-axis, these points can be determined by equating the constant terms of Eq. (2.2-2) and Eq. (2.2-3), which are designated as D_{17} and D_{27} . Thus a simultaneous solution of D_{17} and D_{27} for a particular phase angle gives the points on the real axis for each shift, γ .

2.4 Linear Transformation of $j\omega$ -Axis

The open loop transfer function is rewritten as follows:

$$K\,G(s) = \frac{P(s)}{Q(s)} = \frac{K(a_o s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n)}{(b_o s^m + b_1 s^{m-1} + \ldots + b_{m-1} + b_m)} = e^{\pm j\phi}$$

(2.4-1)

The $j\omega$ -axis is translated parallel by "γ" from its original position as seen in Figure 2. Thus the origin 0 will be shifted to its new origin 0' on the σ -real axis. The new s'-domain can be expressed in terms of the previous s-domain and the amount of shift, "γ".

The coefficients of the numerator and the denominator polynomial can be found for each shift. A relation for finding the new coefficients may be derived as follows:

$$s' = \sigma + j\omega \qquad (2.4\text{-}2)$$

After shifting the $j\omega$ -axis by γ , s is:

$$s = (\sigma + \gamma) + j\omega = s' + \gamma \qquad (2.4\text{-}3)$$

Then substituting s' for $(\sigma + j\omega)$, the s is expressed in terms of s' and γ .

In Eq. (2.4-4), (s) is the complex variable of initial s-domain, (s') is the complex variable of the new s-domain, and γ is the amount of a parallel shift of the $j\omega$ -axis.

The powers of s in terms of s' and γ are given as follows:

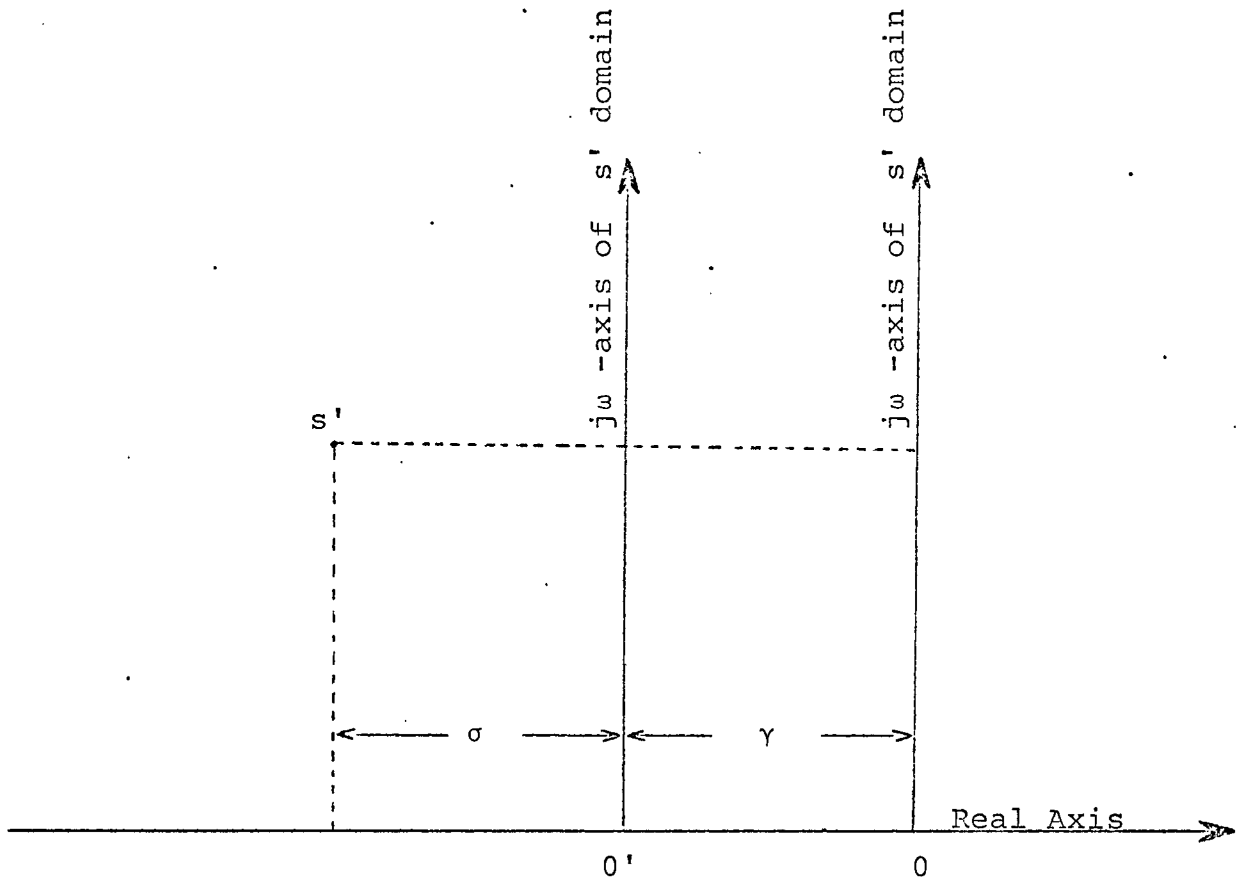


Figure 2. Complex s and s' -plane

$$s = s' + \gamma$$

$$s^2 = s'^2 + 2s'\gamma + \gamma^2$$

$$s^3 = s'^3 + 3s'^2\gamma + 3s'\gamma^2 + \gamma^3$$

$$s^4 = s'^4 + 4s'^3\gamma + 6s'^2\gamma^2 + 4s'\gamma^3 + \gamma^4$$

$$s^n = (s' + \gamma) = \sum_{i=0}^{n} \binom{n}{i} s'^i \gamma^{n-1} \qquad (2.4\text{-}4)$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Substituting $(s' + \gamma)$ for s in the numerator and denominator polynomials of Eq. (2.4-1), one obtains the following results:

$$P(s') = A_o s'^n + A_1 s'^{n-1} + \ldots\ldots + A_{n-1}s' + A_n \qquad (2.4\text{-}5)$$

$$Q(s') = B_o s'^m + B_1 s'^{m-1} + \ldots\ldots + B_{m-1}s' + B_m \qquad (2.4\text{-}6)$$

where

$A_o, A_1, \ldots, A_n$ and $B_o, B_1, \ldots, B_m$ are functions of γ and given as follows:

$$A_n = a_o\gamma^n + a_1\gamma^{n-1} + \ldots\ldots + a_{n-1}\gamma + a_n$$

$$A_{n-1} = na_o\gamma^{n-1} + (n-1)a_1\gamma^{n-2} + \ldots + a_{n-1} \qquad (2.4\text{-}7)$$

$$A_1 = na_o\gamma + a_1$$

$$A_o = a_o$$

and similarly

$$\left.\begin{aligned} B_m &= b_o\gamma^m + b_1\gamma^{m-1} + \ldots + b_{m-1}\gamma + b_m \\ B_{m-1} &= mb_o\gamma^{m-1} + (m-1)b_1\gamma^{m-2} + \ldots + b_{m-1} \\ B_1 &= mb_o\gamma \\ B_o &= b_o \end{aligned}\right\} \qquad (2.4\text{-}8)$$

The coefficients of Eq. (2.4-5) and Eq. (2.4-6) can be determined by taking consecutive derivatives of A_n and B_m with respect to γ. The General formulas are given as follows:

$$A_{n-i} = \frac{1}{i!} \frac{d^i}{d\gamma^i} (A_n) \qquad (2.4\text{-}9)$$

$$B_{m-i} = \frac{1}{i!} \cdot \frac{d^i}{d\gamma^i} (B_m) \qquad (2.4\text{-}10)$$

CHAPTER III

DETERMINATION OF GAIN LOCI

3.1 Derivation of the Gain Loci Formula

The gain loci are a family of curves which is orthogonal to the family of the phase angle loci curves. The gain loci provide important information about the relative effect of gain changes along the various root loci. The gain loci can be constructed by the shifting technique.

The procedure for constructing the gain loci by shifting technique is as follows:

The open loop transfer function after shifting by γ is:

$$K\,G(s') = \frac{K\,P(s')}{Q\,(s')} = \frac{K(A_o s'^n + A_1 s'^{n-1} + \,.\,.\, + A_n)}{(B_o s'^m + B_1 s'^{m-1} + \,.\,.\, + B_m)} =$$

$$= \mathrm{Exp}\ (\pm j\phi) = \mathrm{Cos}\ \phi \pm j\ \mathrm{Sin}\ \phi \qquad (3.1\text{-}1)$$

where

$s' = s - \gamma$, and

$A_o, A_1, A_2, \ldots$ and $B_o, B_1, B_2, \ldots$ are functions of a shift, γ .

Since the gain loci curves are orthogonal to the phase angle loci curves, the gain loci also have the same properties as the phase angle loci (Chapter II, Section 2.1, page 4).

Thus two important properties of the gain loci are:

a) Gain loci are symmetrical with respect to the real axis of the complex s-plane.

b) Shape of gain loci depends on the relative position of the poles and zeros, and is independent of the axes of the complex s-plane.

When a real part of s' is zero, Eq. (3.1-1) may be rewritten for a negative and positive values of s' as follows:

$$K\,G(+j\omega) = \frac{K\,P(+j\omega)}{Q(+j\omega)} = e^{-j\phi}\ , \quad s' > 0\ , \text{ and } \quad \sigma = 0 \qquad (3.1\text{-}2)$$

$$K\,G(-j\omega) = \frac{K\,P(-j\omega)}{Q(-j\omega)} = e^{+j\phi}\ , \quad s' < 0\ , \text{ and } \quad \sigma = 0 \qquad (3.1\text{-}3)$$

where

ω is $j\omega$ -axis frequency and ϕ is phase angle.

The product of Eq. (3.1-2) and Eq. (3.1-3) gives the following result:

$$\frac{K^2\,P(+j\omega)\,P(-j\omega)}{Q(+j\omega)\,Q(-j\omega)} = 1 \qquad (3.1\text{-}4)$$

As it is seen from Eq. (3.1-4), P's and Q's exist in complex conjugate pairs. Thus the product $P(+j\omega)\,P(-j\omega)$ and $Q(+j\omega)\,Q(-j\omega)$ are even functions of ω. Therefore, the construction of the gain loci for a constant value of K is reduced to determining the roots of a polynomial which is obtained from Eq. (3.1-4).

The product $P(+j\omega)\ P(-j\omega)$ and $Q(+j\omega)\ Q(-j\omega)$ may be calculated by a general formula: The product of a complex conjugate polynomial pair $P(+j\omega)$ and $P(-j\omega)$ is

$$P(+j\omega)\ P(-j\omega) = \sum_{i=0}^{n} A_i^{\,2}\ \omega^{2(n-i)} +$$

$$+ (-1)^n \sum_{i=0}^{L} \sum_{j=i}^{L} (-1)^{(n-j-i-1)}\, 2A_{2i}A_{2(j+1)}\omega^{2(n-j-i-1)} -$$

$$- (-1)^n \sum_{i=0}^{M} \sum_{j=i}^{M} (-1)^{(n-j-i-2)}\, 2A_{(2i+1)}A_{(2j+3)}\omega^{2(n-j-i-2)} \qquad (3.1\text{-}5)$$

where

$$\left.\begin{aligned} L &= \frac{n}{2} - 1 \\ M &= (L - 1) \end{aligned}\right\} \quad \text{for } n \text{ is an even integer}$$

$$\left.\begin{aligned} L &= \frac{n+1}{2} - 2 \\ M = L &= \frac{n+1}{2} - 2 \end{aligned}\right\} \quad \text{for } n \text{ is an odd integer}$$

A general formula for the denominator polynomial of Eq. (3.1-1) may be obtained by replacing P with Q, A with B and n with m in Eq. (3.1-5).

Substituting the above result in Eq. (3.1-4), one obtains:

$$\frac{\sum_{i=0}^{n} A_i^{\,2}\ \omega^{2(n-i)} + (-1)^n \sum_{i=0}^{L} \sum_{j=i}^{L} (-1)^{(n-j-i-1)}\, 2A_{2i}A_{2(j+1)}\,\omega^{2(n-j-i-1)} - (-}{\sum_{i=0}^{m} A_i^{\,2}\ \omega^{2(m-i)} + (-1)^m \sum_{i=0}^{L'} \sum_{j=i}^{L'} (-1)^{(m-j-i-1)}\, 2A_{2i}A_{2(j+1)}\,\omega^{2(m-j-i-1)} - (-}$$

where

$$\left.\begin{aligned} L &= \frac{n}{2} - 1 \\ M &= (L - 1) \end{aligned}\right\} \quad \text{for } n \text{ is an even integer}$$

$$L = M = \frac{n-1}{2} - 2 \quad \text{for } n \text{ is an odd integer}$$

$$\left.\begin{aligned} L' &= \frac{m}{2} - 1 \\ M' &= (L' - 1) \end{aligned}\right\} \quad \text{for } m \text{ is an even integer}$$

$$L' = M' = \frac{m-1}{2} - 2 \quad \text{for } m \text{ is an odd integer}$$

Solution of Eq. (3.1-6) for a particular K, gives the points of t

$$\frac{\sum_{i=0}^{M}\sum_{j=i}^{M}(-1)^{(n-j-i-2)}2A_{(2i+1)}A_{(2j+3)}\omega^{2(n-j-i-2)}}{\sum_{i=0}^{M'}\sum_{j=i}^{M'}(-1)^{(m-j-i-2)}2A_{(2i+1)}A_{(2j+3)}\omega^{2(m-j-i-2)}} = 1 \qquad (3.1\text{-}6)$$

in loci on the $j\omega$ -axis.

3.2 Determining Gain Loci by Shifting Techniques

If the open loop transfer function in Eq. (3.1-1) is given in a factored form as follows:

$$K\ G(s) = \frac{K\ P(s)}{Q(s)} = \frac{K\prod^{n}\{s' + (\gamma - z_i)\}}{\prod^{m}\{s' + (\gamma - p_i)\}} \tag{3.1-7}$$

where $s = s' + \gamma$, and

z_i are zeros and p_i are poles,

then the phase angle loci equations are:

$$K\ G(+j\omega) = \frac{K\prod^{n}\{+j\omega + (\gamma - z_i)\}}{\prod^{m}\{+j\omega + (\gamma - P_i)\}} = e^{\pm j\phi}\ ,\ s' > 0 \text{ and } \sigma = 0 \tag{3.1-8}$$

$$K\ G(-j\omega) = \frac{K\prod^{n}\{-j\omega + (\gamma - z_i)\}}{\prod^{m}\{-j\omega + (\gamma - P_i)\}} = e^{\mp j\phi}\ ,\ s' < 0 \text{ and } \sigma = 0 \tag{3.1-9}$$

Then the product of Eq. (3.1-8) and Eq. (3.1-9) is

$$K^2\ G(+j\omega)G(-j\omega) = \frac{K^2\prod^{n}\{w^2 + (\gamma - z_i)^2\}}{\prod^{m}\{\omega^2 + (\gamma - P_i)^2\}} = 1 \tag{3.1-10}$$

Similarly, solution of Eq. (3.1-10) for a particular K , gives the points of the gain loci on the $j\omega$ -axis.

Therefore, shifting the $j\omega$ -axis by γ and then, for particular values of K , finding the corresponding ω for each shift, the gain loci can be constructed on the s-plane.

Example No. 3

Gain Loci

The phase angle loci transfer function, as given in Example No. 1, is:

$$K\,G(s) = \frac{K\,(s + 10)}{s(s + 3)(s + 6)} = e^{\pm j\phi} \qquad (3.1\text{-}11)$$

Substituting $(s' + \gamma)$ for s, one finds the following:

$$K\,G(s') = \frac{K\{s' + (\gamma + 10)\}}{(s'+\gamma)\}\{s'+(\gamma+3)\}\{s'+(\gamma+6)\}} = e^{\pm j\phi} \text{ for } s' > 0 \qquad (3.1\text{-}12)$$

$$K\,G(s') = \frac{K\{-s' + (\gamma + 10)\}}{(-s'+\gamma)\{-s'+(\gamma+3)\}\{-s'+(\gamma+6)\}} = e^{\mp j\phi} \text{ for } s' < 0 \qquad (3.1\text{-}13)$$

The product of Eq. (3.1-12) and Eq. (3.1-13) is

$$\frac{K^2\{-s'^2 + (\gamma + 10)^2\}}{(-s'^2+\gamma^2)\{-s'^2+(\gamma+3)^2\}\{-s'^2+(\gamma+6)^2\}} = 1 \qquad (3.1\text{-}14)$$

Substituting $j\omega$ for s', the following result is obtained:

$$\frac{K^2\{\omega^2+(\gamma+10)^2\}}{(\omega^2+\gamma^2)\{\omega^2+(\gamma+3)^2\}\{\omega^2+(\gamma+6)^2\}} = 1 \qquad (3.1\text{-}15)$$

Simplification of Eq. (3.1-15) gives:

$$K^2\{\omega^2 + (\gamma + 10)^2\} - \{(\omega^2 + \gamma^2)\}\{\omega^2 + (\gamma + 3)^2\}\{\omega^2 + (\gamma + 6)^2\} = 0 \qquad (3.1\text{-}16)$$

Solution of Eq. (3.1-16) for a particular value of K gives the points of the gain loci on the $j\omega$-axis for each shift γ.

Thus determining ω for each shift γ, the gain loci are constructed on the s-plane.

CHAPTER IV

ILLUSTRATIVE EXAMPLES

4.1 Problem No. 1

4.1.1 Construction of Phase Angle Loci

The open loop transfer function is:

$$K\,G(s) = \frac{K}{s + a} \tag{4.1-1}$$

and the phase angle loci is:

$$\frac{K}{s + a} = e^{-j\phi} \tag{4.1-2}$$

where ϕ is a positive phase angle.

After shifting the j -axis by γ , s is:

$$s = \bar{s} + \gamma \tag{4.1-3}$$

where $\bar{s} = j\omega$.

Substituting Eq. (4.1-3) in Eq. (4.1-2), one obtains:

$$K = \omega \text{ Sin } \phi + (\gamma + a) \text{ Cos } \phi \tag{4.1-4}$$

$$0 = \delta \text{ Cos } \phi - (\gamma + a) \text{ Sin } \phi \tag{4.1-5}$$

Then the Continued Fraction Expansion is applied as follows:

$$\begin{array}{l|l} & \quad (\text{Sin} \;\; /\text{Cos} \;\;) \\ \hline \omega \text{ Cos } \phi - (\gamma + a) \text{ Sin } \phi & \omega \text{ Sin } \phi + (\gamma + a) \text{ Cos } \phi - K \\ & \underline{\omega \text{ Sin } \phi - (\gamma + a)(\text{Sin}^2 \phi/\text{Cos } \phi)} \\ & \quad 0 + (\gamma+a)\text{Cos } \phi - K + (\gamma+a)(\text{Sin}^2\phi/\text{Cos } \phi) \end{array}$$

When

$$(\gamma - a) \text{ Cos } \phi - K + \frac{\text{Sin}^2 \phi}{\text{Cos } \phi} (\gamma - a) = 0 \qquad (4.1\text{-}6)$$

the Continual Fraction Expansion ends prematurely. Therefore $\{\gamma \text{ Cos } \phi - (\gamma + a) \text{ Sin } \phi\}$ is a common factor of Eq. (4.1-4) and Eq. (4.1-5). Thus one obtains:

$$K = \frac{(\gamma - a)}{\text{Cos } \phi}, \quad \phi \neq (2n - 1)\pi/2 \qquad (4.1\text{-}7)$$

$$\omega = \frac{\text{Sin } \phi}{\text{Cos } \phi} (\gamma + a), \quad \phi \neq (2n + 1)\pi/2 \qquad (4.1\text{-}8)$$

where $n - 0,1,2,3, \ldots$

For $\phi = \pi/2$, $\gamma \neq a$, K and ω are infinite, for $\phi = \pi/2$, $\gamma = a$, K and ω are indefinite; but substituting $\pi/2$ for ϕ in Eq. (4.1-4) and Eq. (4.1-5), one obtains:

$$\omega = K$$

$$(\gamma + a) = 0$$

Therefore the phase angle loci is a vertical line at $(-a)$ for a $\pi/2$ degree phase angle. The family of curves of the phase angle loci is shown in Figure 3.

4.1.2 Construction of Gain Loci

After shifting the $j\omega$ -axis by γ, the phase angle loci equations for $(j\omega)$ and $(-j\omega)$ values of s are given as follows:

$$K\ G(j\omega) = \frac{K}{j\omega + (\gamma + a)} = e^{-j\phi}\ , \text{ for } \ s > 0 \qquad (4.1\text{-}9)$$

$$K\ G(-j\omega) = \frac{K}{-j\omega + (\gamma + a)} = e^{j\phi}\ , \text{ for } \ s < 0 \qquad (4.1\text{-}10)$$

The product of Eq. (4.1-9) and Eq. (4.1-10) is:

$$\frac{K^2}{\omega^2 + (\gamma + a)^2} = 1 \qquad (4.1\text{-}11)$$

Equation (4.1-11) gives

$$\omega^2 + (\gamma + a)^2 = K^2 \qquad (4.1\text{-}12)$$

Thus Eq. (4.1-12) gives a family of circles centered on the real axis at (-a) as seen in Figure 3.

4.2 Problem No. 2

4.2.1 Construction of Phase Angle Loci

The open loop transfer function is

$$K\ G(s) - \frac{K}{s(s + 3)(s + 5)} = e^{-j\phi} \qquad (4.2\text{-}1)$$

In a polynomial form

$$K\ G(s) \doteq \frac{K}{s^3 + 8s^2 + 15s} = \text{Cos}\ \phi - j\ \text{Sin}\ \phi \qquad (4.2\text{-}2)$$

After shifting the $j\omega$ -axis by γ , the phase angle loci equation is:

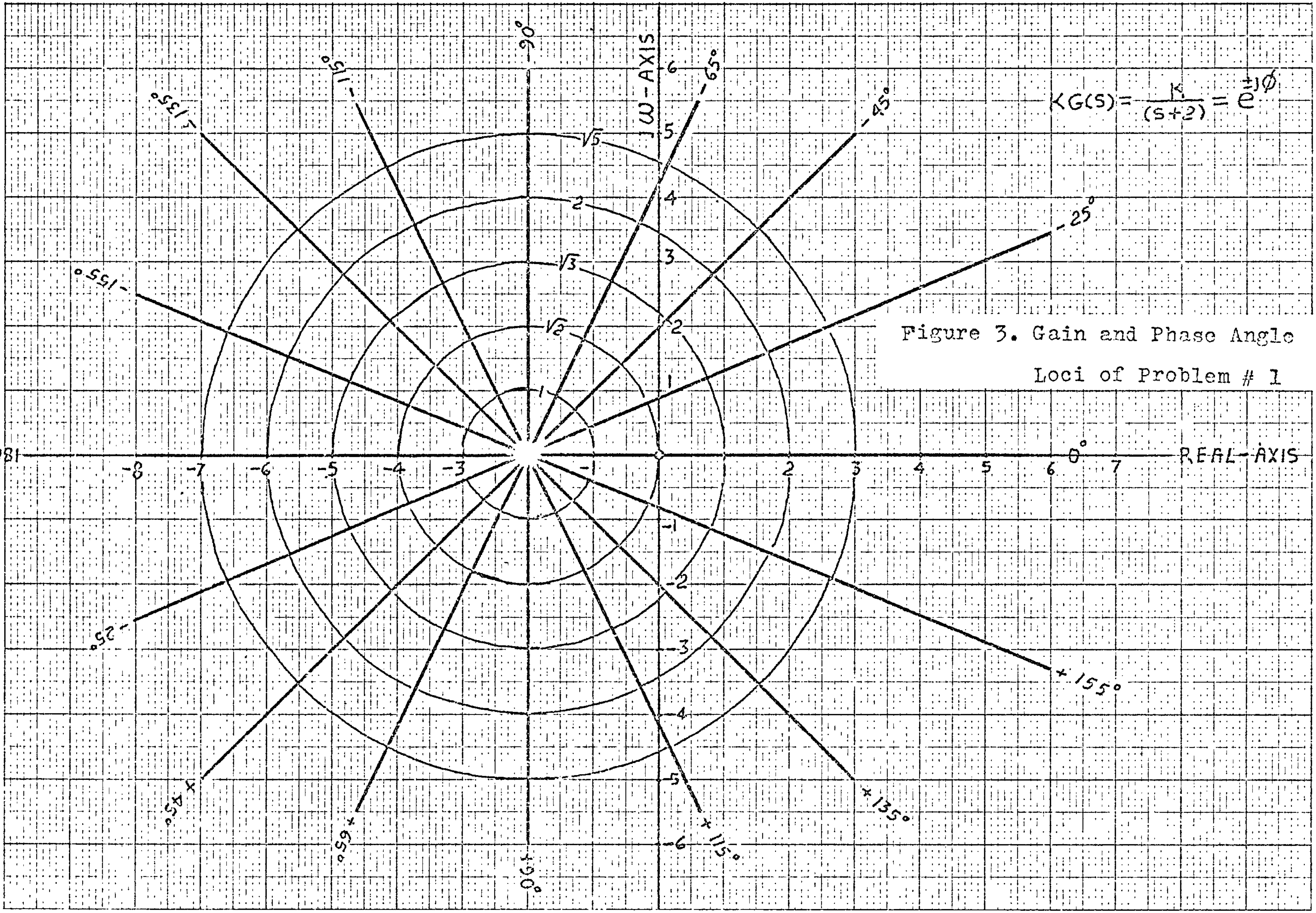


Figure 3. Gain and Phase Angle Loci of Problem # 1

$$\frac{K}{B_o s^3 + B_1 s^2 + B_2 s + B_3} = \text{Cos}\ \phi - j\ \text{Sin}\ \phi \qquad (4.2\text{-}3)$$

Where

$$B_3 = \gamma^3 + 8\gamma^2 + 15\gamma + 0$$

$$B_2 = 3\gamma^2 + 16\gamma + 15$$

$$B_1 = 3\gamma + 8$$

$$B_0 = 1$$

From the real and imaginary part of Eq. (4.2-3), one obtains the following equations:

$$\omega^3 B_o\ \text{Sin}\ \phi + \omega^2 B_1\ \text{Cos}\ \phi - \omega B_2\ \text{Sin}\ \phi - B_3\ \text{Cos}\ \phi + K = 0 \qquad (4.2\text{-}4)$$

$$\omega^3 B_o\ \text{Cos}\ \phi - \omega^2 B_1\ \text{Sin}\ \phi - \omega B_2\ \text{Cos}\ \phi + B_3\ \text{Sin}\ \phi = 0 \qquad (4.2\text{-}5)$$

Applying the Continual Fraction Expansion procedure to Eq. (4.2-4) and Eq. (4.2-5) after substituting a particular value of the phase angle, the oscillatory conditions are obtained.

The calculation fo curves for various phase angles are given on the following pages.

γ	B_o	B_1	B_2	B_3
3	1	17	90	144
2	1	14	59	70
1	1	11	34	24
0	1	8	15	0
-1	1	5	2	-8
-2	1	2	-5	-6
-2.5	1	0.5	-6.25	-3.125
-3	1	-1	-6	0
-3.5	1	-2.5	-1.25	2.625
-4	1	-4	-1	4
-4.5	1	-5.5	-36.75	3.5
-5	1	-7	10	0
-6	1	-10	27	-18
-7	1	-13	50	-71
-8	1	-16	79	-120

$\phi = 90°$ (or phase angle is $-90°$)

$\mathrm{Sin}\ \phi = 1$

$\mathrm{Cos}\ \phi = 0$

Substituting $\mathrm{Sin}\ \phi$ and $\mathrm{Cos}\ \phi$ in Eq. (4.2-4) and Eq. (4.2-5), one obtains

$$-B_1\omega^2 - B_3 = 0$$

$$\omega^3 - B_2\omega + K = 0$$

Applying the procedure, one obtains

$$K^2 = \frac{B_3}{B_1} \quad B_2 - \frac{B_3}{B_1}^{\quad 2}$$

$$\omega = \frac{B_3}{B_1 K} \left(B_2 - \frac{B_3}{B_1}\right)$$

γ	ω	K
0	0	0
1	1.5	47.6
2	2.2	122.6
3	2.9	238.

$\phi = 225°$ (or phase angle $-225°$)

$\cos \phi = - \frac{2}{2}$

$\sin \phi = - \frac{2}{2}$

$$-\omega^3 - \omega^2 B_1 + \omega B_2 + B_3 + \quad 2 K = 0$$

$$-\omega^3 + \omega^2 B_1 + \omega B_2 - B_3 = 0$$

$$L\{2[B_2 + (\frac{2B_3+L}{2B_1})]^2 - B_1 L\} + 4B_3\{B_2 + (\frac{2B_3+L}{2B_1})\}^2 = 0$$

where $L = \quad 2 K$

$$\omega = \frac{-2B_3 \ \{B_2 + \frac{(2B_3+L)}{2B_1}\}}{[2\{B_2 + \frac{(2B_3+L)}{2B_1}\}^2 + LB_1}$$

$\phi = 135°$

$\sin \phi = \frac{2}{2}$

$\cos \phi = - \frac{2}{2}$

$$\omega^3 - \omega^2 B_1 - \omega B_2 + B_3 + \quad 2 K = 0$$

$$-\omega^3 - \omega^2 B_1 + \omega B_2 + B_3 = 0$$

$$L\{[2B_1B_2 + (2B_3 + L)]^2 - 2B_1{}^3L\} + 2B_3\{2B_1B_2+(2B_3 + L)\}^2 = 0$$

where

$$L = 2K$$

$$\omega = \frac{-B_3\{B_2 + \dfrac{(2B_3+L)}{2A_1}\}}{\{2(B_2 + \dfrac{(2B_3 + L)}{2B_1}) - B_1L\}}$$

Root Locus:

$\phi = 180°$ (or phase angle $-180°$)

$\text{Sin}\ \phi = 0$

$\text{Cos}\ \phi = 01$

Substituting Sin ϕ and Cos ϕ in Eq. (4.2-3) and Eq. (4.2-4), one obtains:

$$-\omega^2 B_1 - B_3 - K = 0$$

$$-\omega^3 - \omega B_2 = 0$$

Then the Stieltjes procedure gives:

$$K = B_3 - B_1B_2$$

$$\omega^2 = - B_2$$

On the real axis:

$$K = - B_3$$
$$= 0$$

γ	ω	γ	ω
0	3.87	0	0
1	5.83		
2	7.68	3	0
3	9.48	5	0
-1	1.41		

$\phi = 270°$ (or phase angle $-270°$)

$\text{Cos } \phi = 0$

$\text{Sin } \phi = 01$

Substituting Sin ϕ and Cos ϕ in Eq. (4.2-4) and Eq. (4.2-5), one obtains:

$$-B_o\omega^3 + B_2\omega + K = 0$$

$$B_1\omega^2 - B_3 = 0$$

The the Stieltjes Continual Fraction gives

$$K^2 = \frac{B_3}{B_1} (B_2 - \frac{B_3}{B_1})$$

$$\omega = \frac{B_3}{B_1 K}\left(B_2 - \frac{B_3}{B_1}\right) = K$$

A possible solution is only for = -3. Therefore the curve of the phase angle loci for -270 degrees is a vertical line at (-3).

4.2.2 Construction of Gain Loci

After shifting the $j\omega$ -axis by γ , the phase angle loci equations for $j\omega$ and $-j\omega$ are:

$$K\,G(j\omega) = \frac{K}{(j\omega+\gamma)(J\omega+\gamma+3)(j\omega+\gamma+5)} - e^{-j\phi} \;, \text{ for } \; s > 0 \qquad (4.2\text{-}6)$$

$$K\,G(-j\omega) = \frac{K}{(-j\omega-\gamma)(-j\omega+\gamma+3)(-j\omega+\gamma+5)} - e^{j\phi} \;, \text{ for } \; s < 0 \qquad (4.2\text{-}7)$$

The product of Eq. (4.2-6) and Eq. (4.2-7) is:

$$\frac{K^2}{(\omega^2 + \gamma^2)\{\omega^2 + (\gamma + 3)^2\}\{\omega^2 + (\gamma + 5)^2\}} = 1 \qquad (4.2\text{-}8)$$

Equation (4.2-8) gives a six degree polynomial of ω , and it has only even terms. Thus the construction of the Gain Loci is reduced to finding the roots of this polynomial for each shift.

Gain Loci From Eq. (4.2-1):

K = 100

γ	ω
2	± 1.704
1	± 2.880
0	± 3.528
-1	± 3.924
-2	± 4.138
-3	± 4.186
-4	± 4.066
-5	± 3.757
-6	± 3.200
-7	± 2.210
-7.5	± 1.320

K = 24

γ	ω
0	± 0.129
-1	± 1.397
-2	± 2.064
-3	± 2.184
-4	± 2.210
-5	± 1.720
-6	± 0.795

K = 8

γ	ω
0.5	±0.063
0.3	±0.339
0.	±0.523
-0.3	±0.533
-0.5	±0.491
-0.8	±0.43
-1.	±0.0
-1.5	±0.189
-2	±0.706
-3	±1.098
-3.5	±1.074
-4.	±0.971
-4.5	±0.87
-5.	±0.742
-5.5	±0.294

K = 4

γ	ω
0	±0.265
-0.3	±0.092
-2.5	±0.376
-3.	±0.623
-3.5	±0.448
-4.5	±0.297
-5.	±0.391
-5.3	±0.124
+0.2	±0.134

4.3 Problem No. 3

4.3.1 Construction of Phase Angle Loci

The open loop transfer function is

$$K\ G(s) + \frac{K(s + 3)}{(s + 1)(s + 2)} = e^{-j\phi} \qquad (4.3\text{-}1)$$

After shifting by, the phase angle loci equation is:

$$\frac{K\{s + (\gamma + 3)\}}{\{s + (\gamma + 2)\}\{s + (\gamma + 1)\}} = \text{Cos}\ \phi - j\ \text{Sin}\ \phi \qquad (4.3\text{-}2)$$

From the real imaginary part of Eq. (4.3-2), one obtains the following:

$$\omega^2\text{Cos}\ \phi - \omega(2\gamma - 3)\ \text{Sin}\ \phi - (\gamma+1)(\gamma+2)\ \text{Cos}\ \phi + K(\gamma+3) = 0 \qquad (4.3\text{-}3)$$

$$\omega^2\text{Sin}\ \phi + \{(2\gamma+3)\text{Cos}\ \phi - K\} - (\gamma+1)(\gamma+2)\ \text{Sin}\ \phi = 0 \qquad (4.3\text{-}4)$$

Applying the Continued Fraction Expansion procedure to Eq. (4.3-3) and Eq. (4.3-4) after substituting a particular value of the phase angle, the oscillatory conditions are obtained.

$\phi = 60°$ (or phase angle $-60°$)

$$\omega^2 - 1.74(2\gamma + 3)\omega - (\gamma + 1)(\gamma + 2) + 2K(\gamma + 3) = 0$$

$$\omega^2 + \{0.575(2\gamma + 3) - 1.15K\}\omega - (\gamma + 1)(\gamma + 2) = 0$$

$$\{2.31(2\gamma + 3) - 1.15K\}\omega = +2K(\gamma + 3)$$

$$\omega = \frac{2K(\gamma + 3)}{\{2.315(2\gamma + 3) - 1.15K\}}$$

γ	K	ω
0	1.40	1.45
1	2.6	3.52
2	4.6	4.2
5	10.2	8.8

$\phi = 90°$ (or phase angle $-90°$)

$$\omega^2 - K - (\gamma + 1)(\gamma + 2) = 0$$

$$-\omega(2\gamma + 3) + (\gamma + 3)K = 0$$

The Continued Fraction Expansion procedure gives:

$$A_2 = -K$$

$$A_3 = -(\gamma + 1)(\gamma + 2)$$

$$B_1 = -(2\gamma + 3)$$

$$B_2 = K(\gamma + 3)$$

$$B_2\{A_2B_1 - B_2\} - B_1{}^2A_3 = 0$$

$$\omega = \frac{-A_3B_1}{(A_2B_1 - B_2)}$$

γ	K	ω
0		0
-0.25		3.80
-0.50		1.94
-0.75		0.967
-1.		0.

Root Locus:

$\phi = 180°$ (or phase angle -180°)

$$\omega^2 + (\gamma + 1)(\gamma + 2) - K(\quad - 3) = 0$$

$$- \{(2\gamma + 3) + K\}\ \omega = 0$$

$$K = - (2\gamma + 3)$$

$$\omega^2 = (\gamma + 1)(\gamma + 2) - (2\gamma + 3)(\gamma + 3)$$

On the real axis:

$$\omega = 0$$

$$K = - (\gamma + 1)(\gamma + 2)/(\gamma + 3)$$

γ	K	ω
-1.000		0
-1.586	0.172	0
-2.		0
-3.		0
-3.5		1.32
-2.	1.	1.
-2.5		1.32
-3.	3.	1.414
-4.	5.	1.
-4.414	7.	0.00
		0.00

$\phi = 270°$ (or phase angle -270°)

$$\omega(2\gamma + 3) + K(\gamma - 3) = 0$$

$$-\omega^2 - K\omega + (\gamma - 1)(\gamma - 2) = 0$$

The Continued Fraction Expansion procedure gives:

$$A_2(B_1A_1 + A_2) - B_2A_1^2 = 0$$

$$\omega = \frac{-B_2A_1}{(B_1A_1 + A_2)}$$

where

$A_1 = (2\gamma + 3)$ $\qquad$ $A_2 = K(\gamma + 3)$

$B_1 = - K$ $\qquad$ $B_2 = (\gamma - 1)(\gamma - 2)$

γ	K	ω
-2	0	0
-2.3	0.788	0.345
-2.5	1.55	0.388
-2.8		
-2.8	4.17	0.33
-2.9	6.8	0.244
-3.		0.

4.3.2 Construction of Gain Loci

After shifting the $j\omega$ -axis by, the phase angle loci equations for $j\omega$ and $-j\omega$ are:

$$K\ G(j\omega) = \frac{K\{j\omega + (\gamma + 3)\}}{\{j\omega + (\gamma+1)\}\{j\omega+(\gamma+2)\}} = e^{-j\phi} \quad \text{for} \quad s > 0 \qquad (4.3\text{-}5)$$

$$K\ G(-j\omega) = \frac{K\{-j\omega + (\gamma + 3)\}}{\{-j\omega+(\gamma+1)\}\{-j\omega+(\gamma+2)\}} = e^{j\phi} \quad \text{for} \quad s < 0 \qquad (4.3\text{-}6)$$

The product of Eq. (4.3-5) and Eq. (4.3-6) gives:

$$K^2\{\omega^2 + (\gamma+3)^2\} = \{\omega^2 + (\gamma+1)^2\}\{\omega^2 + (\gamma+2)^2\} \qquad (4.3\text{-}7)$$

K = 0.5		K = 1.0	
γ	ω	γ	ω
-0.8	0.105	0.4	0.19
-0.9	0.76	0.2	0.74
-1.	0.83	0.	1.
-1.3	0.815	-0.5	1.36
-1.5	0.77	-1.	1.414
-1.8	0.704	-1.5	1.32
-2.	0.5	-2.	1.
-2.1	0.406	-2.1	0.89
-2.2	0.243	-2.2	0.74
-2.28	0.	-2.415	0.

$\underline{K = 2}$

γ	ω
1.56	$\pm 0.$
1.	± 1.565
0.	± 2.26
-0.5	± 2.4
-1.	± 2.39
-2.	$\pm 2.$
-2.4	± 2.36
-2.5	± 1.36
-2.56	$\pm 0.$

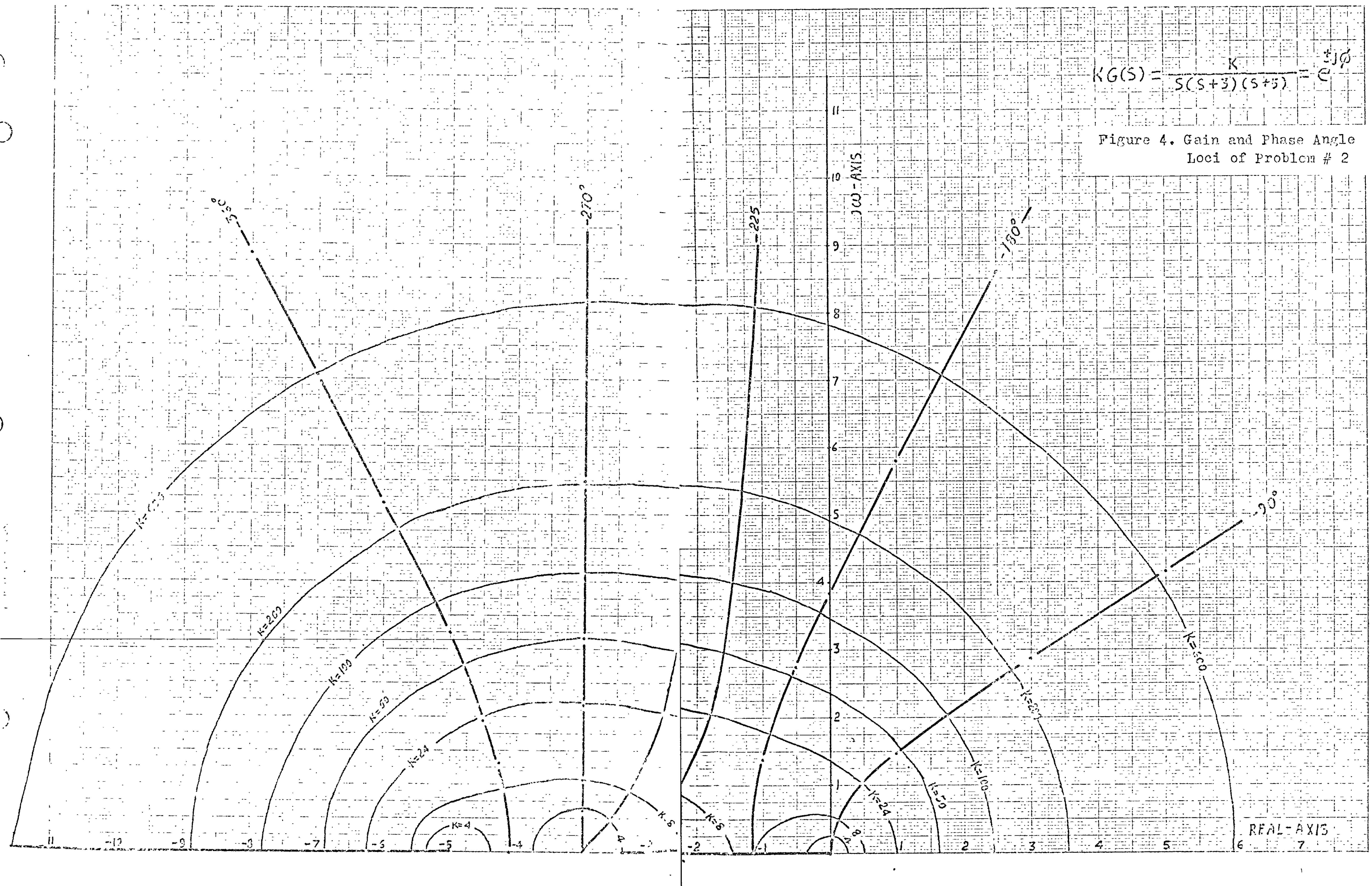


Figure 4. Gain and Phase Angle Loci of Problem # 2

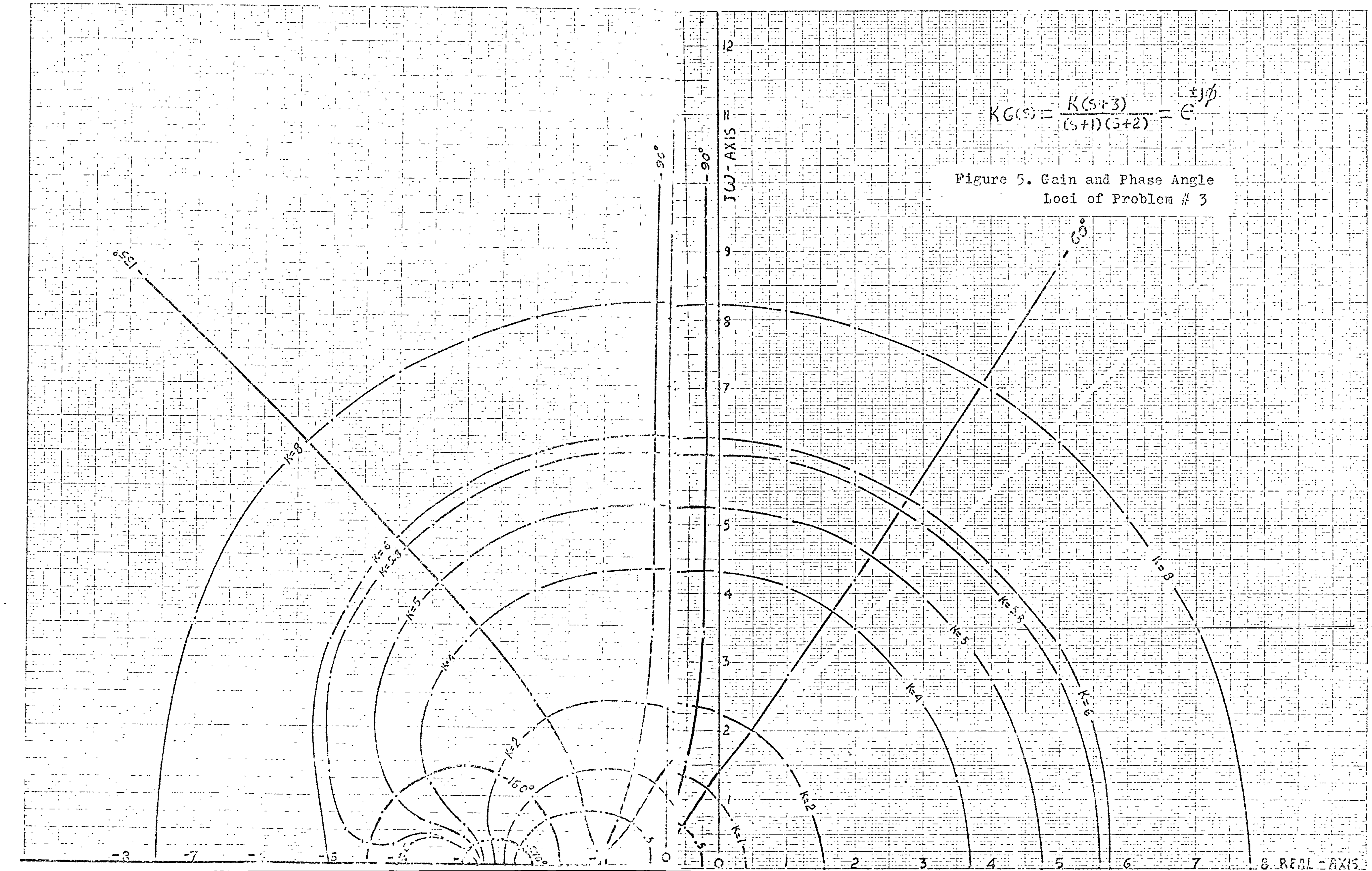


Figure 5. Gain and Phase Angle Loci of Problem # 3

CHAPTER V

CONCLUSION

The formulas derived for the phase angle loci oscillatory condition in Chapter II, Section 2.2.1 and Section 2.2.2, and for the gain loci in Chapter III, Section 3.1, give easy and exact solutions for the $j\omega$ -axis angular frequencies and gains for the phase angle loci, and the $j\omega$ -axis angular frequencies for the gain loci. These formulas are easily applicable to a digital computer. Thus the shifting technique with these formulas provides an easy and exact way for constructing the phase angle loci and the gain loci of a system.

BIBLIOGRAPHY

1. Chu, Yaohan. "Synthesis of Feedback Control System by Phase Angle Loci," Trans. AIEE, Part II, Application and Industry, Vol. 71, 1952, pp. 330-339.

2. Puri, N. N. Linear Control System Design Via Phase Locus Technique. Conference Proceedings, November, 1967.

3. Chen, C. F. and C. Hsu. "Determination of Root Loci Using Routh's Algorithm," Journal of Franklin Institute, Vol. 281, No. 2, February, 1966.

4. Shen, D. W. and C. F. Chen. "Routh's Method for Approximating Oscillatory Modes in Linear Systems," IEEE Inter. Conv. Rec., Part 2, March, 1963, pp. 10-15.

5. Truxal, John G. Control System Synthesis. New York: McGraw Hill, 1955.

6. D'Azzo and Houpis. Control System Analysis and Synthesis. New York: McGraw Hill, 1960.

7. Bendicker, G. A. and K. E. Teodorikik. "The Analytical Theory of the Determination of Root Trajectories," Automatika i Telemechanika, Vol. 20, No. 3, 1959.

8. Van Valkenburg, M. E. Modern Network Synthesis. New York: John Willey and Sons, Inc., 1960.

9. Krishnan, Venkatarama. "Semi-Analytic Approach to Root Locus," IEEE Trans. on Automatic Control, January, 1966.

APPENDIX A

The Continued Fraction Expansion for 5 degree polynomials:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}\omega^5 + D_{12}\omega^4 + D_{13}\omega^3 + D_{14}\omega^2 + D_{15}\omega + D_{16}}{D_{21}\omega^5 + D_{22}\omega^4 + D_{23}\omega^3 + D_{24}\omega^2 + D_{25}\omega + D_{26}} =$$

$$= \frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{21}}{C_{31}}\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\omega + \frac{M_{61}}{C_{51}} + \cfrac{1}{\frac{C_{51}}{C_{71}}\omega + \frac{M_{81}\omega^2 + M_{82}\omega + M_{83}}{C_{71}\omega^2 + C_{72}\omega + C_{73}}}}}$$

The last fraction, $\frac{M_{8,1}\omega^2 - M_{8,2}\omega - M_{8,3}}{C_{7,1}\omega^2 + C_{7,2}\omega + C_{7,3}}$, provides the oscillatory condition for ϕ, $(0,1,2,3, \ldots,)$ when it ends prematurely.

For the phase angles ϕ the Continued Fraction Expansion is carried on until the following fraction is obtained:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{31}}{C_{31}}\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\omega + \frac{M_{61}}{C_{51}} + \cfrac{1}{\frac{C_{51}}{C_{71}}\omega + \frac{M_{81}}{C_{71}} + \cfrac{1}{\frac{C_{71}}{C_{91}}\omega + \frac{M_{10,1}}{C_{91}} + \frac{C_{91}\omega + C_{92}}{C_{11,1}}}}}}$$

The last fraction, $\frac{C_{9,1}\omega - C_{9,2}}{C_{11,1}}$, provides the oscillatory conditions for the phase angle loci for ϕ when it ends prematurely.

$$
\begin{array}{r|l}
 & \dfrac{D_{11}}{D_{21}} \\
\hline
D_{21}\omega^5 + D_{22}\omega^4 + D_{23}\omega^3 + D_{24}\omega^2 + D_{25}\omega + D_{26} & D_{11}\omega^5 + D_{12}\omega^4 + D_{13}\omega^3 + D_{14}\omega^2 + D_{15}\omega + D_{16} \\
 & D_{11}\omega^5 + D_{22}\dfrac{D_{11}}{D_{21}}\omega^4 + D_{23}\dfrac{D_{11}}{D_{21}}\omega^3 + D_{24}\dfrac{D_{11}}{D_{21}}\omega^2 + D_{25}\dfrac{D_{11}}{D_{21}} + D_{26}\dfrac{D_{11}}{D_{21}} \\
\hline
\end{array}
$$

$$
(D_{12}-D_{22}\frac{D_{11}}{D_{21}})\omega^4+(D_{13}-D_{23}\frac{D_{11}}{D_{21}})\omega^3+(D_{14}-D_{24}\frac{D_{11}}{D_{21}})\omega^2+(D_{15}-D_{25}\frac{D_{11}}{D_{21}})\omega+(D_{16}-D_{26}\frac{D_{11}}{D_{21}})
$$

$$
\begin{array}{r|l}
 & \dfrac{D_{21}}{D_{31}}\,\omega \\
\hline
D_{31}\omega^4 + D_{32}\omega^3 + D_{33}\omega^2 + D_{34}\omega + D_{35} & D_{21}\omega^5 + D_{22}\omega^4 + D_{23}\omega^3 + D_{24}\omega^2 + D_{25}\omega + D_{26} \\
 & D_{21}\omega^5 + D_{32}\dfrac{D_{21}}{D_{31}}\omega^4 + D_{33}\dfrac{D_{21}}{D_{31}}\omega^3 + D_{34}\dfrac{D_{21}}{D_{31}}\omega^2 + D_{35}\dfrac{D_{21}}{D_{31}}\omega \\
\hline
\end{array}
$$

$$
(D_{22} - D_{32}\frac{D_{21}}{D_{31}})\omega^4 + (D_{23} - D_{33}\frac{D_{21}}{D_{31}})\omega^3 + D_{24} - D_{34}\frac{D_{21}}{D_{31}})\omega^2 + (D_{25} - D_{35}\frac{D_{21}}{D_{31}})\omega + D_{26}
$$

$$
\begin{array}{r|l}
 & \dfrac{M_{41}}{D_{31}} \\ \hline
D_{31}\omega^4 + D_{32}\omega^3 + D_{33}\omega^2 + D_{34}\omega + D_{35} & M_{41}\omega^4 + M_{42}\omega^3 + M_{43}\omega^2 + M_{44}\omega + M_{45} \\
 & M_{41}\omega^4 + D_{32}\dfrac{M_{41}}{D_{31}}\omega^3 + D_{33}\dfrac{M_{41}}{D_{31}}\omega^2 + D_{34}\dfrac{M_{41}}{D_{31}}\omega + D_{35}\dfrac{M_{41}}{D_{31}} \\ \hline
 & (M_{42} - D_{32}\dfrac{M_{41}}{D_{31}})\omega^3 + (M_{43} - D_{33}\dfrac{M_{41}}{D_{31}})\omega^2 + (M_{44} - D_{34}\dfrac{M_{41}}{D_{31}})\omega + (M_{45} - D_{35}\dfrac{M_{41}}{D_{31}})
\end{array}
$$

$$
\begin{array}{r|l}
 & \dfrac{D_{31}}{C_{51}}\omega \\ \hline
C_{51}\omega^3 + C_{52}\omega^2 + C_{53}\omega + C_{54} & D_{31}\omega^4 + D_{32}\omega^3 + D_{33}\omega^2 + D_{34}\omega + D_{35} \\
 & D_{31}\omega^4 + C_{52}\dfrac{D_{31}}{C_{51}}\omega^3 + C_{53}\dfrac{D_{31}}{C_{51}}\omega^2 + C_{54}\dfrac{D_{31}}{C_{51}}\omega \\ \hline
 & (D_{32} - C_{52}\dfrac{D_{31}}{C_{51}})\omega^3 + (D_{33} - C_{53}\dfrac{D_{31}}{C_{51}})\omega^2 + (D_{34} - C_{54}\dfrac{D_{31}}{C_{51}})\omega + (D_{35} - C_{54}\dfrac{D_{31}}{C_{51}})
\end{array}
$$

$$
\begin{array}{r|l}
 & \dfrac{M_{61}}{C_{51}} \\
\hline
C_{51}\omega^3 + C_{52}\omega^2 + C_{53}\omega + C_{54} & M_{61}\omega^3 + M_{62}\omega^2 + M_{63}\omega + M_{64} \\
 & M_{61}\omega^3 + C_{52}\dfrac{M_{61}}{C_{51}}\omega^2 + C_{53}\dfrac{M_{61}}{C_{51}}\omega + C_{54}\dfrac{M_{61}}{C_{51}} \\
\hline
 & \left(M_{62} - C_{52}\dfrac{M_{61}}{C_{51}}\right)\omega^2 + \left(M_{63} - C_{53}\dfrac{M_{61}}{C_{51}}\right)\omega + \left(M_{64} - C_{54}\dfrac{M_{61}}{C_{51}}\right)
\end{array}
$$

$$
\begin{array}{r|l}
 & \dfrac{C_{51}}{C_{71}}\omega \\
\hline
C_{71}\omega^2 + C_{72}\omega + C_{73} & C_{51}\omega^3 + C_{52}\omega^2 + C_{53}\omega + C_{54} \\
 & C_{51}\omega^3 + C_{72}\dfrac{C_{51}}{C_{71}}\omega^2 + C_{73}\dfrac{C_{51}}{C_{71}}\omega \\
\hline
 & \left(C_{52} - C_{72}\dfrac{C_{51}}{C_{71}}\right)\omega^2 + \left(C_{53} - C_{73}\dfrac{C_{51}}{C_{71}}\right)\omega + C_{54}
\end{array}
$$

$$
\begin{array}{r|l}
 & \dfrac{M_{81}}{C_{71}} \\
\hline
C_{71}\omega^2 + C_{72}\omega + C_{73} & M_{81}\omega^2 + M_{82}\omega + M_{83} \\
 & C_{81}\omega^2 + C_{72}\dfrac{C_{81}}{C_{71}}\omega + C_{73}\dfrac{C_{81}}{C_{71}} \\
\hline
 & \left(M_{82} - C_{72}\dfrac{M_{81}}{C_{71}}\right)\omega + \left(M_{83} - C_{73}\dfrac{M_{81}}{C_{71}}\right)
\end{array}
$$

$$\begin{array}{r|l}
 & \frac{C_{71}}{C_{91}}\,\omega \\ \cline{2-2}
C_{91}\omega + C_{92} & C_{71}\omega^{2} + C_{72}\omega + C_{73} \\
 & C_{71}\omega^{2} + C_{92}\,\frac{C_{71}}{C_{91}}\,\omega \\ \cline{2-2}
 & (C_{72} - C_{92}\,\frac{C_{71}}{C_{91}})\,\omega + C_{73}
\end{array}$$

$$\begin{array}{r|l}
 & \frac{M_{10,1}}{C_{91}} \\ \cline{2-2}
C_{91}\omega + C_{92} & M_{10,1}\omega + M_{10,2} \\
 & M_{10,1}\omega + C_{92}\,\frac{M_{10,1}}{C_{91}} \\ \cline{2-2}
 & (M_{10,2} - C_{92}\,\frac{M_{10,1}}{C_{91}})
\end{array}$$

APPENDIX B

The Continued Fraction Expansion for a 4 degree polynomial is:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}\omega^4 + D_{12}\omega^3 + D_{13}\omega^2 + D_{14}\omega + D_{15}}{D_{21}\omega^4 + D_{22}\omega^3 + D_{23}\omega^2 + D_{24}\omega + D_{25}}$$

$$= \frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{21}}{C_{31}}\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\omega + \frac{M_{61}\omega^2 + M_{62}\omega + M_{63}}{C_{51}\omega^2 + C_{52}\omega + C_{53}}}}$$

The last fraction, $\frac{M_{61}\omega^2 + M_{62}\omega + M_{63}}{C_{51}\omega^2 + C_{52}\omega + C_{53}}$, provides the oscillatory condition for $\phi = \nu\pi$, $(\nu = 0,1,2 \ldots)$ when it ends prematurely.

For the phase angle $\phi \neq \nu\pi$, the Continued Fraction Expansion is carried on until the following fraction is obtained:

$$\frac{F_1(\omega)}{F_2(\omega)} = \frac{D_{11}}{D_{21}} + \cfrac{1}{\frac{D_{21}}{C_{31}}\omega + \frac{M_{41}}{C_{31}} + \cfrac{1}{\frac{C_{31}}{C_{51}}\omega + \frac{M_{61}}{C_{51}} + \cfrac{1}{\frac{C_{51}}{C_{71}}\omega + \frac{M_{81}}{C_{71}} + \frac{C_{71}\omega + C_{72}}{C_{91}}}}}$$

For $\phi = \nu\pi$, $(\nu = 0,1,2,3, \ldots)$:

$$D_{13} = D_{15} = D_{22} = D_{24} = 0$$

When M_{52} is zero, the Continued Fraction Expansion ends prematurely and gives the following results:

$$\left.\begin{aligned} \omega^2 &= -\frac{C_{53}}{C_{51}} \\ M_{62} &= 0 \end{aligned}\right\} \quad \phi = \nu\pi , \ (\nu + 0,1,2,3, \ldots)$$

For $\phi \neq \nu_n$:

When C_{91} is zero, the Continued Fraction Expansion ends prematurely and gives the following results:

$$\left.\begin{aligned} \omega &= -\frac{C_{72}}{C_{71}} \\ C_{91} &= 0 \end{aligned}\right\} \quad \phi \neq \nu\pi , \ (\nu = 0,1,2,3, \ldots)$$

$$\frac{D_{11}}{D_{21}}$$

$$D_{21}\omega^4 + D_{22}\omega^3 + D_{23}\omega^2 + D_{24}\omega + D_{25} \;\Big|\; D_{11}\omega^4 + D_{12}\omega^3 + D_{13}\omega^2 + D_{14}\omega + D_{15}$$

$$D_{11}\omega^4 + D_{22}\frac{D_{11}}{D_{21}}\omega^3 + D_{23}\frac{D_{11}}{D_{21}}\omega^2 + D_{24}\frac{D_{11}}{D_{12}}\omega + D_{25}\frac{D_{11}}{D_{12}}$$

$$(D_{12} - D_{22}\frac{D_{11}}{D_{21}})\omega^3 + (D_{13} - D_{23}\frac{D_{11}}{D_{21}})\omega^2 + (D_{14} - D_{24}\frac{D_{11}}{D_{21}})\omega + (D_{15} - D_{25}\frac{D_{11}}{D_{21}})$$

$$\frac{D_{21}}{C_{31}}\omega$$

$$C_{31}\omega^3 + C_{32}\omega^2 + C_{33}\omega + C_{34} \;\Big|\; D_{21}\omega^4 + D_{22}\omega^3 + D_{23}\omega^2 + D_{24}\omega + D_{25}$$

$$D_{21}\omega^4 + C_{32}\frac{D_{21}}{C_{31}}\omega^3 + C_{33}\frac{D_{21}}{C_{31}}\omega + C_{34}\frac{D_{21}}{C_{31}}$$

$$(D_{22} - C_{32}\frac{D_{21}}{C_{31}})\omega^3 + (D_{23} - C_{33}\frac{D_{21}}{C_{31}})\omega^2 + (D_{24} - C_{34}\frac{D_{21}}{C_{31}})\omega + D_{25}$$

$$\frac{M_{41}}{C_{31}}$$

$$C_{31}\omega^3 + C_{32}\omega^2 + C_{33}\omega + C_{34} \;\Big|\; M_{41}\omega^3 + M_{42}\omega^2 + M_{43}\omega + M_{44}$$

$$M_{41}\omega^3 + C_{32}\frac{M_{41}}{C_{31}}\omega^2 + C_{33}\frac{M_{41}}{C_{31}}\omega + C_{34}\frac{M_{41}}{C_{31}}$$

$$(M_{42} - C_{32}\frac{M_{41}}{C_{31}})\omega^2 + (M_{43} - C_{33}\frac{M_{41}}{C_{31}})\omega + (M_{44} - C_{34}\frac{M_{41}}{C_{31}})$$

$$
\begin{array}{r|l}
 & \frac{C_{31}}{C_{51}}\omega \\
\cline{2-2}
C_{51}\omega^2 + C_{52}\omega + C_{53} & C_{31}\omega^3 + C_{32}\omega^2 + C_{33}\omega + C_{34} \\
 & C_{31}\omega^3 + C_{52}\frac{C_{31}}{C_{51}}\omega^2 + C_{53}\frac{C_{31}}{C_{51}}\omega \\
\cline{2-2}
 & (C_{32} - C_{52}\frac{C_{31}}{C_{51}})\omega^2 + (C_{33} - C_{53}\frac{C_{31}}{C_{51}})\omega + C_{34}
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{M_{61}}{C_{51}} \\
\cline{2-2}
C_{51}\omega^2 + C_{52}\omega + C_{53} & M_{61}\omega^2 + M_{62}\omega + M_{63} \\
 & M_{61}\omega^2 + C_{52}\frac{M_{61}}{C_{51}}\omega + C_{53}\frac{M_{61}}{C_{51}} \\
\cline{2-2}
 & (M_{62} - C_{52}\frac{M_{61}}{C_{51}})\omega + (M_{63} - C_{53}\frac{M_{61}}{C_{51}})
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{C_{51}}{C_{71}}\omega \\
\cline{2-2}
C_{71}\omega + C_{72} & C_{51}\omega^2 + C_{52}\omega + C_{53} \\
 & C_{51}\omega^2 + C_{72}\frac{C_{51}}{C_{71}}\omega \\
\cline{2-2}
 & (C_{52} - C_{72}\frac{C_{51}}{C_{71}})\omega + C_{53}
\end{array}
$$

$$
\begin{array}{r|l}
 & \dfrac{M_{81}}{C_{71}} \\
\hline
C_{71}\omega + C_{72} & M_{81}\omega + M_{82} \\
 & M_{81}\omega + C_{72}\,\dfrac{M_{81}}{C_{71}} \\
\cline{2-2}
 & \left(M_{82} - C_{72}\,\dfrac{M_{82}}{C_{71}}\right)
\end{array}
$$

When C_{91} is forced to be zero, one obtains $C_{71}\omega + C_{72} = 0$ for a premature ending.

Thus:

$$\omega = -\frac{C_{7,2}}{C_{7,1}}$$

$$C_{91} = 0$$

for $\phi \neq \nu\pi$ $\quad (\nu = 0,1,2,3, \ldots)$

For $\phi = \nu\pi$ (Root Locus)

$$D_{22} = D_{24} = D_{11} = D_{13} = D_{15} = 0$$

Then M_{62} is forced to be zero, for expansion to end prematurely. Thus:

$$C_{51}\omega^2 + C_{53} = 0$$

$$M_{62} = 0 \qquad \phi = \nu\pi \qquad (\nu = 0,1,2,3, \ldots)$$

$$\omega^2 = -\frac{C_{5,3}}{C_{5,1}} \qquad \text{For} \quad \phi = \nu\pi \quad (\nu = 0,1,2,3,\ldots$$

$$M_{62} = 0$$